Theory and Application of Optimal Estimation

Lecture 2: Dynamics Modeling With Application to Target Tracking

Linfeng Xu
e-mail: xulinf@gmail.com

School of Automation
Northwestern Polytechnical University
Xi'an 710072, China

For more details, please refer to

Y. Bar-Shalom, X.R. Li, T. Kirubarajan. Estimation with Applications To Tracking and Navigation, John Wiley Inc., 2001.

Fall, 2019

Outline

Outline

Introduction

- Target tracking is basically to use the state estimation tools in realistic environments.
- State estimation algorithms are developed theoretically under strict assumptions, in a real environment those assumptions are commonly violated.
- Some real environment aspects to consider are
  - Missing detections
  - Association uncertainty
  - Model mismatch (maneuvers)
- A combination of control theory and signal processing

Introduction

- Has widespread military and civilian applications:
  - Defense surveillance (radar)
  - Air traffic surveillance (radar)
  - Video surveillance
  - Robotics (video, laser)
- This problem has been studied for quite a long time.
- Estimation algorithms known as Kalman filters are used as the main solutions.
- Still a very active and expanding research area.

Problem (Target Tracking)

Consider the system

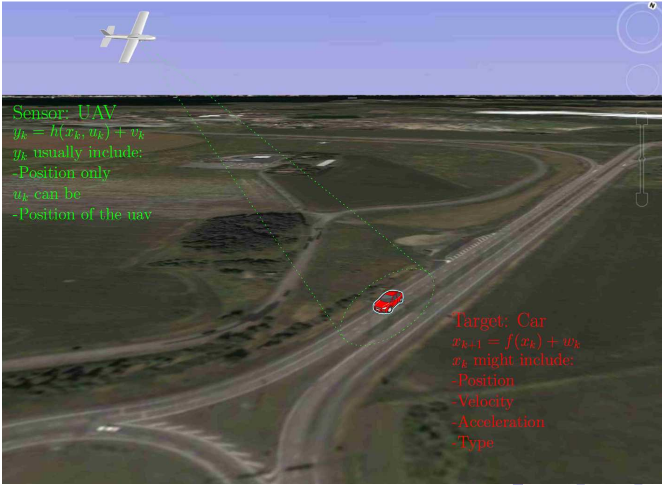
$$x_{k+1} = f_k(x_k, u_k) + w_k$$
$$z_k = h_k(x_k) + v_k$$

where $x_0 \sim p(x_0)$.

Aim: *Find*

$$\hat{x}_{k|k} = E[x_k|Z^k]$$
$$P_{k|k} = \mathrm{cov}[x_k|Z^k]$$

- **Target tracking**
  Incorporates all significant aspects of estimation problems that appear in other applications.



Problem (Target Tracking)

Consider the system

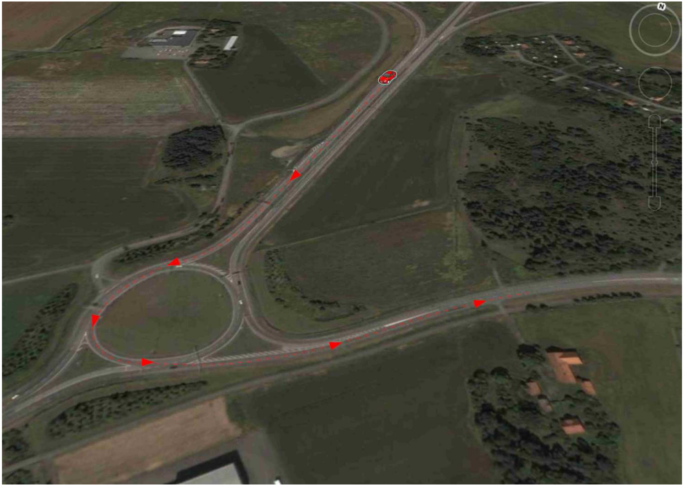
$$x_{k+1} = f_k(x_k, u_k) + w_k \quad \textit{(maneuver)}$$
$$z_k = h_k(x_k) + v_k \quad \textit{(multitargets, extended targets, clutter, false alarms)}$$

where $x_0 \sim p(x_0)$ *(initialization).*
Aim: Find

$$\hat{x}_{k|k} = E[x_k | Z^k] \quad \textit{(multisensor fusion)}$$
$$P_{k|k} = \operatorname{cov}[x_k | Z^k]$$

Introduction

- Systems have a variety of modes
- Maneuvers $x_{k+1} = {\color{red} f_k}(x_k, u_k) + w_k$



Introduction

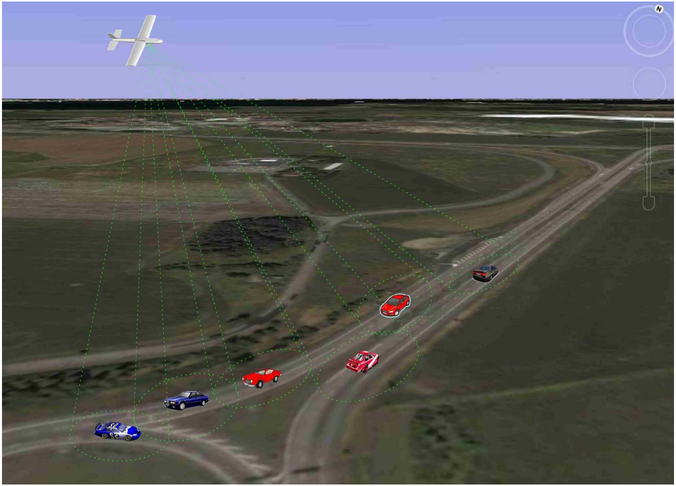
- Multiple model system description

$$x_{k+1} = f^{r_k}(x_k) + w_k$$

  where $r_k \in \{1, 2, ..., M\}$ is the mode variable which determines the target model by selecting among $\{f^1, f^2, ..., f^M\}$.
- Early approaches make decision+tracking (< 1990).
- State of the art: Use all models at the same time (> 1990)

Introduction

- Data origin uncertainty $z_k = h_k(x_k) + v_k$



Introduction

- Define $x_k = [x_k^1, x_k^2, ..., x_k^N]^\top$
- Define $y_k = [y_k^{\sigma(1)}, y_k^{\sigma(2)}, ..., y_k^{\sigma(N)}]^\top$
- Data Association + Single object estimation
- Computation: $N!\times$Single object estimation!
- There are also false alarms and missed detections
- State of the art $(2000 \rightarrow)$:
  - Try to bypass data association
  - Use new modeling methodologies: Random sets, PMHT

- Multiple-Sensor information fusion
  - Sensor 1

$$y_k^1 = h^1(x_k) + v_k^1$$
$$\Rightarrow \hat{x}_{k|k}^1, P_{k|k}^1$$

  - Sensor 2

$$y_k^2 = h^2(x_k) + v_k^2$$
$$\Rightarrow \hat{x}_{k|k}^2, P_{k|k}^2$$

  - Fusion center

$$\begin{aligned}(P_{k|k})^{-1} &= (P_{k|k}^1)^{-1} + (P_{k|k}^2)^{-1} \\ \hat{x}_{k|k} &= P_{k|k}(P_{k|k}^1)^{-1}\hat{x}_{k|k}^1 + P_{k|k}(P_{k|k}^2)^{-1}\hat{x}_{k|k}^2\end{aligned}$$

Introduction

- Target tracking is a vast field and no single course can cover all of its aspects. Below are some limitations on the content of this course.
- We mainly consider radar sensors to be of interest. The issues related to sensors like video or laser are mostly not covered.
- We are mostly constrained to **point** targets.
- Even with the radar sensors, issues like
  - Bias
  - Registration
  - Attribute (Target-Type) Estimation

  are not covered.

Introduction to Dynamic modeling

- For the successful tracking of a moving target it is essential to extract the maximum useful information about the target state from the available observations.
- A good model is worth a thousand pieces of data. This statement has an even stronger positive connotation in target tracking where observation data are rather limited.
- Most tracking algorithms are model based because knowledge of target motion is available and a good model-based tracking algorithm will greatly outperform any model-free tracking algorithm if the underlying model turns out to be a good one.

Introduction to Dynamic modeling

- Target dynamic models and tracking algorithms have intimate ties. The applicability of a target dynamic model for a practical problem can hardly be evaluated without referring to the corresponding tracking algorithms used. Some target models and tracking algorithms work well jointly.
- Understand not only how these models work but also their pros and cons.
- Reveals well the interrelationships among various models.

Outline

Mathematical Models

- Target tracking is to estimate the state trajectories of a target—a moving or movable object.
- Although a target is almost never really a point in space and the information about its orientation is valuable for tracking, a target is usually treated as a point object without a shape in tracking, especially in target dynamic models.
- A target dynamic/motion model describes the evolution of the target state with respect to time.
- Almost all maneuvering target tracking methods are model based. They assume that the target motion and its observations can be represented by some known mathematical models sufficiently accurately.

Mathematical Models

- The most commonly used such models are those known as state-space models,

$$x_{k+1} = f_k(x_k, u_k) + w_k$$
$$z_k = h_k(x_k) + v_k$$

where x_k, z_k, and u_k are the target state, observation, and control input vectors, respectively, at the discrete time t_k; w_k and v_k are process and measurement noise sequences, respectively; and f_k and h_k are some vector-valued (possibly time-varying) functions.

Mathematical Models

- Such a discrete-time model is often obtained by discretizing (sampling) the following continuous-time model

$$
\begin{aligned}
\dot{x} &= f(x(t), u(t), t) + w(t), \qquad x(t_0) = x_0 \\
z(t) &= h(x(t), t) + v(t)
\end{aligned}
$$

  where $x_k = x(t_k)$ and it is usually assumed that $z_k = z(t_k)$, $v_k = v(t_k)$, $h_k(x_k) = h(x(t_k), t_k)$.
- The control input is often assumed (approximately) piecewise constant with $u_k = u(t)$, $t_k \leq t < t_{k+1}$ when discretizing a continuous-time system. The control input u is usually not known.
- Note that

$$
w_k \neq w(t_k), \qquad f_k(x_k, u_k, w_k) \neq f(x(t_k), u(t_k), w(t_k), t_k)
$$

Mathematical Models

- In fact, it is often more appropriate to use the following mixed-time models for most tracking problems

$$\dot{x} = f(x(t), u(t), t) + w(t), \qquad x(t_0) = x_0$$
$$z_k = h_k(x_k) + v_k$$

  because while observations are usually available only at discrete time instants, the target motion is more accurately modeled in continuous time.
- For example, target motions should not depend on how and when samples are taken, which is often the case, however, for a discrete-time model.
- For a similar reason, a discrete-time equivalent model is usually more systematic and consistent, and is in many cases probably preferable to the corresponding direct discrete-time counterpart.

Mathematical Models

The continuous-, discrete-, and mixed-time linear counterparts of the above models are the corresponding pairs of the following equations

$$\begin{aligned} x_{k+1} &= F_k x_k + E_k u_k + G_k w_k \\ \dot{x} &= f(x(t), u(t), t) + w(t), \qquad x(t_0) = x_0 \\ z_k &= H_k x_k + v_k \\ z(t) &= C(t)x(t) + v(t) \end{aligned}$$

Mathematical Models

- Target motion model

$$x_{k+1} = F_k x_k + E_k u_k + G_k w_k$$
$$\dot{x} = f(x(t), u(t), t) + w(t), \qquad x(t_0) = x_0$$

  One of the major challenges for target tracking arises from the target motion uncertainty.
- This uncertainty refers to the fact that an accurate dynamic model of the target being tracked is not available to the tracker. Specifically, although the general form of the model is usually adequate, a tracker lacks knowledge about the actual control input u of the target, and possibly the actual form of f, its parameters, or statistical properties of the noise w for the particular target being tracked.
- Target motion modeling is thus one of the first tasks for maneuvering target tracking. It aims at developing a tractable model that accounts well for the effect of target motion.

Mathematical Models

- Modeling the target motion for tracking a maneuvering target without knowing its true dynamic behavior. Most of these efforts have been made along two lines: 1) approximate the actually nonrandom control input u as a random process of certain properties, and 2) describe typical target trajectories by some representative motion models with properly designed parameters.
- Target motions are normally classified into two classes: maneuver and nonmaneuver.
- A nonmaneuvering motion is the straight and level motion at a constant velocity in an inertial reference system.
- All other motions belong to the maneuvering mode.

Outline

Nonmaneuver Models

- A point moving in our 3D physical world can be described by its 3D position and velocity vectors. For instance, $x = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{z}, \dot{\mathsf{z}}]$ can be used as a state vector of such a point in the Cartesian coordinate system, where $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ are the position coordinates along x, y, and z axes, respectively, and $(\dot{\mathsf{x}}, \dot{\mathsf{y}}, \dot{\mathsf{z}})$ is the velocity vector.
- When a target is treated as a point object, the nonmaneuvering motion is thus described by the vector-valued equation $\dot{x}(t) = 0$, where $x = [\dot{\mathsf{x}}, \dot{\mathsf{y}}, \mathsf{z}]$.
- Note that z direction is treated differently because a nonmaneuvering motion is assumed in the horizontal x–y plane. In practice, this ideal equation is usually modified as

$$\dot{x}(t) = w(t) \approx 0$$

where $w(t)$ is white noise with a "small" effect on x that accounts for unpredictable modeling errors due to turbulence, etc.

- The corresponding state-space model is given by, with state vector $x = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{z}]$,

$$\dot{x}(t) = \text{diag}[A_{cv}, 0]x(t) + \text{diag}[B_{cv}, 1]w(t)$$

where $w(t) = [w_x(t), w_y(t), w_z(t)]$ is a continuous-time vector-valued white noise process with power spectral density matrix $\text{diag}[S_x, S_y, S_z]$, $A_{cv} = \text{diag}[A_2, A_2]$, and $B_{cv} = \text{diag}[B_2, B_2]$ with

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The direct discrete-time counterpart of the above model is

$$
\begin{aligned}
x_{k+1} &= Fx_k + Gw_k = \text{diag}[F_{cv}, 1]x_k + \text{diag}[G_{cv}, T]w_k \\
&= \text{diag}[F_2, F_2, 1]x_k + \text{diag}[G_2, G_2, T]w_k
\end{aligned}
$$

where

$$
F_{cv} = \text{diag}[F_2, F_2], \quad G_{cv} = \text{diag}[G_2, G_2]
$$

$$
F_2 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}
$$

$w_k = [w_x, w_y, w_z]_k$ is a discrete-time white noise sequence and T is the sampling interval.

Nonmaneuver Models

- Discrete-time model is

$$\begin{aligned} x_{k+1} &= Fx_k + Gw_k = \text{diag}[F_{cv}, 1]x_k + \text{diag}[G_{cv}, T]w_k \\ &= \text{diag}[F_2, F_2, 1]x_k + \text{diag}[G_2, G_2, T]w_k \end{aligned}$$

  Note that w_x and w_y correspond to noisy "accelerations" along x and y axes, respectively, while w_z corresponds to noisy "velocity" along z axis. If w is uncoupled across its components, then the nonmaneuvering motion modeled by the above models is uncoupled across x, y, and z directions.

- In this case, the covariance of the noise term in (13) is given by

$$\text{cov}(Gw_k) = \text{diag}[\text{var}(w_x)Q_2, \text{var}(w_y)Q_2, \text{var}(w_z)]$$

$$Q_2 = \begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix}$$

  This model is defined directly in discrete time and is not entirely equivalent to the above continuous-time model.

Nonmaneuver Models

- The discrete-time equivalent of the above continuous-time model is

$$x_{k+1} = \text{diag}[F_2, F_2, 1]x_k + w_k$$

where

$$\text{cov}(w_k) = \text{diag}(\frac{S_x}{T}\tilde{Q}_2, \frac{S_y}{T}\tilde{Q}_2, \frac{S_z}{T})$$

$$Q_2 = \begin{bmatrix} T^4/3 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix}$$

- Note the difference between the discrete-time equivalent and the direct discrete-time counterpart.

Outline

Coordinate-Uncoupled Maneuver Models

- The control input u responsible for a target maneuver is primarily deterministic in nature and most often unknown to the tracker.
- A natural way is to model it as an unknown, deterministic process and estimate this process from measurement data during tracking.
- Such deterministic input models are the basis for the so-called input estimation method. Due to a lack of knowledge of its dynamics, this unknown process is often assumed to be piecewise constant and treated as an unknown time-invariant parameter over a time window.
- The main difficulty then lies in the determination of the input level and the instants at which the input jumps.
- An alternative is to model the input u as a random process, (much more popular than the deterministic modeling).

Coordinate-Uncoupled Maneuver Models

- Models the input u as a random process:
  - White noise models: The control input is modeled as white noise. This includes constant-velocity, constant-acceleration, and polynomial models.
  - Markov process models: The control input is modeled as a Markov process, which has a time autocorrelation. This includes the well-known Singer model, its various extensions, and some other models.
  - Semi-Markov jump process models: The control input is modeled as a semi-Markov jump process.
  - Most target maneuvers are coupled across different coordinates. For simplicity, however, many maneuver models developed assume that this coordinate coupling is weak and can be neglected.

Coordinate-Uncoupled Maneuver Models

- As a consequence, we need to consider only a generic coordinate direction.
- Let x, $\dot{\text{x}}$, and $\ddot{\text{x}}$ be the target position, velocity, and acceleration along a generic direction, respectively. Specifically,

$$\ddot{\text{x}}(t) = a(t)$$

  The models differ in how the function $a(t)$ is defined.
- In this section, the state vector is always taken to be $x = [\text{x}, \dot{\text{x}}, \ddot{\text{x}}]$ along the generic direction.

Coordinate-Uncoupled Maneuver Models

A. White-Noise Acceleration Model

- It is the simplest model for a target maneuver
- $\ddot{\mathrm{x}}(t)$ is an independent process (strictly white noise).
- It differs from the nonmaneuver model in the noise level: the white noise process w used to model the effect of the control input u has a much higher intensity than the one used in a nonmaneuver model.

Coordinate-Uncoupled Maneuver Models

B. Wiener-Process Acceleration Model

- It is the second simplest model for a target maneuver
- It assumes that the acceleration is a Wiener process, or more generally and precisely, the acceleration is a process with independent increments, which is not necessarily a Wiener process.
- It is also referred to simply as the constant-acceleration (CA) model or more precisely "nearly-constant-acceleration model."
- This model has two commonly used versions.

B. Wiener-Process Acceleration Model

- The first one, referred to as the white-noise jerk model, assumes that the acceleration derivative (i.e., "jerk") $\dot{a}(t)$ is an independent process (white noise) $w(t)$:

$$\dot{a}(t) = w(t)$$

with power spectral density S_w. The corresponding state-space representation is $\dot{x}(t) = A_3 x(t) + B_3 w(t)$, where

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

B. Wiener-Process Acceleration Model

- Its discrete-time equivalent is

$$x_{k+1} = F_3 x_k + w_k, \quad F_3 = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$Q = \mathrm{cov}(w_k) = S_w Q_3, \quad Q_3 = \begin{bmatrix} T^5/20 & T^4/8 & T^3/6 \\ T^4/8 & T^3/3 & T^2/2 \\ T^3/6 & T^2/2 & T \end{bmatrix}$$

Note that S_w is the power spectral density, not the variance, of the continuous-time white noise $w(t)$.

B. Wiener-sequence acceleration model

- The second version: assumes that the acceleration increment is an independent (white noise) process. This model is most conveniently expressed in discrete time directly,

$$x_{k+1} = F_3 x_k + G_3 w_k, \quad G_3 = \begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix}$$

  Note that its noise term has a covariance different from that of the white-noise jerk model:

$$Q = \mathrm{cov}(G_3 w_k) = \mathrm{var}(w_k) \begin{bmatrix} T^4/4 & T^3/2 & T^2/2 \\ T^3/2 & T^2/2 & T \\ T^2/2 & T & 1 \end{bmatrix}$$

C. Polynomial Models

- It is well known that any continuous target trajectory can be approximated by a polynomial of a certain degree to an arbitrary accuracy. It is possible to model target motion by an nth-degree polynomial in the Cartesian coordinates:

$$\begin{bmatrix} \mathsf{x}(t) \\ \mathsf{y}(t) \\ \mathsf{z}(t) \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_n \\ c_0 & c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n/n! \end{bmatrix} + \begin{bmatrix} w_x(t) \\ w_y(t) \\ w_z(t) \end{bmatrix}$$

  with a certain choice of the coefficients a_i, b_i, c_i, where $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ are the position coordinates and (w_x, w_y, w_z) are the corresponding noise terms.

- Such an nth-degree polynomial model amounts to assuming the nth time derivative of the position is (nearly) constant

C. Polynomial Models

- The CV and CA models described above are special cases (for $n = 1$, 2, respectively)
- This model in its general setting does not appear very attractive for tracking for several reasons. Such models are usually good for fitting to a set of data, that is, for smoothing problem; however, the primary purpose of tracking is prediction and filtering, rather than fitting or smoothing.

D. Singer Acceleration Model—Zero-Mean First-Order Markov Model

- White noise process → Wiener process → Markov process
- The Singer model assumes that the target acceleration $a(t)$ is a zero-mean first-order stationary Markov process with autocorrelation

$$R_a(\tau) = E[a(t+\tau)a(t)] = \sigma^2 e^{-\alpha|\tau|}$$

or equivalently, power spectrum:

$$S_a(\omega) = 2\alpha\sigma^2/(\omega^2+\alpha^2)$$

Such a process $a(t)$ is the state process of a linear time-invariant system

$$\dot{a}(t) = -\alpha a(t) + w(t), \qquad \alpha > 0$$

where $w(t)$ is zero-mean white noise with constant power spectral density $S_w = 2\alpha\sigma^2/(\omega^2+\alpha^2)$.

D. Singer Acceleration Model—Zero-Mean First-Order Markov Model

- Its discrete-time equivalent is

$$a_{k+1} = \beta a_k + w_k^a$$

where w_k^a is a zero-mean white noise sequence with variance $\sigma^2(1-\beta^2)$ and $\beta = e^{-\alpha T}$. The state-space representation of the continuous-time Singer model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t)$$

Its discrete-time equivalent is

$$x_{k+1} = F_\alpha x_k + w_k = \begin{bmatrix} 1 & T & (\alpha T - 1 + e^{-\alpha T})/\alpha^2 \\ 0 & 1 & (1-e^{-\alpha T})/\alpha \\ 0 & 0 & e^{-\alpha T} \end{bmatrix} x_k + w_k$$

D. Singer Acceleration Model—Zero-Mean First-Order Markov Model

- The success of the Singer model relies on an accurate determination of the parameters α and σ^2. The parameter $\alpha = 1/\tau$ is the reciprocal of the maneuver time constant τ and thus depends on how long the maneuver lasts.
- for an aircraft, $\tau \approx 60$ s for a lazy turn and $\tau \approx 10 - 20$ s for an evasive maneuver. The parameter $\sigma^2 = E[a(t)^2]$ is the "instantaneous variance" of the acceleration.

Coordinate-Uncoupled Maneuver Models

D. Singer Acceleration Model—Zero-Mean First-Order Markov Model

- As the maneuver time constant τ increases (i.e., τT decreases), the Singer model reduces to the CA model. In this case, the acceleration would be Wiener-sequential.
- As the maneuver time constant τ decreases (i.e., τT increases), the Singer model reduces to the CV model. In this case, the acceleration becomes white noise.
- The Singer model is in essence an a priori model since it does not use online information about the target maneuver. The main shortcomings of the Singer model stems from this symmetry; that is, the target acceleration has zero mean at any moment.

E. Mean-Adaptive Acceleration Model

- Also called the "current" model , is in essence a Singer model with an adaptive mean; that is, a Singer model modified to have a non-zero mean of the acceleration: $a(t) = \bar{a}(t) + \tilde{a}(t)$, where $\tilde{a}$ is the zero-mean Singer acceleration process and $\bar{a}(t)$ is the mean of the acceleration, artificially assumed constant over each sampling interval.
- Such a non-zero-mean acceleration satisfies

$$\dot{a}(t) = -\alpha\tilde{a}(t) + w(t) \text{ or } \dot{a}(t) = -\alpha a(t) + \alpha\bar{a}(t) + w(t)$$

The estimate $\hat{a}_k$ of a_k from all available online information is taken to be the "current" value of the mean $\bar{a}_{k+1}$. It is potentially more effective than the Singer model.

E. Mean-Adaptive Acceleration Model

- The state-space representation of this model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \bar{a}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t)$$

  Here, $\bar{a}(t)$ is assumed piecewise constant.

- The discrete-time equivalent is

$$x_{k+1} = F_\alpha x_k + \left(\begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix} - \begin{bmatrix} (\alpha T - 1 + e^{-\alpha T})/\alpha^2 \\ (1 - e^{-\alpha T})/\alpha \\ e^{-\alpha T} \end{bmatrix} \right) \bar{a}_k + w_k$$

- These two equations differ from the Singer model only in the additional terms associated with $\bar{a}(t)$ and $\bar{a}_k$, respectively.

E. Mean-Adaptive Acceleration Model

- A key underlying assumption of the "current" model is that

$$\bar{a}_{k+1} = E[a_{k+1}|Z^k] \approx E[a_k|Z^k] = \hat{a}_k$$

where Z^k stands for all measurements through time t_k. It is not the real "current" model.

- An improved "current" model by replacing $\bar{a}_{k+1} = \hat{a}_k$ with

$$\begin{aligned}\bar{a}_{k+1} &= E[a_{k+1}|Z^k] \\ &= e^{-\alpha T} E[a_k|Z^k] + (1 - e^{-\alpha T})\bar{a}_k \\ &= e^{-\alpha T} \hat{a}_k + (1 - e^{-\alpha T})\bar{a}_k\end{aligned}$$

since

$$a_{k+1} = e^{-\alpha T} a_k + (1 - e^{-\alpha T})\bar{a}_k + w_k^a$$

Coordinate-Uncoupled Maneuver Models

F. Asymmetrically Distributed Normal Acceleration Model

- Target acceleration can be decomposed along two directions: lift (normal to the target velocity and wing directions for an aircraft) and thrust or drag (along the velocity direction).
- For lift, which is usually dominant during a maneuver, its direction is determined by the target aspect angle and its magnitude can be modeled as a colored random process with an asymmetrical distribution.
- The acceleration $a_n(t)$ be modeled as an asymmetric and deterministic function of a zero-mean first-order Gauss-Markov process $b(t)$:

$$a_n(t) = \alpha + \beta e^{\gamma b(t)}$$

where α, β, γ are design parameters; $b(t)$ satisfies

$$\dot{b}(t) = -(1/\tau) b(t) + w(t)$$

where τ is the correlation time constant of $b(t)$.

G. Markov Models for Oscillatory Targets

- In reality, the acceleration along one coordinate direction is oscillatory due to, e.g., wind sway or platform roll. The Singer model is not very suitable for such practical maneuvers.
- The autocorrelation of such acceleration may be described by

$$\begin{aligned} R_a(\tau) &= \sigma_a^2 e^{-\alpha|\tau|} \cos(\omega_c \tau) \\ &= \sigma_a^2 e^{-\zeta\omega_n|\tau|} \cos(\omega_n \sqrt{1-\zeta^2}\tau) \end{aligned}$$

and

$$\omega_n^2 = \alpha^2 + \omega_c^2, \quad \zeta = \alpha/\omega_n$$

where σ_a^2, α, ω_c, ζ, and ω_n are average power, damping coefficient, actual (damped) frequency, damping ratio, and undamped natural frequency of the target acceleration, respectively.

G. Markov Models for Oscillatory Targets

- Such an acceleration process is the response of a second-order prewhitening system to zero-mean white noise input $w(t)$ with power spectral density $2\alpha\sigma_a^2$. Transfer function

$$H(s) = \frac{s + \omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and is described by

$$\begin{bmatrix} \dot{a}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} a(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} 1 \\ (1 - 2\zeta)\omega_n \end{bmatrix} w(t)$$

where $d(t) = \dot{a}(t) - w(t)$ is called acceleration drift.

G. Markov Models for Oscillatory Targets

- The state-space model for $x = [\mathsf{x}, \dot{\mathsf{x}}, \ddot{\mathsf{x}}, d]^\top$ is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ (1-2\zeta)\omega_n \end{bmatrix} w(t)$$

  The discrete-time equivalent of model is

$$x_{k+1} = \begin{bmatrix} 1 & T & F_{13} & F_{14} \\ 0 & & & \\ 0 & & F(\omega_c, \alpha) & \\ 0 & & & \end{bmatrix} x_k + w_k$$

  where the transition functions can be found in the survey paper.

Coordinate-Uncoupled Maneuver Models

H. Markov Acceleration Model for Constant Turns

- A typical target maneuver, such as a turn, often has an approximately constant speed and turn rate.
- $a = [a_x, a_y]^\top$ denotes the acceleration a along the x and y directions, v the constant speed, $\phi(t)$ the (velocity) heading angle, and $\omega = \dot{\phi}$ the constant turn rate.
- From the constant-turn equation $a_x = \ddot{\mathsf{x}} = -\omega\dot{\mathsf{y}} = -\omega v \sin\phi$ and $\phi(t+\tau) = \phi(t) + \omega\tau$ that the autocorrelation of $a_x(t)$ is

$$
\begin{aligned}
&R(t+\tau, t) \\
&= \frac{1}{2}v^2 E\left[\omega^2\left\{\cos\omega\tau\left[1-\cos 2\omega(t)\right] + \sin\omega\tau\sin 2\omega(t)\right\}\right] e^{-\alpha|\tau|} \\
&= \sigma^2(t,\tau)e^{-\alpha|\tau|}
\end{aligned}
$$

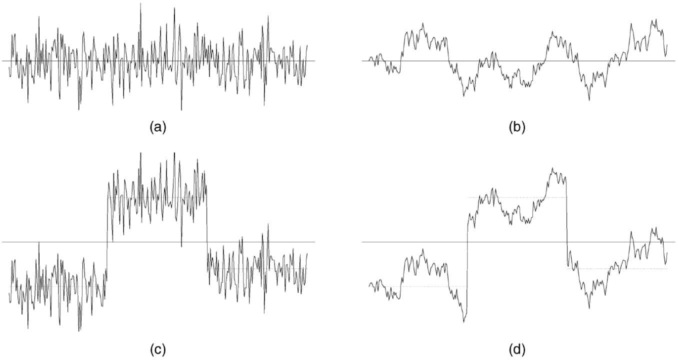
  $a_x(t)$ is nonstationary because $R(t+\tau, t)$ depends on t as well as τ

- More accurate but more complicated state-space form.

Coordinate-Uncoupled Maneuver Models

I. Semi-Markov Jump Process Models

- In practice, many target maneuvers involve an acceleration of a non-zero mean that may be reasonably assumed piecewise constant.



(a) Zero-mean white noise. (b) Zero-mean colored noise. (c) Jump-mean white noise. (d) Jump-mean colored noise

I. Semi-Markov Jump Process Models

- However, the difficulty is that neither the time intervals over which the acceleration mean is piecewise constant nor the corresponding constant levels of the non-zero mean are known to a tracker.
- Markov jump:

$$P_{ij} = P\{u(t_k) = \bar{a}_j | u(t_{k-1}) = \bar{a}_i\}$$

  The past and future states are independent given the present state.
- Semi-Markov jump:

$$P_{ij} = P\{u(t_k) = \bar{a}_j | u(t_{k-1}) = \bar{a}_i, \tau_{ij}\}$$

  where $\tau_{ij} = t_k - t_{k-1}$ is the sojourn time. The future states may be coupled through the time interval it stays in a state (called sojourn time) as well as the present state.

I. Semi-Markov Jump Process Models

- The acceleration $a(t)$ as a combination of the above jump-mean model and the Singer model was proposed:

$$a(t) = \beta v(t) + u(t) + \tilde{a}(t)$$

where $\tilde{a}(t)$ is the Singer acceleration, v is the velocity, $u(t)$ is the unknown acceleration mean, and β is a drag coefficient. The corresponding continuous-time state-space representation is, for $x = [\mathrm{x}, \dot{\mathrm{x}}, \ddot{\mathrm{x}}]^\top$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\beta & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t)$$

J. Jerk Models

- Jerk is the derivative of acceleration.
- A non-zero-mean jerk model was proposed by introducing an additional term in the Singer-model equation

$$\dot{a}(t) = -\alpha a(t) + w(t) + \overline{\dot{a}}$$

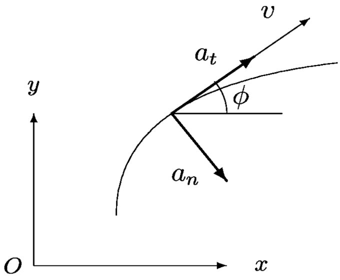
  where $\overline{\dot{a}}$ is a non-zero expected jerk.
- This model and the "current" model are the same in spirit—they aim at improving the Singer model by adding a non-zero-mean term to equations that describe the acceleration.

Outline

2D Horizontal Motion Models

- Most 2D and 3D target maneuver models are naturally turn motion models. These models are usually established relying on target kinematics, in contrast to those of the previous section that are based on random processes. This is understandable since random processes are more natural for modeling time correlation than describing spatial trajectories where kinematics is a more appropriate tool.
- Coordinate-coupled target models are highly dependent on the choice of the state components. The choice of the state components (and implicitly the respective kinematic model) is not a trivial problem, where target dynamics, accuracy of approximations, sensor coordinate system, among others, must be taken into account.

2D Horizontal Motion Models



- Various kinematic models can be comprised from the following standard curvilinear-motion model

$$\dot{\mathsf{x}}(t) = v(t)\cos(\phi(t));\ \dot{\mathsf{y}}(t) = v(t)\sin(\phi(t))$$
$$\dot{v}(t) = a_t(t);\quad \dot{\phi}(t) = a_n(t)/v(t)$$

2D Horizontal Motion Models

- $\Rightarrow$

$$\begin{aligned}\ddot{\mathsf{x}}(t) &= \dot{v}(t)\cos(\phi(t)) - v(t)\sin(\phi(t))\dot{\phi}(t) \\ &= a_t(t)\cos(\phi(t)) - \dot{\mathsf{y}}(t)\dot{\phi}(t)\end{aligned}$$

  where (x, y),v,ϕ are the target position in Cartesian coordinates, ground speed (airspeed plus wind speed), and (velocity) heading angle, respectively, a_t and a_n at and an are the target tangential (along-track) and normal (cross-track) accelerations in the horizontal plane, respectively.

- Reduces to the following special cases:
  1) $a_t(t) = 0$, $a_n(t) = 0 \rightarrow$ rectilinear, CV motion
  2) $a_t(t) \neq 0$, $a_n(t) = 0 \rightarrow$ rectilinear, accelerated motion (CA motion if a_t = constant)
  3) $a_t(t) = 0$, $a_n(t) \neq 0 \rightarrow$ circular, constant-speed motion (CT motion if a_n = constant).

A. CT Models with Known Turn Rate

- This model presumes that the target moves with (nearly) constant speed v and (nearly) constant angular (turn) rate $\omega = \dot{\phi}$.
- Assuming ω is known leads to (only four-dimensional) state vector, e.g., $x = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}]^\top$, in the Cartesian coordinates. It follows that $f(x, u, t) = [\dot{\mathsf{x}}, -\omega\dot{\mathsf{y}}, \dot{\mathsf{y}}, -\omega\dot{\mathsf{x}}]^\top$;

$$\dot{x}(t) = \begin{bmatrix} \dot{\mathsf{x}}(t) \\ -\omega\dot{\mathsf{y}}(t) \\ \dot{\mathsf{y}}(t) \\ \omega\dot{\mathsf{x}}(t) \end{bmatrix} + Bw(t) = A(\omega)x(t) + Bw(t)$$

$$A(\omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\omega & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

where white noise $w = [w_x, w_y]^\top$

A. CT Models with Known Turn Rate

- Its discrete-time equivalent is

$$
\begin{aligned}
x_{k+1} &= F_{ct}(\omega) x_k + w_k \\
&= \begin{bmatrix} 1 & \frac{\sin(\omega T)}{\omega} & 0 & -\frac{1-\cos(\omega T)}{\omega} \\ 0 & \cos(\omega T) & 0 & -\sin(\omega T) \\ 0 & \frac{1-\cos(\omega T)}{\omega} & 1 & \frac{\sin(\omega T)}{\omega} \\ 0 & \sin(\omega T) & 0 & \cos(\omega T) \end{bmatrix} x_k + w_k
\end{aligned}
$$

Its direct discrete-time counterpart is better known.

B. CT Models with Unknown Turn Rate

- These models differ from the above CT models only in that the turn rate is included as a state component, to be estimated.
- The two most popular models for ω are the Wiener process model

$$\dot{\omega}(t) = w_\omega(t);\ \text{in continuous time}$$
$$\omega_{k+1} = \omega_{k+1} + w_k;\ \text{in discrete time}$$

and the first-order Markov process model

$$\dot{\omega}(t) = -\frac{1}{\tau}\omega(t) + w_\omega(t);\ \text{in continuous time}$$
$$\omega_{k+1} = e^{-T/\tau}\omega_k + w_k;\ \text{in discrete time}$$

B. CT Models with Unknown Turn Rate

- In this model, the state vector is chosen to be $x = [\mathrm{x}, \dot{\mathrm{x}}, \mathrm{y}, \dot{\mathrm{y}}, \omega]^\top$; a direct discrete-time version is:

$$x_{k+1} = \begin{bmatrix} F_{ct}(\omega^*) & 0 \\ 0 & \beta \end{bmatrix} x_k + \mathrm{diag}\left[G_2, G_2, 1\right] w_k$$

where $\beta = e^{-\alpha T}$, $\omega^* = \omega_k$, ω_{k+1}, $\bar{\omega}$, or something similar, $w_k = [w_x, w_y, w_\omega]_k^\top$ is zero-mean white noise.

C. Circular Motion Models

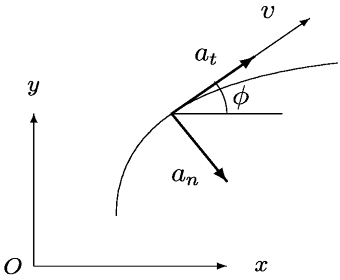
- For a circular motion of a target, if its center were known, the simplest model would be to represent the circle in the polar coordinates and place the origin at the circle center. In this coordinate system, the target dynamic model is linear for $x = [\rho, \theta, \dot{\theta}]^\top$

$$x_{k+1} = \text{diag}[1, F_2]x_k + \text{diag}[1, G_2/T]w_k$$

  an accurate determination of the center of the turn in terms of the sensor coordinate system, which is inherently a nonlinear problem.

D. Curvilinear Motion Model

- This model is more general than those considered so far.
- It accounts for possibly non-zero normal (cross-track) and tangential (along-track) target maneuver accelerations simultaneously.



$$\dot{x}(t) = v(t)\cos(\phi(t)); \;\; \dot{y}(t) = v(t)\sin(\phi(t))$$

D. Curvilinear Motion Model

$$\begin{aligned}\ddot{\mathsf{x}}(t) &= \dot{v}(t)\cos(\phi(t)) + v(t)(-\sin(\phi(t)))\dot{\phi}(t) \\ &= a_t \cos(\phi(t)) - \sin(\phi(t))a_n(t)\end{aligned}$$

For the Cartesian state vector $x = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}]^\top$, it follows from the standard equations of curvilinear motion that this model in continuous time is given by

$$\dot{x}(t) = A_{cv}x(t) + B(x(t))a(t) + w(t)$$

where $a = [a_t, a_n]^\top$ is the acceleration for the maneuver, A_{cv} was given before and

$$B(x(t)) = \begin{bmatrix} 0 & 0 \\ -\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} & -\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ 0 & 0 \\ \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} & \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \end{bmatrix}$$

Outline

3D Motion Models

- Many of the 2D horizontal models reviewed above have been considered for application to 3D tracking of civilian aircraft in ATC systems.
- Such targets maneuver mostly in a horizontal plane with nearly constant speed and turn rate and have little or limited vertical maneuver, usually performed not at the time of a horizontal turn. Thus, the altitude changes are most often modeled independently by a (nearly) CV model or a random walk model along z direction, leading to an acceptable accuracy in practice.
- However, when the task is to track agile military aircraft, capable of performing "high-g" turns in the 3D space (e.g., for tracking in air defense systems) rather than just horizontally, decoupled models may be inadequate. Many efforts have been devoted to solving this problem, and more accurate models have been developed.