Multiscale Methods for Partial Differential Equations

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Outline

- Introduction
- Divide and conquer: subspace correction and subspace corrections
- How to decompose a PDE problem: locality of high frequencies
- Some examples
  - $\star$ Elliptic equations: highly oscillatory coefficients
    Points of view on numerical homogenization:
    If we are able to "record" the fine grid structures everywhere, then a direct simulation is possible by using multigrid method.
    In general, model based on mathematical analysis is required for unscaling, direct simulations may be used near the boundary, and places with special structures

- ⋆ Parallelization:
- ⋆ Symmetrization: Convection-diffusions
- ⋆ Linearization: Nonlinear problems: eigenvalue problem
- ⋆ Convection-dominated case: the Burger's equation
- ⋆ Unstructured grids: algebraic multigrid methods
- ⋆ Nonmatching grids: the method based on partition of unity
- ⋆ Application to PDE theory: coarse grid to take care of compact part of the problem, on the fine grid the equation are dominated by nicer part of the equations

- An industrial application: electrochemical devise simulations

- Concluding remarks

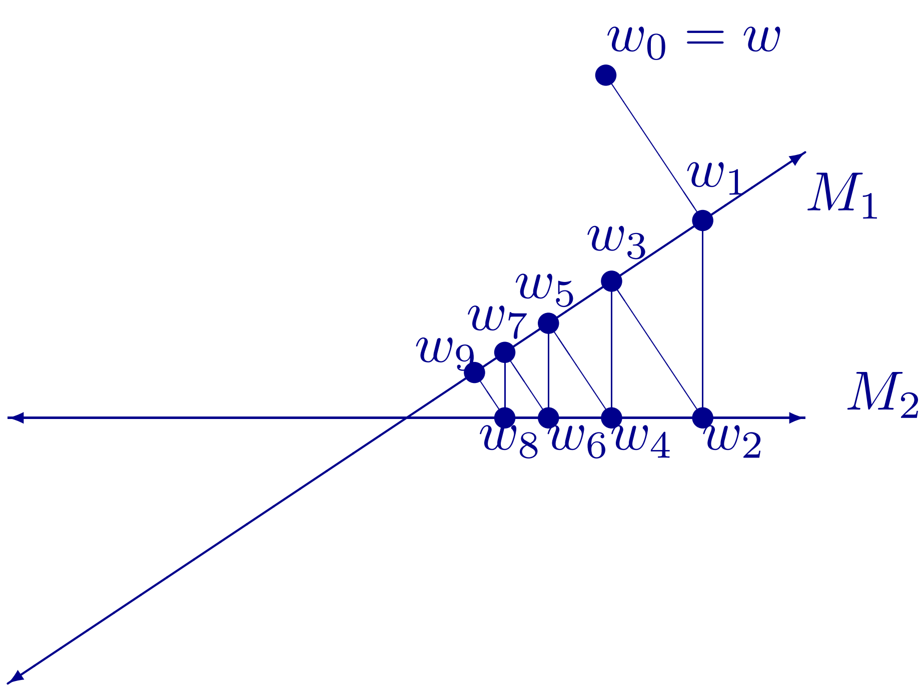
Introduction

Von Neumann's method of alternating projections

- von Neumann (1933): Let M_1 and M_2 be closed subspaces of Hilbert space H, then

$$\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1\cap M_2}.$$

(Note that $P_{M_2}P_{M_1} = P_{M_1\cap M_2}$ if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$)



- Rate of convergence:

$$\|(P_{M_2}P_{M_1})^k - P_{M_1\cap M_2}\| = [c(M_1, M_2)]^{2k-1}$$

where $c(M_1, M_2)$ is the cosine of the angle between M_1 and M_2:

$$c(M_1, M_2) = \sup\left\{\frac{|(u,v)|}{\|u\|\|v\|} : u \in M_1 \cap (M_1 \cap M_2)^{\perp}, v \in M_2 \cap (M_1 \cap M_2)^{\perp}\right\}$$

<u>Fact</u>: $c(M_1, M_2) < 1$ if and only if $M_1 + M_2$ is closed.

$\Rightarrow$ uniform convergence.

Generalizations

- Many subspaces case: given any closed subspaces M_i,

$$\lim_{k\to\infty} w_k = \lim_{k\to\infty} (\prod_{i=1}^{J} P_{M_i})^k w = P_{\cap_{i=1}^{J} M_i} w.$$

- Closed convex sets: M_i
- Nonlinear problems

A framework for the following type of algorithm:

Compute a quantity in a more complicated “small” subspace by solving prolems associated with simpler “large” subspaces

- Some Related References on MAP (theory of approximation ...)
  - ⋆ von Newmann (1933)
  - ⋆ N. Aronszjan (1950)
  - ⋆ I. Halperin (1962)
  - ⋆ F. Deutsch (1982, 1983, 1985, 1992)
  - ⋆ S. Kayalar and H. Weinert (1988)
  - ⋆ H. Bauschke and J. Borwein (1996) (SIAM Review)

- Quantitative convergence estimates:
  - ⋆ Two subspace case can not be generalized to more than two subspaces
  - ⋆ Optimal estimate for more than two subspaces has been an open problem until recently (Xu-Zikatanov 2001)

Another class of methods: space decomposition and subspace correction

Divide and conquer: Compute a quantity in more expensive “large” subspace by means of cheaper “smaller(?)” subspaces.

A general framework

Finding the solution of a linear equation by approximately solving equations restricted on a number of subspaces which make up the entire space.

- Problem: Find $u \in V$ such that $a(u, v) = f(v), \quad \forall v \in V$.
- Space decomposition: $V = \sum_i V_i$

  Different subspaces V_i may correspond different scales, or different subdomains
- Approximate subspace problems: $a_i \approx a$ on $V_i \times V_i$ and $T_i \approx P_i$:

$$a_i(T_i v, v_i) = a(v, v_i), v \in V, v_i \in V_i.$$

with $T_i = P_i$ is a projection if $a_i = a$.

- **Algorthm MSC:** Let $u^0 \in H$ be given.

for $\ell = 1, 2, \ldots$
$u_0^{\ell-1} = u^{\ell-1}$.
for $i = 1 : J$
Let $e_i \in V_i$ solve

$$a_i(e_i, v_i) = f(v_i) - a(u_{i-1}^{\ell-1}, v_i) \quad \forall v_i \in V_i.$$

$u_i^{\ell-1} = u_{i-1}^{\ell-1} + e_i$
endfor
$u^\ell = u_J^{\ell-1}$.
endfor

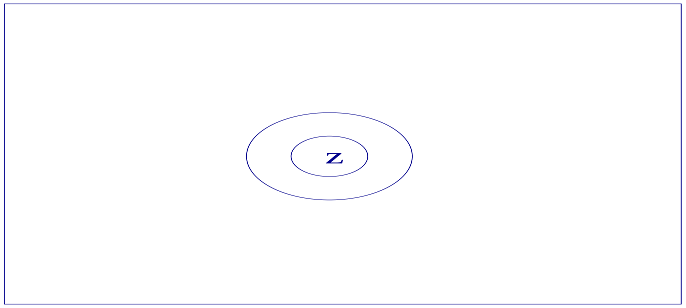
Applications

- $\star$ Classic iterative methods such as Jacobi and Gauss-Seidel
- $\star$ Domain decomposition methods
- $\star$ Multigrid method
- $\star$ . . .

How to decompose for a PDE problem?

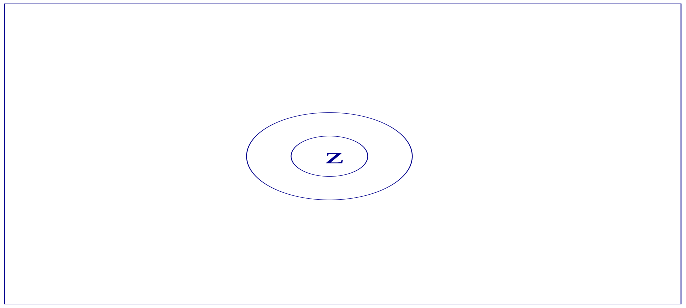
A study of “frequencies”: $u \sim \sum_k a_k e^{ik\cdot x}$

⋆ Low frequency components — smaller k:



- ∗ smooth
- ∗ easy to resolve (approximate)

⋆ High frequency components — larger k:



- ∗ fine structures
- ∗ singularities
- ∗ . . .
- ∗ difficult to resolve (approximate)

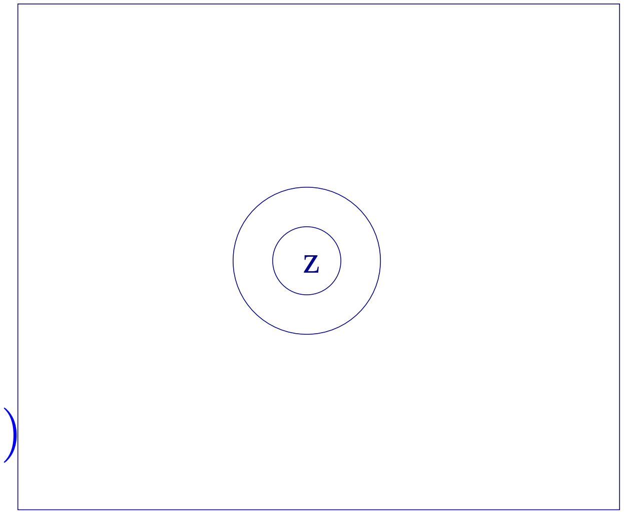
- Difficulties:
  - Resolution of high frequencies: require a lot of grid points
  - Difficult to solve the resulting large system: global coupling
- Remedies:

  For certain partial differential equations, higher frequencies tend to behavior more locally

A simple illustration

Consider the model problem:

$$-\nabla \cdot (a(x)\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Omega$$



Solve this equation locally:

$$-\nabla\cdot(a(x)\nabla u_\delta) = f \text{ in } B_z(\delta), \quad u_\delta = 0 \text{ on } \partial B_\delta(z)$$

The error $e = u - u_\delta$, being "harmonic" inside $B_z(\delta)$, satisfies

$$\int_\Omega a|\nabla(\theta e)|^2 = \int_\Omega a e^2 |\nabla\theta|^2, \quad \theta \in C_0^\infty(\Omega), \text{ hence}$$

$$\int_{B_{\delta/2}(z)} a(x)|\nabla(u-u_\delta)(x)|^2 \leq 4\delta^{-2}\int_{B_\delta(z)} a(x)|(u-u_\delta)(x)|^2$$

Locality of high frequencies

$$\frac{\displaystyle\int_{B_{\delta/2}(z)} a(x)|\nabla(u-u_\delta)(x)|^2}{\displaystyle\int_{B_{\delta}(z)} a(x)|(u-u_\delta)(x)|^2} \le 4\delta^{-2}$$

- ⋆ the relative magnitude of $\nabla(u-u_\delta)$ inside $B_{\delta/2}(z)$ is bounded by $2\delta^{-1}$
- ⋆ $u-u_\delta$ does not contain any significant component that oscillate very much inside $B_{\frac{\delta}{2}}(z)$.
- ⋆ In other words, the local solution has captured the (significant) oscillatory component of u inside $B_{\frac{\delta}{2}}(z)$.
- ⋆ As a consequence, high frequencies can be removed by solving local (small) problems.

An illustration: classic Schwarz overlapping domain decomposition method

Consider a general elliptic boundary value problem:

$$-\nabla \cdot (a(x)\nabla u) = f \text{ for } x \in \Omega, \text{ for } \quad u = 0 \; x \in \partial\Omega$$

where $a = (a_{ij})$ consists measurable function entries satisfying

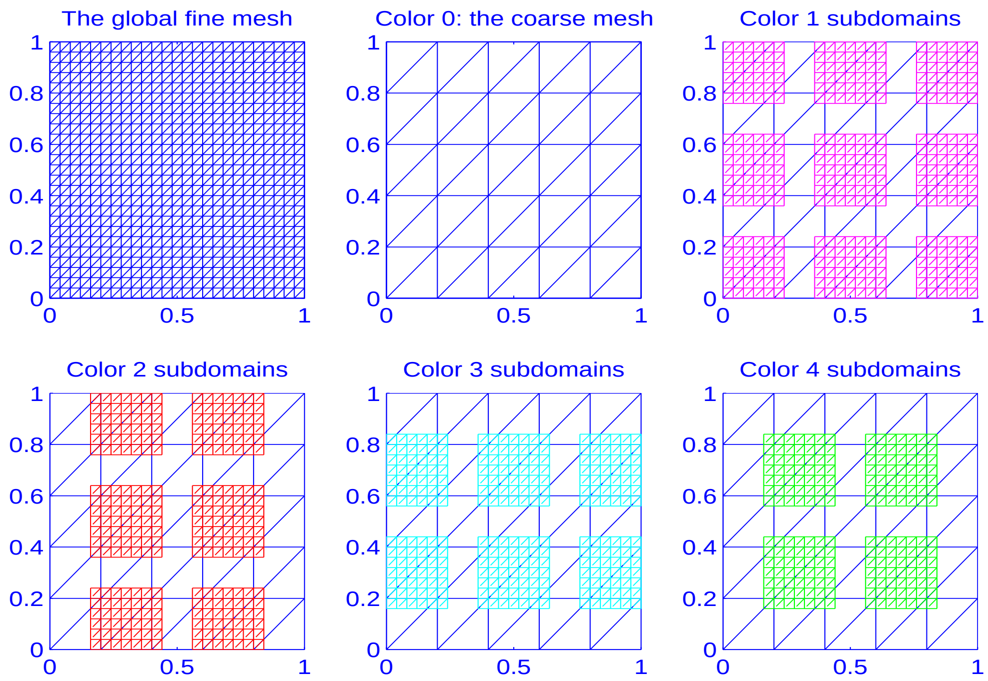
$$\Lambda_0|\xi|^2 \leq \xi^t a\xi \leq \Lambda_1|\xi|^2 \quad \forall \xi \in R^n$$

Example in consideration:

$$a(x) = \frac{2 + P\sin(2\pi x_1/\epsilon)}{2 + P\sin(2\pi x_2/\epsilon)} + \frac{2 + P\sin(2\pi x_2/\epsilon)}{2 + P\cos(2\pi x_1/\epsilon)}$$

A picture of a to be given here

Consider an overlapping domain decomposition: $\Omega = \sum_{i=1}^{J} \Omega_i$.



Define local subsapces: $V_i = \{v \in V : v(x) = 0,\ \forall x \in \Omega \setminus \Omega_i\} \subset V \equiv H_0^1(\Omega)$ $(1 \le i \le J)$

An illustration of effects of local corrections

Initial error:

Correction on a corner subdomain:

Correction on a middle subdomain:

Correction through all subdomains

Introduction of coarser space

$V_0 \subset V:$ finite element space on a mesh of size h_0

The corresponding subspace correction method is the Schwarz overlapping domain decomposition method

Quantitative convergence theory

- ⋆ Gauss-Seidel type methods
- ⋆ Domain decomposition
  - ∗ Original Schwarz alternating iteration (1870)
  - ∗ P.L. Lions (1987) ($J = 2$)
  - ∗ Dryja and Widlund (1991) (additive)
  - ∗ Bramble, Pasciak, Wang and Xu (1991)
- ⋆ Multigrid
  - ∗ Fedorenko (1961), Bachvalov (1966)
  - ∗ Hackbusch (1976), Brandt (1977), Bank-Dupont (1977), Nicolaides(1977)
  - ∗ Braess and Hackbusch (1983)
  - ∗ Bramble, Pasciak, Wang and Xu (1991)
- ⋆ General framework and theory
  - ∗ Xu (1992), SIAM Review
  - ∗ Xu and Zikatanov (2000), J. of AMS (to appear)
- ⋆ An optimal theory ...

- Error equations: $u - u_i^{\ell-1} = (I - T_i)(u - u_{i-1}^{\ell-1})$ and

$$u - u^\ell = E(u - u^{\ell-1}) = \ldots = E^\ell(u - u^0)$$

where

$$E = (I - T_J)(I - T_{J-1}) \ldots (I - T_1).$$

- Convergence:

$$\|E\|_A < 1?$$

A new convergence theory

(Xu and Zikatanov 2001, J. of AMS (to appear))

$$\|E\|^2 \equiv \|(I-T_J)(I-T_{J-1})\dots(I-T_1)\|^2 = \frac{c_0}{1+c_0}$$

where $\bar{T}_i = T_i^* + T_i - T_i^* T_i$

- A trivial case: point Gauss-Seidel method ($A = D - L - U$)

$$\|I-(D-L)^{-1}A\|_A^2 = \frac{c_0}{1+c_0} \quad c_0 = \sup_{(Ax,x)=1} \|D^{-1/2}Ux\|^2.$$

- **a special case** ($T_i = P_i$)

$$\|(I-P_J)\cdots(I-P_1)\|_A^2 = \frac{c_0}{1+c_0}$$

where

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} \|P_i \sum_{j=i+1}^{J} v_j\|^2$$

- most general case:

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} (\bar{T}_i^{-1} T_i^* w_i, T_i^* w_i) \text{ with } w_i = \sum_{j=i}^{J} v_j - T_i^{-1} v_i.$$

Note: $\bar{T}_i \equiv T_i^* + T_i - T_i^* T_i$ SPD $\Longrightarrow c_0 > 0$.

An illustration: a concise convergence analysis for Schwarz domain decomposition method

Main ingredients: (1) Partition of unity:

$\sum_{i=1}^{J}\theta_i(x) \equiv 1, \quad \mathrm{supp}\theta_i \subset \Omega_i \cup \partial\Omega, \quad 0 \le \theta_i \le 1, \quad \max_{x\in\bar{\Omega}_i}|\nabla\theta_i(x)| \le c_1 h_0^{-1}.$

and (2) L^2 projection $Q_0 : V \mapsto V_0$: $h_0^{-1}\|v - Q_0 v\|_{0,\Omega} + |v - Q_0 v|_{1,\Omega} \le c_2|v|_{1,\Omega}$.

Then $c_0 \le \frac{\Lambda_1}{\Lambda_0}C_0$ with $C_0 = C_0(c_1, c_2)$ independent of a_{ij} and J:

$$
\begin{aligned}
\sum_{k=0}^{J}\|P_k\sum_{i=k+1}^{J} v_i\|_{a,\Omega}^2 &\le \|v - Q_0 v\|_{a,\Omega}^2 + \sum_{k=1}^{J}\|(\sum_{i=k+1}^{J}\theta_i)(v - Q_0 v))\|_{a,\Omega_k}^2 \\
&\le \Lambda_1\left(|v - Q_0 v|_{1,\Omega}^2 + \sum_{k=1}^{J}\max_{x\in\bar{\Omega}_k}|\sum_{i=k+1}^{J}\nabla\theta_i(x)|\|v - Q_0 v\|_{0,\Omega_k}^2 + |v - Q_0 v|_{1,\Omega_k}^2\right) \\
&\le \Lambda_1 C_0|v|_{1,\Omega}^2 \le \frac{\Lambda_1}{\Lambda_0}C_0\|v\|_{a,\Omega}^2
\end{aligned}
$$

Some features of multigrid method

- Optimal (or nearly optimal) computational complexity: $O(N)$ (or $O(N \log N)$ operations for a problem of size N.

- Multigrid method is highly problem dependent, but it can be applied to a variety of problems.

- For problems with osscillatory coefficients, if the osscillations can be numerically resolved, the corresponding systems may be solved by multigrid method in an optimal fashion and (in parallel).

- Difficulties:

  ⋆ Strong heterogeneity ($\Lambda_1/\Lambda_0 \gg 1$)
  ⋆ Unstructured grids

A concise convergence analysis using the new theory

Main ingredients: (1) Partition of unity:

$\sum_{i=1}^J \theta_i(x) \equiv 1, \quad \mathrm{supp}\theta_i \subset \Omega_i \cup \partial\Omega, \quad 0 \le \theta_i \le 1, \quad \max_{x\in\bar\Omega_i} |\nabla\theta_i(x)| \le c_1 h_0^{-1}.$

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$$
\begin{aligned}
\sum_{k=0}^J \|P_k \sum_{i=k+1}^J v_i\|_{a,\Omega}^2 &\le \|v - Q_0 v\|_{a,\Omega}^2 + \sum_{k=1}^J \|(\sum_{i=k+1}^J \theta_i)(v - Q_0 v))\|_{a,\Omega_k}^2 \\
&\le \Lambda_1 \left(|v - Q_0 v|_{1,\Omega}^2 + \sum_{k=1}^J \max_{x\in\bar\Omega_k} |\sum_{i=k+1}^J \nabla\theta_i(x)| \|v - Q_0 v\|_{0,\Omega_k}^2 + |v - Q_0 v|_{1,\Omega_k}^2 \right) \\
&\le \Lambda_1 C_0 |v|_{1,\Omega}^2 \le \frac{\Lambda_1}{\Lambda_0} C_0 \|v\|_{a,\Omega}^2
\end{aligned}
$$

A multilevel version: a typical multigrid method

Consider a discrete space V_h on a fine grid $\mathcal{T}_h$:

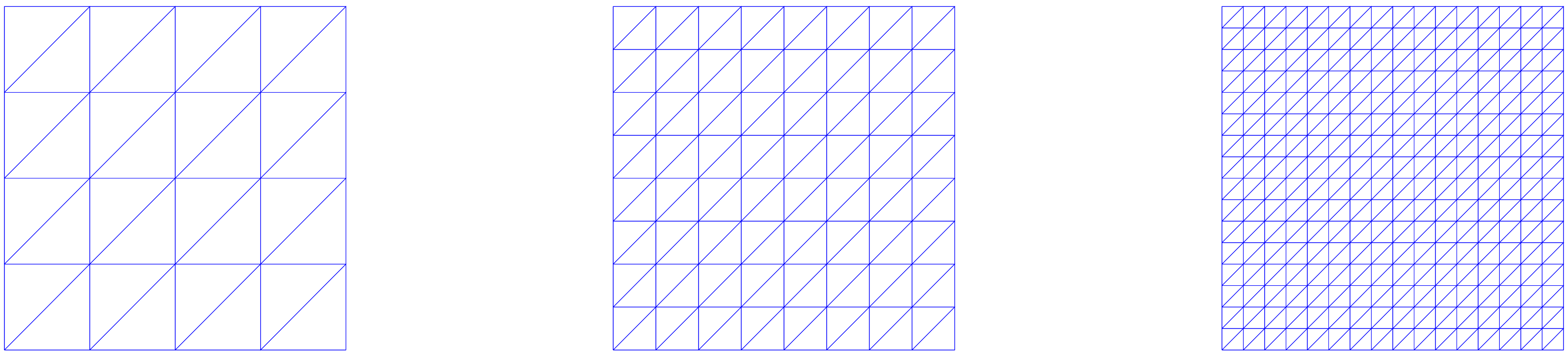
- Consider smallest possible subdomains: dim(V_i)=1. The corresponding Schwarz method (without V_0) is just the point Gauss-Seidel method.

- Choose coarse space $V_0 = V_{2h}$. The corresponding Schwarz method is a two-level multigrid method with Gauss-Seidel as a smoother.

- Repeat the above procedure with V_{2H} gives rise to a multigrid method.

$$
\begin{array}{cccccccc}
V_h & \Rightarrow & (GS)_h & & & & & \\
 & & + & \searrow & & & & \\
 & & V_{2h} & \Rightarrow & (GS)_{2h} & & & \\
 & & & & + & \searrow & & \\
 & & & & V_{4h} & \Rightarrow & (GS)_{4h} & \\
 & & & & & & + & \searrow \\
 & & & & & & V_{8h} & \ldots V_0
\end{array}
$$

How to make the diagram dynamic? How to add the searrow later?



$$M_0 \subset M_1 \subset \ldots M_J = V_h.$$

Use gimp to get rid of the blank margins of the above pictures

Some features of multigrid method

- Optimal (or nearly optimal) computational complexity: $O(N)$ (or $O(N \log N)$ operations for a problem of size N.

- Multigrid method is highly problem dependent, but it can be applied to a variety of problems.

- For problems with osscillatory coefficients, if the osscillations can be numerically resolved, the corresponding systems may be solved by multigrid method in an optimal fashion and (in parallel).

- Difficulties:

  ⋆ Strong heterogeneity ($\Lambda_1/\Lambda_0 \gg 1$)
  ⋆ Unstructured grids

Some remarks on smoothers

- Multiplicative or additive Schwarz domain decomposition methods can often be used as smoother. The point Gauss-Seidel and point Jacobi iterations are exteme cases.

- Point relaxation methods only works well for problems that have point locality for high frequencies (for isotropic coarsening)

- Many steady state problems can be viewed as the time limit of corresponding evolution problems. An explicit solver for the envolution problem has often been used as a smoother.

- Explicit Euler (or multistage Runge-Kutta) for evolution problem is equivalent to the damped point Richardson/Jacobi relaxation for the corresponding steady state equation. The efficiency of this type of smoother should be quite limited for hyperbolic equations whose high frequencies solution do not have point local property.

Strongly discontinuous coefficients

Consider a special case of piecewise constant diffusion coefficients:

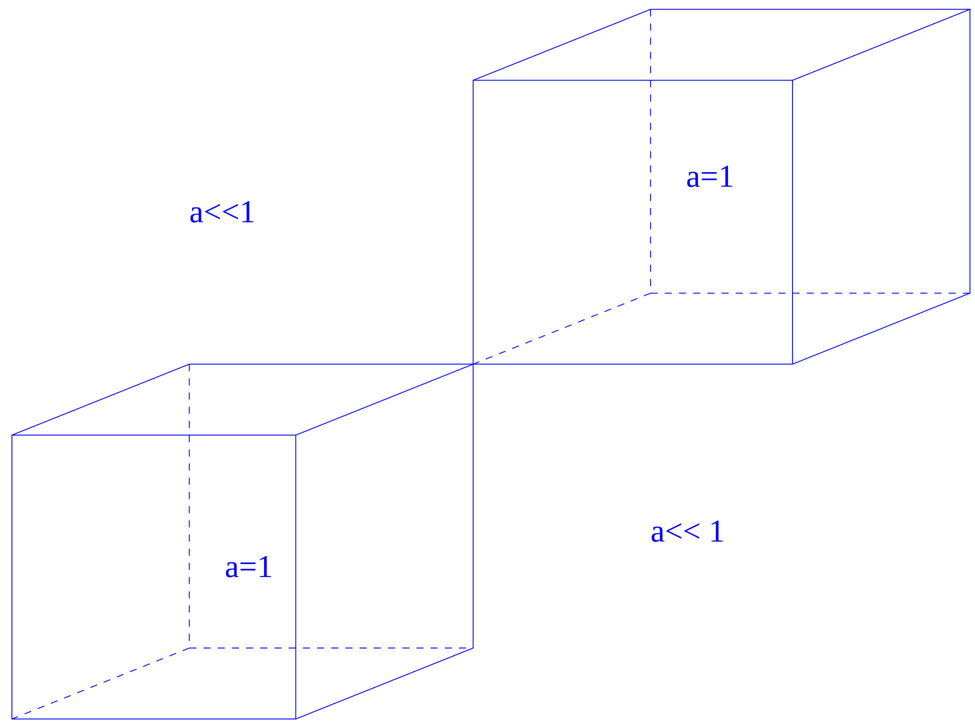
$$a(x) = a_i, \quad x \in \Omega_i$$

It has been numerically observed and theoretical proved that, in three dimensions, the convergence of standard multigrid method deteteriotes as $\max a_i / \min a_i$ gets large and the worst case is

$$\rho \approx 1 - ch$$

This has to do with the fact that, in three dimensions:

$$H^1(\Omega) \subset L^6(\Omega) \text{ and } \|v_h\|_{L^\infty(\Omega)} \le ch^{-1}\|v_h\|_{L^6(\Omega)}.$$


a=1
a<<1
a=1
a<< 1

Technical difficulties

Weighted L^2 inner project and corresponding projection Q_h^a:

$$(v,w)_{L_a^2(\Omega)} = \int_\Omega a(x)v(x)w(x)dx = \sum_{i=1}^m a_i \int_{\Omega_i} vwdx, \quad v,\ w \in L^2(\Omega).$$

If we require uniform estimate (with respect to jumps):

$$\|(I - Q_H^a)v_h\|_{L_a^2(\Omega)} \le \alpha(h,H)H|v_h|_A$$

then (Xu 1992)

$$\alpha(h,H) \ge \frac{H}{h}.$$

In fact the best uniform convergence estimate we can obtain is

$$1 - ch.$$

A new algorithm

Hu and Xu 2002

The new technique is motivated by an estimate of Bramble and Xu (1991):

$$\|(I - Q_k^a)v\|_{L_a^2(\Omega)} \lesssim h_k\|v\|_A + h_k^{\frac{1}{2}}\|v - Q_\Gamma^a v\|_{L_a^2(\Gamma)}, \quad \forall v \in V_h.$$

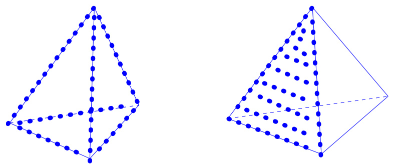
where $\Gamma = \sum \partial\Omega_i \setminus \partial\Omega$. This implies that if $v = 0$ on $\mathcal{W} = \sum \partial^2\Omega_i \setminus \partial\Omega$ (wirebasket), we have

$$\|(I - Q_k^a)v\|_{L_a^2(\Omega)} \lesssim h_k\|v\|_A.$$

We define the following "wirebasket" interpolant used in overlapping DD method:

$$\Pi_0 v = \begin{cases} v(x) & x \in \mathcal{W} \\ \frac{1}{n_i}\sum_{x_i \in \partial\Omega_i} v(x_i) & \text{in } \partial\Omega_i \setminus \mathcal{W} \\ \text{proper extension} & \text{in } \Omega \setminus \Gamma \end{cases}$$

(see Bramble-Pasciak-Schatz (1989), Smith (1991), Chan and Mathew (1994), Xu and Zou (1998)).



Use the range of Π_0 as a coarse subspace together with other standard coarse subspaces, we can prove the corresponding multigrid method converges uniformly with respect to the jumps with a rate estimated by

$$1 - c(\log h)^{-3}.$$

Difficult situations

- Coarse grid elements do not align with discountinuity
- Unstructured grid case: no obvious choice of coarse spaces even for smooth coefficient cases

Construction of coarse subsapces/equations: numerical homogenization

The multigrid method: from analytic/geometric to more algebraic approach

- Analytic/geometric multigrid methods:
  - ⋆ Continuous (PDEs) and discrete problems
  - ⋆ Nested sequence of grids
  - ⋆ Obvious choice of course grid equations
- More algebraic approach: algebraic multigrid method (AMG)
  - ⋆ No obvious coarse level spaces
  - ⋆ No obvious coarse level equations
  - ⋆ No continuous problems available
  - ⋆ Problem independent?

Some examples of works

- AMG (Brandt, Ruge, Stüben, 87)
- Aggregation methods (Mandel, Vanek, Brezina 95, 98)
- Coarsening unstructured meshes (Xu, Bank 95)
- Auxiliary Space method (Xu, 96)
- Harmonic extensions energy minimization (Chan, Smith, Wan, 98)
- AMGe (Brezina, Cleary, Falgout, Henson, Jones, Manteuffel, McCormick, Ruge, 99)
- AMGe (Jones, Vassilevski, 00)
- Agglomerated grids (Chan, Go, Z., 1997, Chan, Xu, Zikatanov 98)
- Stüben (1999, 2000) (commercial version?)
- Many other papers

Design of Algebraic Multigrid methods

Consider a system on a finite dimensional space V_h:

$$A_h u_h = f_h$$

Main considerations:

- Construction of coarser subspaces $V_H \subset V_h$
- Construction of coarse operator A_H

One existing basic theory and beyond

Theorem (Bramble, Pasciak, Wang and Xu'91, Xu 92) Assume
$\exists Q_k : V_J \mapsto V_k$:

$$\|Q_k v\|_A \le C_1 \|v\|_A, np \quad \|(Q_k - Q_{k-1})v\|_0 \le C_2 \lambda_{max}(A_k)^{-1/2} \|v\|_A, \quad \forall v \in V_J$$

Then $\|u - u^k\|_A^2 \le (1 - \frac{1}{CJ})^k \|u - u^0\|_A^2$

$$v_i = (Q_i - Q_{i-1})v \Rightarrow c_0 \le c_1 J$$

- Most existing development and analysis of algebraic multigrid depend on the above theory

- New theory may offer a more precise tool for algorithmic development: $c_0 \longrightarrow$ min!

An energy minimization approach by Chan and Wan

Based on the aforementioned theory, Chan and Wan () formulated the following optimization problem for constructing "coarse" subspace basis functions $\{\phi_i^H\}$ (locally supported):

$$\min \sum_{i=1}^{n_H} |\phi_i^H|_a^2$$

subject to the following constraint:

$$\sum_{i=1}^{n_H} \phi_i^H(x) = 1. \tag{1}$$

- The constraint (1) means that $\{\phi_i^H\}$ preserves constant locally

- For basis functions with minimal overlappings of supports, it is impossible to construct basis functions $\{\phi_i^H\}$ to preserve linear locally on unstructured grids

Features of the approach:

- It was motivated by convergence theory
- It works great numerically for a lot of problems (see Chan and Wan)
- Many other algebraic multigrid methods can be interpreted by a similar minimization problem (Mandel, Vanek ...)

The only problem:

- The minimization problem could be costly?

New observation: the minimization problem can be solved easily/exactly

Xu and Zikatanov (2002)

Theorem. Let $T_i = A_i^{-1} Q_i$ and $T = \sum_{i=1}^{n_H} T_i$. Then the minimizer is given by

$$v_i = T_i T^{-1} 1 \tag{2}$$

and

$$\min \sum_{i=1}^{n_H} |\phi_i^H|_a^2 = (T^{-1}1, 1).$$

Proof. Let $v = 1$ and $w_i = T_i T^{-1} v$ and $\xi_i \in V_i$. Then we obtain

$$
\begin{aligned}
\inf_{\sum v_i = v} \sum_i |v_i|_A^2 &= \inf_{\sum \xi_i = 0} \sum_i |w_i + \xi_i|_A^2 \\
&= \inf_{\sum \xi_i = 0} \sum_i (|w_i|_A^2 + |\xi_i|_A^2) + 2 \sum_i (A w_i, \xi_i) \\
&= \inf_{\sum \xi_i = 0} \sum_i (|w_i|_A^2 + |\xi_i|_A^2) + 2 \sum_i (A_i A_i^{-1} Q_i T^{-1} v, \xi_i) \\
&= \inf_{\sum \xi_i = 0} \sum_i (|w_i|_A^2 + |\xi_i|_A^2) + 2 (T^{-1} v, \sum_i \xi_i) \\
&= \inf_{\sum \xi_i = 0} \sum_i (|w_i|_A^2 + |\xi_i|_A^2).
\end{aligned}
$$

Finally, to get (**??**) we observe that

$$
\begin{aligned}
\sum_i (Aw_i, w_i) &= (A_i w_i, w_i) = (A_i A_i^{-1} Q_i T^{-1} v, w_i) \\
&= (T^{-1} v, \sum_i w_i) = (T^{-1} v, v).
\end{aligned}
$$

One important property

Each basis function ϕ_i^H is harmonic on each coarse "element"

1. $d = 1$: already observed by Chan and Wan

2. $d > 1$: new observation (not obvious)

Related algorithms:

- $d > 1$: it coincides with the generalized finite element by Babŭska and Osborn (1983, 1994)

- $d > 1$: it is analogous to (but different from) the multi-dim generalization of Babŭska-Osborn by Hou et al (1999), see also the Residual-free bubble methods (Brezzi, Hughes, Franca 1993 –)

- AMG: Alcouffe, Brandt, Dendy and Painter (1981), Dendy (1982),
- Hackbusch (1985), Brandt (1996), Zeeuw (1990), Chan, Xu and Zikatanov (1998), Wan (1998), Wan, Chan and Smith (preprint) Mandel, Brezina and Vaněk (preprint).

Discretization Methods Based on Partition of Unity

- Partition of unity methods
- generalized finite element methods
- h-p cloud methods
- ...

Basic idea

Given an overlapping DD: $\Omega = \cup_{i=1}^m \Omega_i$ and various local approximate subspaces, say $V_i \subset H^1(\Omega_i)$. A global conforming space can be obtained by "gluing" all V_i together by means of a partition of unity $\{\phi_i\}$:

$$V_h = \{\sum_{i=1}^m \phi_i(x) v_i(x), v_i \in V_i\}$$

where

$$\mathrm{supp}(\varphi_i) \subset \bar{\Omega}_i, \ \sum_i \varphi_i \equiv 1, \ \|\varphi_i\|_{L^\infty} \lesssim 1, \ \|\nabla \varphi_i\|_{L^\infty} \lesssim (\mathrm{diam}(\Omega_i))^{-1}.$$

A natural and simple choice of partition of unity is the nodal basis functions of the standard linear or bilinear element with $\Omega_i = \mathrm{supp}(\varphi_i)$.

A Conforming Finite Element Method for Overlapping Grids

The main idea of overlapping grids is to divide a physical domain into a set of overlapping subregions which can accommodate smooth, simple, easily generated grids.

- **Why Overlapping Grids?**

- piecewise structured grids to approximate complicated domains
- refinement grids can be added or removed without changing other grids
- different equations/numerical methods may be used on different grids
- better performance in parallel computations
- efficient structured grid solvers may be used?

- **Related Methods:** Finite difference/finite volume methods

- G. Starius. Composite mesh difference methods for elliptic problems. *Numer. Math.*, 28:243–258, 1977.
- W. Henshaw. *Part I: The numerical solution of hyperbolic systems of conservation laws; Part II: Composite overlapping grid techniques*. PhD thesis, Dept. Appl. Math., California Institute of Technology, Pasadena, CA, 1985.
- J. Steger and J. Benek. On the use of composite grid schemes in computational aerodynamics. *Comp. Meth. Appl. Mech. Eng.*, (64):301–320, 1987.
- M. Aftosmis, J. Melton, and M. Berger. Adaptation and surface modeling for Cartesian mesh methods. In *12th AIAA CFD. Conf*, volume AIAA Paper 95-1725, San Diego, CA, June 1995.
- G. Chesshire and W. Henshaw. Composite overlapping meshes for the solution of partial differential equations. *J. Comp. Phys.*, 90:1–64, 1990.
- W. D. Henshaw D. L. Brown and D. J. Quinlan. Overture: An object-oriented framework for solving partial differential equations on overlapping grids. Technical report, UCRL-JC-132017, 1999.
- . . .

- **Related Methods:** Mortar elements (for nonoverlapping grids)

- X.-C. Cai, M. Dryja, and M. Sarkis. Overlapping non-matching grids mortar element methods for elliptic problems. *SIAM J Numer Anal*, 36(2):581–606, 1999.
- Y. Achdou and Y. Maday. The mortar element method with overlapping subdomains. In *Proceedings of the International Conference on Domain Decomposition Methods*. ddm.org.
- X.-C. Cai, T. Mathew, and M. Sarkis. Maximun norm analysis of overlapping non-matching grids discretization of elliptic problems. *SIAM J. on Numerical Analysis*, 37(5):1709–1728.

Finite element space based on the partition of unity

Define

$$
\begin{aligned}
V^h(\Omega) &= \sum_{i=1}^{p} \varphi_i S^{h_i}(\Omega_i) \\
&= \left\{ v = \sum_{i=1}^{p} \varphi_i v_i\,, \quad v_i \in S^{h_i}(\Omega_i) \right\} \subset V(\Omega)
\end{aligned}
$$

where $S^{h_i}(\Omega_i)$ are standard finite elements satisfying

$$
\inf_{v \in S^{h_i}(\Omega_i)} \left(\|u - v\|_{0,\Omega_i} + h_i \|\nabla(u - v)\|_{0,\Omega_i} \right) \lesssim h_i^{m+1} \|u\|_{m+1,\Omega_i}
$$

A basic approximation property

If $d_i \geq h_i$, then

$$\inf_{v_h \in V^h} \|u - v_h\|_{1,\Omega} \leq C \sum_i h_i^m |u|_{m+1,\Omega_i}$$

with C independent of subdomain and mesh parameters.

`Notice the local feature of the estimate!`

Proof. Given $v_h^i \in S^{h_i}(\Omega_i)$, set $v_h = \sum_i \phi_i v_h^i$, then

$$\|\nabla(u - v_h)\|_{0,\Omega} = \|\sum_i \nabla(\phi_i(u - v_h^i))\|_{0,\Omega} \leq \sum_i \|\nabla(\phi_i(u - v_h^i))\|_{0,\Omega_i}$$

$$\leq \sum_i \left(\|\phi_i\|_{0,\infty,\Omega_i} \|\nabla(u - v_h^i)\|_{0,\Omega_i} + \|\nabla\phi_i\|_{0,\infty,\Omega_i} \|u - v_h^i\|_{0,\Omega_i} \right)$$

$$\leq \sum_i \left(\|\nabla(u - v_h^i)\|_{0,\Omega_i} + c d_i^{-1} \|u - v_h^i\|_{0,\Omega_i} \right)$$

POU Finite Element Approximation

Find $u_h \in V^h$ s.t.

$$a(u_h, v) = (f, v) \qquad \forall\, v \in V^h$$

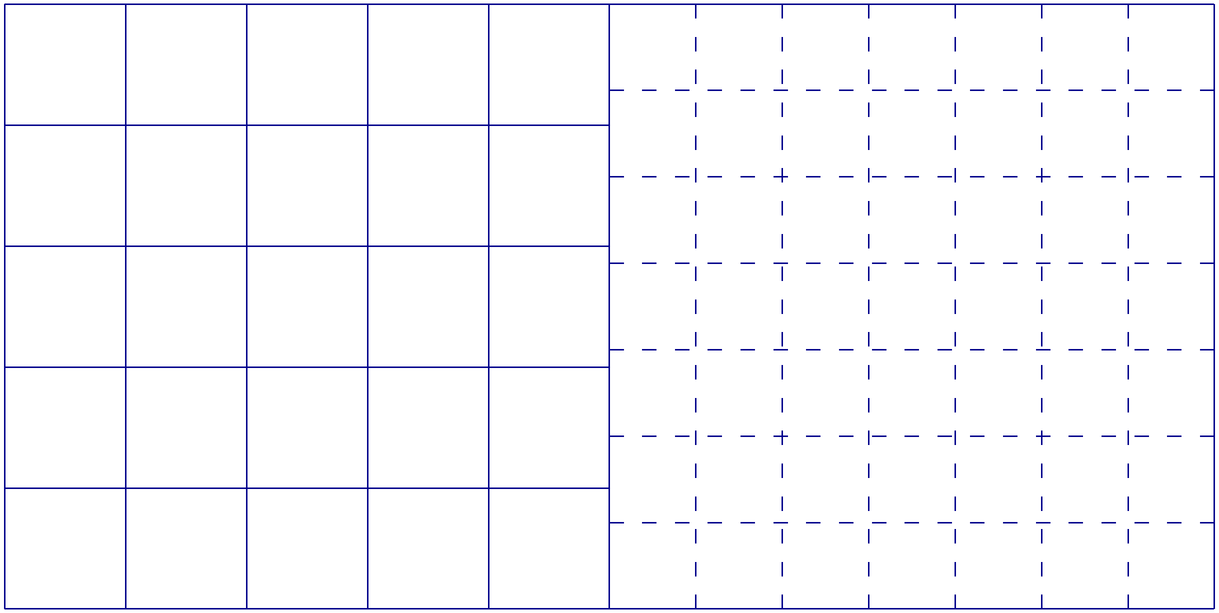
If $d_i \gtrsim h_i$ then

$$\|u - u_h\|_{1,\Omega} \lesssim \sum_{i=1}^{p} h_i^m \|u\|_{m+1,\Omega_i}$$

Furethermore, if Ω is smooth or convex, then

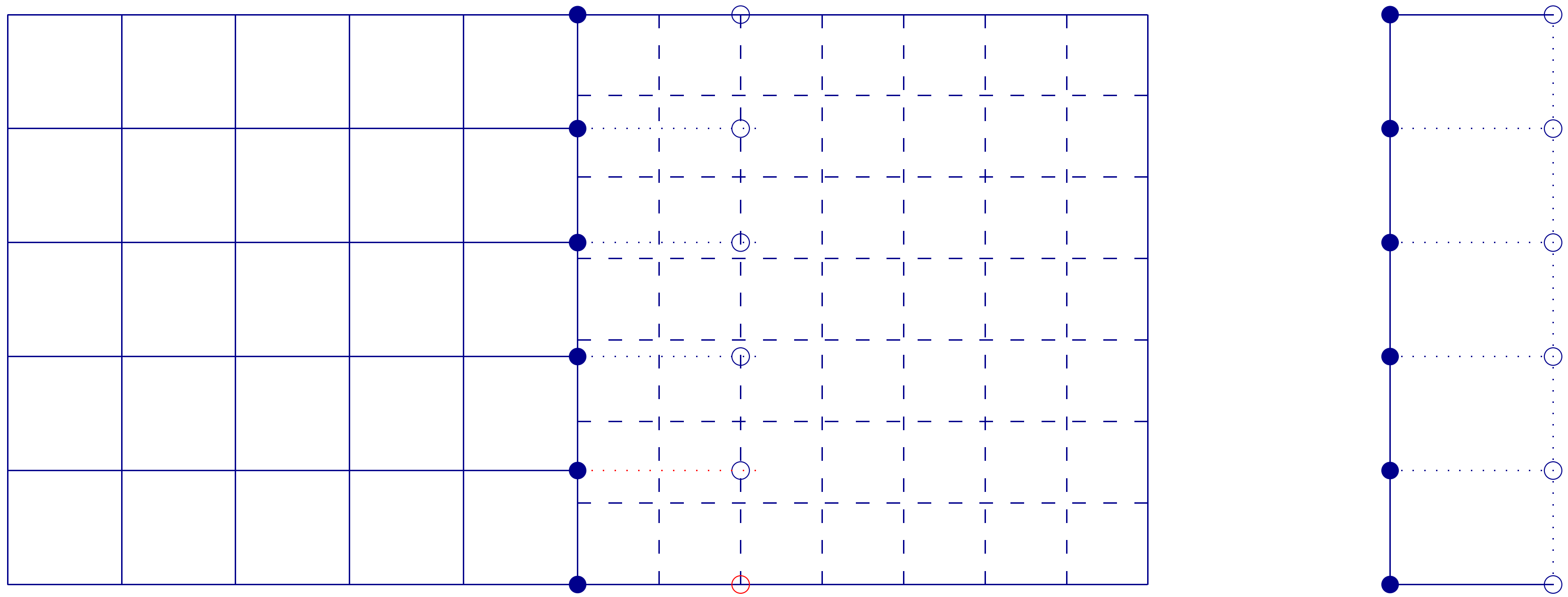
$$\|u - u_h\|_{0,\Omega} \lesssim (\max_j h_j) \sum_{i=1}^{p} h_i^m \|u\|_{m+1,\Omega_i}$$

Nonoverlapping nonmatching grid



There have been a lot of works devoted to study this type of grids recently, mostly using the so-called “mortar” elements or more generally using Lagrange multipliers.

A simple extension to overlapping grids



Make the picture more dynamical: add the extended lines with color late

Nonoverlapping grids can be made to be overlapping by simply extending one or two grid width from one subdomain over to its neighboring subdomain

A special choice of partition of unity

The usual finite element shape functions associated with the grid on the extended region, denoted by G, satisfy:

$$\sum_{x_i \in N_L} \phi_i(x) + \sum_{z_j \in N_L} \phi_j(x) = 1$$

where N_L is the set of grid points on the left and N_R is the set of (auxiliary) grid points on the right.

A special partition of unity can be given by

$$\theta_1(x) = \begin{cases} 1 & x \in \Omega_1 \\ \sum_{x_i \in N_L} \phi_i(x) & x \in G \end{cases} \qquad \theta_2(x) = \begin{cases} 1 & x \in \Omega_1 \setminus G \\ \sum_{z_j \in N_R} \phi_j(x) & x \in G \end{cases}$$

Implementations

- **Main issue: on the construction of partition-of-unity**
  - ⋆ Techniques from overlapping finite difference grid should apply
  - ⋆ Main points:
    - ∗ Each subdomain grid is structured: interaction between neighboring subdomains can be dealt with easily
    - ∗ Overlapping region is one dimension lower, the relevant computation, although maybe complicated, is not expensive.
  - ⋆ Finite element nodal basis functions from each subdomain may be used for partition of unity

- **Resulting linear algebraic systems**
  - ⋆ The matrix may be singular
  - ⋆ Iterative methods apply:
    such as conjugate gradient, Gauss-Seidel, multigrid ...

Some remarks on the method

1. The proposed finite element method provides a viable option for overlapping grids

2. Our method for nonoverlapping grid appears to be an interesting competitor to the mortar finite element method

3. The method has many attractive features

   - most natural and *"perfect"*, at least theoretically, handling of the overlapping regions
   - optimal rate of convergence (better than finite difference or mortar)
   - resulting linear algebraic systems may be naturally solved by modern iterative methods

Application:

Unstructured grids (nonmatching grids)

A simple convection-diffusion example: compact operator and coarse grid

Consider: $L(u) \equiv = -\nabla \cdot (\alpha \nabla u) + \beta \cdot \nabla u + \gamma u = f$ plus boundary conditions

- L is a compact perturbation of an SPD operator $S(u) \equiv -\nabla \cdot (\alpha \nabla u)$
- The discretized L_h is non-SPD and less pleasant to solve. How can we make use of the compactness on discrete level?

A naive iteration: $L = S + N$

$$Su^{\ell} + Nu^{\ell-1} = f$$

This would not converge since S does not dominate N on low frequencies!
But S does dominate N on high frequenices

- Coarse grid discretization:

$$L_H(e_H) = Q_H(f - L(u^{\ell-1}))$$

- Fine grid symmetrization:

$$S(u^\ell) + N(u^{\ell-1} + e_H) = f$$

- This procedure converges with reasonably small H. In fact it gives rise to a contractive iteration with contraction number $O(H)$.

- Using fine h-grid for step 2 and only one iteration (comparing $\|u - u_h\|_1 = O(h)$):

$$\|u - u^h\|_1 = O(h + H^2)$$

- Compact part is significant on low frequencies and can be resolved by a coarse grid

- This procedure may provide a different way for well-posedness studies for partial differential equations in general? (comparing the often-used Galerkin method, or Lerry-Schauder theory)

  Most PDEs are dominated by simple linear operators on high frequencies!

- Other applications

  ⋆ Localization and parallelization
  ⋆ Linearization for nonlinear problems

Example: Fast Poisson solver on unstructured grids

The Poisson equation discretized on general unstructured (shape-regular) grids in both two and three dimensions by finite element, finite volume or finite difference methods can be solved using geometric or algebraic multigrid methods with optimal or nearly optimal compuational complexity.

Example: Fast Stokes solver on unstructured grids

Consider the Stokes equation:

$$\begin{array}{rcll} -\Delta u + \nabla p & = & f & \text{in } \Omega \\ -\nabla \cdot u & = & g & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array}$$

The Stokes operator:

$$\begin{pmatrix} -\Delta & \nabla \\ (\nabla)^* & 0 \end{pmatrix} : [H_0^1(\Omega)]^d \times L_0^2(\Omega) \mapsto [H^{-1}(\Omega)]^d \times L_0^2(\Omega) \text{ is an isomorphism}$$

But the simple diagonal operator Thus

$$\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix} : [H_0^1(\Omega)]^d \times L_0^2(\Omega) \mapsto [H^{-1}(\Omega)]^d \times L_0^2(\Omega) \text{ is also an isomorphism}$$

Thus

$$\begin{pmatrix} (-\Delta)^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} -\Delta & \nabla \\ (\nabla)^* & 0 \end{pmatrix} : [H_0^1(\Omega)]^d \times L_0^2(\Omega) \mapsto [H_0^1(\Omega)]^d \times L_0^2(\Omega)$$

is an isomorphism and hence well-conditioned!

Fast Stokes solvers: Let B_h be a uniform preconditioner for $-\Delta_h$ (given by, for example, AMG), then

$$\begin{pmatrix} B_h & 0 \\ 0 & I_h \end{pmatrix} \begin{pmatrix} -\Delta_h & \nabla_h \\ (\nabla)_h^* & 0 \end{pmatrix}$$

is uniformly well-conditioned and, for example, the corresponding preconditioned *minimal-residual* converges uniformly.

Conclusion:

> The Stokes equation discretized on general unstructured (shape-regular) grids in both two and three dimensions by stable finite element methods

can be solved using geometric or algebraic multigrid methods with optimal or nearly optimal compuational complexity (the preconditioned MINRES method is only one possibility).

An ongoing project: modelling of nonnewtonian fluids

A. Belmonte, A. Jayaramn, Y. Lee and J. Xu

A simpler(?) problem: Navier-Stokes equations

Motivation:

- Guidance for non-newtonian fluid flow
- Proton exchange membrane fuel-cell simulations (with C. Wang and J. Wu)

  coupling of Navier-Stokes equations, nonlinear reaction-diffusion equations and heat equations

Consider the incompressible NS equations:

$$\begin{aligned} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p &= f \quad \text{in } \Omega \\ -\nabla\cdot u &= 0 \quad \text{in } \Omega \end{aligned}$$

together with initial and boundary value problems.

One popular numerical approach is the *projection method* which reduces to solutions of several pressure-Poisson equations (thus *Fast Poisson Solvers* may be used).

Question: With *Fast Stokes Solvers*, can we do better?

One approach based on characteristic method

$$u_t + (u \cdot \nabla)u = \frac{Du}{Dt} = \frac{\partial}{\partial t}(u(x(X,t),t))$$

where

$$\frac{\partial}{\partial t}x(X,t) = u(x(X,t),t).$$

A simple implicit Euler scheme:

$$(u_t + (u \cdot \nabla)u)(x,t^n) \approx \frac{u(x,t^n) - u(x^*,t^{n-1})}{\Delta t}$$

where x^* is the location of the particle x at time t^{n-1}.

Lagrangian-Eulerian discretization scheme:

$$\begin{array}{rcll} \frac{u^n - u(x^*,t^{n-1})}{\Delta t} - \nu\Delta u^n + \nabla p^n & = & f & \text{in } \Omega \\ -\nabla \cdot u^n & = & 0 & \text{in } \Omega \end{array}$$

which can be solved by “Fast Stokes Solvers” at each time step.

References: Pironoun, J. Douglas, G. Yeh,

Some comments

- Can this method be used for NDS of turbulence?
- This method is now being extended to Maxwell upper convective model for non-Newtonian fluid
- The method can be used as an effective smoother for a multigrid procedure to compute steady state solution

A simple illustration

A one-dimensional nonlinear singularly perturbed problem (Osher 1980?):

$$u_t - \varepsilon u_x x + u u_x + u = 0 \quad x \in (0,1), t > 0 u(0,t) = .1, \quad u(1,t) = -2$$

1. A transient computation
2. A multigrid procedure for computing steady-state solution

A simple numerical illustration: a nonlinear convection-dominated problem

Concluding remarks

- Theory
  - ⋆ New theory is optimal and is instrumental in algorithm design
  - ⋆ Given the relationship revealed between MAP and MSC, techniques from both subject areas may be further explored
  - ⋆ Qualitative and quantitative studies of local properties of high frequencies in partial differential equations should be a subject of practical significance
  - ⋆ Two grid method versus Lerry-Schauder theory
  - ⋆ A lot of open problems
- Algorithm and applications
  - ⋆ Fast Poisson solvers on unstructured grids
  - ⋆ Fast Stokes solvers on unstructured grids
  - ⋆ Possible fast solvers for Navier-Stokes equations and other nonnewtonian fluid flows

- $\star$ Algebraic multigrid methods and numerical homogenizations
- $\star$ Energy minimization technique is of great theoretical and practical significance
- $\star$ With appropriate application of multigrid methods, some traditional numerical methods may be viewed differently (such as projection methods for N-S equations, explicit versus implicit methods in general)
- $\star$ Partition of unity method for nonmatching grids is a great competitor with other methods such as martor elements
- $\star$ The great potential of multigrid methods is yet to be realized in applications