Let $J(V,W)$ be any functional on $V \times W$ with V and W be two finite dimensional spaces. Say we have some assumptions on J, V and W to ensure the required minimums can be reached. Then we have

$$\min_{v\in V}\min_{w\in W} J(v,w) = \min_{(v,w)\in V\times W} J(v,w).$$

Proof. Let (v_0,w_0) and (v_1,w_1) be two points in $V\times W$ such that

$$J(v_1,w_1) = \min_{v\in V}\min_{w\in W} J(v,w), \quad J(v_0,w_0) = \min_{(v,w)\in V\times W} J(v,w).$$

Then obviously, $J(v_0,w_0) \le J(v_1,w_1)$. We now prove that $J(v_1,w_1) \le J(v_0,w_0)$.

Let $F(v) = \min_{w\in W} J(v,w)$, then

$$J(v_1,w_1) = \min_{v\in V} F(v) = F(v_1) \le F(v_0) = \min_{w\in W} J(v_0,w) \le J(v_0,w_0),$$

that is the required result.

First note: please replace $\gamma_{\partial F}$ in the definition of the wirebasket coarse interpolant by $\gamma_{h,\partial F}$. All the proofs are still valid by Lemma 4.20.

To solve (8.9), we first express any $w_0 \in V_0(\Gamma)$ into

$$w_0 = \sum_F \gamma_{h,\partial F}(w_0)\phi_F + \sum_E \sum_{x_j\in E} w_0(x_j)\phi_j + \sum_k w_0(x_{v_k})\phi_k,$$

where $\phi_F = I_F^0 1$ and ϕ_j for $x_j \in \mathcal{W}_i$ are natural nodal basis functions on $\mathcal{W}_i$. We see

$$\langle g_0, w_0\rangle = \sum_F \gamma_{h,\partial F}(w_0)\langle g_0,\phi_F\rangle + \sum_E\sum_{x_j\in E} w_0(x_j)\langle g_0,\phi_j\rangle + \sum_k w_0(x_{v_k})\langle g_0,\phi_k\rangle,$$

using $\gamma_{h,\partial F}(w_0) = \frac{1}{n(\partial F)}\sum_{x_j\in\partial F} w_0(x_j)$, $n(\partial F)$ is the number of nodes on ∂F, we can write

$$\langle g_0, w_0\rangle = x^\top b$$

with x a vector consisting of all wirebasket nodal values $w_0(x_j)$. Let x_i be a vector with all components of x corresponding to $\mathcal{W}_i$, and e_i a vector with the same dimension as x_i but with all components 1. Then using the definition of $\|\cdot\|_{h,\mathcal{W}_i}$, (8.9) can be rewritten as follows

$$\min_x \frac{1}{2}\sum_i \rho_i h \min_{\lambda_i\in R^1}(x_i-\lambda_i e_i)^\top(x_i-\lambda_i e_i) - x^\top b,$$

which is equivalent to the following linear systems obtained by taking the derivatives with respect to x and λ_i:

$$\sum_i \rho_i h(x_i - \lambda_i e_i) = b, \tag{1}$$

$$e_i^\top \rho_i h(x_i-\lambda_i e_i) = 0, \quad i = 1,\cdots,p. \tag{2}$$

Let B be the matrix such that $Bx = \sum_i \rho_i h x_i$, then eliminating x from the equation (1), we derive the equation with respect to λ_i:

$$(e_i^\top e_i)\lambda_i - e_i^\top B^{-1}\sum_j(\rho_j h e_j)\lambda_j = e_i^\top B^{-1} b, \quad i = 1,\cdots,p.$$

Once we get λ_i from the above equation, we can then solve (1) to get x.