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For computing $u_i \in V_i$, let $\hat{\mathcal{G}}_i$ denote the union of all subdomains Ω_j satisfying $\Omega_j \cap \Gamma^i \neq \emptyset$, then we know by definition of R_i and Q_i that $u_i = \tilde{u}_i|_\Gamma$ with $\tilde{u}_i$ solving

$$A(\tilde{u}_i, \phi_l) = \langle g, \phi_l \rangle \quad \forall\, \phi_l \in V_0^h(\hat{G}_i).$$

From these, we can state

ALGORITHM 12.1 (INTERFACE ADDITIVE SCHWARZ ALGORITHM). *For any $g \in V^h(\Gamma)$, $u = B_h g \equiv \sum_{i=0}^p u_i$ is calculated as follows:*

1. *Coarse problem: $u_0 \in V_0(\Gamma)$ solves*

$$h_0 \sum_{v_i, v_j \in \Gamma_k} (u_0(v_i) - u_0(v_j))(v_0(v_i) - v_0(v_j)) = (g_0, v_0) \quad \forall v_0 \in V_0(\Gamma).$$

2. *Local problems on $\hat{\mathcal{G}}_i$ $(1 \le i \le p)$: $u_i = \tilde{u}_i|_\Gamma$ with $\tilde{u}_i \in V_0^h(\hat{G}_i)$ solving*

$$A(\tilde{u}_i, \phi_l) = \langle g, \phi_l \rangle_{0,\Gamma}, \ \forall\, \phi_l \in V_0^h(\hat{G}_i),$$

3. *Compute $u = \sum_{i=0}^p u_i$.*

REMARK 12.1. *If we choose some special subdomains $\{\mathcal{G}_i\}$ such that the interface subregions Γ^i are vertex-related, edge-related and face-related overlapping subregions of the interface Γ, then our general algorithms reduce to vertex algorithms proposed by Smith [82, 83].*

12.4. Bibliographic remark. The interface BPX preconditioner is a special restriction of the standard BPX multilevel algorithms on the interface was proposed by Bamble-Pasciak-Xu [16] and generalized by Zhang [103]. Oswald [76] and Zhang [103] proved the optimality of the BPX algorithm. The interface variant of the BPX preconditioner was first considered by Tong-Chan-Kuo [93].

The hierarchical basis methods was initially proposed by Yserentant [101], and studied later by many authors, e.g. Bank-Dupont-Yserentant [3], Yserentant [102], Bank-Xu [4], Xu [95, 96]. The interface hierarchical basis preconditioner was considered by Smith-Widlund [85].

The interface additive Schwarz algorithms is a generalization of the vertex algorithms which was proposed by Smith [82, 83]. Our generalization is not so much of practial application, but just for its own interest. Shao [81] and Chan-Mathew-Shao [29] addressed how to efficiently implement vertex algorithms.

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where all local solvers R_i are taken to be exact, i.e. they are the restrictions of S_h on $V_i(\Gamma)$ defined by

$$\left\langle R_i^{-1} u, v \right\rangle = \langle S_h u, v \rangle \quad \forall\, u, v \in V_i(\Gamma),$$

while the coarse solver $R_0 : V_0 \to V_0$ is defined by

$$\left\langle R_0^{-1} u_0, v_0 \right\rangle = A(I_0 \tilde{u}_0, I_0 \tilde{u}_0). \tag{12.25}$$

Then we have

THEOREM 12.1. *For the additive Schwarz preconditiner B_h defined above,*

$$\kappa(B_h S_h) \lesssim r(\rho).$$

where $r(\rho)$ is the coefficient ratio, i.e. $\max_{1\le i\le p} \rho_i / \min_{1\le i\le p} \rho_i$.

Proof. By Theorem 3.2, we have to estimate three parameters K_0, K_1 and ω_1. We redaily know that $K_1 = q_0$ from the assumption that each point $x \in \Omega$ belonds to at most q_0 subregions from $\{\mathcal{G}_i\}$.

For the estimate of ω_1, it suffices by definition (3.3) to estimate $\lambda_{\max}(R_0 S_0)$ for the coarse subspace as all local solvers are exact. This is easily obtained by noting the fact that $\tilde{u}_0$ are equal to $I_0 \tilde{u}_0$ on the interface for any $u_0 \in V_0(\Gamma)$ and thus

$$\langle S_h u_0, u_0 \rangle = A(\tilde{u}_0, \tilde{u}_0) \le A(I_0 \tilde{u}_0, I_0 \tilde{u}_0) = \left\langle R_0^{-1} u_0, u_0 \right\rangle,$$

which shows $\omega_1 \le 1$.

Finally, we estimate K_0 associated with the partition of $V^h(\Gamma)$. For any $u \in V^h(\Gamma)$, from Lemma 9.1 we have a partition for the harmonic extension $\tilde{u}$ of u: $\tilde{u} = \sum_{i=0}^p u_i$ such that $u_i \in V_i$ and $\sum_i \|u_i\|_{1,\Omega}^2 \lesssim \|u\|_{1,\Omega}$, this with Poincaré inequality implies that

$$\sum_{i=0}^{p} A(u_i, u_i) \lesssim r(\rho)\, A(u, u). \tag{12.26}$$

Obviously, the restrictions w_i of u_i on the interface Γ, i.e. $w_i = u_i|_\Gamma$, belong to $V_i(\Gamma)$ and $u = \sum_{i=0}^p w_i$ by definition of V_i and the choice of u_i. Now we show that

$$\sum_{i=0}^{p} \langle S_h w_i, w_i \rangle \lesssim \langle S_h u, u \rangle. \tag{12.27}$$

Using the fact that the harmonic functions $\tilde{w}_i$ equals to the non-harmonic functions u_i on the interface, we have

$$\langle S_h w_i, w_i \rangle = A(\tilde{w}_i, \tilde{w}_i) \le A(u_i, u_i),$$

from which and (12.26) the inequality (12.27) follows. □

Next we consider how to compute the action $B_h g$ for a given $g \in V^h(\Gamma)$. Let $u = B_h g = \sum_{i=0}^p R_i Q_i g \equiv \sum_{i=0}^p u_i$.

For the coarse solution $u_0 = R_0 Q_0 g$, by definition of R_0 and the equivalency (8.3) we easily know that $u_0 \in V_0(\Gamma)$ solves

$$h_0 \sum_{v_i, v_j \in \Gamma_k} (u_0(v_i) - u_0(v_j))\,(v_0(v_i) - v_0(v_j)) = \langle g_0, v_0 \rangle \quad \forall v_0 \in V_0(\Gamma).$$

Then by Theorem 3.3, the corresponding preconditioner $\hat{B}$ for the operator $\hat{A}$ is

$$\hat{B}\tilde{v} = \hat{P}M_h\hat{P}^*\tilde{v} = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in \Gamma_h} (\tilde{v}, \tilde{\phi}_i^k)\, \tilde{\phi}_i^k \quad \forall\, \tilde{v} \in \tilde{V}^h.$$

The interface preconditioner M_h can thus be defined as in (12.22), and (12.23) with Theorem 3.3 implies that

$$\kappa(B_h S_h) = \kappa(\hat{B}\hat{A}) \le \kappa(M_h A_h) \lesssim 1.$$

12.2. Interface hierarchical basis preconditioner. Let V_k and $\mathcal{N}_k$, $1 \le k \le J$ be defined as in §12.1. One version of the hierarchical basis preconditioner M_h for the stiffness operator A_h can be written as (cf. Xu [95, 96], Yserentant [101]):

$$M_h v = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} (v, \phi_i^k)\, \phi_i^k \quad \forall\, v \in V^h,$$

and $\kappa(M_h A_h) \lesssim \gamma_0(n)$ with $\gamma_0(1) = 1$, $\gamma_0(2) = |\log h|^2$ and $\gamma_0(3) = h^{-1}$.

Then by Theorem 3.3, the corresponding preconditioner $\hat{B}$ for the operator $\hat{A}$ is

$$\hat{B}\tilde{v} = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in N_k \setminus N_{k-1}} (\tilde{v}, \tilde{\phi}_i^k)\, \tilde{\phi}_i^k \quad \forall\, \tilde{v} \in \tilde{V}^h.$$

The interface preconditioner B_h can then be defined as in (12.22), and (12.23) with Theorem 3.3 implies that

$$\kappa(T_h S_h) = \kappa(\hat{B}\hat{A}) \le \kappa(M_h A_h) \lesssim \begin{cases} 1 & \text{if } n = 1 \\ |\log h|^2 & \text{if } n = 2 \\ h^{-1} & \text{if } n = 3. \end{cases}$$

12.3. Interface version of the additive Schwarz method: vertex algorithms. In this section, we consider the interface Additive Schwarz algorithms. As the algorithms can be viewed as restrictions of standard overlapping additive Schwarz algorithms on the interface Γ, we first discuss the standard algorithms defined on the whole domain Ω.

First decompose the domain Ω into p nonoverlapping subregions $\mathcal{G}_i^0$ with diameters of size h_0. Note that $\{\mathcal{G}_i\}$ may be different from the existing subdomains $\{\Omega_i\}$. Then we extend each $\mathcal{G}_i^0$ to a larger subregion $\mathcal{G}_i$ with dist $(\partial\mathcal{G}_i \cap \Omega, \partial\mathcal{G}_i^0) \lesssim h_0$. We assume that each point $x \in \Omega$ belongs to at most q_0 subregions $\{\mathcal{G}_i\}_{i=1}^p$. Restrict these subregions $\mathcal{G}_i$ on the interface, we get

$$\Gamma^i = \Gamma \cap O_i \ne \emptyset, \quad 1 \le i \le p.$$

We define local subspaces $V_i = V^h(\mathcal{G}_i)$ which are associated with subregions $\mathcal{G}_i$, but the coarse subspace V_0 to be the standard coarse subspace defined in §8.1, and by I_0 we denote the corresponding interpolant to V_0.

Now based on the interface subspaces $V_0(\Gamma)$ and $V_i(\Gamma)$ $(1 \le i \le p)$, we define the additive Schwarz preconditioner B_h for the interface operator S_h. As in (3.1), B_h can be defined as follows:

$$B_h = \sum_{i=0}^{p} R_i Q_i,$$

The balancing domain decomposition method was first proposed by Mandel [68], and later its convergence proof was improved by Mandel-Brezina [69]. Cowsar-Mandel-Wheeler [38] extended the BDD method to mixed finite elements. The major idea is to restrict local Neumann subproblems on certain subspaces so that local subproblems are uniquely solvable. Two other ways of overcoming singularities of local Neumann problems are to use different bilinear forms on local subdomains (see §11.1.1) and to use Lagrange multiplier approach by Farhat-Roux [49].

12. Interface preconditioners derived from stiffness preconditioners. We now apply the theory in §3.3 to generate some interface preconditioners from known stiffness preconditioners. Associated with §3.3, we choose the space $V = V^h$ and the subspace $\hat{V}$ of V^h to be the space of discrete harmonic functions, i.e. $\hat{V} = V_H$. As $\hat{P} : V^h \to V_H$ is a orthogonal projection with respect to $\langle S_h \cdot, \cdot \rangle$, it is immidiate to verify that $\hat{P}u$, for any $u \in V^h$, is the discrete harmonic extension of the restriction of u on the interface Γ.

Define an operator $B_h : V^h(\Gamma) \to V^h(\Gamma)$ by

$$B_h = \hat{R}\hat{B}\hat{R}^*. \tag{12.22}$$

where $\hat{R} : \hat{V} \to V^h(\Gamma)$ defined by $\hat{R}\tilde{u} = u$, for any $\tilde{u} \in \hat{V}$, and its adjoint $\hat{R}^*$ by :

$$(\hat{R}^* u, \tilde{v}) = \left\langle u, \hat{R}\tilde{v} \right\rangle.$$

Then we have

$$\kappa(\hat{B}\hat{A}) = \kappa(M_h S_h), \tag{12.23}$$

since it is easy to see $\hat{A} = \hat{R}^* S_h \hat{R}$ and

$$\begin{aligned} \langle B_h S_h u, S_h u \rangle &= \left\langle \hat{R}\hat{B}\hat{R}^* S_h u, S_h u \right\rangle = \left\langle \hat{B}\hat{R}^* S_h \hat{R}\tilde{u}, \hat{R}^* S_h \hat{R}\tilde{u} \right\rangle \\ &= \left\langle \hat{B}\hat{A}\tilde{u}, \hat{A}\tilde{u} \right\rangle. \end{aligned} \tag{12.24}$$

12.1. Interface BPX multilevel preconditioner. Given a sequence of nested piecewise linear finite element subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_J = V^h$$

corresponding to a sequence of nested quasi-uniform triangulations $\{\mathcal{T}^k\}_{k=1}^J$ of Ω with mesh size parameters

$$h_k = \max_{\tau \in \mathcal{T}^k} \operatorname{diam}(\tau) \approx 2^{-k}.$$

Let $\{\phi_i^k\}_{i=1}^{n_k}$ be the set of nodal basis functions of V_k $(1 \le k \le J)$ corresponding to the set $\mathcal{N}_k$ of interior nodal points of the triangulation $\mathcal{T}^k$. The BPX preconditioner M_h for the stiffness operator A_h can be written as (cf. Xu [95, 96]),

$$M_h v = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in \mathcal{N}_k} (v, \phi_i^k)\phi_i^k \quad \forall\, v \in V^h,$$

and $\kappa(M_h A_h) \lesssim 1$.

with α_0 and α_1 appearing in (3.7), i.e.

$$\alpha_0 \langle \hat{S}\hat{u}, \hat{u} \rangle \leq \langle \hat{B}\hat{S}\hat{u}, \hat{u} \rangle \leq \alpha_1 \langle \hat{S}\hat{u}, \hat{u} \rangle \quad \forall\, \hat{u} \in \hat{V},$$

which, from (11.18), is then equivalent to

$$\alpha_0 \langle S_h \hat{u}, \hat{u} \rangle \leq \langle (\sum_{i=1}^{p} \Theta_i S_i^+ \Theta_i^* Q_i) S_h \hat{u}, \hat{u} \rangle \leq \alpha_1 \langle S_h \hat{u}, \hat{u} \rangle \quad \forall\, \hat{u} \in \hat{V}. \tag{11.19}$$

Let $\hat{S}_i : \hat{V}_i \to \hat{V}_i$ be the restriction of S_i on $\hat{V}_i$. Obviously, the inverse $\hat{S}_i^{-1}$ exists and we have by definition of S_i^+ that $S_i^+ u_i = \hat{S}_i^{-1} u_i$, $\forall\, u_i \in \hat{V}_i$.

Let $\check{S}_i$ be defined as in (11.1), i.e. the interface operator on $V^h(\Gamma_i)$ corresponding to the bilinear form $\check{A}_i(\cdot,\cdot) = A_i(\cdot,\cdot) + \rho_i h_0^{-2}(\cdot,\cdot)_{0,\Omega_i}$. Note that the assumptions made on the boundary subdomains anable us to use Friedrichs inequality to obtain

$$\langle S_i^+ u_i, u_i \rangle_{0,\Gamma_i} = \left\langle \hat{S}_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i} \approx \langle \check{S}_i^{-1} u_i, u_i \rangle_{0,\Gamma_i} \quad \forall\, u_i \in \hat{V}_i. \tag{11.20}$$

Now (11.19) holds with $\alpha_1 = \omega_1 = O(\wp^2)$ and $\alpha_0 = K_0^{-1} = O(1)$ by combining (11.20) with (11.21) in the following Remark 11.1, thus $\kappa(B_h S_h) = O(\wp^2)$. □

REMARK 11.1. *Let $N_h = \sum_{i=1}^{p} \Theta_i \check{S}_i^{-1} \Theta_i^* Q_i$, then it is easy to find out by going through the proof of Theorem 11.1 which used Lemma 8.3 that the condition number of $N_h S_h$ restricted on the subspace $\hat{V} = V_0^{\perp}$ is the order $O(K_0\omega) = O(\wp^3)$; but the order $O(\wp^2)$ if each boundary subdomain has a common face with the boundary $\partial\Omega$. More exactly, we have*

$$K_0^{-1} \langle S_h \hat{u}, \hat{u} \rangle \lesssim \langle N_h S_h \hat{u}, S_h \hat{u} \rangle \lesssim \omega_1 \langle S_h \hat{u}, \hat{u} \rangle, \forall\, \hat{u} \in \hat{V}. \tag{11.21}$$

To see this, let $V = V^h(\Gamma)$, $A = S_h$ and define $\hat{P}, \hat{P}^*$ as in Section 3.3, $\hat{S}$ to be the restriction of S_h on $\hat{V}$, and $\hat{B} = \hat{P} B_h \hat{P}^*$. It is straightforward to verify by using $\hat{P}^* \hat{S} \hat{u} = S_h \hat{u}$ that for any $\hat{u} \in \hat{V}$,

$$\begin{aligned} \left\langle \hat{B}_h \hat{S}_h \hat{u}, \hat{S}_h \hat{u} \right\rangle &= \langle M_h S_h \hat{u}, S_h \hat{u} \rangle = \langle R_0 Q_0 S_h \hat{u}, S_h \hat{u} \rangle + \langle N_h S_h \hat{u}, S_h \hat{u} \rangle \\ &= (N_h S_h \hat{u}, S_h \hat{u}) \ (\textit{by } \hat{V} = V_0^{\perp}), \end{aligned}$$

which, combining with Theorem 11.1 implies (11.21).

11.3. Bibliograph remarks. The methods discussed in this section, often known as Neumann-Neumann type of algorithms, can be traced back to the work by Dinh-Glowinski-Périaux [41] and Glowinski-Wheeler[52]. Thereafter there are a few extensions in the theory and algorithms. We refer to Bourgat-Glowinski-Tallec-Vidrascu[6], Roeck-Tallec [77], Tallec-Roeck-Vidrascu [89], Mandel [68, 69], and Dryja-Widlund [45]. For extension of the approach for mixed finite element framework by Glowinski-Wheeler[52] to many subdomain case, see Cowsar-Wheeler[39].

Neumann-Neumann algorithms with weighted coarse subspaces for $n = 3$ was proposed by Dryja-Widlund [45], where the use of standard coarse subspaces (cf. §11.1.2) was also considered for elliptic problems with uniformly bounded coefficients. Here we give a unified presentation for both two and three dimensional cases with the case of large jukmps in coefficients included. In particular, we added the case of using the zero extensions E_i in local solvers R_i instead of weighted operators Θ_i.

Obviously, for a balanced function r^h there exists $u_i \in \hat{V}_i$ such that

$$S_i u_i = \Theta_i^* Q_i r^h,$$

and the solution u_i will be denoted by $u_i = S_i^+ \Theta_i^* Q_i r^h$. Note the inverse of S_i does not exist for interior subdomains Ω_i, and the above $\hat{V}_i$ defined by:

$$\hat{V}_i = \{u_i \in V^h(\Gamma_i); \quad \int_{\Gamma_i} u_i \, dx = 0\}. \tag{11.17}$$

For boundary subdomains $\Omega_i : \bar{\Omega}_i \cap \partial\Omega \neq \emptyset$, we let $\hat{V}_i = V^h(\Gamma_i)$.

Applying **Global Algorithm** in §3.3 to the present case with $A = S_h$, $V = V^h(\Gamma)$, $\hat{V} = V_0^{\perp}$ - the complement of V_0 in the sense of $\langle S_h \cdot, \cdot \rangle$, and as in §3.3, take $\hat{B} = \hat{P}(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i)\hat{P}^*$ as a known preconditioner to $\hat{S}$, then we derive from Theorem 3.4 a preconditioner B_h for S_h:

$$\begin{aligned} B_h S_h &= P_0 + \hat{P}\hat{B}\hat{S}\hat{P} = P_0 + \hat{P}(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i)\hat{P}^* \hat{S} \hat{P} \\ &= P_0 + (I - P_0)(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i) S_h (I - P_0). \end{aligned} \tag{11.18}$$

where $P_0 : V^h(\Gamma) \to V_0$ is the orthogonal projection with respect to $\langle S_h \cdot, \cdot \rangle$. Using (11.18) and the definition of S_i^+, we immediately come to the following algorithm:

ALGORITHM 11.2 (BALANCING DOMAIN DECOMPOSITON ALGORITHM). *For any $g \in V^h(\Gamma)$, $u = B_h g$ is done, step by step, as follows:*

1. *Balancing the original residual by solving*

$$\langle S_h w_0, \phi \rangle = \langle g, \phi \rangle \quad \forall \, \phi \in V_0.$$

Set $r^h = g - S_h w_0$.

2. *Compute u_i, $1 \leq i \leq p$ in parallel: $u_i = \tilde{u}_i|_{\Gamma_i}$ with $\tilde{u}_i \in V^h(\Omega_i)$ solving*

$$A_i(\tilde{u}_i, \phi_l) = \left\langle r^h, \Theta_i \phi_l \right\rangle, \forall \, \phi_l \in V^h(\Omega_i);$$

Compute $\hat{u}_i = \Theta_i u_i$.

3. *Balancing the residual: $w_1 \in V_0$ solves*

$$\langle S_h w_1, \phi \rangle = \left\langle g - S_h \sum_{i=1}^p \hat{u}_i, \phi \right\rangle, \forall \, \phi \in V_0.$$

4. *Compute $u = \sum_{i=1}^p \hat{u}_i + w_1$.*

THEOREM 11.3. *Assume that for each subdomain Ω_i, $\partial\Omega_i \cap \partial\Omega$ is either empty or a face ($n = 3$) or an edge ($n = 2$) of Ω_i. Then for the above BDD algorithm,*

$$\kappa(B_h S_h) \lesssim \log^2(h_0/h).$$

Proof. By using Theorem 3.4, we know

$$\kappa(B_h S_h) \leq \frac{\max(1, \alpha_1)}{\min(1, \alpha_0)}$$

noticing that $\check{w}$ equals to $\tilde{u} - Q^{\rho}_{h_0}\tilde{u}$ on the interface Γ, thus by Lemma 4.10,

$$\check{A}_i(\check{w},\check{w}) \le \check{A}_i(\tilde{u} - Q^{\rho}_{h_0}\tilde{u}, \tilde{u} - Q^{\rho}_{h_0}\tilde{u}) \tag{11.14}$$

then it follows from (11.13)-(11.14) and Lemma 8.1 that

$$\sum_{i=1}^{p} \left\langle R_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i} \lesssim \gamma_0(n) \sum_{i=1}^{p} A_i(\tilde{u},\tilde{u}) = \gamma_0(n) \left\langle S_h u, u \right\rangle, \tag{11.15}$$

with $\gamma_0(2) = \wp$ but $\gamma_0(3) = h_0/h$. Again using Lemmas 4.10 and Lemma 8.1,

$$\begin{aligned} \left\langle R_0^{-1} u_0, u_0 \right\rangle &= \langle S_h u_0, u_0 \rangle = \sum_{i=1}^{p} A_i(\tilde{u}_0, \tilde{u}_0) \le \sum_{i=1}^{p} A_i(Q^{\rho}_{h_0}\tilde{u}, Q^{\rho}_{h_0}\tilde{u}) \\ &\lesssim \gamma(n) A(\tilde{u},\tilde{u}) = \gamma(n) \left\langle S_h u, u \right\rangle, \end{aligned} \tag{11.16}$$

which with (11.15) implies $K_0 = \gamma_0(n)$, that proves (11.12).

(11.11) follows by replacing the operator Q^{ρ}_H in the above proof for the estimation of K_0 by the standard L^2 projection Q_{h_0}.

Finally consider the E_i case, i.e. the local solver $R_i = E_i \check{S}_i^{-1} E_i^*$: the only differece from the Θ_i case is the estimate of K_0, i.e. the derivation of (11.15)-(11.16) from (11.13)-(11.14) with replacing the weighted operator $Q^{\rho}_{h_0}$ by the standard L^2 projection Q_{h_0} here. Then in the present case, we obtain

$$\left\langle R_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i} = \left\langle \check{S}_i E_i^{-1} u_i, E_i^{-1} u_i \right\rangle_{0,\Gamma_i} \lesssim \rho_i \left\| E_i^{-1} u_i \right\|^2_{1/2,\partial\Omega_i} = \rho_i \left\| u_i \right\|^2_{1/2,\partial\Omega_i},$$

now repeating the same decomposition (11.7) and the estimates thereafter, we have

$$\rho_i \|u_i\|^2_{1/2,\partial\Omega_i} \lesssim \wp^2 \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i \right\rangle_{0,\Gamma_i} = \wp^2 \left\langle \check{S}_i w, w \right\rangle_{0,\Gamma_i},$$

the rest is the same as proving (11.15)-(11.16 but with $Q^{\rho}_{h_0}$ replaced by Q_{h_0} here. □

11.2. Balancing Domain Decomposition Method. The balancing domain decomposition method is resulted from another approach to the singularity of S_i on $V(\Gamma_i)$. Rather than modifying the expression of the operator S_i itself as done in the last subsection, S_i can be made nonsingular by removing its null space. In another word, S_i is to be applied on a $V(\Gamma_i)$'s subspace on which S_i is nonsingular. In fact, the null space of S_i is at most a one dimensional space that contains only constant functions. If these contant functions can be annihilated, the operator S_i then becomes nonsingular.

The idea is first to solve the equation on a coarse subspace V_0 so that the resulting residual does not contain any constant component on each Γ_i and then to apply the Neumann-Newmann type algorithm (with a non-modified S_i) to the residual equation. This approach falls into the *local-global* technique described in §3.3.

Let e_0 denote a constant in this subsection. We then define the coarse subspace

$$V_0 = \text{span}\{\Theta_i e_0, \text{ for all interior subdomain } \Omega_i : \bar{\Omega}_i \cap \partial\Omega = \emptyset\}.$$

We say a function $r^h \in V^h(\Gamma)$ is balanced if r^h is orthogonal to V_0, or equivalently,

$$\left\langle \Theta_i^* Q_i r^h, e_0 \right\rangle_{0,\Gamma_i} = 0, \quad i.e. \quad \int_{\Gamma_i} \Theta_i^* Q_i r^h \, dx = 0.$$

$$\lesssim \quad \langle S_h u, u\rangle + \sum_{i=1}^{p} \langle R_i^{-1} u_i, u_i\rangle \lesssim \wp \langle S_h u, u\rangle ,$$

combining this with (11.9) implies $K_0 \lesssim \wp$, which concludes the proof of the first estimate in Theorem 11.1.

The second estimate in the theorem follows by going through the proof and noting that in this case $\omega_1 = O(\wp^2)$ and $K_0 = O(1)$. □

11.1.2. The use of standard coarse subspaces. As an alternative choice, the standard coarse space in §8.1 can also be used as the coarse space to define the preconditioner (11.2). Such a coarse space obviously has a much simpler structure than the weighted space, but it has limitations as it is not efficient for problems wth large discontinuous jumps for $n = 3$.

With V_0 being the standard coarse space and exact coarse solver $R_0 = S_0^{-1}$, the function $w_0 \in V_0$ in (11.3) for the action of M_h can be obtained by solving

$$A(\tilde{w}_0, \phi) = \langle g, \phi\rangle , \ \forall\, \phi \in V_0. \tag{11.10}$$

THEOREM 11.2. *If V_0 is the standard coarse space discussed in §8.1 and $R_0 = S_0^{-1}$, then the preconditioner given by (11.2) satisfies*

$$\kappa(M_h S_h) \lesssim (\max_{1\le i\le p} \rho_i)\wp^2, \tag{11.11}$$

where ρ_i are the coefficients of equation (2.1), or without the coefficients

$$\kappa(M_h S_h) \lesssim \begin{cases} \wp^3 & \text{if } n = 2, \\ \wp^2 h_0/h & \text{if } n = 3. \end{cases} \tag{11.12}$$

Moreover if each Θ_i is replaced by the zero extension operator $E_i : V^h(\Gamma_i) \to V_i$, namely the subspace sover $R_i = E_i \check{S}_i^{-1} E_i^*$, then

$$\kappa(M_h S_h) \lesssim r(\rho)\wp^4.$$

where $r(\rho) = \max_i \rho_i / \min_i \rho_i$.

Proof. By Theorem 3.2, it suffices to estimate K_1, K_0 and ω_1.

The same proof as for Theorem 11.1 gives $K_1 \lesssim 1$ and $\omega_1 \lesssim \wp^2$ for the Θ_i case but $\omega_1 \lesssim r(\rho)\,\wp^2$ for the E_i case. The only difference for the second case is to replace Θ_i by E_i in the proof for Theorem 11.1.

Now we estimate K_0. First consider the Θ_i case, i.e. $R_i = \Theta_i \check{S}_i^{-1} \Theta_i^*$: for any $u \in V^h(\Gamma)$, to define a partition of u, we take $u_0 = (Q^\rho_{h_0} \tilde{u})|_\Gamma \in V_0$ and $u_i = \Theta_i(u - u_0) \in V_i$. Here $Q^\rho_{h_0}$ is the weighted L^2 projection from $L^2(\Omega)$ to V_0 defined in (8.5). Obviously, $u = \sum_{i=0}^p u_i$. Let $w = u - u_0$. Using the properties of Q^ρ_H in Lemma 8.1, we obtain

$$\begin{aligned} \sum_{i=1}^{p} \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i} &= \sum_{i=1}^{p} \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i\right\rangle_{0,\Gamma_i} = \sum_{i=1}^{p} \left\langle \check{S}_i w, w\right\rangle_{0,\Gamma_i} \\ &= \sum_{i=1}^{p} \check{A}_i(\check{w}, \check{w}), \end{aligned} \tag{11.13}$$

We now proceed to establish the estimate that $\omega_1 \lesssim \wp^2$. To this end, it suffices to prove that

$$\langle S_h u_i, u_i\rangle \lesssim \wp^2 \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i}, \ \forall u_i \in V_i. \tag{11.6}$$

By definition of S_h and Lemma 4.9,

$$\langle S_h u_i, u_i\rangle = A(\tilde{u}_i, \tilde{u}_i) \eqsim \sum_m \rho_m \left|u_i\right|^2_{1/2,\partial\Omega_i}, \tag{11.7}$$

where the summation is over all subdomains Ω_m which share either a face, or an edge or a vertex with Ω_i. Let $\gamma_{im} = \partial\Omega_i \cap \partial\Omega_m$, we can write u_i on the interface Γ_m of Ω_m into

$$u_i = \sum_{F\subset\gamma_{im}} I_F^0 u_i + \sum_{E\subset\gamma_{im}} I_E^0 u_i + \sum_{V_k\in\gamma_{im}} I_{V_k}^0 u_i. \tag{11.8}$$

Then

$$\begin{aligned}
&\rho_m \Big|\sum_{F\subset\gamma_{im}} I_F^0 u_i\Big|^2_{1/2,\partial\Omega_m} \\
\lesssim\ & \sum_{F\subset\gamma_{im}} \left|I_F^0(\nu_\rho u_i)\right|^2_{1/2,\partial\Omega_m} \ (\rho_m \le \nu_\rho^2 = \text{Const on } F) \\
\lesssim\ & \sum_{F\subset\gamma_{im}} \left\|I_F^0(\nu_\rho u_i)\right\|^2_{H_{00}^{1/2}(F)} \ (\text{by Lemma 4.6}) \\
\lesssim\ & \wp^2 \left\|\nu_\rho u_i\right\|^2_{1/2,\partial\Omega_i} \ (\text{Lemma 4.14}) \\
=\ & \wp^2 \rho_i \left\|\Theta_i^{-1} u_i\right\|^2_{1/2,\partial\Omega_i} \ (\text{definition of } \Theta_i) \\
\lesssim\ & \wp^2 \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i\right\rangle_{0,\Gamma_i} \ (\text{Lemma 4.10 \& } \check{S}_i\text{'s definition}) \\
=\ & \wp^2 \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i}.
\end{aligned}$$

Conducting the same for the second and third terms in (11.8) with Lemmas 4.10 and 4.13, we obtain (11.6) from (11.7-(11.8) and triangle inequality.

We next estimate K_0. Given any $u \in V^h(\Gamma)$, take $u_0 = \sum_{i=1}^p \Theta_i I_i u \in V_0$ and $u_i = \Theta_i(u - I_i u) \in V_i$. We readily see $u = \sum_{i=0}^p u_i$. Let $w_i = u - I_i u$. We obtain

$$\begin{aligned}
&\sum_{i=1}^p \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i} \\
=\ & \sum_{i=1}^p \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i\right\rangle_{0,\Gamma_i} = \sum_{i=1}^p \left\langle \check{S}_i w_i, w_i\right\rangle_{0,\Gamma_i} \\
=\ & \sum_{i=1}^p \check{A}_i(\check{w}_i, \check{w}_i) \le \sum_{i=1}^p \check{A}_i(\tilde{w}_i, \tilde{w}_i) \ (\text{minimizing of } \check{w}_i) \\
\le\ & \wp \sum_{i=1}^p A_i(\tilde{u}, \tilde{u}) = \wp \left\langle S_h u, u\right\rangle_{0,\Gamma} \ (\text{Lemma 8.3}).
\end{aligned} \tag{11.9}$$

Noting that $u_0 = u - \sum_{i=1}^p u_i$, (11.9), $K_1 \lesssim 1$ and $\omega_1 \lesssim \wp^2$, we deduce that

$$\left\langle R_0^{-1} u_0, u_0\right\rangle \ = \ \wp^{-2} \left\langle S_h u_0, u_0\right\rangle \lesssim \wp^{-2} \left\langle S_h u, u\right\rangle + \wp^{-2} \sum_{i=1}^p \left\langle S_h u_i, u_i\right\rangle$$

With a proper choice of subspace V_0 and a solver R_0, we obtain the following space decomposition

$$V^h(\Gamma) = \sum_{i=0}^{p} V_i$$

and the corresponding PSC preconditioner

$$M_h = R_0 Q_0 + \sum_{i=1}^{p} \Theta_i \check{S}_i^{-1} \Theta_i^* Q_i. \tag{11.2}$$

Thus for any $g \in V^h(\Gamma)$,

$$M_h g = w_0 + \sum_{i=1}^{p} \Theta_i (w_i|_{\Gamma_i}), \tag{11.3}$$

and by means of the definition (11.2), the components w_i can be obtained by

ALGORITHM 11.1 (NEUMANN-NEUMANN ALGORITHM). *The components w_i in (11.3) for $0 \le i \le p$ are calculated as follows:*

1. *$w_i \in V^h(\Omega_i)$, for $1 \le i \le p$, solves the following local Neumman problem*

$$\check{A}_i(w_i, \phi_l) = \langle g, \Theta_i \phi_l \rangle \ , \forall\, \phi_l \in V^h(\Omega_i), \tag{11.4}$$

2. *$w_0 \in V_0$ solves the proper coarse problem depending on V_0 and R_0 to be chosen later, e.g. the subsequent wirebasket coarse problem (11.5) and the standard coarse problem (11.10) to be discussed in §11.1.1 and §11.1.2 respectively.*

11.1.1. The use of weighted coarse space. The method to be discussed now is based on the weighted coarse spaces in §8.3 and the following global coarse solver

$$R_0 = \wp^2 S_0^{-1},$$

concerning the action of the corresponding preconditioner M_h as in (11.2), $w_0 \in V_0$ can be obtained by solving

$$\langle S_h u_0, \phi \rangle = \wp^2 \langle g, \phi \rangle \quad \forall\, \phi \in V_0. \tag{11.5}$$

THEOREM 11.1. *With the aforementioned choice of weighted coarse space and R_0, the preconditioner M_h given by (11.2) satisfies*

$$\kappa(T_h S_h) \lesssim \log^3(h_0/h).$$

Moreover, if each boundary subdomain shares a common face with $\partial\Omega$, then

$$\kappa(T_h S_h) \lesssim \log^2(h_0/h).$$

Proof. By Theorem 3.2, we need to estimate K_1, K_0 and ω_1. Again it is clear that $K_1 \lesssim 1$. Different from all other situations in this paper, the estimate for ω_1 is not that straightforward here.

10.3. Bibliographic remark. The substructuring preconditioners discussed in the section are initiated by Bramble-Pasciak-Schatz [10, 14] and the analogue to wirebasket algorithms on the whole domain. The method in §10.1 is a fundamental algorithm which was applied to generate a lot of similar algorithms, e.g. Bramble-Pasciak-Schatz [12, 13, 14], Cai [18], Cai-Widlund [23], Cai-Gropp-Keyes [19, 22], Liang-Liang [62]

The *wirebasket algorithms* was proposed by Smith [82, 84] (n=3), and later the convergence proofs for elliptic problems with jumps in the coefficients was given by Dryja-Smith-Widlund [43] (n=3); here we add also the 2D case.

11. Algorithms based on local Neumann problems. This section is devoted to another type of preconditioner for the interface operator $S_h : V_\Gamma^h \to V^h(\Gamma)$. These algorithms are based on Neummann problems on subdomains.

The natural space for a Neumann problem on a subdomain, say Ω_i, is $V^h(\Gamma_i)$; nevertheless this is not a subspace of $V^h(\Gamma)$. To overcome this difficulty, for each i, we introduce a subspace V_i consisting of functions in $V^h(\Gamma)$ vanishing at nodes on $\Gamma \setminus \Gamma_i$. The spaces V_i and $V^h(\Gamma_i)$ have the same dimension. Unfortunately the operator S_i is not always invertible. There exist two main approaches to overcome this difficulty. The first approach, to be discussed in the subsection 11.1, is to slightly modify the operator S_i to introduce a nearby nonsingular operator by adding an appropriate lower order term. The first approach leads to the so-called Neumann-Neumann methods. The second approach, to be discussed in the subsection 11.2, is to first solve the coarse grid equation and then solve the residual equation for S_i which is nonsingular as the residual equation can be viewed on the complement of the coarse space in which the kernel of S_i is annihilated. The second approach leads to the so-called balancing domain decomposition method.

11.1. Neumann-Neumann methods. In this subsection, we discuss the methods based on modifying the operator S_i. The modification is based on the following bilinear form:

$$\check{A}_i(u,v) = A_i(u,v) + \rho_i h_0^{-2}(u,v)_{0,\Omega_i},\ \forall\, u, v \in H^1(\Omega_i).$$

Correspondingly, a modified operator $\check{S}_i : V^h(\Gamma_i) \to V^h(\Gamma_i)$ can be defined as follows

$$\left\langle \check{S}_i u, v \right\rangle_{0,\Gamma_i} = \check{A}_i(\check{u}, \check{v}),\ \forall\, u, v \in V^h(\Gamma_i). \tag{11.1}$$

Here "$\check{u}$" denotes the $\check{A}_i$-discrete harmonic extension of u.

Obviously, the modified operator $\check{S}_i$ is invertible. A subspace solver, denoted by R_i, on each V_i is then defined by

$$R_i = \Theta_i \check{S}_i^{-1} \Theta_i^*$$

where Θ_i is defined as in (8.12) which is restated below for convenience

$$\Theta_i u_i = \rho_i^{1/2} I^0_{\Gamma_i}(\nu_\rho^{-1} u_i) \quad \forall\, u_i \in V^h(\Gamma_i);$$

the adjoint $\Theta_i^* : V_i \to V^h(\Gamma_i)$ by

$$\langle \Theta_i^* u_i, v_i \rangle_{0,\Gamma_i} = \langle u_i, \Theta_i v_i \rangle \quad \forall\, v_i \in V^h(\Gamma_i).$$

Let $Q_F : V^h(\Gamma) \to V_0^h(F)$ be the orthogonal projections with respect to $\langle \cdot, \cdot \rangle$, the parallel subspace correction preconditioner for S_h is then given by

$$M_h = R_0 Q_0 + \sum_{F \subset \Gamma} R_F Q_F. \tag{10.5}$$

THEOREM 10.3. *For the preconditioner $B_h = R_0 Q_0 + \sum_F R_F Q_F$, we have*

$$\kappa(B_h S_h) \lesssim (1 + \log(h_0/h))^2.$$

Proof. By Theorem 3.2, we need to estimate K_1, ω_1 and $\rho(\varepsilon)$. Evidently, $K_1 \lesssim 1$ as for each face subspace $V_0^h(F)$, only a fixed number of other face subspaces are not orthogonal to $V_0^h(F)$.

Using Lemma 4.9 and Lemmas 4.6-4.7, we obtain for any $u \in V_0^h(F)$ that

$$\begin{aligned} \langle S_F u, u \rangle &= A_{j_1}(\tilde{u}, \tilde{u}) + A_{j_2}(\tilde{u}, \tilde{u}) \\ &\lesssim \rho_F \|u\|^2_{H^{1/2}_{00}(F)} \eqsim \left\langle R_F^{-1} u, u \right\rangle, \end{aligned}$$

together with Lemma 8.2, we derive $\omega_1 \lesssim 1$.

Finally, we analyse K_0. For any $u \in V^h(\Gamma)$, let $u_0 = I_0 u \in V_0^h$ and $w = u - u_0$. Clearly, $u = u_0 + \sum_F w_F$. We can deduce

$$\begin{aligned} \left\langle R_0^{-1} u_0, u_0 \right\rangle &\lesssim \wp \sum_{i=1}^p \rho_i \|u - \gamma_{\partial\Omega_i}(u)\|^2_{0,\mathcal{W}_i} \text{ (minimizing of } \gamma_{\mathcal{W}_i}(u)) \\ &\lesssim \wp^2 \langle S_h u, u \rangle \ (n = 3) \text{ (Lemmas 4.13 \& 4.10 \& Poincaré ineq)} \\ \left\langle R_0^{-1} u_0, u_0 \right\rangle &= \sum_{i=1}^p A_i(\tilde{u}_0, \tilde{u}_0) \lesssim \wp \sum_{i=1}^p A_i(\tilde{u}, \tilde{u}) = \wp \langle S_h u, u \rangle \ (n = 2) \end{aligned}$$

Consider $n = 3$ and one face F. The same technique as used in (10.3) gives

$$\left\langle R_F^{-1} I_F^0 w, I_F^0 w \right\rangle \eqsim \rho_F \|I_F^0 w\|^2_{H^{1/2}_{00}(F)} \lesssim \wp^2 (A_{j_1}(\tilde{u}, \tilde{u}) + A_{j_2}(\tilde{u}, \tilde{u})),$$

which holds also for $n = 2$ by Lemma 4.19. Therefore $K_0 \lesssim \wp^2$, which ends the proof of Theorem 10.3. □

Recall the **Substructuring Algorithm I** in §10.1, we easily come to

ALGORITHM 10.3 (WIREBASKET ALGORITHM). *For any $g \in V^h(\Gamma)$, $B_h g = R_0 Q_0 g + \sum_f R_F Q_F g = u_0 + \sum_F u_F$ is computed as follows:*

1. *Compute $u_F \in V_0^h(F)$ in parallel:*

$$\left\langle (-\Delta_{F,h})^{1/2} u_F, v_F \right\rangle = \langle g, v_F \rangle, \quad \forall\, v_F \in V_0^h(F).$$

2. *If $n = 3$, compute $u_0 \in V_0$ by solving the minimization problem:*

$$\min_{w_0 \in V_0} \frac{1}{2} \wp \sum_{i=1}^p \rho_i \min_{\lambda_i \in R^1} \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - \langle g, w_0 \rangle;$$

if $n = 2$, compute $u_0 = w_0|_\Gamma \in V_0$ by solving the coarse problem:

$$\gamma(w_0, v_0) = \langle g, v_0 \rangle \quad \forall\, v_0 \in V_0.$$

where $\gamma(\cdot, \cdot)$ is defined in §10.1.

3. *Compute $B_h g = u_0 + \sum_F u_F$.*

Then as $u = u_P + u_H$, we have

$$\begin{aligned}
&\wp^{-2} Q_i(u - \mu_i(u), u - \mu_i(u)) \\
&\quad \leq \quad \wp^{-2} Q_i(u - \gamma_{\Omega_i}(u_H), u - \gamma_{\Omega_i}(u_H)) \ (\mu_i\text{'s minimizing}) \\
&\quad \lesssim \quad \|u - \gamma_{\Omega_i}(u_H)\|^2_{1/2,\partial\Omega_i} \ (\text{by (10.4)}) \\
&\quad \lesssim \quad \|u_H - \gamma_{\Omega_i}(u_H)\|^2_{1,\Omega_i} \ (\text{Lemma 4.9}) \\
&\quad \lesssim \quad |u_H|^2_{1,\Omega_i} \ (\text{Friedrichs ineq}) \\
&\quad \lesssim \quad \|u - \mu_i(u)\|^2_{1/2,\partial\Omega_i} \ (\text{Lemma 4.10}) \\
&\quad \lesssim \quad Q_i(u - \mu_i(u), u - \mu_i(u)) \ (\text{from (10.4)}).
\end{aligned}$$

Now we can define the preconditioner M_h to the stiffness operator A_h by

$$(M_h^{-1} u, v) = A(u_P, v_P) + \sum_{i=1}^{p} \rho_i \, Q_i(u - \mu_i(u), v - \mu_i(v)) \quad \forall\, u, v \in V^h.$$

Then the previous statement shows

LEMMA 10.2.

$$\kappa(M_h A_h) \lesssim (1 + \log(h_0/h))^2.$$

The algorithm for solving problem associated with the corresponding bilinear form is given below.

ALGORITHM 10.2. ***algorithm to be included***

10.2. A variant of the substructuring preconditioner-I. We shall now present a parallel subspace correction version of the substructuring preconditioner-I in §10.1. This variant of substructuring method was first considered by Smith [82, 84], known as wirebasket methods there.

As mentioned in §2, the "breaking" process on the interface Γ gives a natural decomposition of the space $V^h(\Gamma)$ as follows:

$$V^h(\Gamma) = V_0 + \sum_{F \subset \Gamma} V_0^h(F),$$

where $V_0 = I_0 V^h(\Gamma)$ with I_0, the *joint-operator* introduced in §2, being the wirebasket interpolant defined in §8.2 ($n = 3$) and standard coarse space interpolant ($n = 2$) defined in §8.1.

The coarse subspace solver R_0 is chosen to be the interface restriction of the standard coarse solver R_0 defined in §8.1 for $n = 2$ and the wirebasket coarse solver on the interface defined in §8.2 for $n = 3$, i.e.

$$\left\langle R_0^{-1} u_0, v_0 \right\rangle = \wp \sum_{i=1}^{p} \rho_i \left\langle u_0 - \gamma_{h,\mathcal{W}_i}(u_0), v_0 - \gamma_{h,\mathcal{W}_i}(v_0) \right\rangle_{h,\mathcal{W}_i};$$

For each face F, let ρ_F be the average value defined as in §10.1, and we adopt local face solvers $R_F^{-1} = \rho(F)\,(-\Delta_{F,h})^{1/2}$.

where $\gamma(I_0 u_H, I_0 v_H)$ is defined as follows:

$$\begin{aligned}\gamma(I_0 u_H, I_0 v_H) &= h_0 \sum_{v_i, v_j \in \Gamma_k} (u_H(v_i) - u_H(v_j))\,((v_H(v_i) - v_H(v_j)), \qquad \text{if } n = 2,\\ \gamma(I_0 u_H, I_0 v_H) &= \wp \sum_{i=1}^{p} \rho_i \left\langle u_H - \gamma_{h,\mathcal{W}_i}(u_H), v_H - \gamma_{h,\mathcal{W}_i}(v_H)\right\rangle_{h,\mathcal{W}_i}, \qquad \text{if } n = 3.\end{aligned}$$

The above statement shows

THEOREM 10.1. *For any $u \in V^h$,*

$$\kappa(M_h A_h) \lesssim C(n).$$

where $C(2) = C$ independent of the h and h_0, but $C(3) = (1 + \log(h_0/h))^2$.

Recall the standard and wirebasket coarse solvers defined in §8.1 and §8.2 respectively, we have the following algorithm:

ALGORITHM 10.1 (SUBSTRUCTURING ALGORITHM I). *For given $g \in V^h$, let $u = M_h g$, then $u = u_P + u_H$ can be obtained as follows:*

1. *For $1 \le i \le p$, $u_P \in V_0^h(\Omega_i)$ solves*

$$A(u_P, v) = (g, v) \quad \forall v \in V_0^h(\Omega_i).$$

2. *On each face $F \subset \Gamma$, u_E solves*

$$\rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} = (g, v) - A(u_P, v) \quad \forall\, v \in V_0^h(F).$$

3. *If $n = 2$, find $u_0 = \tilde{I}_0 u_H \in V_0$ solving*

$$\gamma(u_0, v) = (g, v) - A(u_P, v) \quad \forall\, v \in V_0;$$

If $n = 3$, find $u_0 \in V_0(\Gamma)$ on Γ by solving

$$\min_{w_0 \in V_0} \frac{1}{2} \sum_{i=1}^{p} \rho_i \min_{\lambda_i \in R^1} \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - \langle g, w_0\rangle - A(u_P, w_0) \quad \forall\, w_0 \in V_0(\Gamma),$$

4. *Extend $u_0 + \sum_F u_E$ harmonically by solving homogeneous Dirichlet problem on each Ω_i.*

Substructuring preconditioner II (n=3). We next present another substructuring technique that is based on the following estimate implied by Lemmas 4.13-4.14:

$$\wp^{-2} Q_i(w, w) \lesssim \|w\|^2_{1/2,\partial\Omega_i} \lesssim Q_i(w, w) \quad \forall\, v \in V^h(\partial\Omega_i). \tag{10.4}$$

where

$$Q_i(u, u) = h \sum_{x_i \in \mathcal{W}_{i,h}} u^2(x_i) + \sum_{F \subset \partial\Omega_i} \left\langle (-\Delta_{F,h})^{1/2} I_F^0 u, I_F^0 u \right\rangle_{0,F} \quad \forall\, u \in V^h(\partial\Omega_i).$$

Here $\mathcal{W}_{i,h}$ is the set of nodes on $\mathcal{W}_i$. Let $\mu_i(v)$ be a constant satisfying that

$$Q_i(v - \mu_i(v), 1) = 0, \forall\, v \in V^h(\partial\Omega_i).$$

10.1. Substructuring preconditioners-I ($n = 2, 3$). At first, we present the technical details on the construction of substructuring preconditioners whose motivation has been discussed in §2. As mentioned before, a proper choice of the joint-operator on the space $V^h(\Gamma)$ is crucial in the construction of such preconditioners.

Let I_0 (resp. V_0) be the standard coarse interpolant (resp. standard coarse subspace) defined in §8.1 for $n = 2$, but the wirebasket interpolant (resp. the wirebasket coarse space) defined in §8.2 for $n = 3$.

As in §2, we split any $u \in V^h$ into $u = u_P + u_H$ with $u_P \in V_0^h(\Omega_i)$ for $1 \le i \le p$ and u_H being a $A(\cdot,\cdot)$-discrete harmonic function in Ω. To decompose u_H, we use an auxiliary operator $\tilde{I}_0$: for $n = 3$, $\tilde{I}_0 u_H = I_0 u_H$; but for $n = 2$, $\tilde{I}_0 u_H$ is the $A(\cdot,\cdot)$-discrete harmonic extension of $(I_0 u_H)|_\Gamma$, then using $\tilde{I}_0$, we can write $u_H = \tilde{I}_0 u_H + u_E$ with $u_E = (u_H - \tilde{I}_0 u_H)$ vanishing on the wirebasket set. By the triangle inequalty,

$$A(u_H, u_H) \le 2A(u_E, u_E) + 2A(\tilde{I}_0 u_H, \tilde{I}_0 u_H). \tag{10.1}$$

As u_E vanishing on the wirebasket set, it follows from Lemmas 4.6-4.7 and 4.9 that

$$\begin{aligned} A(u_E, u_E) &\eqsim \sum_i \rho_i \, |u_E|^2_{1/2,\partial\Omega_i} \lesssim \sum_i \sum_{F\subset\Omega_i} \rho_i \, |u_E|^2_{H_{00}^{1/2}(F)} \\ &\eqsim \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, u_E \right\rangle_{0,F} \end{aligned}$$

where ρ_F is the average value of two coefficients associated with two subdomains sharing the common face F, this combining with (10.1) and $\tilde{I}_0 u_H = I_0 u_H$ on Γ yields

$$A(u_H, u_H) \lesssim \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, u_E \right\rangle_{0,F} + \sum_{i=1}^{p} \rho_i A_i(I_0 u_H, I_0 u_H), \tag{10.2}$$

Note that for $n = 3$,

$$I_F^0 u_E = I_F^0(u - I_0 u) = I_F^0(u - \gamma_{\partial F}(u)) = I_F^0 u - \gamma_{\partial F}(u) I_F^0 1, \tag{10.3}$$

thus we derive by using Lemmas 4.13-4.14 and Lemma 4.20 that

$$\begin{aligned} \rho_F \|I_F^0 u_E\|^2_{H_{00}^{1/2}(F)} &\lesssim \wp^2(\rho_{j_1} |u|^2_{1/2,\partial\Omega_{j_1}} + \rho_{j_2} |u|^2_{1/2,\partial\Omega_{j_2}}) \\ &\eqsim \wp^2(A_{j_1}(\tilde{u},\tilde{u}) + A_{j_2}(\tilde{u},\tilde{u})) \text{ (Lemma 4.9)}, \end{aligned}$$

where Ω_{j_1} and Ω_{j_2} are two subdomains sharing the face F, this along with (10.1)-(10.2) and Lemmas 4.14 and 8.2 implies for $n = 3$ that

$$\begin{aligned} A(u_H, u_H) &\lesssim \wp \sum_{i=1}^{p} \rho_i (\wp \, |u_H|^2_{1,\Omega_i} + \|I_0 u_H - \gamma_{\mathcal{W}_i}(I_0 u_H)\|^2_{0,\mathcal{W}_i}) \\ &\lesssim \wp^2 A(u_H, u_H) \text{ (Lemma 8.2)}. \end{aligned}$$

In summary, for $n = 2$ (noting (8.3)) we define the preconditioner M_h by

$$(M_h^{-1} u, v) = A(u_P, v_P) + \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} + \gamma(I_0 u_H, I_0 v_H),$$

while for $n = 3$ (noting (4.20)) by

$$(M_h^{-1} u, v) = A(u_P, v_P) + \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} + \gamma(I_0 u_H, I_0 v_H).$$

Let $V_i = V_0^h(\mathcal{G}_i)$, for $1 \le i \le p$. The overlapping Schwarz method is based on the additive Schwarz preconditioner for the stiffness operator A_h defined as follows:

$$M_h = R_0 Q_0 + \sum_{i=0}^{p} A_i^{-1} Q_i,$$

where the operator $A_i : V_i \to V_i$ is the restriction of A_h on V_i for $1 \le i \le p$, but $R_0 : V_0 \to V_0$ is the standard coarse solver defined in §8.1 for $n = 2$ and the wirebasket coarse solver $\tilde{R}_0$ defined in §8.2.

We have the following condition number bounds for M_h:

LEMMA 9.2. *For $n = 2, 3$,*

$$\kappa(M_h A_h) \lesssim (1 + \log(h_0/h))^{n-1}.$$

Proof. By Lemma 3.2 it suffices to estimate the parameters K_0, K_1 and ω_1. As we are using exact local and global coarse solver, we know that $\omega_1 = 1$. By the definitions of the local subspaces V_i $(1 \le i \le p)$ and the parameter K_1 in (3.5), we can also readily know that $K_1 \lesssim 1$. What remains is to bound the parameter K_0 which is the smallest constant satisfying

$$\sum_{i=0}^{p} A(u_i, u_i) \le K_0 A(u, u), \quad \forall u \in V^h,\ u = \sum_{i=0}^{p} u_i,\ u_i \in V_i.$$

For this purpose, by Lemma 9.1 we need only to find an operator $Q_0 : V^h \to V_0$ satisfying (8.1). We can take the weighted L^2 projection defined in (8.5) for $n = 2$ but the wirebasket interplant itsself defined in §8.2. The results follow then from Lemma 8.1 and (8.6)-(8.7). □

We have the following algorithm for computing the action of the preconditioner M_h:

ALGORITHM 9.1 (OVERLAPPING SCHWARZ METHOD). *For given $g \in V^h$, let $u = M_h g = R_0 Q_0 + \sum_{i=1}^{p} A_i^{-1} Q_i \equiv \sum_{i=0}^{p} u_i$, and u_i cab be computed as follows:*

1. *For $i = 1, \cdots, p$, $u_i \in V_0^h(O_i)$ solves the Dirichlet problem:*

$$A(u_i, v) = (g, v) \quad \forall\, v \in V_0^h(O_i).$$

2. *u_0 can be obtained for $n = 2$ from (8.4) and for $n = 3$ from (8.10)- (8.11) with $g_0 = Q_0 g$.*

9.1. Bibiographic remarks. The partition lemma was first introduced in the domain decomposition context by Nepomnyaschikh [75] and Lions [64].

The convergence analysis of overlapping Schwarz methods with standard coarse subspaces for elliptic problems with jumps in the coefficients was considered by Dryja-Sarkis-Widlund [42] and Martins in [71] for the case that neighboring coefficients at each vertex of subdomains are monotone in a certain direction.

10. Substructuring method. We shall now discuss the so-called substruturing methods and pay attention to three major algorithms and some of their variants.

8.4. Bibliographic remarks. The approximation results (8.2) for standard coarse interpolants was proved by Bramble-Xu [17].

Wirebasket subspaces was first proposed by Smith [82, 84] in an interface formulation. Here we give its equivalent global formulation.

Weighted coarse subspace was originated from Bramble-Paschiak-Schatz [12, 14] for solving their coarse problem and first used by Dryja-Widlund [45] and Mandel [68] for so-called Neumann-Neumannn type methods (cf. §11.1 and §11.2).

9. A partition lemma and overlapping Schwarz methods for elliptic probems with jumps in the coefficients. It is known that in the three dimensional case, the overlapping Schwarz methods with standard coarse subspaces described in §8.1 are not effective for the problem (2.1), i.e. the elliptic problem with large jumps in the coefficients. In this section, we will show that with the help of the wirebasket subspace given in §8.2, the overlapping Schwarz methods will work pretty well also for the problem (2.1).

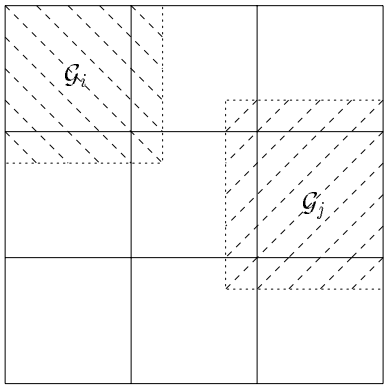

FIG. 5. *overlapping subdomains* $\{\mathcal{G}_i\}$

We first introduce the partition lemma which says that any function of the finite element space V^h defined on the triangulation $\mathcal{T}^h$ (cf. §2) can be decomposed into a sum of functions in a few of subspaces of V^h and the partition is stable in the $H^1(\Omega)$-norm.

Suppose we are now given p non-overlapping subregions $\mathcal{G}_i^0$ $(1 \le i \le p)$ of Ω, then each $\mathcal{G}_i'$ is extended to a larger $\mathcal{G}_i$ with dist $(\partial\mathcal{G}_i \cap \Omega, \partial\mathcal{G}_i^0) \lesssim h_0$, cf. Fig. 5. Assume that $\partial\mathcal{G}_i'$ and $\partial\mathcal{G}_i$ align with elements of $\mathcal{T}^h$ and each point $x \in \Omega$ belongs to at most q_0 subdomains of $\{\mathcal{G}_i\}_{i=1}^p$, with $q_0 < p$ a positive integer. We have

LEMMA 9.1. *Let V_0 be a subspace of V^h such that $V_0 = I_0 V^h$. We assume that there exists another projection $Q_0 : V^h \to V_0$ satisfying the L^2 approximation and H^1 stability (8.1). Then for any $u \in V^h$, there exist elements $u_0 \in V_0$ and $u_i \in V_0^h(\mathcal{G}_i)$ such that $u = \sum_{i=0}^p u_i$ and*

$$\sum_{i=0}^{p} \|u_i\|^2_{H^1_\rho(\Omega)} \le \max\{\alpha_1^2, \alpha_0^2\} \|u\|^2_{H^1_\rho(\Omega)},$$

where α_0 and α_1 are constants appearing in (8.1).

Overlapping Schwarz methods. Let V_0 be the standard coarse subspace of V^h described in §8.1 for $n = 2$, but the wirebasket subspace defined in §8.2 for $n = 3$.

8.3. Weighted coarse subspaces. This coarse space is a subspace of $V^h(\Gamma)$ defined on the interface Γ, and it uses at most one degree of freedom per subdomain.

Let χ_K be the characteristic function on a subset $K \subset \Gamma$ and for each local interface Γ_i, let V_i be a subspace of $V^h(\Gamma)$ with functions vanishing at nodes in $\Gamma \setminus \Gamma_i$.

Using a weighted counting function

$$\nu_\rho(x) = \sum_{i=1}^{p} \rho_i^{1/2}\, \chi_{\partial\Omega_i}(x),\ \forall\, x \in \Gamma \cup \partial\Omega,$$

and a special partition of unity on the interface Γ: $\sum_{i=1}^{p} \Theta_i \equiv 1$ on Γ with $\Theta_i : V^h(\Gamma_i) \to V_i$ given by

$$\Theta_i u_i = \rho_i^{1/2}\, I^0_{\Gamma_i}(\nu_\rho^{-1} u_i),\ \forall\, u_i \in V^h(\Gamma_i), \tag{8.12}$$

we define the weighted coarse subspace as

$$V_0 = \text{span}\{\Theta_i I_i u;\ 1 \le i \le p,\ \ u \in V^h(\Gamma)\},$$

where $I_i : V^h(\Gamma_i) \to V^h(\Gamma_i)$, is defined, for any $u \in V^h(\Gamma_i)$, by $I_i u(x) = \gamma_{\partial\Omega_i}(u)$ for $x \in \Gamma_{i,h}$ and $I_i u(x) = 0$ for $x \in \partial\Omega_i \cap \partial\Omega$; but $I_i \equiv 0$ if $\partial\Omega_i \cap \partial\Omega$ is a face or an adge.

LEMMA 8.3. *For any $u \in V^h(\Gamma_i)$, we have*

$$|\widetilde{I_i u}|^2_{1,\Omega_i} \lesssim \begin{cases} \wp\, |\tilde{u}|^2_{1,\Omega_i} & \text{if } \partial\Omega_i \cap \partial\Omega \text{ is a vertex} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\|\tilde{u} - \widetilde{I_i u}\|^2_{0,\Omega_i} \lesssim \begin{cases} h_0^2\, |\tilde{u}|^2_{1,\Omega_i} & \text{if } \partial\Omega_i \cap \partial\Omega \text{ is empty or a face} \\ h_0^2 \wp\, |\tilde{u}|^2_{1,\Omega_i} & \text{otherwise.} \end{cases}$$

Proof. The case that $\partial\Omega_i \cap \partial\Omega$ is empty or a face, is the consequence of Friedrichs' and Poincare's inequalities, while the one that $\partial\Omega_i \cap \partial\Omega$ is an adge follows from Lemma 4.16.

We next prove the case that $\partial\Omega_i \cap \partial\Omega$ is a vertex, say x_k. We can write $I_i u = \gamma_{\partial\Omega_i}(u) - \gamma_{\partial\Omega_i}(u)\phi_k$ with ϕ_k the nodal basis function of $V^h(\Gamma)$ at the vertex x_k. It follows from Lemma 4.13 that for any constant c,

$$|\gamma_{\partial\Omega_i}(u)|^2 \lesssim \max_{v_j \in W_{i,h}} |u(v_j)|^2 \lesssim \max_{v_j \in W_{i,h}} |u(v_j) - c|^2 \lesssim h^{2-n}\, \wp\, \|\tilde{u} - c\|^2_{1,\Omega_i},$$

from this, Lemma 4.13 and Poincaré inequality we have,

$$\begin{aligned} |\widetilde{I_i u}|^2_{1,\Omega_i} &= |\gamma_{\partial\Omega_i}(u)|^2 |\phi_k|^2_{1,\Omega_i} \lesssim \wp\, |\tilde{u}|^2_{1,\Omega_i}, \\ \|\widetilde{I_i u} - \tilde{u}\|^2_{0,\Omega_i} &\lesssim \|\tilde{u} - \gamma_{\partial\Omega_i}(u)\|^2_{0,\Omega_i} + |\gamma_{\partial\Omega_i}(u)|^2 |\phi_k|^2_{0,\Omega_i} \\ &\lesssim h_0^2 |\tilde{u}|^2_{1,\Omega_i} + h^2\, \wp\, |\tilde{u}|^2_{1,\Omega_i} \lesssim h_0^2\, \wp\, |\tilde{u}|^2_{1,\Omega_i}, \end{aligned}$$

this completes the proof of Lemma 8.3. □

REMARK 8.1. *If one replaces the power $1/2$ of ρ_i by any real number $t \ge 1/2$ in the definitions of ν_ρ, the coarse subspace V_0 and the operators Θ_i, all related results in the subsequent sections are still true with little modification.*

$$\lesssim \quad h_0\wp\sum_{i=1}^{p}\rho_i\sum_{F\subset\partial\Omega_i}|\gamma_{\partial F}(w_0)|^2+\wp\sum_{i=1}^{p}\rho_i\,\|u_0-\gamma_{\mathcal{W}_i}(u_0)\|^2_{0,\mathcal{W}_i}$$

$$\lesssim \quad \wp\sum_{i=1}^{p}\rho_i\,\|w_0\|^2_{0,\mathcal{W}_i}\ \text{(by Lemma 4.20 for } |\gamma_{\partial f}(w_0)|).$$

The first inequality follows from Lemma 4.13 and Lemma 4.20,

$$\wp^{-1}|||u_0|||^2\lesssim\wp^{-1}\sum_{i=1}^{p}\rho_i\,\|u_0\|^2_{0,\mathcal{W}_i}\lesssim\sum_{i=1}^{p}\rho_i\,\|u_0\|^2_{1/2,\partial\Omega_i}\lesssim A(u_0\,,u_0).$$

□

Lemma 8.2 actually implies an equivalent coarse solvers both for the global coarse operator A_0 and the interface operator S_0. For the sake of implementation, we can replace the integration appearing in the norm $|||\cdot|||_{0,\mathcal{W}_i}$ by the numerical counterpart. Using Lemma 4.20 and Lemma 8.2, we obtain

$$\wp^{-1}|||u_0|||_h^2\lesssim A(u_0\,,u_0)=\langle S_h\,u_0\,,u_0\rangle\lesssim\wp|||u_0|||_h^2\,.$$

We now decsribe these two solvers.

Wirebasket coarse solver R_0 on the interface Γ. Define R_0 by

$$\langle R_0^{-1}u_0\,,v_0\rangle=\sum_{i=1}^{p}\rho_i\,\langle u_0-\gamma_{h,\mathcal{W}_i}(u_0),v_0-\gamma_{h,\mathcal{W}_i}(v_0)\rangle_{h,\mathcal{W}_i}\quad\forall\,u_0\,,v_0\in V_0(\Gamma).\tag{8.8}$$

It is readily to see that $u_0=R_0g_0$ for given $g_0\in V_0(\Gamma)$ can be obtained by solving the minimization problem:

$$\min_{w_0\in V_0(\Gamma)}\frac{1}{2}\sum_{i=1}^{p}\rho_i\min_{\lambda_i\in R^1}\|w_0-\lambda_i\|^2_{h,\mathcal{W}_i}-\langle g,w_0\rangle\tag{8.9}$$

by using the symmetry of the operator R_0 and the minimizing property of an average value in the constant space.

Wirebasket coarse solver $\tilde{R}_0$ on the domain Ω. Define $\tilde{R}_0v_0$, for any $v_0\in V_0$, to be a $A(\cdot,\cdot)$-discrete harmonic function which equals to R_0v_0 on the interface Γ with R_0 defined in (8.8). Then we know $u_0=\tilde{R}_0g_0$ for given $g_0\in V_0$ can be obtained by first solving the minimization problem:

$$\min_{w_0\in V_0(\Gamma)}\frac{1}{2}\sum_{i=1}^{p}\rho_i\min_{\lambda_i\in R^1}\|w_0-\lambda_i\|^2_{h,\mathcal{W}_i}-\langle g,w_0\rangle\,,\tag{8.10}$$

and then solving the Dirichlet problems in parallel on each subdomain Ω_i:

u_0 given on the interface Γ by (8.10) and

$$A(u_0\,,v_i)=0\quad v_i\in V_0^h(\Omega_i).\tag{8.11}$$

values on the wirebasket sets. That is, define $I_0 u$, for any $u \in V^h$, to be a discrete harmonic funtion in each Ω_i with respect to $A(\cdot,\cdot)$, which takes the values of u on the wirebasket sets, but a constant value, the average $\gamma_{\partial F}(u)$ of u on the boundary of a face F, at all nodes of a face F (cf. Fig. 4). Therefore, we can represent $I_0 u$ on the interface Γ as follows:

$$I_0 u = \sum_F \gamma_{\partial F}(u)\, I_F^0 1 + \sum_E I_E^0 u + \sum_k I_{v_k}^0 u \quad \forall\, u \in V^h.$$

Then we define the coarse space V_0 to be the range of I_0, i.e. $V_0 = I_0 V^h$.

This interpolant, called *wirebasket interpolant*, has the following properties that for any $u \in V^h$,

$$|u - I_0 u|_{1,\Omega_i} \lesssim (1 + \log(h_0/h))|u|_{1,\Omega_i}, \tag{8.6}$$

$$\|u - I_0 u\|_{0,\Omega_i} \lesssim h_0\,(1 + \log(h_0/h))|u|_{1,\Omega_i}. \tag{8.7}$$

Proof. To see (8.6), let $u_0 = I_0 u$, we obtain by Lemma 4.9 and definition of I_0 that

$$|u - u_0|^2_{1,\Omega_i} \eqsim \|u - u_0\|^2_{1/2,\partial\Omega_i} = \|\sum_{F\subset\partial\Omega_i} I_F^0(u - u_0)\|^2_{1/2,\partial\Omega_i},$$

now using Lemma 4.6 and the fact that $I_F^0(u - u_0) = I_F^0(u - \gamma_{\partial F}(u))$ first and then Lemmas 4.14-4.15 and Lemma 4.20 yields

$$|u - u_0|^2_{1,\Omega_i} \lesssim \sum_{F\subset\partial\Omega_i} \|I_F^0 u - \gamma_{\partial F}(u) I_F^0 1\|^2_{H_{00}^{1/2}(F)} \lesssim \wp^2\, |u|^2_{1,\Omega_i}.$$

(8.7) follows easily by doing the same as the above on a reference domain of a unit size instead of Ω_i, then scaling G back to Ω_i. □

Define a norm $|||\cdot|||$ and its numerical form $|||\cdot|||_h$ on the coarse space V_0 by

$$|||u_0||| = \sum_{i=1}^p \rho_i \|u_0 - \gamma_{\mathcal{W}_i}(u_0)\|^2_{0,\mathcal{W}_i}, \quad |||u_0|||_h = \sum_{i=1}^p \rho_i \|u_0 - \gamma_{h,\mathcal{W}_i}(u_0)\|^2_{h,\mathcal{W}_i}.$$

Then the wirebasket space V_0 has the following property:

LEMMA 8.2. *Let $n = 3$. For any $u_0 \in V_0$,*

$$\wp^{-1}|||u_0|||^2 \lesssim A(u_0,u_0) = \langle S_h\, u_0, u_0\rangle \lesssim \wp |||u_0|||^2.$$

Proof. For the 2nd inequality: let $w_0 = u_0 - \gamma_{\mathcal{W}_i}(u_0)$ on $\bar\Omega_i$. By the definition of V_0, we can express w_0 on $\partial\Omega_i$ as

$$w_0 = \sum_{F\subset\partial\Omega_i} \gamma_{\partial F}(w_0) I_F^0 1 + \sum_{E\subset\partial\Omega_i} I_E^0 w_0 + \sum_{v_k\in\partial\Omega_i} I_{v_k}^0 w_0,$$

thus the result follows from the triangle inequality and Lemmas 4.13-4.14:

$$A(u_0,u_0) \;=\; \sum_{i=1}^p A_i(\tilde w_0, \tilde w_0) \eqsim \sum_{i=1}^p \rho_i\, |w_0|^2_{1/2,\partial\Omega_i}$$

In the remainder of this subsection, we recall the L^2 and weighted L^2 projection operator $Q^\rho_{h_0}$ associated with the above standard coarse subspace V_0.

The weighted L^2 projection $Q^\rho_{h_0} : L^2(\Omega) \to V_0$ is defined by

$$(Q^\rho_{h_0} u, v)_{L^2_\rho(\Omega)} = (u, v)_{L^2_\rho(\Omega)} \quad \forall u \in L^2(\Omega),\ v \in V_0, \tag{8.5}$$

where $(\cdot,\cdot)_{L^2_\rho(\Omega)}$ is the scalar product related to the norm $\|\cdot\|_{L^2_\rho(\Omega)}$. We will denote $Q_{h_0} = Q^\rho_{h_0}$ if $\rho = 1$.

Bramble-Xu [17] proved

LEMMA 8.1. *For any $u \in V^h$, we have for $n = 2$ that*

$$\begin{aligned} \|u - Q^\rho_{h_0} u\|_{L^2_\rho(\Omega)} &\lesssim h_0(1 + \log(h_0/h))^{1/2} |u|_{H^1_\rho(\Omega)}, \\ |Q^\rho_{h_0} u|_{H^1_\rho(\Omega)} &\lesssim (1 + \log(h_0/h)^{1/2} |u|_{H^1_\rho(\Omega)}; \end{aligned}$$

but for $n = 3$ that

$$\begin{aligned} \|u - Q^\rho_{h_0} u\|_{L^2_\rho(\Omega)} &\lesssim h_0 (h_0/h)^{1/2} |u|_{H^1_\rho(\Omega)}, \\ |Q^\rho_{h_0} u|_{H^1_\rho(\Omega)} &\lesssim (h_0/h)^{1/2} |u|_{H^1_\rho(\Omega)}. \end{aligned}$$

If all the coefficients $\rho_i = 1$, $1 \le i \le p$, for both $n = 2$ and $n = 3$ we have

$$\|u - Q_{h_0} u\|_{0,\Omega} \lesssim h_0 |u|_{1,\Omega}, \quad |Q_{h_0} u|_{1,\Omega} \lesssim |u|_{1,\Omega} \quad \forall\, u \in H^1_0(\Omega).$$

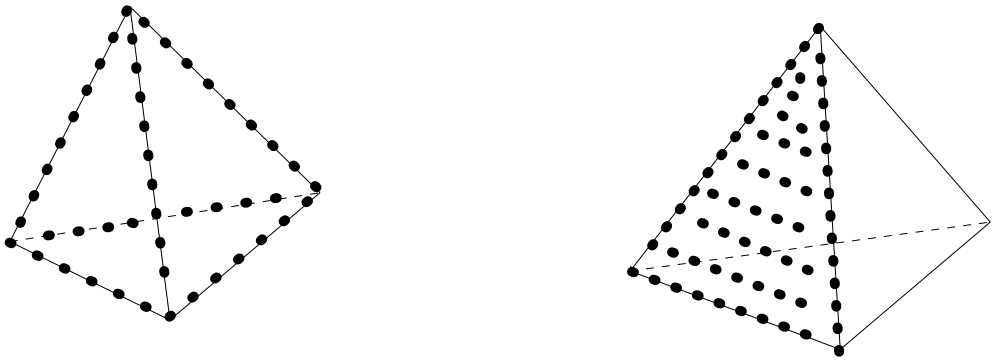

FIG. 4. *Left: nodes on the wirebasket W; right: nodes on a face F and its boundary ∂F*

8.2. Wirebasket subspaces. From (8.2) in §8.1, we know that for $n = 2$ the energy of the coarse interpolant $I_0 u$ for any u in V^h by using only the values of u at vertices of Ω_i exceeds that of u by at most a factor $\log(h_0/h)$. However for $n = 3$, it may result in a h_0/h increasing in the energy. In fact, a simple example can show this point. Take a nodal basis function $\phi_k \in V^h$ which equals to 1 at one vertex v_k of Ω_i, but vanishes at all other nodes on $\bar\Omega_i$. It is readily to see that

$$|I_0 \phi_k|^2_{1,\Omega_i} \eqsim h_0^{-2} |\Omega_i| \eqsim h_0 = \frac{h_0}{h} h \eqsim \frac{h_0}{h} |\phi_k|^2_{1,\Omega_i}.$$

Therefore interpolating the value of a function in V^h by exploiting only the vertex values is not reasonable for $n = 3$. Smith [82, 84] thus suggested to use all the nodal

effective, we observe that the interpolant I_0 associated with the coarse subspace has to possess two properties, i.e. L^2- optimal or almost optimal approximation and the stability of the energy norm. More specifically, let V_0 be the range of the interpolant I_0 and V_0 is a subspace of V^h, then I_0 should have the following two properties: for any $u \in V^h$,

$$|I_0 u|_{H^1_\rho(\Omega)} \lesssim \alpha_1 |u|_{H^1_\rho(\Omega)}, \quad \|u - I_0 u\|_{L^2_\rho(\Omega)} \lesssim \alpha_0 h_0 \, |u|_{H^1_\rho(\Omega)}, \tag{8.1}$$

where α_0 and α_1 are two constants depending usually on h and h_0, and

$$|v|^2_{H^1_\rho(\Omega)} = \sum_{i=1}^{p} \rho_i |v|^2_{1,\Omega_i}, \quad \|v\|^2_{L^2_\rho(\Omega)} = \sum_{i=1}^{p} \rho_i \, \|v\|^2_{0,\Omega_i} \quad \forall\, v \in H^1(\Omega).$$

Equivalently, on the interface, let $V_0 = I_0 V^h(\Gamma) \subset V^h(\Gamma)$, then the interpolant I_0 should satisfy that

$$|\widetilde{I_0 u}|_{H^1_\rho(\Omega)} \lesssim \alpha_1 |\widetilde{u}|_{H^1_\rho(\Omega)}, \quad \|\tilde{u} - \widetilde{I_0 u}\|_{L^2_\rho(\Omega)} \lesssim \alpha_0 h_0 \, |\widetilde{u}|_{H^1_\rho(\Omega)}.$$

8.1. Standard coarse subspaces. The most natural and simple coarse subspace of V^h is the standard coarse subspace V_0 generated by the subdomains $\{\Omega_i\}$. We now describe this coarse space.

If all subdomains Ω_i are simplices, then define V_0 to be the space of continuous piecewise linear functions defined on $\{\Omega_i\}$. If some subdomain Ω_i is a quadrilateral ($n = 2$) or a hexahedron ($n = 3$), we subdivide Ω_i into two triangles ($n = 2$) or four simplices ($n = 3$) and denote the resulting triangulation by $\mathcal{T}^0$. Then we define V_0 to be the space of continuous piecewise linear functions defined on $\mathcal{T}^0$.

Let I_0 be the nodal value interpolant associated with the coarse space V_0. Obviously, $V_0 = I_0 V^h$. We have for any $u \in V^h$ and $s = 0, 1$ that

$$\begin{array}{lll} |u - I_0 u|_{s,\Omega_i} \lesssim h_0^{1-s}(1 + \log(h_0/h))^{1/2} |u|_{1,\Omega_i}, & \text{if } n = 2, \\ |u - I_0 u|_{s,\Omega_i} \lesssim h_0^{1-s}(h_0/h)^{1/2} |u|_{1,\Omega_i}, & \text{if } n = 3. \end{array} \tag{8.2}$$

Standard coarse solver R_0. Let A_0 be the restriction of the stiffness operator A_h on the coarse subspace V_0. In order to avoid using the exact solver A_0^{-1}, we can replace A_0 by any of its equivalent form. Here we consider one of such options. Recall that V_0 is a piecewise linear space defined on the simplicial traingulation $\mathcal{T}^0$, one can readily obtain the equivalency by some basic calculations: for $u_0 \in V_0$,

$$A(u_0, u_0) \eqsim \sum_{k=1}^{p} A_k(u_0, u_0) \eqsim h_0 \sum_{v_i, v_j \in \Gamma_k} (u_0(v_i) - u_0(v_j))^2. \tag{8.3}$$

The standard coarse solver R_0 is therefore defined by

$$(R_0^{-1} u_0, v_0) = h_0 \sum_{v_i, v_j \in \Gamma_k} (u_0(v_i) - u_0(v_j))\,(v_0(v_i) - v_0(v_j)) \quad \forall\, u_0, v_0 \in V_0.$$

Thus for any given $g_0 \in V_0$, $u_0 = R_0 g_0$ can be calculated as follows:

$$h_0 \sum_{v_i, v_j \in \Gamma_k} (u_0(v_i) - u_0(v_j))\,(v_0(v_i) - v_0(v_j)) = (g_0, v_0) \quad \forall v_0 \in V_0. \tag{8.4}$$

7.3. Neumann-Neumann algorithm ($n = 2, 3$)**.** The preconditioner B_h to be constructed is for the interface operator S_h. As $S_h = S_1 + S_2$, it is natural to define the so-called Neumann-Neumann preconsitioner B_h by

$$B_h = S_1^{-1} + S_2^{-1}.$$

For this preconditioner, we have

LEMMA 7.3.

$$\kappa(B_h S_h) \lesssim 1.$$

Proof. By Lemmas 4.9 and 4.6, we have for any $u \in V^h(\Gamma)$,

$$\langle S_1 u, u\rangle_{0,\Gamma_1} = A_1(\tilde{u}, \tilde{u}) \eqsim |u|^2_{1/2,\partial\Omega_1} \eqsim \|u\|^2_{H^{1/2}_{00}(\Gamma)} \eqsim |u|^2_{1/2,\partial\Omega_2} \lesssim \langle S_2 u, u\rangle . \tag{7.2}$$

Thus,

$$\begin{aligned}
\langle B_h S_h u, S_h u\rangle &= \left\langle (2I + S_1^{-1}S_2 + S_2^{-1}S_1)u, S_h u\right\rangle \ (B_h\text{'s definition}) \\
&= 3\langle S_h u, u\rangle + \left\langle S_1^{-1}S_2 u, S_2 u\right\rangle + \left\langle S_2^{-1}S_1 u, S_1 u\right\rangle \ (S_h = S_1 + S_2) \\
&\eqsim 3\langle S_h u, u\rangle + \langle S_2 u, u\rangle + \langle S_1 u, u\rangle \ (\text{by (7.2)}) \\
&\eqsim 4\langle S_h u, u\rangle \ (S_h = S_1 + S_2),
\end{aligned}$$

that completes our proof. ∎

For a given $g \in V^h(\Gamma)$, let $u_i = S_i^{-1} g$ $(i = 1, 2)$, then $\langle S_i u_i, v\rangle = \langle g, v\rangle \quad \forall\, v \in V^h(\Gamma)$, which implies using the definition of S_i that

$$A_i(\tilde{u}_i, \phi) = \langle g, v\rangle \quad \phi \in V^h(\Omega_i).$$

Thus we have the following

ALGORITHM 7.3 (NEUMANN-NEUMANN ALGORITHM). *For a given $g \in V^h(\Gamma)$, let $u = B_h g \equiv u_1 + u_2$, then u_1 and u_2 can be obtained by solving independent mixed Neumann problems on two subdomains: $u_i = \tilde{u}_i|_\Gamma$ with $\tilde{u}_i \in V^h(\Omega_i)$ satisfying*

$$A_i(\tilde{u}_i, \phi) = \langle g, v\rangle \quad \phi \in V^h(\Omega_i).$$

7.4. Bibliograpic remarks. The Dirichlet-Neumann-Dirichlet method was proposed by Bramble-Pasciak-Schatz [11], whose ideas were then developed and extended to generate many important algorithms, e.g. Bramble-Pasciak-Schatz [10, 12, 13, 14].

The Neumann-Neumann algorithm was proposed by Bourgat-Glowinski-Tallec-Vidrascu [6] which is a standard variational extension of an earlier approach by Glowinski-Wheeler [52] within a mixed finite element framework for two subdomain case.

8. Coarse subspaces. To extend the algorithms discussed in the last section to multiple subdomains, it proves important to introduce a global coarse space into the algorithm to avoid the deterioration of the algorthms as the number of subdomains get large. How to properly construct such a coarse subspace, denoted by V_0, is one of the major interest of domain decomposition algorithms. We devote this section to address various coarse subspaces and their properties. To make a coarse subspace

1. *The component u_P on Ω_2 solves the Dirichlet problem:*
   $u_P \in V_0^h(\Omega_2)$ *and*

$$A_2(u_P, \phi) = (g, \phi)_{0,\Omega_2}, \ \forall\, \phi \in V_0^h(\Omega_2).$$

2. *The u on $\bar{\Omega}_1$ solves the mixed Neumann problem:*

$$A_1(u_1, \phi) = (g, \phi) - A_2(u_P, \phi), \ \forall\, \phi \in V^h(\Omega_1).$$

3. *The component u_H in Ω_2 solves the mixed Neumann problem:*
   $u_H - u_1 \in V_0^h(\Omega_2)$ *and*

$$A_2(u_H, \phi) = 0, \ \forall\, \phi \in V_0^h(\Omega_2).$$

7.2. Dirichlet-Dirichlet algorithm ($n = 2, 3$)**.** As in the last subsection, we split any u in V^h as $u = u_P + u_H$. Applying Lemmas 4.9, 4.7 and 4.14 to the discrete harmonic component u_H, we see

$$\begin{aligned} A(u_H, u_H) &\eqsim \sum_{i=1}^{2} |u_H|^2_{1/2,\partial\Omega_i} \lesssim \|u_H\|^2_{H_{00}^{1/2}(F)} \\ &\eqsim \left\langle (-\Delta_{F,h})^{1/2} u_H, u_H \right\rangle_{0,F} \lesssim \wp^2 A(u_H, u_H). \end{aligned} \tag{7.1}$$

From this, we can naturally define the preconditioner M_h for the stiffness operator A_h by

$$(M_h^{-1} u, v) = \tilde{A}(u, v) \quad \equiv \quad A(u_P, v_P) + \sum_F \left\langle (-\Delta_{F,h})^{1/2} u_H, v_H \right\rangle_{0,F},$$

and from (7.1) we have

THEOREM 7.2. *For any $u \in V^h$,*

$$\kappa(M_h A_h) \lesssim (1 + \log(H/h))^2.$$

We have the following algorithm:

ALGORITHM 7.2 (DIRICHLET-DIRICHLET ALGORITHM). *Let $u = M_h g = u_P + u_H$, for a given $g \in V^h$. The components u_P and u_H can be computed as follows:*

1. *The component u_P in Ω_i for $i = 1, 2$ solves the Dirichlet problems:*
   $u_P \in V_0^h(\Omega_i)$ *and*

$$A(u_P, v) = (g, v) \quad \forall v \in V_0^h(\Omega_i).$$

2. *On the face $F \subset \bar{\Omega}_1 \cap \bar{\Omega}_2$, u_H solves*

$$\left\langle (-\Delta_{F,h})^{1/2} u_H, v_H) \right\rangle_{0,F} = (g, v) - A(u_P, v) \quad \forall\, v \in V^h(\Gamma).$$

3. *The component u_H in Ω_i for $i = 1, 2$ solves the mixed Neumann problem:*
   u_H *is given on F by Step 2 and*

$$A(u_H, v) = 0 \quad \forall v \in V_0^h(\Omega_i).$$

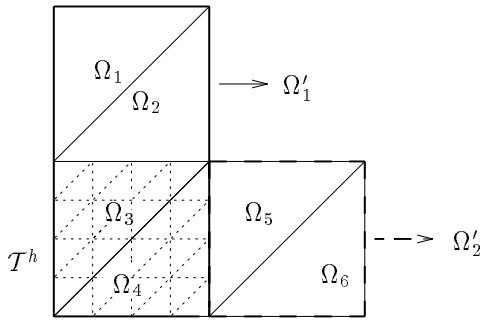

FIG. 3. *Subdomains* $\Omega_1' = \cup_{i=1}^4 \Omega_i$ *and* $\Omega_2' = \cup_{i=5}^6 \Omega_i$ *with simplicial* Ω_i*'s*

M_h usually corresponding to a bilinear form $\tilde{A}(\cdot,\cdot)$, we decompose V^h into two parts, i.e. for any $u \in V^h$, we can split $u = u_P + u_H$ such that $u_P \in V_P$ solves the Dirichlet problem on Ω_i for $i = 1, 2$:

$$A_i(u_P, \phi_i) = A_i(u, \phi_i), \forall\, \phi_i \in V_0^h(\Omega_i),$$

while u_H is discrete harmonic in Ω, or $u_H \in V_H$. We can then define the preconditioner M_h or the bilinear form $\tilde{A}(\cdot,\cdot)$ by

$$(M_h^{-1}u, v) = \tilde{A}(u, v) \equiv A_1(u, v) + A_2(u_P, v_P), \forall\, u, v \in V^h.$$

THEOREM 7.1. *We have*

$$\kappa(M_h A_h) \lesssim 1.$$

Proof. By Lemma 3.1, it suffices to prove

$$A(u, u) \eqsim \tilde{A}(u, u), \forall\, u \in V^h.$$

By Lemmas 4.6 and 4.10,

$$A_1(u_H, u_H) \eqsim \|u_H\|^2_{H_{00}^{1/2}(\Gamma)} \eqsim A_2(u_H, u_H).$$

Thus from this and the definition of $\tilde{A}(\cdot,\cdot)$,

$$\begin{aligned} A(u,u) &= A(u_P, u_P) + A(u_H, u_H) \\ &= A_1(u_P, u_P) + A_2(u_P, u_P) + A_1(u_H, u_H) + A_2(u_H, u_H) \\ &\eqsim A_1(u_P, u_P) + A_2(u_P, u_P) + A_1(u_H, u_H) \\ &\eqsim A_1(u, u) + A_2(u_P, u_P) = \tilde{A}(u, u). \end{aligned}$$

□

Now we state the algorithm for computing $M_h g$, for a given $g \in V^h$. Let $u = M_h g$. As above, we write $u = u_P + u_H$. Obviously, u solves the following problem:

$$A_1(u, v) + A_2(u_P, v_P) = (g, v) \quad \forall\, u, v \in V^h,$$

we then have the algorithm:

ALGORITHM 7.1 (DIRICHLET-NEUMANN-DIRICHLET ALGORITHM). $u = M_h g = u_P + u_H$ *can be calculated as follows:*

5.1. Bibliographic remarks. Lemma 5.2 was proved by Bjorstad-Widlund [5] in two dimensions for many subdomains case without crosspoints, and by Mansfield [70] in both two and three dimensions for general domain decompositions. Our proof is simpler.

6. Basic assumptions and notation. We will use the following notation throughout the paper:

$\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ – the interface of Ω_i;

$\Gamma = \cup_{i=1}^{p}\Gamma_i$ – interface among all the subdomains $\{\Omega_i\}$;

$\mathcal{W}_i$ – the wirebasket set of the subdomain Ω_i, whose points belonging to more than two subdomains;

E, F and v_k – edges, faces and vertices of subdomains Ω_i $(1 \le i \le p)$ resp.;

$(\cdot,\cdot)_{0,K}$ – the scalar product in $L^2(K)$, where K is a subset of Ω; we omit the subscript when $K = \Omega$;

$\langle\cdot,\cdot\rangle_{0,K}$ – the scalar product in $L^2(K)$, where K is a subset of Γ; we omit the subscript when $K = \Gamma$;

$\tilde{u}$ – the discrete harmonic extension of $u \in V^h(\Gamma)$, i.e. $A(\tilde{u}, v) = 0 \quad \forall\, v \in V_0^h(\Omega_i)$ $(1 \le i \le p)$;

$I_K^0 u$ – the finite element function belonging to V_Γ^h and equal to u at any node on K but vanishing at nodes of $(\Gamma \setminus K)$. Here K is any open subset of Γ and u any function in V_Γ^h; e.g., $I_F^0 u$ denotes the function belonging to V_Γ^h and equal to u at any node on F while vanishing at nodes on $(\Gamma \setminus F)$.

$V_0^h(K)$ – the subspace of V^h consisting of functions vanishing on $(\Omega \setminus K)$. Here K is any union of finite elements in $\mathcal{T}^h$.

$\gamma_K(w)$ – the average value of the function w on a set K, i.e. $\gamma_K(\omega) = (int_K \omega\, dx)/|\omega|$;

$\wp$ – the constant $1 + \log(h_0/h)$;

$r(\rho)$ – the coefficient ratio of the model problem, i.e. $r(\rho) = (\max_{1\le i\le p}\rho_i)/(\min_{1\le i\le p}\rho_i)$.

Assumptions on the subdomains. We make the following assumptions on the subdomains $\{\Omega_i\}_{i=1}^p$ throughout the paper:

(A1) For $n = 2$, each subdomain Ω_i is either a triangle or a quadrilateral.

(A2) For $n = 3$, all subdomains Ω_i are tetrahedra or all of them are hexahedra.

(A3) Each Ω_i is assumed to be the image of either a reference simplex G, or a unit n-cube G under an one-to-one mapping $T_i : G \to \Omega_i$, which satisfies

$$\Omega_i = T_i(G), \quad \|\partial T_i\| \lesssim 1, \quad \|\partial T_i^{-1}\| \lesssim 1.$$

Here ∂T_i is the Jacobian and $\|\cdot\|$ the Euclidean matrix norm.

7. Examples on two subdomains. To illustrate the main idea behind the more complicated domain decomposition on multiple subdomains discussed later, a brief discussion will be given here to some examples of algorithms on two subdomains for uniformly elliptic problems, i.e. we assume that the coefficients $\rho_i = 1$ on each subdomain Ω_i. We also assume that all vertices of each subdomain Ω_i for $i = 1, \cdots, p$ lie on the boundary $\partial\Omega$ (cf. Fig. 3). Let $\Omega = \Omega_1' \cup \Omega_2'$ and $\Gamma = \Omega_1' \cap \Omega_2'$. Here we allow each Ω_1' and Ω_2' to be the union of some subdomains from $\{\Omega_i\}_{i=1}^p$. But for ease of notation we still use Ω_i instead of Ω_i'. We will discuss three preconditioners, two for the stiffness operator A_h, one for the interface operator S_h.

7.1. Dirichlet-Neumann-Dirichlet algorithm ($n = 2, 3$). As stated in the outline (§2), in order to find a preconditioner M_h for the stiffness operator A_h, with

Proof. Let $(\cdot,\cdot)$ be the Euclidean inner product either in R^{n_2} or in R^n. For any $\mu_2 \in R^{n_2} \neq 0$, let $\mu = (\mu_1, \mu_2) \in R^n$ with $\mu_1 = -\mathcal{A}_{11}^{-1}\mathcal{A}_{12}\mu_2$, then

$$(\mathcal{S}\mu_2, \mu_2) = (\mathcal{A}\mu, \mu),$$

that implies

$$\sup_{\mu_2 \neq 0} \frac{(\mathcal{S}\mu_2, \mu_2)}{(\mu_2, \mu_2)} \leq \sup_{\mu \neq 0} \frac{(\mathcal{A}\mu, \mu)}{(\mu, \mu)}, \quad \inf_{\mu_2 \neq 0} \frac{(\mathcal{S}\mu_2, \mu_2)}{(\mu_2, \mu_2)} \geq \inf_{\mu \neq 0} \frac{(\mathcal{A}\mu, \mu)}{(\mu, \mu)},$$

which proves Lemma 5.1. □

Stiffness operator. Let the operator $S_h : V^h(\Gamma) \to V^h(\Gamma)$, called as a *stiffness operator*, be defined as in (2.10), i.e.

$$\langle S_h u_H, v_H \rangle = A(u_H, v_H) \quad u_H, v_H \in V_H,$$

and V_H is the finite element space of discrete harmonic functions defined in §2. It is well-known that the condition number of the stiffness operator A_h is the order $O(h^{-2})$. Here the operator $A_h : V^h \to V^h$ is defined by

$$(A_h u, v) = A(u, v) \quad u, v \in V^h.$$

The next lemma will show that the condition lemma of the interface operator S_h is the order $O((h_0 h)^{-1})$.

LEMMA 5.2. *For the interface operator S_h, we have*

$$\kappa(S_h) = O(\frac{1}{h_0 h}).$$

Proof. For any $u \in V^h(\Gamma)$, we deduce by definition and Poincaré inequality that

$$\langle S_h u, u \rangle = A(\tilde{u}, \tilde{u}) \gtrsim |\tilde{u}|^2_{1,\Omega} \gtrsim \|\tilde{u}\|^2_{1,\Omega}, \tag{5.18}$$

but $\|\tilde{u}\|^2_{1,\Omega} = \sum_{i=1}^p (\|\tilde{u}\|^2_{0,\Omega_i} + |\tilde{u}|^2_{1,\Omega_i})$, thus applying the trace lemma 4.2 with $\varepsilon = h_0$, we obtain

$$\langle S_h u, u \rangle \geq h_0 \sum_{i=1}^p \|u\|^2_{0,\partial\Omega_i} = h_0 \langle u, u \rangle,$$

that implies $\lambda_{min}(S_h) \gtrsim h_0$. On the other hand, by Lemma 4.10 we have

$$\langle S_h u, u \rangle = A(\tilde{u}, \tilde{u}) \lesssim \sum_{i=1}^p |\tilde{u}|^2_{1,\Omega_i} \eqsim \sum_{i=1}^p |u|^2_{1/2,\partial\Omega_i},$$

then applying Lemma 4.17 gives

$$\langle S_h u, u \rangle \lesssim h^{-1} \sum_{i=1}^p \|u\|^2_{0,\partial\Omega_i} = h^{-1} \langle u, u \rangle,$$

this shows $\lambda_{max}(S_h) \lesssim h^{-1}$. Therefore we get $\kappa(S_h) = \lambda_{max}/\lambda_{min} \lesssim (h_0 h)^{-1}$. □

LEMMA 4.20. *Let $n = 2, 3$. For any $v \in V^h(\partial G)$*

$$d^{-1/2}\|v - \gamma_{h,\partial G}(v)\|_{h,\partial G} \lesssim |v|_{1/2,\partial G} \lesssim h^{-1/2}\|v - \gamma_{h,\partial G} v\|_{h,\partial G},$$
$$\|v - \gamma_{h,K}(v)\|_{h,K} \eqsim \|v - \gamma_K(v)\|_{0,K},$$
$$|\gamma_{h,K}(v)| \lesssim d^{-\alpha/2}\|v\|_{0,K}, \quad |\gamma_K(v)| \lesssim d^{-\alpha/2}\|v\|_{0,K}.$$

Proof. The last two inequalities follow from the definition of the discrete average, Cauchy-Schwarz inequality and (4.17). Next we prove the first two relations. The minimization property of the average value in the constant space and (4.17) implies

$$\begin{aligned} \|v - \gamma_{h,K}(v)\|_{h,K} &\leq \|v - \gamma_K(v)\|_{h,K} \lesssim \|v - \gamma_K(v)\|_{0,K} \\ &\leq \|v - \gamma_{h,K}(v)\|_{0,K} \lesssim \|v - \gamma_{h,K}(v)\|_{h,K}, \end{aligned}$$

hence $\|v - \gamma_{h,K}(v)\|_{h,K} \eqsim \|v - \gamma_K(v)\|_{0,K}$. The proof is completed by combining Lemma 4.18. ∎

4.6. Bibliographic remarks. The average nodal value interpolant was proposed by Scott-Zhang [79] and more general results than Lemma 4.8 was given there. Other similar interpolants can be found, e.g. in Clément[36]. The properties of the standard nodal value interpolant are discussed in many books, e.g. in Ciarlet [35], where most standard finite element convergence theory can be found.

Lemma 4.11 was proved by Bramble [7], Bramble-Pasciak-Schatz [10] and Xu [95]. Our proof follows the one by Xu [95].

5. Schur compliments. In this section, we discuss briefly the Schur compliment matrix for general SPD matrix and its interface operator form corresponding to the domain decomposition $\{\Omega_i\}_{i=1}^p$.

Schur compliment matrix. Let $\mathcal{A} \in R^{n\times n}$ be any SPD matrix, and we write blockwise as follows

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^t & \mathcal{A}_{22} \end{pmatrix}$$

where $\mathcal{A}_{12} \in R^{n_1\times n_2}$, $\mathcal{A}_{22} \in R^{n_2\times n_2}$. By one step of block LU factorization for $\mathcal{A}$, the resultant matrix is:

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{S} \end{pmatrix}$$

where the matrix $\mathcal{S}$ is called *Schur complement* of $\mathcal{A}$ with respect to $\mathcal{A}_{22}$, and

$$\mathcal{S} = \mathcal{A}_{22} - \mathcal{A}_{12}^t \mathcal{A}_{11}^{-1} \mathcal{A}_{12}.$$

Then we know that the condition number of the Schur compliment $\mathcal{S}$ is always less than the one of the matrix $\mathcal{A}$, i.e.

LEMMA 5.1.

$$\kappa(\mathcal{S}) \leq \kappa(\mathcal{A}).$$

□

LEMMA 4.18. *Let $n = 2, 3$. For any $v \in V^h(\partial G)$,*

$$d^{-1/2}\|v - \gamma_{\partial G}(v)\|_{0,\partial G} \lesssim |v|_{1/2,\partial G} \lesssim h^{-1/2}\|v - \gamma_{\partial G}(v)\|_{0,\partial G}.$$

Proof. The second inequality follows directly from Lemma 4.17, while the first is obtained by Lemma 4.4 and Friedrichs inequality:

$$d^{-1/2}\|v - \gamma_{\partial G}(v)\|_{0,\partial G} \lesssim \|\tilde{v} - \gamma_{\partial G}(v)\|_{1,G} \lesssim |\tilde{v}|_{1,G} \lesssim |v|_{1/2,\partial G}$$

where $\tilde{v}$ is the $D(\cdot,\cdot)$-discrete harmonic extension of v with $D(u,v) = (\nabla u, \nabla v)_{0,G}$. □

LEMMA 4.19. *Let $n = 2$. For any $v \in V^h(\partial G)$ and any edge $E \subset \partial G$*

$$\|v - I_{\partial G} v\|_{H^{1/2}_{00}(E)} \lesssim (1 + \log \frac{d}{h})|v|_{1/2,\partial G}.$$

where $I_{\partial G} v$ equals to v on the corner nodes of ∂G and linear on each edge of ∂G.

Proof. Consider only $E = \{(x, 0) : 0 \le x \le d\}$ and $0 \le x_0 < \ldots < x_k = d$ are the mesh points on E. Let $w = v - I_{\partial G} v$. Note $w(0) = 0$, we easily come to

$$\begin{aligned}
\int_0^d \frac{|w(x)|^2}{x} dx &= \int_0^{x_1} \frac{|w(x)|^2}{x} dx + \int_{x_1}^d \frac{|w(x)|^2}{x} dx \\
&\lesssim \|\frac{\partial w}{\partial x}\|^2_{L^\infty(0,d)} \int_0^{x_1} x\, dx + \|w\|^2_{L^\infty(0,d)} \int_{x_1}^d \frac{dx}{x} \\
&\lesssim \|w\|^2_{L^\infty(0,d)}(1 + \log \frac{d}{h}).
\end{aligned}$$

Analygously,

$$\int_0^d \frac{|w(x)|^2}{1 - x} dx \lesssim \|w\|^2_{L^\infty(0,d)}(1 + \log \frac{d}{h}).$$

But by Lemma 4.12,

$$\|w\|_{L^\infty(0,d)} \lesssim (1 + \log \frac{d}{h})^{1/2} \|v\|_{1/2,\partial G},$$

the desired estimate then follows. □

Next we introduce the discrete L^2 inner product on ∂G.

Let K be ∂G, or the wirebasket set W of ∂G or one face F of ∂G. We define

$$\langle v, w \rangle_{h,K} = h^\alpha \sum_{x_i \in K_h} v(x_i) w(x_i)$$

and the corresponding norm $\|v\|_{h,K} = \langle v, v \rangle^{1/2}_{h,K}$. Here $\alpha = n - 1$ if $K = \partial G$ or $K = F$; $\alpha = n - 2$ if $K = W$.

Then by the quasi-uniformity of $\mathcal{T}^h$ and straightforward computations, we get

$$\|v\|_{h,K} \eqsim \|v\|_{0,K}, \forall\, v \in V^h(K). \tag{4.17}$$

Similarly to $\gamma_K(v)$, we define the discrete average value of v on K by $\gamma_{h,K}(v) = \langle v, 1 \rangle_{h,K} / \langle 1, 1 \rangle_{h,K}$.

Similarly,

$$\int_0^d \frac{|I_F^0 u|^2}{1-x} dx \lesssim (1+\log\frac{d}{h})\|u\|^2_{L^\infty(F)}.$$

Combining the last two inequalities with Lemma 4.12 gives the desired result. □

LEMMA 4.15. *Let $n = 2, 3$. For any $u \in V^h(G)$ and a face F of G,*

$$\|I_F^0 1\|^2_{H^{1/2}_{00}(F)} \lesssim d^{n-2}(1+\log\frac{d}{h}). \tag{4.16}$$

Proof. The conclusions come immediately by going through the proof of Lemma 4.14. □

LEMMA 4.16. *Let $n = 3$. For any $u \in V^h(G)$ vanishing on one edge of G, then*

$$\|u\|^2_{0,G} \lesssim d^2\,(1+\log\frac{d}{h})\,|u|^2_{1,G}.$$

Proof. The proof follows [45]. Without loss of generality, we assume $G = (0,d)^3$ and u vanishes on the edge with $x = y = 0$. Let $\Delta_s = G \cap \{z = s\}$. By the assumption, one knows that for $z \in (0,d)$, u vanishes at least at one point in Δ_z. So for any constant c,

$$\max_{(x,y)\in\Delta_z} |u(x,y,z)| \le 2 \max_{(x,y)\in\Delta_z} |u(x,y,z) - c|,$$

thus by Lemma 4.11 and Friedrichs' inequality (choosing $c = \gamma_G(u)$),

$$\begin{aligned}
\|u\|^2_{0,G} &\le d^2 \int_0^d \max_{(x,y)\in\Delta_z} |u(x,y,z)|^2 dz \le 4\,d^2 \int_0^d \max_{(x,y)\in\Delta_z} |u(x,y,z)-c|^2 dz \\
&\le 4d^2 \int_0^d (1+\log\frac{d}{h})\,\|u-c\|^2_{1,\Delta_z} dz \\
&= 4d^2\,(1+\log\frac{d}{h})\,\|u-c\|^2_{1,G} \lesssim d^2\,(1+\log\frac{d}{h})\,|u|^2_{1,G}.
\end{aligned}$$

□

The next lemma is the inverse inequality for finite element functions defined on the boundary of G.

LEMMA 4.17. *Let $n = 2, 3$. For any $v \in V^h(\partial G)$*

$$|v|_{1/2,\partial G} \lesssim h^{-1/2}\|v\|_{0,\partial G}.$$

Proof. Let $\hat{v}$ be the function in $V^h(G)$ which vanishes at all interior nodes of G, but equals to v at the boundary nodes of G. Then we first use Lemma 4.5 and then the inverse inequality for $\hat{v}$ to obtain that

$$|v|^2_{1/2,\partial G} \lesssim d^{-2}\|\hat{v}\|^2_{0,G} + |\hat{v}|^2_{1,G} \lesssim h^{-2}\|\hat{v}\|^2_{0,G}.$$

Now using the quasi-uniformity of the mesh $\mathcal{T}^h$, we have

$$|v|^2_{1/2,\partial G} \lesssim h^{-1}\Big(\sum_{x_i\in\partial G_h} h^{n-1} v^2(x_i)\Big) \eqsim h^{-1}\|v\|^2_{0,\partial G}.$$

Using $u = \sum_{F' \subset \partial G} I^0_{F'} u + \sum_{E \subset \partial G} I^0_E u + \sum_{v_k \in \partial G} I^0_{v_k} u$ on ∂G, it is readily to see by the definition of $|\cdot|_{1/2,\partial G}$ and Lemma 4.13 that

$$\begin{aligned} |I^0_F u|^2_{1/2,\partial G} &\lesssim |\sum_{F' \subset \partial G} I^0_{F'} u|^2_{1/2,\partial G} \\ &\lesssim |u|^2_{1/2,\partial G} + \sum_E |I^0_E u|^2_{1/2,\partial G} + \sum_{v_k} |I^0_{v_k} u|^2_{1/2,\partial G} \\ &\lesssim (1 + \log \frac{d}{h}) \|u\|^2_{1/2,\partial G}. \end{aligned} \tag{4.15}$$

Thus I_1 is bounded as required. It remains to estimate I_2. Due to the similarity, we consider a specific face, say the face $F = G \cap \{z = 0\}$ and the term

$$\begin{aligned} \int_F \frac{(I^0_F u)^2(x,y,0)}{x}\, dxdy &= \int_0^d \int_0^{x_1} \frac{(I^0_F u)^2(x,y,0)}{x}\, dxdy + \int_0^d \int_{x_1}^d \frac{(I^0_F u)^2(x,y,0)}{x}\, dxdy \\ &\equiv I_3 + I_4. \end{aligned}$$

Here x_1 is the first node on x axis. For I_3, let $\Delta_s = G \cap \{y = s\}$ and $v = I^0_F u - \gamma_G(u)$, then

$$\begin{aligned} I_3 &\lesssim h^2 \int_0^d \|\frac{\partial v(\cdot,y,0)}{\partial x}\|^2_{L^\infty(0,x_1)} dy \lesssim \int_0^d \|v(\cdot,y,0)\|^2_{L^\infty(0,x_1)} dy \text{ (inverse ineq)} \\ &\lesssim (1 + \log \frac{d}{h}) \int_0^d \|\tilde{v}(\cdot,y,\cdot)\|^2_{1,\Delta_y} dy \text{ (Lemma 4.11 for } \Delta_y) \\ &\lesssim (1 + \log \frac{d}{h}) \|\tilde{v}\|^2_{1,G} \lesssim (1 + \log \frac{d}{h})^2 \|u\|^2_{1/2,\partial G} \text{ (Lemma 4.10(a) \& (4.15))}. \end{aligned}$$

For I_4, by direct computing and Lemmas 4.11 and 4.13, we obtain

$$\begin{aligned} I_4 &\lesssim \log \frac{d}{h} \int_0^d \|I^0_F u(\cdot,y,0)\|^2_{L^\infty(0,1)} dy \\ &\lesssim \log \frac{d}{h} \Big(\int_0^d \|u(\cdot,y,0)\|^2_{L^\infty(0,1)} dy + \int_0^d u^2(0,y,0) dy + \int_0^d u^2(1,y,0) dy \Big) \\ &\lesssim (1 + \log \frac{d}{h})^2 \|u\|^2_{1/2,\partial G}. \end{aligned}$$

Hence I_2 is also bounded as required that ends the proof of $n = 3$.

Next we prove for $n = 2$. We assume $F = \{(x,0) : 0 \le x \le d\}$ and $0 = x_0 < \ldots < x_k = d$ are the mesh points on F. By the mean value theorem and inverse inequality,

$$\int_0^{x_1} \frac{|I^0_F u|^2}{x} dx \le x_1^2 \, |I^0_F u|^2_{W^{1,\infty}(F)} \lesssim \|u\|_{L^\infty(F)}, \quad \int_{x_{k-1}}^d \frac{|I^0_F u|^2}{x} \lesssim \|u\|_{L^\infty(F)}.$$

Moreover

$$\int_{x_1}^{x_{k-1}} \frac{|I^0_F u|^2}{x} dx \lesssim \|u\|^2_{L^\infty(F)} \int_{x_1}^{x_{k-1}} \frac{1}{x} dx \lesssim (1 + \log \frac{d}{h}) \|u\|^2_{L^\infty(F)}.$$

Consequently

$$\int_0^d \frac{|I^0_F u|^2}{x} dx \lesssim (1 + \log \frac{d}{h}) \|u\|^2_{L^\infty(F)}.$$

Proof. Let $\tilde{u}$ be the discrete harmonic extension of u in G with respect to $D(\cdot,\cdot)$. Here $D(u,v) = d^{-2}(u,v)_{0,G} + (\nabla u, \nabla v)_{0,G}$. Then by Lemma 4.11, we obtain

$$\|u\|^2_{L^\infty(\partial G)} \lesssim \|\tilde{u}\|^2_{L^\infty(G)} \lesssim (1+\log\frac{d}{h}) D(\tilde{u},\tilde{u}),$$

which combining with Lemma 4.10 give the desired results. □

LEMMA 4.13. *Let E be any edge of ∂G ($n=3$) or any vertex of $\partial G (n=2)$ and E_h be the set of nodes belonging to E. Then for any $u \in V^h(\partial G)$,*

$$\|I^0_E u\|^2_{1/2,\partial G} \lesssim h^{n-2} \sum_{x_i \in E_h} u^2(x_i) \eqsim \|u\|^2_{0,E} \lesssim (1+\log\frac{d}{h}) \|u\|^2_{1/2,\partial G}. \tag{4.12}$$

Proof. Let $\hat{v}$ be the function in $V^h(G)$ which equals to u at interior nodes of E but vanishes at other nodes on $\bar{G}$. Then we first use Lemma 4.5 and then the inverse inequality for $\hat{v}$ and $h \lesssim d$ to obtain that

$$\|I^0_E v\|^2_{1/2,\partial G} \quad \lesssim \quad d^{-2}\|\hat{u}\|^2_{0,G} + |\hat{u}|^2_{1,G} \lesssim h^{-2}\|\hat{u}\|_{0,G}.$$

Now using the quasi-uniformity of the mesh $\mathcal{T}^h$, we have

$$\|I^0_E v\|^2_{1/2,\partial G} \lesssim h^{n-2} \sum_{x_i \in E_h} u^2(x_i) \eqsim \|u\|^2_{0,E},$$

hence we proved (4.12) except the last inequality.

For the last inequality of (4.12): the case $n=2$ is a direct consequence of Lemmas 4.12 and for $n=3$ it suffices to show by Lemma 4.11 that

$$\|I^0_E u\|^2_{0,e} \lesssim (1+\log(h^{-1}))(d^{-2}\|\tilde{u}\|^2_{0,G} + |\tilde{u}|^2_{1,G}) \tag{4.13}$$

where $\tilde{u}$ is the discrete harmonic extension of $u \in V^h(\partial G)$ into G. Without loss of generality, assume $G = (0,1)^3$ and consider a specific edge of G, say the edge E corresponding to $x = y = 0$. Lemma 4.11 implies that

$$\int_0^1 u^2(0,0,z)\,dz \lesssim (1+\log(h^{-1})) \int_0^1 \|\tilde{u}\|^2_{1,\Delta_z} dz \lesssim (1+\log(h^{-1}))\|\tilde{u}\|^2_{1,G},$$

where $\Delta_s = G \cap \{z = s\}$, that gives (4.13). □

LEMMA 4.14. *Let F be a face of ∂G (n=3) or an edge of ∂G (n=2). Then*

$$\|I^0_F u\|_{H^{1/2}_{00}(F)} \lesssim (1+\log\frac{d}{h})\|u\|_{1/2,\partial G}.$$

Proof. We show for the case $G = (0,d)^n$. First we follow Lemma 4.3 in Bramble et al. [14] to prove for $n=3$. By Lemma 4.6, we have

$$\begin{aligned} \|I^0_F u\|^2_{H^{1/2}_{00}(F)} \quad &\eqsim \quad |I^0_F u|^2_{1/2,F} + \int_F \frac{I^0_F u^2(x)}{\text{dist }(x,\partial F)} ds(x) \\ &\equiv \quad I_1 + I_2. \end{aligned} \tag{4.14}$$

then by Lemma 4.4, we have $|U|_{1,G} \eqsim |u^h|_{1/2,\partial G}$. Now use the operator Π_h defined in (4.8) and Lemma 4.8(d) to conclude that

$$\begin{aligned} |u^h|_{1,G} &= \inf_{v-u^h \in V_0^h(G)} |v|_{1,G} \lesssim |\Pi_h U|_{1,G} \\ &\lesssim |U|_{1,G} \eqsim |u^h|_{1/2,\partial G}. \end{aligned}$$

□

LEMMA 4.10. *Let $D(u,v) = d^{-2}(u,v)_{0,G} + (\nabla u, \nabla v)_{0,G}$, for any $u, v \in H^1(G)$. Assume that $u^h \in V^h(G)$ is $D(\cdot,\cdot)$-discrete harmonic, i.e. $D(u^h, v) = 0, \forall\, v \in V_0^h(G)$. Then*

$$D(u^h, u^h) = \inf_{v-u^h \in V_0^h(G)} D(v,v) \eqsim d^{-1}\|u^h\|_{0,\partial G}^2 + |u^h|_{1/2,\partial G}^2. \tag{4.10}$$

If in addition, the mean value $\gamma_{\partial G}(u^h)$ of u^h on ∂G vanishes or u^h vanishes on one face of G, then

$$|u^h|_{1,G} \eqsim |u^h|_{1/2,\partial G}. \tag{4.11}$$

Proof. (4.10) can be proved in the same way as for Lemma 4.9 but with the help of Lemmas 4.5 and Lemma 4.8 (c)-(d), while (4.11) follows readily from (4.10) and the Friedrichs inequality. □

It is known that $H^1(G)$ can not be imbedded into $C^0(\bar{G})$. But for finite element functions, we have the following lemma:

LEMMA 4.11. *Let $n = 2$. For any $u \in V^h(G)$,*

$$\|u\|_{L^\infty(G)}^2 \lesssim (1 + \log \frac{d}{h})(d^{-2}\|u\|_{0,G}^2 + |u|_{1,G}^2).$$

Proof. By scaling, it suffices to prove for the diameter $d = 1$. We know the following inequality, cf. Corollary 2.1 in Xu [95],

$$\|u\|_{L^\infty(G)} \lesssim |\log \varepsilon|^{1/2}\|u\|_{1,G} + \varepsilon\|u\|_{W^{1,\infty}(G)}, \forall\, u \in W^{1,\infty}(G).$$

Then for any $u \in V^h(G)$, taking $\varepsilon = h$ and using the inverse inequality for finite element functions gives

$$\|u\|_{L^\infty(G)}^2 \lesssim |\log \varepsilon| \|u\|_{1,G}^2 + \varepsilon^2\, h^{-2}\|u\|_{1,G}^2 \lesssim (1 + \log(h^{-1}))\|u\|_{1,G}^2.$$

□

The next lemma is an analogue to Lemma 4.11 on the boundary of G.

LEMMA 4.12. *Let $n = 2$. For any $u \in V^h(\partial G)$,*

$$\|u\|_{L^\infty(\partial G)}^2 \lesssim (1 + \log \frac{d}{h})(d^{-1}\|u\|_{0,\partial G} + |u|_{1/2,\partial G}^2).$$

Furthermore, if $\gamma_{\partial G}(u) = 0$, then

$$\|u\|_{L^\infty(\partial G)} \lesssim (1 + \log \frac{d}{h})^{1/2}|u|_{1/2,\partial G}.$$

where $K(\tau) = x_\tau + \gamma(\tau - x_\tau)$, cf. Fig. 2, (with x_τ being the barycenter of τ) is the simplex that is similar to τ (with similarity constant being γ) and has the same barycenter x_τ. Let $\tilde{\tau}$ be the reference element and $K(\tilde{\tau}) = \tilde{x}_\tau + \gamma(\tilde{\tau} - \tilde{x}_\tau)$ (with $\tilde{x}_\tau$ being the barycenter of $\tilde{\tau}$). Let $F_\tau : K(\tilde{\tau}) \to K(\tau)$ be the affine mapping and we denote $\tilde{\tau}' = F_\tau^{-1}(\tau')$ for any $\tau' \in T(\tau)$.

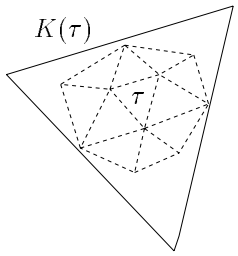

FIG. 2. *A specific element τ and $K(\tau)$*

Now given $\tau \in \mathcal{T}^h$, then

$$|v - \Pi_h v|_{t,\tau} \lesssim h^{n/2-t} |\hat{v} - \widehat{\Pi_h v}|_{t,\hat{\tau}}.$$

Define $F(\hat{v}) = |\hat{v} - \widehat{\Pi_h v}|_{t,\hat{\tau}}$.

Obviously $F(p) = 0, \quad \forall\, p \in \mathcal{P}_{s-1}(\tilde{\tau})$. Using (4.7) and Lemma 4.2, we can easily deduce that

$$F(\hat{v}) \lesssim \|\hat{v}\|_{1,K(\tilde{\tau})} \lesssim \|\hat{v}\|_{s,K(\tilde{\tau})}, \; \forall\, \hat{v} \in H^1(K(\tilde{\tau})).$$

We conclude from Bramble-Hilbert Lemma that

$$F(\hat{v}) \lesssim |\hat{v}|_{s,K(\tilde{\tau})}.$$

Therefore

$$|v - \Pi_h v|_{t,\tau} \lesssim h^{n/2-t} |\hat{v}|_{s,K(\tilde{\tau})} \lesssim h^{s-t} |v|_{s,K(\tau)}.$$

The desired result then follows easily. □

4.5. Norm estimates and equivalencies for finite element functions. This section will focus on all norm bounds and norm equivalencies for finite element functions. Let $V^h(G)$ be any Lagrangian finite element space (cf. Ciarlet [35]) defined on a quasi-uniform simplicial triangulation of G with each simplicial element having a diameter of size h.

The first lemma describes the norm relations of the space $H^1(G)$ and the trace space $H^{1/2}(\partial G)$ for finite element functions.

LEMMA 4.9. *Let $D(u,v) = (\nabla u, \nabla v)_{0,G}$, for any $u, v \in H^1(G)$. Assume that $u^h \in V^h(G)$ is $D(\cdot,\cdot)$-discrete harmonic, i.e. $D(u^h, v) = 0, \forall\, v \in V_0^h(G)$. Then*

$$|u^h|_{1,G} = \inf_{v - u^h \in V_0^h(G)} |v|_{1,G} \eqsim |u^h|_{1/2,\partial G}. \tag{4.9}$$

Proof. The equality in (4.9) follows immediately from the orthogonality: $D(u^h, v - u^h) = 0, \; \forall v \in V^h(G) : v - u^h \in V_0^h(G)$. Let $U \in H^1(G)$ be $D(\cdot,\cdot)$-harmonic in G,

4.3. Preconditioner for the discrete $H_{00}^{1/2}$ norm. The most basic result concerning the $H_{00}^{1/2}$ norm for finite element functions is as follows.

LEMMA 4.7.

$$\|v\|_{H_{00}^{1/2}(G)} \eqsim ((-\Delta_{G,h})^{1/2} v, v)_{0,G} \quad \forall v \in V_0^h(G). \tag{4.6}$$

Proof. By Lemma 4.5, it suffices to prove that

$$\|(-\Delta)^{1/4} v\|_{0,G} \eqsim \|(-\Delta_{G,h})^{1/4} v\|_{0,G} \quad \forall v \in V_0^h(G).$$

But this estimate is well-known, see Bank-Dupont [2], Xu [95], Xu [97]. □

4.4. Average interpolant. We introduce the average nodal value interpolant which not only keeps all approximation properties of the standard nodal value interpolant, but it can be also applied to functions with much less regularity than the standard interpolant requires.

Given $x_i \in \mathcal{T}^h$, let τ_i be an $n-1$ simplex from the triangulation $\mathcal{T}^h$ with vertices z_l $(l = 1, \cdots, n)$ such that $z_1 = x_i$. The choice of τ_i is not unique, but if $x_i \in \partial\Omega$, we take $\tau_i \subset \partial\Omega$. Let $\theta_i \in \mathcal{P}_1(\tau_i)$ be the unique function satisfying

$$(\theta_i, \lambda_l)_{0,\tau_i} = \delta_{l1}, \quad l = 1, \cdots, n$$

where λ_l is the barycentric coordinate of τ_i with respect to z_l. It is easy to see that

$$|(\theta_i, v)_{0,\tau_i}| \lesssim h^{1-n} \int_{\tau_i} |v|\, dx \tag{4.7}$$

and

$$(\theta_i, v)_{0,\tau_i} = v(x_i) \quad \text{if } v \in \mathcal{P}_1(\tau_i).$$

The average nodal vaule interpolant Π_h is then defined by

$$(\Pi_h v)(x) = \sum_{i=1}^{N_h} (\theta_i, v)_{0,\tau_i} \phi_i(x). \tag{4.8}$$

Let $\bar{V}^h \subset H^1(\Omega)$ be a finite element space consisting of piecewise linear functions with respect to the triangulation $\mathcal{T}^h$. We have

LEMMA 4.8. *The operator Π_h defined above satisfies*

- ***(a)*** $\Pi_h : H^1(\Omega) \to \bar{V}^h$ *and* $\Pi_h : H_0^1(\Omega) \to V^h$.
- ***(b)*** $(I - \Pi_h)v \in H_0^1(\Omega)$ *if* $v|_{\partial\Omega} \in \bar{V}^h(\partial\Omega)$.
- ***(c)*** $|(I - \Pi_h)v|_t \lesssim h^{s-t}|v|_s \qquad \forall\, v \in H^s(\Omega)$ $(s = 1, 2, t = 0, 1)$.
- ***(d)*** $|\Pi_h v|_1 \lesssim |v|_1 \qquad \forall\, v \in H^1(\Omega)$.

Proof. The proof of (a) or (b) is trivial by definition. (d) follows from (c). It remains to prove (c).

By Sobolev extension theorem (cf. Stein [88]), it suffices to prove that

$$\|v - \Pi_h v\|_{t,R^n} \lesssim h^{s-t} |v|_{s,R^n}, \quad v \in H^1(R^n)(s = 1, 2,\ t = 0, 1).$$

By quasiuniformity, there exists a constant $\gamma > 0$ indepenedent of h such that

$$T(\tau) \equiv \cup_{\partial\tau' \cap \partial\tau \neq \emptyset} \tau' \subset K(\tau), \quad \tau \in \mathcal{T}^h$$

Sobolev space $H_{00}^{1/2}(F)$**.** For a face F of G, we define a special space

$$H_{00}^{1/2}(F) = \{v \in L^2(\partial G);\ supp\ v \subset F,\quad \hat{v} \in H^{1/2}(\partial G)\}$$

where $\hat{v}$ is the zero extensin of v onto ∂G, and its norm

$$\|v\|_{H_{00}^{1/2}(F)} = (d^{-1}\|\hat{v}\|_{0,\partial G}^2 + |\hat{v}|_{1/2,\partial G}^2)^{1/2}.$$

We have

LEMMA 4.6. *Let $u \in H^{1/2}(\partial G)$ and supp $u \subset F$, F being a face of G. Then*

$$H_{00}^{1/2}(F) \underset{\sim}{=} \Big(\int_F \int_F \frac{(u(x)-u(y))^2}{|x-y|^n} ds(x) ds(y) + \int_F \frac{u^2(x)}{dist\ (x, \partial F)} ds(x) \Big)^{1/2}.$$

Proof. Because of the similarity, we prove only for $n = 3$ and $G = (0,d)^3$. By the assumption, we see the double integral in the norm $\|u\|_{H_{00}^{1/2}(F)}$ can be reduced into

$$|u|_{1/2,\partial G}^2 = \int_F \int_F \frac{(u(x)-u(y))^2}{|x-y|^3} ds(x) ds(y) + 2 \int_F \int_{\partial G \setminus F} \frac{u^2(x)}{|x-y|^3} ds(x).$$

By straightforward computations, we derive

$$\int_{\partial G \setminus F} \frac{1}{|x-y|^3} ds(x) \underset{\sim}{=} \sum_{l=1}^{3} \frac{1}{\text{dist}\ (y, e_l)} \underset{\sim}{=} \frac{1}{\text{dist}\ (y, \partial F)}$$

where e_1, e_2 and e_3 are three edges of the face F. So we see from above that

$$|u|_{1/2,\partial G}^2 \underset{\sim}{=} \|u\|_{H_{00}^{1/2}(F)}^2.$$

Then Lemma 4.6 follows easily. □

Interpolation property of $H_{00}^{1/2}$ ***space.*** Let $-\Delta_G : H_0^1(G) \mapsto H^{-1}(G)$ be the Laplacian operator. It is well-known that

$$\|v\|_{1,G} \underset{\sim}{=} \|(-\Delta_G)^{1/2} v\|_{0,\Omega} \quad \forall v \in H_0^1(G). \tag{4.4}$$

For the space $H_{00}^{1/2}(G)$, the following equivalence holds:

$$\|v\|_{H_{00}^{1/2}(G)} \underset{\sim}{=} \|(-\Delta_G)^{1/4} v\|_{0,\Omega} \quad \forall v \in H_{00}^{1/2}(G). \tag{4.5}$$

The above result is well-know for smooth domains (cf. Lions-Magenes [63]). The justification of (4.5) depends on a Sobolev extension theorem for the space $H^{1/2}$. A complete proof of a general fractional order Sobolev extension theorem for a Lipschitz domain can be found in DeVore-Sharpley [40], although a proof for $H^{1/2}$ space can be in principle obtained by using local Lipschitz charts. With the Sobolev extension theorem in hand, the proof of (4.5) is easy. For example, a technique in Bramble [8] can be naturally modified to compose a simple proof for (4.5).

By the well-known equivalence between Hilbert scale and real method of interpolation (cf. Lions-Magenes [63]), (4.5) is equivalent to the statement that $H_{00}^{1/2}(G)$ is the interpolated space half way between $H_0^1(G)$ and $L^2(G)$ spaces. And this fact is widely used in the literature and will also be used in this paper later.

The next two lemmas give more accurate relations between the Sobolev space $H^1(G)$ and the trace space $H^{1/2}(\partial G)$.

LEMMA 4.4. *Let $D(u,v) = (\nabla u, \nabla v)_{0,G}$, for any $u, v \in H^1(G)$. Assume that $u \in H^1(G)$ is $D(\cdot,\cdot)$-harmonic, i.e. $D(u,v) = 0, \forall\, v \in H_0^1(G)$. Then*

$$|u|_{1,G} = \inf_{v-u \in H_0^1(G)} |v|_{1,G} \eqsim |u|_{1/2,\partial G}. \tag{4.1}$$

Proof. The equality in (4.1) follows from the orthogonality that $D(u, v-u) = 0, \forall\, v - u \in H_0^1(G)$:

$$D(u,u) \le D(u,u) + D(v-u, v-u) = D(v,v).$$

For the equivalancy in (4.1), first using a priori estimates for the Dirichlet problem (cf. Grisvard [56]):

$$u = u|_{\partial G}, \quad D(u,v) = 0, \ \forall\, v \in H_0^1(G),$$

we know that

$$|u|_{1,G}^2 \lesssim \|u\|_{1/2,\partial G}^2 = d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2,$$

from which we obtain by replacing u by $u - \gamma_{\partial G}(u)$ that

$$|u|_{1,G}^2 \lesssim d^{-1}\|u - \gamma_{\partial G}(u)\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2. \tag{4.2}$$

Now let $d = 1$ and it is easy to verify by contradiction that $\|u - \gamma_{\partial G}(u)\|_{0,\partial G}^2 \lesssim |u|_{1/2,\partial G}^2$, which implies by scaling that for $d \neq 1$, $d^{-1}\|u - \gamma_{\partial G}(u)\|_{0,\partial G}^2 \lesssim |u|_{1/2,\partial G}^2$, thus combining with (4.2) gives

$$|u|_{1,G} \lesssim |u|_{1/2,\partial G}.$$

On the other hand, by the trace theorem 4.3,

$$|u|_{1/2,\partial G}^2 \lesssim \|u\|_{1/2,\partial G}^2 \lesssim d^{-2}\|u\|_{0,G}^2 + |u|_{1,G}^2,$$

which together with Friedrichs inquality yields

$$|u|_{1/2,\partial G} \lesssim |u|_{1,G}.$$

Therefore we proved $|u|_{1/2,\partial G} \eqsim |u|_{1,G}$. ∎

LEMMA 4.5. *Let $D(u,v) = d^{-2}(u,v)_{0,G} + (\nabla u, \nabla v)_{0,G}$, for $u, v \in H^1(G)$. Assume that $u \in H^1(G)$ is $D(\cdot,\cdot)$-harmonic, i.e. $D(u,v) = 0, \forall\, v \in H_0^1(G)$. Then*

$$D(u,u) = \inf_{v-u \in H_0^1(G)} D(v,v) \eqsim d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2 \tag{4.3}$$

Proof. The lemma can be proved in the same way as for Lemma 4.4, but no need for Friedrichs inequality here. ∎

The following lemma states the well-known Poincaré and Friedrichs inequalities.

LEMMA 4.1. *Let $\gamma(u)$ be either $\gamma_\Omega(u)$ or $\gamma_{\Gamma_0}(u)$, with $\gamma_\Omega(u)$ and $\gamma_{\Gamma_0}(u)$ the averages of u on Ω and $\Gamma_0 \subset \partial\Omega$ resp., and meas$(\Gamma_0) > 0$. Then*

$$\begin{aligned} \|u\|_{0,\Omega} &\lesssim dist(\Omega)\,|u|_{1,\Omega}, \forall\, u \in H_0^1(\Omega), \quad \textit{(Poincaré inequality)} \\ \|u-\gamma(u)\|_{0,\Omega} &\lesssim dist(\Omega)\,|u|_{1,\Omega}, \forall\, u \in H^1(\Omega), \quad \textit{(Friedrichs inequality).} \end{aligned}$$

The next is a trace theorem, the proof follows Grisvard [56].

LEMMA 4.2. *For any $u \in H^1(G)$ and $\varepsilon \in (0,1)$,*

$$\|u\|_{0,\partial G} \lesssim \varepsilon^{-1}\,\|u\|_{0,G} + \varepsilon\,|u|_{1,G}.$$

Proof. By the assumption on the boundary ∂G, there exists a constant δ depending only on the Lipschitz parameter of the boundary ∂G and a function $\vec{\mu} \in (C^1(\bar{G}))^n$ (cf. Lemma 1.5.1.9 in [56]) such that

$$\vec{\mu}(x)\cdot\vec{n}(x) \geq \delta \quad a.e.\ on\ \partial G.$$

Using this, we get

$$\begin{aligned} \int_{\partial G} u^2 dx &\lesssim \int_{\partial G} u^2 \vec{\mu}(x)\cdot\vec{n}(x) dx \\ &= \int_G u^2 \mathrm{div}\vec{\mu} dx + \int_G 2u\vec{\mu}\cdot\nabla u dx \text{ (Green's formula)} \\ &\lesssim \int_G u^2 dx + \int_G |u|\,|\nabla u| dx \text{ (by } \vec{\mu} \in (C^1(\bar{G}))^n) \\ &\lesssim (1+\varepsilon^{-2})\int_G u^2 dx + \varepsilon^2 \int_G |\nabla u|^2 dx \text{ (by } ab \leq \varepsilon a^2 + b^2/(4\varepsilon)). \end{aligned}$$

□

4.2. Sobolev spaces $H^{1/2}$ and $H_{00}^{1/2}$. Because of the extraordinary importance of these two spaces, in this subsection, we shall give a careful description for them.

Sobolev space $H^{1/2}(\partial G)$**.** Define the trace Sobolev space $H^{1/2}(\partial G)$ by

$$H^{\frac{1}{2}}(\partial G) = \{u \in L^1(\partial G);\ d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2 < \infty\}$$

where, let ds denote the surface element on ∂G,

$$|u|_{1/2,\partial G}^2 = \int_{\partial G}\int_{\partial G} \frac{(u(x)-u(y))^2}{|x-y|^n} ds(x) ds(y), \quad \|u\|_{0,\partial G}^2 = \int_{\partial G} u^2 ds,$$

and the norm of $H^{\frac{1}{2}}(\partial G)$ is always understood as $(d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2)^{1/2}$.

The following theorem is a spacial case of a result by Gagliardo stating the relation between the trace space $H^{1/2}(\partial G)$ and the space $H^1(G)$ (cf. Theorem 1.5.1.3 in Grisvard [56]).

THEOREM 4.3. *The mapping $u \to u|_{\partial G}$ which is defined for $u \in C^{0,1}(\bar{G})$, has a unique continuous extension as an operator from $H^1(G)$ onto $H^{1/2}(\partial G)$. This operator has a right continous inverse.*

The history of the theoretical development of PSC method can be traced in Nepomnyaschikh [75], Dryja-Widlund [44], Bramble-Pasciak-Xu [16], Bramble-Pasciak-Xu-Wang [15] and Xu[96]. Addtive Schwarz method was initiated by the classic alternating or multiplicative Schwarz algorithm, which was first proposed by Schwarz [78] in 1870. Variants and applications of the alternating Schwarz method were later studied by Sobolev [87], Morgenstern [72], Babuška [1], Courant-Hilbert [37], Kang-Sun-Chen [59]. The first fundamental variational formulation of the overlapping Schwarz method was done by Lions [64, 65, 66, 67]. More quantitative convergence analysis for special shaped domains was studied by Kang [47], Tang [92, 91] and Chan-Hou-Lions [34]. Recently the additive Schwarz methods were showed to be very efficient as well for solving elliptic or parabolic problems on unstructured meshes which don't require any matching between coarse and fine grids (cf. Chan-Zou [31, 32, 33], Chan-Smith-Zou [30] and Cai [21].

4. Preliminary II: Sobolev spaces and finite element spaces. In contrast to the algebraic nature of the previous section, this section is devoted to the analytical tools for analyzing domain decomposition methods. Some basic facts on Sobolev spaces and finite element spaces will be carefully reviewed here.

We shall use the notation G to denote a reference domain (polygonal for $n = 2$ and polyhedral for $n = 3$) unless otherwise specified, with each edge length of d instead of the unit length as usual. For simplicity, we assume that G is a n-dimensional simplex, or a n-cube with each edge length of d, the boundary ∂G consisting of faces $\{F\}$, edges $\{E\}$ and vertices $\{v\}$.

The domain G will play the role of a general subdomain in a domain decomposition method, all the inequalities of this section will be given the explicit dependence on the diameter d of G. This is done by first establishing the inequality for $d = 1$ and the general result is then obtained by a scaling. We have also implicitly used the fact that the constants in certain Sobolev inequalities for a unit size domain will be uniform for a reasonable class of such domains.

4.1. Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. For any open bounded domain Ω in R^3, we define $H^1(\Omega)$ by

$$H^1(\Omega) = \{u \in L^2(\Omega); \nabla u \in (L^2(\Omega))^n\}$$

with its semi-norm and norm defined as

$$|u|_{1,\Omega} = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}, \quad \|u\|_{1,\Omega} = (\|u\|_{0,\Omega}^2 + |u|_{1,\Omega}^2)^{1/2}$$

where $\|u\|_{0,\Omega} = (\int_\Omega u^2 dx)^{1/2}$, $\nabla u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$, and the derivatives are to be understood in the sense of distributions (cf. Lions-Magenes [63]), and $|\nabla u|$ is the Euclidean norm of ∇u in R^n.

The Sobolev space $H_0^1(\Omega) \subset H^1(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{1,\Omega}$, $C_0^\infty(\Omega)$ consisting of infinitely differentiable functions with compact support in Ω.

We will also mention the Sobolev space $W^{1,\infty}(\Omega)$, which is defined by

$$W^{1,\infty}(\Omega) = \{u \in L^\infty(\Omega);\ \nabla u \in (L^\infty(\Omega))^n\}$$

with its norm defined by

$$\|u\|_{W^{1,\infty}(\Omega)} = \max_{1\le i\le n} \|\partial u/\partial x_i\|_{L^\infty(\Omega)}, \quad \|v\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|.$$

Proof. By noting $\hat{A}\hat{P} = \hat{Q}A$, we have on the subspace $\hat{V}$,

$$\hat{B}\hat{A} = \hat{P}B\hat{P}^*\hat{A} = \hat{P}B(\hat{A}\hat{P})^* = \hat{P}B(\hat{Q}A)^* = \hat{P}BA\hat{Q}^* = \hat{P}BA.$$

□

Local-global technique. This is a technique for constructing preconditioners on the space V by using some known local preconditioners on a subspaces $\hat{V}$ of V.

Given a preconditioner $\hat{B} : \hat{V} \to \hat{V}$ for the operator $\hat{A}$ on $\hat{V}$. We assume that there exist two constants α_0 and α_1 such that

$$\alpha_0(\hat{A}\hat{u}, \hat{u}) \leq (\hat{B}\hat{A}\hat{u}, \hat{A}\hat{u}) \leq \alpha_1(\hat{A}\hat{u}, \hat{u}), \forall\, \hat{u} \in \hat{V}, \tag{3.7}$$

then the following algorithm provides a preconditioner B for A on the space V.

ALGORITHM 3.2 (GLOBAL ALGORITHM). *Given $g \in V$, $u = Bg = u_P + u_R$ is computed as follows:*

1. Solving the local problem: $u_P \in \hat{V}^\perp$ satisfying

$$(Au_P, v) = (g, v), \forall\, v \in \hat{V}^\perp.$$

2. Compute u_R by

$$u_R = \hat{B}\hat{Q}(g - Au_P).$$

THEOREM 3.4. *For **Global algorithm** defined above, we have*

$$B = (\hat{P}^\perp + \hat{P}\hat{B}\hat{A}\hat{P})A^{-1}$$

where $\hat{P}^\perp : V \to \hat{V}^\perp$ is an orthogonal projection with respect to $(A\cdot, \cdot)$, and

$$\kappa(BA) \leq \frac{\max(1, \alpha_1)}{\min(1, \alpha_0)}.$$

Proof. By definition we see from Step 1

$$u_P = \hat{P}^\perp A^{-1} g,$$

and substituting u_P into u_R in Step 2 gives

$$u_R = \hat{B}\hat{Q}(AA^{-1}g - A\hat{P}^\perp A^{-1}g) = \hat{B}\hat{Q}A\hat{P}A^{-1}g.$$

Theorem 3.4 then follows immediately from (3.7) and the expressions of u_P and u_R. □

3.4. Bibliographic remarks. The CG method was proposed by Hestenes-Stiefel in [73], see also Hestenes [58]. The analysis of the CG method can be found in many books, e.g. Hageman-Young [57], Golub-Van Loan [53] and Bramble [9]. For a history and extensive bibliography on the CG and PCG method, we refer to the survey paper Golub-O'Leary [54].

The concept of *parallel subspace correction* was formulated in Xu [96] and it is a generalization of the *additive Schwarz method*. The term "addtive Schwarz method" is attributed to Dryja-Widlund [44] and the basic idea is due to Nepomnyaschikh [74, 75].

decomposition methods. In most cases, it can be reduced to estimate an constant C_0 satisfying, for any $v \in V$, there exists a decomposition $v = \sum_{i=0}^{p} v_i$, $v_i \in V_i$ such that

$$\sum_{i=0}^{p} (Av_i, v_i) \leq C_0 \, (Av, v).$$

It is easy to see that

$$K_0 \leq \frac{C_0}{\omega_0} \tag{3.6}$$

where

$$\omega_0 = \min_{0 \leq i \leq p} \lambda_{\min}(R_i A_i).$$

In domain decomposition methods, K_1 measures the degree of overlapping among the subspaces V_i for $1 \leq i \leq p$. The space V_i is said to overlap with V_j if V_i is not orthogonal with V_j with respect to $(A\cdot, \cdot)$. Hence K_1 is the largest number of subspaces that a subspace can overlap with (with the exclusion of the subspace V_0). The term "overlapping" comes from the fact that, in domain decompositin methods, V_i overlaps with V_j if and only if the two subdomains that define these two subspaces do not overlap with each other. As in all the domain decomposition method considered in this paper, each subdomain overlaps with only a fixed number of other subdomains, hence K_1 is always bounded by a fixed constant. It deserves remark that, for other applications such as multigrid methods, the parameter K_1 should be defined in a more precise way than here, and for example, K_1 can be defind as in (3.5) but with ϵ_{ij} defined by

$$(AT_i u, T_j v) \leq \omega_1 \epsilon_{ij} (T_i u, u)_A^{1/2} (T_j v, v)_A^{1/2} \quad \forall 1 \leq i, j \leq p, u, v \in V.$$

3.3. Two special techniques. For investigating the relationship between different domain decomposition preconditioners, two special techniques formulated here will prove to be instrumental. These techniques are based on a single subspace $\hat{V}$ of V.

Let $\hat{A}$ be the restriction of A on $\hat{V}$ defined by

$$(\hat{A}\hat{u}, \hat{v}) = (A\hat{u}, \hat{v}), \quad \forall \hat{u}, \hat{v} \in \hat{V},$$

and $\hat{Q}, \hat{P} : V \to \hat{V}$ be two orthogonal projections with respect to $(\cdot, \cdot)$ and $(A\cdot, \cdot)$ respectively. Let $\hat{P}^*$ be the adjoint of $\hat{P}$ with respect to $(\cdot, \cdot)$.

Global-local technique. This simple technique, known as *global-local technique*, is for constructing preconditioner on a subspace $\hat{V}$ of V from a known preconditioner on the space V.

THEOREM 3.3. *Given that B is an SPD operator on V, define $\hat{B} = \hat{P} B \hat{P}^*$. Then $\hat{B}$ is SPD on $\hat{V}$ and on the subspace $\hat{V}$,*

$$\hat{B}\hat{A} = \hat{P}BA.$$

As a consequence,

$$\kappa(\hat{B}\hat{A}) \leq \kappa(BA).$$

Proof. If $v = \sum_{i=1}^{p} v_i$ is a decomposition that satisfies (3.4), then

$$(v, v)_A = \sum_{i=1}^{p} (v_i, v)_A = \sum_{i=1}^{p} (v_i, P_i v)_A,$$

and we can derive

$$\begin{aligned}
&\sum_{i=1}^{p} (v_i, P_i v)_A \\
&= \sum_{i=1}^{p} (v_i, A_i P_i v) \text{ (definite of } A_i) \\
&\leq \sum_{i=1}^{p} (R_i^{-1} v_i, v_i)^{1/2} (R_i A_i P_i v, v)_A^{1/2} \text{ (Cauchy inequality)} \\
&\leq (\sum_{i=1}^{p} (R_i^{-1} v_i, v_i))^{1/2} (\sum_{i=1}^{p} (T_i v, v)_A)^{1/2} \text{ (Cauchy inequality)} \\
&\leq \sqrt{K_0} \|v\|_A (Tv, v)_A \text{ (definition of } K_0).
\end{aligned}$$

Consequently

$$\|v\|_A^2 \leq K_0 (Tv, v)_A$$

which implies that

$$\lambda_{\min}(BA) \geq K_0^{-1}.$$

Let $\tilde{T} = \sum_{i=1}^{p} T_i$. By definition of K_1,

$$\begin{aligned}
\|\tilde{T} v\|_A^2 &= \sum_{i,j=1}^{p} (T_i v, T_j v)_A \\
&\leq \sum_{i,j=1}^{p} \omega_1 \epsilon_{ij} (T_i v, v)_A^{1/2} (T_j v, v)_A^{1/2} \\
&\leq \omega_1 K_1 (Tv, v)_A,
\end{aligned}$$

which implies that

$$\lambda_{\max}(\tilde{T}) \leq \omega_1 K_1.$$

Note $(T_0 u, u) = (Q_0 A u, R_0 u) = (A_0 u, R_0 u) = (R_0 A_0 u, u)$, for any $u \in V$, thus

$$\lambda_{\max}(BA) \leq \lambda_{\max}(T_0) + \lambda_{\max}(\tilde{T}) \leq \omega_1 (1 + K_1).$$

□

A few comments on the parameters ω_1, K_0 and K_1 are in order. The parameter ω_1 measures the resolution of R_i for the upper spectrum of A_i and $\omega_1 = 1$ if exact solvers are used on subspaces, namely $R_i = A_i^{-1}$ for all $0 \leq i \leq p$. In most of the domain decompsotion methods (with only one exception), the boundedness of ω_1 comes as an assumption. The estimate for K_0 often dominates the analysis in domain

For each i, we define two orthogonal projections $Q_i, P_i : V \to V_i$ by

$$(Q_i u, v_i) = (u, v_i), \quad (AP_i u, v_i) = (Au, v_i), \quad u \in V,\ v_i \in V_i,$$

and the restriction A_i of A on V_i by

$$(A_i u_i, v_i) = (Au_i, v_i), \quad u_i, v_i \in V_i.$$

It follows from the definition that

$$A_i P_i = Q_i A.$$

Let $R_i : V_i \to V_i$ be an SPD operator that approximates to the inverse of A_i in some sense. The PSC preconditioner for A is formulated as follows:

$$B = \sum_{i=0}^{p} R_i Q_i \tag{3.1}$$

Set $T_i = R_i Q_i A$, $0 \le i \le p$. Note that $T_i : V \to V_i$ is symmetric with respect to $(A\cdot, \cdot)$ and non-negative. We obtain from above that

$$BA = \sum_{i=0}^{p} T_i = \sum_{i=0}^{p} R_i Q_i A = \sum_{i=0}^{p} R_i A_i P_i.$$

If the i-th subproblem solver is exact, i.e. $R_i = A_i^{-1}$, then $T_i = R_i Q_i A = P_i$ and

$$BA = \sum_{i=0}^{p} P_i. \tag{3.2}$$

THEOREM 3.2. *The preconditioner B given by (3.1) satisfies*

$$\kappa(AB) \le \omega_1 K_0 (1 + K_1)$$

where

$$\omega_1 = \max_{0 \le i \le p} \lambda_{\max}(R_i A_i), \tag{3.3}$$

K_0 is the smallest positive constant satisfying, for any $v \in V$, there exists a decomposition $v = \sum_{i=0}^{p} v_i$, $v_i \in V_i$ such that

$$\sum_{i=0}^{p} (R_i^{-1} v_i, v_i) \le K_0 (Av, v), \tag{3.4}$$

and

$$K_1 = \max_{1 \le i \le p} \sum_{i=1}^{p} \varepsilon_{ij} \tag{3.5}$$

where $\varepsilon_{ij} = 0$ if $P_i P_j = 0$ (namely $V_i \perp V_j$) and $\varepsilon_{ij} = 1$ otherwise.

The above theorem is a special case of the general theory in Xu [96], for completeness, a proof is included here.

3.1. Conjugate gradient method and preconditioning. The CG method is the basis of all the preconditioning techniques to be studied in this paper. The preconditioned conjugate gradient (PCG) method can be viewed as a conjugate gradient method applied to a preconditioned system as follows:

$$BAu = Bf. \tag{3.2}$$

Here $B : V \mapsto V$ is another SPD operator and known as a preconditioner for A. Note that BA is symmetric with respect to the inner product $(B^{-1}\cdot, \cdot)$. The detailed algorithm is as follows:

ALGORITHM 3.1 (PCG). *Given u_0; $r_0 = f - Au_0$; $p_0 = Br_0$; For $k = 1, 2, \ldots,$*

$$\begin{aligned}
&\alpha_k = (Br_{k-1}, r_{k-1})/(Ap_{k-1}, p_{k-1}),\\
&u_k = u_{k-1} + \alpha_k p_{k-1},\\
&r_k = r_{k-1} - \alpha_k A p_{k-1},\\
&\beta_k = (Br_k, r_k)/(Br_{k-1}, r_{k-1}),\\
&p_k = Br_k + \beta_k p_{k-1}.
\end{aligned}$$

It is well-known that (e.g. Bramble [9])

$$\|u - u_k\|_A \leq 2\Big(\frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1}\Big)^k \|u - u_0\|_A, \tag{3.3}$$

which implies that PCG converges faster with smaller condition number $\kappa(BA)$.

Observing the formulae in the PCG method and the convergence estimate (3.3), one sees that the efficiency of a PCG method depends on two main factors: the action of B and the size of $\kappa(BA)$. Hence, a good preconditioner should have the properties that the action of B is relatively easy to compute and that $\kappa(BA)$ is relatively small (at least smaller than $\kappa(A)$).

The following result is often useful in the estimate of the condition number $\kappa(BA)$.

LEMMA 3.1. *Assume that A and B are both SPD with respect to $(\cdot, \cdot)$ and μ_0 and μ_1 are two positive constants. The following, which hold for all $v \in V$, are equivalent:*

$$\begin{array}{lcccl}
\mu_0(Av, v) & \leq & (ABAv, v) & \leq & \mu_1(Av, v),\\
\mu_0(Bv, v) & \leq & (BABv, v) & \leq & \mu_1(Bv, v),\\
\mu_1^{-1}(Av, v) & \leq & (B^{-1}v, v) & \leq & \mu_0^{-1}(Av, v),\\
\mu_1^{-1}(Bv, v) & \leq & (A^{-1}v, v) & \leq & \mu_0^{-1}(Bv, v).
\end{array}$$

If any of the above inequalities holds, then $\kappa(BA) \leq \mu_1/\mu_0$.

3.2. Preconditioning by parallel subspace correction. All the domain decomposition preconditioners in this paper will be interpreted and analyzed within the framework of *parall subspace correction method* which was formulated by Xu in [96].

The parallel subspace correction (PSC in short) method is based on a sequence of subspace V_i, $0 \leq i \leq p$ of V such that

$$V = \sum_{i=1}^{p} V_i.$$

The above *space decomposition* is understood in the way that for any $v \in V$, there exist $v_i \in V_i$ such that $v = \sum_{i=1}^{p} v_i$.

not a subspace of $V^h(\Gamma)$. This difficulty can be overcome by introducing a subspace $V_i \subset V^h(\Gamma)$ that is "closest" to $V^h(\partial\Omega_i)$. Let $\Theta_i : V^h(\partial\Omega_i) \to V_i$ be an appropriate linear operator that links these two spaces. Using the operator Θ_i, a solver on V_i can be obtained from a solver on $V^h(\Gamma_i)$. But it is not difficult to see that the most natural operator S_i on $V^h(\Gamma_i)$ given by

$$\langle S_i u_H, v_H \rangle_{0,\Gamma_i} = A_i(u_H, v_H) = (\nabla u_H, \nabla v_H)_{0,\Omega_i}, \tag{2.11}$$

is not always invertible. There are two approaches to get around this problem. The first approach is to slightly modify S_i by adding a lower order term to make it non-singular, which results in the so-called Neumann-Neumann method (see §11). The second approach is to remove the null part of S_i in $V^h(\Gamma_i)$ by first solving a proper coarse space, which results in the so-called balancing domain decomposition method (see §11.2). The balancing domain decomposition method is viewed as a local-global approach that is discussed in §3.3.

In order to get the optimal efficiency for this type of methods, special caution needs to be taken for the construction of Θ_i and the coarse space. These issues are discussed in §8.3.

Variants and coarse spaces. Substructuring methods and methods based on local Neumann problems mentioned above represent two major kinds of methods for domain decomposition without overlappings. There are numerous variants of these two types of methods presented in the literature. It can be easily observed that many variants come from the different choices of coarse spaces. One special section (§8) will be devoted to the discussion of the main coarse spaces that can be used in domain decomposition methods. These coarse spaces can all be viewed as the range of certain linear operator, say I_0. It is observed in this paper that the analysis of a efficient coarse grid space is reduced to the analysis of certain approximation and stability properties of the associated operator I_0. As a consequence, many different coarse spaces for different algorithms can be treated in a rather unified fashion.

Other methods. A few other methods that are related to the aforementioned two major classes of methods will also be discussed in this paper for the preconditioning of the interface system. These methods, to be discussed in §12, are obtained by taking the restrictions on the interface Γ from some well-known preconditioners on Ω such as BPX preconditioner, hierarchical basis preconditioner and overlapping additive Schwatz methods. The technical tool in studying these methods is the global-local technique that is presented in §??.

3. Preliminary I: preconditioning techniques. All the domain decomposition methods discussed in this paper is based on a very important algebraic method: preconditioned conjugate gradient method (PCG in short). This section contains some basic facts for the PCG method and some basic techniques for constructing and analyzing preconditioners. The presentation in this section is purely algebraic.

Given a finite dimensional vector space V and linear operator $A : V \mapsto V$, consider the following linear equation on V:

$$Au = f. \tag{3.1}$$

We assume that A is symmetric positive definite (SPD) with respect to an inner product $(\cdot, \cdot)$ on V. Next, we shall give a brief overview of the conjugate gradient method (CG) or preconditioned conjugate gradient method (PCG), and methods of constructing and analyzing preconditioners for the above linear equation.

I_0, often known as a coarse space, to be simple and to have a very small degree of freedom. If the operator I_0 has an appropriate boundedness property, we then have

$$\begin{aligned}\sum_{i=1}^{p} |u_H|^2_{1/2,\partial\Omega_i} &\sim \sum_{i=1}^{p} \rho_i |u_H - I_0 u_H|^2_{1/2,\partial\Omega_i} + \sum_{i=1}^{p} \rho_i |I_0 u_H|^2_{1/2,\partial\Omega_i} \\ &\sim \sum_{F\subset\Gamma\setminus\mathcal{W}} \rho_F |u_H - I_0 u_H|^2_{H_{00}^{1/2}(F)} + \sum_{i=1}^{p} \rho_i |I_0 u_H|^2_{1/2,\partial\Omega_i}, \end{aligned} \tag{2.9}$$

where $F \subset \Gamma \setminus \mathcal{W}$ is an edge for $n = 2$ and a face for $n = 3$ and, roughly speaking, if $F \subset \partial\Omega_i$ and χ_F is the characteristic function for the set F,

$$|u_H - I_0 u_H|^2_{H_{00}^{1/2}(F)} \sim |\chi_F (u_H - I_0 u_H)|^2_{1/2,\partial\Omega_i}.$$

The Sobolev space $H_{00}^{1/2}(F)$ is very important in the theory of domain decomposition and it will be discussed in detail in §4.2.

The expression in (2.9) naturally leads to a preconditioner that depends on local solvers and a small global solver. In fact, the term $|u_H - I_0 u_H|^2_{H_{00}^{1/2}(F)}$ can be replaced by computationally more efficient quadratic term, which will be discussed in §4.3. The resulting preconditioner in such a process is often known as *substructuring* preconditioner.

As we note, the crucial technical aspect in a substructuring method is the operator I_0. Based on the above discussions, we expect that I_0 has the following features: it is invariant on the joint-set $\mathcal{W}$, it is bounded in certain $H^{1/2}$ norm and has a simple range with a relatively small degree of freedom. Such an operator may be known as a *joint-operator* whereas $I - I_0$ may be known as a *decomposition- operator*. The construction of I_0 will be discussed in detail in §8.

The above "breaking" or decomposition process naturally leads to a decomposition of $V^h(\Gamma)$ into a sum of subspaces as follows:

$$V^h(\Gamma) = V_0 + \sum_{F\subset\Gamma\setminus\mathcal{W}} V_F$$

with

$$V_0 = \text{Range}(I_0), \quad V_F = \{v \in V^h(\Gamma); \text{supp } v \subset F\}.$$

As a result, the framework of space decomposition and subspace correction (see discussions in §3.2) can be applied to construct parallel subspace correction preconditioners. Such type of preconditioners will be discussed in Section 11.

Methods based on local Neumann problems. As we see above, the function u_H is uniquely determined by its value on Γ and hence it is convenient to view the function in the space $V^h(\Gamma)$. If we define an operator $S_h : V^h(\Gamma) \to V^h(\Gamma)$ as follows:

$$\langle S_h u_H, v_H\rangle_{0,\Gamma} = A(u_H, v_H) \quad \forall\, u_H, v_H \in V_H, \tag{2.10}$$

then the question is reduced to constructing preconditioners for the operator S_h. A natural way to decompose the space $V^h(\Gamma)$ into subspaces is to use its restriction on each $\partial\Omega_i$: $V^h(\partial\Omega_i)$. The natural solver on $V^h(\partial\Omega_i)$ can be obtained by solving a Neumann problem on Ω_i. But there is a "catch" in this approach, namely, $V^h(\partial\Omega_i)$ is

Therefore u_P can be obtained by solving Dirichlet probelms on subdomains concurrently. The remaining part of the solution $u_H = u - u_P$ lies in the orthogonal compliment of V_P in V^h:

$$V_H = \{v \in V^h;\ A(v, \chi) = 0, \quad \chi \in V_P\},$$

and u_H obviously satisfies:

$$A(u_H, v) = (f, v) - A(u_P, v) \quad \forall\, v \in V^h \tag{2.6}$$

and

$$A(u, u) = A(u_P, u_P) + A(u_H, u_H). \tag{2.7}$$

The function $u_H \in V_H$ is called a *discrete harmonic* function. In fact

$$A(u_H, \chi) = 0 \quad \forall \chi \in V_0^h(\Omega_i),$$

and furthermore the value of u_H in Ω is uniquely determined by its value on $\partial\Omega_i$ $(1 \le i \le p)$.

As we shall see later, the condition number of the underlying matrix from (2.6) is of order $O((h_0 h)^{-1})$, where h_0 is roughly the diameter of each subdomain. The core of a nonoverlapping domain decomposition is to precondition the euqation (2.6) or, in another word, to find a computationally more efficient bilinear form that is spectrally equivalent or almost equivalent to $A(u_H, u_H)$. To this end, we first note

$$|u_H|_{1,\Omega_i} \eqsim |u_H|_{1/2,\partial\Omega_i}.$$

(the norms above are associated with Sobolev spaces $H^1(\Omega_i)$ and $H^{1/2}(\partial\Omega_i)$ which will be discussed in details in §4.2). As a result, we have

$$A(u_H, u_H) \eqsim \sum_{i=1}^{p} \rho_i |v|^2_{1/2,\partial\Omega_i}. \tag{2.8}$$

The key task is to break the coupling among subdomain boundaries as shown in (2.8). There exist two main approaches to achieve this purpose.

Substructuring method. The first approach, known as *substructuring* method, is to break the interface Γ into small and local substructures. The set that connects these substructures will be called *joint-set* and denoted by $\mathcal{W}$. The joint-set consists of those points belonging to more than two subdomains. For $n = 3$, the joint-set $\mathcal{W}$ looks like a wirebasket (cf. Fig 3) and is actually known as *wirebasket* set in some papers.

The "breaking" of Γ is realized by a properly constructred linear operator I_0 : $V^h(\Gamma) \to V^h(\Gamma)$ that satisfies

$$(I_0 v)(x) = v(x) \quad x \in \mathcal{W}.$$

By means of this operator, we may further split u_H as $u_H = (u_H - I_0 u_H) + I_0 u_H$. Note that $u_H - I_0 u_H$ vanishing on the joint-set $\mathcal{W}$ can be "localized". Although the function $I_0 u_H$ is still global, it is not a big problem if we can make the range of

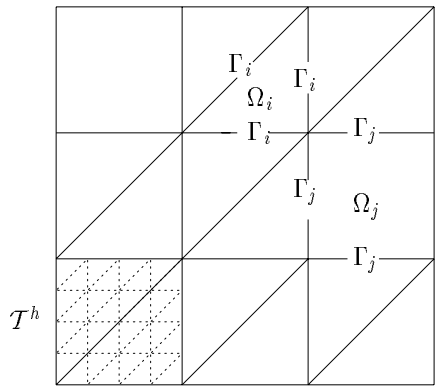

FIG. 1. *Fine mesh $\mathcal{T}^h$ and subdomains $\{\Omega_i\}_{i=1}^p$: some Ω_i are simplices and some quadrilaterals*

boundaries of elements in $\mathcal{T}^h$ (see Fig. 1). Detailed assumptions on the regularity of these subdomains Ω_i will be given in §6. We later often use the notation

$$A_i(u,v) = \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v dx \quad \forall\, u, v \in H^1(\Omega),$$

we see $A(u,v) = \sum_{i=1}^p A_i(u,v)$.

Nonoverlapping domain decomposition methods discussed in this paper are to design preconditioners for the system (2.2). A preconditioner of this kind often corresponds to a bilinear form $\tilde{A}(\cdot,\cdot)$ that, roughly speaking, has the following features. First of all, solution of the problem associated with $\hat{A}(\cdot,\cdot)$ can be reduced to smaller problems defined on subdomains. Secondly, there exist constants α_0 and α_1 such that α_1/α_0 is relatively small and

$$\alpha_0 A(v,v) \le \hat{A}(v,v) \le \alpha_1 A(v,v) \quad \forall v \in V^h. \tag{2.4}$$

For localization, we introduce the following space

$$V_P = \{v \in V^h; v(x) = 0, \quad \forall\, x \in \Gamma\}.$$

Here $\Gamma = \cup_{i=1}^p \Gamma_i$ is the interface among all the subdomains $\{\Omega_i\}$, and $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ is the interface of the subdomain Ω_i (cf. Fig 1).

The finite element equation (2.2) can be splitted into two steps. First, find $u_P \in V_P$ such that

$$A(u_P, v) = (f, v) \quad \forall\, v \in V_P. \tag{2.5}$$

Equivalently, on each subdomain Ω_i for $1 \le i \le p$, $u_P \in V_0^h(\Omega_i)$ solves the Dirichlet problem:

$$A(u_P, v) = (f, v) \quad \forall\, v \in V_0^h(\Omega_i),$$

where

$$V_0^h(\Omega_i) = \{v \in V^h;\ v(x) = 0, \quad \forall\, x \in \Omega \setminus \Omega_i\}.$$

rithm exploiting the wirebasket coarse subspace will be shown to be as efficient as the most non-overlapping methods for elliptic problems with large jumps in coefficients in both two and three dimensions.

In §10 we address the substructurng algorithms based on local Dirichlet problems initiated by Bramble-Pasciak-Schatz, with the wirebasket algorithms proposed by Smith discussed also there. Then in §11, we present the non-overlapping domain decomposition algorithms based on local Neumann problems, among them are so-called Neumann-Neumann algorithms and the balancing domain decomosition algorithm. Finally in §12, we derive a couple of interface preconditioners from the existing stiffness preconditioners by using our two techniques addressed in §3.3, for example, the interface versions of the BPX multilevel preconditioner, of the hierarchical basis preconditioner and of additive schwarz methods; moreover, the so-called vertex method will be a special case of our general formulation.

2. An outline. In this section, we shall give an outline on the algorithms presented in the paper and their motivations.

We shall consider the following model boundary value problem:

$$\begin{aligned} -\nabla \cdot (\rho(x)\nabla U) &= F(x) \quad \text{in } \Omega \\ U &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here $\Omega \subset R^n$ is a polygon for $n = 2$ or a polyhedron for $n = 3$ and $\rho(x)$ is piecewise constant in Ω or $\rho(x) \equiv 1$.

From a preconditioning point of view, the above model problem is sufficient for the study of general symmetric, positive and definite (SPD in short) second order elliptic boundary value problems. Problems with slowly changing variable coefficients can be preconditioned by the Poisson equation, namely $\rho_i = 1$ for all i in (2.1); problems with large jumps in coefficients between subdomains can be modelled by the above problem.

Assume that we are given a quasi-uniform triangulation $\mathcal{T}^h = \{\tau^h\}$ of Ω with each τ^h being a simplex of size $O(h)$. Let V^h be a finite element space consisting of continuous piecewise linear functions that vanish on $\partial\Omega$. A finite element approximation for (2.1) is to find $u \in V^h$ such that

$$A(u, v) = (f, v) \quad \forall\, v \in V^h, \tag{2.2}$$

where

$$A(u, v) = \int_\Omega \rho(x)\nabla u \cdot \nabla v \; dx.$$

A domain decomposition without overlapping consists of a number of mutually disjoint open subdomains Ω_i such that

$$\bar{\Omega} = \sum_{i=1}^{p} \bar{\Omega}_i. \tag{2.3}$$

When the coefficient $\rho(x)$ is discontinuous, each subdomain Ω_i is chosen in such a way that $\rho(x)$ equals to a constant ρ_i in Ω_i. We assume that the triangulation $\mathcal{T}^h$ is consistent with (2.3) in the sense that each $\partial\Omega_i$ can be written as a union of

computations, one needs only to represent the involved operators and finite element functions in the basis of corresponding finite element spaces. These are purely certain algebraic operations, no special background of knowledge is required. In fact, one can figure out a few basic transformations between operators and matrices, which were formulated in the paper by Xu [96].

As our focus is on non-overlapping DDMs, overlapping type methods will be very little mentioned. Only we will devote a brief section (§9 to study an interesting two-level additive Schwarz method using the wirebasket coarse subspace instead of the standard coarse subspace defined by the vertices of the subdomains. There have existed a few references on overlapping DDMs, e.g. Dryja-Widlund [46], Bramble-Pasciak-Wang-Xu [15], Xu [96], Griebel-Oswald [55], Smith-Bjørstad-Gropp [86], Le Tallec [90] for self-adjoint elliptic problems. For non-selfadjoint or indefinite elliptic problems, we refer to Cai [24], Cai-Widlund [20], Xu-Cai [100], Xu [96, 99], and Ernst-Golub [48]. For nonlinear elliptic problems, we refer to Xu [98, 94], Scroggs [80] and Zou-Huang [104].

We are not attempting to present a detailed history and literature of the concerned methods. For more thorough references, the readers are referred to recent survey papers by Chan-Mathew [28], Le Tallec [90], and the domain decomposition proceedings [50, 25, 26, 51, 27, 61, 60].

The content of the article is arranged as follows. First in the next section, we outline all algorithms which will be discussed in the paper. In §3, we formulate the preconditioned conjugate gradient algorithm and some useful equivalent relations for estimating the condition number of the product of two symmetric positive definite operators, and a general principle of the construction of additive type preconditioners (known as *parallel subspace correction* preconditioners) and converegnce analyses of these preconditioners. Then in the remainder of §3, we propose two crucial techniques, i.e. *global-local* and *local-global* techniques for constructing global preconditioners on a larger dimensional vector space V by using local preconditioners defined on a subspace of V, and vice versa. A lot of existing non-overlapping domain decomposition algorithms can be easily derived using these two techniques.

In §4, we address all fundamental tools for the converegence analysis of non-overlapping DDMs, e.g. properties of Sobolev spaces on the boundary of a domain and of continuous and discrete harmonic functions, norm estimates and equivalencies for finite element functions. The readers who are not interested in the Sobolev space and so the convergence theory of algorithms can then skip this section. Section 5 is a brief section addressing the Schur complement matrix and its generalization to the interface corresponding to a domain decomposition. In §6 we list the basic notation and concepts used in the paper.

For better understanding more general domain decomposition algorithms, in §7 we discuss non-overlapping two-subdomain algorithms and their convergence, which in fact may give us an insight into the algorithms and convergence analysis in many subdomains case.

As it is well-known, a global coarse solver is necessary for achieving an optimal or almost optimal domain decomposition algorithm. Therefore we dedicate a whole section (§8) to the investigation of properties of existing coarse subspaces in the literature and observe that their efficiencies lie in the inherent fact that the corresponding interpolants induced by the coarse spaces satisfy two properties: the H^1 stability and L^2 optimal or almost optimal approximation. Section 9 is the only one in the paper addressing overlapping DDM, where the two-level overlapping additive Schwarz algo-

NON-OVERLAPPING DOMAIN DECOMPOSITION METHODS

JINCHAO XU * AND JUN ZOU †

Abstract. We give a thorough investigation on the existing non-overlapping domain decomposition methods for solving second order elliptic problems in two and three dimensions, and also propose a couple of new algorithms. After addressing some fundamental tools for the convergence analysis of domain decomposition methods and formulating two techniques for constructing preconditioners, i.e. *local-global* and *global-local* techniques, most existing methods are easily derived and their convergence proofs follow simply. In particular, the two and three dimensional algorithms and proofs can be formulated and conducted jointly. We shall indicate that the convergence analysis for many known methods can be reduced to verfifying the L^2-optimal approximation and H^1-stability of the interpolants corresponding to global coarse subspaces.

Key Words. Non-overlapping domain decomposition, interface preconditioners, stiffness preconditoners, local-global and global-local techniques, jumps in coeffcients.

1. Introduction. In this paper, we are interested in the convergence theory of non-overlapping domain decomposition methods (DDMs) for solving large sparse linear systems of equations which arise from finite element discretizations of second order self-adjoint elliptic problems defined on a domain $\Omega \subset R^n$ $(n = 2, 3)$. Our major aim is to give a unifiied convergence analysis for existing non-overlapping DDMs, and address in detail some fundamental theoretical tools often used in the domain decomposition context.

Non-overlapping DDMs don't require any overlapp between neighboring subdomains or substructures, which seems to be able to have more flexible substructures, compared to overlapping DDMs. And this flexibility enables non-overlapping DDMs efficiently to handle elliptic problems with jumps in coefficients, when large coefficient jumps happen only subdomain by subdomain. As most overlapping DDMs, all non-overlapping DDMs discussed in the paper are based on a very important iterative method: preconditioned conjugate gradient method. And the actual aim of DDMs here is to construct a good preconditioner either for the stiffness operator A_h corresponding to the finite element equation or for the Schur complement operator S_h (we later call it *interface operator*) corresponding to the reduced interface equation. By good preconditioner we mean that the action of preconditioner is not expensive to calculate, about the same as the action of A_h or S_h. For every algorithm to be discussed, we shall present in very detail how to implement the corresponding preconditioning process required by PCG iteration, the rest part of each algorithm is standard.

Throughout the paper, we shall use operator languages or variational presentations, which might give us a deeper insight into some essentials of the algorithms and their convergence properties. The implementation of algorithms will be also formulated in variational framework. When applying these algorithms in practical

* Department of Mathematics, Penn State University, University Park, PA 16802, <xu@math.psu.edu>. The work of this author was partially supported by National Science Foundation

† Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024-1555 and Computing Center, the Chinese Academy of Sciences, Beijing 100080, P. R. China. <zou@math.ucla.edu>. The work of this author was partially supported by the National Science Foundation under contract ASC 92-01266, the Army Research Office under contract DAAL03-91-G-0150, and subcontract DAAL03-91-C-0047.