The balancing domain decomposition method was first proposed by Mandel [?], and later its convergence proof was improved by Mandel-Brezina [?]. Cowsar-Mandel-Wheeler [?] extended the BDD method to mixed finite elements. The major idea is to restrict local Neumann subproblems on certain subspaces so that local subproblems are uniquely solvable. Two other ways of overcoming singularities of local Neumann problems are to use different bilinear forms on local subdomains (see §3.1.1) and to use Lagrange multiplier approach by Farhat-Roux [?].

4. Interface preconditioners derived from stiffness preconditioners. We now apply the theory in §?? to generate some interface preconditioners from known stiffness preconditioners. Associated with §??, we choose the space $V = V^h$ and the subspace $\hat{V}$ of V^h to be the space of discrete harmonic functions, i.e. $\hat{V} = V_H$. As $\hat{P} : V^h \rightarrow V_H$ is a orthogonal projection with respect to $\langle S_h \cdot, \cdot \rangle$, it is immidiate to verify that $\hat{P}u$, for any $u \in V^h$, is the discrete harmonic extension of the restriction of u on the interface Γ.

Define an operator $B_h : V^h(\Gamma) \rightarrow V^h(\Gamma)$ by

$$B_h = \hat{R}\hat{B}\hat{R}^*. \tag{4.22}$$

where $\hat{R} : \hat{V} \rightarrow V^h(\Gamma)$ defined by $\hat{R}\tilde{u} = u$, for any $\tilde{u} \in \hat{V}$, and its adjoint $\hat{R}^*$ by :

$$(\hat{R}^* u, \tilde{v}) = \left\langle u, \hat{R}\tilde{v} \right\rangle.$$

Then we have

$$\kappa(\hat{B}\hat{A}) = \kappa(M_h S_h), \tag{4.23}$$

since it is easy to see $\hat{A} = \hat{R}^* S_h \hat{R}$ and

$$\begin{aligned} \langle B_h S_h u, S_h u \rangle &= \left\langle \hat{R}\hat{B}\hat{R}^* S_h u, S_h u \right\rangle = \left\langle \hat{B}\hat{R}^* S_h \hat{R}\tilde{u}, \hat{R}^* S_h \hat{R}\tilde{u} \right\rangle \\ &= \left\langle \hat{B}\hat{A}\tilde{u}, \hat{A}\tilde{u} \right\rangle. \end{aligned} \tag{4.24}$$

with α_0 and α_1 appearing in (??), i.e.

$$\alpha_0 \langle \hat{S}\hat{u}, \hat{u}\rangle \leq \langle \hat{B}\hat{S}\hat{u}, \hat{u}\rangle \leq \alpha_1 \langle \hat{S}\hat{u}, \hat{u}\rangle \quad \forall\, \hat{u} \in \hat{V},$$

which, from (3.18), is then equivalent to

$$\alpha_0 \langle S_h\hat{u}, \hat{u}\rangle \leq \big\langle (\sum_{i=1}^{p} \Theta_i S_i^+ \Theta_i^* Q_i) S_h \hat{u}, \hat{u}\big\rangle \leq \alpha_1 \langle S_h\hat{u}, \hat{u}\rangle \quad \forall\, \hat{u} \in \hat{V}. \tag{3.19}$$

Let $\hat{S}_i : \hat{V}_i \longrightarrow \hat{V}_i$ be the restriction of S_i on $\hat{V}_i$. Obviously, the inverse $\hat{S}_i^{-1}$ exists and we have by definition of S_i^+ that $S_i^+ u_i = \hat{S}_i^{-1} u_i$, $\forall\, u_i \in \hat{V}_i$.

Let $\breve{S}_i$ be defined as in (3.1), i.e. the interface operator on $V^h(\Gamma_i)$ corresponding to the bilinear form $\breve{A}_i(\cdot,\cdot) = A_i(\cdot,\cdot) + \rho_i h_0^{-2}(\cdot,\cdot)_{0,\Omega_i}$. Note that the assumptions made on the boundary subdomains anable us to use Friedrichs inequality to obtain

$$\langle S_i^+ u_i, u_i\rangle_{0,\Gamma_i} = \left\langle \hat{S}_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i} \approx \langle \breve{S}_i^{-1} u_i, u_i\rangle_{0,\Gamma_i} \quad \forall\, u_i \in \hat{V}_i. \tag{3.20}$$

Now (3.19) holds with $\alpha_1 = \omega_1 = O(\log(h_0/h)^2)$ and $\alpha_0 = K_0^{-1} = O(1)$ by combining (3.20) with (3.21) in the following Remark 3.1, thus $\kappa(B_h S_h) = O(\log(h_0/h)^2)$. □

Remark 3.1. *Let $N_h = \sum_{i=1}^{p} \Theta_i \breve{S}_i^{-1} \Theta_i^* Q_i$, then it is easy to find out by going through the proof of Theorem 3.1 which used Lemma ?? that the condition number of $N_h S_h$ restricted on the subspace $\hat{V} = V_0^{\perp}$ is the order $O(K_0\omega) = O(\log(h_0/h)^3)$; but the order $O(\log(h_0/h)^2)$ if each boundary subdomain has a common face with the boundary $\partial\Omega$. More exactly, we have*

$$K_0^{-1} \langle S_h\hat{u}, \hat{u}\rangle \lesssim \langle N_h S_h\hat{u}, S_h\hat{u}\rangle \lesssim \omega_1 \langle S_h\hat{u}, \hat{u}\rangle, \forall\, \hat{u} \in \hat{V}. \tag{3.21}$$

To see this, let $V = V^h(\Gamma)$, $A = S_h$ and define $\hat{P}, \hat{P}^*$ as in Section ??, $\hat{S}$ to be the restriction of S_h on $\hat{V}$, and $\hat{B} = \hat{P} B_h \hat{P}^*$. It is straightforward to verify by using $\hat{P}^*\hat{S}\hat{u} = S_h\hat{u}$ that for any $\hat{u} \in \hat{V}$,

$$\begin{aligned} \left\langle \hat{B}_h \hat{S}_h \hat{u}, \hat{S}_h \hat{u}\right\rangle &= \langle M_h S_h \hat{u}, S_h\hat{u}\rangle = \langle R_0 Q_0 S_h \hat{u}, S_h\hat{u}\rangle + \langle N_h S_h\hat{u}, S_h\hat{u}\rangle \\ &= (N_h S_h \hat{u}, S_h \hat{u}) \text{ (by } \hat{V} = V_0^{\perp}), \end{aligned}$$

which, combining with Theorem 3.1 implies (3.21).

3.3. Bibliograph remarks. The methods discussed in this section, often known as Neumann-Neumann type of algorithms, can be traced back to the work by Dinh-Glowinski-Périaux [?] and Glowinski-Wheeler[?]. Thereafter there are a few extensions in the theory and algorithms. We refer to Bourgat-Glowinski-Tallec-Vidrascu[?], Roeck-Tallec [?], Tallec-Roeck-Vidrascu [?], Mandel [?, ?], and Dryja-Widlund [?]. For extension of the approach for mixed finite element framework by Glowinski-Wheeler[?] to many subdomain case, see Cowsar-Wheeler[?].

Neumann-Neumann algorithms with weighted coarse subspaces for $n = 3$ was proposed by Dryja-Widlund [?], where the use of standard coarse subspaces (cf. §3.1.2) was also considered for elliptic problems with uniformly bounded coefficients. Here we give a unified presentation for both two and three dimensional cases with the case of large jukmps in coefficients included. In particular, we added the case of using the zero extensions E_i in local solvers R_i instead of weighted operators Θ_i.

Obviously, for a balanced function r^h there exists $u_i \in \hat{V}_i$ such that

$$S_i u_i = \Theta_i^* Q_i r^h,$$

and the solution u_i will be denoted by $u_i = S_i^+ \Theta_i^* Q_i r^h$. Note the inverse of S_i does not exist for interior subdomains Ω_i, and the above $\hat{V}_i$ defined by:

$$\hat{V}_i = \{u_i \in V^h(\Gamma_i); \quad \int_{\Gamma_i} u_i \, dx = 0\}. \tag{3.17}$$

For boundary subdomains $\Omega_i : \bar{\Omega}_i \cap \partial\Omega \neq \emptyset$, we let $\hat{V}_i = V^h(\Gamma_i)$.

Applying **Global Algorithm** in §?? to the present case with $A = S_h$, $V = V^h(\Gamma)$, $\hat{V} = V_0^\perp$ - the complement of V_0 in the sense of $\langle S_h \cdot, \cdot \rangle$, and as in §??, take $\hat{B} = \hat{P}(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i)\hat{P}^*$ as a known preconditioner to $\hat{S}$, then we derive from Theorem **??** a preconditioner B_h for S_h:

$$\begin{aligned} B_h S_h &= P_0 + \hat{P}\hat{B}\hat{S}\hat{P} = P_0 + \hat{P}(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i)\hat{P}^* \hat{S}\hat{P} \\ &= P_0 + (I - P_0)(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i) S_h (I - P_0). \end{aligned} \tag{3.18}$$

where $P_0 : V^h(\Gamma) \to V_0$ is the orthogonal projection with respect to $\langle S_h \cdot, \cdot \rangle$. Using (3.18) and the definition of S_i^+, we immediately come to the following algorithm:

Algorithm 3.2 (Balancing Domain Decompositon Algorithm). *For any $g \in V^h(\Gamma)$, $u = B_h g$ is done, step by step, as follows:*

1. Balancing the original residual by solving

$$\langle S_h w_0, \phi \rangle = \langle g, \phi \rangle \quad \forall \phi \in V_0.$$

Set $r^h = g - S_h w_0$.

2. Compute u_i, $1 \le i \le p$ in parallel: $u_i = \tilde{u}_i|_{\Gamma_i}$ with $\tilde{u}_i \in V^h(\Omega_i)$ solving

$$A_i(\tilde{u}_i, \phi_l) = \left\langle r^h, \Theta_i \phi_l \right\rangle, \forall \phi_l \in V^h(\Omega_i);$$

Compute $\hat{u}_i = \Theta_i u_i$.

3. Balancing the residual: $w_1 \in V_0$ solves

$$\langle S_h w_1, \phi \rangle = \big\langle g - S_h \sum_{i=1}^p \hat{u}_i, \phi \big\rangle, \forall \phi \in V_0.$$

4. Compute $u = \sum_{i=1}^p \hat{u}_i + w_1$.

Theorem 3.3. *Assume that for each subdomain Ω_i, $\partial\Omega_i \cap \partial\Omega$ is either empty or a face ($n = 3$) or an edge ($n = 2$) of Ω_i. Then for the above BDD algorithm,*

$$\kappa(B_h S_h) \lesssim \log^2(h_0/h).$$

Proof. By using Theorem **??**, we know

$$\kappa(B_h S_h) \le \frac{\max(1, \alpha_1)}{\min(1, \alpha_0)}$$

noticing that $\breve{w}$ equals to $\tilde{u} - Q^{\rho}_{h_0}\tilde{u}$ on the interface Γ, thus by Lemma ??,

$$\breve{A}_i(\breve{w}, \breve{w}) \le \breve{A}_i(\tilde{u} - Q^{\rho}_{h_0}\tilde{u}, \tilde{u} - Q^{\rho}_{h_0}\tilde{u}) \tag{3.14}$$

then it follows from (3.13)-(3.14) and Lemma 1.1 that

$$\sum_{i=1}^{p} \left\langle R_i^{-1}u_i, u_i \right\rangle_{0,\Gamma_i} \lesssim \gamma_0(n)\sum_{i=1}^{p} A_i(\tilde{u},\tilde{u}) = \gamma_0(n)\left\langle S_h u, u\right\rangle, \tag{3.15}$$

with $\gamma_0(2) = \log(h_0/h)$ but $\gamma_0(3) = h_0/h$. Again using Lemmas ?? and Lemma 1.1,

$$\begin{aligned}
\left\langle R_0^{-1}u_0, u_0\right\rangle &= \left\langle S_h u_0, u_0\right\rangle = \sum_{i=1}^{p} A_i(\tilde{u}_0, \tilde{u}_0) \le \sum_{i=1}^{p} A_i(Q^{\rho}_{h_0}\tilde{u}, Q^{\rho}_{h_0}\tilde{u}) \\
&\lesssim \gamma(n)A(\tilde{u},\tilde{u}) = \gamma(n)\left\langle S_h u, u\right\rangle,
\end{aligned} \tag{3.16}$$

which with (3.15) implies $K_0 = \gamma_0(n)$, that proves (3.12).

(3.11) follows by replacing the operator Q^{ρ}_{H} in the above proof for the estimation of K_0 by the standard L^2 projection Q_{h_0}.

Finally consider the E_i case, i.e. the local solver $R_i = E_i\breve{S}_i^{-1}E_i^*$: the only differece from the Θ_i case is the estimate of K_0, i.e. the derivation of (3.15)-(3.16) from (3.13)-(3.14) with replacing the weighted operator $Q^{\rho}_{h_0}$ by the standard L^2 projection Q_{h_0} here. Then in the present case, we obtain

$$\left\langle R_i^{-1}u_i, u_i\right\rangle_{0,\Gamma_i} = \left\langle \breve{S}_i E_i^{-1}u_i, E_i^{-1}u_i\right\rangle_{0,\Gamma_i} \lesssim \rho_i \left\|E_i^{-1}u_i\right\|^2_{1/2,\partial\Omega_i} = \rho_i \left\|u_i\right\|^2_{1/2,\partial\Omega_i},$$

now repeating the same decomposition (3.7) and the estimates thereafter, we have

$$\rho_i\|u_i\|^2_{1/2,\partial\Omega_i} \lesssim \log(h_0/h)^2 \left\langle \breve{S}_i\Theta_i^{-1}u_i, \Theta_i^{-1}u_i\right\rangle_{0,\Gamma_i} = \log(h_0/h)^2\left\langle \breve{S}_i w, w\right\rangle_{0,\Gamma_i},$$

the rest is the same as proving (3.15)-(3.16 but with $Q^{\rho}_{h_0}$ replaced by Q_{h_0} here. □

3.2. Balancing Domain Decomposition Method. The balancing domain decomposition method is resulted from another approach to the singularity of S_i on $V(\Gamma_i)$. Rather than modifying the expression of the operator S_i itself as done in the last subsection, S_i can be made nonsingular by removing its null space. In another word, S_i is to be applied on a $V(\Gamma_i)$'s subspace on which S_i is nonsingular. In fact, the null space of S_i is at most a one dimensional space that contains only constant functions. If these contant functions can be annihilated, the operator S_i then becomes nonsingular.

The idea is first to solve the equation on a coarse subspace V_0 so that the resulting residual does not contain any constant component on each Γ_i and then to apply the Neumann-Newmann type algorithm (with a non-modified S_i) to the residual equation. This approach falls into the *local-global* technique described in §??.

Let e_0 denote a constant in this subsection. We then define the coarse subspace

$$V_0 = \text{span}\{\Theta_i e_0, \text{ for all interior subdomain } \Omega_i : \ \bar{\Omega}_i \cap \partial\Omega = \emptyset\}.$$

We say a function $r^h \in V^h(\Gamma)$ is balanced if r^h is orthogonal to V_0, or equivalently,

$$\left\langle \Theta_i^* Q_i r^h, e_0\right\rangle_{0,\Gamma_i} = 0, \quad i.e. \ \int_{\Gamma_i} \Theta_i^* Q_i r^h \, dx = 0.$$

Noting that $u_0 = u - \sum_{i=1}^{p} u_i$, (3.9), $K_1 \lesssim 1$ and $\omega_1 \lesssim \log(h_0/h)^2$, we deduce that

$$\begin{aligned}\left\langle R_0^{-1}u_0, u_0\right\rangle &= \log(h_0/h)^{-2}\left\langle S_h u_0, u_0\right\rangle \lesssim \log(h_0/h)^{-2}\left\langle S_h u, u\right\rangle + \log(h_0/h)^{-2}\sum_{i=1}^{p}\left\langle S_h u_i, u_i\right\rangle \\ &\lesssim \left\langle S_h u, u\right\rangle + \sum_{i=1}^{p}\left\langle R_i^{-1}u_i, u_i\right\rangle \lesssim \log(h_0/h)\left\langle S_h u, u\right\rangle,\end{aligned}$$

combining this with (3.9) implies $K_0 \lesssim \log(h_0/h)$, which concludes the proof of the first estimate in Theorem 3.1.

The second estimate in the theorem follows by going through the proof and noting that in this case $\omega_1 = O(\log(h_0/h)^2)$ and $K_0 = O(1)$. □

3.1.2. The use of standard coarse subspaces. As an alternative choice, the standard coarse space in §?? can also be used as the coarse space to define the preconditioner (3.2). Such a coarse space obviously has a much simpler structure than the weighted space, but it has limitations as it is not efficient for problems wth large discontinuous jumps for $n = 3$.

With V_0 being the standard coarse space and exact coarse solver $R_0 = S_0^{-1}$, the function $w_0 \in V_0$ in (3.3) for the action of M_h can be obtained by solving

$$A(\tilde{w}_0, \phi) = \langle g, \phi\rangle, \ \forall\, \phi \in V_0. \tag{3.10}$$

THEOREM 3.2. *If V_0 is the standard coarse space discussed in §?? and $R_0 = S_0^{-1}$, then the preconditioner given by (3.2) satisfies*

$$\kappa(M_h S_h) \lesssim (\max_{1\le i\le p} \rho_i)\log(h_0/h)^2, \tag{3.11}$$

where ρ_i are the coefficients of equation (??), or without the coefficients

$$\kappa(M_h S_h) \lesssim \begin{cases} \log(h_0/h)^3 & \text{if } n = 2, \\ \log(h_0/h)^2 h_0/h & \text{if } n = 3. \end{cases} \tag{3.12}$$

Moreover if each Θ_i is replaced by the zero extension operator $E_i : V^h(\Gamma_i) \to V_i$, namely the subspace sover $R_i = E_i \check{S}_i^{-1} E_i^*$, then

$$\kappa(M_h S_h) \lesssim r(\rho)\log(h_0/h)^4.$$

where $r(\rho) = \max_i \rho_i / \min_i \rho_i$.

Proof. By Theorem ??, it suffices to estimate K_1, K_0 and ω_1.

The same proof as for Theorem 3.1 gives $K_1 \lesssim 1$ and $\omega_1 \lesssim \log(h_0/h)^2$ for the Θ_i case but $\omega_1 \lesssim r(\rho)\log(h_0/h)^2$ for the E_i case. The only difference for the second case is to replace Θ_i by E_i in the proof for Theorem 3.1.

Now we estimate K_0. First consider the Θ_i case, i.e. $R_i = \Theta_i \check{S}_i^{-1}\Theta_i^*$: for any $u \in V^h(\Gamma)$, to define a partition of u, we take $u_0 = (Q_{h_0}^{\rho}\tilde{u})|_\Gamma \in V_0$ and $u_i = \Theta_i(u - u_0) \in V_i$. Here $Q_{h_0}^{\rho}$ is the weighted L^2 projection from $L^2(\Omega)$ to V_0 defined in (1.1). Obviously, $u = \sum_{i=0}^{p} u_i$. Let $w = u - u_0$. Using the properties of Q_H^{ρ} in Lemma 1.1, we obtain

$$\begin{aligned}\sum_{i=1}^{p}\left\langle R_i^{-1}u_i, u_i\right\rangle_{0,\Gamma_i} &= \sum_{i=1}^{p}\left\langle \check{S}_i\Theta_i^{-1}u_i, \Theta_i^{-1}u_i\right\rangle_{0,\Gamma_i} = \sum_{i=1}^{p}\left\langle \check{S}_i w, w\right\rangle_{0,\Gamma_i} \\ &= \sum_{i=1}^{p}\check{A}_i(\check{w}, \check{w}),\end{aligned} \tag{3.13}$$

Proof. By Theorem ??, we need to estimate K_1, K_0 and ω_1. Again it is clear that $K_1 \lesssim 1$. Different from all other situations in this paper, the estimate for ω_1 is not that straightforward here.

We now proceed to establish the estimate that $\omega_1 \lesssim \log(h_0/h)^2$. To this end, it suffices to prove that

$$\langle S_h u_i, u_i \rangle \lesssim \log(h_0/h)^2 \left\langle R_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i}, \forall u_i \in V_i. \tag{3.6}$$

By definition of S_h and Lemma ??,

$$\langle S_h u_i, u_i \rangle = A(\tilde{u}_i, \tilde{u}_i) \eqsim \sum_m \rho_m \, |u_i|^2_{1/2,\partial\Omega_i}, \tag{3.7}$$

where the summation is over all subdomains Ω_m which share either a face, or an edge or a vertex with Ω_i. Let $\gamma_{im} = \partial\Omega_i \cap \partial\Omega_m$, we can write u_i on the interface Γ_m of Ω_m into

$$u_i = \sum_{F \subset \gamma_{im}} I_F^0 u_i + \sum_{E \subset \gamma_{im}} I_E^0 u_i + \sum_{V_k \in \gamma_{im}} I_{V_k}^0 u_i. \tag{3.8}$$

Then

$$\begin{aligned}
&\rho_m | \sum_{F \subset \gamma_{im}} I_F^0 u_i |^2_{1/2,\partial\Omega_m} \\
\lesssim &\sum_{F \subset \gamma_{im}} |I_F^0(\nu_\rho u_i)|^2_{1/2,\partial\Omega_m} \ (\rho_m \le \nu_\rho^2 = \text{Const on } F) \\
\lesssim &\sum_{F \subset \gamma_{im}} \|I_F^0(\nu_\rho u_i)\|^2_{H_{00}^{1/2}(F)} \ (\text{by Lemma ??}) \\
\lesssim &\log(h_0/h)^2 \|\nu_\rho u_i\|^2_{1/2,\partial\Omega_i} \ (\text{Lemma ??}) \\
= &\log(h_0/h)^2 \rho_i \|\Theta_i^{-1} u_i\|^2_{1/2,\partial\Omega_i} \ (\text{definition of } \Theta_i) \\
\lesssim &\log(h_0/h)^2 \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i \right\rangle_{0,\Gamma_i} \ (\text{Lemma ?? \& } \check{S}_i\text{'s definition}) \\
= &\log(h_0/h)^2 \left\langle R_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i}.
\end{aligned}$$

Conducting the same for the second and third terms in (3.8) with Lemmas ?? and ??, we obtain (3.6) from (3.7-(3.8) and triangle inequality.

We next estimate K_0. Given any $u \in V^h(\Gamma)$, take $u_0 = \sum_{i=1}^p \Theta_i I_i u \in V_0$ and $u_i = \Theta_i(u - I_i u) \in V_i$. We readily see $u = \sum_{i=0}^p u_i$. Let $w_i = u - I_i u$. We obtain

$$\begin{aligned}
&\sum_{i=1}^p \left\langle R_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i} \\
= &\sum_{i=1}^p \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i \right\rangle_{0,\Gamma_i} = \sum_{i=1}^p \left\langle \check{S}_i w_i, w_i \right\rangle_{0,\Gamma_i} \\
= &\sum_{i=1}^p \check{A}_i(\breve{w}_i, \breve{w}_i) \le \sum_{i=1}^p \check{A}_i(\tilde{w}_i, \tilde{w}_i) \ (\text{minimizing of } \breve{w}_i) \\
\le &\log(h_0/h) \sum_{i=1}^p A_i(\tilde{u}, \tilde{u}) = \log(h_0/h) \left\langle S_h u, u \right\rangle_{0,\Gamma} \ (\text{Lemma ??}). \qquad (3.9)
\end{aligned}$$

the adjoint $\Theta_i^*: V_i \longrightarrow V^h(\Gamma_i)$ by

$$\langle \Theta_i^* u_i, v_i \rangle_{0,\Gamma_i} = \langle u_i, \Theta_i v_i \rangle \quad \forall\, v_i \in V^h(\Gamma_i).$$

With a proper choice of subspace V_0 and a solver R_0, we obtain the following space decomposition

$$V^h(\Gamma) = \sum_{i=0}^{p} V_i$$

and the corresponding PSC preconditioner

$$M_h = R_0 Q_0 + \sum_{i=1}^{p} \Theta_i \check{S}_i^{-1} \Theta_i^* Q_i. \tag{3.2}$$

Thus for any $g \in V^h(\Gamma)$,

$$M_h g = w_0 + \sum_{i=1}^{p} \Theta_i (w_i|_{\Gamma_i}), \tag{3.3}$$

and by means of the definition (3.2), the components w_i can be obtained by

ALGORITHM 3.1 (NEUMANN-NEUMANN ALGORITHM). *The components w_i in (3.3) for $0 \le i \le p$ are calculated as follows:*

1. *$w_i \in V^h(\Omega_i)$, for $1 \le i \le p$, solves the following local Neumman problem*

$$\check{A}_i(w_i, \phi_l) = \langle g, \Theta_i \phi_l \rangle\,, \; \forall\, \phi_l \in V^h(\Omega_i), \tag{3.4}$$

2. *$w_0 \in V_0$ solves the proper coarse problem depending on V_0 and R_0 to be chosen later, e.g. the subsequent wirebasket coarse problem (3.5) and the standard coarse problem (3.10) to be discussed in §3.1.1 and §3.1.2 respectively.*

3.1.1. The use of weighted coarse space. The method to be discussed now is based on the weighted coarse spaces in §?? and the following global coarse solver

$$R_0 = \log(h_0/h)^2 S_0^{-1},$$

concerning the action of the corresponding preconditioner M_h as in (3.2), $w_0 \in V_0$ can be obtained by solving

$$\langle S_h u_0, \phi \rangle = \log(h_0/h)^2 \langle g, \phi \rangle \quad \forall\, \phi \in V_0. \tag{3.5}$$

THEOREM 3.1. *With the aforementioned choice of weighted coarse space and R_0, the preconditioner M_h given by (3.2) satisfies*

$$\kappa(T_h S_h) \lesssim \log^3(h_0/h).$$

Moreover, if each boundary subdomain shares a common face with $\partial\Omega$, then

$$\kappa(T_h S_h) \lesssim \log^2(h_0/h).$$

if $n=2$, compute $u_0 = w_0|_\Gamma \in V_0$ by solving the coarse problem:

$$\gamma(w_0, v_0) = \langle g, v_0 \rangle \quad \forall\, v_0 \in V_0.$$

where $\gamma(\cdot,\cdot)$ is defined in §2.1.

3. *Compute $B_h g = u_0 + \sum_F u_F$.*

2.3. Bibliographic remark. The substructuring preconditioners discussed in the section are initiated by Bramble-Pasciak-Schatz [?, ?] and the analogue to wirebasket algorithms on the whole domain. The method in §2.1 is a fundamental algorithm which was applied to generate a lot of similar algorithms, e.g. Bramble-Pasciak-Schatz [?, ?, ?], Cai [?], Cai-Widlund [?], Cai-Gropp-Keyes [?, ?], Liang-Liang [?]

The *wirebasket algorithms* was proposed by Smith [?, ?] (n=3), and later the convergence proofs for elliptic problems with jumps in the coefficients was given by Dryja-Smith-Widlund [?] (n=3); here we add also the 2D case.

3. Algorithms based on local Neumann problems. This section is devoted to another type of preconditioner for the interface operator $S_h : V_\Gamma^h \to V^h(\Gamma)$. These algorithms are based on Neummann problems on subdomains.

The natural space for a Neumann problem on a subdomain, say Ω_i, is $V^h(\Gamma_i)$; nevertheless this is not a subspace of $V^h(\Gamma)$. To overcome this difficulty, for each i, we introduce a subspace V_i consisting of functions in $V^h(\Gamma)$ vanishing at nodes on $\Gamma \setminus \Gamma_i$. The spaces V_i and $V^h(\Gamma_i)$ have the same dimension. Unfortunately the operator S_i is not always invertible. There exist two main approaches to overcome this difficulty. The first approach, to be discussed in the subsection 3.1, is to slightly modify the operator S_i to introduce a nearby nonsingular operator by adding an appropriate lower order term. The first approach leads to the so-called Neumann-Neumann methods. The second approach, to be discussed in the subsection 3.2, is to first solve the coarse grid equation and then solve the residual equation for S_i which is nonsingular as the residual equation can be viewed on the complement of the coarse space in which the kernel of S_i is annihilated. The second approach leads to the so-called balancing domain decomposition method.

3.1. Neumann-Neumann methods. In this subsection, we discuss the methods based on modifying the operator S_i. The modification is based on the following bilinear form:

$$\breve{A}_i(u,v) = A_i(u,v) + \rho_i h_0^{-2}(u,v)_{0,\Omega_i}, \; \forall\, u, v \in H^1(\Omega_i).$$

Correspondingly, a modified operator $\breve{S}_i : V^h(\Gamma_i) \to V^h(\Gamma_i)$ can be defined as follows

$$\left\langle \breve{S}_i u, v \right\rangle_{0,\Gamma_i} = \breve{A}_i(\breve{u}, \breve{v}), \; \forall\, u, v \in V^h(\Gamma_i). \tag{3.1}$$

Here "$\breve{u}$" denotes the $\breve{A}_i$-discrete harmonic extension of u.

Obviously, the modified operator $\breve{S}_i$ is invertible. A subspace solver, denoted by R_i, on each V_i is then defined by

$$R_i = \Theta_i \breve{S}_i^{-1} \Theta_i^*$$

where Θ_i is defined as in (**??**) which is restated below for convenience

$$\Theta_i u_i = \rho_i^{1/2} I^0_{\Gamma_i}(\nu_\rho^{-1} u_i) \quad \forall\, u_i \in V^h(\Gamma_i);$$

For each face F, let ρ_F be the average value defined as in §2.1, and we adopt local face solvers $R_F^{-1} = \rho(F)\,(-\Delta_{F,h})^{1/2}$.

Let $Q_F : V^h(\Gamma) \to V_0^h(F)$ be the orthogonal projections with respect to $\langle\cdot,\cdot\rangle$, the parallel subspace correction preconditioner for S_h is then given by

$$M_h = R_0 Q_0 + \sum_{F\subset\Gamma} R_F Q_F. \tag{2.5}$$

THEOREM 2.3. *For the preconditioner $B_h = R_0Q_0 + \sum_F R_F Q_F$, we have*

$$\kappa(B_h S_h) \lesssim (1 + \log(h_0/h)\,)^2.$$

Proof. By Theorem ??, we need to estimate K_1, ω_1 and $\rho(\varepsilon)$. Evidently, $K_1 \lesssim 1$ as for each face subspace $V_0^h(F)$, only a fixed number of other face subspaces are not orthogonal to $V_0^h(F)$.

Using Lemma ?? and Lemmas ??-??, we obtain for any $u \in V_0^h(F)$ that

$$\begin{aligned}\langle S_F u, u\rangle &= A_{j_1}(\tilde u,\tilde u) + A_{j_2}(\tilde u,\tilde u)\\ &\lesssim \rho_F \|u\|^2_{H^{1/2}_{00}(F)} \eqsim \left\langle R_F^{-1}u, u\right\rangle,\end{aligned}$$

together with (??), we derive $\omega_1 \lesssim 1$.

Finally, we analyse K_0. For any $u \in V^h(\Gamma)$, let $u_0 = I_0 u \in V_0^h$ and $w = u - u_0$. Clearly, $u = u_0 + \sum_F w_F$. We can deduce

$$\begin{aligned}\left\langle R_0^{-1}u_0, u_0\right\rangle &\lesssim \log(h_0/h)\sum_{i=1}^{p}\rho_i\|u - \gamma_{\partial\Omega_i}(u)\|^2_{0,\mathcal{W}_i} \text{ (minimizing of } \gamma_{\mathcal{W}_i}(u))\\ &\lesssim \log(h_0/h)^2\,\langle S_h u, u\rangle\ \ (n=3) \text{ (Lemmas ?? \& ?? \& Poincaré ineq)}\\ \left\langle R_0^{-1}u_0, u_0\right\rangle &= \sum_{i=1}^{p} A_i(\tilde u_0,\tilde u_0) \lesssim \log(h_0/h)\sum_{i=1}^{p} A_i(\tilde u,\tilde u) = \log(h_0/h)\,\langle S_h u, u\rangle\ \ (n=2)\end{aligned}$$

Consider $n = 3$ and one face F. The same technique as used in (2.3) gives

$$\left\langle R_F^{-1} I_F^0 w, I_F^0 w\right\rangle \eqsim \rho_F \|I_F^0 w\|^2_{H^{1/2}_{00}(F)} \lesssim \log(h_0/h)^2 (A_{j_1}(\tilde u,\tilde u) + A_{j_2}(\tilde u,\tilde u)),$$

which holds also for $n = 2$ by Lemma ??. Therefore $K_0 \lesssim \log(h_0/h)^2$, which ends the proof of Theorem 2.3. $\square$

Recall the **Substructuring Algorithm I** in §2.1, we easily come to

ALGORITHM 2.3 (WIREBASKET ALGORITHM). *For any $g \in V^h(\Gamma)$, $B_h g = R_0Q_0 g + \sum_f R_F Q_F g = u_0 + \sum_F u_F$ is computed as follows:*

1. *Compute $u_F \in V_0^h(F)$ in parallel:*

$$\left\langle (-\Delta_{F,h})^{1/2} u_F, v_F\right\rangle = \langle g, v_F\rangle, \quad \forall\, v_F \in V_0^h(F).$$

2. *If $n = 3$, compute $u_0 \in V_0$ by solving the minimization problem:*

$$\min_{w_0\in V_0} \frac{1}{2}\log(h_0/h)\sum_{i=1}^{p}\rho_i \min_{\lambda_i\in R^1} \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - \langle g, w_0\rangle\,;$$

Here $\mathcal{W}_{i,h}$ is the set of nodes on $\mathcal{W}_i$. Let $\mu_i(v)$ be a constant satisfying that

$$Q_i(v-\mu_i(v),1)=0,\ \forall\, v\in V^h(\partial\Omega_i).$$

Then as $u=u_P+u_H$, we have

$$\begin{aligned}
&\log(h_0/h)^{-2}Q_i(u-\mu_i(u),u-\mu_i(u))\\
&\le \log(h_0/h)^{-2}Q_i(u-\gamma_{\Omega_i}(u_H),u-\gamma_{\Omega_i}(u_H))\ (\mu_i\text{'s minimizing})\\
&\lesssim \|u-\gamma_{\Omega_i}(u_H)\|^2_{1/2,\partial\Omega_i}\ (\text{by (2.4)})\\
&\lesssim \|u_H-\gamma_{\Omega_i}(u_H)\|^2_{1,\Omega_i}\ (\text{Lemma ??})\\
&\lesssim |u_H|^2_{1,\Omega_i}\ (\text{Friedrichs ineq})\\
&\lesssim \|u-\mu_i(u)\|^2_{1/2,\partial\Omega_i}\ (\text{Lemma ??})\\
&\lesssim Q_i(u-\mu_i(u),u-\mu_i(u))\ (\text{from (2.4)}).
\end{aligned}$$

Now we can define the preconditioner M_h to the stiffness operator A_h by

$$(M_h^{-1}u,v)=A(u_P,v_P)+\sum_{i=1}^{p}\rho_i\,Q_i(u-\mu_i(u),v-\mu_i(v))\quad \forall\, u,v\in V^h.$$

Then the previous statement shows

LEMMA 2.2.

$$\kappa(M_hA_h)\lesssim(1+\log(h_0/h))^2.$$

The algorithm for solving problem associated with the corresponding bilinear form is given below.

ALGORITHM 2.2. ***algorithm to be included***

2.2. A variant of the substructuring preconditioner-I. We shall now present a parallel subspace correction version of the substructuring preconditioner-I in §2.1. This variant of substructuring method was first considered by Smith [?, ?], known as wirebasket methods there.

As mentioned in §??, the "breaking" process on the interface Γ gives a natural decomposition of the space $V^h(\Gamma)$ as follows:

$$V^h(\Gamma)=V_0+\sum_{F\subset\Gamma}V_0^h(F),$$

where $V_0=I_0V^h(\Gamma)$ with I_0, the *joint-operator* introduced in §??, being the wirebasket interpolant defined in §?? ($n=3$) and standard coarse space interpolant ($n=2$) defined in §??.

The coarse subspace solver R_0 is chosen to be the interface restriction of the standard coarse solver R_0 defined in §?? for $n=2$ and the wirebasket coarse solver on the interface defined in §?? for $n=3$, i.e.

$$\left\langle R_0^{-1}u_0,v_0\right\rangle=\log(h_0/h)\sum_{i=1}^{p}\rho_i\left\langle u_0-\gamma_{h,\mathcal{W}_i}(u_0),v_0-\gamma_{h,\mathcal{W}_i}(v_0)\right\rangle_{h,\mathcal{W}_i};$$

In summary, for $n=2$ (noting (??)) we define the preconditioner M_h by

$$(M_h^{-1}u,v) = A(u_P,v_P) + \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} + \gamma(I_0 u_H, I_0 v_H),$$

while for $n=3$ (noting (??)) by

$$(M_h^{-1}u,v) = A(u_P,v_P) + \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} + \gamma(I_0 u_H, I_0 v_H).$$

where $\gamma(I_0u_H, I_0v_H)$ is defined as follows:

$$\begin{array}{lll} \gamma(I_0u_H, I_0v_H) & = h_0 \sum_{v_i,v_j\in\Gamma_k} (u_H(v_i)-u_H(v_j))\left((v_H(v_i)-v_H(v_j)\right), & \text{if } n=2, \\ \gamma(I_0u_H, I_0v_H) & = \log(h_0/h) \sum_{i=1}^{p} \rho_i \left\langle u_H - \gamma_{h,\mathcal{W}_i}(u_H), v_H - \gamma_{h,\mathcal{W}_i}(v_H) \right\rangle_{h,\mathcal{W}_i}, & \text{if } n=3. \end{array}$$

The above statement shows

THEOREM 2.1. *For any $u\in V^h$,*

$$\kappa(M_hA_h) \lesssim C(n).$$

where $C(2)=C$ independent of the h and h_0, but $C(3)=(1+\log(h_0/h))^2$.

Recall the standard and wirebasket coarse solvers defined in §?? and §?? respectively, we have the following algorithm:

ALGORITHM 2.1 (SUBSTRUCTURING ALGORITHM I). *For given $g\in V^h$, let $u = M_hg$, then $u=u_P+u_H$ can be obtained as follows:*

1. *For $1\le i\le p$, $u_P\in V_0^h(\Omega_i)$ solves*

$$A(u_P,v) = (g,v) \quad \forall v\in V_0^h(\Omega_i).$$

2. *On each face $F\subset\Gamma$, u_E solves*

$$\rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} = (g,v) - A(u_P,v) \quad \forall\, v\in V_0^h(F).$$

3. *If $n=2$, find $u_0=\tilde{I}_0u_H\in V_0$ solving*

$$\gamma(u_0,v) = (g,v) - A(u_P,v) \quad \forall\, v\in V_0;$$

If $n=3$, find $u_0\in V_0(\Gamma)$ on Γ by solving

$$\min_{w_0\in V_0} \frac{1}{2}\sum_{i=1}^{p} \rho_i \min_{\lambda_i\in R^1} \|w_0-\lambda_i\|_{h,\mathcal{W}_i}^2 - \langle g, w_0\rangle - A(u_P,w_0) \quad \forall\, w_0\in V_0(\Gamma),$$

4. *Extend $u_0+\sum_F u_E$ harmonically by solving homogeneous Dirichlet problem on each Ω_i.*

Substructuring preconditioner II (n=3). We next present another substructuring technique that is based on the following estimate implied by Lemmas ??-??:

$$\log(h_0/h)^{-2} Q_i(w,w) \lesssim \|w\|_{1/2,\partial\Omega_i}^2 \lesssim Q_i(w,w) \quad \forall\, v\in V^h(\partial\Omega_i). \tag{2.4}$$

where

$$Q_i(u,u) = h\sum_{x_i\in\mathcal{W}_{i,h}} u^2(x_i) + \sum_{F\subset\partial\Omega_i} \left\langle (-\Delta_{F,h})^{1/2} I_F^0 u, I_F^0 u \right\rangle_{0,F} \quad \forall\, u\in V^h(\partial\Omega_i).$$

1.1. Bibiographic remarks. The partition lemma was first introduced in the domain decomposition context by Nepomnyaschikh [?] and Lions [?].

The convergence analysis of overlapping Schwarz methods with standard coarse subspaces for elliptic problems with jumps in the coefficients was considered by Dryja-Sarkis-Widlund [?] and Martins in [?] for the case that neighboring coefficients at each vertex of subdomains are monotone in a certain direction.

2. Substructuring method. We shall now discuss the so-called substruturing methods and pay attention to three major algorithms and some of their variants.

2.1. Substructuring preconditioners-I ($n = 2, 3$). At first, we present the technical details on the construction of substructuring preconditioners whose motivation has been discussed in §??. As mentioned before, a proper choice of the joint-operator on the space $V^h(\Gamma)$ is crucial in the construction of such preconditioners.

Let I_0 (resp. V_0) be the standard coarse interpolant (resp. standard coarse subspace) defined in §?? for $n = 2$, but the wirebasket interpolant (resp. the wirebasket coarse space) defined in §?? for $n = 3$.

As in §??, we split any $u \in V^h$ into $u = u_P + u_H$ with $u_P \in V_0^h(\Omega_i)$ for $1 \le i \le p$ and u_H being a $A(\cdot,\cdot)$-discrete harmonic function in Ω. To decompose u_H, we use an auxiliary operator $\tilde{I}_0$: for $n = 3$, $\tilde{I}_0 u_H = I_0 u_H$; but for $n = 2$, $\tilde{I}_0 u_H$ is the $A(\cdot,\cdot)$-discrete harmonic extension of $(I_0 u_H)|_\Gamma$, then using $\tilde{I}_0$, we can write $u_H = \tilde{I}_0 u_H + u_E$ with $u_E = (u_H - \tilde{I}_0 u_H)$ vanishing on the wirebasket set. By the triangle inequalty,

$$A(u_H, u_H) \le 2A(u_E, u_E) + 2A(\tilde{I}_0 u_H, \tilde{I}_0 u_H). \tag{2.1}$$

As u_E vanishing on the wirebasket set, it follows from Lemmas ??-?? and ?? that

$$\begin{aligned} A(u_E, u_E) &\eqsim \sum_i \rho_i \, |u_E|^2_{1/2,\partial\Omega_i} \lesssim \sum_i \sum_{F\subset\Omega_i} \rho_i \, |u_E|^2_{H_{00}^{1/2}(F)} \\ &\eqsim \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, u_E \right\rangle_{0,F} \end{aligned}$$

where ρ_F is the average value of two coefficients associated with two subdomains sharing the common face F, this combining with (2.1) and $\tilde{I}_0 u_H = I_0 u_H$ on Γ yields

$$A(u_H, u_H) \lesssim \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, u_E \right\rangle_{0,F} + \sum_{i=1}^p \rho_i A_i(I_0 u_H, I_0 u_H), \tag{2.2}$$

Note that for $n = 3$,

$$I_F^0 u_E = I_F^0 (u - I_0 u) = I_F^0 (u - \gamma_{\partial F}(u)\,) = I_F^0 u - \gamma_{\partial F}(u) I_F^0 1, \tag{2.3}$$

thus we derive by using Lemmas ??-?? and Lemma ?? that

$$\begin{aligned} \rho_F \|I_F^0 u_E\|^2_{H_{00}^{1/2}(F)} &\lesssim \log(h_0/h)^2 (\rho_{j_1} |u|^2_{1/2,\partial\Omega_{j_1}} + \rho_{j_2} |u|^2_{1/2,\partial\Omega_{j_2}}) \\ &\eqsim \log(h_0/h)^2 (A_{j_1}(\tilde{u},\tilde{u}) + A_{j_2}(\tilde{u},\tilde{u})) \text{ (Lemma ??)}, \end{aligned}$$

where Ω_{j_1} and Ω_{j_2} are two subdomains sharing the face F, this along with (2.1)-(2.2) and Lemma ?? and (??) implies for $n = 3$ that

$$\begin{aligned} A(u_H, u_H) &\lesssim \log(h_0/h) \sum_{i=1}^p \rho_i (\log(h_0/h)\, |u_H|^2_{1,\Omega_i} + \|I_0 u_H - \gamma_{\mathcal{W}_i}(I_0 u_H)\|^2_{0,\mathcal{W}_i}) \\ &\lesssim \log(h_0/h)^2 A(u_H, u_H) \text{ ((??))}. \end{aligned}$$

LEMMA 1.2. *Let V_0 be a subspace of V^h. Suppose that there exists a linear operator $Q_0 : V^h \to V_0$ satisfying the L^2 approximation and H^1 stability (??). Then for any $u \in V^h$, there exist elements $u_0 \in V_0$ and $u_i \in V_0^h(\O_i)$ such that $u = \sum_{i=0}^p u_i$ and*

$$\sum_{i=0}^p \|u_i\|^2_{H^1_\rho(\Omega)} \le \max\{\alpha_1^2, \alpha_0^2\}\|u\|^2_{H^1_\rho(\Omega)},$$

where α_0 and α_1 are constants appearing in (??).

Overlapping Schwarz methods. Let V_0 be the standard coarse subspace of V^h described in §?? for $n = 2$, but the wirebasket subspace defined in §?? for $n = 3$. Let $V_i = V_0^h(\O_i)$, for $1 \le i \le p$. The overlapping Schwarz method is based on the additive Schwarz preconditioner for the stiffness operator A_h defined as follows:

$$M_h = R_0 Q_0 + \sum_{i=0}^p A_i^{-1} Q_i,$$

where the operator $A_i : V_i \to V_i$ is the restriction of A_h on V_i for $1 \le i \le p$, but $R_0 : V_0 \to V_0$ is the standard coarse solver defined in §?? for $n = 2$ and the wirebasket coarse solver $\tilde{R}_0$ defined in §??.

We have the following condition number bounds for M_h:

LEMMA 1.3. *For $n = 2, 3$,*

$$\kappa(M_h A_h) \lesssim (1 + \log(h_0/h))^{n-1}.$$

Proof. By Lemma ?? it suffices to estimate the parameters K_0, K_1 and ω_1. As we are using exact local and global coarse solver, we know that $\omega_1 = 1$. By the definitions of the local subspaces V_i ($1 \le i \le p$) and the parameter K_1 in (??), we can also readily know that $K_1 \lesssim 1$. What remains is to bound the parameter K_0 which is the smallest constant satisfying

$$\sum_{i=0}^p A(u_i, u_i) \le K_0 A(u, u), \quad \forall u \in V^h, \ u = \sum_{i=0}^p u_i, \ u_i \in V_i.$$

For this purpose, by Lemma 1.2 we need only to find an operator $Q_0 : V^h \to V_0$ satisfying (??). We can take the weighted L^2 projection defined in (1.1) for $n = 2$ but the wirebasket interplant itsself defined in §??. The results follow then from Lemma 1.1 and (??). □

We have the following algorithm for computing the action of the preconditioner M_h:

ALGORITHM 1.1 (OVERLAPPING SCHWARZ METHOD). *For given $g \in V^h$, let $u = M_h g = R_0 Q_0 + \sum_{i=1}^p A_i^{-1} Q_i \equiv \sum_{i=0}^p u_i$, and u_i cab be computed as follows:*

1. *For $i = 1, \cdots, p$, $u_i \in V_0^h(O_i)$ solves the Dirichlet problem:*

$$A(u_i, v) = (g, v) \quad \forall\, v \in V_0^h(O_i).$$

2. *u_0 can be obtained for $n = 2$ from (??) and for $n = 3$ from (??)- (??) with $g_0 = Q_0 g$.*

1. A partition lemma and overlapping additive Schwarz methods for elliptic probems with jumps in the coefficients. It is known that in the three dimension, the overlapping additive Schwarz method with standard coarse subspaces described in §?? are not effective for the problem (??) with large jumps in the coefficients. In this section, we will show that with the help of the wirebasket subspace given in §??, the overlapping additive Schwarz methods will work pretty well in both two and three dimensions.

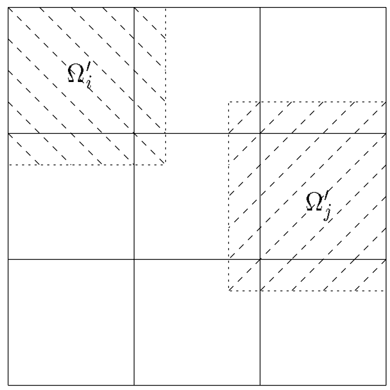

FIG. 1. *Overlapping subdomains* $\{\Omega_i'\}$

Based on the given p non-overlapping subdomains Ω_i $(1 \le i \le p)$, we extend each Ω_i to a larger Ω_i' with dist $(\partial\Omega_i' \cap \Omega, \partial\Omega_i') \lesssim h_0$, cf. Fig. 1. Assume that $\partial\Omega_i'$ and $\partial\Omega_i$ align with elements of $\mathcal{T}^h$ and each point $x \in \Omega$ belongs to at most q_0 subdomains of $\{\Omega_i'\}_{i=1}^p$, with $q_0 < p$ a positive integer. For the sake of explanation, we allow in this section the subdomains Ω_i to be only simplicial.

Before discussing the algorithm, we first recall L^2 and weighted L^2 projection operator $Q_{h_0}^\rho$. Let V_0 be the piecewise linear finite element space related to the triangulation of the non-overlapping subdomains $\{\Omega_i\}$.

The weighted L^2 projection $Q_{h_0}^\rho : L^2(\Omega) \to V_0$ is defined by

$$(Q_{h_0}^\rho u, v)_{L^2_\rho(\Omega)} = (u, v)_{L^2_\rho(\Omega)} \quad \forall u \in L^2(\Omega),\ v \in V_0, \tag{1.1}$$

where $(\cdot,\cdot)_{L^2_\rho(\Omega)}$ is the scalar product related to the norm $\|\cdot\|_{L^2_\rho(\Omega)}$. We will denote $Q_{h_0} = Q_{h_0}^\rho$ if $\rho = 1$.

Bramble-Xu [?] proved

LEMMA 1.1. *For any $u \in V^h$, we have for $n = 2$ that*

$$\begin{aligned} \|u - Q_{h_0}^\rho u\|^2_{L^2_\rho(\Omega)} &\lesssim h_0^2 \log(h_0/h)\, |u|^2_{H^1_\rho(\Omega)}, \\ |Q_{h_0}^\rho u|^2_{H^1_\rho(\Omega)} &\lesssim \log(h_0/h)\, |u|^2_{H^1_\rho(\Omega)}; \end{aligned}$$

If all the coefficients $\rho_i = 1$ $(1 \le i \le p)$, then for both $n = 2$ and $n = 3$ we have

$$\|u - Q_{h_0} u\|_{0,\Omega} \lesssim h_0 |u|_{1,\Omega}, \quad |Q_{h_0} u|_{1,\Omega} \lesssim |u|_{1,\Omega} \quad \forall\, u \in H_0^1(\Omega).$$

We have