MATHEMATICS

Question about the proof of Stone-Weierstrass theorem (Weierstrass approximation theorem) in Rudin

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In Rudin's *Principles of Mathematical Analysis*, a proof of the Stone-Weierstrass theorem in its original statement is included (3ed, p159):

THE STONE-WEIERSTRASS THEOREM

7.26 Theorem *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \le x \le 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

(47) $$Q_n(x) = c_n(1 - x^2)^n \qquad (n = 1, 2, 3, \ldots),$$

where c_n is chosen so that

(48) $$\int_{-1}^{1} Q_n(x)\, dx = 1 \qquad (n = 1, 2, 3, \ldots).$$

We need some information about the order of magnitude of c_n. Since

$$\int_{-1}^{1} (1 - x^2)^n\, dx = 2\int_0^1 (1 - x^2)^n\, dx \geq 2\int_0^{1/\sqrt{n}} (1 - x^2)^n\, dx$$
$$\geq 2\int_0^{1/\sqrt{n}} (1 - nx^2)\, dx$$
$$= \frac{4}{3\sqrt{n}}$$

it follows from (48) that

(49) $$c_n < \sqrt{n}.$$

The inequality $(1 - x^2)^n \geq 1 - nx^2$ which we used above is easily shown to be true by considering the function

$$(1 - x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive in $(0, 1)$.

For any $\delta > 0$, (49) implies

(50) $$Q_n(x) \leq \sqrt{n}\,(1 - \delta^2)^n \qquad (\delta \leq |x| \leq 1),$$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$.

Now set

(51) $$P_n(x) = \int_{-1}^{1} f(x + t) Q_n(t)\, dt \qquad (0 \leq x \leq 1).$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x} f(x+t)Q_n(t)\,dt = \int_0 f(t)Q_n(t-x)\,dt,$$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (48), (50), and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x+t) - f(x)]Q_n(t)\,dt \right| \\
&\leq \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)\,dt \\
&\leq 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,dt + 2M\int_{\delta}^{1} Q_n(t)\,dt \\
&\leq 4M\sqrt{n}\,(1-\delta^2)^n + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

for all large enough n, which proves the theorem.

It is instructive to sketch the graphs of Q_n for a few values of n; also, note that we needed uniform continuity of f to deduce uniform convergence of $\{P_n\}$.

My question is about the step after (51), $P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\,\mathrm{d}\,t$. How does one proceed from this, by a change of variable, to the next step, namely $P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)\,\mathrm{d}\,t$?

And another question is why $P_n(x) = \int_0^1 f(t)Q_n(t-x)\,\mathrm{d}\,t$ is a polynomial.

real-analysis complex-analysis proof-explanation

edited Dec 14 '18 at 12:08
Scientifica
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asked Dec 14 '18 at 12:05
Sayako Hoshimiya
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1 Answer

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Well the first equality, namely $\int_{-1}^{1} f(x+t)Q_n(t)dt = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$ follows just from the fact that f is 0 outside [0, 1] which is one of the simplificating assumptions Rudin makes.

Now $\int_{-x}^{1-x} f(x+t)Q_n(t)dt = \int_0^1 f(t)Q_n(t-x)dt$ follows by the substitution t = t-x.

The fact that $\int_0^1 f(t)Q_n(t-x)dt$ is a poly in x follows from writing $Q_n(t+x) = \sum_{k=0}^{n} a_i(t+x)^k = \sum_{k=0}^{n} b_i(t)x^k$ and now $\int_0^1 f(t)Q_n(t-x)dt = \sum_{k=0}^{n}(\int_0^1 b_i(t)dt)x^k$, where $b_i(t)$ are just the functions(polys) obtained by expanding each $(t+x)^k$.

answered Dec 14 '18 at 12:15

Sorin Tirc

1,905 3 13

Okay, now I see why that is a polynomial: apply binomial expansion multiple times, first on $(1-(t-x)^2)^n$ and on those $(t-x)^{2i}$, then the integral will become

$$\sum_{i=0}^{2n} c_n k(i) \left(\int_0^1 f(t) \cdot t^{2n-i} \, \mathrm{d}\, t \right) x^i$$

where $k(i)$ are the merged binomial coefficients. – Sayako Hoshimiya Dec 16 '18 at 11:46

But I am still wondering about the $\int_{-1}^{1} f(x+t)Q_n(t)dt = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$ part, would you explain this part with a little bit more details? Thanks. – Sayako Hoshimiya Dec 16 '18 at 11:50

1 Ok, so the integral has t varying from -1 to 1, but in fact f(x+t) is 0 for t < -x so the integral from -1 to -x of f(x+t)$Q_n(t)$ will be 0. Is this clear? – Sorin Tirc Dec 16 '18 at 16:04

Yes, now I get it. Thank you. – Sayako Hoshimiya Dec 17 '18 at 2:18