(0.0.1) Lemma. *For all* $v_H \in \mathcal{V}_H$

$$\|v_H - I_h v_H\| \lesssim h(h/H)^{d/2}\|v_H\|_1, \quad \|I_h v_H\|_1 \leq \left(1 + C(h/H)^{d/2}\right)\|v_H\|_1.$$

Proof. Given $\tau \in \mathcal{T}_h$, let $\tilde{\tau}$ be the union of elements in $\mathcal{T}_H$ that intersect with τ. It follows that

$$\begin{aligned}
\|v_H - I_h v_H\|_{0,\tau} &\lesssim |\tau|\,\|v_H - I_h v_H\|_{0,\infty,\tau} \\
&\lesssim h^{d/2}h\|v_H\|_{1,\infty,\tau} \quad \text{(standard error estimate)} \\
&\lesssim h^{d/2}hH^{-d/2}\|v_H\|_{1,\tilde{\tau}} \quad \text{(inverse inequality)} \\
&\lesssim h(h/H)^{d/2}\|v_H\|_{1,\tilde{\tau}}.
\end{aligned}$$

Summing over all τ then yields the first estimate. A similar and in fact a slightly simpler argument shows that

$$\|v_H - I_h v_H\|_1 \lesssim (h/H)^{d/2}\|v_H\|_1.$$

The second estimate then follows. $\square$