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We define

$$Ev = \sum_{j=1}^{J} \tilde{v}_j .$$

It can be proved that

$$|Ev|_{1,G} \leq c_0 |v|_{1/2,\partial G}$$

with a constant c_0 independent of h.

A multigrid preconditioner. As mentioned before, the BPX preconditioner is not always efficient for three dimensional problems with large discontinuous jumps. We shall now briefly describe a procedure proposed in Xu [?]. With all the machinaries described earlier, the construction of such a preconditioner is relatively easy. Basically, it is the preconditioner (2.15) with the following choices:

1. The subdomain preconditioner B^0 is the BPX preconditioner such as (??) on each subdomain.
2. The interface preconditioner T is, for example, the preconditioner derived in §?? restricted on Γ.
3. Preconditioners used in constructing the aforementioned T for operator like $(-\Delta)^{1/2}$ are given by the BPX preconditioners (2.10).
4. The extension operator E is the multigrid extension operator discussed above.

It is easy to see that the resulting preconditioner admit the following estimate:

$$\kappa(BA) \leq C |\log(H/h)|^{\alpha}$$

with $\alpha = 2$ or 3 and some constant C independent of h and the discontinuous jump $\kappa(a)$.

We would like to mention the relevant methods discussed by Wang-Xie [?] and by Dryja-Sarkis-Widlund [42] and we note, however, that their approach requires exact subdomain sovers.

12.5. Bibliographic remark. The interface BPX preconditioner is a special restriction of the standard BPX multilevel algorithms on the interface was proposed by Bamble-Pasciak-Xu [16] and generalized by Zhang [103]. Oswald [76] and Zhang [103] proved the optimality of the BPX algorithm. The interface variant of the BPX preconditioner was first considered by Tong-Chan-Kuo [93].

The hierarchical basis methods was initially proposed by Yserentant [101], and studied later by many authors, e.g. Bank-Dupont-Yserentant [3], Yserentant [102], Bank-Xu [4], Xu [95, 96]. The interface hierarchical basis preconditioner was considered by Smith-Widlund [85].

The interface additive Schwarz algorithms is a generalization of the vertex algorithms which was proposed by Smith [82, 83]. Our generalization is not so much of practial application, but just for its own interest. Shao [81] and Chan-Mathew-Shao [29] addressed how to efficiently implement vertex algorithms.

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For computing $u_i \in V_i$, let $\hat{\mathcal{G}}_i$ denote the union of all subdomains Ω_j satisfying $\Omega_j \cap \Gamma^i \neq \emptyset$, then we know by definition of R_i and Q_i that $u_i = \tilde{u}_i|_\Gamma$ with $\tilde{u}_i$ solving

$$A(\tilde{u}_i, \phi_l) = \langle g, \phi_l \rangle \quad \forall\, \phi_l \in V_0^h(\hat{G}_i).$$

From these, we can state

ALGORITHM 12.1 (INTERFACE ADDITIVE SCHWARZ ALGORITHM). *For any $g \in V^h(\Gamma)$, $u = B_h g \equiv \sum_{i=0}^p u_i$ is calculated as follows:*

1. *Coarse problem: $u_0 \in V_0(\Gamma)$ solves*

$$h_0 \sum_{v_i, v_j \in \Gamma_k} (u_0(v_i) - u_0(v_j))\,(v_0(v_i) - v_0(v_j)) = (g_0, v_0) \quad \forall v_0 \in V_0(\Gamma).$$

2. *Local problems on $\hat{\mathcal{G}}_i$ $(1 \le i \le p)$: $u_i = \tilde{u}_i|_\Gamma$ with $\tilde{u}_i \in V_0^h(\hat{G}_i)$ solving*

$$A(\tilde{u}_i, \phi_l) = \langle g, \phi_l \rangle_{0,\Gamma}, \ \forall\, \phi_l \in V_0^h(\hat{G}_i),$$

3. *Compute $u = \sum_{i=0}^p u_i$.*

REMARK 12.1. *If we choose some special subdomains $\{\mathcal{G}_i\}$ such that the interface subregions Γ^i are vertex-related, edge-related and face-related overlapping subregions of the interface Γ, then our general algorithms reduce to vertex algorithms proposed by Smith [82, 83].*

12.4. A new 3-d multigrid preconditioner.

Multigrid extension operator. We shall now briefly describe a multigrid extension. The basic idea is similar to [?]. The technical details for this construction can be found in Xu [?].

Obviously it suffices to discuss the extension on each subdomain. Let G be any of the subdomain Ω_j. For simplicity of exposition, we shall not explicitly worry about the possible small size of the the subdomain G.

Assume that $V(\partial G)$ has a sequence of nested multilevel subspaces:

$$V_1(G) \subset V_2(G) \subset \ldots \subset V_J(G) = V(G).$$

Correspondingly,

$$V_1(\partial G) \subset V_2(\partial G) \subset \ldots \subset V_J(\partial G) = V(\partial G)$$

The construction is based on some average nodal value interpolant (see Clemént or Scott-Zhang): $\Pi_j : V(\partial G) \to V_j(\partial G)$ which satisfy

$$\|v - \Pi_j v\|_{0,\partial G} + h_j^{1/2} |\Pi_j v|_{1/2,\partial G} \lesssim h_j^{1/2} |\Pi_j v|_{1/2,\partial G} \quad \forall v \in V(\partial G).$$

Given $v \in V(\partial G)$, we consider the following decomposition

$$v = \sum_{j=1}^{J} v_j$$

with $v_j = (\Pi_j - \Pi_{j-1})v \in V_j(\partial G)$. Let $\tilde{v}_j \in V_j(G)$ be the unique function that equals to v on ∂G and vanishes at all the nodal points in $\mathcal{N}_j$ that are not on ∂G.

where all local solvers R_i are taken to be exact, i.e. they are the restrictions of S_h on $V_i(\Gamma)$ defined by

$$\left\langle R_i^{-1} u, v \right\rangle = \langle S_h u, v \rangle \quad \forall\, u, v \in V_i(\Gamma),$$

while the coarse solver $R_0 : V_0 \to V_0$ is defined by

$$\left\langle R_0^{-1} u_0, v_0 \right\rangle = A(I_0 \tilde{u}_0, I_0 \tilde{u}_0). \tag{12.25}$$

Then we have

THEOREM 12.1. *For the additive Schwarz preconditiner B_h defined above,*

$$\kappa(B_h S_h) \lesssim r(\rho).$$

where $r(\rho)$ is the coefficient ratio, i.e. $\max_{1\le i\le p} \rho_i / \min_{1\le i\le p} \rho_i$.

Proof. By Theorem 3.1, we have to estimate three parameters K_0, K_1 and ω_1. We redaily know that $K_1 = q_0$ from the assumption that each point $x \in \Omega$ belonds to at most q_0 subregions from $\{\mathcal{G}_i\}$.

For the estimate of ω_1, it suffices by definition (3.3) to estimate $\lambda_{\max}(R_0 S_0)$ for the coarse subspace as all local solvers are exact. This is easily obtained by noting the fact that $\tilde{u}_0$ are equal to $I_0 \tilde{u}_0$ on the interface for any $u_0 \in V_0(\Gamma)$ and thus

$$\langle S_h u_0, u_0 \rangle = A(\tilde{u}_0, \tilde{u}_0) \le A(I_0 \tilde{u}_0, I_0 \tilde{u}_0) = \left\langle R_0^{-1} u_0, u_0 \right\rangle,$$

which shows $\omega_1 \le 1$.

Finally, we estimate K_0 associated with the partition of $V^h(\Gamma)$. For any $u \in V^h(\Gamma)$, from Lemma ?? we have a partition for the harmonic extension $\tilde{u}$ of u: $\tilde{u} = \sum_{i=0}^p u_i$ such that $u_i \in V_i$ and $\sum_i \|u_i\|_{1,\Omega}^2 \lesssim \|u\|_{1,\Omega}$, this with Poincaré inequality implies that

$$\sum_{i=0}^{p} A(u_i, u_i) \lesssim r(\rho)\, A(u, u). \tag{12.26}$$

Obviously, the restrictions w_i of u_i on the interface Γ, i.e. $w_i = u_i|_\Gamma$, belong to $V_i(\Gamma)$ and $u = \sum_{i=0}^p w_i$ by definition of V_i and the choice of u_i. Now we show that

$$\sum_{i=0}^{p} \langle S_h w_i, w_i \rangle \lesssim \langle S_h u, u \rangle. \tag{12.27}$$

Using the fact that the harmonic functions $\tilde{w}_i$ equals to the non-harmonic functions u_i on the interface, we have

$$\langle S_h w_i, w_i \rangle = A(\tilde{w}_i, \tilde{w}_i) \le A(u_i, u_i),$$

from which and (12.26) the inequality (12.27) follows. □

Next we consider how to compute the action $B_h g$ for a given $g \in V^h(\Gamma)$. Let $u = B_h g = \sum_{i=0}^p R_i Q_i g \equiv \sum_{i=0}^p u_i$.

For the coarse solution $u_0 = R_0 Q_0 g$, by definition of R_0 and the equivalency (8.7) we easily know that $u_0 \in V_0(\Gamma)$ solves

$$h_0 \sum_{v_i, v_j \in \Gamma_k} (u_0(v_i) - u_0(v_j))\,(v_0(v_i) - v_0(v_j)) = \langle g_0, v_0 \rangle \quad \forall v_0 \in V_0(\Gamma).$$

Then by Theorem 3.2, the corresponding preconditioner $\hat{B}$ for the operator $\hat{A}$ is

$$\hat{B}\,\tilde{v} = \hat{P} M_h \hat{P}^* \tilde{v} = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in \Gamma_h} (\tilde{v}, \tilde{\phi}_i^k)\, \tilde{\phi}_i^k \quad \forall\, \tilde{v} \in \tilde{V}^h.$$

The interface preconditioner M_h can thus be defined as in (12.22), and (12.23) with Theorem 3.2 implies that

$$\kappa(B_h S_h) = \kappa(\hat{B}\hat{A}) \le \kappa(M_h A_h) \lesssim 1.$$

12.2. Interface hierarchical basis preconditioner. Let V_k and $\mathcal{N}_k$, $1 \le k \le J$ be defined as in §12.1. One version of the hierarchical basis preconditioner M_h for the stiffness operator A_h can be written as (cf. Xu [95, 96], Yserentant [101]):

$$M_h\, v = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} (v, \phi_i^k)\, \phi_i^k \quad \forall\, v \in V^h,$$

and $\kappa(M_h A_h) \lesssim \gamma_0(n)$ with $\gamma_0(1) = 1$, $\gamma_0(2) = |\log h|^2$ and $\gamma_0(3) = h^{-1}$.

Then by Theorem 3.2, the corresponding preconditioner $\hat{B}$ for the operator $\hat{A}$ is

$$\hat{B}\,\tilde{v} = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in N_k \setminus N_{k-1}} (\tilde{v}, \tilde{\phi}_i^k)\, \tilde{\phi}_i^k \quad \forall\, \tilde{v} \in \tilde{V}^h.$$

The interface preconditioner B_h can then be defined as in (12.22), and (12.23) with Theorem 3.2 implies that

$$\kappa(T_h S_h) = \kappa(\hat{B}\hat{A}) \le \kappa(M_h A_h) \lesssim \begin{cases} 1 & \text{if } n = 1 \\ |\log h|^2 & \text{if } n = 2 \\ h^{-1} & \text{if } n = 3. \end{cases}$$

12.3. Interface version of the additive Schwarz method: vertex algorithms. In this section, we consider the interface Additive Schwarz algorithms. As the algorithms can be viewed as restrictions of standard overlapping additive Schwarz algorithms on the interface Γ, we first discuss the standard algorithms defined on the whole domain Ω.

First decompose the domain Ω into p nonoverlapping subregions $\mathcal{G}_i^0$ with diameters of size h_0. Note that $\{\mathcal{G}_i\}$ may be different from the existing subdomains $\{\Omega_i\}$. Then we extend each $\mathcal{G}_i^0$ to a larger subregion $\mathcal{G}_i$ with dist $(\partial\mathcal{G}_i \cap \Omega, \partial\mathcal{G}_i^0) \lesssim h_0$. We assume that each point $x \in \Omega$ belongs to at most q_0 subregions $\{\mathcal{G}_i\}_{i=1}^p$. Restrict these subregions $\mathcal{G}_i$ on the interface, we get

$$\Gamma^i = \Gamma \cap O_i \ne \emptyset, \quad 1 \le i \le p.$$

We define local subspaces $V_i = V^h(\mathcal{G}_i)$ which are associated with subregions $\mathcal{G}_i$, but the coarse subspace V_0 to be the standard coarse subspace defined in §8.1, and by I_0 we denote the corresponding interpolant to V_0.

Now based on the interface subspaces $V_0(\Gamma)$ and $V_i(\Gamma)$ $(1 \le i \le p)$, we define the additive Schwarz preconditioner B_h for the interface operator S_h. As in (3.1), B_h can be defined as follows:

$$B_h = \sum_{i=0}^{p} R_i Q_i,$$

The balancing domain decomposition method was first proposed by Mandel [68], and later its convergence proof was improved by Mandel-Brezina [69]. Cowsar-Mandel-Wheeler [38] extended the BDD method to mixed finite elements. The major idea is to restrict local Neumann subproblems on certain subspaces so that local subproblems are uniquely solvable. Two other ways of overcoming singularities of local Neumann problems are to use different bilinear forms on local subdomains (see §11.1.1) and to use Lagrange multiplier approach by Farhat-Roux [49].

12. Interface preconditioners derived from stiffness preconditioners. We now apply the theory in §3.3 to generate some interface preconditioners from known stiffness preconditioners. Associated with §3.3, we choose the space $V = V^h$ and the subspace $\hat{V}$ of V^h to be the space of discrete harmonic functions, i.e. $\hat{V} = V_H$. As $\hat{P} : V^h \to V_H$ is a orthogonal projection with respect to $\langle S_h \cdot, \cdot \rangle$, it is immidiate to verify that $\hat{P}u$, for any $u \in V^h$, is the discrete harmonic extension of the restriction of u on the interface Γ.

Define an operator $B_h : V^h(\Gamma) \to V^h(\Gamma)$ by

$$B_h = \hat{R}\hat{B}\hat{R}^*. \tag{12.22}$$

where $\hat{R} : \hat{V} \to V^h(\Gamma)$ defined by $\hat{R}\tilde{u} = u$, for any $\tilde{u} \in \hat{V}$, and its adjoint $\hat{R}^*$ by :

$$(\hat{R}^* u, \tilde{v}) = \left\langle u, \hat{R}\tilde{v} \right\rangle.$$

Then we have

$$\kappa(\hat{B}\hat{A}) = \kappa(M_h S_h), \tag{12.23}$$

since it is easy to see $\hat{A} = \hat{R}^* S_h \hat{R}$ and

$$\begin{aligned} \langle B_h S_h u, S_h u \rangle &= \left\langle \hat{R}\hat{B}\hat{R}^* S_h u, S_h u \right\rangle = \left\langle \hat{B}\hat{R}^* S_h \hat{R}\tilde{u}, \hat{R}^* S_h \hat{R}\tilde{u} \right\rangle \\ &= \left\langle \hat{B}\hat{A}\tilde{u}, \hat{A}\tilde{u} \right\rangle. \end{aligned} \tag{12.24}$$

12.1. Interface BPX multilevel preconditioner. Given a sequence of nested piecewise linear finite element subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_J = V^h$$

corresponding to a sequence of nested quasi-uniform triangulations $\{\mathcal{T}^k\}_{k=1}^J$ of Ω with mesh size parameters

$$h_k = \max_{\tau \in \mathcal{T}^k} \operatorname{diam}(\tau) \approx 2^{-k}.$$

Let $\{\phi_i^k\}_{i=1}^{n_k}$ be the set of nodal basis functions of V_k $(1 \le k \le J)$ corresponding to the set $\mathcal{N}_k$ of interior nodal points of the triangulation $\mathcal{T}^k$. The BPX preconditioner M_h for the stiffness operator A_h can be written as (cf. Xu [95, 96]),

$$M_h v = \sum_{k=1}^{J} h_k^{2-n} \sum_{x_i^k \in \mathcal{N}_k} (v, \phi_i^k)\phi_i^k \quad \forall\, v \in V^h,$$

and $\kappa(M_h A_h) \lesssim 1$.

with α_0 and α_1 appearing in (3.6), i.e.

$$\alpha_0 \langle \hat{S}\hat{u}, \hat{u} \rangle \leq \langle \hat{B}\hat{S}\hat{u}, \hat{u} \rangle \leq \alpha_1 \langle \hat{S}\hat{u}, \hat{u} \rangle \quad \forall\, \hat{u} \in \hat{V},$$

which, from (11.18), is then equivalent to

$$\alpha_0 \langle S_h \hat{u}, \hat{u} \rangle \leq \langle (\sum_{i=1}^{p} \Theta_i S_i^+ \Theta_i^* Q_i) S_h \hat{u}, \hat{u} \rangle \leq \alpha_1 \langle S_h \hat{u}, \hat{u} \rangle \quad \forall\, \hat{u} \in \hat{V}. \tag{11.19}$$

Let $\hat{S}_i : \hat{V}_i \to \hat{V}_i$ be the restriction of S_i on $\hat{V}_i$. Obviously, the inverse $\hat{S}_i^{-1}$ exists and we have by definition of S_i^+ that $S_i^+ u_i = \hat{S}_i^{-1} u_i$, $\forall\, u_i \in \hat{V}_i$.

Let $\check{S}_i$ be defined as in (11.1), i.e. the interface operator on $V^h(\Gamma_i)$ corresponding to the bilinear form $\check{A}_i(\cdot,\cdot) = A_i(\cdot,\cdot) + \rho_i h_0^{-2}(\cdot,\cdot)_{0,\Omega_i}$. Note that the assumptions made on the boundary subdomains anable us to use Friedrichs inequality to obtain

$$\langle S_i^+ u_i, u_i \rangle_{0,\Gamma_i} = \left\langle \hat{S}_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i} \approx \langle \check{S}_i^{-1} u_i, u_i \rangle_{0,\Gamma_i} \quad \forall\, u_i \in \hat{V}_i. \tag{11.20}$$

Now (11.19) holds with $\alpha_1 = \omega_1 = O(\wp^2)$ and $\alpha_0 = K_0^{-1} = O(1)$ by combining (11.20) with (11.21) in the following Remark 11.1, thus $\kappa(B_h S_h) = O(\wp^2)$. □

REMARK 11.1. *Let $N_h = \sum_{i=1}^{p} \Theta_i \check{S}_i^{-1} \Theta_i^* Q_i$, then it is easy to find out by going through the proof of Theorem 11.1 which used Lemma 8.2 that the condition number of $N_h S_h$ restricted on the subspace $\hat{V} = V_0^{\perp}$ is the order $O(K_0\omega) = O(\wp^3)$; but the order $O(\wp^2)$ if each boundary subdomain has a common face with the boundary $\partial\Omega$. More exactly, we have*

$$K_0^{-1} \langle S_h \hat{u}, \hat{u} \rangle \lesssim \langle N_h S_h \hat{u}, S_h \hat{u} \rangle \lesssim \omega_1 \langle S_h \hat{u}, \hat{u} \rangle, \forall\, \hat{u} \in \hat{V}. \tag{11.21}$$

To see this, let $V = V^h(\Gamma)$, $A = S_h$ and define $\hat{P}, \hat{P}^*$ as in Section 3.3, $\hat{S}$ to be the restriction of S_h on $\hat{V}$, and $\hat{B} = \hat{P} B_h \hat{P}^*$. It is straightforward to verify by using $\hat{P}^* \hat{S} \hat{u} = S_h \hat{u}$ that for any $\hat{u} \in \hat{V}$,

$$\begin{aligned} \left\langle \hat{B}_h \hat{S}_h \hat{u}, \hat{S}_h \hat{u} \right\rangle &= \langle M_h S_h \hat{u}, S_h \hat{u} \rangle = \langle R_0 Q_0 S_h \hat{u}, S_h \hat{u} \rangle + \langle N_h S_h \hat{u}, S_h \hat{u} \rangle \\ &= (N_h S_h \hat{u}, S_h \hat{u}) \ (\textit{by } \hat{V} = V_0^{\perp}), \end{aligned}$$

which, combining with Theorem 11.1 implies (11.21).

11.3. Bibliograph remarks. The methods discussed in this section, often known as Neumann-Neumann type of algorithms, can be traced back to the work by Dinh-Glowinski-Périaux [41] and Glowinski-Wheeler[52]. Thereafter there are a few extensions in the theory and algorithms. We refer to Bourgat-Glowinski-Tallec-Vidrascu[6], Roeck-Tallec [77], Tallec-Roeck-Vidrascu [89], Mandel [68, 69], and Dryja-Widlund [45]. For extension of the approach for mixed finite element framework by Glowinski-Wheeler[52] to many subdomain case, see Cowsar-Wheeler[39].

Neumann-Neumann algorithms with weighted coarse subspaces for $n = 3$ was proposed by Dryja-Widlund [45], where the use of standard coarse subspaces (cf. §11.1.2) was also considered for elliptic problems with uniformly bounded coefficients. Here we give a unified presentation for both two and three dimensional cases with the case of large jukmps in coefficients included. In particular, we added the case of using the zero extensions E_i in local solvers R_i instead of weighted operators Θ_i.

Obviously, for a balanced function r^h there exists $u_i \in \hat{V}_i$ such that

$$S_i u_i = \Theta_i^* Q_i r^h,$$

and the solution u_i will be denoted by $u_i = S_i^+ \Theta_i^* Q_i r^h$. Note the inverse of S_i does not exist for interior subdomains Ω_i, and the above $\hat{V}_i$ defined by:

$$\hat{V}_i = \{u_i \in V^h(\Gamma_i); \quad \int_{\Gamma_i} u_i \, dx = 0\}. \tag{11.17}$$

For boundary subdomains $\Omega_i : \bar{\Omega}_i \cap \partial\Omega \neq \emptyset$, we let $\hat{V}_i = V^h(\Gamma_i)$.

Applying **Global Algorithm** in §3.3 to the present case with $A = S_h$, $V = V^h(\Gamma)$, $\hat{V} = V_0^\perp$ - the complement of V_0 in the sense of $\langle S_h \cdot, \cdot \rangle$, and as in §3.3, take $\hat{B} = \hat{P}(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i)\hat{P}^*$ as a known preconditioner to $\hat{S}$, then we derive from Theorem 3.3 a preconditioner B_h for S_h:

$$\begin{aligned} B_h S_h &= P_0 + \hat{P}\hat{B}\hat{S}\hat{P} = P_0 + \hat{P}(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i)\hat{P}^* \hat{S} \hat{P} \\ &= P_0 + (I - P_0)(\sum_{i=1}^p \Theta_i S_i^+ \Theta_i^* Q_i) S_h (I - P_0). \end{aligned} \tag{11.18}$$

where $P_0 : V^h(\Gamma) \to V_0$ is the orthogonal projection with respect to $\langle S_h \cdot, \cdot \rangle$. Using (11.18) and the definition of S_i^+, we immediately come to the following algorithm:

ALGORITHM 11.2 (BALANCING DOMAIN DECOMPOSITON ALGORITHM). *For any $g \in V^h(\Gamma)$, $u = B_h g$ is done, step by step, as follows:*

1. *Balancing the original residual by solving*

$$\langle S_h w_0, \phi \rangle = \langle g, \phi \rangle \quad \forall \, \phi \in V_0.$$

Set $r^h = g - S_h w_0$.

2. *Compute u_i, $1 \leq i \leq p$ in parallel: $u_i = \tilde{u}_i|_{\Gamma_i}$ with $\tilde{u}_i \in V^h(\Omega_i)$ solving*

$$A_i(\tilde{u}_i, \phi_l) = \left\langle r^h, \Theta_i \phi_l \right\rangle, \forall \, \phi_l \in V^h(\Omega_i);$$

Compute $\hat{u}_i = \Theta_i u_i$.

3. *Balancing the residual: $w_1 \in V_0$ solves*

$$\langle S_h w_1, \phi \rangle = \left\langle g - S_h \sum_{i=1}^p \hat{u}_i, \phi \right\rangle, \forall \, \phi \in V_0.$$

4. *Compute $u = \sum_{i=1}^p \hat{u}_i + w_1$.*

THEOREM 11.3. *Assume that for each subdomain Ω_i, $\partial\Omega_i \cap \partial\Omega$ is either empty or a face ($n = 3$) or an edge ($n = 2$) of Ω_i. Then for the above BDD algorithm,*

$$\kappa(B_h S_h) \lesssim \log^2(h_0/h).$$

Proof. By using Theorem 3.3, we know

$$\kappa(B_h S_h) \leq \frac{\max(1, \alpha_1)}{\min(1, \alpha_0)}$$

noticing that $\check{w}$ equals to $\tilde{u} - Q^{\rho}_{h_0}\tilde{u}$ on the interface Γ, thus by Lemma 4.10,

$$\check{A}_i(\check{w},\check{w}) \le \check{A}_i(\tilde{u} - Q^{\rho}_{h_0}\tilde{u}, \tilde{u} - Q^{\rho}_{h_0}\tilde{u}) \tag{11.14}$$

then it follows from (11.13)-(11.14) and Lemma 9.1 that

$$\sum_{i=1}^{p} \left\langle R_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i} \lesssim \gamma_0(n) \sum_{i=1}^{p} A_i(\tilde{u},\tilde{u}) = \gamma_0(n) \left\langle S_h u, u \right\rangle, \tag{11.15}$$

with $\gamma_0(2) = \wp$ but $\gamma_0(3) = h_0/h$. Again using Lemmas 4.10 and Lemma 9.1,

$$\begin{aligned} \left\langle R_0^{-1} u_0, u_0 \right\rangle &= \langle S_h u_0, u_0 \rangle = \sum_{i=1}^{p} A_i(\tilde{u}_0, \tilde{u}_0) \le \sum_{i=1}^{p} A_i(Q^{\rho}_{h_0}\tilde{u}, Q^{\rho}_{h_0}\tilde{u}) \\ &\lesssim \gamma(n) A(\tilde{u},\tilde{u}) = \gamma(n) \left\langle S_h u, u \right\rangle, \end{aligned} \tag{11.16}$$

which with (11.15) implies $K_0 = \gamma_0(n)$, that proves (11.12).

(11.11) follows by replacing the operator Q^{ρ}_H in the above proof for the estimation of K_0 by the standard L^2 projection Q_{h_0}.

Finally consider the E_i case, i.e. the local solver $R_i = E_i \check{S}_i^{-1} E_i^*$: the only differece from the Θ_i case is the estimate of K_0, i.e. the derivation of (11.15)-(11.16) from (11.13)-(11.14) with replacing the weighted operator $Q^{\rho}_{h_0}$ by the standard L^2 projection Q_{h_0} here. Then in the present case, we obtain

$$\left\langle R_i^{-1} u_i, u_i \right\rangle_{0,\Gamma_i} = \left\langle \check{S}_i E_i^{-1} u_i, E_i^{-1} u_i \right\rangle_{0,\Gamma_i} \lesssim \rho_i \left\| E_i^{-1} u_i \right\|^2_{1/2,\partial\Omega_i} = \rho_i \left\| u_i \right\|^2_{1/2,\partial\Omega_i},$$

now repeating the same decomposition (11.7) and the estimates thereafter, we have

$$\rho_i \|u_i\|^2_{1/2,\partial\Omega_i} \lesssim \wp^2 \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i \right\rangle_{0,\Gamma_i} = \wp^2 \left\langle \check{S}_i w, w \right\rangle_{0,\Gamma_i},$$

the rest is the same as proving (11.15)-(11.16 but with $Q^{\rho}_{h_0}$ replaced by Q_{h_0} here. □

11.2. Balancing Domain Decomposition Method. The balancing domain decomposition method is resulted from another approach to the singularity of S_i on $V(\Gamma_i)$. Rather than modifying the expression of the operator S_i itself as done in the last subsection, S_i can be made nonsingular by removing its null space. In another word, S_i is to be applied on a $V(\Gamma_i)$'s subspace on which S_i is nonsingular. In fact, the null space of S_i is at most a one dimensional space that contains only constant functions. If these contant functions can be annihilated, the operator S_i then becomes nonsingular.

The idea is first to solve the equation on a coarse subspace V_0 so that the resulting residual does not contain any constant component on each Γ_i and then to apply the Neumann-Newmann type algorithm (with a non-modified S_i) to the residual equation. This approach falls into the *local-global* technique described in §3.3.

Let e_0 denote a constant in this subsection. We then define the coarse subspace

$$V_0 = \text{span}\{\Theta_i e_0, \text{ for all interior subdomain } \Omega_i : \ \bar{\Omega}_i \cap \partial\Omega = \emptyset\}.$$

We say a function $r^h \in V^h(\Gamma)$ is balanced if r^h is orthogonal to V_0, or equivalently,

$$\left\langle \Theta_i^* Q_i r^h, e_0 \right\rangle_{0,\Gamma_i} = 0, \quad i.e. \quad \int_{\Gamma_i} \Theta_i^* Q_i r^h \, dx = 0.$$

$$\lesssim \quad \langle S_h u, u\rangle + \sum_{i=1}^{p} \langle R_i^{-1} u_i, u_i\rangle \lesssim \wp \langle S_h u, u\rangle ,$$

combining this with (11.9) implies $K_0 \lesssim \wp$, which concludes the proof of the first estimate in Theorem 11.1.

The second estimate in the theorem follows by going through the proof and noting that in this case $\omega_1 = O(\wp^2)$ and $K_0 = O(1)$. □

11.1.2. The use of standard coarse subspaces. As an alternative choice, the standard coarse space in §8.1 can also be used as the coarse space to define the preconditioner (11.2). Such a coarse space obviously has a much simpler structure than the weighted space, but it has limitations as it is not efficient for problems wth large discontinuous jumps for $n = 3$.

With V_0 being the standard coarse space and exact coarse solver $R_0 = S_0^{-1}$, the function $w_0 \in V_0$ in (11.3) for the action of M_h can be obtained by solving

$$A(\tilde{w}_0, \phi) = \langle g, \phi\rangle , \ \forall\, \phi \in V_0. \tag{11.10}$$

THEOREM 11.2. *If V_0 is the standard coarse space discussed in §8.1 and $R_0 = S_0^{-1}$, then the preconditioner given by (11.2) satisfies*

$$\kappa(M_h S_h) \lesssim (\max_{1\le i\le p} \rho_i)\wp^2, \tag{11.11}$$

where ρ_i are the coefficients of equation (2.1), or without the coefficients

$$\kappa(M_h S_h) \lesssim \begin{cases} \wp^3 & \text{if } n = 2, \\ \wp^2 h_0/h & \text{if } n = 3. \end{cases} \tag{11.12}$$

Moreover if each Θ_i is replaced by the zero extension operator $E_i : V^h(\Gamma_i) \to V_i$, namely the subspace sover $R_i = E_i \check{S}_i^{-1} E_i^*$, then

$$\kappa(M_h S_h) \lesssim r(\rho)\wp^4.$$

where $r(\rho) = \max_i \rho_i / \min_i \rho_i$.

Proof. By Theorem 3.1, it suffices to estimate K_1, K_0 and ω_1.

The same proof as for Theorem 11.1 gives $K_1 \lesssim 1$ and $\omega_1 \lesssim \wp^2$ for the Θ_i case but $\omega_1 \lesssim r(\rho)\,\wp^2$ for the E_i case. The only difference for the second case is to replace Θ_i by E_i in the proof for Theorem 11.1.

Now we estimate K_0. First consider the Θ_i case, i.e. $R_i = \Theta_i \check{S}_i^{-1} \Theta_i^*$: for any $u \in V^h(\Gamma)$, to define a partition of u, we take $u_0 = (Q^\rho_{h_0} \tilde{u})|_\Gamma \in V_0$ and $u_i = \Theta_i(u - u_0) \in V_i$. Here $Q^\rho_{h_0}$ is the weighted L^2 projection from $L^2(\Omega)$ to V_0 defined in (9.18). Obviously, $u = \sum_{i=0}^p u_i$. Let $w = u - u_0$. Using the properties of Q^ρ_H in Lemma 9.1, we obtain

$$\begin{aligned} \sum_{i=1}^{p} \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i} &= \sum_{i=1}^{p} \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i\right\rangle_{0,\Gamma_i} = \sum_{i=1}^{p} \left\langle \check{S}_i w, w\right\rangle_{0,\Gamma_i} \\ &= \sum_{i=1}^{p} \check{A}_i(\check{w}, \check{w}), \end{aligned} \tag{11.13}$$

We now proceed to establish the estimate that $\omega_1 \lesssim \wp^2$. To this end, it suffices to prove that

$$\langle S_h u_i, u_i\rangle \lesssim \wp^2 \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i}, \ \forall u_i \in V_i. \tag{11.6}$$

By definition of S_h and Lemma 4.9,

$$\langle S_h u_i, u_i\rangle = A(\tilde{u}_i, \tilde{u}_i) \eqsim \sum_m \rho_m \left|u_i\right|^2_{1/2,\partial\Omega_i}, \tag{11.7}$$

where the summation is over all subdomains Ω_m which share either a face, or an edge or a vertex with Ω_i. Let $\gamma_{im} = \partial\Omega_i \cap \partial\Omega_m$, we can write u_i on the interface Γ_m of Ω_m into

$$u_i = \sum_{F\subset\gamma_{im}} I_F^0 u_i + \sum_{E\subset\gamma_{im}} I_E^0 u_i + \sum_{V_k\in\gamma_{im}} I_{V_k}^0 u_i. \tag{11.8}$$

Then

$$\begin{aligned}
&\rho_m \Big|\sum_{F\subset\gamma_{im}} I_F^0 u_i\Big|^2_{1/2,\partial\Omega_m} \\
\lesssim\ & \sum_{F\subset\gamma_{im}} \left|I_F^0(\nu_\rho u_i)\right|^2_{1/2,\partial\Omega_m} \ (\rho_m \le \nu_\rho^2 = \text{Const on } F) \\
\lesssim\ & \sum_{F\subset\gamma_{im}} \left\|I_F^0(\nu_\rho u_i)\right\|^2_{H_{00}^{1/2}(F)} \ (\text{by Lemma 4.6}) \\
\lesssim\ & \wp^2 \left\|\nu_\rho u_i\right\|^2_{1/2,\partial\Omega_i} \ (\text{Lemma 4.14}) \\
=\ & \wp^2 \rho_i \left\|\Theta_i^{-1} u_i\right\|^2_{1/2,\partial\Omega_i} \ (\text{definition of } \Theta_i) \\
\lesssim\ & \wp^2 \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i\right\rangle_{0,\Gamma_i} \ (\text{Lemma 4.10 \& } \check{S}_i\text{'s definition}) \\
=\ & \wp^2 \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i}.
\end{aligned}$$

Conducting the same for the second and third terms in (11.8) with Lemmas 4.10 and 4.13, we obtain (11.6) from (11.7-(11.8) and triangle inequality.

We next estimate K_0. Given any $u \in V^h(\Gamma)$, take $u_0 = \sum_{i=1}^p \Theta_i I_i u \in V_0$ and $u_i = \Theta_i(u - I_i u) \in V_i$. We readily see $u = \sum_{i=0}^p u_i$. Let $w_i = u - I_i u$. We obtain

$$\begin{aligned}
&\sum_{i=1}^p \left\langle R_i^{-1} u_i, u_i\right\rangle_{0,\Gamma_i} \\
=\ & \sum_{i=1}^p \left\langle \check{S}_i \Theta_i^{-1} u_i, \Theta_i^{-1} u_i\right\rangle_{0,\Gamma_i} = \sum_{i=1}^p \left\langle \check{S}_i w_i, w_i\right\rangle_{0,\Gamma_i} \\
=\ & \sum_{i=1}^p \check{A}_i(\check{w}_i, \check{w}_i) \le \sum_{i=1}^p \check{A}_i(\tilde{w}_i, \tilde{w}_i) \ (\text{minimizing of } \check{w}_i) \\
\le\ & \wp \sum_{i=1}^p A_i(\tilde{u}, \tilde{u}) = \wp \left\langle S_h u, u\right\rangle_{0,\Gamma} \ (\text{Lemma 8.2}).
\end{aligned} \tag{11.9}$$

Noting that $u_0 = u - \sum_{i=1}^p u_i$, (11.9), $K_1 \lesssim 1$ and $\omega_1 \lesssim \wp^2$, we deduce that

$$\left\langle R_0^{-1} u_0, u_0\right\rangle \ = \ \wp^{-2} \left\langle S_h u_0, u_0\right\rangle \lesssim \wp^{-2} \left\langle S_h u, u\right\rangle + \wp^{-2} \sum_{i=1}^p \left\langle S_h u_i, u_i\right\rangle$$

With a proper choice of subspace V_0 and a solver R_0, we obtain the following space decomposition

$$V^h(\Gamma) = \sum_{i=0}^{p} V_i$$

and the corresponding PSC preconditioner

$$M_h = R_0 Q_0 + \sum_{i=1}^{p} \Theta_i \check{S}_i^{-1} \Theta_i^* Q_i. \tag{11.2}$$

Thus for any $g \in V^h(\Gamma)$,

$$M_h g = w_0 + \sum_{i=1}^{p} \Theta_i (w_i|_{\Gamma_i}), \tag{11.3}$$

and by means of the definition (11.2), the components w_i can be obtained by

ALGORITHM 11.1 (NEUMANN-NEUMANN ALGORITHM). *The components w_i in (11.3) for $0 \le i \le p$ are calculated as follows:*

1. *$w_i \in V^h(\Omega_i)$, for $1 \le i \le p$, solves the following local Neumman problem*

$$\check{A}_i(w_i, \phi_l) = \langle g, \Theta_i \phi_l \rangle \ , \forall\, \phi_l \in V^h(\Omega_i), \tag{11.4}$$

2. *$w_0 \in V_0$ solves the proper coarse problem depending on V_0 and R_0 to be chosen later, e.g. the subsequent wirebasket coarse problem (11.5) and the standard coarse problem (11.10) to be discussed in §11.1.1 and §11.1.2 respectively.*

11.1.1. The use of weighted coarse space. The method to be discussed now is based on the weighted coarse spaces in §8.3 and the following global coarse solver

$$R_0 = \wp^2 S_0^{-1},$$

concerning the action of the corresponding preconditioner M_h as in (11.2), $w_0 \in V_0$ can be obtained by solving

$$\langle S_h u_0, \phi \rangle = \wp^2 \langle g, \phi \rangle \quad \forall\, \phi \in V_0. \tag{11.5}$$

THEOREM 11.1. *With the aforementioned choice of weighted coarse space and R_0, the preconditioner M_h given by (11.2) satisfies*

$$\kappa(T_h S_h) \lesssim \log^3(h_0/h).$$

Moreover, if each boundary subdomain shares a common face with $\partial\Omega$, then

$$\kappa(T_h S_h) \lesssim \log^2(h_0/h).$$

Proof. By Theorem 3.1, we need to estimate K_1, K_0 and ω_1. Again it is clear that $K_1 \lesssim 1$. Different from all other situations in this paper, the estimate for ω_1 is not that straightforward here.

10.4. Bibliographic remark. The substructuring preconditioners discussed in the section are initiated by Bramble-Pasciak-Schatz [10, 14] and the analogue to wirebasket algorithms on the whole domain. The method in §10.1 is a fundamental algorithm which was applied to generate a lot of similar algorithms, e.g. Bramble-Pasciak-Schatz [12, 13, 14], Cai [18], Cai-Widlund [23], Cai-Gropp-Keyes [19, 22], Liang-Liang [62]

The *wirebasket algorithms* was proposed by Smith [82, 84] (n=3), and later the convergence proofs for elliptic problems with jumps in the coefficients was given by Dryja-Smith-Widlund [43] (n=3); here we add also the 2D case.

11. Algorithms based on local Neumann problems. This section is devoted to another type of preconditioner for the interface operator $S_h : V_\Gamma^h \to V^h(\Gamma)$. These algorithms are based on Neummann problems on subdomains.

The natural space for a Neumann problem on a subdomain, say Ω_i, is $V^h(\Gamma_i)$; nevertheless this is not a subspace of $V^h(\Gamma)$. To overcome this difficulty, for each i, we introduce a subspace V_i consisting of functions in $V^h(\Gamma)$ vanishing at nodes on $\Gamma \setminus \Gamma_i$. The spaces V_i and $V^h(\Gamma_i)$ have the same dimension. Unfortunately the operator S_i is not always invertible. There exist two main approaches to overcome this difficulty. The first approach, to be discussed in the subsection 11.1, is to slightly modify the operator S_i to introduce a nearby nonsingular operator by adding an appropriate lower order term. The first approach leads to the so-called Neumann-Neumann methods. The second approach, to be discussed in the subsection 11.2, is to first solve the coarse grid equation and then solve the residual equation for S_i which is nonsingular as the residual equation can be viewed on the complement of the coarse space in which the kernel of S_i is annihilated. The second approach leads to the so-called balancing domain decomposition method.

11.1. Neumann-Neumann methods. In this subsection, we discuss the methods based on modifying the operator S_i. The modification is based on the following bilinear form:

$$\check{A}_i(u,v) = A_i(u,v) + \rho_i h_0^{-2}(u,v)_{0,\Omega_i}, \ \forall\, u, v \in H^1(\Omega_i).$$

Correspondingly, a modified operator $\check{S}_i : V^h(\Gamma_i) \to V^h(\Gamma_i)$ can be defined as follows

$$\left\langle \check{S}_i u, v \right\rangle_{0,\Gamma_i} = \check{A}_i(\check{u}, \check{v}), \ \forall\, u, v \in V^h(\Gamma_i). \tag{11.1}$$

Here "$\check{u}$" denotes the $\check{A}_i$-discrete harmonic extension of u.

Obviously, the modified operator $\check{S}_i$ is invertible. A subspace solver, denoted by R_i, on each V_i is then defined by

$$R_i = \Theta_i \check{S}_i^{-1} \Theta_i^*$$

where Θ_i is defined as in (8.17) which is restated below for convenience

$$\Theta_i u_i = \rho_i^{1/2} I^0_{\Gamma_i}(\nu_\rho^{-1} u_i) \quad \forall\, u_i \in V^h(\Gamma_i);$$

the adjoint $\Theta_i^* : V_i \to V^h(\Gamma_i)$ by

$$\langle \Theta_i^* u_i, v_i \rangle_{0,\Gamma_i} = \langle u_i, \Theta_i v_i \rangle \quad \forall\, v_i \in V^h(\Gamma_i).$$

Let $Q_F : V^h(\Gamma) \to V_0^h(F)$ be the orthogonal projections with respect to $\langle \cdot, \cdot \rangle$, the parallel subspace correction preconditioner for S_h is then given by

$$M_h = R_0 Q_0 + \sum_{F \subset \Gamma} R_F Q_F. \tag{10.6}$$

THEOREM 10.3. *For the preconditioner $B_h = R_0 Q_0 + \sum_F R_F Q_F$, we have*

$$\kappa(B_h S_h) \lesssim (1 + \log(h_0/h))^2.$$

Proof. By Theorem 3.1, we need to estimate K_1, ω_1 and $\rho(\varepsilon)$. Evidently, $K_1 \lesssim 1$ as for each face subspace $V_0^h(F)$, only a fixed number of other face subspaces are not orthogonal to $V_0^h(F)$.

Using Lemma 4.9 and Lemmas 4.6-4.7, we obtain for any $u \in V_0^h(F)$ that

$$\begin{aligned} \langle S_F u, u \rangle &= A_{j_1}(\tilde{u}, \tilde{u}) + A_{j_2}(\tilde{u}, \tilde{u}) \\ &\lesssim \rho_F \|u\|^2_{H^{1/2}_{00}(F)} \eqsim \left\langle R_F^{-1} u, u \right\rangle, \end{aligned}$$

together with (8.1), we derive $\omega_1 \lesssim 1$.

Finally, we analyse K_0. For any $u \in V^h(\Gamma)$, let $u_0 = I_0 u \in V_0^h$ and $w = u - u_0$. Clearly, $u = u_0 + \sum_F w_F$. We can deduce

$$\begin{aligned} \left\langle R_0^{-1} u_0, u_0 \right\rangle &\lesssim \wp \sum_{i=1}^{p} \rho_i \|u - \gamma_{\partial\Omega_i}(u)\|^2_{0,\mathcal{W}_i} \text{ (minimizing of } \gamma_{\mathcal{W}_i}(u)) \\ &\lesssim \wp^2 \langle S_h u, u \rangle \ (n = 3) \text{ (Lemmas 4.13 \& 4.10 \& Poincaré ineq)} \\ \left\langle R_0^{-1} u_0, u_0 \right\rangle &= \sum_{i=1}^{p} A_i(\tilde{u}_0, \tilde{u}_0) \lesssim \wp \sum_{i=1}^{p} A_i(\tilde{u}, \tilde{u}) = \wp \langle S_h u, u \rangle \ (n = 2) \end{aligned}$$

Consider $n = 3$ and one face F. The same technique as used in (10.4) gives

$$\left\langle R_F^{-1} I_F^0 w, I_F^0 w \right\rangle \eqsim \rho_F \|I_F^0 w\|^2_{H^{1/2}_{00}(F)} \lesssim \wp^2 (A_{j_1}(\tilde{u}, \tilde{u}) + A_{j_2}(\tilde{u}, \tilde{u})),$$

which holds also for $n = 2$ by Lemma 4.19. Therefore $K_0 \lesssim \wp^2$, which ends the proof of Theorem 10.3. □

Recall the **Substructuring Algorithm I** in §10.1, we easily come to

ALGORITHM 10.3 (WIREBASKET ALGORITHM). *For any $g \in V^h(\Gamma)$, $B_h g = R_0 Q_0 g + \sum_f R_F Q_F g = u_0 + \sum_F u_F$ is computed as follows:*

1. *Compute $u_F \in V_0^h(F)$ in parallel:*

$$\left\langle (-\Delta_{F,h})^{1/2} u_F, v_F \right\rangle = \langle g, v_F \rangle, \quad \forall\, v_F \in V_0^h(F).$$

2. *If $n = 3$, compute $u_0 \in V_0$ by solving the minimization problem:*

$$\min_{w_0 \in V_0} \frac{1}{2} \wp \sum_{i=1}^{p} \rho_i \min_{\lambda_i \in R^1} \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - \langle g, w_0 \rangle;$$

if $n = 2$, compute $u_0 = w_0|_\Gamma \in V_0$ by solving the coarse problem:

$$\gamma(w_0, v_0) = \langle g, v_0 \rangle \quad \forall\, v_0 \in V_0.$$

where $\gamma(\cdot, \cdot)$ is defined in §10.1.

3. *Compute $B_h g = u_0 + \sum_F u_F$.*

Here $\mathcal{W}_{i,h}$ is the set of nodes on $\mathcal{W}_i$. Let $\mu_i(v)$ be a constant satisfying that

$$Q_i(v - \mu_i(v), 1) = 0 \quad forall\, v \in V^h(\partial\Omega_i).$$

We can deduce that

$$\begin{aligned}
&(\wp)^{-2} Q_i(u - \mu_i(u), u - \mu_i(u)) \\
&\quad \leq \quad (\wp)^{-2} Q_i(u - \gamma_{\partial\Omega_i}(u), u - \gamma_{\partial\Omega_i}(u)) \ (\mu_i\text{'s minimizing}) \\
&\quad \lesssim \quad \|u - \gamma_{\partial\Omega_i}(u)\|^2_{1/2,\partial\Omega_i} \ (\text{by (10.5)}) \\
&\quad \lesssim \quad |u|^2_{1/2,\partial\Omega_i} \ (\text{Lemma 4.18}) \\
&\quad \lesssim \quad Q_i(u - \mu_i(u), u - \mu_i(u)) \ (\text{from (10.5)}).
\end{aligned}$$

Using this and (10.1), we can define the preconditioner M_h to the stiffness operator A_h by

$$(M_h^{-1} u, v) = A(u_P, v_P) + \sum_{i=1}^{p} \rho_i\, Q_i(u - \mu_i(u), v - \mu_i(v)) \quad \forall\, u, v \in V^h.$$

Then the previous statement shows

LEMMA 10.2.

$$\kappa(M_h A_h) \lesssim \log^2(h_0/h).$$

The algorithm for solving problem associated with the corresponding bilinear form is given below.

ALGORITHM 10.2. ***algorithm to be included***

10.3. A variant of the substructuring preconditioner-I. We shall now present a parallel subspace correction version of the substructuring preconditioner-I in §10.1. This variant of substructuring method was first considered by Smith [82, 84], known as wirebasket methods there.

As mentioned in §2, the "breaking" process on the interface Γ gives a natural decomposition of the space $V^h(\Gamma)$ as follows:

$$V^h(\Gamma) = V_0 + \sum_{F \subset \Gamma} V_0^h(F),$$

where $V_0 = I_0 V^h(\Gamma)$ with I_0, the *joint-operator* introduced in §2, being the wirebasket interpolant defined in §8.2 ($n = 3$) and standard coarse space interpolant ($n = 2$) defined in §8.1.

The coarse subspace solver R_0 is chosen to be the interface restriction of the standard coarse solver R_0 defined in §8.1 for $n = 2$ and the wirebasket coarse solver on the interface defined in §8.2 for $n = 3$, i.e.

$$\left\langle R_0^{-1} u_0, v_0 \right\rangle = \wp \sum_{i=1}^{p} \rho_i \left\langle u_0 - \gamma_{h,\mathcal{W}_i}(u_0), v_0 - \gamma_{h,\mathcal{W}_i}(v_0) \right\rangle_{h,\mathcal{W}_i};$$

For each face F, let ρ_F be the average value defined as in §10.1, and we adopt local face solvers $R_F^{-1} = \rho(F)\,(-\Delta_{F,h})^{1/2}$.

while for $n = 3$ (noting (4.20)) by

$$(M_h^{-1}u, v) = A(u_P, v_P) + \sum_{F\subset\Gamma} \rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} + \gamma(I_0 u_H, I_0 v_H).$$

where $\gamma(I_0 u_H, I_0 v_H)$ is defined as follows:

$$\gamma(I_0 u_H, I_0 v_H) = \begin{cases} \sum_{E_{ij}\subset\Gamma} (u_H(v_i) - u_H(v_j))\,((v_H(v_i) - v_H(v_j)), & \text{if } n = 2, \\ \wp \sum_{i=1}^{p} \rho_i \left\langle u_H - \gamma_{h,\mathcal{W}_i}(u_H), v_H - \gamma_{h,\mathcal{W}_i}(v_H) \right\rangle_{h,\mathcal{W}_i}, & \text{if } n = 3. \end{cases}$$

And the above derivations show

THEOREM 10.1. *For teh condition number of $M_h A_h$, we have*

$$\kappa(M_h A_h) \lesssim C(n).$$

where $C(2) = C$ independent of the h and h_0, but $C(3) = \log^2(h_0/h)$.

Recall the standard and wirebasket coarse solvers defined in §8.1 and §8.2 respectively, we have the following algorithm:

ALGORITHM 10.1 (SUBSTRUCTURING ALGORITHM I). *For given $g \in V^h$, let $u = M_h g$, then $u = u_P + u_H$ can be obtained as follows:*

1. *For $1 \le i \le p$, $u_P \in V_0^h(\Omega_i)$ solves*

$$A(u_P, v) = (g, v) \quad \forall v \in V_0^h(\Omega_i).$$

2. *On each face $F \subset \Gamma$, u_E solves*

$$\rho_F \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} = (g, v) - A(u_P, v) \quad \forall\, v \in V_0^h(F).$$

3. *If $n = 2$, find $u_0 = I_0 u_H \in V_0(\Gamma)$ solving*

$$\gamma(u_0, v) = (g, v) - A(u_P, v) \quad \forall\, v \in V_0(\Gamma);$$

If $n = 3$, find $u_0 \in V_0(\Gamma)$ on Γ by solving

$$\min_{w_0 \in V_0} \frac{1}{2} \sum_{i=1}^{p} \rho_i \min_{\lambda_i \in R^1} \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - (g, w_0) - A(u_P, w_0) \quad \forall\, w_0 \in V_0(\Gamma),$$

4. *Calculate u_H on each subdomain concurrently by solving a Dirichlet problem with the boundary value given by $u_0 + u_E$ on $\partial\Omega_i$.*

10.2. Substructuring preconditioner-II ($n = 3$). We next present another substructuring technique that is based on the following estimate implied by Lemmas 4.13-4.14:

$$(\wp)^{-2} Q_i(w, w) \lesssim \|w\|^2_{1/2,\partial\Omega_i} \lesssim Q_i(w, w) \quad \forall\, v \in V^h(\partial\Omega_i). \tag{10.5}$$

where

$$Q_i(u, u) = h \sum_{x_i \in \mathcal{W}_{i,h}} u^2(x_i) + \sum_{F\subset\partial\Omega_i} \left\langle (-\Delta_{F,h})^{1/2} I_F^0 u, I_F^0 u \right\rangle_{0,F} \quad \forall\, u \in V^h(\partial\Omega_i).$$

an $A(\cdot,\cdot)$-discrete harmonic function in Ω and uniquely determined by its interface values. Note that

$$A(u,u) = A(u_P,u_P) + A(u_H,u_H) \eqsim A(u_P,u_P) + S(u_H,u_H), \tag{10.1}$$

so to find a preconditioner for A, our tast becomes breaking u_H on the interface Γ. As pointed out in §2, a proper choice of the joint-operator on the space $V^h(\Gamma)$ is crucial to reach this aim effectively.

We define the joint-operator I_0 here to be the interpolant corresponding to the standard coarse subspace $V_0(\Gamma)$ defined in §8.1 for $n=2$, but the interpolant corresponding to the wirebasket coarse space $V_0(\Gamma)$ defined in §8.2 for $n=3$. Using I_0, we can write

$$u_H = I_0 u_H + u_E, \quad u_E = u_H - I_0 u_H$$

with u_E vanishing on the wirebasket set. By the triangle inequalty,

$$S(u_H,u_H) \le 2S(u_E,u_E) + 2S(I_0 u_H, I_0 u_H). \tag{10.2}$$

As u_E vanishing on the wirebasket set $\mathcal{W}$, it follows from Lemmas 4.6-4.7 that

$$\begin{aligned} S(u_E,u_E) &= \sum_{i=1}^{p} \rho_i \, |u_E|^2_{1/2,\partial\Omega_i} \lesssim \sum_{i=1}^{p} \sum_{F\subset\partial\Omega_i} \rho_i \, |u_E|^2_{H^{1/2}_{00}(F)} \\ &\eqsim \sum_{F\subset\Gamma} \rho_F \, \left\langle (-\Delta_{F,h})^{1/2} u_E, u_E \right\rangle_{0,F} \end{aligned}$$

where ρ_F is the average value of two coefficients associated with two subdomains sharing the common face F. Combining with (10.2) yields

$$S(u_H,u_H) \lesssim \sum_{F\subset\Gamma} \rho_F \, \left\langle (-\Delta_{F,h})^{1/2} u_E, u_E \right\rangle_{0,F} + S(I_0 u_H, I_0 u_H). \tag{10.3}$$

Note that for $n=3$,

$$I^0_F u_E = I^0_F(u - I_0 u) = I^0_F(u - \gamma_{\partial F}(u)\,) = I^0_F u - \gamma_{\partial F}(u) I^0_F 1, \tag{10.4}$$

thus we derive by using Lemmas 4.13-4.14 and Lemma 4.20 that

$$\rho_F \|I^0_F u_E\|^2_{H^{1/2}_{00}(F)} \lesssim \log^2(h_0/h)(\rho_{j_1}|u|^2_{1/2,\partial\Omega_{j_1}} + \rho_{j_2}|u|^2_{1/2,\partial\Omega_{j_2}})$$

where Ω_{j_1} and Ω_{j_2} are two subdomains sharing the face F, this along with (10.2)-(10.3) and Lemma 4.14 implies for $n=3$ that

$$\begin{aligned} S(u_H,u_H) &\lesssim (\log\frac{h_0}{h}) \sum_{i=1}^{p} \rho_i \Big((\log\frac{h_0}{h}) |u_H|^2_{1,\Omega_i} + \|I_0 u_H - \gamma_{\mathcal{W}_i}(I_0 u_H)\|^2_{0,\mathcal{W}_i} \Big) \\ &\lesssim (\log\frac{h_0}{h})^2 \, S(u_H,u_H) \end{aligned}$$

In summary, for $n=2$ (noting (8.7)) we define the preconditioner M_h by

$$(M_h^{-1}u, v) = A(u_P, v_P) + \sum_{F\subset\Gamma} \rho_F \, \left\langle (-\Delta_{F,h})^{1/2} u_E, v_E \right\rangle_{0,F} + \gamma(I_0 u_H, I_0 v_H),$$

Here $R_0 : V_0(\Gamma) \to V_0(\Gamma)$ is the standard coarse solver in §8.1 for $n = 2$ and the wirebasket coarse solver in §8.2.

In summary, our additive Schwarz preconditioner is then formulated as

$$M_h = \tilde{R}_0 \tilde{Q}_0 + \sum_{i=1}^{p} A_i^{-1} Q_i.$$

We have the following condition number bounds for M_h:

LEMMA 9.2. *For $n = 2, 3$,*

$$\kappa(M_h A_h) \lesssim (1 + \log(h_0/h))^{n-1}.$$

Proof. By Lemma 3.1 it suffices to estimate the parameters K_0, K_1 and ω_1. As we are using exact local and global coarse solver, we know that $\omega_1 = 1$. By the definitions of the local subspaces V_i $(1 \le i \le p)$ and the parameter K_1 in (3.5), we can also readily know that $K_1 \lesssim 1$. What remains is to bound the parameter K_0 which is the smallest constant satisfying

$$\sum_{i=0}^{p} A(u_i, u_i) \le K_0 A(u, u), \quad \forall u \in V^h,\ u = \sum_{i=0}^{p} u_i,\ u_i \in V_i.$$

For this purpose, by Lemma **??** we need only to find an operator $Q_0 : V^h \to V_0$ satisfying (8.1). We can take the weighted L^2 projection defined in (9.18) for $n = 2$ but the wirebasket interplant itsself defined in §8.2. The results follow then from Lemma 9.1 and (8.10). □

We have the following algorithm for computing the action of the preconditioner M_h:

ALGORITHM 9.1 (OVERLAPPING SCHWARZ METHOD). *For given $g \in V^h$, let $u = M_h g = R_0 Q_0 + \sum_{i=1}^{p} A_i^{-1} Q_i \equiv \sum_{i=0}^{p} u_i$, and u_i cab be computed as follows:*

1. *For $i = 1, \cdots, p$, $u_i \in V_0^h(O_i)$ solves the Dirichlet problem:*

$$A(u_i, v) = (g, v) \quad \forall\, v \in V_0^h(O_i).$$

2. *u_0 can be obtained for $n = 2$ from (8.8) and for $n = 3$ from (??)- (??) with $g_0 = Q_0 g$.*

9.1. Bibiographic remarks. The partition lemma was first introduced in the domain decomposition context by Nepomnyaschikh [75] and Lions [64].

The convergence analysis of overlapping Schwarz methods with standard coarse subspaces for elliptic problems with jumps in the coefficients was considered by Dryja-Sarkis-Widlund [42] and Martins in [71] for the case that neighboring coefficients at each vertex of subdomains are monotone in a certain direction.

10. Substructuring method. We shall now discuss the so-called substruturing methods and pay attention to three major algorithms and some of their variants.

10.1. Substructuring preconditioners-I ($n = 2, 3$). At first, we present the technical details on the construction of substructuring preconditioners whose motivation has been discussed in the outline (§2). As we know, any function u in V^h can be split into $u = u_P + u_H$ with $u_P \in V_0^h(\Omega_i)$ which can be obtained concurrently on each subdomain by solving a homogeneous Dirichlet problem, while $u_H \in V_H$ is

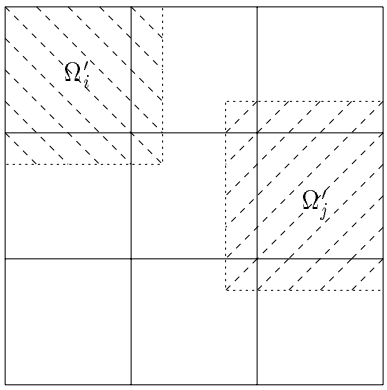

FIG. 5. *Overlapping subdomains* $\{\Omega_i'\}$

The weighted L^2 projection $Q_{h_0}^{\rho} : L^2(\Omega) \to V_0$ is defined by

$$(Q_{h_0}^{\rho} u, v)_{L^2_{\rho}(\Omega)} = (u, v)_{L^2_{\rho}(\Omega)} \quad \forall u \in L^2(\Omega),\ v \in V_0, \tag{9.18}$$

where $(\cdot,\cdot)_{L^2_{\rho}(\Omega)}$ is the scalar product related to the norm $\|\cdot\|_{L^2_{\rho}(\Omega)}$. We will denote $Q_{h_0} = Q_{h_0}^{\rho}$ if $\rho = 1$.

Bramble-Xu [17] proved

LEMMA 9.1. *For any* $u \in V^h$, *we have for* $n = 2$ *that*

$$\begin{aligned} \|u - Q_{h_0}^{\rho} u\|^2_{L^2_{\rho}(\Omega)} &\lesssim h_0^2\, \log(h_0/h)\, |u|^2_{H^1_{\rho}(\Omega)}, \\ |Q_{h_0}^{\rho} u|^2_{H^1_{\rho}(\Omega)} &\lesssim \log(h_0/h)\, |u|^2_{H^1_{\rho}(\Omega)}; \end{aligned}$$

If all the coefficients $\rho_i = 1$ $(1 \le i \le p)$, *then for both* $n = 2$ *and* $n = 3$ *we have*

$$\|u - Q_{h_0} u\|_{0,\Omega} \lesssim h_0 |u|_{1,\Omega}, \quad |Q_{h_0} u|_{1,\Omega} \lesssim |u|_{1,\Omega} \quad \forall\, u \in H_0^1(\Omega).$$

Overlapping Schwarz methods. As usual, for each overlapping subdomain Ω_i', the natural local subspace is $V_i = V_0^h(\Omega')$. To construct a global coarse subspace of V^h, we use the harmonic extension of the standard coarse space in §8.1 for $n = 2$, but of the wirebasket space in §8.2 for $n = 3$ as our global coarse space $\tilde{V}_0$, i.e.

$$\tilde{V}_0 = \{\tilde{v} \in V_H;\ \ \tilde{v} = \text{ discrete harmonic extension of } v \in V_0(\Gamma)\}.$$

Then the local solver on each subdomain Ω_i' is defined to be the restriction A_i of the stiffness operator A_h on V_i, i.e.

$$(A_i u, v)_{0,\Omega_i'} = A(u, v) \quad \forall\, u, v \in V_i;$$

while the global coarse solver $\tilde{R}_0$ is defined by

$$(\tilde{R}_0^{-1} \tilde{u}, \tilde{v}) = \left\langle R_0^{-1} u, v \right\rangle \quad \forall\, u, v \in \tilde{V}_0$$

with $\tilde{R}_0$ approximating the restriction $A_0 : \tilde{V}_0 \to \tilde{V}_0$ defined by

$$(A_0 \tilde{u}, \tilde{v}) = A(u, v) \quad \forall\, u, v \in \tilde{V}_0.$$

Proof. The case that $\partial\Omega_i \cap \partial\Omega$ is empty or a face, is the consequence of Lemma 4.18.

For the case that $\partial\Omega_i \cap \partial\Omega$ is an adge: we know $I_i u = 0$, applying Lemma 4.16 to the discrete harmonic extension $\tilde{u}$ of u into Ω_i, we derive

$$\begin{aligned} \|u\|^2_{0,\partial\Omega} &\eqsim h_0^2 \sum_{x_i \in \partial\Omega} u^2(x_i) \lesssim h_0 \, \|\tilde{u}\|^2_{0,\Omega_i} \\ &\lesssim h_0^3 \, \wp \, |u|^2_{1/2,\partial\Omega}. \end{aligned}$$

We next prove the case that $\partial\Omega_i \cap \partial\Omega$ is a vertex, say v_k. We can write $I_i u = \gamma_{\partial\Omega_i}(u) - \gamma_{\partial\Omega_i}(u)\phi_k$ with ϕ_k the nodal basis function of $V^h(\Gamma)$ at the vertex v_k. It follows from Lemma 4.13 that for any constant c,

$$\begin{aligned} |\gamma_{\partial\Omega_i}(u)|^2 &\lesssim \max_{v_j \in W_{i,h}} |u(v_j)|^2 \lesssim \max_{v_j \in W_{i,h}} |u(v_j) - c|^2 \\ &\lesssim h^{2-n} \, \wp \, \|u - c\|^2_{1/2,\partial\Omega_i}, \end{aligned}$$

from this, Lemma 4.13 and Poincaré inequality we have,

$$\begin{aligned} |I_i u|^2_{1/2,\partial\Omega_i} &= |\gamma_{\partial\Omega_i}(u)|^2 |I^0_{v_k}\phi_k|^2_{1/2,\partial\Omega_i} \lesssim \wp \, |u|^2_{1/2,\partial\Omega_i}, \\ \|I_i u - u\|^2_{0,\partial\Omega_i} &\lesssim \|u - \gamma_{\partial\Omega_i}(u)\|^2_{0,\partial\Omega_i} + |\gamma_{\partial\Omega_i}(u)|^2 |\phi_k|^2_{0,\partial\Omega_i} \\ &\lesssim h_0 \, |u|^2_{1/2,\partial\Omega_i} + h \, \wp \, |u|^2_{1/2,\partial\Omega_i}. \end{aligned}$$

this completes the proof of Lemma 8.2. □

REMARK 8.1. *If one replaces the power* $1/2$ *of* ρ_i *by any real number* $t \geq 1/2$ *in the definitions of* ν_ρ*, the coarse subspace* V_0 *and the operators* Θ_i*, all related results in the subsequent sections are still true with little modification.*

8.4. Bibliographic remarks. The approximation results (8.2) for standard coarse interpolants was proved by Bramble-Xu [17].

Wirebasket subspaces was first proposed by Smith [82, 84] in an interface formulation. Here we give its equivalent global formulation.

Weighted coarse subspace was originated from Bramble-Paschiak-Schatz [12, 14] for solving their coarse problem and first used by Dryja-Widlund [45] and Mandel [68] for so-called Neumann-Neumannn type methods (cf. §11.1 and §11.2).

9. A partition lemma and overlapping additive Schwarz methods for elliptic probems with jumps in the coefficients. It is known that in the three dimension, the overlapping additive Schwarz method with standard coarse subspaces described in §8.1 are not effective for the problem (2.1) with large jumps in the coefficients. In this section, we will show that with the help of the wirebasket subspace given in §8.2, the overlapping additive Schwarz methods will work pretty well in both two and three dimensions.

Based on the given p non-overlapping subdomains Ω_i $(1 \leq i \leq p)$, we extend each Ω_i to a larger Ω_i' with dist $(\partial\Omega_i' \cap \Omega, \partial\Omega_i') \gtrsim h_0$, cf. Fig. 5. Assume that $\partial\Omega_i'$ and $\partial\Omega_i$ align with elements of $\mathcal{T}^h$ and each point $x \in \Omega$ belongs to at most q_0 subdomains of $\{\Omega_i'\}_{i=1}^p$, with $q_0 < p$ a positive integer. For the sake of explanation, we allow in this section the subdomains Ω_i to be only simplicial.

Before discussing the algorithm, we first recall L^2 and weighted L^2 projection operator $Q^\rho_{h_0}$. Let V_0 be the piecewise linear finite element space related to the triangulation of the non-overlapping subdomains $\{\Omega_i\}$.

and 0 otherwise. Let μ be the vector with components $u_0(x_k)$ for all wirebasket nodes x_k. Then the aforementioned Euler equation can be written as

$$\sum_{i=1}^{p} \rho_i \mathcal{I}_i(\mu - \lambda_i e) = b \tag{8.13}$$

for some appropriate vector b, and

$$e^t I_i(\mu - \lambda_i e) = 0 \quad \forall 1 \le i \le p. \tag{8.14}$$

Denote $\mathcal{D} = \sum_{i=1}^{p} \rho_i \mathcal{I}_i$ which is obviously a nonsingular diagonal matrix. By (8.13) we have

$$\mu = \mathcal{D}^{-1} \sum_{i=1}^{p} \rho_i \mathcal{I}_i \lambda_i e + \mathcal{D}^{-1} b. \tag{8.15}$$

Substituting (8.15) to (8.14) yields

$$e^t \mathcal{I}_i (\mathcal{D}^{-1} \sum_{j=1}^{p} \rho_j \mathcal{I}_j \lambda_j e - \lambda_i e) = -e^t I_i \mathcal{D}^{-1} b \quad \forall 1 \le i \le p. \tag{8.16}$$

Solving the equation (8.16), we get λ_i, and then we can calculate μ from (8.15).

8.3. Weighted coarse subspaces. This coarse space is a subspace of $V^h(\Gamma)$ defined on the interface Γ, and it uses at most one degree of freedom per subdomain.

Let χ_K be the characteristic function on a subset $K \subset \Gamma$ and for each local interface Γ_i, let V_i be a subspace of $V^h(\Gamma)$ with functions vanishing at nodes in $\Gamma \setminus \Gamma_i$.

Using a weighted counting function

$$\nu_\rho(x) = \sum_{i=1}^{p} \rho_i^{1/2} \, \chi_{\partial\Omega_i}(x), \, \forall \, x \in \Gamma \cup \partial\Omega,$$

and a special partition of unity on the interface Γ: $\sum_{i=1}^{p} \Theta_i \equiv 1$ on Γ with $\Theta_i : V^h(\Gamma_i) \to V_i$ given by

$$\Theta_i u_i = \rho_i^{1/2} \, I^0_{\Gamma_i}(\nu_\rho^{-1} u_i), \, \forall \, u_i \in V^h(\Gamma_i), \tag{8.17}$$

we define the weighted coarse subspace as

$$V_0 = \text{span}\{\Theta_i I_i u; \; 1 \le i \le p, \;\; u \in V^h(\Gamma)\},$$

where $I_i : V^h(\Gamma_i) \to V^h(\Gamma_i)$, is defined, for any $u \in V^h(\Gamma_i)$, by $I_i u(x) = \gamma_{\partial\Omega_i}(u)$ for $x \in \Gamma_{i,h}$ and $I_i u(x) = 0$ for $x \in \partial\Omega_i \cap \partial\Omega$; but $I_i \equiv 0$ if $\partial\Omega_i \cap \partial\Omega$ is a face or an adge.

LEMMA 8.2. *For any $u \in V^h(\Gamma_i)$, we have*

$$|I_i u|^2_{1/2,\partial\Omega_i} \lesssim \begin{cases} \wp \, |u|^2_{1/2,\partial\Omega_i} & \text{if } \partial\Omega_i \cap \partial\Omega \text{ is a vertex} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\|u - I_i u\|^2_{0,\partial\Omega_i} \lesssim \begin{cases} h_0 \, |u|^2_{1/2,\partial\Omega_i} & \text{if } \partial\Omega_i \cap \partial\Omega \text{ is empty or a face} \\ h_0 \, \wp \, |u|^2_{1/2,\partial\Omega_i} & \text{otherwise.} \end{cases}$$

We next show that for any $u_0 \in V_0$,

$$(\log \frac{h_0}{h})^{-1} |u_0|^2_{1/2,\partial\Omega_i} \lesssim \|u_0\|^2_{h,\mathcal{W}_i}, \tag{8.11}$$

then the desired first inequality follows by replacing u_0 by $u_0 - \gamma_{h,\mathcal{W}_i}(u_0)$ in (8.11). By the definition of V_0, we can express u_0 on $\partial\Omega_i$ as

$$u_0 = \sum_{F \subset \partial\Omega_i} \gamma_{\partial F}(u_0) I^0_F 1 + I^0_{\mathcal{W}_i} u_0,$$

thus (8.11) follows from the triangle inequality and Lemmas 4.13-4.14:

$$\begin{aligned} |u_0|^2_{1/2,\partial\Omega_i} &\lesssim h_0 \wp \sum_{F \subset \partial\Omega_i} |\gamma_{\partial F}(u_0)|^2 + \wp \|u_0 - \gamma_{h,\mathcal{W}_i}(u_0)\|^2_{h,\mathcal{W}_i} \\ &\lesssim \wp \|u_0\|^2_{h,\mathcal{W}_i} \text{ (by Lemma 4.20 for } |\gamma_{\partial F}(u_0)|). \end{aligned}$$

□

Lemma 8.1 actually implies an equivalent coarse solver for the interface operator S_0 by replacing it by the discrete norm defined on the wirebasket. We now decsribe this solver.

Wirebasket coarse solver R_0 on the interface Γ. Define R_0 by

$$\left\langle R_0^{-1} u_0, v_0 \right\rangle = \sum_{i=1}^{p} \rho_i \left\langle u_0 - \gamma_{h,\mathcal{W}_i}(u_0), v_0 - \gamma_{h,\mathcal{W}_i}(v_0) \right\rangle_{h,\mathcal{W}_i} \quad \forall\, u_0, v_0 \in V_0(\Gamma).$$

It is readily to see that $u_0 = R_0 g_0$ for given $g_0 \in V_0(\Gamma)$ can be obtained by solving the minimization problem:

$$\min_{w_0 \in V_0(\Gamma)} \frac{1}{2} \sum_{i=1}^{p} \rho_i \min_{\lambda_i \in R^1} \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - \langle g, w_0 \rangle \tag{8.12}$$

By the symmetry of the operator R_0 and the minimizing property of an average value in the constant space, the above problem is obviously equivalent to

$$\min_{w_0 \in V_0(\Gamma), \Lambda \in R^p} \frac{1}{2} \sum_{i=1}^{p} \rho_i \|w_0 - \lambda_i\|^2_{h,\mathcal{W}_i} - \langle g, w_0 \rangle$$

where $\Lambda = (\lambda_1, \ldots, \lambda_p)$.

The Euler equation for the above problem is

$$\sum_{i=1}^{p} \rho_i (u_0 - \lambda_i, v_0 - \lambda_i)_{h,\mathcal{W}_i} = \langle g, v_0 \rangle \quad \forall v_0 \in V_0$$

and

$$(u_0 - \lambda_i, 1)_{h,\mathcal{W}_i} = 0.$$

With certain ordering of the nodes $\{x_k\}$ on the wirebasket, for each i, let $\mathcal{I}_i$ be a matrix with zero off diagonal entries and its kth diagonal entry equals to 1 if $x_k \in W_i$

8.2. Wirebasket subspaces. The estimates (8.2)-(8.3) indicate that for $n = 2$ the energy of the coarse interpolant $I_0 u$ for any u in V^h by using only the values of u at vertices of Ω_i exceeds that of u by at most a factor $\log(h_0/h)$. However for $n = 3$, it may result in a (h_0/h) increasing in the energy. In fact, a simple example can show this can not be improved. Taking a nodal basis function $\phi_k \in V^h$ which equals to 1 at one vertex v_k of a single simplex Ω_i, but vanishes at all other nodes on $\bar{\Omega}_i$. It is readily to see that

$$|I_0\phi_k|^2_{1,\Omega_i} \eqsim h_0^{-2}|\Omega_i| \eqsim h_0 = \frac{h_0}{h}\, h \eqsim \frac{h_0}{h}\, |\phi_k|^2_{1,\Omega_i}.$$

Therefore interpolating the value of a function in V^h by exploiting only the vertex values is not reasonable in three dimension.

To motivate the construction of an effective I_0 in three dimensions, recall the first requirement **(C1)**, for any $v \in V_\Gamma$, we define $I_0 v$ to be equal to v at the nodes on the joint-set (which is called the wirebasket set). In regard to the fourth requirement, it is desireable not to introduce any more degree of freedom. Hence we need to extend the value on the joint-set inside each face set F. One possible way, as implicitly used in Bramble-Pasciak-Schatz-I, is to use discrete harmonic extension but this approach is computationally a little too expensive for this purpose. Following Smith [82, 84], we define $I_0 v$ to be a constant value at all interior nodes of each face F, namely the arithmetic average, $\gamma_{h,\partial F}$, of the nodal values of v on ∂F.

In summary, the operator I_0 is defined, for any $v \in V_\Gamma$ and nodal point x, by

$$(I_0 v)(x) = \begin{cases} v(x) & \text{if } x \in \mathcal{W} \\ \gamma_{h,\partial F}(v) & \text{if } x \in F \quad \text{for each (open) face set } F. \end{cases} \tag{8.9}$$

This interpolant, called *wirebasket interpolant*, satisfies, any $u \in V^h$,

$$|u - I_0 u|_{0,\partial\Omega_i} + h_0^{1/2}|I_0 u|_{1/2,\partial\Omega_i} \lesssim h_0^{1/2}\log(h_0/h)|u|_{1/2,\partial\Omega_i}. \tag{8.10}$$

In fact, by the obvious identity

$$u - I_0 u = \sum_{F\subset\partial\Omega_i} I_F^0(u - I_0 u) = \sum_{F\subset\partial\Omega_i} \left(I_F^0 u - \gamma_{h,\partial F}(u) I_F^0 1\right)$$

and by Lemmas 4.14, 4.15 and 4.20, it follows that

$$|u - I_0 u|^2_{1/2,\partial\Omega_i} \lesssim \sum_{F\subset\partial\Omega_i} \|I_F^0 u - \gamma_{h,\partial F}(u) I_F^0 1\|^2_{H_{00}^{1/2}(F)} \lesssim \log^2(h_0/h)\, |u|^2_{1/2,\partial\Omega_i},$$

The estimate of $\|u - I_0 u\|_{0,\partial\Omega_i}$ can be done the same way as above but first on the reference domain G and then scaling back to the subddomain Ω_i.

Next we show that the wirebasket space $V_0 = I_0 V^h(\Gamma)$ has the property:

LEMMA 8.1. *For any $u \in V^h(\Gamma)$, $u_0 = I_0 u$ satisfies*

$$(\log\frac{h_0}{h})^{-1}|u_0|^2_{1/2,\partial\Omega_i} \lesssim \|u_0 - \gamma_{h,\mathcal{W}_i}(u_0)\|^2_{h,\mathcal{W}_i} \lesssim \log\frac{h_0}{h}|u_0|^2_{1/2,\partial\Omega_i}.$$

Proof. The second inequality is an immediate consequence of Lemma 4.13 and Lemma 4.18.

The inequality (8.2) is a consequence of the following equivalancy which will be used later to define the coarse solver:

$$|u|^2_{1/2,\partial\Omega_i} \eqsim \sum_{E_{ij}\subset\partial\Omega_i} (u(v_i)-u(v_j))^2 \quad \forall\, u \in V_0(\partial\Omega_i). \tag{8.4}$$

To see (8.4), for any $E_{ij} \subset \partial\Omega_i$, using the linearity of u on E_{ij} we obtain

$$\frac{|u(v_i)-u(v_j)|}{h_0} = \frac{|u(x)-u(y)|}{|x-y|}\frac{|x-y|}{h_0} \lesssim \frac{|u(x)-u(y)|}{|x-y|}, \quad \forall\, x,y \in E_{ij}, \tag{8.5}$$

then integrating in both sides over E_{ij} and using the defintion of $|\cdot|_{1/2,\partial\Omega_i}$ implies

$$(u(v_i)-u(v_j))^2 \lesssim \int_{E_{ij}}\int_{E_{ij}} \frac{(u(x)-u(y))^2}{|x-y|^2} ds(x)\, ds(y) \le |u|^2_{1/2,\partial\Omega_i},$$

therefore we have

$$\sum_{E_{ij}\subset\partial\Omega_i} (u(v_i)-u(v_j))^2 \lesssim |u|^2_{1/2,\partial\Omega_i}. \tag{8.6}$$

To get the reverse inequality of (8.6), we first subdivide Ω_i into two triangles by connecting one diagonal in the case that Ω_i is a quadrilatral, then we naturally extend u to a function $\hat{u}$ defined on Ω_i such that $\hat{u}$ is linear on each triangle. By Lemma 4.4,

$$|u|^2_{1/2,\partial\Omega_i} \lesssim |\hat{u}|^2_{1,\Omega_i},$$

combining the quasi-uniformity of the triangulation and the triangle inequality gives

$$|u|^2_{1/2,\partial\Omega_i} \lesssim \sum_{v_i,v_j\in\partial\Omega_i} (u(v_i)-u(v_j))^2 \eqsim \sum_{E_{ij}\subset\partial\Omega_i} (u(v_i)-u(v_j))^2,$$

this with (8.6) implies (8.4).

Standard coarse solver R_0. Let S_0 be the restriction of the interface operator S_h on the coarse subspace $V_0(\Gamma)$. In order to avoid using the exact solver S_0^{-1}, we can replace S_0 by any of its equivalent form. Here we consider one of such options. It follows from (8.4) that

$$\langle S_h u_0, u_0\rangle \eqsim \sum_{E_{ij}\subset\partial\Omega_i} \rho_i (u_0(v_i)-u_0(v_j))^2. \tag{8.7}$$

The standard coarse solver R_0 is therefore defined by

$$\left\langle R_0^{-1}u_0, v_0\right\rangle = \sum_{E_{ij}\subset\partial\Omega_i} \rho_i (u_0(v_i)-u_0(v_j))\,(v_0(v_i)-v_0(v_j)) \quad \forall\, u_0, v_0 \in V_0(\Gamma).$$

Thus for any given $g_0 \in V_0(\Gamma)$, $u_0 = R_0 g_0$ can be calculated as follows:

$$\sum_{E_{ij}\subset\partial\Omega_i} \rho_i (u_0(v_i)-u_0(v_j))\,(v_0(v_i)-v_0(v_j)) = \langle g_0, v_0\rangle \quad \forall v_0 \in V_0(\Gamma). \tag{8.8}$$

8. Coarse subspaces defined on the interface. To design domain decomposition algorithms for multiple subdomains, it proves important to introduce a global coarse space into the algorithm to avoid the deterioration of the algorthms as the number of subdomains becomes large. How to properly construct such a coarse subspace, denoted by V_0, is one of the major interest of domain decomposition algorithms. We devote this section to address various coarse subspaces and their properties.

The coarse space $V_0(\Gamma)$ will be viewed as the range of a properly defined linear operator I_0 on V_Γ, namely

$$V_0(\Gamma) = \text{range of } \ I_0 = I_0(V^h(\Gamma)).$$

The effective application of a coarse subspace in domain decomposition, as we shall observe, depends on its approximation property in L^2 norm and its stability property in $H^{1/2}$ norm. Namely it depends on the size of α_0 and α_1 (which usually depend on h and h_0) in the following estimates

$$\|u - I_0 u\|_{L^2_\rho(\Gamma)} \lesssim \alpha_0 h_0^{1/2} |u|_{H^{1/2}_\rho(\Gamma)}, \quad |I_0 u|_{H^{1/2}_\rho(\Gamma)} \lesssim \alpha_1 |u|_{H^{1/2}_\rho(\Gamma)} \tag{8.1}$$

where

$$|v|^2_{H^{1/2}_\rho(\Gamma)} = \sum_{i=1}^p \rho_i |v|^2_{1/2,\Gamma_i}, \quad \|v\|^2_{L^2_\rho(\Gamma)} = \sum_{i=1}^p \rho_i \|v\|^2_{0,\Gamma_i} \quad \forall\, v \in V^h(\Gamma).$$

To motivate the construction of an effective I_0, let us state the desired properties of this operator:

(C1) It is invariant on the joint-set $\mathcal{W}$.
(C2) It is invariant on constant functions.
(C3) Its bounds α_0 and α_1 in (8.1) are not large.
(C4) Its range is simple, and with relatively small degrees of freedom.

8.1. Standard coarse subspaces ($n = 2$). The most natural and effective coarse subspace of $V^h(\Gamma)$ in two dimension is the standard coarse subspace $V_0(\Gamma)$ generated by the subdomains $\{\Omega_i\}$. This coarse subspace $V_0(\Gamma)$ is defined to be the space consisting of all continuos functions on the interface Γ which are linear on each edge of all subdomains Ω_i. We refer to a simple example in § 8.2 for the ineffectiveness of the coarse sspace in three diemnsion.

Let I_0 be the nodal value interpolant associated with the coarse space $V_0(\Gamma)$. We have for any $u \in V^h(\Gamma)$ that

$$|I_0 u|^2_{1/2,\partial\Omega_i} \lesssim \log(h_0/h)\, |u|^2_{1/2,\partial\Omega_i}, \tag{8.2}$$

$$\|u - I_0 u\|^2_{0,\partial\Omega_i} \lesssim h_0(\log(h_0/h))|u|^2_{1/2,\partial\Omega_i}. \tag{8.3}$$

The inequality (8.3) is derived by the definition of $\|\cdot\|_{1/2}$, the vanishing of u at vertices of $\partial\Omega_i$ and Lemma 4.19:

$$\begin{aligned} h_0^{-1}\|u - I_0 u\|^2_{0,\partial\Omega_i} &\lesssim \|u - I_0 u\|^2_{1/2,\partial\Omega_i} \lesssim \sum_{E_{ij}\subset\partial\Omega_i} \|u - I_0 u\|^2_{H^{1/2}_{00}(E_{ij})} \\ &\lesssim \log^2(h_0/h)\, |u|^2_{1/2,\partial\Omega_i}. \end{aligned}$$

Here E_{ij} denotes the edge of $\partial\Omega_i$ connecting two vertices v_i and v_j.

2. *On the face* $F \subset \bar{\Omega}_1 \cap \bar{\Omega}_2$, u_H *solves*

$$\left\langle (-\Delta_{F,h})^{1/2} u_H, v_H) \right\rangle_{0,F} = (g, v) - A(u_P, v) \quad \forall\, v \in V^h(\Gamma).$$

3. *The component* u_H *in* Ω_i *for* $i = 1, 2$ *solves the mixed Neumann problem:* u_H *is given on* F *by Step 2 and*

$$A(u_H, v) = 0 \quad \forall v \in V_0^h(\Omega_i).$$

7.3. Neumann-Neumann algorithm $(n = 2, 3)$**.** The preconditioner B_h to be constructed is for the interface operator S_h. As $S_h = S_1 + S_2$, it is natural to define the so-called Neumann-Neumann preconsitioner B_h by

$$B_h = S_1^{-1} + S_2^{-1}.$$

For this preconditioner, we have

LEMMA 7.3.

$$\kappa(B_h S_h) \lesssim 1.$$

Proof. By Lemmas 4.9 and 4.6, we have for any $u \in V^h(\Gamma)$,

$$\langle S_1 u, u \rangle_{0,\Gamma_1} = A_1(\tilde{u}, \tilde{u}) \eqsim |u|^2_{1/2,\partial\Omega_1} \eqsim \|u\|^2_{H^{1/2}_{00}(\Gamma)} \eqsim |u|^2_{1/2,\partial\Omega_2} \lesssim \langle S_2 u, u \rangle . \tag{7.2}$$

Thus,

$$\begin{aligned}
\langle B_h S_h u, S_h u \rangle &= \left\langle (2I + S_1^{-1} S_2 + S_2^{-1} S_1) u, S_h u \right\rangle \ (B_h\text{'s definition}) \\
&= 3 \langle S_h u, u \rangle + \left\langle S_1^{-1} S_2 u, S_2 u \right\rangle + \left\langle S_2^{-1} S_1 u, S_1 u \right\rangle \ (S_h = S_1 + S_2) \\
&\eqsim 3 \langle S_h u, u \rangle + \langle S_2 u, u \rangle + \langle S_1 u, u \rangle \ (\text{by } (7.2)) \\
&\eqsim 4 \langle S_h u, u \rangle \ (S_h = S_1 + S_2),
\end{aligned}$$

that completes our proof. □

For a given $g \in V^h(\Gamma)$, let $u_i = S_i^{-1} g$ $(i = 1, 2)$, then $\langle S_i u_i, v \rangle = \langle g, v \rangle \quad \forall\, v \in V^h(\Gamma)$, which implies using the definition of S_i that

$$A_i(\tilde{u}_i, \phi) = \langle g, v \rangle \quad \phi \in V^h(\Omega_i).$$

Thus we have the following

ALGORITHM 7.3 (NEUMANN-NEUMANN ALGORITHM). *For a given* $g \in V^h(\Gamma)$, *let* $u = B_h g \equiv u_1 + u_2$, *then* u_1 *and* u_2 *can be obtained by solving independent mixed Neumann problems on two subdomains:* $u_i = \tilde{u}_i|_\Gamma$ *with* $\tilde{u}_i \in V^h(\Omega_i)$ *satisfying*

$$A_i(\tilde{u}_i, \phi) = \langle g, v \rangle \quad \phi \in V^h(\Omega_i).$$

7.4. Bibliograpic remarks. The Dirichlet-Neumann-Dirichlet method was proposed by Bramble-Pasciak-Schatz [11], whose ideas were then developed and extended to generate many important algorithms, e.g. Bramble-Pasciak-Schatz [10, 12, 13, 14].

The Neumann-Neumann algorithm was proposed by Bourgat-Glowinski-Tallec-Vidrascu [6] which is a standard variational extension of an earlier approach by Glowinski-Wheeler [52] within a mixed finite element framework for two subdomain case.

□

Now we state the algorithm for computing $M_h g$, for a given $g \in V^h$. Let $u = M_h g$. As above, we write $u = u_P + u_H$. Obviously, u solves the following problem:

$$A_1(u, v) + A_2(u_P, v_P) = (g, v) \quad \forall\, u, v \in V^h,$$

we then have the algorithm:

ALGORITHM 7.1 (DIRICHLET-NEUMANN-DIRICHLET ALGORITHM). *$u = M_h g = u_P + u_H$ can be calculated as follows:*

1. *The component u_P on Ω_2 solves the Dirichlet problem:*
   $u_P \in V_0^h(\Omega_2)$ and

$$A_2(u_P, \phi) = (g, \phi)_{0,\Omega_2},\ \forall\, \phi \in V_0^h(\Omega_2).$$

2. *The u on $\bar{\Omega}_1$ solves the mixed Neumann problem:*

$$A_1(u_1, \phi) = (g, \phi) - A_2(u_P, \phi),\ \forall\, \phi \in V^h(\Omega_1).$$

3. *The component u_H in Ω_2 solves the mixed Neumann problem:*
   $u_H - u_1 \in V_0^h(\Omega_2)$ and

$$A_2(u_H, \phi) = 0,\ \forall\, \phi \in V_0^h(\Omega_2).$$

7.2. Dirichlet-Dirichlet algorithm ($n = 2, 3$)**.** As in the last subsection, we split any u in V^h as $u = u_P + u_H$. Applying Lemmas 4.9, 4.7 and 4.14 to the discrete harmonic component u_H, we see

$$\begin{aligned} A(u_H, u_H) &\eqsim \sum_{i=1}^{2} |u_H|^2_{1/2,\partial\Omega_i} \lesssim \|u_H\|^2_{H_{00}^{1/2}(F)} \\ &\eqsim \left\langle (-\Delta_{F,h})^{1/2} u_H, u_H \right\rangle_{0,F} \lesssim \wp^2 A(u_H, u_H). \end{aligned} \tag{7.1}$$

From this, we can naturally define the preconditioner M_h for the stiffness operator A_h by

$$(M_h^{-1} u, v) = \tilde{A}(u, v) \quad \equiv \quad A(u_P, v_P) + \sum_F \left\langle (-\Delta_{F,h})^{1/2} u_H, v_H \right\rangle_{0,F},$$

and from (7.1) we have

THEOREM 7.2. *For any $u \in V^h$,*

$$\kappa(M_h A_h) \lesssim (1 + \log(H/h))^2.$$

We have the following algorithm:

ALGORITHM 7.2 (DIRICHLET-DIRICHLET ALGORITHM). *Let $u = M_h g = u_P + u_H$, for a given $g \in V^h$. The components u_P and u_H can be computed as follows:*

1. *The component u_P in Ω_i for $i = 1, 2$ solves the Dirichlet problems:*
   $u_P \in V_0^h(\Omega_i)$ and

$$A(u_P, v) = (g, v) \quad \forall v \in V_0^h(\Omega_i).$$

lie on the boundary $\partial\Omega$ (cf. Fig. 4). Let $\Omega = \Omega_1' \cup \Omega_2'$ and $\Gamma = \Omega_1' \cap \Omega_2'$. Here we allow each Ω_1' and Ω_2' to be the union of some subdomains from $\{\Omega_i\}_{i=1}^p$. But for ease of notation we still use Ω_i instead of Ω_i'. We will discuss three preconditioners, two for the stiffness operator A_h, one for the interface operator S_h.

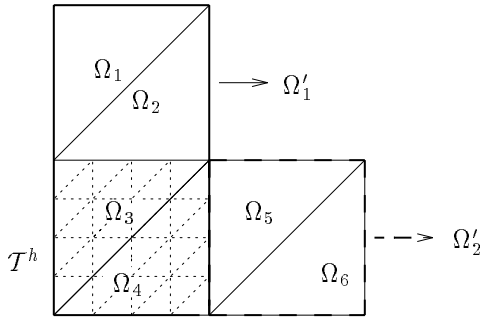

FIG. 4. *Subdomains* $\Omega_1' = \cup_{i=1}^4 \Omega_i$ *and* $\Omega_2' = \cup_{i=5}^6 \Omega_i$ *with simplicial* Ω_i*'s*

7.1. Dirichlet-Neumann-Dirichlet algorithm ($n = 2, 3$). As stated in the outline (§2), in order to find a preconditioner M_h for the stiffness operator A_h, with M_h usually corresponding to a bilinear form $\tilde{A}(\cdot,\cdot)$, we decompose V^h into two parts, i.e. for any $u \in V^h$, we can split $u = u_P + u_H$ such that $u_P \in V_P$ solves the Dirichlet problem on Ω_i for $i = 1, 2$:

$$A_i(u_P, \phi_i) = A_i(u, \phi_i),\ \forall\, \phi_i \in V_0^h(\Omega_i),$$

while u_H is discrete harmonic in Ω, or $u_H \in V_H$. We can then define the preconditioner M_h or the bilinear form $\tilde{A}(\cdot,\cdot)$ by

$$(M_h^{-1}u, v) = \tilde{A}(u, v) \equiv A_1(u, v) + A_2(u_P, v_P),\ \forall\, u, v \in V^h.$$

THEOREM 7.1. *We have*

$$\kappa(M_h A_h) \lesssim 1.$$

Proof. By Lemma ??, it suffices to prove

$$A(u, u) \eqsim \tilde{A}(u, u),\ \forall\, u \in V^h.$$

By Lemmas 4.6 and 4.10,

$$A_1(u_H, u_H) \eqsim \|u_H\|^2_{H_{00}^{1/2}(\Gamma)} \eqsim A_2(u_H, u_H).$$

Thus from this and the definition of $\tilde{A}(\cdot,\cdot)$,

$$\begin{aligned} A(u,u) &= A(u_P, u_P) + A(u_H, u_H) \\ &= A_1(u_P, u_P) + A_2(u_P, u_P) + A_1(u_H, u_H) + A_2(u_H, u_H) \\ &\eqsim A_1(u_P, u_P) + A_2(u_P, u_P) + A_1(u_H, u_H) \\ &\eqsim A_1(u,u) + A_2(u_P, u_P) = \tilde{A}(u,u). \end{aligned}$$

then applying Lemma 4.17 gives

$$\langle S_h u, u\rangle \lesssim h^{-1} \sum_{i=1}^{p} \|u\|^2_{0,\partial\Omega_i} = h^{-1} \langle u, u\rangle ,$$

this shows $\lambda_{max}(S_h) \lesssim h^{-1}$. Therefore we get $\kappa(S_h) = \lambda_{max}/\lambda_{min} \lesssim (h_0 h)^{-1}$. □

5.1. Bibliographic remarks. Lemma 5.2 was proved by Bjorstad-Widlund [5] in two dimensions for many subdomains case without crosspoints, and by Mansfield [70] in both two and three dimensions for general domain decompositions. Our proof is simpler.

6. Basic assumptions and notation. We will use the following notation throughout the paper:

$\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ – the interface of Ω_i;

$\Gamma = \cup_{i=1}^{p} \Gamma_i$ – interface among all the subdomains $\{\Omega_i\}$;

$\mathcal{W}_i$ – the wirebasket set of the subdomain Ω_i, whose points belonging to more than two subdomains;

E, F and v_k – edges, faces and vertices of subdomains Ω_i $(1 \le i \le p)$ resp.;

$(\cdot,\cdot)_{0,K}$ – the scalar product in $L^2(K)$, where K is a subset of Ω; we omit the subscript when $K = \Omega$;

$\langle\cdot,\cdot\rangle_{0,K}$ – the scalar product in $L^2(K)$, where K is a subset of Γ; we omit the subscript when $K = \Gamma$;

$\tilde{u}$ – the discrete harmonic extension of $u \in V^h(\Gamma)$, i.e. $A(\tilde{u}, v) = 0 \quad \forall\, v \in V_0^h(\Omega_i)$ $(1 \le i \le p)$;

$I_K^0 u$ – the finite element function belonging to V_Γ^h and equal to u at any node on K but vanishing at nodes of $(\Gamma \setminus K)$. Here K is any open subset of Γ and u any function in V_Γ^h; e.g., $I_F^0 u$ denotes the function belonging to V_Γ^h and equal to u at any node on F while vanishing at nodes on $(\Gamma \setminus F)$.

$V_0^h(K)$ – the subspace of V^h consisting of functions vanishing on $(\Omega \setminus K)$. Here K is any union of finite elements in $\mathcal{T}^h$.

$\gamma_K(w)$ – the average value of the function w on a set K, i.e. $\gamma_K(\omega) = (int_K \omega dx)/|\omega|$;

$\wp$ – the constant $1 + \log(h_0/h)$;

$r(\rho)$ – the coefficient ratio of the model problem, i.e. $r(\rho) = (\max_{1\le i\le p} \rho_i)/(\min_{1\le i\le p} \rho_i)$.

Assumptions on the subdomains. We make the following assumptions on the subdomains $\{\Omega_i\}_{i=1}^p$ throughout the paper:

(A1) For $n = 2$, each subdomain Ω_i is either a triangle or a quadrilateral.

(A2) For $n = 3$, all subdomains Ω_i are tetrahedra or all of them are hexahedra.

(A3) Each Ω_i is assumed to be the image of either a reference simplex G, or a unit n-cube G under an one-to-one mapping $T_i : G \to \Omega_i$, which satisfies

$$\Omega_i = T_i(G), \quad \|\partial T_i\| \lesssim 1, \quad \|\partial T_i^{-1}\| \lesssim 1.$$

Here ∂T_i is the Jacobian and $\|\cdot\|$ the Euclidean matrix norm.

7. Examples on two subdomains. To illustrate the main idea behind the more complicated domain decomposition on multiple subdomains discussed later, a brief discussion will be given here to some examples of algorithms on two subdomains for uniformly elliptic problems, i.e. we assume that the coefficients $\rho_i = 1$ on each subdomain Ω_i. We also assume that all vertices of each subdomain Ω_i for $i = 1, \cdots, p$

where the matrix $\mathcal{S}$ is called *Schur complement* of $\mathcal{A}$ with respect to $\mathcal{A}_{22}$, and

$$\mathcal{S} = \mathcal{A}_{22} - \mathcal{A}_{12}^t \mathcal{A}_{11}^{-1} \mathcal{A}_{12}.$$

Then we know that the condition number of the Schur compliment $\mathcal{S}$ is always less than the one of the matrix $\mathcal{A}$, i.e.

LEMMA 5.1.

$$\kappa(\mathcal{S}) \leq \kappa(\mathcal{A}).$$

Proof. Let $(\cdot,\cdot)$ be the Euclidean inner product either in R^{n_2} or in R^n. For any $\mu_2 \in R^{n_2} \neq 0$, let $\mu = (\mu_1, \mu_2) \in R^n$ with $\mu_1 = -\mathcal{A}_{11}^{-1}\mathcal{A}_{12}\mu_2$, then

$$(\mathcal{S}\mu_2, \mu_2) = (\mathcal{A}\mu, \mu),$$

that implies

$$\sup_{\mu_2 \neq 0} \frac{(\mathcal{S}\mu_2, \mu_2)}{(\mu_2, \mu_2)} \leq \sup_{\mu \neq 0} \frac{(\mathcal{A}\mu, \mu)}{(\mu, \mu)}, \quad \inf_{\mu_2 \neq 0} \frac{(\mathcal{S}\mu_2, \mu_2)}{(\mu_2, \mu_2)} \geq \inf_{\mu \neq 0} \frac{(\mathcal{A}\mu, \mu)}{(\mu, \mu)},$$

which proves Lemma 5.1. □

Stiffness operator. Let the operator $S_h : V^h(\Gamma) \to V^h(\Gamma)$, called as a *stiffness operator*, be defined as in (??), i.e.

$$\langle S_h u_H, v_H \rangle = A(u_H, v_H) \quad u_H, v_H \in V_H,$$

and V_H is the finite element space of discrete harmonic functions defined in §2. It is well-known that the condition number of the stiffness operator A_h is the order $O(h^{-2})$. Here the operator $A_h : V^h \to V^h$ is defined by

$$(A_h u, v) = A(u, v) \quad u, v \in V^h.$$

The next lemma will show that the condition lemma of the interface operator S_h is the order $O((h_0 h)^{-1})$.

LEMMA 5.2. *For the interface operator S_h, we have*

$$\kappa(S_h) = O(\frac{1}{h_0 h}).$$

Proof. For any $u \in V^h(\Gamma)$, we deduce by definition and Poincaré inequality that

$$\langle S_h u, u \rangle = A(\tilde{u}, \tilde{u}) \gtrsim |\tilde{u}|^2_{1,\Omega} \gtrsim \|\tilde{u}\|^2_{1,\Omega}, \tag{5.18}$$

but $\|\tilde{u}\|^2_{1,\Omega} = \sum_{i=1}^p (\|\tilde{u}\|^2_{0,\Omega_i} + |\tilde{u}|^2_{1,\Omega_i})$, thus applying the trace lemma 4.2 with $\varepsilon = h_0$, we obtain

$$\langle S_h u, u \rangle \geq h_0 \sum_{i=1}^p \|u\|^2_{0,\partial\Omega_i} = h_0 \langle u, u \rangle,$$

that implies $\lambda_{min}(S_h) \gtrsim h_0$. On the other hand, by Lemma 4.10 we have

$$\langle S_h u, u \rangle = A(\tilde{u}, \tilde{u}) \lesssim \sum_{i=1}^p |\tilde{u}|^2_{1,\Omega_i} \eqsim \sum_{i=1}^p |u|^2_{1/2,\partial\Omega_i},$$

Let K be ∂G, or the wirebasket set W of ∂G or one face F of ∂G. We define

$$\langle v, w\rangle_{h,K} = h^{\alpha} \sum_{x_i \in K_h} v(x_i) w(x_i)$$

and the corresponding norm $\|v\|_{h,K} = \langle v, v\rangle_{h,K}^{1/2}$. Here $\alpha = n-1$ if $K = \partial G$ or $K = F$; $\alpha = n-2$ if $K = W$.

Then by the quasi-uniformity of $\mathcal{T}^h$ and straightforward computations, we get

$$\|v\|_{h,K} \eqsim \|v\|_{0,K}, \forall\, v \in V^h(K). \tag{4.17}$$

Similarly to $\gamma_K(v)$, we define the discrete average value of v on K by $\gamma_{h,K}(v) = \langle v, 1\rangle_{h,K} / \langle 1, 1\rangle_{h,K}$.

LEMMA 4.20. *Let $n = 2, 3$. For any $v \in V^h(\partial G)$*

$$\begin{aligned}
&d^{-1/2}\|v - \gamma_{h,\partial G}(v)\|_{h,\partial G} \lesssim |v|_{1/2,\partial G} \lesssim h^{-1/2}\|v - \gamma_{h,\partial G} v\|_{h,\partial G},\\
&\|v - \gamma_{h,K}(v)\|_{h,K} \eqsim \|v - \gamma_K(v)\|_{0,K},\\
&|\gamma_{h,K}(v)| \lesssim d^{-\alpha/2}\|v\|_{0,K}, \quad |\gamma_K(v)| \lesssim d^{-\alpha/2}\|v\|_{0,K}.
\end{aligned}$$

Proof. The last two inequalities follow from the definition of the discrete average, Cauchy-Schwarz inequality and (4.17). Next we prove the first two relations. The minimization property of the average value in the constant space and (4.17) implies

$$\begin{aligned}
\|v - \gamma_{h,K}(v)\|_{h,K} &\leq \|v - \gamma_K(v)\|_{h,K} \lesssim \|v - \gamma_K(v)\|_{0,K}\\
&\leq \|v - \gamma_{h,K}(v)\|_{0,K} \lesssim \|v - \gamma_{h,K}(v)\|_{h,K},
\end{aligned}$$

hence $\|v - \gamma_{h,K}(v)\|_{h,K} \eqsim \|v - \gamma_K(v)\|_{0,K}$. The proof is completed by combining Lemma 4.18. □

4.6. Bibliographic remarks. The average nodal value interpolant was proposed by Scott-Zhang [79] and more general results than Lemma 4.8 was given there. Other similar interpolants can be found, e.g. in Clément[36]. The properties of the standard nodal value interpolant are discussed in many books, e.g. in Ciarlet [35], where most standard finite element convergence theory can be found.

Lemma 4.11 was proved by Bramble [7], Bramble-Pasciak-Schatz [10] and Xu [95]. Our proof follows the one by Xu [95].

5. Schur compliments. In this section, we discuss briefly the Schur compliment matrix for general SPD matrix and its interface operator form corresponding to the domain decomposition $\{\Omega_i\}_{i=1}^p$.

Schur compliment matrix. Let $\mathcal{A} \in R^{n\times n}$ be any SPD matrix, and we write blockwise as follows

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^t & \mathcal{A}_{22} \end{pmatrix}$$

where $\mathcal{A}_{12} \in R^{n_1\times n_2}$, $\mathcal{A}_{22} \in R^{n_2\times n_2}$. By one step of block LU factorization for $\mathcal{A}$, the resultant matrix is:

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{S} \end{pmatrix}$$

Proof. Let $\hat{v}$ be the function in $V^h(G)$ which vanishes at all interior nodes of G, but equals to v at the boundary nodes of G. Then we first use Lemma 4.5 and then the inverse inequality for $\hat{v}$ to obtain that

$$|v|^2_{1/2,\partial G} \lesssim d^{-2}\|\hat{v}\|^2_{0,G} + |\hat{v}|^2_{1,G} \lesssim h^{-2}\|\hat{v}\|^2_{0,G}.$$

Now using the quasi-uniformity of the mesh $\mathcal{T}^h$, we have

$$|v|^2_{1/2,\partial G} \lesssim h^{-1}\Big(\sum_{x_i\in\partial G_h} h^{n-1}v^2(x_i)\Big) \eqsim h^{-1}\|v\|^2_{0,\partial G}.$$

□

LEMMA 4.18. *Let $n = 2, 3$. For any $v \in V^h(\partial G)$,*

$$d^{-1/2}\|v - \gamma_{\partial G}(v)\|_{0,\partial G} \lesssim |v|_{1/2,\partial G} \lesssim h^{-1/2}\|v - \gamma_{\partial G}(v)\|_{0,\partial G}.$$

Proof. The second inequality follows directly from Lemma 4.17, while the first is obtained by Lemma 4.4 and Friedrichs inequality:

$$d^{-1/2}\|v - \gamma_{\partial G}(v)\|_{0,\partial G} \lesssim \|\tilde{v} - \gamma_{\partial G}(v)\|_{1,G} \lesssim |\tilde{v}|_{1,G} \lesssim |v|_{1/2,\partial G}$$

where $\tilde{v}$ is the $D(\cdot,\cdot)$-discrete harmonic extension of v with $D(u,v) = (\nabla u, \nabla v)_{0,G}$. □

LEMMA 4.19. *Let $n = 2$. For any $v \in V^h(\partial G)$ and any edge $E \subset \partial G$*

$$\|v - I_{\partial G}v\|_{H^{1/2}_{00}(E)} \lesssim (1 + \log\frac{d}{h})|v|_{1/2,\partial G}.$$

where $I_{\partial G}v$ equals to v on the corner nodes of ∂G and linear on each edge of ∂G.

Proof. Consider only $E = \{(x,0) : 0 \le x \le d\}$ and $0 \le x_0 < \ldots < x_k = d$ are the mesh points on E. Let $w = v - I_{\partial G}v$. Note $w(0) = 0$, we easily come to

$$\begin{aligned}\int_0^d \frac{|w(x)|^2}{x}dx &= \int_0^{x_1}\frac{|w(x)|^2}{x}dx + \int_{x_1}^d \frac{|w(x)|^2}{x}dx \\ &\lesssim \|\frac{\partial w}{\partial x}\|^2_{L^\infty(0,d)}\int_0^{x_1} x\,dx + \|w\|^2_{L^\infty(0,d)}\int_{x_1}^d \frac{dx}{x} \\ &\lesssim \|w\|^2_{L^\infty(0,d)}(1 + \log\frac{d}{h}).\end{aligned}$$

Analygously,

$$\int_0^d \frac{|w(x)|^2}{1-x}dx \lesssim \|w\|^2_{L^\infty(0,d)}(1 + \log\frac{d}{h}).$$

But by Lemma 4.12,

$$\|w\|_{L^\infty(0,d)} \lesssim (1 + \log\frac{d}{h})^{1/2}\|v\|_{1/2,\partial G},$$

the desired estimate then follows. □

Next we introduce the discrete L^2 inner product on ∂G.

Moreover

$$\int_{x_1}^{x_{k-1}} \frac{|I_F^0 u|^2}{x} dx \lesssim \|u\|^2_{L^\infty(F)} \int_{x_1}^{x_{k-1}} \frac{1}{x} dx \lesssim (1+\log\frac{d}{h})\|u\|^2_{L^\infty(F)}.$$

Consequently

$$\int_0^d \frac{|I_F^0 u|^2}{x} dx \lesssim (1+\log\frac{d}{h})\|u\|^2_{L^\infty(F)}.$$

Similarly,

$$\int_0^d \frac{|I_F^0 u|^2}{1-x} dx \lesssim (1+\log\frac{d}{h})\|u\|^2_{L^\infty(F)}.$$

Combining the last two inequalities with Lemma 4.12 gives the desired result. □

LEMMA 4.15. *Let $n = 2, 3$. For any $u \in V^h(G)$ and a face F of G,*

$$\|I_F^0 1\|^2_{H_{00}^{1/2}(F)} \lesssim d^{n-2}(1+\log\frac{d}{h}). \tag{4.16}$$

Proof. The conclusions come immediately by going through the proof of Lemma 4.14. □

LEMMA 4.16. *Let $n = 3$. For any $u \in V^h(G)$ vanishing on one edge of G, then*

$$\|u\|^2_{0,G} \lesssim d^2\,(1+\log\frac{d}{h})\,|u|^2_{1,G}.$$

Proof. The proof follows [45]. Without loss of generality, we assume $G = (0,d)^3$ and u vanishes on the edge with $x = y = 0$. Let $\Delta_s = G \cap \{z = s\}$. By the assumption, one knows that for $z \in (0,d)$, u vanishes at least at one point in Δ_z. So for any constant c,

$$\max_{(x,y)\in\Delta_z} |u(x,y,z)| \le 2 \max_{(x,y)\in\Delta_z} |u(x,y,z) - c|,$$

thus by Lemma 4.11 and Friedrichs' inequality (choosing $c = \gamma_G(u)$),

$$\begin{aligned}
\|u\|^2_{0,G} &\le d^2 \int_0^d \max_{(x,y)\in\Delta_z} |u(x,y,z)|^2 dz \le 4\,d^2 \int_0^d \max_{(x,y)\in\Delta_z} |u(x,y,z) - c|^2 dz \\
&\le 4d^2 \int_0^d (1+\log\frac{d}{h})\,\|u-c\|^2_{1,\Delta_z} dz \\
&= 4d^2\,(1+\log\frac{d}{h})\,\|u-c\|^2_{1,G} \lesssim d^2\,(1+\log\frac{d}{h})\,|u|^2_{1,G}.
\end{aligned}$$

□

The next lemma is the inverse inequality for finite element functions defined on the boundary of G.

LEMMA 4.17. *Let $n = 2, 3$. For any $v \in V^h(\partial G)$*

$$|v|_{1/2,\partial G} \lesssim h^{-1/2}\|v\|_{0,\partial G}.$$

Proof. We show for the case $G = (0,d)^n$. First we follow Lemma 4.3 in Bramble et al. [14] to prove for $n = 3$. By Lemma 4.6, we have

$$\begin{aligned} \|I_F^0 u\|^2_{H^{1/2}_{00}(F)} &\eqsim |I_F^0 u|^2_{1/2,F} + \int_F \frac{I_F^0 u^2(x)}{\text{dist }(x,\partial F)} ds(x) \\ &\equiv I_1 + I_2. \end{aligned} \tag{4.14}$$

Using $u = \sum_{F' \subset \partial G} I^0_{F'} u + \sum_{E \subset \partial G} I^0_E u + \sum_{v_k \in \partial G} I^0_{v_k} u$ on ∂G, it is readily to see by the definition of $|\cdot|_{1/2,\partial G}$ and Lemma 4.13 that

$$\begin{aligned} |I_F^0 u|^2_{1/2,\partial G} &\lesssim |\sum_{F' \subset \partial G} I^0_{F'} u|^2_{1/2,\partial G} \\ &\lesssim |u|^2_{1/2,\partial G} + \sum_E |I^0_E u|^2_{1/2,\partial G} + \sum_{v_k} |I^0_{v_k} u|^2_{1/2,\partial G} \\ &\lesssim (1 + \log \frac{d}{h}) \|u\|^2_{1/2,\partial G}. \end{aligned} \tag{4.15}$$

Thus I_1 is bounded as required. It remains to estimate I_2. Due to the similarity, we consider a specific face, say the face $F = G \cap \{z = 0\}$ and the term

$$\begin{aligned} \int_F \frac{(I_F^0 u)^2(x,y,0)}{x} dxdy &= \int_0^d \int_0^{x_1} \frac{(I_F^0 u)^2(x,y,0)}{x} dxdy + \int_0^d \int_{x_1}^d \frac{(I_F^0 u)^2(x,y,0)}{x} dxdy \\ &\equiv I_3 + I_4. \end{aligned}$$

Here x_1 is the first node on x axis. For I_3, let $\Delta_s = G \cap \{y = s\}$ and $v = I_F^0 u - \gamma_G(u)$, then

$$\begin{aligned} I_3 &\lesssim h^2 \int_0^d \|\frac{\partial v(\cdot,y,0)}{\partial x}\|^2_{L^\infty(0,x_1)} dy \lesssim \int_0^d \|v(\cdot,y,0)\|^2_{L^\infty(0,x_1)} dy \text{ (inverse ineq)} \\ &\lesssim (1 + \log \frac{d}{h}) \int_0^d \|\tilde{v}(\cdot,y,\cdot)\|^2_{1,\Delta_y} dy \text{ (Lemma 4.11 for } \Delta_y) \\ &\lesssim (1 + \log \frac{d}{h}) \|\tilde{v}\|^2_{1,G} \lesssim (1 + \log \frac{d}{h})^2 \|u\|^2_{1/2,\partial G} \text{ (Lemma 4.10(a) \& (4.15))}. \end{aligned}$$

For I_4, by direct computing and Lemmas 4.11 and 4.13, we obtain

$$\begin{aligned} I_4 &\lesssim \log \frac{d}{h} \int_0^d \|I_F^0 u(\cdot,y,0)\|^2_{L^\infty(0,1)} dy \\ &\lesssim \log \frac{d}{h} \Big(\int_0^d \|u(\cdot,y,0)\|^2_{L^\infty(0,1)} dy + \int_0^d u^2(0,y,0) dy + \int_0^d u^2(1,y,0) dy \Big) \\ &\lesssim (1 + \log \frac{d}{h})^2 \|u\|^2_{1/2,\partial G}. \end{aligned}$$

Hence I_2 is also bounded as required that ends the proof of $n = 3$.

Next we prove for $n = 2$. We assume $F = \{(x,0) : 0 \le x \le d\}$ and $0 = x_0 < \ldots < x_k = d$ are the mesh points on F. By the mean value theorem and inverse inequality,

$$\int_0^{x_1} \frac{|I_F^0 u|^2}{x} dx \le x_1^2 \, |I_F^0 u|^2_{W^{1,\infty}(F)} \lesssim \|u\|_{L^\infty(F)}, \quad \int_{x_{k-1}}^d \frac{|I_F^0 u|^2}{x} \lesssim \|u\|_{L^\infty(F)}.$$

LEMMA 4.12. *Let $n = 2$. For any $u \in V^h(\partial G)$,*

$$\|u\|^2_{L^\infty(\partial G)} \lesssim (1 + \log \frac{d}{h})(d^{-1}\|u\|_{0,\partial G} + |u|^2_{1/2,\partial G}).$$

Furthermore, if $\gamma_{\partial G}(u) = 0$, then

$$\|u\|_{L^\infty(\partial G)} \lesssim (1 + \log \frac{d}{h})^{1/2}|u|_{1/2,\partial G}.$$

Proof. Let $\tilde{u}$ be the discrete harmonic extension of u in G with respect to $D(\cdot,\cdot)$. Here $D(u,v) = d^{-2}(u,v)_{0,G} + (\nabla u, \nabla v)_{0,G}$. Then by Lemma 4.11, we obtain

$$\|u\|^2_{L^\infty(\partial G)} \lesssim \|\tilde{u}\|^2_{L^\infty(G)} \lesssim (1 + \log \frac{d}{h})D(\tilde{u},\tilde{u}),$$

which combining with Lemma 4.10 give the desired results. □

LEMMA 4.13. *Let E be any edge of ∂G ($n = 3$) or any vertex of ∂G($n = 2$) and E_h be the set of nodes belonging to E. Then for any $u \in V^h(\partial G)$,*

$$\|I^0_E u\|^2_{1/2,\partial G} \lesssim h^{n-2} \sum_{x_i \in E_h} u^2(x_i) \eqsim \|u\|^2_{0,E} \lesssim (1 + \log \frac{d}{h})\|u\|^2_{1/2,\partial G}. \tag{4.12}$$

Proof. Let $\hat{v}$ be the function in $V^h(G)$ which equals to u at interior nodes of E but vanishes at other nodes on $\bar{G}$. Then we first use Lemma 4.5 and then the inverse inequality for $\hat{v}$ and $h \lesssim d$ to obtain that

$$\|I^0_E v\|^2_{1/2,\partial G} \quad \lesssim \quad d^{-2}\|\hat{u}\|^2_{0,G} + |\hat{u}|^2_{1,G} \lesssim h^{-2}\|\hat{u}\|_{0,G}.$$

Now using the quasi-uniformity of the mesh $\mathcal{T}^h$, we have

$$\|I^0_E v\|^2_{1/2,\partial G} \lesssim h^{n-2} \sum_{x_i \in E_h} u^2(x_i) \eqsim \|u\|^2_{0,E},$$

hence we proved (4.12) except the last inequality.

For the last inequality of (4.12): the case $n = 2$ is a direct consequence of Lemmas 4.12 and for $n = 3$ it suffices to show by Lemma 4.11 that

$$\|I^0_E u\|^2_{0,e} \lesssim (1 + \log(h^{-1}))(d^{-2}\|\tilde{u}\|^2_{0,G} + |\tilde{u}|^2_{1,G}) \tag{4.13}$$

where $\tilde{u}$ is the discrete harmonic extension of $u \in V^h(\partial G)$ into G. Without loss of generality, assume $G = (0,1)^3$ and consider a specific edge of G, say the edge E corresponding to $x = y = 0$. Lemma 4.11 implies that

$$\int_0^1 u^2(0,0,z)\,dz \lesssim (1 + \log(h^{-1})) \int_0^1 \|\tilde{u}\|^2_{1,\Delta_z} dz \lesssim (1 + \log(h^{-1}))\|\tilde{u}\|^2_{1,G},$$

where $\Delta_s = G \cap \{z = s\}$, that gives (4.13). □

LEMMA 4.14. *Let F be a face of ∂G (n=3) or an edge of ∂G (n=2). Then*

$$\|I^0_F u\|_{H^{1/2}_{00}(F)} \lesssim (1 + \log \frac{d}{h})\|u\|_{1/2,\partial G}.$$

LEMMA 4.9. *Let $D(u,v) = (\nabla u, \nabla v)_{0,G}$, for any $u, v \in H^1(G)$. Assume that $u^h \in V^h(G)$ is $D(\cdot,\cdot)$-discrete harmonic, i.e. $D(u^h, v) = 0, \forall\, v \in V_0^h(G)$. Then*

$$|u^h|_{1,G} = \inf_{v - u^h \in V_0^h(G)} |v|_{1,G} \eqsim |u^h|_{1/2,\partial G}. \tag{4.9}$$

Proof. The equality in (4.9) follows immediately from the orthogonality: $D(u^h, v - u^h) = 0,\ \forall v \in V^h(G) : v - u^h \in V_0^h(G)$. Let $U \in H^1(G)$ be $D(\cdot,\cdot)$-harmonic in G, then by Lemma 4.4, we have $|U|_{1,G} \eqsim |u^h|_{1/2,\partial G}$. Now use the operator Π_h defined in (4.8) and Lemma 4.8(d) to conclude that

$$\begin{aligned} |u^h|_{1,G} &= \inf_{v - u^h \in V_0^h(G)} |v|_{1,G} \lesssim |\Pi_h U|_{1,G} \\ &\lesssim |U|_{1,G} \eqsim |u^h|_{1/2,\partial G}. \end{aligned}$$

□

LEMMA 4.10. *Let $D(u,v) = d^{-2}(u,v)_{0,G} + (\nabla u, \nabla v)_{0,G}$, for any $u, v \in H^1(G)$. Assume that $u^h \in V^h(G)$ is $D(\cdot,\cdot)$-discrete harmonic, i.e. $D(u^h, v) = 0, \forall\, v \in V_0^h(G)$. Then*

$$D(u^h, u^h) = \inf_{v - u^h \in V_0^h(G)} D(v,v) \eqsim d^{-1}\|u^h\|_{0,\partial G}^2 + |u^h|_{1/2,\partial G}^2. \tag{4.10}$$

If in addition, the mean value $\gamma_{\partial G}(u^h)$ of u^h on ∂G vanishes or u^h vanishes on one face of G, then

$$|u^h|_{1,G} \eqsim |u^h|_{1/2,\partial G}. \tag{4.11}$$

Proof. (4.10) can be proved in the same way as for Lemma 4.9 but with the help of Lemmas 4.5 and Lemma 4.8 (c)-(d), while (4.11) follows readily from (4.10) and the Friedrichs inequality. □

It is known that $H^1(G)$ can not be imbedded into $C^0(\bar{G})$. But for finite element functions, we have the following lemma:

LEMMA 4.11. *Let $n = 2$. For any $u \in V^h(G)$,*

$$\|u\|_{L^\infty(G)}^2 \lesssim (1 + \log \frac{d}{h})(d^{-2}\|u\|_{0,G}^2 + |u|_{1,G}^2).$$

Proof. By scaling, it suffices to prove for the diameter $d = 1$. We know the following inequality, cf. Corollary 2.1 in Xu [95],

$$\|u\|_{L^\infty(G)} \lesssim |\log \varepsilon|^{1/2}\|u\|_{1,G} + \varepsilon\|u\|_{W^{1,\infty}(G)},\ \forall\, u \in W^{1,\infty}(G).$$

Then for any $u \in V^h(G)$, taking $\varepsilon = h$ and using the inverse inequality for finite element functions gives

$$\|u\|_{L^\infty(G)}^2 \lesssim |\log \varepsilon|\|u\|_{1,G}^2 + \varepsilon^2\, h^{-2}\|u\|_{1,G}^2 \lesssim (1 + \log(h^{-1}))\|u\|_{1,G}^2.$$

□

The next lemma is an analogue to Lemma 4.11 on the boundary of G.

4.3. Preconditioner for the discrete $H_{00}^{1/2}$ norm. The most basic result concerning the $H_{00}^{1/2}$ norm for finite element functions is as follows.

LEMMA 4.7.

$$\|v\|_{H_{00}^{1/2}(G)} \eqsim ((-\Delta_{G,h})^{1/2} v, v)_{0,G} \quad \forall v \in V_0^h(G). \tag{4.6}$$

Proof. By Lemma 4.5, it suffices to prove that

$$\|(-\Delta)^{1/4} v\|_{0,G} \eqsim \|(-\Delta_{G,h})^{1/4} v\|_{0,G} \quad \forall v \in V_0^h(G).$$

But this estimate is well-known, see Bank-Dupont [2], Xu [95], Xu [97]. □

4.4. Average interpolant. We introduce the average nodal value interpolant which not only keeps all approximation properties of the standard nodal value interpolant, but it can be also applied to functions with much less regularity than the standard interpolant requires.

Given $x_i \in \mathcal{T}^h$, let τ_i be an $n-1$ simplex from the triangulation $\mathcal{T}^h$ with vertices z_l $(l = 1, \cdots, n)$ such that $z_1 = x_i$. The choice of τ_i is not unique, but if $x_i \in \partial\Omega$, we take $\tau_i \subset \partial\Omega$. Let $\theta_i \in \mathcal{P}_1(\tau_i)$ be the unique function satisfying

$$(\theta_i, \lambda_l)_{0,\tau_i} = \delta_{l1}, \quad l = 1, \cdots, n$$

where λ_l is the barycentric coordinate of τ_i with respect to z_l. It is easy to see that

$$|(\theta_i, v)_{0,\tau_i}| \lesssim h^{1-n} \int_{\tau_i} |v|\, dx \tag{4.7}$$

and

$$(\theta_i, v)_{0,\tau_i} = v(x_i) \quad \text{if } v \in \mathcal{P}_1(\tau_i).$$

The average nodal vaule interpolant Π_h is then defined by

$$(\Pi_h v)(x) = \sum_{i=1}^{N_h} (\theta_i, v)_{0,\tau_i} \phi_i(x). \tag{4.8}$$

Let $\bar{V}^h \subset H^1(\Omega)$ be a finite element space consisting of piecewise linear functions with respect to the triangulation $\mathcal{T}^h$. We have

LEMMA 4.8. *The operator Π_h defined above satisfies*

- ***(a)*** $\Pi_h : H^1(\Omega) \to \bar{V}^h$ *and* $\Pi_h : H_0^1(\Omega) \to V^h$.
- ***(b)*** $(I - \Pi_h)v \in H_0^1(\Omega)$ *if* $v|_{\partial\Omega} \in \bar{V}^h(\partial\Omega)$.
- ***(c)*** $|(I - \Pi_h)v|_t \lesssim h^{s-t}|v|_s \qquad \forall\, v \in H^s(\Omega)$ $(s = 1, 2, t = 0, 1)$.
- ***(d)*** $|\Pi_h v|_1 \lesssim |v|_1 \qquad \forall\, v \in H^1(\Omega)$.

4.5. Norm estimates and equivalencies for finite element functions. This section will focus on all norm bounds and norm equivalencies for finite element functions. Let $V^h(G)$ be any Lagrangian finite element space (cf. Ciarlet [35]) defined on a quasi-uniform simplicial triangulation of G with each simplicial element having a diameter of size h.

The first lemma describes the norm relations of the space $H^1(G)$ and the trace space $H^{1/2}(\partial G)$ for finite element functions.

Sobolev space $H_{00}^{1/2}(F)$**.** For a face F of G, we define a special space

$$H_{00}^{1/2}(F) = \{v \in L^2(\partial G);\ supp\ v \subset F,\quad \hat{v} \in H^{1/2}(\partial G)\}$$

where $\hat{v}$ is the zero extensin of v onto ∂G, and its norm

$$\|v\|_{H_{00}^{1/2}(F)} = (d^{-1}\|\hat{v}\|_{0,\partial G}^2 + |\hat{v}|_{1/2,\partial G}^2)^{1/2}.$$

We have

LEMMA 4.6. *Let $u \in H^{1/2}(\partial G)$ and supp $u \subset F$, F being a face of G. Then*

$$H_{00}^{1/2}(F) \overline{\underset{\sim}{=}} \Big(\int_F \int_F \frac{(u(x)-u(y))^2}{|x-y|^n} ds(x) ds(y) + \int_F \frac{u^2(x)}{dist\ (x, \partial F)} ds(x) \Big)^{1/2}.$$

Proof. Because of the similarity, we prove only for $n = 3$ and $G = (0,d)^3$. By the assumption, we see the double integral in the norm $\|u\|_{H_{00}^{1/2}(F)}$ can be reduced into

$$|u|_{1/2,\partial G}^2 = \int_F \int_F \frac{(u(x)-u(y))^2}{|x-y|^3} ds(x) ds(y) + 2 \int_F \int_{\partial G \setminus F} \frac{u^2(x)}{|x-y|^3} ds(x).$$

By straightforward computations, we derive

$$\int_{\partial G \setminus F} \frac{1}{|x-y|^3} ds(x) \overline{\underset{\sim}{=}} \sum_{l=1}^{3} \frac{1}{\text{dist}\ (y, e_l)} \overline{\underset{\sim}{=}} \frac{1}{\text{dist}\ (y, \partial F)}$$

where e_1, e_2 and e_3 are three edges of the face F. So we see from above that

$$|u|_{1/2,\partial G}^2 \overline{\underset{\sim}{=}} \|u\|_{H_{00}^{1/2}(F)}^2.$$

Then Lemma 4.6 follows easily. □

Interpolation property of $H_{00}^{1/2}$ ***space.*** Let $-\Delta_G : H_0^1(G) \mapsto H^{-1}(G)$ be the Laplacian operator. It is well-known that

$$\|v\|_{1,G} \overline{\underset{\sim}{=}} \|(-\Delta_G)^{1/2} v\|_{0,\Omega} \quad \forall v \in H_0^1(G). \tag{4.4}$$

For the space $H_{00}^{1/2}(G)$, the following equivalence holds:

$$\|v\|_{H_{00}^{1/2}(G)} \overline{\underset{\sim}{=}} \|(-\Delta_G)^{1/4} v\|_{0,\Omega} \quad \forall v \in H_{00}^{1/2}(G). \tag{4.5}$$

The above result is well-know for smooth domains (cf. Lions-Magenes [63]). The justification of (4.5) depends on a Sobolev extension theorem for the space $H^{1/2}$. A complete proof of a general fractional order Sobolev extension theorem for a Lipschitz domain can be found in DeVore-Sharpley [40], although a proof for $H^{1/2}$ space can be in principle obtained by using local Lipschitz charts. With the Sobolev extension theorem in hand, the proof of (4.5) is easy. For example, a technique in Bramble [8] can be naturally modified to compose a simple proof for (4.5).

By the well-known equivalence between Hilbert scale and real method of interpolation (cf. Lions-Magenes [63]), (4.5) is equivalent to the statement that $H_{00}^{1/2}(G)$ is the interpolated space half way between $H_0^1(G)$ and $L^2(G)$ spaces. And this fact is widely used in the literature and will also be used in this paper later.

The next two lemmas give more accurate relations between the Sobolev space $H^1(G)$ and the trace space $H^{1/2}(\partial G)$.

LEMMA 4.4. *Let $D(u,v) = (\nabla u, \nabla v)_{0,G}$, for any $u, v \in H^1(G)$. Assume that $u \in H^1(G)$ is $D(\cdot,\cdot)$-harmonic, i.e. $D(u,v) = 0, \forall\, v \in H_0^1(G)$. Then*

$$|u|_{1,G} = \inf_{v-u \in H_0^1(G)} |v|_{1,G} \eqsim |u|_{1/2,\partial G}. \tag{4.1}$$

Proof. The equality in (4.1) follows from the orthogonality that $D(u, v-u) = 0, \forall\, v - u \in H_0^1(G)$:

$$D(u,u) \le D(u,u) + D(v-u, v-u) = D(v,v).$$

For the equivalancy in (4.1), first using a priori estimates for the Dirichlet problem (cf. Grisvard [56]):

$$u = u|_{\partial G}, \quad D(u,v) = 0, \ \forall\, v \in H_0^1(G),$$

we know that

$$|u|_{1,G}^2 \lesssim \|u\|_{1/2,\partial G}^2 = d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2,$$

from which we obtain by replacing u by $u - \gamma_{\partial G}(u)$ that

$$|u|_{1,G}^2 \lesssim d^{-1}\|u - \gamma_{\partial G}(u)\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2. \tag{4.2}$$

Now let $d = 1$ and it is easy to verify by contradiction that $\|u - \gamma_{\partial G}(u)\|_{0,\partial G}^2 \lesssim |u|_{1/2,\partial G}^2$, which implies by scaling that for $d \neq 1$, $d^{-1}\|u - \gamma_{\partial G}(u)\|_{0,\partial G}^2 \lesssim |u|_{1/2,\partial G}^2$, thus combining with (4.2) gives

$$|u|_{1,G} \lesssim |u|_{1/2,\partial G}.$$

On the other hand, by the trace theorem 4.3,

$$|u|_{1/2,\partial G}^2 \lesssim \|u\|_{1/2,\partial G}^2 \lesssim d^{-2}\|u\|_{0,G}^2 + |u|_{1,G}^2,$$

which together with Friedrichs inquality yields

$$|u|_{1/2,\partial G} \lesssim |u|_{1,G}.$$

Therefore we proved $|u|_{1/2,\partial G} \eqsim |u|_{1,G}$. ∎

LEMMA 4.5. *Let $D(u,v) = d^{-2}(u,v)_{0,G} + (\nabla u, \nabla v)_{0,G}$, for $u, v \in H^1(G)$. Assume that $u \in H^1(G)$ is $D(\cdot,\cdot)$-harmonic, i.e. $D(u,v) = 0, \forall\, v \in H_0^1(G)$. Then*

$$D(u,u) = \inf_{v-u \in H_0^1(G)} D(v,v) \eqsim d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2 \tag{4.3}$$

Proof. The lemma can be proved in the same way as for Lemma 4.4, but no need for Friedrichs inequality here. ∎

LEMMA 4.1. *Let $\gamma(u)$ be either $\gamma_\Omega(u)$ or $\gamma_{\Gamma_0}(u)$, with $\gamma_\Omega(u)$ and $\gamma_{\Gamma_0}(u)$ the averages of u on Ω and $\Gamma_0 \subset \partial\Omega$ resp., and meas$(\Gamma_0) > 0$. Then*

$$\begin{aligned} \|u\|_{0,\Omega} &\lesssim dist(\Omega)\,|u|_{1,\Omega}, \forall\, u \in H_0^1(\Omega), \quad \textit{(Poincaré inequality)} \\ \|u-\gamma(u)\|_{0,\Omega} &\lesssim dist(\Omega)\,|u|_{1,\Omega}, \forall\, u \in H^1(\Omega), \quad \textit{(Friedrichs inequality).} \end{aligned}$$

The next is a trace theorem, the proof follows Grisvard [56].

LEMMA 4.2. *For any $u \in H^1(G)$ and $\varepsilon \in (0,1)$,*

$$\|u\|_{0,\partial G} \lesssim \varepsilon^{-1}\,\|u\|_{0,G} + \varepsilon\,|u|_{1,G}.$$

Proof. By the assumption on the boundary ∂G, there exists a constant δ depending only on the Lipschitz parameter of the boundary ∂G and a function $\vec{\mu} \in (C^1(\bar{G}))^n$ (cf. Lemma 1.5.1.9 in [56]) such that

$$\vec{\mu}(x)\cdot\vec{n}(x) \geq \delta \quad a.e.\ on\ \partial G.$$

Using this, we get

$$\begin{aligned} \int_{\partial G} u^2 dx &\lesssim \int_{\partial G} u^2 \vec{\mu}(x)\cdot\vec{n}(x) dx \\ &= \int_G u^2 \mathrm{div}\vec{\mu} dx + \int_G 2u\vec{\mu}\cdot\nabla u dx \ \text{(Green's formula)} \\ &\lesssim \int_G u^2 dx + \int_G |u|\,|\nabla u| dx \ (\text{by } \vec{\mu} \in (C^1(\bar{G}))^n) \\ &\lesssim (1+\varepsilon^{-2})\int_G u^2 dx + \varepsilon^2 \int_G |\nabla u|^2 dx \ (\text{by } ab \leq \varepsilon a^2 + b^2/(4\varepsilon)). \end{aligned}$$

□

4.2. Sobolev spaces $H^{1/2}$ and $H_{00}^{1/2}$. Because of the extraordinary importance of these two spaces, in this subsection, we shall give a careful description for them.

Sobolev space $H^{1/2}(\partial G)$**.** Define the trace Sobolev space $H^{1/2}(\partial G)$ by

$$H^{\frac{1}{2}}(\partial G) = \{u \in L^1(\partial G);\ d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2 < \infty\}$$

where, let ds denote the surface element on ∂G,

$$|u|_{1/2,\partial G}^2 = \int_{\partial G}\int_{\partial G} \frac{(u(x)-u(y))^2}{|x-y|^n} ds(x) ds(y), \quad \|u\|_{0,\partial G}^2 = \int_{\partial G} u^2 ds,$$

and the norm of $H^{\frac{1}{2}}(\partial G)$ is always understood as $(d^{-1}\|u\|_{0,\partial G}^2 + |u|_{1/2,\partial G}^2)^{1/2}$.

The following theorem is a spacial case of a result by Gagliardo stating the relation between the trace space $H^{1/2}(\partial G)$ and the space $H^1(G)$ (cf. Theorem 1.5.1.3 in Grisvard [56]).

THEOREM 4.3. *The mapping $u \to u|_{\partial G}$ which is defined for $u \in C^{0,1}(\bar{G})$, has a unique continuous extension as an operator from $H^1(G)$ onto $H^{1/2}(\partial G)$. This operator has a right continous inverse.*

by Sobolev [87], Morgenstern [72], Babuška [1], Courant-Hilbert [37], Kang-Sun-Chen [59]. The first fundamental variational formulation of the overlapping Schwarz method was done by Lions [64, 65, 66, 67]. More quantitative convergence analysis for special shaped domains was studied by Kang [47], Tang [92, 91] and Chan-Hou-Lions [34]. Recently the additive Schwarz methods were showed to be very efficient as well for solving elliptic or parabolic problems on unstructured meshes which don't require any matching between coarse and fine grids (cf. Chan-Zou [31, 32, 33], Chan-Smith-Zou [30] and Cai [21].

4. Preliminary II: Sobolev spaces and finite element spaces. In contrast to the algebraic nature of the previous section, this section is devoted to the analytical tools for analyzing domain decomposition methods. Some basic facts on Sobolev spaces and finite element spaces will be carefully reviewed here.

We shall use the notation G to denote a reference domain (polygonal for $n = 2$ and polyhedral for $n = 3$) unless otherwise specified, with each edge length of d instead of the unit length as usual. For simplicity, we assume that G is a n-dimensional simplex, or a n-cube with each edge length of d, the boundary ∂G consisting of faces $\{F\}$, edges $\{E\}$ and vertices $\{v\}$.

The domain G will play the role of a general subdomain in a domain decomposition method, all the inequalities of this section will be given the explicit dependence on the diameter d of G. This is done by first establishing the inequality for $d = 1$ and the general result is then obtained by a scaling. We have also implicitly used the fact that the constants in certain Sobolev inequalities for a unit size domain will be uniform for a reasonable class of such domains.

4.1. Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. For any open bounded domain Ω in R^3, we define $H^1(\Omega)$ by

$$H^1(\Omega) = \{u \in L^2(\Omega); \nabla u \in (L^2(\Omega))^n\}$$

with its semi-norm and norm defined as

$$|u|_{1,\Omega} = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}, \quad \|u\|_{1,\Omega} = (\|u\|_{0,\Omega}^2 + |u|_{1,\Omega}^2)^{1/2}$$

where $\|u\|_{0,\Omega} = (\int_\Omega u^2 dx)^{1/2}$, $\nabla u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$, and the derivatives are to be understood in the sense of distributions (cf. Lions-Magenes [63]), and $|\nabla u|$ is the Euclidean norm of ∇u in R^n.

The Sobolev space $H_0^1(\Omega) \subset H^1(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{1,\Omega}$, $C_0^\infty(\Omega)$ consisting of infinitely differentiable functions with compact support in Ω.

We will also mention the Sobolev space $W^{1,\infty}(\Omega)$, which is defined by

$$W^{1,\infty}(\Omega) = \{u \in L^\infty(\Omega);\ \nabla u \in (L^\infty(\Omega))^n\}$$

with its norm defined by

$$\|u\|_{W^{1,\infty}(\Omega)} = \max_{1\le i\le n} \|\partial u/\partial x_i\|_{L^\infty(\Omega)}, \quad \|v\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|.$$

The following lemma states the well-known Poincaré and Friedrichs inequalities.

a linear operator $Q : V \mapsto \mathcal{W}$ *and positive constants* $\alpha_0, \alpha_1, \lambda_0, \lambda_1, \beta_0, \beta_1$ *and* γ_0 *that satisfy, for all* $v \in V$ *and* $w \in \mathcal{W}$

$$\alpha_0 \rho_A^{-1}(v,v) \leq (Rv,v) \leq \alpha_1 \rho_A^{-1}(v,v), \lambda_0 [w,w]_S \leq [TSw,w]_S \leq \lambda_1 [w,w]_S$$

and

$$\|Qv\|_S^2 \leq \beta_0^{-1} \|v\|_A^2, \ \|\Pi w\|_A^2 \leq \beta_1 \|w\|_S^2, \quad \textit{and} \quad \|(I - \Pi Q)v\|^2 \leq \gamma_0^{-1} \rho_A^{-1} \|v\|_A^2$$

where ρ_A *is the spectral radius of* A. *let* $\Pi^t : V \mapsto \mathcal{W}$ *be given by* $[\Pi^t v, w] = (v, \Pi w)$ *for* $v \in V, w \in \mathcal{W}$. *Then the preconditioner given by*

$$B = R + \Pi T \Pi^t \tag{3.7}$$

admits the following estimate

$$\kappa(BA) \leq (\alpha_1 + \beta_1 \lambda_1)((\alpha_0 \gamma_0)^{-1} + (\beta_0 \lambda_0)^{-1}).$$

In particular, if Q *is a right inverse of* Π *namely* $\Pi Q = I$, *then*

$$\kappa((\Pi T \Pi^t)A) \leq \frac{\beta_1}{\beta_0} \frac{\lambda_1}{\lambda_0}.$$

The last estimate in the above theorem corresponds to the " *Fictitious Space Lemma*" (see Nepomnyaschikh [**?**]). In this case, the space $\mathcal{W}$ has to be at least as rich as the original space V and the constructin of Π needs more caution.

Of course, we may consider, as a direct generalization of the parallel subspace correction method, more general preconditioner such as $B = \sum_i \Pi_i R_i \Pi_i^t$; but we will not address it here.

The above theorem was applied in Xu [**?**] to analyze a preconditioner for general unstructured grids (see (**??**) below). There are also some other obvious applications. For example, we may take V to be a higher order finite element space and $\mathcal{W}$ to be a lower order space, or V to be a conforming space and $\mathcal{W}$ to be a nonconforming space. For fourth order (such as biharmonic) equations, as another example, we may take V to be the fifth order conforming elements and $\mathcal{W}$ to be the second order nonconforming Morley elements. Roughly speaking, any two pair of reasonable finite element spaces can be used to precondition one aonther; of course, only the simpler ones are used to precondition the more complicated ones.

3.5. Bibliographic remarks. The CG method was proposed by Hestenes-Stiefel in [73], see also Hestenes [58]. The analysis of the CG method can be found in many books, e.g. Hageman-Young [57], Golub-Van Loan [53] and Bramble [9]. For a history and extensive bibliography on the CG and PCG method, we refer to the survey paper Golub-O'Leary [54].

The concept of *parallel subspace correction* was formulated in Xu [96] and it is a generalization of the *additive Schwarz method*. The term "addtive Schwarz method" is attributed to Dryja-Widlund [44] and the basic idea is due to Nepomnyaschikh [74, 75]. The history of the theoretical development of PSC method can be traced in Nepomnyaschikh [75], Dryja-Widlund [44], Bramble-Pasciak-Xu [16], Bramble-Pasciak-Xu-Wang [15] and Xu[96]. Addtive Schwarz method was initiated by the classic alternating or multiplicative Schwarz algorithm, which was first proposed by Schwarz [78] in 1870. Variants and applications of the alternating Schwarz method were later studied

Proof. By noting $\hat{A}\hat{P} = \hat{Q}A$, we have on the subspace $\hat{V}$,

$$\hat{B}\hat{A} = \hat{P}B\hat{P}^*\hat{A} = \hat{P}B(\hat{A}\hat{P})^* = \hat{P}B(\hat{Q}A)^* = \hat{P}BA\hat{Q}^* = \hat{P}BA.$$

□

Local-global technique. This is a technique for constructing preconditioners on the space V by using some known local preconditioners on a subspaces $\hat{V}$ of V.

Given a preconditioner $\hat{B} : \hat{V} \to \hat{V}$ for the operator $\hat{A}$ on $\hat{V}$. We assume that there exist two constants α_0 and α_1 such that

$$\alpha_0(\hat{A}\hat{u}, \hat{u}) \leq (\hat{B}\hat{A}\hat{u}, \hat{A}\hat{u}) \leq \alpha_1(\hat{A}\hat{u}, \hat{u}), \forall\, \hat{u} \in \hat{V}, \tag{3.6}$$

then the following algorithm provides a preconditioner B for A on the space V.

ALGORITHM 3.2 (GLOBAL ALGORITHM). *Given $g \in V$, $u = Bg = u_P + u_R$ is computed as follows:*

1. *Solving the local problem: $u_P \in \hat{V}^\perp$ satisfying*

$$(Au_P, v) = (g, v), \forall\, v \in \hat{V}^\perp.$$

2. *Compute u_R by*

$$u_R = \hat{B}\hat{Q}(g - Au_P).$$

THEOREM 3.3. *For **Global algorithm** defined above, we have*

$$B = (\hat{P}^\perp + \hat{P}\hat{B}\hat{A}\hat{P})A^{-1}$$

where $\hat{P}^\perp : V \to \hat{V}^\perp$ is an orthogonal projection with respect to $(A\cdot, \cdot)$, and

$$\kappa(BA) \leq \frac{\max(1, \alpha_1)}{\min(1, \alpha_0)}.$$

Proof. By definition we see from Step 1

$$u_P = \hat{P}^\perp A^{-1} g,$$

and substituting u_P into u_R in Step 2 gives

$$u_R = \hat{B}\hat{Q}(AA^{-1}g - A\hat{P}^\perp A^{-1}g) = \hat{B}\hat{Q}A\hat{P}A^{-1}g.$$

Theorem 3.3 then follows immediately from (3.6) and the expressions of u_P and u_R. □

3.4. Auxiliary and/or fictitious space methods. We shall now present a technique that can be applied in both multigrid and domain decomposition methods. This method may be viewed as a two level nonnested multigrid preconditioner or parallel subspace correction method with two nonnested subspaces.

THEOREM 3.4. *Assume that V and $\mathcal{W}$ are two linear inner product spaces, $A, R : V \mapsto V$ are SPD with respect to an inner product $(\cdot, \cdot)$ on V, $S, T : \mathcal{W} \mapsto \mathcal{W}$ are SPD with respect to an inner product $[\cdot, \cdot]$ on $\mathcal{W}$, and $\Pi : \mathcal{W} \mapsto V$. Assume that there exists*

which implies that

$$\lambda_{\max}(\tilde{T}) \leq \omega_1 K_1 .$$

Note $(T_0 u, u) = (Q_0 A u, R_0 u) = (A_0 u, R_0 u) = (R_0 A_0 u, u)$, for any $u \in V$, thus

$$\lambda_{\max}(BA) \leq \lambda_{\max}(T_0) + \lambda_{\max}(\tilde{T}) \leq \omega_1 (1 + K_1).$$

□

A few comments on the parameters ω_1, C_0 and K_1 are in order. The parameter ω_1 measures the resolution of R_i for the upper spectrum of A_i and $\omega_1 = 1$ if exact solvers are used on subspaces, namely $R_i = A_i^{-1}$ for all $0 \leq i \leq p$. In most of the domain decompsotion methods (with only one exception), the boundedness of ω_1 comes as an assumption. The estimate for C_0 often dominates the analysis in domain decomposition methods.

In domain decomposition methods, K_1 measures the degree of overlapping among the subspaces V_i for $1 \leq i \leq p$. The space V_i is said to overlap with V_j if V_i is not orthogonal with V_j with respect to $(A\cdot, \cdot)$. Hence K_1 is the largest number of subspaces that a subspace can overlap with (with the exclusion of the subspace V_0). The term "overlapping" comes from the fact that, in domain decompositin methods, V_i overlaps with V_j if and only if the two subdomains that define these two subspaces do not overlap with each other. As in all the domain decomposition method considered in this paper, each subdomain overlaps with only a fixed number of other subdomains, hence K_1 is always bounded by a fixed constant. It deserves remark that, for other applications such as multigrid methods, the parameter K_1 should be defined in a more precise way than here, and for example, K_1 can be defind as in (3.5) but with ϵ_{ij} defined by

$$(AT_i u, T_j v) \leq \omega_1 \epsilon_{ij} (T_i u, u)_A^{1/2} (T_j v, v)_A^{1/2} \quad \forall 1 \leq i, j \leq p, u, v \in V.$$

3.3. Two special techniques. For investigating the relationship between different domain decomposition preconditioners, two special techniques formulated here will prove to be instrumental. These techniques are based on a single subspace $\hat{V}$ of V.

Let $\hat{A}$ be the restriction of A on $\hat{V}$ defined by

$$(\hat{A}\hat{u}, \hat{v}) = (A\hat{u}, \hat{v}), \quad \forall \hat{u}, \hat{v} \in \hat{V},$$

and $\hat{Q}, \hat{P} : V \to \hat{V}$ be two orthogonal projections with respect to $(\cdot, \cdot)$ and $(A\cdot, \cdot)$ respectively. Let $\hat{P}^*$ be the adjoint of $\hat{P}$ with respect to $(\cdot, \cdot)$.

Global-local technique. This simple technique, known as *global-local technique*, is for constructing preconditioner on a subspace $\hat{V}$ of V from a known preconditioner on the space V.

THEOREM 3.2. *Given that B is an SPD operator on V, define $\hat{B} = \hat{P} B \hat{P}^*$. Then $\hat{B}$ is SPD on $\hat{V}$ and on the subspace $\hat{V}$,*

$$\hat{B}\hat{A} = \hat{P} B A.$$

As a consequence,

$$\kappa(\hat{B}\hat{A}) \leq \kappa(BA).$$

K_0 is the smallest positive constant satisfying, for any $v \in V$, there exists a decomposition $v = \sum_{i=0}^{p} v_i$, $v_i \in V_i$ such that

$$\sum_{i=0}^{p} (v_i, v_i)_A \le C_0 \, (v, v)_A, \tag{3.4}$$

and

$$K_1 = \max_{1 \le j \le p} \sum_{i=1}^{p} \varepsilon_{ij} \tag{3.5}$$

where $\varepsilon_{ij} = 0$ if $P_i P_j = 0$ (namely $V_i \perp V_j$) and $\varepsilon_{ij} = 1$ otherwise.

The above theorem is a special case of the general theory in Xu [96], for completeness, a proof is included here.

Proof. If $v = \sum_{i=1}^{p} v_i$ is a decomposition that satisfies (3.4), then

$$(v, v)_A = \sum_{i=1}^{p} (v_i, v)_A = \sum_{i=1}^{p} (v_i, P_i v)_A,$$

and we can derive

$$\begin{aligned}
&\sum_{i=1}^{p} (v_i, P_i v)_A \\
&= \sum_{i=1}^{p} (v_i, A_i P_i v) \text{ (definite of } A_i) \\
&\le \sum_{i=1}^{p} (R_i^{-1} v_i, v_i)^{1/2} (R_i A_i P_i v, v)_A^{1/2} \text{ (Cauchy inequality)} \\
&\le (\sum_{i=1}^{p} (R_i^{-1} v_i, v_i))^{1/2} (\sum_{i=1}^{p} (T_i v, v)_A)^{1/2} \text{ (Cauchy inequality)} \\
&\le \sqrt{\frac{K_0}{\omega_0}} \|v\|_A (Tv, v)_A \text{ (definition of } C_0).
\end{aligned}$$

Consequently

$$\|v\|_A^2 \le \frac{C_0}{\omega_0} (Tv, v)_A$$

which implies that

$$\lambda_{\min}(BA) \ge \omega_0 / C_0.$$

Let $\tilde{T} = \sum_{i=1}^{p} T_i$. By definition of K_1,

$$\begin{aligned}
\|\tilde{T} v\|_A^2 &= \sum_{i,j=1}^{p} (T_i v, T_j v)_A \\
&\le \sum_{i,j=1}^{p} \omega_1 \epsilon_{ij} (T_i v, v)_A^{1/2} (T_j v, v)_A^{1/2} \\
&\le \omega_1 K_1 (Tv, v)_A,
\end{aligned}$$

Observing the formulae in the PCG method and the convergence estimate (3.3), one sees that the efficiency of a PCG method depends on two main factors: the action of B and the size of $\kappa(BA)$. Hence, a good preconditioner should have the properties that the action of B is relatively easy to compute and that $\kappa(BA)$ is relatively small (at least smaller than $\kappa(A)$).

3.2. Preconditioning by parallel subspace correction. All the domain decomposition preconditioners in this paper will be interpreted and analyzed within the framework of *parall subspace correction method* or additive Schwarz method.

The parallel subspace correction (PSC in short) method is based on a sequence of subspace V_i, $0 \leq i \leq p$ of V such that

$$V = \sum_{i=1}^{p} V_i.$$

The above *space decomposition* is understood in the way that for any $v \in V$, there exist $v_i \in V_i$ such that $v = \sum_{i=1}^{p} v_i$.

For each i, we define two orthogonal projections $Q_i, P_i : V \to V_i$ by

$$(Q_i u, v_i) = (u, v_i), \quad (AP_i u, v_i) = (Au, v_i), \quad u \in V,\ v_i \in V_i,$$

and the restriction A_i of A on V_i by

$$(A_i u_i, v_i) = (Au_i, v_i), \quad u_i, v_i \in V_i.$$

It follows from the definition that

$$A_i P_i = Q_i A.$$

Let $R_i : V_i \to V_i$ be an SPD operator that approximates to the inverse of A_i in some sense. The PSC preconditioner for A is formulated as follows:

$$B = \sum_{i=0}^{p} R_i Q_i \tag{3.1}$$

Set $T_i = R_i Q_i A$, $0 \leq i \leq p$. Note that $T_i : V \to V_i$ is symmetric with respect to $(A\cdot, \cdot)$ and non-negative. We obtain from above that

$$BA = \sum_{i=0}^{p} T_i = \sum_{i=0}^{p} R_i Q_i A = \sum_{i=0}^{p} R_i A_i P_i.$$

If the i-th subproblem solver is exact, i.e. $R_i = A_i^{-1}$, then $T_i = R_i Q_i A = P_i$ and

$$BA = \sum_{i=0}^{p} P_i. \tag{3.2}$$

THEOREM 3.1. *The preconditioner B given by (3.1) satisfies*

$$\kappa(BA) \leq \frac{\omega_1}{\omega_0} C_0 (1 + K_1)$$

where

$$\omega_0 = \min_{0 \leq i \leq p} \lambda_{\min}(R_i A_i), \omega_1 = \max_{0 \leq i \leq p} \lambda_{\max}(R_i A_i), \tag{3.3}$$

Theorem ??, we obtain a preconditioner for the original operator A as follows:

$$B = B^0 Q_0 + ETE^t Q_E. \tag{2.15}$$

What becomes crucial is then the extension operator E. The mathematically easiest is the discrete harmonic extension, but this extension requires exact subdomain solvers. The algorithms discussed earlier pretty much correspond to such type of extension.

Non-harmonic extensions are also possible to obtain, but they are in general quite technical, we refer to Nepomnyaschikh [?] (and the references cited therein). If the finite element space has a nested multilevel struture, the extension operator can be more easily constructed, see Haase-Langer-Meyer-Nepomnyaschikh [?].

3. Preliminary I: preconditioning techniques. All the domain decomposition methods discussed in this paper is based on a very important algebraic method: preconditioned conjugate gradient method (PCG in short). This section contains some basic facts for the PCG method and some basic techniques for constructing and analyzing preconditioners. The presentation in this section is purely algebraic.

Given a finite dimensional vector space V and linear operator $A : V \mapsto V$, consider the following linear equation on V:

$$Au = f. \tag{3.1}$$

We assume that A is symmetric positive definite (SPD) with respect to an inner product $(\cdot, \cdot)$ on V. Next, we shall give a brief overview of the conjugate gradient method (CG) or preconditioned conjugate gradient method (PCG), and methods of constructing and analyzing preconditioners for the above linear equation.

3.1. Conjugate gradient method and preconditioning. The CG method is the basis of all the preconditioning techniques to be studied in this paper. The preconditioned conjugate gradient (PCG) method can be viewed as a conjugate gradient method applied to a preconditioned system as follows:

$$BAu = Bf. \tag{3.2}$$

Here $B : V \mapsto V$ is another SPD operator and known as a preconditioner for A. Note that BA is symmetric with respect to the inner product $(B^{-1}\cdot, \cdot)$. The detailed algorithm is as follows:

ALGORITHM 3.1 (PCG). *Given u_0; $r_0 = f - Au_0$; $p_0 = Br_0$; For $k = 1, 2, \ldots,$*

$$\begin{aligned}
\alpha_k &= (Br_{k-1}, r_{k-1})/(Ap_{k-1}, p_{k-1}),\\
u_k &= u_{k-1} + \alpha_k p_{k-1},\\
r_k &= r_{k-1} - \alpha_k A p_{k-1},\\
\beta_k &= (Br_k, r_k)/(Br_{k-1}, r_{k-1}),\\
p_k &= Br_k + \beta_k p_{k-1}.
\end{aligned}$$

It is well-known that (e.g. Bramble [9])

$$\|u - u_k\|_A \le 2\Big(\frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1}\Big)^k \|u - u_0\|_A, \tag{3.3}$$

which implies that PCG converges faster with smaller condition number $\kappa(BA)$.

2.4. Variants and coarse spaces. Substructuring methods and methods based on local Neumann problems mentioned above represent two major kinds of methods for domain decomposition without overlappings. There are numerous variants of these two types of methods presented in the literature. It can be easily observed that many variants come from the different choices of coarse spaces. One special section (§8) will be devoted to the discussion of the main coarse spaces that can be used in domain decomposition methods. These coarse spaces can all be viewed as the range of certain linear operator, say I_0. It is observed in this paper that the analysis of a efficient coarse grid space is reduced to the analysis of certain approximation and stability properties of the associated operator I_0. As a consequence, many different coarse spaces for different algorithms can be treated in a rather unified fashion.

A few other methods that are related to the aforementioned two major classes of methods will also be discussed in this paper for the preconditioning of the interface system. These methods, to be discussed in §12, are obtained by taking the restrictions on the interface Γ from some well-known preconditioners on Ω such as BPX preconditioner, hierarchical basis preconditioner and overlapping additive Schwatz methods. The technical tool in studying these methods is the global-local technique that is presented in §??.

2.5. Applications of the basic preconditioners. The preconditioners described earlier can be applied in various ways for the solution of the original system (2.2). We shall now describe a few such applications.

2.5.1. Implementation with exact subdomain solvers. Assume that T is a preconditioner for the interface operator S. The basic application of T is given in the following algorithm.

ALGORITHM 2.1.

1. *Solve for u_P.*
2. *Solve the boundary system by the PCG method using the preconditioner T.*
3. *Compute the harmonic extension of the boundary value into the subdomains.*

2.5.2. Inexact subdomain solvers and discrete extension operators. The algorithms described above require an exact solution of the relevant subdomain problems; we shall now discuss the possibility of using inexact subdomain solvers. Such type of algorithms depend on the construction of some appropriate operator that extend a finite element function defined on the boundary of a domain into the inside of the domain in a properly bounded fashion.

Assume that $E : V_\Gamma \to V$ is a linear operator satisftying

$$A(Ev, Ev) \lesssim S(v, v) \quad \forall v \in V_\Gamma. \tag{2.14}$$

For any $v \in V$, denote $v_\Gamma = v|_\Gamma$. We write

$$v = (v - Ev_\Gamma) + Ev_\Gamma.$$

Note that $v - Ev_\Gamma \in V_P$. By (2.14), we then obtain a "stable" space decomposition:

$$V = V_P + V_E \quad \text{with} \quad V_E = \text{range of } E.$$

By the Fictitious Space Lemma (see §3.4), if T is a good preconditioner of S, then ETE^t is also a preconditioner for the operator associated with the space V_E. Therefore, if B^0 is a good preconditioner for the operator associated with the space V_P, by

problems. Again, let us first look at the two subdomain case. We first observe that $S_h = S_1 + S_2$, where $S_i : V_\Gamma \to V_\Gamma$ for $i = 1, 2$ is defined by

$$\langle S_i u, v\rangle = A_{\Omega_i}(\tilde{u}, \tilde{u}).$$

Note that, by Lemmas 4.9 and 4.6, we have for any $u \in V^h(\Gamma)$,

$$\langle S_i u, u\rangle_{0,\Gamma_i} \eqsim |u|^2_{1/2,\partial\Omega_i} \eqsim A_{\Omega_i}(\tilde{u}, \tilde{u}) \eqsim \|u\|^2_{H^{1/2}_{00}(\Gamma)} \lesssim \langle Su, u\rangle .$$

This means that either S_1^{-1} or S_2^{-1} is an optimal preconditioner for the interface operator S. As a result, their summation

$$T_h = S_1^{-1} + S_2^{-1}. \tag{2.12}$$

is also an optimal preconditioner.

The action of T_h in (2.12) amounts to the solution of two local Neumann problems in Ω_i with $i = 1, 2$. For this reason, this type of preconditioning technique is sometimes known as a Neumann-Neumann method.

The use of the summation of S_i^{-1} as in (2.12) would give a ballance of two subdomains and it also generalizes to multiple subdomains in a more natural way. Its advantange is even clearer when applied to problem with jump coefficients; in this case

$$S = \rho_1 S_1 + \rho_2 S_2.$$

We shall now discuss the generalization of the above procedure to multiple subdomain case. A natural attemp is perhaps to define S_i on each subdpace $V(\partial\Omega_i)$ as

$$\langle S_i u_H, v_H\rangle_{0,\Gamma_i} = A_i(u_H, v_H) = (\nabla u_H, \nabla v_H)_{0,\Omega_i}, \tag{2.13}$$

and then, as in two subdomain case, take the sum of their inverses. But there are at least two difficulties in this approach. First of all, $V^h(\partial\Omega_i)$ is not a subspace of $V^h(\Gamma)$; secondly not all S_i is invertible. To overcome the first difficulty, we introduce a subspace $V_i \subset V^h(\Gamma)$ that in some sense is closest to $V^h(\partial\Omega_i)$, namely the smallest subspace of $V^h(\Gamma)$ that contains the degree of freedom of $V(\partial\Omega)$. Let $\Theta_i : V^h(\partial\Omega_i) \to V_i$ be an appropriate linear operator that links these two spaces (the construction of such Θ_i's will be discussed in §8.3). Using the operator Θ_i, a solver on V_i can be obtained from a solver on $V^h(\Gamma_i)$. But as mentioned above, the most natural operator S_i is not always invertible. To overcome this difficulty, we introduce a modified operator $\hat{S}_i$ obtained from S_i by adding a lower order term. With an appropriate coarse subspace V_0 (which will be discussed in § 8.3) and a solver R_0, we obtain a preconditioner in the form of

$$T_h = R_0 Q_0 + \sum_{j=1}^{J} \Theta_i \hat{S}_i^{-1} \Theta_i^t.$$

As mentioned earlier, such type of preconditioner is sometimes called a Neumann-Neumann method (see §11).

Another effective approach of avoiding the possible singularity of S_i is to remove the null part of S_i in $V^h(\Gamma_i)$ by first solving a proper coarse space, which results in the so-called balancing domain decomposition method (see §11.2). The balancing domain decomposition method is viewed as a local-global approach that is discussed in §3.3.

$V^h(\Gamma) \to V^h(\Gamma)$ that satisfies

$$(I_0 v)(x) = v(x) \quad x \in \mathcal{W}.$$

By means of this operator, we may further split u_H as $u_H = (u_H - I_0 u_H) + I_0 u_H$. Note that $u_H - I_0 u_H$ vanishing on the joint-set $\mathcal{W}$ can be "localized". Although the function $I_0 u_H$ is still global, it is not a big problem if we can make the range of I_0, often known as a coarse space, to be simple and to have a very small degree of freedom. If the operator I_0 has an appropriate boundedness property, we then have

$$\begin{aligned} \sum_{i=1}^{p} \rho_i |u_H|^2_{1/2,\partial\Omega_i} &\sim \sum_{i=1}^{p} \rho_i |u_H - I_0 u_H|^2_{1/2,\partial\Omega_i} + \sum_{i=1}^{p} \rho_i |I_0 u_H|^2_{1/2,\partial\Omega_i} \\ &\sim \sum_{F \subset \Gamma \setminus \mathcal{W}} \rho_F |u_H - I_0 u_H|^2_{H_{00}^{1/2}(F)} + \sum_{i=1}^{p} \rho_i |I_0 u_H|^2_{1/2,\partial\Omega_i}, \end{aligned} \tag{2.11}$$

where $F \subset \Gamma \setminus \mathcal{W}$ is an edge for $n = 2$ and a face for $n = 3$ and, roughly speaking, if $F \subset \partial\Omega_i$ and χ_F is the characteristic function for the set F,

$$|u_H - I_0 u_H|^2_{H_{00}^{1/2}(F)} \sim |\chi_F (u_H - I_0 u_H)|^2_{1/2,\partial\Omega_i}.$$

The Sobolev space $H_{00}^{1/2}(F)$ is very important in the theory of domain decomposition and it will be discussed in detail in §4.2.

The expression in (2.11) naturally leads to a preconditioner that depends on local solvers and a small global solver. In fact, the term $|u_H - I_0 u_H|^2_{H_{00}^{1/2}(F)}$ can be replaced by computationally more efficient quadratic term, which will be discussed in §4.3. The resulting preconditioner in such a process is often known as *substructuring* preconditioner.

As we note, the crucial technical aspect in a substructuring method is the operator I_0. Based on the above discussions, we expect that I_0 has the following features: it is invariant on the joint-set $\mathcal{W}$, it is bounded in certain $H^{1/2}$ norm and has a simple range with a relatively small degree of freedom. Such an operator may be known as a *joint-operator* whereas $I - I_0$ may be known as a *decomposition-operator*. The construction of I_0 will be discussed in detail in §8.

The above "breaking" or decomposition process naturally leads to a decomposition of $V^h(\Gamma)$ into a sum of subspaces as follows:

$$V^h(\Gamma) = V_0 + \sum_{F \subset \Gamma} V_F$$

with

$$V_0 = \text{Range}(I_0), \quad V_F = \{v \in V^h(\Gamma); \text{supp } v \subset F\}.$$

As a result, the framework of space decomposition and subspace correction (see discussions in §3.2) can be applied to construct parallel subspace correction preconditioners. Such type of preconditioners will be discussed in Section 11.

2.3. Methods based on local Neumann problems. We shall now briefly discuss another class of technique for preconditioning S based on local Neumann

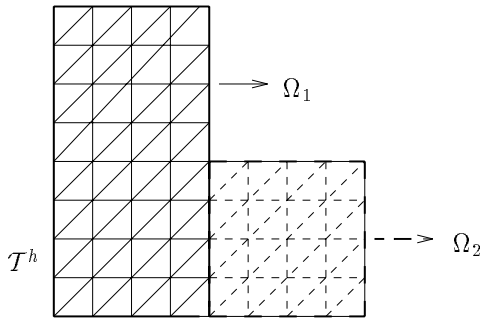

FIG. 2. *Subdomains* Ω_1 *and* Ω_2

where $-\Delta_{\Gamma,h}$ is the discrete Laplacian on the set Γ. Therefore, the operator S_h can be preconditioned by

$$T_h = (-\Delta_{\Gamma,h})^{-1/2}. \tag{2.10}$$

The above operator can be evaluated, when possible, by the FFT (fast Fourier transform) method. Other methods for preconditioning S is also possible, details can be found in ???.

The above procedure extends directly to the decomposition consisting of multiple subdomain without cross points. If cross points are present in the decomposition, however, the situation is much more complicated. For one thing, the interface set Γ is unusual and the relation like (2.10) is no more readily available. In this case, the main idea is to break the interface Γ into small and local substructures on which relation like (2.10) may be used.

The substructures we expect to use are obviously those edge sets in two dimensions and those face sets in three dimensions. For convenience we shall use the term *joint-set* and the notation $\mathcal{W}$ to represent the set that connects these substructures. The joint-set consists of those points belonging to more than two subdomains. For $n = 3$, the joint-set $\mathcal{W}$ looks like a wirebasket (cf. Fig 3) and is actually known as *wirebasket* set in some papers.

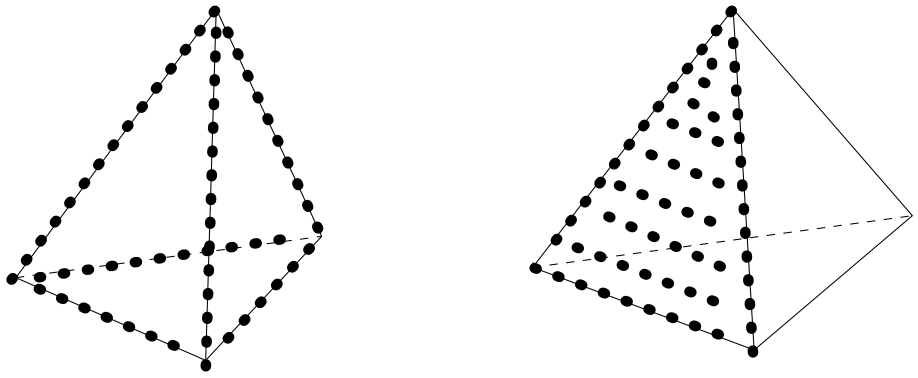

FIG. 3. *Left: nodes on the wirebasket* $\mathcal{W}$*; right: nodes on a face* F *and its boundary* ∂F

The "breaking" of Γ is realized by a properly constructred linear operator I_0 :

but $\|\tilde{u}\|^2_{1,\Omega} = \sum_{i=1}^{p}(\|\tilde{u}\|^2_{0,\Omega_i} + |\tilde{u}|^2_{1,\Omega_i})$, thus applying the trace lemma 4.2 with $\varepsilon = h_0$, we obtain

$$\langle S_h u, u\rangle \geq h_0 \sum_{i=1}^{p} \|u\|^2_{0,\partial\Omega_i} = h_0 \langle u, u\rangle ,$$

that implies $\lambda_{min}(S_h) \gtrsim h_0$. On the other hand, by Lemma 4.10 we have

$$\langle S_h u, u\rangle = A(\tilde{u}, \tilde{u}) \lesssim \sum_{i=1}^{p} |\tilde{u}|^2_{1,\Omega_i} \eqsim \sum_{i=1}^{p} |u|^2_{1/2,\partial\Omega_i},$$

then applying Lemma 4.17 gives

$$\langle S_h u, u\rangle \lesssim h^{-1} \sum_{i=1}^{p} \|u\|^2_{0,\partial\Omega_i} = h^{-1} \langle u, u\rangle ,$$

this shows $\lambda_{max}(S_h) \lesssim h^{-1}$. Therefore we get $\kappa(S_h) = \lambda_{max}/\lambda_{min} \lesssim (h_0 h)^{-1}$. □

Let $\mathcal{S}$ be the stiffness matrix associated with the bilinear form $S(\cdot,\cdot)$ under the nodal basis functions in V_Γ. One important obervation is that $\mathcal{S}$ is a Schur complement of the stiffness matrix $\mathcal{A}$ associated with $A(\cdot,\cdot)$ under the nodal basis functions in V. More specifically, if we write the stiffness matrix $\mathcal{A} \in R^{n\times n}$ blockwise

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}^t & \mathcal{A}_{22} \end{pmatrix}$$

where $\mathcal{A}_{11} \in R^{n_1\times n_1}$ is the stiffness matrix associated with the nodes in $\Omega \setminus \Gamma$ and $\mathcal{A}_{22} \in R^{n_2\times n_2}$ is the stiffness matrix associated with the nodes on Γ, then

$$\mathcal{S} = \mathcal{A}_{22} - \mathcal{A}_{12}^t \mathcal{A}_{11}^{-1} \mathcal{A}_{12}.$$

It is well know that the condition number of the Schur compliment $\mathcal{S}$ is always less than the one of the matrix $\mathcal{A}$. namely $\kappa(\mathcal{S}) \leq \kappa(\mathcal{A})$. But, by the conditioning estimate of the operator S, we can easily see that

$$\kappa(\mathcal{S}) = O\Big((h_0 h)^{-1}\Big).$$

2.2. Methods based on local Dirichlet problems. Methods based on local Dirichlet problems are often known as substructuring methods. For clarity, we shall first look at a simple example based on the simplest domain decomposition consisting of two subdomains. Assume that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ with $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ (cf. Fig.2).

In this case, we have, for $u \in V_\Gamma$,

$$S(u,u) \eqsim |u|^2_{H^{1/2}_{00}(\Gamma)}.$$

The interface Γ is either an interval for $n = 2$ or a polygonal domain for $n = 3$. In both these cases, the following spectral equivalence holds:

$$S(u,u) \eqsim |u|^2_{H^{1/2}_{00}(\Gamma)} \eqsim \Big\langle (-\Delta_{\Gamma,h})^{1/2} u, u \Big\rangle_{0,\Gamma},$$

Of course, u_P is only "part" of the solution u that we are seeking. The remaining part of the solution $u_H = u - u_P$ lies in the orthogonal compliment of V_P in V^h:

$$V_H = \{v \in V^h;\ A(v, \chi) = 0, \quad \chi \in V_P\},$$

and u_H obviously satisfies:

$$A(u_H, v) = (f, v) - A(u_P, v) \quad \forall\, v \in V^h$$

or equivalently

$$A(u_H, v_H) = (f, v) - A(u_P, v) \quad \forall\, v \in V^h. \tag{2.5}$$

Here $v_H \in V_H$ is understood similarly as u_H. We note that

$$A(u, u) = A(u_P, u_P) + A(u_H, u_H). \tag{2.6}$$

The function $u_H \in V_H$ is called a *discrete harmonic* function which satisfies

$$A_{\Omega_i}(u_H, \chi) = 0 \quad \forall \chi \in V_0^h(\Omega_i),$$

The value of u_H in Ω is clearly uniquely determined by its value on $\partial\Omega_i$ $(1 \le i \le p)$ and in fact

$$|u_H|_{1,\Omega_i} \eqsim |u_H|_{1/2,\partial\Omega_i}.$$

(the norms above are associated with Sobolev spaces $H^1(\Omega_i)$ and $H^{1/2}(\partial\Omega_i)$ which will be discussed in details in §4.2).

The main concern of this paper is to construct preconditioners for the system (2.5). We first note that

$$A(u_H, v_H) \eqsim S(u, u) \tag{2.7}$$

where $S(u, v)$ is the bilinear form defined on the interface Γ:

$$S(v, v) = \sum_{i=1}^{p} \rho_i |v|^2_{1/2,\partial\Omega_i} \quad \forall v \in V_\Gamma. \tag{2.8}$$

Let $S : V_\Gamma \to V_\Gamma$ be defined by

$$\langle Su, v\rangle_{0,\Gamma} = S(u, v) \quad \forall u, v \in V_\Gamma.$$

Our task is therefore to construct preconditioners for the interface operator S. We shall now briefly discuss some properties of the operator S.

LEMMA 2.1. *For the interface operator S_h, we have*

$$\kappa(S_h) = O(\frac{1}{h_0 h}).$$

Proof. For any $u \in V^h(\Gamma)$, we deduce by definition and Poincaré inequality that

$$\langle S_h u, u\rangle = A(\tilde{u}, \tilde{u}) \gtrsim |\tilde{u}|^2_{1,\Omega} \gtrsim \|\tilde{u}\|^2_{1,\Omega}, \tag{2.9}$$

When the coefficient $\rho(x)$ is discontinuous, each subdomain Ω_i is chosen in such a way that $\rho(x)$ equals to a constant ρ_i in Ω_i. We assume that the triangulation $\mathcal{T}^h$ is consistent with (2.3) in the sense that each $\partial\Omega_i$ can be written as a union of boundaries of elements in $\mathcal{T}^h$ (see Fig. 1). Detailed assumptions on the regularity of these subdomains Ω_i will be given in §6. We later often use the notation

$$A_i(u,v) = \int_{\Omega_i} \rho_i \nabla u \cdot \nabla v dx \quad \forall\, u, v \in H^1(\Omega),$$

we see $A(u,v) = \sum_{i=1}^{p} A_i(u,v)$.

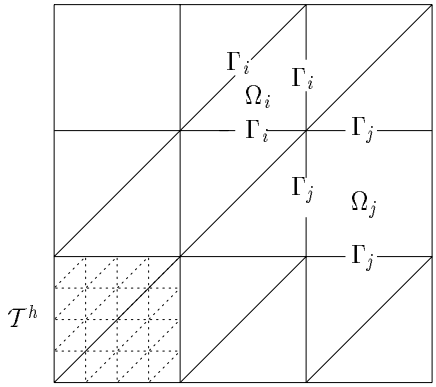

FIG. 1. *Fine mesh $\mathcal{T}^h$ and subdomains $\{\Omega_i\}_{i=1}^p$: some Ω_i are simplices and some quadrilaterals*

Nonoverlapping domain decomposition methods discussed in this paper correspond to preconditioners for the system (2.2) obtained by solving certain small problems associated with the subdomains from the domain decomposition. Perhaps the most natural step is to solve for $u_{P,i} \in V_0^h(\Omega_i)$ on each subdomain Ω_i the following local homogeneous Dirichlet problem:

$$A(u_{P,i}, v) = (f, v) \quad \forall\, v \in V_0^h(\Omega_i),$$

where the local subspace $V_0^h(\Omega_i)$ is defined by

$$V_0^h(\Omega_i) = \{v \in V^h;\ v(x) = 0, \quad \forall\, x \in \Omega \setminus \Omega_i\}.$$

Notice that the computation of $u_{P,i}$ can be carried out on each subdomain concurrently. This is typically how the parallelization is realized in domain decomposition methods.

If $u_P \in V$ is the function that is equal to $u_{P,i}$ on Ω_i, then u_P belongs to the following subspace:

$$V_P = \{v \in V^h; v(x) = 0, \quad \forall\, x \in \Gamma\}.$$

Here $\Gamma = \cup_{i=1}^{p} \Gamma_i$ is the interface among all the subdomains $\{\Omega_i\}$, and $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ is the interface of the subdomain Ω_i (cf. Fig 1).

Obviously $u_P \in V_P$ is the solution of the following problem:

$$A(u_P, v) = (f, v) \quad \forall\, v \in V_P. \tag{2.4}$$

rithm exploiting the wirebasket coarse subspace will be shown to be as efficient as the most non-overlapping methods for elliptic problems with large jumps in coefficients in both two and three dimensions.

In §10 we address the substructurng algorithms based on local Dirichlet problems initiated by Bramble-Pasciak-Schatz, with the wirebasket algorithms proposed by Smith discussed also there. Then in §11, we present the non-overlapping domain decomposition algorithms based on local Neumann problems, among them are so-called Neumann-Neumann algorithms and the balancing domain decomosition algorithm. Finally in §12, we derive a couple of interface preconditioners from the existing stiffness preconditioners by using our two techniques addressed in §3.3, for example, the interface versions of the BPX multilevel preconditioner, of the hierarchical basis preconditioner and of additive schwarz methods; moreover, the so-called vertex method will be a special case of our general formulation.

2. Basic ideas and outline. Because of the technical nature of the nonoverlapping domain decomposition methods, we shall devote this section, for the clarity of presentation, to the basic ideas and motivation of the main algorithms to be discussed. Meanwhile, an outline of the rest of the paper will be laid out.

The presentation of this section is quite loose with very little technical detail. Also very few references will be cited. Technical details and relavant literature are to be given in the following sections.

2.1. Model problem and basic idea. We shall consider the following model boundary value problem:

$$\begin{aligned} -\nabla \cdot (\rho(x)\nabla U) &= F(x) \quad \text{in } \Omega \\ U &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here $\Omega \subset R^n$ is a polygon for $n = 2$ or a polyhedron for $n = 3$ and $\rho(x)$ is piecewise constant in Ω or $\rho(x) \equiv 1$.

From a preconditioning point of view, the above model problem is sufficient for the study of general symmetric, positive and definite (SPD in short) second order elliptic boundary value problems. Problems with slowly changing variable coefficients can be preconditioned by the Poisson equation, namely $\rho_i = 1$ for all i in (2.1); problems with large jumps in coefficients between subdomains can be modelled by the above problem.

Assume that we are given a quasi-uniform triangulation $\mathcal{T}^h = \{\tau^h\}$ of Ω with each τ^h being a simplex of size $O(h)$. Let V^h be a finite element space consisting of continuous piecewise linear functions that vanish on $\partial\Omega$. A finite element approximation for (2.1) is to find $u \in V^h$ such that

$$A(u, v) = (f, v) \quad \forall\, v \in V^h, \tag{2.2}$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$, and

$$A(u, v) = \int_\Omega \rho(x)\nabla u \cdot \nabla v \; dx.$$

A domain decomposition without overlapping consists of a number of mutually disjoint open subdomains Ω_i such that

$$\bar{\Omega} = \sum_{i=1}^{p} \bar{\Omega}_i. \tag{2.3}$$

computations, one needs only to represent the involved operators and finite element functions in the basis of corresponding finite element spaces. These are purely certain algebraic operations, no special background of knowledge is required. In fact, one can figure out a few basic transformations between operators and matrices, which were formulated in the paper by Xu [96].

As our focus is on non-overlapping DDMs, overlapping type methods will be very little mentioned. Only we will devote a brief section (§9 to study an interesting two-level additive Schwarz method using the wirebasket coarse subspace instead of the standard coarse subspace defined by the vertices of the subdomains. There have existed a few references on overlapping DDMs, e.g. Dryja-Widlund [46], Bramble-Pasciak-Wang-Xu [15], Xu [96], Griebel-Oswald [55], Smith-Bjørstad-Gropp [86], Le Tallec [90] for self-adjoint elliptic problems. For non-selfadjoint or indefinite elliptic problems, we refer to Cai [24], Cai-Widlund [20], Xu-Cai [100], Xu [96, 99], and Ernst-Golub [48]. For nonlinear elliptic problems, we refer to Xu [98, 94], Scroggs [80] and Zou-Huang [104].

We are not attempting to present a detailed history and literature of the concerned methods. For more thorough references, the readers are referred to recent survey papers by Chan-Mathew [28], Le Tallec [90], and the domain decomposition proceedings [50, 25, 26, 51, 27, 61, 60].

The content of the article is arranged as follows. First in the next section, we outline all algorithms which will be discussed in the paper. In §3, we formulate the preconditioned conjugate gradient algorithm and some useful equivalent relations for estimating the condition number of the product of two symmetric positive definite operators, and a general principle of the construction of additive type preconditioners (known as *parallel subspace correction* preconditioners) and converegnce analyses of these preconditioners. Then in the remainder of §3, we propose two crucial techniques, i.e. *global-local* and *local-global* techniques for constructing global preconditioners on a larger dimensional vector space V by using local preconditioners defined on a subspace of V, and vice versa. A lot of existing non-overlapping domain decomposition algorithms can be easily derived using these two techniques.

In §4, we address all fundamental tools for the converegence analysis of non-overlapping DDMs, e.g. properties of Sobolev spaces on the boundary of a domain and of continuous and discrete harmonic functions, norm estimates and equivalencies for finite element functions. The readers who are not interested in the Sobolev space and so the convergence theory of algorithms can then skip this section. Section 5 is a brief section addressing the Schur complement matrix and its generalization to the interface corresponding to a domain decomposition. In §6 we list the basic notation and concepts used in the paper.

For better understanding more general domain decomposition algorithms, in §7 we discuss non-overlapping two-subdomain algorithms and their convergence, which in fact may give us an insight into the algorithms and convergence analysis in many subdomains case.

As it is well-known, a global coarse solver is necessary for achieving an optimal or almost optimal domain decomposition algorithm. Therefore we dedicate a whole section (§8) to the investigation of properties of existing coarse subspaces in the literature and observe that their efficiencies lie in the inherent fact that the corresponding interpolants induced by the coarse spaces satisfy two properties: the H^1 stability and L^2 optimal or almost optimal approximation. Section 9 is the only one in the paper addressing overlapping DDM, where the two-level overlapping additive Schwarz algo-

NON-OVERLAPPING DOMAIN DECOMPOSITION METHODS

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Abstract. We give a thorough investigation on the existing non-overlapping domain decomposition methods for solving second order elliptic problems in two and three dimensions, and also propose a couple of new algorithms. After addressing some fundamental tools for the convergence analysis of domain decomposition methods and formulating two techniques for constructing preconditioners, i.e. *local-global* and *global-local* techniques, most existing methods are easily derived and their convergence proofs follow simply. In particular, the two and three dimensional algorithms and proofs can be formulated and conducted jointly. We shall indicate that the convergence analysis for many known methods can be reduced to verfifying the L^2-optimal approximation and H^1-stability of the interpolants corresponding to global coarse subspaces.

Key Words. Non-overlapping domain decomposition, interface preconditioners, stiffness preconditoners, local-global and global-local techniques, jumps in coeffcients.

1. Introduction. In this paper, we are interested in the convergence theory of non-overlapping domain decomposition methods (DDMs) for solving large sparse linear systems of equations which arise from finite element discretizations of second order self-adjoint elliptic problems defined on a domain $\Omega \subset R^n$ $(n = 2, 3)$. Our major aim is to give a unifiied convergence analysis for existing non-overlapping DDMs, and address in detail some fundamental theoretical tools often used in the domain decomposition context.

Non-overlapping DDMs don't require any overlapp between neighboring subdomains or substructures, which seems to be able to have more flexible substructures, compared to overlapping DDMs. And this flexibility enables non-overlapping DDMs efficiently to handle elliptic problems with jumps in coefficients, when large coefficient jumps happen only subdomain by subdomain. As most overlapping DDMs, all non-overlapping DDMs discussed in the paper are based on a very important iterative method: preconditioned conjugate gradient method. And the actual aim of DDMs here is to construct a good preconditioner either for the stiffness operator A_h corresponding to the finite element equation or for the Schur complement operator S_h (we later call it *interface operator*) corresponding to the reduced interface equation. By good preconditioner we mean that the action of preconditioner is not expensive to calculate, about the same as the action of A_h or S_h. For every algorithm to be discussed, we shall present in very detail how to implement the corresponding preconditioning process required by PCG iteration, the rest part of each algorithm is standard.

Throughout the paper, we shall use operator languages or variational presentations, which might give us a deeper insight into some essentials of the algorithms and their convergence properties. The implementation of algorithms will be also formulated in variational framework. When applying these algorithms in practical

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