Proof of Theorem 1.7.8 We choose a sequence Ω_m, $m = 1, 2, \dots$ of convex subsets of Ω with C^2 bondaries $\partial\Omega_m$ so that dist $(\partial\Omega_m, \partial\Omega)$ tends to zero as m tends to infinity. By the well-known regularity theorem for smooth domain (cf. Gilbarg and Trudinger), for each m, there exists $u_m \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $-\Delta u_m = F$ in Ω_m. By Lemma 1.7.12, there exists a constant C such that $\|u_m\|_{2,\Omega_m} \le C$. This implies that $\tilde{u}_m$ is a bounded sequence in $H^1(\mathbb{R}^d)$ and $v_{m,ij} = \partial_i \widetilde{\partial_j u_m}$ are bounded sequences in $L^2(\mathbb{R}^d)$ for $1 \le i, j \le d$. Consequently there exit $V \in H^1(\mathbb{R}^d)$ and $V_{ij} \in L^2(\mathbb{R}^d)$ and a suitable increasing sequence of integers m_k (k=1,2,...) such that, as $k \to \infty$

$$\tilde{u}_{m_k} \to V \text{ weakly in } H^1(\mathbb{R}^d), \quad \tilde{v}_{m_k,i,j} \to V_{i,j} \text{ weakly in } L^2(\mathbb{R}^d).$$

Let $u = V|_\Omega$. It is easy to check that $u \in H_0^1(\Omega)$ and satifies $(\nabla u, \nabla\phi) = (F, \phi)$ for all $\phi \in H_0^1(\Omega)$. It can also be easily checked, by definition, that $\partial_i \partial_j u = V_{ij}|_\Omega \in L^2(\Omega)$. Thus $u \in H^2(\Omega)$. This completes the proof.

We shall prove the result under a slightly stronger assumption that Ω is piecewise smooth. Given a smooth point $x_0 \in \partial\Omega$. Assume that, around x_0, Γ is given by the graph of the function $x_d = g(x_1, \cdots, x_{d-1})$, $|x_i - x_i^0| < \delta, 0 \leq i \leq n-1$ for some $\delta > 0$. Since both Δu and $|\nabla u|$ are invariant under both rotation and translation, we may assume that $x^0 = 0$ and $\nu == (0, \ldots, 0, 1)^T$.

Let $\Phi = x_d - g(x_1, \cdots, x_{d-1})$. Then

$$\nu = \nabla\Phi/|\nabla\Phi|, \quad \partial_i g = -\partial_i \Phi = \nu_i/\nu_d = 0, \quad 1 \leq i \leq d-1.$$

Since $u \in H_0^1(\Omega)$, we have $u(x_1, \cdot, x_{d-1}, g(x_1, \cdots, x_{d-1}) \equiv 0$. Differentiating this identity and using the chain rule, we deduce that, for $1 \leq i \leq d-1$

$$\partial_i u = 0$$

and, for $1 \leq i, j \leq d-1$

$$\partial_{ij}^2 u = \partial_{ij}^2 \Phi \partial_d u.$$

It follows that

$$\Delta u = \text{tr } (D^2\Phi)\partial_d u$$

and

$$\frac{1}{2}\partial_\nu |\nabla u|^2 = \sum_{i=1}^d \partial_i u \sum_{j=1}^d \partial_{ij}^2 u \nu_j = \partial_n u \partial_{nn}^2 u.$$

Consequently,

$$\begin{aligned} \Delta u \partial_\nu u - \frac{1}{2}\partial_\nu |\nabla u|^2 &= (\partial_n u)(\partial_{nn}^2 u + \text{tr } (D^2\Phi)\partial_n u) - \partial_n u \partial_{nn}^2 u \\ &= (\partial_n u)^2 \text{tr } (D^2\Phi) \geq 0. \end{aligned}$$

The desired estimate then follows. □

(1.7.11) Remark. We have in fact proved the following identity

$$\int_\Omega |\Delta u|^2 - \sum_{i,j=1}^d \int_\Omega |\partial_{ij}^2 u|^2 = \int_{\partial\Omega} H_{\partial\Omega} |\nabla u|^2 \quad \forall\ u \in H_0^1(\Omega) \cap H^2(\Omega)$$

where $H_{\partial\Omega}$ is the mean curvature function for the boundary of Ω.

(1.7.12) Lemma. *Let Ω be a convex, bounded domain of $\mathbb{R}^d$. Then for any $u \in H_0^1(\Omega) \cap H^2(\Omega)$*

$$\|u\|_{2,\Omega} \leq C(\Omega)\|\Delta u\|_{0,\Omega}$$

Proof. By a standard energy argument, $|v|_{1,\Omega} \leq \|\Delta u\|_{0,\Omega}$. By Poincare inequality, $\|u\|_{1,\Omega} \leq C(\text{diam } \Omega)|v|_{1,\Omega}$. The desired estiamte then follows by combining the previous lemma. □

Let $v = r^\beta \sin(\beta\theta)$. Being the imaginary part of the complex analytic functin z^β, v is harmonic in Ω. Define $u = (1 - r^2)v$. A direct calculation shows that

$$-\Delta u = 4(1+\beta)v \text{ in } \Omega \text{ and } u|_{\partial\Omega} = 0.$$

Note that $4(1+\beta)v \in L^\infty(\Omega) \subset L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Nevertheless, a slightly weaker result does hold for general Lipschitz domains.

(1.7.6) Theorem. *Assume that Ω is a bounded Lipschitz domain. Then there exists a constant $\alpha \in (0, 1]$ such that*

$$\|U\|_{1+\alpha} \le C\|F\|_{\alpha-1}, \tag{1.7.7}$$

for the solution U of (1.7.4)*, where C is a constant depending on the domain Ω and the coefficients defining $\mathcal{L}$.*

Again the proof for the above theorem is quite complicated, we refer to Grisvard.

A remarkable fact is that we can take $\alpha = 1$ in the above theorem for convex domains. This means that Theorem 1.7.5 can be extended to convex domains. Because of the importance of this result and the fact the proof of this result is not widely available in the text books, we shall now restate this result and give a complete proof based on Theorem 1.7.5

(1.7.8) Theorem. *Let Ω be a convex, bounded domain of $\mathbb{R}^d$. Then for each $F \in L^2(\Omega$, there exists a unique $U \in H^2(\Omega)$, the solution of* (1.7.1)*, that satisfies*

$$\|U\|_{2,\Omega} \le C\|F\|_{0,\Omega}$$

where C is a positive constant depending only on the diameter of Ω and the coefficients of $\mathcal{L}$.

(1.7.9) Lemma. *Let Ω be a convex, bounded domain of $\mathbb{R}^d$. Then for any $u \in H_0^1(\Omega) \cap H^2(\Omega)$*

$$\sum_{i,j=1}^{d} \int_\Omega |\partial_{ij}^2 u|^2 \le \int_\Omega |\Delta u|^2.$$

Proof. We first establish the inequality for $u \in H_0^1(\Omega) \cap C^3(\bar{\Omega})$. It follows from the Green formula that

$$\int_\Omega |\Delta u|^2 - \sum_{i,j=1}^{d} \int_\Omega |\partial_{ij}^2 u|^2 = \int_{\partial\Omega} (\Delta u \partial_\nu u - \frac{1}{2}\partial_\nu |\nabla u|^2). \tag{1.7.10}$$

The H^1 solution of (1.7.4) is often called a weak solution of (1.7.1). This solution U can be proved to be a solution of (1.7.1) in a more classic sense if U is smooth enough. The theory for proving the smoothness of the weak solution is called *regularity* theory. This sort of theory is often not very straightforward. We shall only give a brief account of this theory. But this theory is very important for the theory of finite element approximation and convergence of multigrid methods.

To get a rough idea of regularity theory, let us study a property of Laplacian operator on the whole space by using Fourier transform. Given a distribution v defined on $\mathbb{R}^n$ such that $\Delta v \in L^2$, by the properties of Fourier transform, we have

$$\widehat{D^\alpha v}(\xi) = (i\xi)^\alpha \hat{v}(\xi) = -(i\xi)^\alpha |\xi|^{-2} \widehat{\Delta v}(\xi).$$

The function $(i\xi)^\alpha|\xi|^{-2}$ is bounded by 1 if $|\alpha| = 2$, hence

$$||\widehat{D^\alpha v}||_{0,\mathbb{R}^d} \le ||\widehat{\Delta v}||_{0,\mathbb{R}^d}.$$

By Planchel identidy, we have

$$||D^\alpha v||_{0,\mathbb{R}^d} \le ||\Delta v||_{0,\mathbb{R}^d} \quad \forall\ |\alpha| = 2.$$

The above inequality illustrates an important fact that if v is a function such that $\Delta v \in L^2$, then all its second order derivatives are also in L^2. If we think about it a little, this is a rather significant fact since Δv is a very special combination of the second order derivatives of v. It is not easy to see this property of Laplacian also holds for the elliptic operator given (1.7.2) if the coefficient functions are sufficently smooth.

This property of elliptic operator can be extended to bounded domains with smooth boundary, but such an extension is not trivial. The following theorem is well-known and it can be found in most of the text books on elliptic boundary value problems.

(1.7.5) Theorem. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$. Then for each $F \in L^2(\Omega$, there exists a unique $U \in H^2(\Omega)$, the solution of* (1.7.1), *that satisfies*

$$||U||_{2,\Omega} \le C||F||_{0,\Omega}$$

where C is a positive constant depending on Ω and the coefficients of $\mathcal{L}$.

The above regularity theorem, however, does not hold on general Lipschitz domains. To see this, let us give a simple counter example. Given $\beta \in (0, 1)$, consider the following nonconvex domain

$$\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < \pi/\beta\}.$$

where $\vec{n}(x)$ is the outer normal direction of $\partial\Omega$ at x, we have

$$\begin{aligned}
\int_{\partial\Omega} u^2 dx &\leq \int_{\partial\Omega} u^2 \vec{\rho}\cdot\vec{n} dx \\
&= \int_\Omega div\vec{\rho} u^2 dx + \int_\Omega 2u\vec{\rho}\cdot\nabla u dx \\
&\lesssim \int_\Omega u^2 dx + \int_\Omega |u|\,|\nabla u| dx \\
&\lesssim (1+\epsilon^{-2})\int_\Omega u^2 dx + \epsilon^2 \int_\Omega |\nabla u|^2 dx,
\end{aligned}$$

as desired. □

1.7 Elliptic Boundary Value Problems

We shall consider the problem of approximating the solution U of

$$\begin{aligned}
\mathcal{L}U &= F \quad in\ \Omega, \\
U &= 0 \quad on\ \partial\Omega.
\end{aligned} \tag{1.7.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^d$ and $\mathcal{L}$ is given by

$$\mathcal{L}v = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial v}{\partial x_j}) + a_0 v, \tag{1.7.2}$$

with $\{a_{ij}\}$ uniformly positive definite and bounded on Ω and a_0 is non-negative. The bilinear form corresponding to the operator $\mathcal{L}$ is defined by

$$A(v,w) = \sum_{i,j=1}^{d} \int_\Omega (a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + a_0 vw)dx. \tag{1.7.3}$$

This form is defined for all v and w in the Sobolev space $H^1(\Omega)$. Clearly, $U \in H_0^1(\Omega)$ is the solution of

$$A(U,\chi) = (F,\chi) \quad \forall \chi \in H_0^1(\Omega). \tag{1.7.4}$$

The existence of the above variational problem is guarrenteed by the well-known Riesz representation theorem in Hilbert space since it is easy to see that $A(\cdot,\cdot)$ defines an inner product on the Hilbert space $H_0^1(\Omega)$ and $(F,\cdot)$ can be viewed as a continuous linear functional on $H_0^1(\Omega)$.

(1.6.21) Corollary. *Assume that Γ is a measurable subset of $\partial\Omega$ with positive measure. Then*

$$\|v\|_{1,p} \lesssim |v|_{0,p} + \left|\int_\Gamma v ds\right| \quad \forall\, v \in W^{1,p}(\Omega).$$

The above results can obviously be extended to fractional order Sobolev space.

(1.6.22) Theorem. *1. If G is a l.s.c. seminorm on V such that for $\phi \in \mathcal{P}_m(\Omega)$, $G(\phi) = 0$ iff $\phi = 0$, then*

$$\|v\|_{m+\sigma,p} \eqsim G(v) + |v|_{m+\sigma,p} \quad \forall\, v \in W^{m+\sigma,p}(\Omega), \tag{1.6.23}$$

2.

$$\inf_{\phi\in\mathcal{P}_m(\Omega)} \|v+\phi\|_{m+\sigma,p} \eqsim |v|_{m+\sigma,p}, \quad \forall\, v \in W^{m+\sigma,p}(\Omega), \tag{1.6.24}$$

3. If B is a l.s.c. seminorm on V such that for all $\phi \in \mathcal{P}_m(\Omega)$, $B(\phi) = 0$, then

$$B(v) \lesssim |v|_{m+\sigma,p}, \quad \forall\, v \in W^{m+\sigma,p}, \tag{1.6.25}$$

As we know, the trace of the function in $H^\sigma(\Omega)$ is well-defined if $\sigma > \frac{1}{2}$, hence we can take $G(v) = |\int_\Gamma v dx|$ in (1.6.18) and have a special case of (1.6.18) as follows:

(1.6.26) Theorem (Poincaré Inequality). *Assume that $\Gamma \subset \partial\Omega$ is such that meas$(\Gamma) > 0$ and $\frac{1}{2} < \sigma \le 1$, then*

$$\|v\|_\sigma \lesssim \left|\int_\Gamma v dx\right| + |v|_\sigma, \quad \forall\, v \in H^\sigma(\Omega).$$

Consequently

$$\|v\|_\sigma \lesssim |v|_\sigma, \quad \forall\, v \in H^\sigma_\Gamma(\Omega).$$

(1.6.27) Lemma.

$$\|u\|_{0,\partial\Omega} \lesssim \epsilon^{-1}\|u\| + \epsilon\|u\|_1,$$

for any $u \in H^1(\Omega)$ and $\epsilon \in (0,1)$.

Proof. It can be proved that (see Grisvard) that there exists a function $\vec{\rho} \in [C^1(\bar{\Omega})]^2$ such that

$$\vec{\rho}(x)\cdot\vec{n}(x) \ge 1, \quad \forall\, x \in \partial\Omega,$$

Namely

$$G(v + \phi_v) = 0.$$

Since G is obviously a l.s.c. seminar on V and also $\ker(F) \cap \ker(G) = \{0\}$, we conclude from (i) that

$$\|v + \phi_v\|_V \lesssim F(v + \phi_v) + G(v + \phi_v) = F(v + \phi_v) = F(v).$$

which implies (1.6.16) and completes the proof of (ii).

We are now in a position to prove (iii). By Lemma 1.6.11, for any $v \in V$ and $\phi \in \ker(F)$

$$B(v + \phi) \lesssim \|v + \phi\|_V.$$

By hypothesis that $\ker(F) \subset \ker(B)$, we have $\phi \in \ker(B)$, thus $B(v) = B(v + \phi)$, therefore, using (ii), we get that

$$B(v) \lesssim \inf_{\phi \in \ker(F)} \|v + \phi\|_V \lesssim F(v),$$

completing the proof. $\square$

Given $m \geq 0$ and $p \geq 1$, we then take $V = W^{m+1,p}(\Omega)$ and $F(v) = |v|_{m+1,p}$, $T(v) = \|v\|_{m,p}$, we can see that (1.6.13) is trivially satisfied. Obviously, F is a l.s.c. seminorm by definition. As $W^{m+1,p}(\Omega)$ is compactly imbedded into $W^{m,p}$, T is a compact functional. Also it is straightforward to check that $\ker F = \mathcal{P}_m(\Omega)$. Therefore we can apply Theorem 1.6.12 to deduce the following (generalized) well-known results:

(1.6.17) Theorem. *1. If G is a l.s.c. seminorm on V such that for $\phi \in \mathcal{P}_m(\Omega)$, $G(\phi) = 0$ iff $\phi = 0$, then*

$$\|v\|_{m+1,p} \eqsim G(v) + |v|_{m+1,p}, \quad \forall\ v \in W^{m+1,p}(\Omega), \tag{1.6.18}$$

2.

$$\inf_{\phi \in \mathcal{P}_m(\Omega)} \|v + \phi\|_{m+1,p} \eqsim |v|_{m+1,p}, \quad \forall\ v \in W^{m+1,p}(\Omega), \tag{1.6.19}$$

3. If B is a l.s.c. seminorm on V such that for all $\phi \in \mathcal{P}_m(\Omega)$, $B(\phi) = 0$, then

$$B(v) \lesssim |v|_{m+\sigma,p}, \quad \forall\ v \in W^{m+\sigma,p}, \tag{1.6.20}$$

: (1.6.18) is often called *Sobolev norm equivalence theorem* in which G usually takes the form of $G(v) = \sum_{i=1}^{m} |f_i(v)|$, for some bounded linear functionals f_i's. (1.6.20) is often called the Bramble-Hilbert Lemma.

A trivial consquence of the above theorem is

Since $\{v_n\}$ is bounded, by the compactness of the functional T, we may assume that

(1.6.15) $$T(v_n - v_m) \to 0, \quad \text{as } n, m \to \infty.$$

It follows from the hypothesis (1.6.13), (1.6.14) and (1.6.15) that

$$\begin{aligned} \|v_n - v_m\|_V &\lesssim F(v_n - v_m) + T(v_n - v_m) \\ &\leq F(v_n) + F(v_m) + T(v_n - v_m) \to 0, \quad \text{as } n, m \to \infty. \end{aligned}$$

This means that $\{v_n\}$ is a Cauchy sequence on V. But V is a Banach space, hence there exists $v \in V$ so that

$$\lim_{n\to\infty} \|v_n - v\|_V = 0.$$

Since both F and G are l.s.c. seminorms, we conclude that

$$F(v) + G(v) \leq \lim_{n\to\infty} \sup F(v_n) + \lim_{n\to\infty} \sup G(v_n) = 0.$$

Hence $F(v) = G(v) = 0$, i.e., $v \in \ker(F) \cap \ker(G)$. By hypothesis, $v = 0$, but this contradicts to (1.6.14) which implies $\|v\|_V = 1$. This complete the proof of (i).

To prove (ii), we should first point out that $\ker(F)$ is obviously a subspace of V. Furthermore we note that $\ker(F)$ is finite dimensional, in fact, by the hypothesis (1.6.13), we can easily see that the unit ball in $\ker(F)$ is compact.

We need to show that

(1.6.16) $$\inf_{\phi\in\ker(F)} \|v + \phi\|_V \lesssim F(v), \quad \forall\, v \in V.$$

as the other direction of the above inequality follows easily from Lemma 1.6.11. As mentioned above $m \stackrel{\text{def}}{=} \dim(\ker(F)) < \infty$, hence we can choose m functionals $\{f_k : k = 1, 2, \cdots, m\}$ over $\ker(F)$ that forms a basis for the dual space $(\ker(F))^*$. By Hahn-Banach theorem, we may assume that f_k's are all defined on the whole of V. Set

$$G(v) = \sum_{k=1}^{m} |f_k(v)|.$$

Using the fact that $\{f_k\}$ forms a basis of $(\ker(F))^*$, for any $v \in V$, by solving a linear system with a nonsingular Gramm matrix, we can find a $\phi_v \in \ker(F)$ such that

$$f_k(\phi_v) = -f_k(v), \quad k = 1, 2, \cdots, m.$$

1.6 Norm equivalence theorem

(1.6.11) Lemma. *Any l.s.c. (lower-semi-continuous) semi-norm $F(\cdot)$ on a Banach space V is bounded, namely*

$$F(v) \lesssim \|v\|_V \quad \forall\, v \in V.$$

This result is sometimes known as Gelfand Lemma and its proof is similar to that of *uniform boundedness principle* in Banach space theory (c.f. [?]). The idea is to consider the decomposition $V = \cup_{k=1}^{\infty}\{v \in V : F(v) \leq k\|v\|_V\}$ and apply the Baire category theorem. The details of the proof are left to the readers.

(1.6.12) Theorem. *Assume that V is a Banach space, normed with $\|\cdot\|_V$, satisfying*

$$\|v\|_V \lesssim F(v) + T(v), \quad \forall\, v \in V, \tag{1.6.13}$$

for some l.s.c seminorm $F(\cdot)$ on V and some functional T which is compact in the sense that any bounded and infinite set in V contains an infinite sequence $\{v_n\}$ so that $T(v_n - v_m) \to 0$ as $n, m \to \infty$. Then the following are true:

(i) *For any other l.s.c. seminorm G over V, as long as $\ker(F) \cap \ker(G) = \{0\}$, then*

$$\|v\|_V \eqsim F(v) + G(v) \quad \forall\, v \in V.$$

(ii) *Let $V/\ker(F)$ be the ordinary quotient space and $\|\cdot\|_{V/\ker(F)}$ the corresponding quotient norm, then*

$$\|v\|_{V/\ker(F)} \eqsim F(v), \quad \forall\, v \in V.$$

(iii) *For any other l.s.c. seminorm B over V, as long as $\ker(F) \subset \ker(B)$, then*

$$B(v) \lesssim F(v), \quad \forall\, v \in V.$$

Proof. An application of Lemma 1.6.11 gives that

$$F(v) + G(v) \lesssim \|v\|_V, \quad \forall\, v \in V.$$

To see the other direction of (i), we use a contradiction argument, namely we assume if what we want to show were not true, there would exist $\{v_n\} \subset V$ such that

$$\|v_n\|_V = 1, \quad \text{and} \quad F(v_n) + G(v_n) \to 0 \quad (\text{as } n \to \infty). \tag{1.6.14}$$

Taking the $L^1(D_\epsilon)$-norm on both hand sides of above inequality, we obtain

$$|D_\epsilon||u(x_0)| \lesssim \|u\|_{0,1,D_\epsilon} + \epsilon^\lambda |D_\epsilon| \|u\|_{C^{0,\lambda}(\bar\Omega)},$$

hence

$$|u(x_0)| \lesssim \epsilon^{-1}\|u\|_{0.1,D_\epsilon} + \epsilon^\lambda \|u\|_{C^{0,\lambda}(\bar\Omega)}.$$

An application of Hölder's inequality with $r \geq 1$ yields

$$\begin{aligned} \|u\|_{0,1,D_\epsilon} &\leq |D_\epsilon|^{1-\frac{1}{r}}\|u\|_{0,r,\Omega} \\ &\lesssim \epsilon^{1-\frac{d}{r}}\|u\|_{0,r,\Omega} \\ &\lesssim \epsilon^{1-\frac{d}{r}} r^\alpha \|u\|_{0,(\alpha)}. \end{aligned}$$

Taking $r = \log\frac{1}{\epsilon}$, we get

$$\begin{aligned} \|u\|_{0,1,D_\epsilon} &\lesssim \epsilon^{d-\frac{d}{r}} r^\alpha \|u\|_{0,(\alpha)} \\ &\lesssim \epsilon^d |\log\epsilon|^\alpha \|u\|_{0,(\alpha)}. \end{aligned}$$

The first inequality in the theorem then follows.

The second inequality can be obtained from the first one by choosing ϵ as the unique positive root of the following algebraic equation:

$$\epsilon^{-\lambda}|\log\epsilon|^{1-\frac{1}{d}} = 1 + \frac{\|u\|_{C^{0,\lambda}(\bar\Omega)}}{\|u\|_{0,(\alpha)}}.$$

This completes the proof. □

(1.5.8) Remark. If we define

$$L^{(\alpha)}(\Omega) = \{v \in L^1(\Omega), \|v\|_{0,(\alpha)} < \infty\}$$

we then have

$$C^{0,\lambda}(\bar\Omega) \subset L^\infty(\Omega) \subset L^{(\alpha)}(\Omega)$$

The above inequality may be viewed as certain interpolation inequality of $L^\infty(\Omega)$ between the space $C^{0,\lambda}(\bar\Omega)$ and $L^{(\alpha)}(\Omega)$.

(1.5.9) Theorem. *For any* $\epsilon > 0$,

$$\|u\|_{C(\bar\Omega)} \lesssim |\log\epsilon|^{1-1/n}\|u\|_{1,n,\Omega} + \epsilon^\lambda \|u\|_{C^{0,\lambda}(\bar\Omega)}.$$

and

$$\|u\|_{C(\bar\Omega)} \lesssim \lambda^{1-1/n}\|u\|_{1,n,\Omega}\log^\alpha\left(1 + \frac{\|u\|_{C^{0,\lambda}(\bar\Omega)}}{\|u\|_{1,n,\Omega}}\right).$$

(1.5.10) EXERCISE. Prove that, if $mp > d$, then

$$\|uv\|_{m,p,\Omega} \lesssim \|u\|_{m,p,\Omega}\|v\|_{m,p,\Omega}.$$

On the other hand,

$$\begin{aligned}\int_{|\xi|\geq r} |\hat{v}_{m_k} - \hat{v}_{m_l}|^2 d\xi &\leq r^{-s}\int_{|\xi|\geq r}(1+|\xi|^2)^s |\hat{v}_{m_k} - \hat{v}_{m_l}|^2 d\xi \\ &\leq 2r^{-s}(\|v_{m_k}\|^2_{s,\mathbb{R}^n} + \|v_{m_l}\|^2_{s,\mathbb{R}^n}) \leq 2r^{-s}\end{aligned}$$

which approachs to zero as $r \to \infty$. Consequently, as

$$\begin{aligned}\|v_{m_k} - v_{m_l}\|_{0,\Omega} &\leq \|v_{m_k} - v_{m_l}\|_{0,\mathbb{R}^n} \\ &= \int_{|\xi|\leq r} |\hat{v}_{m_k} - \hat{v}_{m_l}|^2 d\xi + \int_{|\xi|\geq r} |\hat{v}_{m_k} - \hat{v}_{m_l}|^2 d\xi,\end{aligned}$$

we conclude that $\{v_{m_k}\}$ is a Cauchy sequence in $L^2(\Omega)$, as desired. □

As we pointed out in the foregoing section, the space $W^{\frac{d}{p},p}(\Omega)$ $(1 < p < \infty)$ generally fails to be imbedded into $C(\bar{\Omega})$. The following inequality will be useful to explain some further relationship between these two spaces.

(1.5.7) Lemma. *Assume $\Omega \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz continuous boundary and $p > 1$, then there exists a constant $C \equiv C(n, \Omega)$ such that*

$$\|u\|_{0,\infty} \leq C\{|\log\epsilon|^\alpha \|u\|_{0,(\alpha)} + \epsilon^\lambda \|u\|_{C^{0,\lambda}(\bar{\Omega})}\},$$

for any $\alpha, \epsilon > 0, \lambda \in (0, 1]$ and $u \in C^{0,\lambda}(\bar{\Omega})$. Here

$$\|u\|_{0,(\alpha)} = \sup_{p\geq 1}\left(p^{-\alpha}\|u\|_{0,p,\Omega}\right)$$

Furthermore, we have

$$\|u\|_{0,\infty} \leq C\lambda^\alpha \|u\|_{0,(n)} \log^\alpha\left(1 + \frac{\|u\|_{C^{0,\lambda}(\bar{\Omega})}}{\|u\|_{0,(\alpha)}}\right).$$

Proof. Without loss of generality, we may assume that $0 < \epsilon \leq 3^{-p}$. Given $u \in C^{0,\lambda}(\bar{\Omega})$, let $x_0 \in \bar{\Omega}$ be such that

$$|u(x_0)| = \|u\|_{0,\infty}.$$

Let $D_\epsilon \equiv \{x \in \Omega : |x - x_0| < \epsilon\}$, then $|D_\epsilon| \equiv meas(D_\epsilon) \gtrsim \epsilon^d$ since $\partial\Omega$ is Lipschitz continuous by hypothesis.

Note, for any $x \in D_\epsilon$, that

$$|u(x_0)| \lesssim |u(x)| + \epsilon^\lambda \|u\|_{C^{0,\lambda}(\bar{\Omega})}.$$

The desired estimate then follows by choosing $r = \frac{(n-1)p}{n-p}$ and observing that

$$\frac{rn}{n-1} = \frac{(r-1)p}{p-1} = \frac{np}{n-p}.$$

□

(1.5.5) Corollary.

$$\sup_{q\geq 1} (q^{1/n-1}\|v\|_{0,q,\Omega} \lesssim \|v\|_{1,n,\Omega} \quad \forall\, v \in W^{1,n}(\Omega).$$

Compact embedding All the embeddings theorems proved above are on continuous embeddings. We shall present a compact embedding theorem.

(1.5.6) Theorem. *Assume that Ω is a bounded Lipschitz domain, then for any $s > t \geq 0$, $H^s(\Omega)$ is compactly imbedded in $H^t(\Omega)$.*

Proof. It suffies to prove the theorem for $t = 0$. Given a bounded sequence $\{v_m\}$, we need to show it has a subsequence that is convergent in $L^2(\Omega)$. Without loss of generality, by the Sobolev extension theorem, we may assume that each v_m is defined on the whole $\mathbb{R}^n$ and satisfy

$$\|v_m\|_{s,\mathbb{R}^n} \leq 1 \quad m \geq 1.$$

As Ω is bounded, by mulplying some cut-off function, we may assume that, for some compact set $K \subset \mathbb{R}^n$,

$$\text{supp }(v_m) \subset K \quad \forall\, m \geq 1.$$

As a bounded sequence in $L^2(K)$, $\{v_m\}$ has a subsequence $\{v_{m_k}\}$ that is weakly convergent in $L^2(K)$, therefore, for each $\xi \in \mathbb{R}^n$, the sequence

$$\hat{v}_{m_k}(\xi) = (2\pi)^{\frac{-n}{2}} \int_K v_{m_k}(x) e^{-ix\dot{\xi}} d\xi$$

is convergent as $k \to \infty$. Furthermore,

$$|\hat{v}_{m_k}(\xi)| \leq |K|^{\frac{1}{2}} \|\hat{v}_m\|_{0,\mathbb{R}^n} = |K|^{\frac{1}{2}} \|v_m\|_{0,\mathbb{R}^n} \leq |K|^{\frac{1}{2}}.$$

Thus, by Lebesgue dominating convergent theorem, we have, for any $r > 0$,

$$\int_{|\xi|\leq r} |\hat{v}_{m_k} - \hat{v}_{m_l}|^2 d\xi \to 0 \quad \text{as } k, l \to \infty.$$

For $p < n$, we first consider the case $p = 1$. By the identity $v(x) = \int_{-\infty}^{x_i} D_i v dx_i$, we have

$$|v(x)| \le \int_{-\infty}^{\infty} |D_i v| dx_i.$$

Multiplying the above inequalities with $i = 1 : n$ and then taking the $n-1$st root give that

$$|v(x)|^{\frac{n}{n-1}} \le (\prod_{i=1}^{n} \int_{-\infty}^{\infty} |D_i v| dx_i)^{\frac{1}{n-1}}.$$

Integrating with respect to x_1 gives that

$$\begin{aligned}
&\int_{-\infty}^{\infty} |v(x)|^{\frac{n}{n-1}}\, dx \\
\le\ & \left(\int_{-\infty}^{\infty} |D_1 v| dx_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |D_i v| dx_i\right)^{\frac{1}{n-1}} dx_1 \\
\le\ & \left(\int_{-\infty}^{\infty} |D_1 v| dx_1\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty} \prod_{i=1}^{n} \int_{-\infty}^{\infty} |D_i v| dx_i dx_1\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Integrating with respect to $x_2, x_3, \ldots, x_n$ in a similar fashion successively yields

$$\int_\Omega |v|^{\frac{n}{n-1}} dx \le \left(\prod_{i=1}^{n} \int_\Omega |D_i v| dx\right)^{\frac{1}{n-1}}.$$

Consequently

$$\|v\|_{0,\frac{n}{n-1},\Omega} dx \le \left(\prod_{i=1}^{n} \int_\Omega |D_i v| dx\right)^{\frac{1}{n}} \le \frac{1}{n} \sum_{i=1}^{n} \int_\Omega |D_i v| dx.$$

This is the desired estimate for $p = 1$.

Applying the above estimate with $|v|^r$ in place of v for some $r > 0$ gives

$$\begin{aligned}
\||v|^r\|_{0,\frac{n}{n-1},\Omega} dx &\le \frac{r}{n} \sum_{i=1}^{n} |v|^{r-1} \int_\Omega |D_i v| dx \\
&\le \frac{r}{n} \left(\int_\Omega |v|^{\frac{(r-1)p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\sum_{i=1}^{n} \int_\Omega |D_i v|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

($\omega \in \mathbb{R}^n$) and making a change of variable, we obtain that

$$\begin{aligned} v(x) &= -\frac{1}{\omega_n}\int_0^\infty \int_{|\omega|=1} D_r u(x+r\omega)dr\, d\omega \\ &= -\frac{1}{\omega_n}\int_\Omega \sum_{i=1}^n \frac{x_i-y_i}{|x-y|^n} D_i u(y)dy. \end{aligned}$$

It then follows that

$$|v(x)| \le \frac{n}{\omega_n}\int_\Omega \frac{|Dv(y)|}{|x-y|^{n-1}}dy.$$

If $p > n$, by Hölder inequality and Lemma 1.5.3, we have

$$\begin{aligned} |v(x)| &\le \frac{n}{\omega_n}(\int_\Omega |x-y|^{-\frac{(n-1)p}{p-1}})^{1-\frac{1}{p}}|v|_{1,p,\Omega} \\ &\le C(n)(p-n)^{\frac{1}{p}-1}|\Omega|^{\frac{1}{n}-\frac{1}{p}}|v|_{1,p,\Omega}. \end{aligned}$$

If $p = n$, we may assume that $q > n$. By Hölder inequality and Lemma 1.5.3, we have

$$\begin{aligned} |v(x)| &\le \frac{n}{\omega_n}\int_\Omega \left[|x-y|^{-\frac{(n-1)p}{1+1/q-1/n}}\right]^{1-\frac{1}{n}} \cdot [|Dv|^n]^{\frac{1}{n}-\frac{1}{q}} \\ &\quad \cdot [|x-y|^{-\frac{(n-1)p}{1+1/q-1/n}})^{1-\frac{1}{n}}|Dv|^n]^{\frac{1}{q}}dy \\ &\le \frac{n}{\omega_n}[\int_\Omega |x-y|^{-\frac{(n-1)p}{1+1/q-1/n}}dy]^{1-\frac{1}{n}} \cdot [\int_\Omega |Dv|^n dy]^{\frac{1}{n}-\frac{1}{q}} \\ &\quad \cdot [\int_\Omega |x-y|^{-\frac{(n-1)p}{1+1/q-1/n}})^{1-\frac{1}{n}}|Dv|^n dy]^{\frac{1}{q}}dy \\ &\le \frac{n}{\omega_n}(1+(1-\frac{1}{n})q)^{1-\frac{1}{n}}|\Omega||^{1-\frac{1}{n}}|v|_{1,n,\Omega}^{1-\frac{n}{q}} \\ &\quad \cdot [\int_\Omega |x-y|^{-\frac{(n-1)p}{1+1/q-1/n}})^{1-\frac{1}{n}}|Dv|^n dy]^{\frac{1}{q}}dy. \end{aligned}$$

Taking the L^q norm on both hand sides of the above inequality, changing the order of integration and appying Lemma 1.5.3, we obtain

$$\begin{aligned} \|v\|_{0,q,\Omega} &\le \frac{n}{\omega_n}(1+(1-\frac{1}{n})q)^{1-\frac{1}{n}}|\Omega||^{1-\frac{1}{n}}|v|_{1,n,\Omega} \\ &\quad \sup_{y\in\Omega}[\int_\Omega |x-y|^{-\frac{n-1}{1+1/q-1/n}}dx]^{\frac{1}{q}} \\ &\le q^{1-\frac{1}{n}}|\Omega|^{\frac{1}{q}}|v|_{1,n,\Omega}. \end{aligned}$$

(1.5.3) Lemma. *There exists a constant $C(n)$ depending only on n such that for any $0 \le \lambda < n$,*

$$\max_{x\in\Omega} \int_\Omega |x-y|^{-\lambda} dy \le C(n)(n-\lambda)^{-1}|\Omega|^{1-\lambda/n}.$$

Proof. Let $R > 0$ satisfying $|\Omega| = |B(x,R)| = \frac{\omega_n}{n} R^n$. Thus $|\Omega \setminus B| = |B \setminus \Omega|$. It follows that

$$\begin{aligned}
\int_\Omega |x-y|^{-\lambda} dy &\le \int_{\Omega\cap B} |x-y|^{-\lambda} dy + \int_{\Omega\setminus B} |x-y|^{-\lambda} dy \\
&\le \int_{\Omega\cap B} |x-y|^{-\lambda} dy + R{-}\lambda|\Omega \setminus B| \\
&\le \int_B |x-y|^{-\lambda} dy \\
&= \omega_n (n-\lambda)^{-1} R^{n-\lambda} \\
&\le C(n)(n-\lambda)^{-1}|\Omega|^{1-\lambda/n}.
\end{aligned}$$

□

(1.5.4) Theorem. *There exists a constant $C(n)$ depending only on n such that, for all $v \in W_0^{1,p}(\Omega)$,*

$$\|v\|_{C(\bar\Omega)} \le C(n)(p-n)^{\frac{1}{p}-1}|\Omega|^{\frac{1}{n}-\frac{1}{p}}|v|_{1,p,\Omega} \quad \textit{if } p > n;$$

$$\|v\|_{0,q,\Omega} \le C(n)q^{1-\frac{1}{n}}|\Omega|^{\frac{1}{q}}|v|_{1,n,\Omega} \quad \textit{if } p = n,\ q \ge 1;$$

$$\|v\|_{0,\frac{np}{n-p},\Omega} \le C(n)(n-p)^{-1}|v|_{1,p,\Omega} \quad \textit{if } 1 \le p < n.$$

Furthermore, there exists a constant $C(n,\Omega)$ depending on n and Ω such that, for all $v \in W^{1,p}(\Omega)$,

$$\|v\|_{C(\bar\Omega)} \le C(n,\Omega)(p-n)^{\frac{1}{p}-1}\|v\|_{1,p,\Omega} \quad \textit{if } p > n;$$

$$\|v\|_{0,q,\Omega} \le C(n,\Omega)q^{1-\frac{1}{n}}\|v\|_{1,p,\Omega} \quad \textit{if } p = n,\ q \ge 1;$$

$$\|v\|_{0,\frac{np}{n-p},\Omega} \le C(n,\Omega)(n-p)^{-1}\|v\|_{1,p,\Omega} \quad \textit{if } 1 \le p < n.$$

Proof. It suffices to prove the results for $v \in C_0^\infty(\mathbb{R}^n)$ with supp $(v) \subset \Omega$. Integrating the obvious identity $v(x) = -\int_0^\infty D_r u(x+r\omega)dr$ with $|\omega| = 1$

1.5 Embedding theorems

Embedding theorems of Sobolev spaces are what make the Sobolev spaces interesting and important. The main result is the following theorem.

(1.5.1) Theorem (General Sobolev embedding).

Case 1. $sp > n$

$$W^{s,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Here the symble $\hookrightarrow$ denotes the continuous embedding.

Case 2. $sp = n$

$$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega);$$

Furthermore

$$W^{n,1}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 3. $sp < n$

$$W^{s,p}(\Omega) \hookrightarrow L^{\frac{np}{n-sp}}(\Omega);$$

We shall present proof for some important special cases of the above theorem. We pay particular attention to the dependence of the embedding constants with respect to the Sobolev space indices as such dependence has applications in finite element analysis.

(1.5.2) Theorem. *If $s > n/2$, then for all $v \in H^s(\mathbb{R}^n)$,*

$$\|v\|_{C_B(\mathbb{R}^n)} \leq C(n)(2s-n)^{-\frac{1}{2}}\|v\|_{s,\mathbb{R}^n}.$$

If $s > n/2$ and Ω is a bounded Lipschitz domain, then for all $v \in H^s(\Omega)$,

$$\|v\|_{C(\bar{\Omega}} \leq C(n,\Omega)(2s-n)^{-\frac{1}{2}}\|v\|_{s,\Omega}.$$

Proof. By the Fourier inversion formular

$$v(x) = (2\pi)^{-\frac{n}{2}}\int_{R^n}\hat{v}(\xi)e^{-ix\cdot\xi}d\xi,$$

and Cauchy inequality, it follows that

$$|v(x)| \leq (2\pi)^{-\frac{n}{2}}\|(1+|\xi|^2)^{-\frac{s}{2}}\|_{0,R^n}\|v\|_{s,\mathbb{R}^n} \leq C(n)(2s-n)^{-\frac{1}{2}}\|v\|_{s,\mathbb{R}^n}.$$

The proof for the General Lipschitz domain follows by the extension theorem. $\square$

Hence

$$\|Sv\|_{[H_0^{s_1}(\Omega),H_0^{s_0}(\Omega)]_\theta} \lesssim \|v\|_{s_\theta,\mathbb{R}^n}.$$

But, if $v \in H_0^{s_\theta}(\Omega)$, then $\tilde{v} \in H^{s_\theta}(\mathbb{R}^n)$ and $S\tilde{v} = v$. Consequently

$$\|v\|_{[H_0^{s_1}(\Omega),H_0^{s_0}(\Omega)]} = \|S\tilde{v}\|_{[H_0^{s_1}(\Omega),H_0^{s_0}(\Omega)]} \lesssim \|\tilde{v}\|_{s_\theta,\mathbb{R}^n} \lesssim \|v\|_{0,(\alpha)}s_\theta, \Omega.$$

This proves that $\tilde{H}_0^{s_\theta}(\Omega \subset [H_0^{s_0}(\Omega), H_0^{s_1}(\Omega)]$. □

The following special case will be often used in finite element analysis.

(1.4.14) Corollary.

$$[H_0^1(\Omega), L^2(\Omega)]_s = \begin{cases} H^s(\Omega) & \textit{if } 0 \le s < 1/2 \\ H_{00}^{1/2}(\Omega) & \textit{if } s = 1/2 \\ H_0^s(\Omega) & \textit{if } 1/2 < s < 1. \end{cases}$$

Furthermore

$$[H_0^1(\Omega), L^2(\Omega)]_s = \|(-\Delta)^{s/2}v\|.$$

(1.4.15) EXERCISE. Prove that

$$[L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega)]_{1/2} = H_{00}^{1/2}(\Omega) \cap H^1(\Omega).$$

1.4-d On the bounds of the gradient operator

(1.4.16) Theorem.

$$\|\nabla u\|_\alpha \lesssim \|u\|_{1+\alpha} \quad (-1 \le \alpha \le 1).$$

Proof. Let $D_i = \partial/\partial x_i$. Obviously $D_i : H_0^1(\Omega) \mapsto L^2(\Omega)$ is continuous, and by duality, so is $D_i : L^2(\Omega) \mapsto H^{-1}(\Omega)$. By interpolation, for $\alpha \in (0, \frac{1}{2})$, $D_i : H^{1-\alpha}(\Omega) \mapsto H^{-\alpha}(\Omega)$ is also continuous. Consequently $\|\nabla v\|_{H^{-\alpha}(\Omega)} \lesssim \|v\|_{H^{1-\alpha}(\Omega)}$, for $v \in H_0^1(\Omega)$. □

(1.4.17) Corollary.

$$(\nabla u, \nabla v) \lesssim \|u\|_{1+\alpha}\|v\|_{1-\alpha} \quad (0 < \alpha < 1).$$

The interpolation property for Sobolev spaces defined on a bounded Lipschitz domain is more interesting. For spaces $H^s(\Omega)$, the interpolation property follows directly from that for $H^s(\mathbb{R}^n)$ by using Sobolev extension theorem.

(1.4.12) Theorem. *Assume that Ω is a bounded Lipschitz domain. If $s_1 > s_0$ and $\theta \in (0,1)$, then*

$$[H^{s_1}(\Omega), H^{s_0}(\Omega)]_\theta = H^{s_\theta}(\Omega), \text{ with } s_\theta = (1-\theta)s_0 + \theta s_1.$$

Proof. By extension theorem. $\square$

Interpolation property for $H_0^s(\mathbb{R}^n)$ is a little more subtle. The following result is well known for smooth domains (see Lions and Magenes (1972)). The proof gives below follows lines of Bramble (1994) for his proof of a special case of such a result.

(1.4.13) Theorem. *Assume that Ω is a bounded Lipschitz domain. If s_1 and s_0 are not integers with $s_1 > s_0$. Then for any $\theta \in (0,1)$, with $s_\theta = (1-\theta)s_0 + \theta s_1$*

$$\begin{aligned} &[H_0^{s_1}(\Omega), H_0^{s_0}(\Omega)]_\theta = \tilde{H}_0^{s_\theta}(\Omega) \\ &= \begin{cases} H_0^{s_\theta}(\Omega) & \text{if } s_\theta - 1/2 \text{ is not an integer} \\ H_{00}^{s_\theta}(\Omega) & \text{if } s_\theta - 1/2 \text{ is an integer} \end{cases} \end{aligned}$$

Proof. Denote $\tilde{E}v = \tilde{v}$. By definition

$$\|\tilde{E}v\|_{s_j,\mathbb{R}^n} = \|\tilde{v}\|_{s_j,\mathbb{R}^n} \lesssim \|v\|_{s_j,\Omega} \quad \forall\, v \in H_0^{s_j}(\Omega), j = 1,2.$$

Hence, by interpolation of operators, with $s_\theta = (1-\theta)s_0 + \theta s_1$

$$\|\tilde{E}v\|_{s_\theta,\mathbb{R}^n} = \|\tilde{E}v\|_{[H^{s_1}(\mathbb{R}^n), H^{s_0}(\mathbb{R}^n)]_\theta} = \|\tilde{v}\|_{[H_0^{s_1}(\Omega), H_0^{s_0}(\Omega)]_\theta}.$$

This proves that $[H_0^{s_1}(\Omega), H_0^{s_0}(\Omega)]_\theta \subset \tilde{H}_0^{s_\theta}(\Omega)$.

To prove the inclusion in the oposite direction, we shall construct an operator $S : H^{s_j}(\mathbb{R}^n) \mapsto H_0^{s_j}(\Omega)$. For this purpose, we take an open ball B such that $\bar{\Omega} \subset B$ and consider a continuous extension operator for $s_1 \le s \le s_0$. The existence of E is guarenteed by the Sobolev extension theorem since Ω is Lipschitz. Let $\chi_{B\setminus\Omega}$ denote the characteristic function of the set $B \setminus \Omega$, define $S = I - E\chi_{B\setminus\Omega}$. It is easy to see that $Sv \in H_0^{s_j}(\Omega)$ for $u \in H^{s_j}(\mathbb{R}^n)$. Furthermore

$$\|Sv\|_{s_j,\Omega} \lesssim \|v\|_{s_j,\Omega} + \|E\chi_{B\setminus\Omega}v\|_{s_j,\Omega} \lesssim \|v\|_{s_j,\mathbb{R}^n} \quad \forall\, v \in H^{s_j}(\mathbb{R}^n), j = 1,2.$$

since $Lu = Lu_0 + Lu_1$ is a decomposition of Lu with $Lu_0 \in \widetilde{H}_0$ and $Lu_1 \in \widetilde{H}_1$. Using (1.4.8) we have

$$\begin{aligned}
\|Lu\|_{\widetilde{H}_s} &\leq \left(\int_0^\infty t^{-2s} \inf_{u_0+u_1=u} (C_0^2\|u_0\|_{H_0}^2 + t^2 C_1^2 \|u_1\|_{H_1}^2)\frac{dt}{t}\right)^{1/2} \\
&= C_0^{1-s} C_1^s \left(\int_0^\infty \left(\frac{C_1 t}{C_0}\right)^{-2s} K^2\left(\frac{C_1 t}{C_0}, u\right)\frac{dt}{t}\right)^{1/2} \\
&= C_0^{1-s} C_1^s \|u\|_{H_s}.
\end{aligned}$$

□

(1.4.10) EXERCISE. Let B_0 and B_1 be two Banach spaces with B_1 continuously embedded and dense in B_0, definite for $p \in [1, \infty)$

$$K(t,u) = \inf_{u_0+u_1=u} (\|u_0\|_{H_0}^p + t^p \|u_1\|_{H_1}^p)^{1/p} \text{ with } u_0 \in B_0, u_1 \in B_1.$$

The intermediate space $B_{s,p}$ is defined by

$$B_{s,p} = \{u \in B_0 : \|u\|_{B_{s,p}} < \infty\}$$

where

$$\|u\|_{B_{s,p}} = \left(\int_0^\infty t^{-sp} K^p(t,u)\frac{dt}{t}\right)^{1/p}$$

1. Prove that, for any $u \in B_1$

$$\|u\|_{B_{s,p}} \leq C_s \|u\|_{B_0}^{1-s} \|u\|_{B_1}^s$$

   with $C_s = (ps(1-s))^{-1/p}$.

2. State and proof a result analogous to Theorem 1.4.9.

1.4-c Hilbert scales for Sobolev spaces

We first present a result on the interpolation for Sobolev spaces defined on $\mathbb{R}^n$. Result of this type is more or less trivial and the following result that follows almost directly from the definition of Hilbert scale is in fact contained in Example 1.4-a.

(1.4.11) Theorem. *If $s_1 > s_0$ and $\theta \in (0,1)$, then*

$$[H^{s_1}(\mathbb{R}^n), H^{s_0}(\mathbb{R}^n)]_\theta = H^{s_\theta}(\mathbb{R}^n) \textit{ with } s_\theta = (1-\theta)s_0 + \theta s_1.$$

To minimize $[(a_i - b_i)^2 + t^2\lambda_i^2 b_i^2]$, we choose $b_i = a_i(t^2\lambda_i^2+1)^{-1}$. Hence

$$K^2(t,u) = \sum_{i=1}^{\infty} t^2\lambda_i^2(t^2\lambda_i^2+1)^{-1}a_i^2.$$

Now

$$\begin{aligned}\int_0^\infty t^{-2s}K^2(t,u)\frac{dt}{t} &= \sum_{i=1}^{\infty}\left(\int_0^\infty t^{1-2s}\lambda_i^2(t^2\lambda_i^2+1)^{-1}dt\right)a_i^2 \\ &= \int_0^\infty \frac{t^{1-2s}}{1+t^2}\sum_{i=1}^{\infty}\lambda_i^{2s}a_i^2 = \frac{\pi}{2\sin\pi s}\|u\|_{H_s}^2.\end{aligned}$$

□

(1.4.6) EXERCISE. Prove Theorem 1.4.5 for the general case by using spectral representation theory for self-adjoint operator. (c.f. Lions and Magenes (1972).

(1.4.7) Remark. The K-functor technique can be easily used to define interpolated spaces between two Banach spaces (see the next exercise).

1.4-b Interpolation of operators

The next result is the important property of "interpolation of operators". Suppose we have spaces $\widetilde{H}_0$ and $\widetilde{H}_1$ analogous to H_0 and H_1; i.e., $\widetilde{H}_1$ is continuously embedded and dense in $\widetilde{H}_0$. Define $\widetilde{K}(t,\cdot)$ analogously. Further let L be a linear operator such that $L: H_i \to \widetilde{H}_i$ and constants C_i with

(1.4.8) $$\|Lu\|_{\widetilde{H}_i} \le C_i\|u\|_{H_i}$$

for $u \in H_i$, $i = 1, 2$. Here $\|\cdot\|_H$ is the norm on the generic Banach space H.

(1.4.9) Theorem. *Suppose that H_i and $\widetilde{H}_i$ are as above and $L: H_i \to \widetilde{H}_i$ satisfies* (1.4.8). *Then*

$$\|Lu\|_{\widetilde{H}_s} \le C_0^{1-s}C_1^s\|u\|_{H_s}.$$

Proof. Given $0 < s < 1$, then

$$\begin{aligned}\|Lu\|_{\widetilde{H}_s} &= \left(\int_0^\infty t^{-2s}\widetilde{K}^2(t,Lu)\frac{dt}{t}\right)^{1/2} \\ &\le \left(\int_0^\infty t^{-2s}\inf_{u_0+u_1=u}(\|Lu_0\|_{\widetilde{H}_0}^2 + t^2\|Lu_1\|_{\widetilde{H}_1}^2)\frac{dt}{t}\right)^{1/2}\end{aligned}$$

The interpolated spaces $[H_0^1(\Omega), L^2(\Omega)]_s$, the domain of $(-\Delta)^{s/2}$ will be characterized later.

A simple application of Hölder inequality gives that

(1.4.4) Theorem. *For* $u \in H_1$

$$\|u\|_{H_s} \leq \|u\|_{H_0}^{1-s}\|u\|_{H_1}^s$$

An equivalent and intrinsic definition For a given pair of space H_0 and H_1, the operator Λ that connects these two spaces are not unique in general. It is then natural to ask if the definition of the intermediate H_s is intrinsic with respect to the choice of the Λ (and even the choice of (equivalent) norms in H_0 and H_1). Next, we shall show that the norm in H_s and hence the definition of H_s can be given by an alternative formulation. With this alternative formulation, the answer to the above quetions on the well-definedness of H_s is then transparent. More importantly, this alternative formulation is an important technical tool.

(1.4.5) Theorem. *For* $u \in H_s,\ 0 < s < 1$,

$$\|u\|_{H_s} = c_s \left(\int_0^\infty t^{-2s} K^2(t,u)\frac{dt}{t}\right)^{1/2} \qquad \text{with } c_s = \left(\frac{2}{\pi}\sin \pi s\right)^{\frac{1}{2}},$$

where for each $t > 0$ *and* $u \in H_0$

$$K(t,u) = \inf_{u_0+u_1=u}(\|u_0\|_{H_0}^2 + t^2\|u_1\|_{H_1}^2)^{1/2} \text{ with } u_0 \in H_0, u_1 \in H_1.$$

Proof. For clarity, we shall present the proof for the special case that Λ has a discrete and complete spectrum, the general case can be proved similary by using the spectral decomposition for self-adjoint operator (see Lions and Magenes (1972)). To begin the proof, we write

$$K^2(t,u) = \inf_{u_1 \in H_1}(\|u-u_1\|_{H_0}^2 + t^2\|u_1\|_{H_1}^2).$$

We solve the minimization problem. Let

$$u = \sum_{i=1}^\infty a_i\varphi_i \quad \text{and} \quad u_1 = \sum_{i=1}^\infty b_i\varphi_i.$$

Then

$$\|u-u_1\|_{H_0}^2 + t^2\|u_1\|_{H_1}^2 = \sum_{i=1}^\infty[(a_i-b_i)^2 + t^2\lambda_i^2 b_i^2].$$

The existence of the operator Λ is clear in our applications given later but the existence in general will not be discussed here (for details, see Riesz-Nagy (1952)).

Given $s \in (0,1)$, we define the intermediate spaces H_s to be the domain of Λ^s with a norm given by

$$\|u\|_{H_s} = \|\Lambda^s u\|_{H_0}.$$

(1.4.1) Example. Given $-\infty < s_0 < s_1 < \infty$, let $H_0 = H^{s_0}(\mathbb{R}^n)$ and $H_1 = H^{s_1}(\mathbb{R}^n)$. The operator that connects H_0 and H_1 is as follows

$$\Lambda = \mathcal{F}^{-1}(1 + |\xi|^2)^{(s_1 - s_0)/2}\mathcal{F}.$$

In fact, by definition

$$\|v\|_{s_1,\mathbb{R}^n} = \|\Lambda v\|_{s_0,\mathbb{R}^n}.$$

By Hilbert scale, we have

$$[H^{s_1}(\mathbb{R}^n), H^{s_0}(\mathbb{R}^n)]_\theta = D(\Lambda^\theta) = H^{s_\theta}(\mathbb{R}^n) \text{ with } s_\theta = (1-\theta)s_0 + \theta s_1.$$

To help those readers who are not that familar with the spectral theory for unbounded operators, we now discuss a special case that the spectrum of Λ is discrete (λ_i) and the eigenvectors (ϕ_i) form a complete orthonormal basis for H_0. Then we may expand any element of H_0 as

$$u = \sum_{i=1}^{\infty} (u, \varphi_i)\varphi_i.$$

If $u \in H_1$, then

$$\Lambda u = \sum_{i=1}^{\infty} \lambda_i (u, \varphi_i)\varphi_i \quad \text{and} \quad \|u\|_{H_1}^2 = \sum_{i=1}^{\infty} \lambda_i^2 (u, \varphi_i)^2.$$

In this case, the intermediate spaces H_s consists of those elements of H_0 for which the norm

$$\|u\|_{H_s} = \left(\sum_{i=1}^{\infty} \lambda_i^{2s} (u, \varphi_i)^2 \right)^{1/2} \tag{1.4.2}$$

is finite.

(1.4.3) Example. Assume Ω is a bounded Lipschitz domain. Let $H_1 = H_0^1(\Omega)$ and $H_0 = L^2(\Omega)$. Then H_1 can be viewed as the domain of $\Lambda = (-\Delta)^{1/2}$ with a norm given by

$$\|\nabla v\| = \|\Lambda v\| \quad \forall \text{ for } v \in H_0^1(\Omega).$$

Proof. Without loss of generality, let $F = K \cap \{x_n = 0\}$. For $x', y' \in F$, let $\hat{x}_i = (y_1, \cdots, y_i, x_{i+1}, \cdots, x_{n-1})$ $(1 \le i \le n-1)$. Denote $K_i = (0,1)^i$. Then

$$
\begin{aligned}
&\int_F \int_F \frac{|v(x',0) - v(y',0)|^p}{|x'-y'|^{n-2+p}} dx dy \\
\le\ & \sum_{i=1}^{n-1} \int_F \int_F \frac{|v(\hat{x}_i,0) - v(\hat{x}_{i-1},0)|^p}{|x'-y'|^{n-2+p}} dx' dy' \\
\le\ & \sum_{i=1}^{n-1} \int_F \int_{K_2} |v(\hat{x}_i,0) - v(\hat{x}_{i-1},0)|^p dx dy_i^- \int_{K_{n-2}} \frac{dy_i^- dy_i^+}{|x'-y'|^{n-2+p}} \\
\lesssim\ & \sum_{i=1}^{n-1} \int_{K_{n-2}} dx_i^- dx_i^+ \int_{K_2} \frac{|v(x_i^-, x_i, x_i^+, 0) - v(x_i^-, y_i, x_i^+, 0)|^p}{|x_i - y_i|^p} dx_i dy_i \\
\lesssim\ & \sum_{i=1}^{n-1} \int_{K_{n-2}} |v(x_i^-, \cdot, x_i^+, \cdot)|_{1,p,K_2}^p dx_i^- dx_i^+ \quad \text{(by Lemma 1.3.37)} \\
\lesssim\ & |v|_{1,p,K}^p
\end{aligned}
$$

Consider for example □

1.4 Hilbert Scale for Sobolev spaces

In this section we study the Hilbert scale technique with applications to Sobolev spaces.

1.4-a Basic definition of Hilbert scale

Let H_0 and H_1 be two Hilbert spaces with H_1 continuously embedded and dense in H_0. An *intermediate space* H is a space satisfying

$$H_1 \subset H \subset H_0.$$

Hilbert scale is a technique to define a family of intermediate spaces between H_1 and H_0.

Assume that $(\cdot,\cdot)$ is the inner product on H_0. By a classical result, the space H_1 may be defined as the domain of an (unbounded) positive selfadjoint operator $\Lambda : H_1 \to H_0$ connecting the norms as follows:

$$||u||_{H_1} = ||\Lambda u||_{H_0}.$$

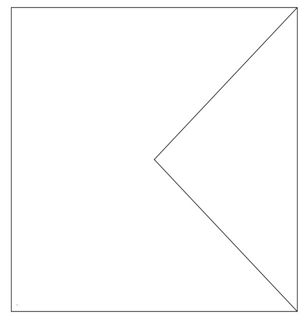

Figure 1.1: viewing the bottom edge as an edge of a right triangle

we get

$$\begin{aligned}\left|\frac{v(t,t)-v(s,s)}{t-s}\right|^p &\leq 2^{p-1}\left[\frac{1}{(t-s)^p}\Big(\int_s^t |\partial_1 v(x_1,\tau)|dx_1\Big)^p\right.\\ &+ \left.\frac{1}{(t-s)^p}\Big(\int_s^t |\partial_2 v(t,x_2)|dx_2\Big)^p\right],\end{aligned}$$

Integrating for t with $s<t<1$ and for s with $0<s<1$ and applying the Hardy inequality in Lemma 1.2.28 yields

$$\int_0^1\int_s^1\left|\frac{v(t,t)-v(s,s)}{t-s}\right|^p dtds \leq 2^{p-1}\Big(\frac{p}{p-1}\Big)^p |v|^p_{1,p,\Delta}.$$

Similarly

$$\int_0^1\int_0^s\left|\frac{v(t,t)-v(s,s)}{t-s}\right|^p dtds \leq 2^{p-1}\Big(\frac{p}{p-1}\Big)^p |v|^p_{1,p,\Delta}.$$

The desired estimate then follows. □

(1.3.37) Lemma.

$$\int_0^1\int_0^1\left|\frac{v(t,0)-v(s,0)}{t-s}\right|^p dtds \lesssim |v|^p_{1,p,(0,1)^2} \quad \forall\ v\in C^1([0,1]^2).$$

Proof. The estimate follows from the preceding lemma by viewing the edge $\{(t,0):\ 0<t<1\}$ as an edge of a right trianlge inside the unit square (see Fig. 1.1) and the using appropriate rotation. □

(1.3.38) Lemma. *Let $K=(0,1)^n$ and F be any face of K, then for any $v\in C^1(\bar K)$,*

$$|v|_{1-\frac{1}{p},p,F} \lesssim |v|_{1,p,K}.$$

1.3 Trace theorems

The main purpose of this section is to prove the following *trace theorem.*

(1.3.34) Theorem. *The mapping $u \to \gamma u \equiv u|_{\partial\Omega}$ (the restriction of u on $\partial\Omega$) which is defined for $u \in C^1(\bar{\Omega})$ has a unique continuous extention as an operator from $W^{1,p}(\Omega)$ onto $W^{1-\frac{1}{p},p}(\partial\Omega)$. This operator has a right continuous inverse independent of p.*

The particular useful case of the above theorem is when $p = 2$. Because of its extraordinary importance, let us restate the theorem and some its consequence.

(1.3.35) Theorem.

1. *The mapping $u \to \gamma u \equiv u|_{\partial\Omega}$ which is defined for $u \in C^1(\bar{\Omega})$ has a unique continuous extention as an operator from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$.*

2. *The operator $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ has a right continuous inverse, namely there exists a constant c_0 such that for any $g \in H^{1/2}(\partial\Omega)$, there corresponds to a function $v_f \in H^1(\Omega)$ such that*

$$f = \gamma v_f \quad \textit{and} \quad \|v_f\|_{1,\Omega} \le c_0 \|f\|_{1/2,\partial\Omega}.$$

3. *For any $u \in H^1(\Omega)$,*

$$\|u\|_{\frac{1}{2},\partial\Omega} \eqsim \inf_{v \in u + H_0^1(\Omega)} \|v\|_{1,\Omega} \le \|u\|_{1,\Omega}.$$

For smooth domains, the above trace theorem and its proof can be found in most standard text books on Sobolev spaces. But for Lipschitz domains, the proof is much less well-known and it is not widely available. We shall give a complete proof for such a case. Our presentation here follows closely Necas [..]

(1.3.36) Lemma. *Let $\Delta = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < x_1\}$, then*

$$\int_0^1 \int_0^1 \left| \frac{v(t,t) - v(s,s)}{t-s} \right|^p dtds \lesssim |v|_{1,p,\Delta}^p \quad \forall\ v \in C^1(\bar{\Delta}).$$

Proof. By the following obvious inequality

$$\left| \frac{v(t,t) - v(s,s)}{t-s} \right| \le \frac{1}{t-s} \int_s^t |\partial_1 v(x_1, \tau)| dx_1 + \frac{1}{t-s} \int_s^t |\partial_2 v(t, x_2)| dx_2,$$

This is the desired result in $W_0^{s,p}(\mathbb{R}_+)$ provided that $s - 1/p$ is not an integer.

We now extend the result to the general Ω. By a standard use of partition of unity, it suffices to consider the following special domain:

$$\Omega = \{(y_1, \cdots, y_n) : \ 0 < y_n < \phi(y_1, \cdots, y_{n-1}), |y_i| < a_i(1 \le i \le n-1), \},$$

for some positive Lipschitz continuous function ϕ and postive constants a_i $(1 \le i \le n-1)$.

Set $y' = (y_1, \cdots, y_{n-1})$ and

$$v_{y'}(t) = v(y', \phi(y') - t).$$

By what we have just proved, we have

$$\|t^{-s} v_{y'}\|_{0,p,\mathbb{R}_+}^p \le C^p \|v_{y'}\|_{s,p,\mathbb{R}_+}^p.$$

Integrating this inequality with respect to y' leads to

$$\|[\phi(y') - y_n]^{-s} v_{y'}\|_{0,p,\Omega} \le C\|v\|_{s,p,\Omega}.$$

Since ϕ is a Lipschitz function, the weight $\phi(y') - y_n$ is equivalent to dist (y, Ω). This completes the proof. $\square$

(1.2.31) EXERCISE. Verify (1.2.24).

Negative order Sobolev spaces

Given $s > 0$, the Sobolev space $H^{-s}(\Omega)$ is defined to be the dual space of $H^s(\Omega)$, namely

$$H^{-s}(\Omega) = (H_0^s(\Omega))^*.$$

(1.2.32) Lemma. *The space $H^{-s}(\Omega)$ defined above is identical to the space that is the completion of the space $L(\Omega)$ with respect to the following Lax negative norm:*

$$\|v\|_{-s,\Omega} = \sup_{\phi \in H^s(\Omega)} \frac{(v, \phi)_{0,\Omega}}{\|\phi\|_{s,\Omega}}.$$

Furthermore the above norm is equivalent to the dual norm for $H^{-s}(\Omega)$.

(1.2.33) EXERCISE. Prove Lemma 1.2.32.

Now, we consider the noninteger case $s = m + \sigma$ with $\sigma \in (0, 1)$. We consider now $v = u^{(m)} \in H_0^\sigma(\mathbb{R}_+)$. We make use of the following tricky identity

$$v(x) = -w(x) + \int_x^\infty \frac{w(y)}{y} dy, \text{ with } w(x) = \frac{1}{x}\int_0^x [v(t) - v(x)]dt.$$

We first show that $x^{-\sigma} w \in L^p(\mathbb{R}_+)$. Indeed, by Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty x^{-\sigma p}\left(\frac{1}{x}\int_0^x [v(t) - v(x)]dt\right)^p dx \\
&\leq \int_0^\infty x^{-\sigma p - 1}\int_0^x |v(t) - v(x)|^p dt dx \\
&\leq \int_0^\infty \int_0^x \frac{|v(t) - v(x)|^p}{|x - t|^{1+\sigma p} dt dx} \\
&\leq \int_0^\infty \int_0^\infty \frac{|v(t) - v(x)|^p}{|x - t|^{1+\sigma p} dt dx} \\
&\leq \|v\|^p_{\sigma,(\mathbb{R}_+)}.
\end{aligned}$$

Then Hardy's inequality shows that, when $\sigma < 1/p$,

$$\begin{aligned}
\|x^{-\sigma}\int_x^\infty \frac{w(y)}{y} dy\|_{0,p,\mathbb{R}_+} &\leq \|x^{-\sigma+1}\frac{1}{x}\int_x^\infty \frac{w(y)}{y} dy\|_{0,p,\mathbb{R}_+} \\
&\leq (1/p - \sigma)^{-1}\|x^{-\sigma} w\|_{0,p,\mathbb{R}_+}
\end{aligned}$$

For $\sigma > 1/p$, we make use of the following identity

$$v(x) = -w(x) - \int_0^x \frac{w(y)}{y} dy$$

with the same w given earlier. Now, Hardy's inequality shows that

$$\|x^{-\sigma}\int_0^x \frac{w(y)}{y} dy\|_{0,p,\mathbb{R}_+} \leq (\sigma - 1/p)^{-1}\|x^{-\sigma} w\|_{0,p,\mathbb{R}_+}$$

Now inequality (1.2.30) and one more application of Hardy's inequality yield that, if $\sigma > 1/p$

$$\|x^{-m-\sigma} u\|_{0,p,\mathbb{R}_+} \leq \frac{1}{(m-1)!}\|x^{-\sigma} u^{(m)}\|_{0,p,\mathbb{R}_+} \leq \frac{1}{(m-1)!}|\sigma - 1/p|^{-1}|u|_{s,p,\Omega}.$$

Similar estimate can also be obtained if $\sigma < 1/p$ by using, in place of (1.2.30), the following identity

$$\frac{|u(x)|}{x^m} \leq \frac{1}{(m-1)!}\frac{1}{x}\int_x^\infty |u^{(m)}(y)|dy.$$

The above result is given in Theorem 1.4.4.5 of Grivsard. Because of its importance in our later applications, we shall give a complete proof of this result. We follow closely the presentation of Grisvard.

(1.2.27) EXERCISE. Prove that $\tilde{W}_0^{s,p}(\Omega)$ is a Banach space.

(1.2.28) Lemma (HARDY'S INEQUALITY). *If $\alpha + 1/p < 1$, then*

$$\|(t-a)^{\alpha}\frac{1}{t-a}\int_a^t v(s)ds\|_{0,p,(a,b)} \leq |\alpha + 1/p - 1|^{-1}\|(t-a)^{\alpha}v\|_{0,p,(a,b)};$$

If $\alpha + 1/p > 1$, then

$$\|(t-a)^{\alpha}\frac{1}{t-a}\int_t^b v(s)ds\|_{0,p,(a,b)} \leq |\alpha + 1/p - 1|^{-1}\|(t-a)^{\alpha}v\|_{0,p,(a,b)};$$

The following result and its proof are taken from Grisvard (1985).

(1.2.29) Lemma. *If $s - 1/p$ is not an integer, then*

$$\|v\|^{\sim}_{s,p,\Omega} \lesssim \|v\|_{s,p,\Omega} \quad \forall\, v \in \mathcal{D}(\Omega).$$

Proof.

$$\rho^{-s+|\alpha|}D^{\alpha}v \in L^p(\Omega) \quad \forall\, v \in H_0^s(\Omega)$$

hold for all $|\alpha| \leq s$.

It suffices to prove the above result for $|\alpha| = 0$.

We first consider the case that $\Omega = \mathbb{R}_+ = (0, \infty)$ and $s = m$ is an integer. By integration by parts that, for $u \in \mathcal{D}'(\mathbb{R}_+)$, we have

$$u(x) = \int_0^x \frac{(x-y)^{m-1}}{(m-1)!}u^{(m)}(y)dy$$

and consequently

$$\frac{|u(x)|}{x^m} \leq \frac{1}{(m-1)!}\frac{1}{x}\int_0^x |u^{(m)}(y)|dy. \tag{1.2.30}$$

An application of Hardy's inequality implies that

$$\|x^{-m}u\|_{0,p,(\mathbb{R}_+)} \leq \frac{p}{(m-1)!(p-1)}\|u^{(m)}\|_{0,p,(\mathbb{R}_+)}.$$

By density, the desired estimate then holds for $H_0^m(\mathbb{R}_+)$.

The above space is closely related to the following better known space.

$$W_0^{s,p}(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ in } W^{s,p}(\Omega). \tag{1.2.22}$$

(1.2.23) Theorem. *1. $C_c^\infty(\Omega)$, the space of all functins defined in Ω which are restrictions to Ω by C^∞ functions with compact support in $\mathbb{R}^n$, is dense in $W_p^s(\Omega)$ for all $s > 0$.*

2. The space $C_0^\infty(\Omega)$ is dense in $\tilde{W}_p^s(\Omega)$ for all $s > 0$.

3. The space $C_0^\infty(\Omega)$ is dense in $W_p^s(\Omega)$ for $0 < s \le 1/p$.

For a proof of the above density theory, we refer to [...].

The spaces $W_0^{s,p}(\Omega)$ and $\tilde{W}_0^{s,p}(\Omega)$ are conceivably related intimately. For example, $\|v\|^\sim_{s,p,\Omega} = \|v\|_{s,p,\Omega}$ if $s = m$ is an integer.

If $s = m+\sigma$ for an integer $m \ge 0$ and $\sigma \in (0,1)$, we have, by definition,

$$\begin{aligned}(\|v\|^\sim_{s,p,\Omega})^p &= \|v\|^p_{s,p,\Omega} + 2\sum_{|\alpha|=m}\int_\Omega\int_{\Omega^c}\frac{|D^\alpha\tilde{v}(x) - D^\alpha\tilde{v}(y)|^p}{|x-y|^{n+\sigma p}}dxdy \\ &= \|v\|^p_{s,p,\Omega} + 2\sum_{|\alpha|=m}\int_\Omega|D^\alpha\tilde{v}(x) - D^\alpha\tilde{v}(y)|^p dx\int_{\Omega^c}\frac{dy}{|x-y|^{n+\sigma p}}.\end{aligned}$$

Using the following ralation

$$\int_{\Omega^c}\frac{dy}{|x-y|^{n+\sigma p}} \approxeq \frac{1}{\text{dist }(x,\partial\Omega)^{\sigma p}}. \tag{1.2.24}$$

we conclude that

$$\|v\|^\sim_{s,p,\Omega} \approxeq \left(\|v\|^p_{s,p,\Omega} + (|v|^\sim_{s,\Omega})^p\right)^{1/p} \tag{1.2.25}$$

where

$$(|v|^\sim_{s,p})^p = \sum_{|\alpha|=m}\int_\Omega\frac{|D^\alpha v|^p}{\text{dist }(x,\partial\Omega)^{\sigma p}}dx.$$

We have the following theorem.

(1.2.26) Theorem. *If $s - 1/p$ is not an integer, then*

$$\tilde{W}_0^{s,p}(\Omega) = W_0^{s,p}(\Omega);$$

furthermore, if $s < 1/p$, then

$$\tilde{W}_0^{s,p}(\Omega) = W_0^{s,p}(\Omega) = W^{s,p}(\Omega);$$

(1.2.15) EXERCISE. Assume that $\Omega = \Omega_1 \cup \Omega_2$ with $d = \text{dist}\,(\Omega_1 \backslash \Omega_2, \Omega_2 \backslash \Omega_1) > 0$. Then

$$\|v\|_{s,p,\Omega} \lesssim d^{-(s+\frac{n}{p})}(\|v\|_{s,p,\Omega_1} + \|v\|_{s,p,\Omega_2})$$

(1.2.16) EXERCISE. Prove (1.2.7).

(1.2.17) EXERCISE. Prove (1.2.10).

(1.2.18) EXERCISE. Prove Lemma 1.2.

Extension theorems

The extension theorem presented below is a fundamental result for Sobolev spaces. It is most often used to extend a result proved on the whole R^n to a bounded domain Ω.

(1.2.19) Theorem. *For any bounded Lipschitz domain Ω, there exits a linear operator E (depending only on Ω) such that $E : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^n)$ is a bounded linear operator for any $s \in [0, \infty)$ and $p \in [1, \infty]$. Furthermore there exists constant $C(s, \Omega)$ which is increasing with respect to $s \geq 0$ such that, for all $1 \leq p \leq \infty$,*

$$\|Ev\|_{s,p,\mathbb{R}^n} \leq C(s, \Omega)\|v\|_{s,p,\Omega} \quad \forall\, v \in W^{s,p}(\Omega).$$

The above Theorem is well-known for integer order Sobolev spaces defined on smooth domains and the corresonding proof can be found in most text books on Sobolev spaces. The Theorem for Lipschitz domain and especially for fractional order spaces is less well-known and the the proof of the theorem for these cases is quite complicated. For integer order Sobolev spaces, we refer to Stein. For factional order Sobolev spaces, we refer to a recent paper by [...].

Sobolev spaces with certain zero boundary conditions

We shall now introduce a class of Sobolev spaces with certain boundary condtions. For a function v defined in Ω, we shall use the notation $\tilde{v}$ to denote the zero extension of v to $\mathbb{R}^n$, namely $\tilde{v}$ equals to v in Ω and equal to zero on $\mathbb{R}^n \setminus \Omega$.

$$\tilde{W}_0^{s,p}(\Omega) = \{v \in W^{s,p}(\Omega) : \tilde{v} \in W^{s,p}(\mathbb{R}^n)\}, \tag{1.2.20}$$

with a norm:

$$\|v\|^{\sim}_{s,p,\Omega} \stackrel{\text{def}}{=} \|\tilde{v}\|_{s,p,\mathbb{R}^n}. \tag{1.2.21}$$

where $C(d,s)$ is a positive constant depending on d and s.

It follows that

$$
\begin{aligned}
& C(d,s)\int_{\mathbb{R}^n} |\xi|^{2s}|\hat{v}(\xi)|^2 d\xi \\
= & \int_{\mathbb{R}^n} \frac{d\eta}{|\eta|^{n+2s}} \int_{\mathbb{R}^n} |e^{i\xi\cdot\eta}-1|^2|\hat{v}(\xi)|^2 d\xi \\
= & \int_{\mathbb{R}^n} \frac{d\eta}{|\eta|^{n+2s}} \int_{\mathbb{R}^n} |v(x+\eta)-v(x)|^2 dx \quad \text{(Plancherel's theorem)} \\
= & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x+\eta)-v(x)|^2}{|\eta|^{n+2s}} dx d\eta \quad \text{(Fubini's theorem)} \\
= & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dx dy \quad \text{(changing variable } y=x+\eta)
\end{aligned}
$$

This yields that, if $s=m+\sigma$ with $\sigma\in(0,1)$,

$$
\|v\|_{s,\mathbb{R}^n}^2 \eqsim \|v\|_{m,\mathbb{R}^n}^2 + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{|\alpha|=m} \frac{|D^\alpha v(x)-D^\alpha v(y)|^2}{|x-y|^{n+2\sigma}} dx dy.
$$

The above equivalent relation is the basis for the definition of fractional order Sobolev space for more general cases.

Given $s=m+\sigma$ with $\sigma\in(0,1)$ and integer $m\geq 0$, define

$$
W^{s,p}(\Omega)=\{v\in W^{m,p}(\Omega):\ |v|_{s,p}<\infty\}
$$

where

$$
|v|_{s,p,\Omega}^p = \sum_{|\alpha|=m}\int_\Omega\int_\Omega \frac{|D^\alpha v(x)-D^\alpha v(y)|^p}{|x-y|^{n+\sigma p}} dx\, dy \tag{1.2.11}
$$

with a norm $\|\cdot\|_{s,p}$ given by

$$
\|v\|_{s,p,\Omega}^p = \|v\|_{m,p}^p + |v|_{s,p}^p. \tag{1.2.12}
$$

(1.2.13) EXERCISE. Prove that $W^{s,p}(\Omega)$ is a Banach space.

When $p=2$ the corresponding spaces are Hilbert spaces and will be most frequently used in this book. In this case we denote $H^s(\Omega)=W^{s,2}(\Omega)$ and $\|\cdot\|_{s,\Omega}=\|\cdot\|_{s,2,\Omega}$.

(1.2.14) Remark. For convenience, when the domain Ω is clear from the context, we shall use the abbreviated notation such as $\|\cdot\|_{s,p}=\|\cdot\|_{s,p,\Omega}$ and $\|\cdot\|_s=\|\cdot\|_{s,\Omega}$.

Fractional order Sobolev spaces

To introduce the definition of Sobolev spaces of fractional order, we first study the Sobolev space $H^m(\mathbb{R}^n)$ defined on the entire space and introduce an equivalent norm. Given a $v \in H^m(\mathbb{R}^n)$, by the obvious identity $\widehat{D^\alpha v} = (i\xi)^\alpha \hat{v}$ and the Plancherel's theorem, we deduce that

$$\|D^\alpha v\|_{0,2,\mathbb{R}^n} = \|\xi^\alpha \hat{v}\|_{0,2,\mathbb{R}^n}.$$

Thus, by definition

$$\|v\|^2_{m,\mathbb{R}^n} = \int_{\mathbb{R}^n} (\sum_{|\alpha|\le m} \xi^{2\alpha})|\hat{v}(\xi)|^2.$$

Using the elementary inequalities

$$(1+|\xi|^2)^m \le \sum_{|\alpha|\le m} \xi^{2\alpha} \lesssim (1+|\xi|^2)^m, \tag{1.2.7}$$

we conclude that

$$\|v\|_{m,\mathbb{R}^n} \eqsim \|(1+|\cdot|^2)^{m/2}\hat{v}\|_{0,2,\mathbb{R}^n}.$$

This relation shows that the Sobolev space $H^m(\mathbb{R}^n)$ may be equivalently defined by

$$H^m(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n), (1+|\xi|^2)^{m/2}\hat{v} \in L^2(\mathbb{R}^n)\}.$$

This alternative definition can be used to extend the definition of Sobolev spaces to the ones of non-integer order. Namely, for any given $s \in [0,\infty)$, the Sobolev space $H^s(\mathbb{R}^n)$ can be defined as follows.

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n), (1+|\xi|^2)^{m/2}\hat{v} \in L^2(\mathbb{R}^n)\} \quad \forall\, s \in \mathbb{R} \tag{1.2.8}$$

with a norm defined by

$$\|v\|_{s,\mathbb{R}^n} \stackrel{\text{def}}{=} \|(1+|\cdot|^2)^{s/2}\hat{v}\|_{0,2,\mathbb{R}^n}. \tag{1.2.9}$$

As the Fourier transformation is sometimes inconvenient to use, we shall derive an equivalent norm for $H^s(\Omega)$ without using the Fourier transformation.

The derivation that follows is based on the following elementary identity

$$\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot\eta}-1|^2}{|\eta|^{n+2s}} d\eta = C(d,s)|\xi|^{2s}, \quad 0 < s < 1 \tag{1.2.10}$$

obviously defines a distribution $T_u \in \mathcal{D}'(\Omega)$. The correspondence $u \mapsto T_u$ is often used to identify an "ordinary" function as a distribution. A distribution is often also known as a *generalized function* as the concept of distribution is a more general than concept of the classic function.

One of the basic distribution which is not an "ordinary" function is the Dirac δ-function:

$$\delta(\phi) = \phi(0) \quad \forall\, \phi \in \mathcal{D}(\mathbb{R}^n).$$

If u is a smooth function, it follows from integration by parts that, for any $\alpha \in \mathbb{Z}_+^n$

$$\int_\Omega D^\alpha u(x)\phi(x) = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \phi(x) \quad \forall\, \in \phi\mathcal{D}(\Omega).$$

The above identity is the basis for defining derivatives for a distribution. Namely, if $T \in \mathcal{D}'(\Omega)$, then for any $\alpha \in \mathbb{Z}_+^n$, $D^\alpha T$ is the distribution given by

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi) \quad \forall\, \phi \in \mathcal{D}(\Omega).$$

Integer order Soboleve spaces

The Sobolev space of index (m, p) is defined by

$$W^{m,p}(\Omega) \stackrel{\text{def}}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \le m\},$$

with a norm $\|\cdot\|_{m,p,\Omega}$ given by

$$\|v\|_{m,p,\Omega}^p \stackrel{\text{def}}{=} \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p, \tag{1.2.5}$$

We will have occasions to use the seminorm $|\cdot|_{m,p,\Omega}$ given by

$$|v|_{m,p,\Omega}^p \stackrel{\text{def}}{=} \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p.$$

(1.2.6) EXERCISE. Prove that $W^{m,p}(\Omega)$ is a Banach space.

For $p = 2$, $H^m(\Omega) \stackrel{\text{def}}{=} W^{m,2}(\Omega)$ is a Hilbert space together with an inner product as follows

$$(u, v) = \sum_{|\alpha|\le m} (D^\alpha u, D^\alpha v)$$

and the corresponding norm is denoted by $\|v\|_{m,\Omega} = \|v\|_{m,2,\Omega}$.

Green's identities For a bounded Lipschitz domain Ω, the outer normal vector $\nu = (\nu_i)$ of $\partial\Omega$ can be well defined almost everywhere on $\partial\Omega$. The following identities will be frequently used in this book.

$$\int_\Omega (D_i u)v = -\int_\Omega u D_i v + \int_{\partial\Omega} uv\nu_i. \tag{1.1.2}$$

$$\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega (\Delta u)v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v. \tag{1.1.3}$$

$$\int_\Omega \text{div } w = \int_{\partial\Omega} w \cdot \nu \tag{1.1.4}$$

1.2 Definition of Sobolev spaces

Sobolev spaces will be first defined here for integer orders using the concept of distributions and their weak derivatives. The fractional order Sobolev spaces will be introduced by looking at some special Sobolev norms in terms of Fourier transform. Definitions will also be given to Sobolev spaces satisfying certain zero boundary conditions. An extension theorem and some density theorem will also be presented.

Distributions and weak derivatives

The definition of distributions is based on the functional space $C_0^\infty(\Omega)$. The space $C_0^\infty(\Omega)$ equipped with a proper topology is denoted by $\mathcal{D}(\Omega)$, namely a sequence of $\{\phi_n\} \subset C_0^\infty(\Omega)$ is said to be convergent to a function $\phi \in C_0^\infty(\Omega)$ in the space $\mathcal{D}(\Omega)$ if

1. there exists a compact set $K \subset \Omega$ such that for all n, supp $(\phi_n - \phi) \subset K$, and
2. for any $\alpha \in \mathbb{Z}_+^n$, $\lim_{k\to\infty} \phi_n(x) = \phi(x)$ uniformly on K.

The space, denoted by $\mathcal{D}'(\Omega)$, of all continuous linear functionals on $\mathcal{D}(\Omega)$ is called the (Schwarz) *distribution space*. $\mathcal{D}'(\Omega)$ will be equipped with the weak $*$ topology, namely, in $\mathcal{D}'(\Omega)$, $T_n \to T$ if and only if $T_n(\phi) \to T(\phi)$ for all $\phi \in \mathcal{D}(\Omega)$.

One important class of functions that can be identified as distributions are functions locally integral functions. Given any such a function u, the following linear functional

$$T_u(\phi) = \int_\Omega u(x)\phi(x), \quad \phi \in \mathcal{D}(\Omega)$$

Notation of Schwarz Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ be a vector of nonnegative integers, denote

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \quad \text{with } |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

And, for a smooth function v, we have

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Also

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Some basic functional spaces Several basic Banach spaces will often be used in this book. $C(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$ with the usual maximum norm

$$\|v\|_{C(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |v(x)|.$$

$C_0^\infty(\Omega)$ denotes the space of infinitely differential functions in Ω that vanish in some neighborhood of $\partial\Omega$; namely any $v \in C_0^\infty(\Omega)$ satisfies supp $(v) \subset \Omega$, where

$$\text{supp } (v) = \text{closure of } \{x \in \Omega : \ v(x) \neq 0\}.$$

For $\lambda \in (0, 1]$, $C^{0,\lambda}(\bar{\Omega})$ denotes the space of the functions satisfying the following Hölder condition

$$|v|_{C^{0,\lambda}(\bar{\Omega})} \stackrel{\text{def}}{=} \sup_{x \neq y \in \Omega} \frac{|v(x) - v(y)|}{|x - y|^\lambda} < \infty.$$

and the corresponding norm is given by

$$\|v\|_{C^{0,\lambda}(\bar{\Omega})} = \|v\|_{C(\bar{\Omega})} + |v|_{C^{0,\lambda}(\Omega)}.$$

Given $1 \leq p < \infty$, $L^p(\Omega)$ denotes the Banach space consisting of $p-$th power integrable functions, with a norm given by

$$\|v\|_{0,p,\Omega} = \Big(\int_\Omega |v|^p \Big)^{1/p};$$

and, for $p = \infty$, $L^\infty(\Omega)$ denotes the Banach space of all essentially bounded functions with a norm given by

$$\|v\|_{0,\infty,\Omega} = \text{ess sup}\{|v(x)| : \ x \in \Omega\}.$$

(1.1.1) EXERCISE. Prove that all the function spaces defined above are Banach spaces.

Chapter 1

Sobolev Spaces and Boundary Value Problems

Sobolev spaces are fundamental in the study of partial differential equations and their numerical approximations. In this chapter, we shall give brief discussions on the basic concepts, Sobolev embedding theorems, extension theorems and interpolation properties. We shall also give a brief introduction to the regularity theory for elliptic boundary value problems.

1.1 Preliminaries

Lipschitz domains Our presentions here will almost exclusively be for bounded Lipschitz domains. Roughly speaking, a bounded domain $\Omega \subset \mathbb{R}^n$ is called a Lipschitz domain if its boundary $\partial\Omega$ can be locally represented by Lipschitz continuous function; namely for any $x \in \partial\Omega$, there exists a neighborhood of x, $G \subset \mathbb{R}^n$, such that $G \cap \partial\Omega$ is the graph of a Lipschitz continuous function under a proper local coordinate system.

Of course, all the smooth domains are Lipschitz. In particular, a domain with C^1-smooth boundary is Lipschitz. A very siginificant non-smooth example is that every polygonal domain in $\mathbb{R}^2$ or polyhedral domain in $\mathbb{R}^3$ is Lipschitz. A more interesting example is that every convex domain in $\mathbb{R}^n$ is Lipschitz.