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METHODOLOGIES AND APPLICATION

Revised convexity, normality and stability properties of the dynamical feedback fuzzy state space model (FFSSM) of insulin–glucose regulatory system in humans

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Abstract
This research tries to explore more important structural properties of the insulin–glucose regulatory system in humans. Consequently, an important theorem, namely "revised modified optimized defuzzified value theorem" for feedback systems is derived and then proved. Moreover, the properties concerning the convexity, normality and the bounded-input bounded-output stability of the induced solution of FFSSM are researched. The proposed theorems and lemmas are successfully implemented and verified for the insulin–glucose system in humans. The successful and promising results and proofs of the theorems of the relevant properties improve the credibility and reliability of the FFSSM model of the insulin–glucose regulatory system in humans.

Keywords Insulin–glucose regulations · Feedback systems · Fuzzy state space model (FSSM) · Inverse modeling · Dynamical systems · Modern control theory

1 Introduction

Control theory is the study of controlling dynamical systems/machines operating continuously. The objective of the theory is to develop a model for controlling dynamical systems/machines with optimal outputs without any delay, overshoot and controllability. In order to ensure these objectives, a control mechanism with relevant corrective behavior is required to be designed. This control mechanism monitors the controlled state variables (SV) at every instant and compares it with the reference set point (SP). Furthermore, the difference between the actual and the desired values (called the error) is calculated. The error applied as a feedback is used to generate a control action to eliminate/reduce the difference between the states variable and the reference set point (Stuart 1992). Thus, a feedback continuously takes measurements and makes requisite adjustments to keep the values of a variable within a narrow range (Otto 1970; Franklin et al. 2018).

The history of control theory can be traced to old Babylonians; water clocks were used by them. A centrifugal governor was used by Maxwell (1868) for controlling the speed of a windmill. Hurwitz (1895) contributed the control stability criteria. Remarkable advancements in control theory are due to Edward Routh 1874, Charles Sturm 1895 (Routh and Fuller 1975; Routh 1977). Control theory experienced a paradigmatic shift with the development of the PID controllers for industrial process control by the Nicolas Minorsky (Ang et al. 2005). Notable applications of the control theory range from the water tank controllers of the flash toilet to the flight control of an air plane in the air and the control of a submarine in the midst of the sea.

A mathematical modeling of a physical system as a set of inputs, states and output variables in the form of first-order differential equations or difference equations is called state space modeling. The values of the state variables evolve through time depending on their values at any given time and imposed values of inputs. The values of the

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outputs depend on the values of the state variables (Hangos et al. 2001, 2004).

Moreover, for a linear, time-invariant, finite-dimensional dynamical system, the differential equations can be represented in matrix form (Vasilyev and Ushakoy 2015). One advantage of the state space model/time domain approach is an easy and compact model to analyze multi-inputs multi-output (MIMO) systems. Otherwise, in the frequency domain/Laplace transforms it is required to encode all the inputs/outputs information of the system. Another advantage of the state space modeling is that its implementation is not limited to linear systems and requires no zero initial conditions. Moreover, it is not only useful in control engineering; it is frequently used in many different fields of studies. An evident use is in econometrics, where it can be used to forecast stock prices in a capital market and other many variables (Durbin and Koopman 2001; Rita et al. 2016).

The internal state variables consist of the smallest possible states, representing the entire state of the system at any given time (Nise 2010). The minimum number of states required to represent a system is called the order of the system. If the system is represented as a transfer function, the order of the system is the order of the denominator of the transfer function. Converting the state space model to a transfer function, some important internal information of the system may lose and thus description of system stability is needed to be furnished. A fuzzy state space model for pressurizer in a power plant was formulated in Ashaari et al. (2015). Aminu et al. (2017) represented fuzzy state space model as a multi-connected network system. Todorov et al. (2017) and Todorov and Terziyska (2018) proposed neural network predictive model of a fuzzy state space model. A genetic algorithm-based fuzzy state space approach was adopted in Huu et al. (2017). Recently, Yu et al. (2018) have presented an identification procedure for a structural state space modeling.

Fuzzy modeling has been extensively used in optimal decision making. Fahmi et al. (2017a) proposed aggregation operators using cubic information. Fahmi et al. (2017b) used an extended TOPSIS with cubic information and applied it to the selection of solgel titanium carbide. Einstein and Hybrid Einstein operators for cubic information were proposed in Fahmi et al. (2018a, b). Mean expected aggregation operators on fuzzy triangular numbers were adopted in Fahmi et al. (2018c, d), and the same operators with trapezoidal numbers in Fahmi et al. (2018e). Linguistic hesitant fuzzy triangular operators for group decision making were adopted in Amin et al. (2018a, b). Amin et al. (2018c) used trapezoidal fuzzy cubic TOPSIS for dealer selection in supply chain.

The series solution of the fuzzy differential equations was studied in Abu Arqub (2013), and an analytical solution was studied using HAM in Abu Arqub et al. (2013). Abu Arqub et al. (2015) commented on the existence and uniqueness of the solution of the fuzzy integrodifferential equations of the Volterra type. Abu Arqub et al. (2016) presented a numerical approach for solving fuzzy differential equations with the help of reproducing kernel. Further, the same technique was used in Abu Arqub et al. (2017a) to solve second-order, two-point fuzzy boundary value problems. The reproducing kernel technique was also used in Abu Arqub (2017) to find the exact and numerical solution of the Fuzzy Fredholm–Volterra integrodifferential equations. An iterative numerical approach was adopted in Abu Arqub et al. (2017b) to solve fuzzy initial value problems.

Insulin–glucose regulatory system is a system primarily responsible for maintaining glucose level in the body within a narrow range. These regulations are also referred to as glucose homeostasis. The action of insulin is to lower blood sugar level, and that of glucagon to raise the blood sugar level (Aronoff et al. 2004).

To keep the body blood glucose levels balanced, the system is regulated by a feedback process (Soman 2009). Although many tissues monitor the glucose level in the body, the alpha and beta cells in the islets of the Langerhans of the pancreas plays a vital role in balancing the glucose levels in the body (Romere et al. 2016). If the sugar level in blood falls, the alpha cells of the islets of the Langerhans secrete glucagon to balance the level. Hypoglycemia is a situation, when the blood glucose level dangerously falls. A minor case of hypoglycemia can be overcome by ingestion of dextrose foods, whereas in severe circumstances it can be treated by injection/infusion of glucagons (Zhang and Lin 2004; Yang et al. 2007).

For high blood glucose level, the beta cells of the islets inhibit the secretion of insulin in the blood stream to balance the blood glucose level (Kim et al. 2005). Insulin as a main regulator of the cell metabolism inhibits and signals other several body metabolic/cell reactions. Other known causes of blood glucose level increase are stress hormones, steroids, trauma, infections and food enriched with carbohydrates. Diabetes can be categorized into two types: type 1 and type 2. Type 1 is characterized by the insufficient or non-production of insulin. Type 2 is due to the non-responsiveness to the insulin tissues in the blood. Type 2 is called insulin-resistant and is curable. Both result in high blood glucose profiles in the blood streams and if not properly treated lead to hyperglycemia.

Meszéna et al. (2014) studied the responsiveness of the blood glucose regulatory system. Trevitt et al. (2016) commented on the development of feedback artificial pancreas. Sankaranarayanan et al. (2017) presented a minimization model for patient with hyperglycemia and hypoglycemia. Sensitivity of the Hovorka model was comprehensively studied in Radomski and Glowacka (2018). A new mathematical model is presented based on the predator prey model (Shabestari et al. 2018). The results help understand insulin regulatory system, diabetes, hyperinsulinemia and hypoglycemia. Priyadharsini et al. (2018) presented stability of their presented mathematical model of glucose regulatory system.

Khan et al. (2013) proposed a feedback fuzzy states space model for the insulin–glucose system in humans by using the models (Sturis 1991; Sturis et al. 1991; Mosekilde 1996; Keener and Sneyd 1998; Tolić et al. 2000). The model insulin–glucose of Sturis (1991), Sturis et al. (1991), Mosekilde (1996), Keener and Sneyd (1998) and Tolić et al. (2000) is presented as:

$$\frac{\mathrm{d}I_\mathrm{p}}{\mathrm{d}t} = f_1(G) - E\left(\frac{I_\mathrm{p}}{V_\mathrm{p}} - \frac{I_\mathrm{i}}{V_\mathrm{i}}\right) - \frac{I_\mathrm{p}}{t_\mathrm{p}} \tag{1}$$

$$\frac{\mathrm{d}I_\mathrm{i}}{\mathrm{d}t} = E\left(\frac{I_\mathrm{p}}{V_\mathrm{p}} - \frac{I_\mathrm{i}}{V_\mathrm{i}}\right) - \frac{I_\mathrm{i}}{t_\mathrm{i}} \tag{2}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = G_\mathrm{in} - f_2(G) - f_3(G)f_4(I_\mathrm{i}) + f_5(x_3) \tag{3}$$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{3}{t_\mathrm{d}}(I_\mathrm{p} - x_1) \tag{4}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{3}{t_\mathrm{d}}(x_1 - x_2) \tag{5}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{3}{t_\mathrm{d}}(x_2 - x_3) \tag{6}$$

The variables I_i, I_p and G denote the concentrations of insulin in intercellular spaces, insulin in plasma and glucose in plasma, respectively. The time lags between hepatic production of glucose and the insulin in plasma are denoted by x_1, x_2 and x_3, respectively. Insulin secreted is either degraded or transported to intercellular space. The transportation occurs at a diffusion rate E (Polonsky et al. 1986). Insulin is degraded exponentially at time constants t_p and t_i [Eqs. (1) and (2)], in plasma and intercellular spaces, respectively. The appearance of insulin based on the concentration of glucose is by Polonsky et al. (1988) and Shapiro et al. (1988).

$$f_1(G) = \frac{R_\mathrm{m}}{1 + \exp((C_1 - G/V_\mathrm{g})/a_1)} \tag{7}$$

The glucose consumed by brain and nerve cells (Verdonk et al. 1981) is:

$$f_2(G) = U_\mathrm{b}(1 - \exp(-G/(C_2 V_\mathrm{g}))) \tag{8}$$

Verdonk et al. (1981) and Rizza et al. (1981) proposed the term depending on glucose as given in (9), whereas (Verdonk et al. 1981) the term depending on insulin is given in Eq. (10).

$$f_3(G) = \frac{G}{C_3 V_\mathrm{g}} \tag{9}$$

$$f_4(G) = U_0 + \frac{U_\mathrm{m} - U_0}{1 + \exp(-\beta \ln(I_\mathrm{i}/C_4(1/V_\mathrm{i} + 1/Et_\mathrm{i})))} \tag{10}$$

Rizza et al. (1981) proposed the hepatic production of glucose due to the effect of insulin as follows:

$$f_5(x_3) = \frac{R_\mathrm{g}}{1 + \exp(\alpha(x_3/V_\mathrm{p} - C_5))} \tag{11}$$

Khan et al. (2013) then presented the states space model for the insulin–glucose system as below.

$$\begin{bmatrix} I'_\mathrm{p} \\ I'_\mathrm{i} \\ G'_\mathrm{in} \\ x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & \left(\frac{3I_\mathrm{p}}{x_1 t_\mathrm{d}} - \frac{3}{t_\mathrm{d}}\right) & 0 & 0 \\ -1 & 1 & -1 & \frac{3}{t_\mathrm{d}} & \frac{-3}{t_\mathrm{d}} & 0 \\ 1 & -1 & -1 & 0 & \frac{3}{t_\mathrm{d}} & \frac{-3}{t_\mathrm{d}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$+ \begin{bmatrix} -\left(\frac{E}{V_\mathrm{p}} + \frac{1}{t_\mathrm{p}}\right) & \frac{E}{V_\mathrm{i}} & \frac{f_1(G)}{G_\mathrm{in}} & 1 & 1 & 1 \\ \frac{E}{V_\mathrm{p}} & -\left(\frac{1}{t_\mathrm{i}} + \frac{E}{V_\mathrm{i}}\right) & 0 & 1 & 1 & 1 \\ 0 & \frac{-f_3(G)f_4(I_\mathrm{i})}{I_\mathrm{i}} & \left(1 - \frac{f_2(G)}{G_\mathrm{in}} + \frac{f_5(x_3)}{G_\mathrm{in}}\right) & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} I_\mathrm{p} \\ I_\mathrm{i} \\ G_\mathrm{in} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [G^{+-}, I^{+-}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_\mathrm{p} \\ I_\mathrm{i} \\ G_\mathrm{in} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

Here x_1, x_2 and x_3 are states variables, I_i, I_p and G_in input variables, I_p and $G(t)$ are output and feedback responses, respectively. There are three feedback loops. 1: glucose concentration->insulin secretion->insulin concentration->glucose uptake->glucose concentration. 2: glucose concentration->glucose uptake->glucose concentration. 3: glucose concentration->insulin secretion->insulin concentration->glucose production->glucose concentration. All of them are operative at all times, but depending on time

Table 1 Parameters of the model

Parameter	Value	Parameter	Value
V_p	3	t_p	6 (min)
V_i	11	t_i	100 (min)
V_g	10	t_d	36 (min)
U_b	72 (mg min^{-1})	E	0.2 (min)
U_0	40 (mg min^{-1})	a_1	300 (mg l^{-1})
U_m	940 (mg min^{-1})	α	0.29 (l mU^{-1})
β	1.77	C_1	2000 (mg l^{-1})
R_m	210 (mU min^{-1})	C_2	144 (mg l^{-1})
R_g	180 (mg min^{-1})	C_3	1000 (mg l^{-1})
C_4	80 (mU l^{-1})	C_5	26 (mU l^{-1})

and concentrations, one particular feedback loop may contribute more or less to the fluxes than the others.

For example, for a high glucose level, the β-cells are stimulated to secrete insulin. Likewise, for low glucose level, the α-cells secrete glucagons to inhibit the hepatic glucose production. The responses of the model are denoted as $[G^{+-}, I^{+-}]$. Here, the negative and positive signs indicate whether the glucose or the insulin loop is active at any time. Table 1 gives the model parameters (Sturis 1991; Sturis et al. 1991; Tolić et al. 2000).

Khan et al. (2013) proposed a feedback fuzzy states space model (FFSSM) for the insulin–glucose system in humans. The idea of an artificial pancreas controlling the insulin–glucose dynamics using modern control theory was presented in Mythreyi et al. (2014), Nimri and Phillip (2014), Russell et al. (2014) and Turksoy et al. (2014). Recently, Gonz'alez and Cipriano (2016) have used an Extended Kalman Filter to control insulin infusion in type 1 diabetic subjects with a fuzzy controller.

The objectives of the current research are listed below.

1. To propose a revised feedback fuzzy states space model for the insulin–glucose regulatory system in humans.
2. To study the important structural properties of the proposed model.
3. To propose the revised convexity property of the feedback fuzzy states space model.
4. To propose the revised normality property of the feedback fuzzy states space model.
5. To propose the revised modified optimized defuzzified value theorem and implement it on the insulin–glucose regulatory system in humans.
6. To propose the revised stability property of the FFSSM.
7. To propose the revised bounded-input stability property of the FFSSM and implement it on the feedback insulin–glucose regulatory system in humans.
8. To propose the revised bounded-output stability property of the FFSSM and implement it on the feedback insulin–glucose regulatory system in humans.

The motivation behind this work is to study the structural properties of the insulin–glucose regulatory system of the humans. We study the normality, convexity, bounded-input stability, bounded-output stability, states stability, and, the revised modified optimized defuzzified value theorem and check the mentioned properties for the insulin–glucose regulatory system in the humans. The successful implementation of these properties shows the credibility, usefulness and robustness of the feedback fuzzy states space model of the insulin–glucose regulatory system in humans.

The novelty/superiority of the research can be counted as follows:

1. First of all a new revised feedback fuzzy states space model for the insulin–glucose regulatory system in humans is formulated.
2. After the formulation the revised convexity property of the feedback fuzzy states space model is studied comprehensively. The property is documented as Theorem 1.
3. The revised normality property of the feedback fuzzy states space model results in the formulation and proof of Theorem 2.
4. Revised modified optimized defuzzified value theorem is formulated and proved as documented in Theorem 3.
5. The proposed revised stability property of the FFSSM can be seen as Theorem 4.
6. The proposed revised bounded-input stability property of the FFSSM is formulated and proved in Lemma 2.
7. The proposed revised bounded-output stability property of the FFSSM is formulated and proved in Theorem 5.
8. All the above-proposed properties are then implemented on the feedback insulin–glucose regulatory system in humans. The successful implementation and promising results show the credibility and robustness of the FFSSM model of the insulin–glucose regulatory system.

In this paper, more structural properties of the FFSSM model proposed in Khan et al. (2013) are investigated. The motive is to gain more knowledge, expertise and insight into the FFSSM model of insulin–glucose regulatory system proposed in Khan et al. (2013). These properties are related to revised modified optimized defuzzified value theorem (RMODVT), revised convexity, revised normality and revised BIBO stability properties of the FFSSM. The successful implementation and proofs of these properties of FFSSM model of insulin–glucose regulatory system in human beings reflect the credibility of the model. Moreover, the model becomes more flexible, effective and reasonable because of modeling uncertainty in the system construction.

This paper is organized as follows. After Introduction in Sect. 1, Sect. 2 consists of four subsections, in which the revised feedback (FFSSM), the revised convexity/normality properties, revised modified optimized defuzzified value theorem and properties related to revised stability of feedback (FFSSM) are presented. The proposed properties presented in Sect. 2 are then proved for FFSSM for the insulin–glucose system in humans in Sect. 3. Finally, the conclusion is given in Sect. 4.

2 Materials and methods

This section consists of four subsections. In Sect. 2.1, the revised feedback fuzzy state space model (FFSSM) algorithm is presented. This is followed by three sections: Sects. 2.2, 2.3 and 2.4 in which the revised convexity and normality properties, revised modified optimized defuzzified value theorem and properties related to revised stability of the feedback (FFSSM) are comprehensively studied.

2.1 Revised feedback fuzzy state space model (FFSSM) algorithm

To develop an inverse feedback fuzzy state space model (FFSSM) for a dynamical system, the system is first transformed to a state space model. The state space model conserves the dynamical properties (Cao and Rees 1995). The hallmark of the (FFSSM) model is that there is a direct transmission between inputs and outputs.

Definition 1 A feedback fuzzy state space model (FFSSM) of a dynamical system is defined as:

$$SgF: \quad \begin{aligned} \dot{x}(t) &= Ax(t) + B\tilde{u}(t), \\ \tilde{y}(t) &= Cx(t) + D\tilde{u}(t). \end{aligned} \tag{13}$$

with the feedback law, $\tilde{u}(t) = K\tilde{y}(t)$.

Where, "K" is the feedback matrix, $\tilde{u}(t) = [u_1, u_2, \ldots, u_n]^{\mathrm{T}}$ and $\tilde{y}(t) = [y_1, y_2, \ldots, y_m]^{\mathrm{T}}$ denote the fuzzified input and fuzzified output matrixes with conditions $t_0 = 0$ and $x_0 = x(t_0) = 0$. Matrices $A_{p\times p}$, $B_{p\times m}$, $C_{m\times p}$, $D_{p\times n}$ denote the states, input, output and feedback matrixes, respectively.

Definition 2 A feedback fuzzy state space model (FFSSM) of a system can be defined as $SgF: \Re^n \to \Re^2$ with feed forward response $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$ and feedback response $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$. The system has two responses, one of which is a feedback response.

For inputs u_{i1} and u_{i2} that belong to intervals I_{i1} and I_{i2}, assume that the preferences for these inputs be presented by fuzzy sets $F_{I_{i1}}$ and $F_{I_{i2}}$, such that $F_{I_{i1}} \in I_{i1}$ and $F_{I_{i2}} \in I_{i2}$. The fuzzy preference parameters for the inputs $F_{I_{i1}}$ and $F_{I_{i2}}$, for SgF_{FF} and SgF_{FB} being fuzzy sets over $u_{i1}, u_{i2} \in I_{i1}, I_{i2} \subset \Re(i1, i2 \in \aleph)$, are normal and convex. Let SgF_{FF} and SgF_{FB} denote the preferred values for these responses, and its fuzzy values are $FSgF_{FF}$ and $FSgF_{FB}$, respectively. Note that preferred values are preferred more than the other ones by the expert opinion. The fuzzy induced values of SgF_{FF} and SgF_{FB} are represented as F_{Ind_1} and F_{Ind_2}, respectively. These values are actually induced by the model. The algorithm determines the optimal values between $\left(F_{Ind_1} \text{ and } F_{SgF_{FF}}\right)$ and $\left(F_{Ind_2} \text{ and } F_{SgF_{FB}}\right)$, using Theorem 3, for SgF_{FF} and SgF_{FB}, respectively.

2.2 Properties related to revised convexity and normality of feedback fuzzy state space model

Yang (1995) and Yang and Yang (2002) presented the convexity and normality properties of fuzzy sets. The idea of fuzzy functions was proposed in Syau (2000). Saade (1996) presented the idea of convex functions on normal fuzzy sets. Pearson et al. (1997) studied the normality of a system $\dot{x}(t) = Ax(t)$, $x \in \Re^n, A : \Re^n \to \Re^n$, when the initial state x_0 was known with uncertainly. The convexity and normality of FSSM were studied in Razidah (2005).

Definition 3 In Zadeh (1965), A fuzzy set "A" is convex if and only if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), x_1, x_2 \in \Re^n, \quad \lambda \in [0,1]. \tag{14}$$

Definition 4 A fuzzy set $A \in \Re^n$, is normal iff $\operatorname*{Sup}_{x\in X} \mu_A(x) = 1$ in Ahmad (1998).

Lemma 1 *In* Klir and Yuan (1995)*, if a fuzzy set* $A \subset F(x), A \in \Re^n$ *is convex, then* $[A]_{\alpha_2} \subseteq [A]_{\alpha_1}, \forall \alpha_2 \geq \alpha_1$.

Theorem 1 (Revised Convexity Property of FFSSM) *Let a feedback fuzzy state space model (FFSSM) be defined as* $SgF: \Re^n \to \Re^2$ *and have feed forward response* $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$ *and the feedback response* $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$. *Let all the inputs* $u_{i1}, u_{i2} \in I_{i1}, I_{i2} \subset \Re(i1, i2 \in \aleph)$ *be convex, and then the induced solutions of* SgF *for the feed forward and the feedback responses are convex.*

Proof We prove for $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$ first, provided that the inputs $u_{11}, u_{21}, \ldots, u_{n1}$ are convex. From

Lemma 1, $[u_{i1}]_{\alpha_{j+1}} \subseteq [u_{i1}]_{\alpha_j}, \forall \alpha_{j+1} \geq \alpha_j$, $\alpha_{j+1} \in [0,1]$ for $i = 1,2,3,\ldots,n$ when $j = 1,2,3,\ldots,k$.

For SgF_{FF}, determine all endpoints of inputs $[u_{i1}]_{\alpha_{j+1}}$ and denote $\{u_{i1\min}, u_{i1\max}\}_{\alpha_{j+1}}$.

Determine all the end points of the inputs $[u_{i1}]$ for $\alpha_{j+1} \in [0,1]$ and denote as $\left\{(u_{11}, u_{21}, \ldots, u_{n1})_{\alpha_{j+1}}\right\}$.

Determine values of $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1})_{\alpha_{j+1}}$ at $\alpha_{j+1} \in [0,1]$ and denote its maximum/minimum values as.

$$y_{1\min_{\alpha_{j+1}}} = \min\left\{SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1})_{\alpha_{j+1}}\right\}. \tag{15}$$

$$y_{1\max_{\alpha_{j+1}}} = \max\left\{SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1})_{\alpha_{j+1}}\right\}. \tag{16}$$

The α_{j+1} cut interval for SgF_{FF} is $y_{1\alpha_{j+1}} = [y_{1\min\alpha_{j+1}}, y_{1\max\alpha_{j+1}}]$, for all $\alpha_{j+1} \in [0,1]$.

For convexity, we need to prove that $y_{1\alpha_{j+1}} \subseteq y_{1\alpha_j}$, for all $\alpha_{j+1} \geq \alpha_j$

Let $y_1 \in [y_1]_{\alpha_{j+1}} \Rightarrow y_1 \in [y_{1\min\alpha_{j+1}}, y_{1\max\alpha_{j+1}}]$ and $y_1 = SgF_{FF}(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})_{\alpha_{j+1}}$ be the optimal feed forward response value for some of the inputs $(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})$, where $u^*_{i1} \in [u^*_{i1}]_{\alpha_{j+1}}$ for all $i = 1,2,3,\ldots n$.

Now since u_{i1} are convex for all $i = 1,2,3,\ldots n$, $u^*_{i1} \in [u^*_{i1}]_{\alpha_{j+1}} \subseteq [u^*_{i1}]_{\alpha_j}$, by Razidah (2005).

$$y_1 = SgF_{FF}\left(u^*_{11}, u^*_{21}, \ldots, u^*_{n1}\right)_{\alpha_{j+1}} \in \left[y_{1\min\alpha_{j+1}}, y_{1\max\alpha_{j+1}}\right] \subseteq \left[y_{1\min\alpha_j}, y_{1\max\alpha_j}\right]. \tag{17}$$

$$y_1 = SgF_{FF}\left(u^*_{11}, u^*_{21}, \ldots, u^*_{n1}\right)_{\alpha_{j+1}} \in \left[y_{1\min\alpha_j}, y_{1\max\alpha_j}\right], \text{alternatively, } y_1 \in [y_1]_{\alpha_j}. \tag{18}$$

Hence, $[y_1]_{\alpha_{j+1}} \subseteq [y_1]_{\alpha_j}$ and so induced solution of the feed forward response SgF_{FF} is convex.

Moreover, from Lemma 1, the inputs $u_{12}, u_{22}, \ldots, u_{n2}$ are convex, for SgF_{FB}. Thus, from Lemma 1.

$$[u_{i2}]_{\alpha_{j+1}} \subseteq [u_{i2}]_{\alpha_j}, \forall \alpha_{j+1} \geq \alpha_j, \quad \alpha_{j+1} \in [0,1] \text{ for } i = 1,2,3,\ldots,n \text{ when } j = 1,2,3,\ldots,k. \tag{19}$$

For SgF_{FB}, determine all endpoints of $[u_{i2}]_{\alpha_{j+1}}$ and denote them as $\{u_{i2\min}, u_{i2\max}\}_{\alpha_{j+1}}$.

Determine all the end points of $[u_{i2}]$ for $\alpha_{j+1} \in [0,1]$ and let denote them as $\left\{(u_{12}, u_{22}, \ldots, u_{n2})_{\alpha_{j+1}}\right\}$.

Find the maximum/minimum values $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2})_{\alpha_{j+1}}$ at $\alpha_{j+1} \in [0,1]$.

$$y_{2\min_{\alpha_{j+1}}} = \min\left\{SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2})_{\alpha_{j+1}}\right\}. \tag{20}$$

$$y_{2\max_{\alpha_{j+1}}} = \max\left\{SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2})_{\alpha_{j+1}}\right\}. \tag{21}$$

The α_{j+1} interval for $SgF_{FB}(u_{i2})$ is $\left[y_{2\alpha_{j+1}}\right] = \left[y_{2\min_{\alpha_{j+1}}}, y_{2\max_{\alpha_{j+1}}}\right]$, for all $\alpha_{j+1} \in [0,1]$.

To prove that the convexity of the induced solution of SgF_{FB}, need to prove $[y_2]_{\alpha_{j+1}} \subseteq [y_2]_{\alpha_j}, \forall \alpha_{j+1} \geq \alpha_j$.

Let $y_2 \in [y_2]_{\alpha_{j+1}} \Rightarrow y_2 \in [y_{2\min\alpha_{j+1}}, y_{2\max\alpha_{j+1}}]$, where $y_2 = SgF_{FB}(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})_{\alpha_{j+1}}$ denote optimal value of SgF_{FB}, for the inputs $(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})$ where $u^*_{i2} \in [u_{i2}]_{\alpha_{j+1}}$ for all $i = 1,2,3,\ldots,n$.

As u_{i2} are convex $\forall\ i = 1,2,3,\ldots,n$, $u^*_{i2} \in [u^*_{i2}]_{\alpha_{j+1}} \subseteq [u^*_{i2}]_{\alpha_j}$.

$$y_2 = SgF_{FB}\left(u^*_{12}, u^*_{22}, \ldots, u^*_{n2}\right)_{\alpha_{j+1}} \in \left[y_{2\min\alpha_{j+1}}, y_{2\max\alpha_{j+1}}\right] \subseteq \left[y_{2\min\alpha_j}, y_{2\max\alpha_j}\right]. \tag{22}$$

$$y_2 = SgF_{FB}\left(u^*_{12}, u^*_{22}, \ldots, u^*_{n2}\right)_{\alpha_{j+1}} \in \left[y_{2\min\alpha_j}, y_{2\max\alpha_j}\right] \text{ or } y_2 \in [y_2]_{\alpha_j}, [y_2]_{\alpha_{j+1}} \subseteq [y_2]_{\alpha_j}. \tag{23}$$

Thus, the SgF_{FB} is convex. Thus, the convexity of the induced solution of SgF_{FF} and SgF_{FB} implies the convexity of the induced solution of SgF.

Theorem 2 (Theorem Related to the Revised Normality Property of FFSSM) *Let the performance of a multi-input single-output (MISO) feedback fuzzy state space model (FFSSM) be defined as* $SgF : \Re^n \to \Re^2$, *composed of the feed forward response* $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$ *and the feedback response* $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$. *Provided that all the inputs* $u_{i1}, u_{i2} \in I_{i1}, I_{i2} \subset \Re(i1, i2 \in \aleph)$ *are normal, then the induced solutions of SgF for the feed forward and the feedback responses are also normal.*

Proof Given that the inputs $u_{i1}, u_{i2} \in I_{i1}, I_{i2} \subset \Re(i1, i2 \in \aleph)$ for the feed forward and feedback responses, respectively, are normal.

For the feed forward response, there exist $u^*_{i1} \in u_{i1}$ such that $\mu(u^*_{i1}) = 1$ for all $i = 1,2,3,\ldots,n$.

Let $y_1 = SgF_{FF}(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})_{\alpha=1}$, so $\mu(y_1) = \mu(SgF_{FF}(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})_{\alpha=1}) = 1$.

Showing that the induced solution of SgF_{FF} is normal.

For feedback response, there exist $u^*_{i2} \in u_{i2}$ such that $\mu(u^*_{i2}) = 1$ for all $i = 1,2,3,\ldots,n$.

Let $y_2 = SgF_{FB}(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})_{\alpha=1}$, so $\mu(y_2) = \mu(SgF_{FB}(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})_{\alpha=1}) = 1$

This shows that SgF_{FB} is normal. Thus, SgF are normal for SgF_{FF} and SgF_{FB}.

2.3 Revised modified optimized defuzzified value theorem for FFSSM

The proposed revised modified optimized defuzzified value theorem (RMODVT) gives the optimal values of the inputs and outputs if implemented on a particular real-world problem. The role of the RMODVT is in the process of defuzzification. In the modeling scenario, inputs are given to the system, the inputs are then fuzzified, and RMODVT is then used to select the optimal values of the inputs and outputs. Furthermore, it defuzzify the inputs and outputs and gives optimal values of the inputs and outputs. In this research, RMODVT is applied to the real-world model of the insulin–glucose regulatory system in humans. The optimal defuzzified results of RMODVT are presented in Tables 2 and 3, for optimal outputs and inputs, respectively. The promising results and successful implementation of the RMODVT on the real-world scenario of the insulin–glucose regulatory system of the humans show the importance, credibility and superior ability of the RMODVT.

The basic characteristics of the revised optimized defuzzified value theorem are twofold.

1. Choose the optimal values of the inputs and outputs for a states space model.
2. Defuzzify the model, and give the defuzzified values of the inputs and outputs.

RMODVT when implemented on the insulin–glucose regulatory system in humans gives promising and successful results as shown in Tables 2 and 3.

This section formulates and proves the important part of the FFSSM. The role of revised modified optimized defuzzified value theorem is in the process of defuzzification. Ahmad (1998) proposed a credible theorem named "Modified Optimized Defuzzified Value Theorem." Moreover, Razidah et al. (2009), Khan et al. (2012a b) and Khan (2013) employed MODVT to study the FSSM of the furnace system of power generation plant, and Normah (2005) used it in modeling of a single drop in a RDC column. In the current work, the MODVT of Razidah (2005) is revised to address feedback system.

Theorem 3 (Revised Modified Optimized Defuzzified Value Theorem RMODVT) *Let a feedback fuzzy state space model (FFSSM)* $SgF : \Re^n \to \Re^2$ *have feed forward response* $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$ *and the feedback response* $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$.

(a) For the feed forward response,
$SgF^*_{FF}(u_{i1}) = r^*_j = \min(r_j)$ such that $\mu(r^*_j) = f^*_1$ and $(r_j, f^*_1) \in F_{Ind_1}$, $j = 1, 2, 3, \ldots, 2^n$.
Then, $r^*_j = SgF^*_{FF}(u_{i1}) = \min\|SgF_{FF}(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})\|$ where $\mu(u^*_{i1}) = f^*_1$, $i = 1, 2, 3, \ldots, n$.
For $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$, feedback response,
$SgF^*_{FB}(u_{i2}) = d^*_j = \min(d_j)$ such that $\mu(d^*_j) = f^*_2$ when $(d_j, f^*_2) \in F_{Ind_2}$, $j = 1, 2, 3, \ldots, 2^n$.
Thus, $d^*_j = SgF^*_{FB}(u_{i2}) = \min\|SgF_{FB}(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})\|$, where $\mu(u^*_{i2}) = f^*_2$, $i = 1, 2, 3, \ldots, n$.

(b) For the feed forward $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$ response,
$SgF^*_{FF}(u_{i1}) = r^*_j = \max(r_j)$ so that $\mu(r^*_j) = f^*_1$ and $(r_j, f^*_1) \in F_{Ind_1}$, $j = 1, 2, 3, \ldots, 2^n$.
Then, $r^*_j = SgF^*_{FF}(u_{i1}) = \max\|SgF_{FF}(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})\|$ where $\mu(u^*_{i1}) = f^*_1$, $i = 1, 2, 3, \ldots, n$.
Again for $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$,
$SgF^*_{FB}(u_{i2}) = d^*_j = \max(d_j)$ so that $\mu(d^*_j) = f^*_2$ when $(d_j, f^*_2) \in F_{Ind_2}$, $j = 1, 2, 3, \ldots, 2^n$.
Then, $d^*_j = SgF^*_{FB}(u_{i2}) = \max\|SgF_{FB}(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})\|$, where $\mu(u^*_{i2}) = f^*_2$, $i = 1, 2, 3, \ldots, n$.

*Proof of Case (a): Fuzzy Preferred Vales on the Right (Maximum) Side of Induced Values*Let us consider the $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$ first.

Table 2 Proposed outputs

		Outputs			
		Preferred range	Preferred value	Calculated	% Difference
Insulin (feed forward)	I_p	[11.0566–18.3554]	18	18.0158	0.0876
Glucose (feedback)	G	[71.9518–103.8242]	100	102.5601	2.5601

Table 3 Proposed input parameters and its percentage difference in calculated and preferred values

Inputs	Plasma insulin (feed forward)			Plasma glucose (feedback)		
	Preferred value	Calculated value	% Difference	Preferred value	Calculated value	% Difference
I_i	19	21.8612	15.0592	19	21.6741	14.0741
I_p	20	25.4306	27.1531	20	25.3370	26.6852
G_{in}	125	143.8612	15.0890	125	143.6741	14.9393

$SgF^*_{FF}(u_{i1}) = r^*_j = \min(r_j), j = 1, 2, 3, \ldots, 2^n$. So that $\mu(r^*_j) = f^*_1$ where $(r_j, f^*_1) \in F_{Ind_1}$. Determine the f^*_1 cuts for $F_{I_{i1}}$. This becomes n-tuple $(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})$. Find $r^*_j = SgF^*_{FF}(u_{i1}) = \min\|SgF_{FF}(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})\|$, when $\mu(u^*_{i1}) = f^*_1$, $i = 1, 2, 3, \ldots, n$.Knowing that F_{Ind_1} is normal and convex, so $r_j = r^*_j$, $j = 1, 2, 3, \ldots 2^n$.Furthermore, for $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$,$SgF^*_{FB}(u_{i2}) = d^*_j = \min(d_j), j = 1, 2, 3, \ldots, 2^n$. So that $\mu(d^*_j) = f^*_2$ when $(d_j, f^*_2) \in F_{Ind_2}$. Determine the f^*_2 cuts for $F_{I_{i2}}$. This again becomes n-tuple $(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})$. Determine $d^*_j = SgF^*_{FB}(u_{i2}) = \min\|SgF_{FB}(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})\|$, when $\mu(u^*_{i2}) = f^*_2$, $i = 1, 2, 3, \ldots, n$. Since F_{Ind_2} is normal and convex, $d_j = d^*_j$, $j = 1, 2, 3, \ldots, 2^n$. The case is shown in Fig. 1.

Case (b) Fuzzy Preferred Values on the Left (Minimum) Side of the Induced Values

For $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = r$,

$SgF^*_{FF}(u_{i1}) = r^*_j = \max(r_j), j = 1, 2, 3, \ldots, 2^n$. such that $\mu(r^*_j) = f^*_1$ and $(r_j, f^*_1) \in F_{Ind_1}$.

Determine the f^*_1 cuts for $F_{I_{i1}}$. This becomes the n-tuple $(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})$.

Find $r^*_j = SgF^*_{FF}(u_{i1}) = \max\|SgF_{FF}(u^*_{11}, u^*_{21}, \ldots, u^*_{n1})\|$, when $\mu(u^*_{i1}) = f^*_1$, $i = 1, 2, 3, \ldots, n$.

But F_{Ind_1} is normal and convex, so $r_j = r^*_j$, $j = 1, 2, 3, \ldots 2^n$.

Further, for $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = d$,

$SgF^*_{FB}(u_{i2}) = d^*_j = \max(d_j), j = 1, 2, 3, \ldots, 2^n$. so that $\mu(d^*_j) = f^*_2$ when $(d_j, f^*_2) \in F_{Ind_2}$.

Determine the f^*_2 cuts for $F_{I_{i2}}$. Again it becomes n-tuple $(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})$.

Set $d^*_j = SgF^*_{FB}(u_{i2}) = \max\|SgF_{FB}(u^*_{12}, u^*_{22}, \ldots, u^*_{n2})\|$, where $\mu(u^*_{i2}) = f^*_2$, $i = 1, 2, 3, \ldots, n$.

Since F_{Ind_2} is normal and convex, $d_j = d^*_j$, $j = 1, 2, 3, \ldots 2^n$.. This is shown in Fig. 2.

2.4 Revised states stability property of feedback FSSM

The feedback response imposes new restrictions on the states stability of the feedback system. The following theorem establishes the stability of FFSSM.

Theorem 4 (Theorem Related to the Revised Stability Property of FFSSM) *Let the performance of a multi-input single-output (MISO) feedback fuzzy state space model (FFSSM) be defined as $SgF : \Re^n \to \Re^2$, composed of the feed forward response $SgF_{FF} : \dot{x} = Ax + Bu$, $y = Cx + Du$. and the feedback response $SgF_{FB} : \dot{x} = Ax + Bu$, $y = Cx + Du$. with the feedback dynamical law $u = -Ky = -K(Cx + Du)$ is the feedback law and "K" the feedback matrix. Then, for stability of $SgF : \Re^n \to \Re^2$ the following conditions hold. Case a: For feed forward*

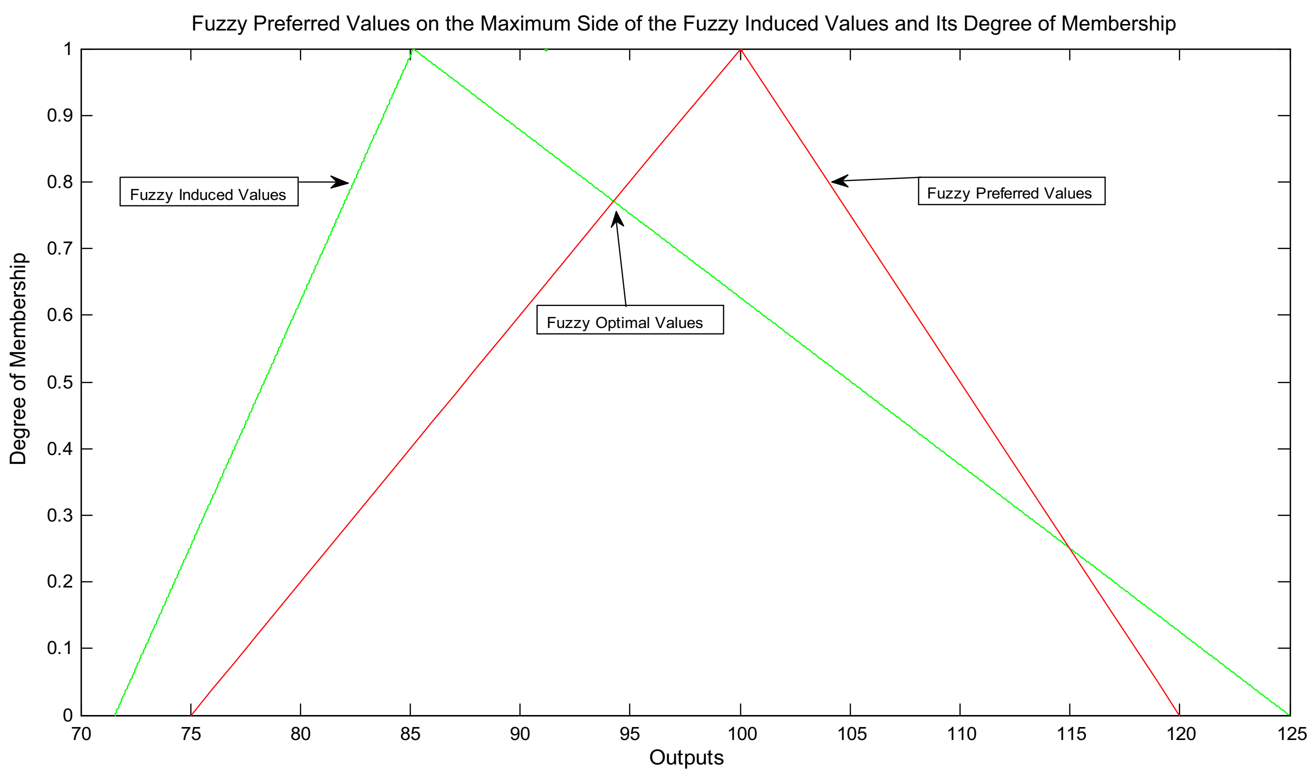

Fig. 1 Fuzzy preferred values on the right (maximum) side of fuzzy induced values

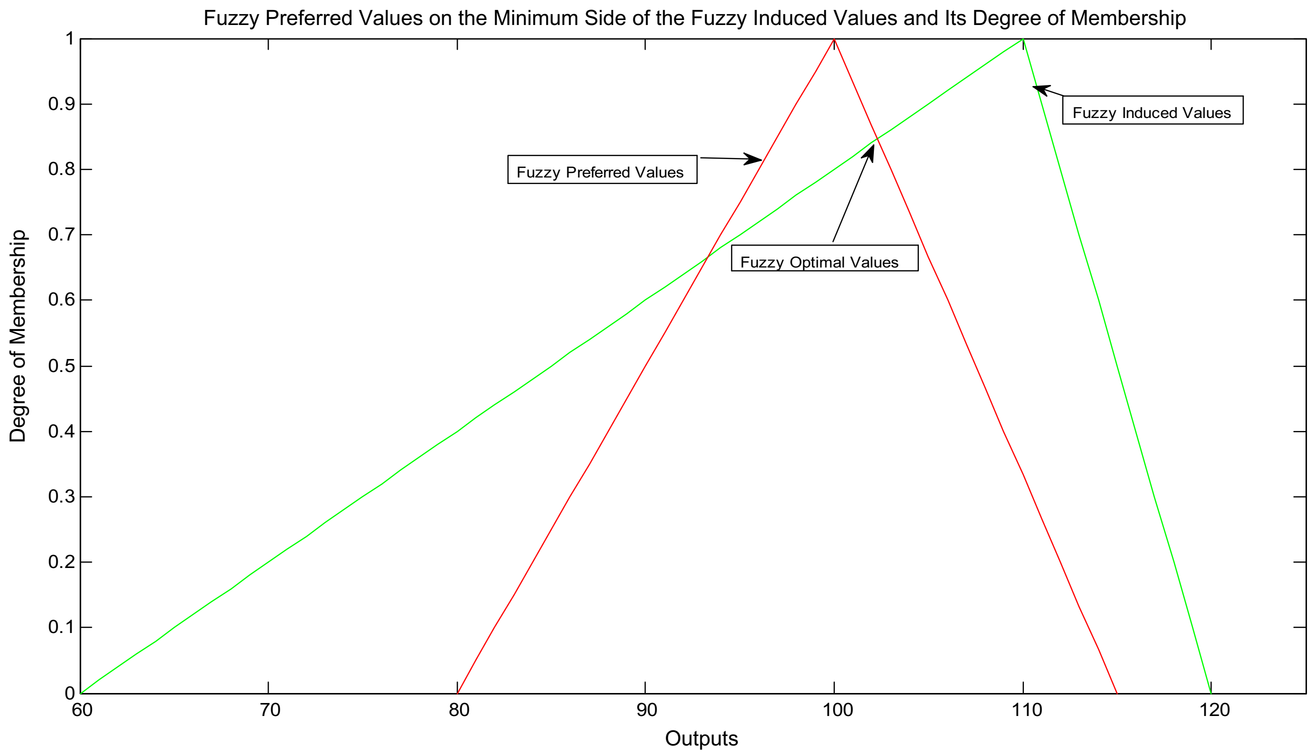

Fig. 2 Fuzzy preferred values on the left (minimum) side of induced values

response:The system is marginally stable if the eigenvalues of (A) have real parts equal to zero. For unstable system, at least one eigenvalue of (A) has a positive real part and for the stable system eigenvalues of (A) have negative real part.Case b: For feedback response:The system is marginally stable if eigenvalues of (A-BCK) have real parts equal to zero. For unstable system, at least one eigenvalue of (A-BCK) has a positive real part and for the stable system, eigenvalues of (A-BCK) have negative real part.

Proof *Case a* Let us consider the feed forward response (24).

$$\begin{aligned} SgF_{FF}: \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned} \tag{24}$$

Take Laplace transforms of (24), with zero initial conditions, i.e., $X(0) = 0$.

$$\begin{aligned} sX(s) &= AX(s) + BU(s), \\ Y(s) &= CX(s) + DU(s). \end{aligned} \tag{25}$$

$$\begin{aligned} (sI - A)X(s) &= BU(s), \\ Y(s) &= CX(s) + DU(s). \end{aligned} \tag{26}$$

$$\begin{aligned} X(s) &= (sI - A)^{-1}BU(s), \\ Y(s) &= C(sI - A)^{-1}BU(s) + DU(s). \end{aligned} \tag{27}$$

$$\begin{aligned} U(s) &= (sI - A)B^{-1}X(s), \\ Y(s) &= C(sI - A)^{-1}BU(s) + DU(s). \end{aligned} \tag{28}$$

Thus, the transfer function becomes.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{C(sI - A)^{-1}BU(s) + DU(s)}{(sI - A)B^{-1}X(s)} \tag{29}$$

Since $X(s) = (sI - A)^{-1}BU(s)$, Eq. (29) becomes.

$$\begin{aligned} T(s) &= \frac{Y(s)}{U(s)} = \frac{C(sI - A)^{-1}BU(s) + DU(s)}{(sI - A)B^{-1}(sI - A)^{-1}BU(s)} \\ &= C(sI - A)^{-1}B + D \end{aligned} \tag{30}$$

Equation (30) shows stability of the transfer function depends on the characteristic polynomial det(sI–A) = 0. Thus, the system is marginally stable, unstable or stable depending on the characteristics values of (sI–A). The system is marginally stable if the eigenvalues of (A) have real parts equal to zero. For unstable system, at least one eigenvalue of (A) has a positive real part, and for the stable system, eigenvalues of (A) have negative real part.-*Case b* Let us consider the feedback response (31).

$$\begin{aligned} SgF_{FB}: \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned} \tag{31}$$

where $u = -Ky = -K(Cx + Du)$ is the feedback law and "K" the feedback matrix. Take Laplace transforms of (31) with the zero initial conditions, i.e., $X(0) = 0$.

$$\begin{aligned} sX(s) &= AX(s) + BK[CX + DU], \\ Y(s) &= CX(s) + DK[CX + DU]. \end{aligned} \tag{32}$$

$$\begin{aligned} (sI - (A - BKC))X(s) &= BKDU(s), \\ Y(s) &= (C + DKC)X(s) + DKDU(s). \end{aligned} \tag{33}$$

$$\begin{aligned} X(s) &= (sI - (A - BKC))^{-1}BKDU(s), \\ Y(s) &= (C + DKC)(sI - (A - BKC))^{-1}BKDU(s) \\ &\quad + DKDU(s). \end{aligned} \tag{34}$$

$$\begin{aligned} U(s) &= (sI - (A - BKC))B^{-1}K^{-1}D^{-1}X(s), \\ Y(s) &= (C + DKC)(sI - (A - BKC))^{-1}BKDU(s) \\ &\quad + DKDU(s). \end{aligned} \tag{35}$$

Thus, the transfer function becomes.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{(C + DKC)(sI - (A - BKC))^{-1}BKDU(s) + DKDU(s)}{(sI - (A - BKC))B^{-1}K^{-1}D^{-1}X(s)} \tag{36}$$

Since $X(s) = (sI - (A - BKC))^{-1}BKDU(s)$, Eq. (36) becomes.

$$\begin{aligned} T(s) &= \frac{(C + DKC)(sI - (A - BKC))^{-1}BKDU(s) + DKDU(s)}{(sI - (A - BKC))B^{-1}K^{-1}D^{-1}(sI - (A - BKC))^{-1}BKDU(s)}, \\ &= (C + DKC)(sI - (\mathrm{A\text{ - }BCK}))^{-1}BKD + DKD = G(s) + DKD. \end{aligned} \tag{37}$$

Equation (37) shows that the states stability of a feedback FSSM having a feedback law $\tilde{u}(t) = K\tilde{y}(t) = K(Cx(t) + D\tilde{u}(t))$ depends on the characteristic values of (A-BCK). The characteristic polynomial of (37) is det(sI–(A-BCK)) = 0.

For marginally stable system, the eigenvalues of (A-BCK) have real parts equal to zero. For unstable system, at least one eigenvalue of (A-BCK) has a positive real part, and for the stable system, eigenvalues of (A-BCK) have negative real parts. The stability of the feed forward and feedback responses ensures the stability of $SgF : \Re^n \to \Re^2$.

Apart from states stability, another structural property is input stability. A system is input bounded; for an input $U(t)$, there are upper bounds U_M, so that $|U(t)| \le U_M < \infty$ (Bay 1999). For input stability, Lemma 2 is formulated and proved.

Lemma 2 (Revised Bounded-Input Stability of FFSSM) *Let a feedback fuzzy state space model (FFSSM) $SgF : \Re^n \to \Re^m$ have a feed forward response $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = (r_1, r_2, \ldots, r_m)$ and a feedback response $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = (d_1, d_2, \ldots, d_m)$. Provided that the fuzzy sets of inputs SgF_{FF} and SgF_{FB} are bounded, then SgF is bounded input.*

Proof Consider a FFSSM $SgF : \Re^n \to \Re^m$ with feed forward response $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = (r_1, r_2, \ldots, r_m)$ and feedback response $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = (d_1, d_2, \ldots, d_m)$, respectively. Let us consider the feedback law $\tilde{u}(t) = K\tilde{y}(t)$, with total "$m = p + n$" responses. Thus, for a system response there are "m" possibilities to be a feedback or a feed forward response. Let us assume that "o" out of "n," $o \le n$, responses are feedback and then the remaining (m–o) responses become feed forward.

Let us denote fuzzy inputs as G_{InFF} and G_{InFB} for "(m–o)" feed forward and "o, ($o \le n$)" for the feedback responses, respectively.

$$G_{InFF} = \left\{ \begin{pmatrix} u_{11} \\ u_{21} \\ . \\ . \\ . \\ u_{(m-o)1} \end{pmatrix}_{\alpha} ; \right. \tag{38}$$

$(o \le n), u_{i1} \in [a_i, b_i]$ and $\alpha \in [0, 1]$.

$$G_{InFB} = \left\{ \begin{pmatrix} u_{11} \\ u_{22} \\ . \\ . \\ . \\ u_{o2} \end{pmatrix}_{\alpha} \right. ; (o \le n), u_{i2} \in [a_j, b_j] \text{ and } \alpha \in [0, 1]. \tag{39}$$

It needs to be proven that there are upper bounds M_{FF} and M_{FB}, such that

$$\left\| \begin{pmatrix} u_{11} \\ u_{21} \\ . \\ . \\ . \\ u_{(m-o)1} \end{pmatrix}_{\alpha} \right\| \le M_{FF}, \forall \left\| \begin{pmatrix} u_{11} \\ u_{21} \\ . \\ . \\ . \\ u_{(m-o)1} \end{pmatrix}_{\alpha} \right\| \in G_{InFF}, (o \le n)$$
$$\text{and} \left\| \begin{pmatrix} u_{12} \\ u_{22} \\ . \\ . \\ . \\ u_{o2} \end{pmatrix}_{\alpha} \right\| \le M_{FB}, \forall \left\| \begin{pmatrix} u_{12} \\ u_{22} \\ . \\ . \\ . \\ u_{o2} \end{pmatrix}_{\alpha} \right\| \in G_{InFB}, (o \le n). \tag{40}$$

Pick

$$\begin{pmatrix} u_{11} \\ u_{21} \\ \cdot \\ \cdot \\ \cdot \\ u_{(m-o)1} \end{pmatrix}_{\alpha} \in G_{InFF}, \ni u_{i1} \in [a_i, b_i], (o \le n). \quad (41)$$

Let the fuzzy sets $[a_i,\ b_i]$ for the SgF_{FF} be bounded.

Then, there are upper bounds $M_{FFi} \in \Re, \ni \|u_{i1}\| \le M_{FFi}\, \forall\, i = 1, 2, 3, \ldots, (m-o), (o \le n)$.

$$\left\| \begin{pmatrix} u_{11} \\ u_{21} \\ \cdot \\ \cdot \\ \cdot \\ u_{(m-o)1} \end{pmatrix} \right\| = \sqrt{u_{11}^2 + u_{21}^2 + \cdots + u_{(m-o)1}^2} \quad (42)$$

$$\le u_{11}^2 + u_{21}^2 + \cdots + u_{(m-o)1}^2$$

$$\le M_{FF1}^2 + M_{FF2}^2 + \cdots + M_{FF(m-o)}^2 \le M_{FF}.$$

Furthermore, for "o" feedback responses, ($o \le n$),

$$\begin{pmatrix} u_{12} \\ u_{22} \\ \cdot \\ \cdot \\ \cdot \\ u_{o2} \end{pmatrix}_{\alpha} \in G_{InFB}, \ni u_{i2} \in [a_j, b_j], (o \le n). \quad (43)$$

Let the sets of inputs $[a_j,\ b_j]$ for SgF_{FB} be bounded.

Then, there are upper bounds $M_{FBj} \in \Re, \ni \|u_{j2}\| \le M_{FBj}\, \forall\, j = 1, 2, 3, \ldots, o, (o \le n)$.

$$\left\| \begin{pmatrix} u_{12} \\ u_{22} \\ \cdot \\ \cdot \\ \cdot \\ u_{o2} \end{pmatrix} \right\| = \sqrt{u_{11}^2 + u_{22}^2 + \cdots + u_{o2}^2} \le u_{12}^2 + u_{22}^2 + \cdots + u_{o2}^2$$

$$\le M_{FB1}^2 + M_{FB2}^2 + \cdots + M_{FBo}^2 \le M_{FB}. \quad (44)$$

Thus, FFSSM is input bounded.

Another structural property of the system is output stability. A system is output stable; for every bounded input, there is a bounded output. In other words, for input $U(t)$, $|U(t)| \le U_M < \infty$ and $x_0 = x(t_0)$, there is an output $N(U_M,\ x_0,\ t_0)$, $\ni$ $\|y(t)\| \le$ N, $\forall\, t \ge t_0$ (Razidah 2005). For output and BIBO stability, Theorem 5 is formulated and proved.

Theorem 5 (Revised Output Stability of FFSSM) *Let a feedback fuzzy state space model (FFSSM)* $SgF : \Re^n \to \Re^m$ *have a feed forward response* $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = (r_1, r_2, \ldots, r_m)$ *and a feedback response* $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = (d_1, d_2, \ldots, d_m)$. *If fuzzy sets of inputs for* SgF_{FF} *and* SgF_{FB} *are bounded, then* SgF *is bounded output.*

Proof Consider Fuzzy State Space Model $SgF : \Re^n \to \Re^m$ having feed forward response $SgF_{FF}(u_{11}, u_{21}, \ldots, u_{n1}) = (r_1, r_2, \ldots, r_m)$ and feedback response $SgF_{FB}(u_{12}, u_{22}, \ldots, u_{n2}) = (d_1, d_2, \ldots, d_m)$. Let us consider the feedback law $\tilde{u}(t) = K\tilde{y}(t)$, with total "$m = p + n$" responses. Thus, for a system response there are "m" possibilities to be a feedback or a feed forward response. Let us assume that "o" out of "n," ($o \le n$) responses are feedback and then the remaining (m–o) responses become feed forward.

By Lemma 2, the fuzzy sets of inputs for the feed forward and feedback responses are bounded input. Thus, there exists least upper bound $M_{FF_i} \in \Re_i \ni \|u_{i1}\| \le M_{FF_i}, i = 1, 2, 3, \ldots, (m-o), (o \le n)$ for the feed forward response and $M_{FB_j} \in \Re_i \ni \|u_{i2}\| \le M_{FB_j}, \forall\, j = 1, 2, 3, \ldots, o, (o \le n)$ for the feedback response.

Let us denote G_{OutFF} as "m–o" feed forward responses and G_{OutFB} as "o, ($o \le n$)" feedback responses.

$$G_{OutFF} = k_{11}u_{11} + k_{21}u_{21} + \cdots + k_{(m-o)1}u_{(m-o)1},\ (o \le n). \quad (45)$$

$$G_{OutFB} = k_{12}u_{12} + k_{22}u_{22} + \cdots + k_{o2}u_{o2},\ (o \le n). \quad (46)$$

First prove for the "m–o" feed forward responses,

$$G_{OutFF} = k_{11}u_{11} + k_{21}u_{21} + \cdots + k_{(m-o)1}u_{(m-o)1} \quad (47)$$

$$\|G_{OutFF}\| = \sqrt{\left(k_{11}u_{11} + k_{21}u_{21} + \cdots + k_{(m-o)1}u_{(m-o)1}\right)^2} = k_{11}u_{11} + k_{21}u_{21} + \cdots + k_{(m-o)1}u_{(m-o)1} \quad (48)$$

$$\le \|k_{11}u_{11}\| + \|k_{21}u_{21}\| + \cdots + \left\|k_{(m-o)1}u_{(m-o)1}\right\|, \text{ By Lemma 2.}$$
$$\le \|k_{11}\|M_{FF1} + \|k_{21}\|M_{FF2} + \cdots + \left\|k_{(m-o)1}\right\|M_{FF(m-o)} \le M_{OutFF} \quad (49)$$

Fuzzy induced output for the feed forward response SgF_{FF} is bounded output.

For the "o, ($o \le n$)" feedback responses,

$$G_{OutFB} = k_{12}u_{12} + k_{22}u_{22} + \cdots + k_{o2}u_{o2} \quad (50)$$

$$\|G_{OutFB}\| = \sqrt{\left(k_{12}u_{12} + k_{22}u_{22} + \cdots + k_{o2}u_{o2}\right)^2} = k_{12}u_{12} + k_{22}u_{22} + \cdots + k_{o2}u_{o2} \quad (51)$$

$$\le \|k_{12}u_{12}\| + \|k_{22}u_{22}\| + \cdots + \|k_{o2}u_{o2}\|, \text{ By Lemma 2.}$$
$$\le \|k_{12}\|M_{FB1} + \|k_{22}\|M_{FB2} + \cdots + \|k_{o2}\|M_{FBo} \le M_{OutFB} \quad (52)$$

Table 4 Based on Lemma 2, input stability of the insulin–glucose system

Random inputs	$\begin{vmatrix} u_{11} \\ u_{21} \\ u_{31} \end{vmatrix}$	Max input on α-cuts	$\begin{vmatrix} u_{11} \\ u_{21} \\ u_{31} \end{vmatrix}_{\text{Max}}$
$\begin{pmatrix} 20 \\ 18 \\ 140 \end{pmatrix}_{\alpha=0}$	$\sqrt{(20)^2+(18)^2+(140)^2} = \sqrt{400+324+19600} = \sqrt{20324} = 142.56$	$\begin{pmatrix} 20 \\ 23 \\ 145 \end{pmatrix}_{\alpha=0}$	$\sqrt{(26)^2+(23)^2+(145)^2} = \sqrt{676+529+21025} = \sqrt{22230} = 149.09$
$\begin{pmatrix} 22.2 \\ 21 \\ 108 \end{pmatrix}_{\alpha=0.2}$	$\sqrt{(22.2)^2+(21)^2+(108)^2} = \sqrt{492.84+441+11664} = \sqrt{12597.84} = 112.24$	$\begin{pmatrix} 24.8 \\ 22.2 \\ 141 \end{pmatrix}_{\alpha=0.2}$	$\sqrt{(24.8)^2+(22.2)^2+(141)^2} = \sqrt{615.04+492.84+19881} = \sqrt{20988.88} = 144.87$
$\begin{pmatrix} 20.4 \\ 19.6 \\ 115.6 \end{pmatrix}_{\alpha=0.4}$	$\sqrt{(20.4)^2+(19.6)^2+(115.6)^2} = \sqrt{416.16+384.16+13363.36} = \sqrt{14163.68} = 119.01$	$\begin{pmatrix} 23.6 \\ 21.4 \\ 137 \end{pmatrix}_{\alpha=0.4}$	$\sqrt{(23.6)^2+(21.4)^2+(137)^2} = \sqrt{547.56+457.96+18769} = \sqrt{19774.52} = 140.62$
$\begin{pmatrix} 21.4 \\ 18.6 \\ 130 \end{pmatrix}_{\alpha=0.6}$	$\sqrt{(21.4)^2+(18.6)^2+(130)^2} = \sqrt{457.96+345.96+16900} = \sqrt{17703.92} = 133.05$	$\begin{pmatrix} 22.4 \\ 20.6 \\ 133 \end{pmatrix}_{\alpha=0.6}$	$\sqrt{(22.4)^2+(20.6)^2+(133)^2} = \sqrt{501.76+424.36+17689} = \sqrt{18615.12} = 136.43$
$\begin{pmatrix} 20.4 \\ 19.4 \\ 128 \end{pmatrix}_{\alpha=0.8}$	$\sqrt{(20.4)^2+(19.4)^2+(128)^2} = \sqrt{416.16+376.36+16384} = \sqrt{17176.52} = 131.05$	$\begin{pmatrix} 21.2 \\ 19.8 \\ 129 \end{pmatrix}_{\alpha=0.8}$	$\sqrt{(21.2)^2+(19.8)^2+(129)^2} = \sqrt{449.44+392.04+16641} = \sqrt{17482.48} = 132.22$
$\begin{pmatrix} 20 \\ 19 \\ 125 \end{pmatrix}_{\alpha=1}$	$\sqrt{(20)^2+(19)^2+(125)^2} = \sqrt{400+361+15625} = \sqrt{16386} = 128.00$	$\begin{pmatrix} 20 \\ 19 \\ 125 \end{pmatrix}_{\alpha=1}$	$\sqrt{(20)^2+(19)^2+(125)^2} = \sqrt{400+361+15625} = \sqrt{16386} = 128.00$

Fuzzy induced output for the feedback response SgF_{FB} is bounded output. Thus, the system of SgF is bounded output.

3 Results and discussion

The states space model of the insulin–glucose system was formulated first. Then, its input/output parameters were fuzzified. This was followed by the defuzzification process using a revised version of the "revised modified optimized defuzzified value theorem." The proposed values for the output and for the feedback transmission [Eqs. (53) and (54)] response are given in Table 2.

$$I_\mathrm{p} = (0.2669)I_\mathrm{p} + (-0.0780)I_\mathrm{i} + (0.0868)G_\mathrm{in} \tag{53}$$

$$G = (0.6587)I_\mathrm{p} + (0.0310)I_\mathrm{i} + (0.5930)G_\mathrm{in} \tag{54}$$

Tables 2 and 3 show optimal values of the inputs $I_\mathrm{p} = 25.4306$ mU l^{-1}, $I_\mathrm{i} = 21.8612$ mU l^{-1} and $G_\mathrm{in} = 143.8612$ mg dl^{-1} with plasma insulin level of $I_\mathrm{p} = 18.0158$ mU l^{-1} and plasma glucose $G = 102.5601$ mg dl^{-1}. Moreover, the preferred values are those that are preferred more than other values for the insulin regulatory system. The algorithm when programmed on Matlab® software takes a CPU time of 0.452 s. The promising results show that the proposed method models uncertainty in the complex insulin–glucose system in a more effective manner. The bounded-input bounded-output stability of the system based on Lemma 2 and Theorem 5 is shown in Tables 4, 5 and 6.

4 Conclusions

First of all some new structural properties of the feedback fuzzy states space model were formulated and proved. In this flow, revised convexity property of the feedback fuzzy states space model was studied comprehensively which resulted in formulation of Theorem 1. This was followed by the revised normality property of the feedback fuzzy states space model that resulted in the formulation and Proof of Theorem 2. The most important theorem related to the defuzzification process named "revised modified

Random inputs	Max inputs	$G_{OutFF} \le \|k_{11}\|\|u_{11}\| + \|k_{21}\|\|u_{21}\| + \|k_{31}\|\|u_{31}\| \le \|k_{11}\|M_{FF1} + \|k_{21}\|M_{FF2} + \|k_{31}\|M_{FF3}$
$\begin{pmatrix} 20 \\ 18 \\ 140 \end{pmatrix}_{\alpha=0}$	$\begin{pmatrix} 20 \\ 23 \\ 145 \end{pmatrix}_{\alpha=0}$	$G_{OutFF} \le \|k_{11}\|\|20\| + \|k_{21}\|\|18\| + \|k_{31}\|\|140\| = 0.2669\|20\| - 0.0780\|18\| + 0.0868\|140\| \le 0.2669 * 26 - 0.0780 * 23 + 0.0868 * 145 = 16.69$
$\begin{pmatrix} 22.2 \\ 21 \\ 108 \end{pmatrix}_{\alpha=0.2}$	$\begin{pmatrix} 24.8 \\ 22.2 \\ 141 \end{pmatrix}_{\alpha=0.2}$	$G_{OutFF} \le \|k_{11}\|\|22.2\| + \|k_{21}\|\|21\| + \|k_{31}\|\|108\| = 0.2669\|22.2\| - 0.0780\|21\| + 0.0868\|108\| \le 0.2669 * 24.8 - 0.0780 * 22.2 + 0.0868 * 141 = 16.13$
$\begin{pmatrix} 20.4 \\ 19.6 \\ 115.6 \end{pmatrix}_{\alpha=0.4}$	$\begin{pmatrix} 23.6 \\ 21.4 \\ 137 \end{pmatrix}_{\alpha=0.4}$	$G_{OutFF} \le \|k_{11}\|\|20.4\| + \|k_{21}\|\|19.6\| + \|k_{31}\|\|115.6\| = 0.2669\|20.4\| - 0.0780\|19.6\| + 0.0868\|115.6\| \le 0.2669 * 23.6 - 0.0780 * 21.4 + 0.0868 * 137 = 15.57$
$\begin{pmatrix} 21.4 \\ 18.6 \\ 130 \end{pmatrix}_{\alpha=0.6}$	$\begin{pmatrix} 22.4 \\ 20.6 \\ 133 \end{pmatrix}_{\alpha=0.6}$	$G_{OutFF} \le \|k_{11}\|\|21.4\| + \|k_{21}\|\|18.6\| + \|k_{31}\|\|130\| = 0.2669\|21.4\| - 0.0780\|18.6\| + 0.0868\|130\| \le 0.2669 * 22.4 - 0.0780 * 20.6 + 0.0868 * 133 = 15.02$
$\begin{pmatrix} 20.4 \\ 19.4 \\ 128 \end{pmatrix}_{\alpha=0.8}$	$\begin{pmatrix} 21.2 \\ 19.8 \\ 129 \end{pmatrix}_{\alpha=0.8}$	$G_{OutFF} \le \|k_{11}\|\|20.4\| + \|k_{21}\|\|19.4\| + \|k_{31}\|\|128\| = 0.2669\|20.4\| - 0.0780\|19.4\| + 0.0868\|128\| \le 0.2669 * 21.2 - 0.0780 * 19.8 + 0.0868 * 129 = 14.46$
$\begin{pmatrix} 20 \\ 19 \\ 125 \end{pmatrix}_{\alpha=1}$	$\begin{pmatrix} 20 \\ 19 \\ 125 \end{pmatrix}_{\alpha=1}$	$G_{OutFF} \le \|k_{11}\|\|20\| + \|k_{21}\|\|19\| + \|k_{31}\|\|125\| = 0.2669\|20\| - 0.0780\|19\| + 0.0868\|125\| \le 0.2669 * 20 - 0.0780 * 19 + 0.0868 * 125 = 13.90$

Table 6 Output stability for the feedback responses

Random inputs	Max inputs	$G_{OutFB} \le \|k_{12}\|\|u_{12}\| + \|k_{22}\|\|u_{22}\| + \|k_{32}\|\|u_{32}\| \le \|k_{12}\|M_{FB1} + \|k_{22}\|M_{FB2} + \|k_{32}\|M_{FB3}$
$\begin{pmatrix} 20 \\ 18 \\ 140 \end{pmatrix}_{\alpha=0}$	$\begin{pmatrix} 20 \\ 23 \\ 145 \end{pmatrix}_{\alpha=0}$	$G_{OutFB} \le \|k_{12}\|\|20\| + \|k_{22}\|\|18\| + \|k_{32}\|\|140\| = 0.6587\|20\| + 0.0310\|18\| + 0.5930\|140\| \le 0.6587 * 26 + 0.0310 * 23 + 0.5930 * 145 = 103.82$
$\begin{pmatrix} 22.2 \\ 21 \\ 108 \end{pmatrix}_{\alpha=0.2}$	$\begin{pmatrix} 24.8 \\ 22.2 \\ 141 \end{pmatrix}_{\alpha=0.2}$	$G_{OutFB} \le \|k_{12}\|\|22.2\| + \|k_{22}\|\|21\| + \|k_{32}\|\|108\| = 0.6587\|22.2\| + 0.0310\|21\| + 0.5930\|108\| \le 0.6587 * 24.8 + 0.0310 * 22.2 + 0.5930 * 141 = 100.63$
$\begin{pmatrix} 20.4 \\ 19.6 \\ 115.6 \end{pmatrix}_{\alpha=0.4}$	$\begin{pmatrix} 23.6 \\ 21.4 \\ 137 \end{pmatrix}_{\alpha=0.4}$	$G_{OutFB} \le \|k_{12}\|\|20.4\| + \|k_{22}\|\|19.6\| + \|k_{32}\|\|115.6\| = 0.6587\|20.4\| + 0.0310\|19.6\| + 0.5930\|115.6\| \le 0.6587 * 23.6 + 0.0310 * 21.4 + 0.5930 * 137 = 97.44$
$\begin{pmatrix} 21.4 \\ 18.6 \\ 130 \end{pmatrix}_{\alpha=0.6}$	$\begin{pmatrix} 22.4 \\ 20.6 \\ 133 \end{pmatrix}_{\alpha=0.6}$	$G_{OutFB} \le \|k_{12}\|\|21.4\| + \|k_{22}\|\|18.6\| + \|k_{32}\|\|130\| = 0.6587\|21.4\| + 0.0310\|18.6\| + 0.5930\|130\| \le 0.6587 * 22.4 + 0.0310 * 20.6 + 0.5930 * 133 = 94.26$
$\begin{pmatrix} 20.4 \\ 19.4 \\ 128 \end{pmatrix}_{\alpha=0.8}$	$\begin{pmatrix} 21.2 \\ 19.8 \\ 129 \end{pmatrix}_{\alpha=0.8}$	$G_{OutFB} \le \|k_{12}\|\|20.4\| + \|k_{22}\|\|19.4\| + \|k_{32}\|\|128\| = 0.6587\|20.4\| + 0.0310\|19.4\| + 0.5930\|128\| \le 0.6587 * 21.2 + 0.0310 * 19.8 + 0.5930 * 129 = 91.07$
$\begin{pmatrix} 20 \\ 19 \\ 125 \end{pmatrix}_{\alpha=1}$	$\begin{pmatrix} 20 \\ 19 \\ 125 \end{pmatrix}_{\alpha=1}$	$G_{OutFB} \le \|k_{12}\|\|20\| + \|k_{22}\|\|19\| + \|k_{32}\|\|125\| = 0.6587\|20\| + 0.0310\|19\| + 0.5930\|125\| \le 0.6587 * 20 + 0.0310 * 19 + 0.5930 * 125 = 87.88$

optimized defuzzified value theorem" was formulated and proved as Theorem 3. This was followed by revised stability property of the FFSSM as Theorem 4. The proposed revised bounded-input stability property and bounded-output stability of the FFSSM were formulated and proved in Lemma 2 and Theorem 5, respectively. All the formulated proposed structural properties were then implemented and verified on the Feedback Insulin–Glucose Regulatory System in Humans. The successful implementation and promising results showed the credibility, reliability and robustness of the FFSSM model of the insulin–glucose regulatory system.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors. The data presented are obtained from the widely accepted published research Sturis (1991), Sturis et al. (1991) and Tolić et al. (2000).

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