Part II: Dynamical Systems - Revision Notes

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1. Basic Definitions

Definition: (ω-limit set)

$\omega(\mathbf{x}) = \{\mathbf{y} : \exists$ infinite sequence $t_1, t_2, ... \to \infty$ with $\phi_{t_n}(\mathbf{x}) \to \mathbf{y}\}$

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2. Fixed Points

2.1. Linearisation

Classifying fixed points

1. **Saddle point:** $\lambda_1 < 0 < \lambda_2$
2. **Stable node:** $\lambda_1, \lambda_2 < 0$
3. **Unstable node:** $\lambda_1, \lambda_2 > 0$
4. **Stable focus:**
5. **Unstable focus:**
6. **Stella node:**
7. **Improper node:**
8. **Centre:**

Hamiltonian systems:

Definition: The system that can be written as $\dot{x} = \frac{\partial H}{\partial y}$ and $\dot{y} = -\frac{\partial H}{\partial x}$ is called the Hamiltonian system.

Hamiltonian systems are always centres or saddles.

$\dot{\mathbf{x}} \cdot \nabla H = 0 \implies$ trajectories are contours of $H(x, y)$.

Definition: (*Hyperbolic fixed point*) If none of the eigenvalues of the Jacobian at this fixed point has zero real part.

Definition: (*Hyperbolic sink*) If **all** eigenvalues have negative real parts.

Definition: (*Hyperbolic source*) If **all** eigenvalues have positive real parts.

Definition: (*Stable, unstable and centre subspaces*) The stable, unstable and centre subspaces of the linearisation of $\mathbf{f}$ at the FPs $\mathbf{x}_0$ are the 3 linear subspaces E^s, E^u and E^c spanned by the subset of (possibly generalised) eigenvectors of $\mathbf{A}$, whose eigenvalues have real parts < 0, > 0 and $= 0$, respectively.

Note: Hyperbolic points do not have a E^c

Theorem: (*Stable Manifold Theorem*) Suppose $\mathbf{0}$ is a hyperbolic fixed point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and E^u and E^s are the unstable and stable subspaces of the linearisation of $\mathbf{f}$ about $\mathbf{0}$. Then $\exists$ local unstable and stable manifolds $W^u_{\text{loc}}(\mathbf{0})$ and $W^s_{\text{loc}}(\mathbf{0})$ which have the same dimension as E^u and E^s and are tangent to E^u and E^s at $\mathbf{0}$ s.t. for $\mathbf{x} \neq \mathbf{0}$ but in a sufficiently small neighborhood of $\mathbf{0}$,

$$W^u_{\text{loc}} = \{\mathbf{x} : \phi_t(\mathbf{x}) \to \mathbf{0} \text{ as } t \to -\infty\}$$

$$W^s_{\text{loc}} = \{\mathbf{x} : \phi_t(\mathbf{x}) \to \mathbf{0} \text{ as } t \to \infty\}$$

Finding Stable and unstable manifold

To find W^u_{loc}, write $y = U(x) = a_2x^2 + a_3x^3 + ...$
with $U(0) = U'(0) = 0$.

To find W^s_{loc}, write $x = S(y) = b_2y^2 + b_3y^3 + ...$
with $S(0) = S'(0) = 0$.

Then take derivatives both sides and substitute $\dot{x}$ and $\dot{y}$ and compare coefficients.

3. Stability

3.1. Stability definitions

Definition: A fixed point $\mathbf{x_0}$ is **Lyapunov stable (LS)** if

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \text{s.t } |\mathbf{x} - \mathbf{x_0}| < \delta,\ |\phi_t(\mathbf{x}) - \mathbf{x_0}| < \epsilon,\ \forall t > 0$$

Informally, we can say "starts near, stays near".

Definition: A fixed point $\mathbf{x_0}$ is **Quasi-asymptotically stable (QAS)** if

$$\exists \delta > 0,\ |\mathbf{x} - \mathbf{x_0}| < 0,\ \phi_t(\mathbf{x}) \to \mathbf{x_0} \text{ as } t \to \infty$$

Informally, we can say "orbit tends to ".

Definition: A fixed point $\mathbf{x_0}$ is **asymptotically stable (AS)** if it is both LS and QAS.

3.2. Lyapunov functions

Definition: A continuous differentiable function $V(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is a **Lyapunov function** for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on a domain D containing neighbourhood of $\mathbf{0}$ if

(i) $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ in D. (positive definite)

(ii) $\dot{V} = \mathbf{f} \cdot \nabla V \leq 0$ in D. (non-increasing)

Strict Lyapunov function: If inequality in condition (ii) is strict apart from $\mathbf{x} = \mathbf{0}$.

Lyapunov first theorem: If a Lyapunov function V exists, then $\mathbf{0}$ is Lyapunov stable.

Proof. Let ϵ be small enough, so $\{|\mathbf{x}| \leq \epsilon\} \subseteq D$.

$$m := \min\{V(\mathbf{x}) : |\mathbf{x}| = \epsilon\}$$

$$C_{m,\epsilon} := \{\mathbf{x} : V(\mathbf{x}) < m,\ |\mathbf{x}| < \epsilon\}$$

Choose $\delta > 0$ s.t. $\{|\mathbf{x}| < \delta\} \subset C_{m,\epsilon}$
Then $\{|\mathbf{x}| < \delta\} \subset C_{m,\epsilon} \subseteq \{|\mathbf{x}| < \epsilon\}$.

Lyapunov second theorem: If a strict Lyapunov function exists, then $\mathbf{0}$ is asymptotically stable.

La Salle's invariance principle: If V is a Lyapunov function on domain D which is compact (closed and bounded) and forward invariant ($\mathbf{x} \in D \implies \phi_t(\mathbf{x}) \in D\ \forall t > 0$), then

$$\omega(\mathbf{x}) \subseteq \{\mathbf{y} : V(\phi_t(\mathbf{y})) = V_0\ \forall t\} \text{ for some } V_0$$

More usefully: $\phi_t(\mathbf{x})$ tends to an invariant subset of $\{\mathbf{y} : \dot{V}(\mathbf{y}) = 0\} \cap D$.

Definition: The **domain of stability (DoS)** of an AS invariant set Λ is

$$\{\mathbf{x} : \phi_t(\mathbf{x}) \to \Lambda\}$$

If DoS is $\mathbb{R}^n$, then we say Λ is globally stable.

General method for finding DoS:

1. Find Lyapunov function V and domain D containing neighbourhood of $\mathbf{0}$ s.t.
   - $V \geq 0$ on D and $V = 0$ only at $\mathbf{x} = 0$.
   - $\dot{V} \leq 0$ on D.
2. Find k s.t. $C_k = \{\mathbf{x} : V(\mathbf{x}) \leq k\} \subseteq D$
3. Adjust k or V so that only invariant subset of $\{\dot{V} = 0\} \cap C_k$ is $\{\mathbf{0}\}$. Then La Salle's $\implies C_k \subseteq$ DoS.

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4. Periodic orbit

4.1. Poincare index test

Definition: (*Poincare index of a curve*)

Definition: (*Poincare index of a fixed point*)

Properties of Poincare index:
1. Integral form: $I(\Gamma) = \frac{1}{2\pi} \oint d\psi = \frac{1}{2\pi} \oint d(\tan^{-1}(\frac{f_2}{f_1}))$
   $= \frac{1}{2\pi} \oint \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$
2. If $I(\Gamma)$ encloses no FPs, then $I(\Gamma) = 0$.
3. Index of a closed trajectory is +1.
4. $I(\Gamma)$ is the sum of indices of all FPs enclosed by Γ.
5. Index of hyperbolic sink/source is +1, hyperbolic saddle is -1.

Poincare index test: (Test 1) Any periodic orbit must contain one or more FPs and sum of their indices is +1.

Note: POs cannot cross any invariant axes.

4.2. Dulac's criterion

Dulac's criterion: (Test 2) If there is a continuously differentiable function $\phi(x, y)$ s.t. $\nabla \cdot (\phi \mathbf{f}) \neq 0$ on a simply connected domain $D \subseteq \mathbb{R}^2$, then there are no periodic orbit that lie entirely in D.

Proof. By contradiction and divergence theorem.

Note: Often use $\phi = 1$ (called divergence test).

Corollary: (Test 3) If $\nabla \cdot (\phi \mathbf{f}) \neq 0$ on some doubly-connected domain $D \subset \mathbb{R}^2$, then there is at most one PO entirely in D (and must enclose the hole).

Note: Can apply in damped pendulum, where there is a cylinder coordinate.

Gradient criterion: (Test 4) If $\exists$ positive function $\rho(x, y)$ s.t. $\rho \mathbf{f} = \nabla \psi$ for some single-valued function ψ on some simply connected domain D, then there are no POs entirely in D.

4.3. Poincare-Bendixson Theorem

Theorem: (Test 5) If the forward orbit $O^+(\mathbf{x})$ of some point $\mathbf{x}$ remains in a compact set (closed and bounded) $K \subseteq \mathbb{R}^2$ that contains no fixed points, then $\omega(\mathbf{x})$ is a periodic orbit.

4.4. Near-Hamiltonian flows

Energy-balance method to find limit cycle in nearly-Hamiltonian system:

$$\dot{x} = f_1(x,y) + \epsilon g_1(x,y)$$

$$\dot{y} = f_2(x,y) + \epsilon g_2(x,y)$$

with $f_1 = \frac{\partial H}{\partial y}$, $f_2 = -\frac{\partial H}{\partial x}$.

– If $\epsilon = 0$, then $\dot{H} = 0$. Trajectories lie on contours of H. Many POs.

– If $\epsilon \neq 0$, then $\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \epsilon(g_2 f_1 - g_1 f_2)$.

There is a P.O. Γ if $\oint_\Gamma dH = 0$.

– If $0 < \epsilon \ll 1$,

$$\begin{aligned}\Delta H(H_0) &= \oint_{H_0} dH \\ &= \oint_{H_0} \dot{H} dt \\ &= \epsilon \oint_{H_0} (g_2 f_1 - g_1 f_2)dt \ + \ O(\epsilon^2) \\ &= \epsilon \oint_{H_0} (g_2 dx - g_1 dy) + O(\epsilon^2) \\ &= 0\end{aligned}$$

4.5. Stability of periodic orbit

Suppose dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with PO $\mathbf{x} = \mathbf{X}(t)$, period T and let $\mathbf{X}_0 = \mathbf{X}(0) = \mathbf{X}(T)$.

Consider small perturbation: $\mathbf{x} = \mathbf{X}(t) + \eta(t)$, then

$$\dot{\eta} = (\eta \cdot \nabla)\mathbf{f}(\mathbf{x}) + O(\eta^2)$$

So to linear order: $\dot{\eta} = \mathbf{A}(t)\eta$, where $A_{ij}(t) = \frac{\partial f_i}{\partial x_j}|_{\mathbf{X}(t)}$.

Solution: $\eta(nT) = [\Phi(T)]^n \eta(0)$, where $\dot{\Phi}_{ij} = A_{ij}\Phi_{kj}$ and $\Phi(0) = I$.

Definition: Floquet multipliers of a periodic orbit are the eigenvalues λ of the matrix $\Phi(T)$ except 1.

1 is always an eigenvalue because perturbation can be on the direction of PO, which leads to unit eigenvalue.

Definition: Floquet exponents are $\mu_i = \frac{1}{T} ln(\lambda_i)$.

- If $|\lambda_i| \neq 1 \ \forall i$, then say PO is **hyperbolic**.
- If all $|\lambda_i| < 1$, PO is **asymptotically stable**.
- If any $|\lambda_i| > 1$, PO is **not (Lyapunov) stable**.

Determine Floquet multiplier

Note: $\dot{\Phi} = \mathbf{A}\Phi$, then $(\dot{\det\Phi}) = \text{tr}A \det\Phi$; and $\text{tr}\mathbf{A} = \nabla \cdot \mathbf{f}$,

$$\implies \quad \lambda = \det\mathbf{\Phi}(T) = \exp(\int_0^T \text{tr}\mathbf{A} dt) = \exp(\int_0^T \nabla \cdot \mathbf{f} dt)$$

Stability of limit cycle:

- If $\int_0^T \nabla \cdot \mathbf{f} dt < 0$: stable
- If $\int_0^T \nabla \cdot \mathbf{f} dt > 0$: unstable
- If $\int_0^T \nabla \cdot \mathbf{f} dt = 0$: non-hyperbolic

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4.6. The Van der Pol Oscillator

Sketch shape of the limit cycle as $\mu \gg 1$.

Calculate the leading-order approximation of the period.

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5. Bifurcation

Definition: System $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \mu)$ that depends continuously on a parameter μ. A bifurcation is a change in the topological structure of the flow as μ passes through some critical value μ_0.

5.1. Centre Manifold Theorem

Theorem: (CMT) If $\mathbf{0}$ is a non-hyperbolic fixed point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with linear stable, unstable and centre subspaces E^s, E^u and E^c, then there exists stable, unstable and centre manifolds W^s, W^u and W^c that have the same dimension as E^s, E^u and E^c, and tangent to them at $\mathbf{0}$, and are invariant.

Example: finding extended centre manifold (ECM)

5.2. Stationary bifurcations ($\lambda = 0$)

(i) **Saddle-node bifurcation:** When stable and unstable fixed points collide and annihilate. Structurally stable. $\dot{x} = \mu - x^2$.

(ii)**Transcritical bifurcation:** Two lines of fixed points. They cross each other and exchange stability. Not structurally stable. $\dot{x} = \mu x - x^2$.

(iii) **Pitchfork bifurcation:** supercritical/subcritical pitchfork. When three fixed points coming together on one side of critical μ and the other side. $\dot{x} = \mu x - a x^3$

Effects on bifurcation when adding small perturbation:

(i) Saddle-node bifurcation will be unchanged as it is struc- -turally stable.

(ii) Transcritical bifurcation will be divided into two saddle-node bifurcations.

(iii) Pitchfork bifurcation will be divided into a stable fixed node and a saddle-node bifurcation.

5.3. Oscillatory/Hopf bifurcations

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6. Maps

6.1. Key examples of maps

(i) Logistic map: $x_{n+1} = \mu x_n(1-x_n)$, with $0 \le \mu \le 4$, $x \in [0,1]$.

(ii) Tent map:

$$x_{n+1} = \begin{cases} \mu x_n & 0 \le x_n \le \frac{1}{2} \\ \mu(1-x_n) & \frac{1}{2} < x_n \le 1 \end{cases}$$

where $0 \le \mu \le 2$, and $x \in [0,1]$.

(iii) Rotation map: $x_{n+1} = x_n + w \pmod 1$

(iv) Sawtooth map/Bernoulli shift map: $x_{n+1} = 2x_n \pmod 1$

(v) Baker's map: $x_{n+1} = 2x_n \pmod 1$

$$y_{n+1} = \begin{cases} \frac{1}{2}y_n & 0 \le x_n < \frac{1}{2} \\ \frac{1}{2}(y_n+1) & \frac{1}{2} \le x_n < 1 \end{cases}$$

6.2. Fixed points, cycles, and stability

- **Fixed point:** x_0 s.t. $F(x_0) = x_0$
- **Periodic point with period** n**:** x_0 if $\mathbf{F}^n(\mathbf{x}_0) = \mathbf{x}_0$ and $\mathbf{F}^k(\mathbf{x}_0) \ne \mathbf{x}_0$ for $k = 1, 2, ..., n-1$.
- **N-cycle:** If $\mathbf{x}_0$ period N: $\{\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{N-1}\}, \mathbf{F}(\mathbf{x}_i) = \mathbf{x}_{i+1}$.
- **Invariant set:** Λ s.t. $\mathbf{x} \in \Lambda \implies \mathbf{F}(\mathbf{x}) \in \Lambda$.
- **Forward orbit:** $O(\mathbf{x}) = \{\mathbf{x}, \mathbf{F}(\mathbf{x}), \mathbf{F}^2(\mathbf{x}), ...\}$

Stability of a FP $\mathbf{x}_0 = \mathbf{F}(\mathbf{x}_0)$ is determined by the Jacobian $A_{ij} = \frac{\partial F_i}{\partial x_j}|_{\mathbf{x}_0}$.

Classification of FP of a map:

- **Asymptotically stable** if **all** eigenvalues of A have $|\lambda| < 1$.
- **Unstable** if **any** eigenvalue has $|\lambda| > 1$.
- **Non-hyperbolic** if **any** eigenvalue is on the unit circle $|\lambda| = 1$ (except bifurcation).

6.3. Bifurcation of 1D maps

Consider $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n; \mu)$, $\lambda = \frac{\partial F}{\partial x}|_{\text{FP}}$. Bifurcation occurs when $\lambda = 1$ or $\lambda = -1$. Wlog, say FP is at $\mathbf{x} = 0$ when $\mu = 0$, and bifurcation when $\mu = 0$.

- Saddle-node ($\lambda = 1$): $x_{n+1} = x_n + \mu - x_n^2$.
- Transcritical ($\lambda = 1$): $x_{n+1} = x_n + x_n(\mu - x_n)$.
- Pitchfork ($\lambda = 1$): $x_{n+1} = x_n + x_n(\mu - ax_n^2)$.
- Period-doubling bifurcation ($\lambda = -1$):

$$x_{n+1} = -x_n + Bx_n^2 + (A\mu x_n + Cx_n^3)$$

Example: Sketching bifurcation diagram.

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7. Chaos

7.1. Introduction

Consider map $F : \Lambda \to \Lambda$:

(i) F has **sensitive dependence on initial conditions (SDIC)** on Λ if $\exists\ \delta > 0$ s.t. for any $x \in \Lambda$ and $\epsilon > 0$, $\exists\ y \in \Lambda$ and $n > 0$, s.t. $|y - x| < \epsilon$ and $|F^n(y) - F^n(x)| > \delta$.

(ii) F is **topologically transitive (TT)** on Λ if for any non-empty open sets $U, V \subseteq \Lambda$, $\exists\ n > 0$ s.t. $F^n(U)$

Definition: *(D-chaos)* $F : \Lambda \to \Lambda$ is chaotic on Λ if
(i) F has SDIC on Λ;
(ii) F has TT on Λ;
(iii) Periodic points are dense on Λ.

Definition: *(horseshoe)* $F : I \to I$ has horseshoe if $\exists$ **open** interval $J \subseteq I$ and **disjoint open** subintervals $K_0, K_1 \subset J$ s.t. $F(K_0) = F(K_1) = J$.

Definition: (*G-chaos*) F is chaotic if F^n has a horseshoe for some $n \ge 1$.

7.2. Sawtooth map (Bernoulli shift)

$x_{n+1} = 2x_n \pmod 1$, $x \in [0,1]$.

Equivalently, binary shift: $0.a_1a_2a_3... \to 0.a_2a_3a_4...$

This is both G-chaotic and D-chaotic.

7.3. Horseshoes + symbolic dynamics

7.4. Period 3 implies chaos

Theorem: If continuous map has a period 3 orbit (i.e. 3-cycle), then F^2 has a horseshoe and hence F is chaotic.

7.5. N-cylces and directed graphs

Lemma 1: If F has an N-cycle, then F must have a FP.

Proof. Apply IVT on $F(x) - x$ on $[x_a, x_b]$. □

Lemma 2: If F continuous on **closed** bounded interval I and $I \subseteq F(I)$, then there is a FP in I.

Proof. Apply IVT. □

Example: Finding period N of a cycle.

7.6. Tent map