Law of Large Numbers

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0.1 Equivalent Sequences, Technique of Truncation

Definition 0.1 (equivalent sequences)**.** Two sequences of r.v.s $\{X_n\}$ and $\{Y_n\}$ are said to be *equivalent* if

$$\sum_{n=1}^{\infty} \mathrm{P}(X_n \neq Y_n) < \infty.$$

Remark 0.1.1 (easy consequences when equivalence). With Borel-Cantelli Lemma, it is immediate that $\mathrm{P}(X_n \neq Y_n, i.o.) = 0$, i.e. $X_n = Y_n$ ultimately a.s. Therefore

(i) $\sum_{n=1}^{\infty}(X_n - Y_n)$ converges a.s. Hence $\sum_{n=1}^{\infty} X_n$ behaves in the same way as $\sum_{n=1}^{\infty} Y_n$ a.s.

(ii) $\frac{1}{a_n}\sum_{n=1}^{\infty}(X_n - Y_n) \to 0$ a.s. if $a_n \to \infty$. Hence $\frac{1}{a_n}\sum_{n=1}^{\infty} X_n = \frac{1}{a_n}\sum_{n=1}^{\infty} Y_n$ a.s.

(iii) By (ii), if $\frac{1}{a_n}\sum_{n=1}^{\infty} X_n$ converges in probability so does $\frac{1}{a_n}\sum_{n=1}^{\infty} Y_n$ and to the same value.

Example 0.1.2. For a given sequence $\{X_n\}$, it is equivalent to its truncated sequence provided that $\{X_n\}$ is identically distributed and $\mathrm{E}\,|X_1| < \infty$. Indeed, $\mathrm{E}\,|X_1| < \infty$ iff

$$\sum_{n=1}^{\infty} \mathrm{P}(|X_1| > n) = \sum_{n=1}^{\infty} \mathrm{P}(|X_n| > n) < \infty.$$

Therefore $\{Y_n\}$ defined by $Y_n \overset{\text{def}}{=} X_n \mathbf{1}_{|X_n| \leq n}$ and $\{X_n\}$ are equivalent since $\{X_n \neq Y_n\} = \mathrm{P}(|X_n| > n)$.

Hence, due to Remark 0.1.1, $\bar{X}_n$ and $\bar{Y}_n$ converge or diverge at the same time to the same value.

1 Weak Law of Large Numbers

Theorem 1.1 (WLLN, i.i.d. case)**.** *Let $\{X_i\}$ be pairwise independent and identically distributed r.v.s with $\mathrm{E}\,|X_1| < \infty$ and $\mathrm{E}\,X_1 = \mu$. Then $\bar{X}_n \to \mu$ in probability.*

Proof. Using the construction in Example 0.1.2, it suffices to show $\bar{Y}_n \to \mu$ in probability. By Markov inequality, it is enough to show (a) $\mathrm{E}\,\bar{Y}_n \to \mu$ and (b) $\mathrm{Var}\,\bar{Y}_n \to 0$. See the proof of SLLN, Theorem 2.2. Alternatively, we may use another level of truncation to finish the proof [1, Note 4, Theorem 7.2.1]. $\square$

2 Strong Law of Large Numbers

2.1 SLLN for Independent Sequences, First Look

Theorem 2.1 (variance criterion, Kolmogorov)**.** *Let $X_1, X_2, \dots$ be independent r.v.s with finite first and second moments, i.e., $\mathrm{E}\,|X_k| < \infty$ and $\mathrm{E}\,|X_k|^2 < \infty$. If $\sum_{k=1}^{\infty} \operatorname{Var} X_k < \infty$, then $\sum_{k=1}^{\infty}(X_k - \mathrm{E}\,X_k)$ converges a.s.*

Remark 2.1.1. Equivalently speaking, the theorem says that, for independent sequence, convergence of L^2 series $\sum_{n=1}^{\infty} \|X_k - \mathrm{E}\,X_k\|_2^2$ implies convergence of series $\sum_{n=1}^{\infty}(X_k - \mathrm{E}\,X_k)$ a.s.

Remark 2.1.2. As $\operatorname{Var} X_k \le \mathrm{E}\,|X_k|^2$, the condition $\sum_{k=1}^{\infty} \mathrm{E}\,|X_k|^2 < \infty$ is of course enough.

Proof. Assume $\mathrm{E}\,X_k \equiv 0$ WLOG. Write $S_n = \sum_{k=1}^{n} X_k$. Then

- S_n converges a.s. iff
- S_n is Cauchy a.s. iff
- for any $\epsilon > 0$, $\lim_{M\to\infty} \mathrm{P}\left(\bigcup_{n,m>M}\{|S_n - S_m| > \epsilon\}\right) = 0$.

Hence it suffices to show the last statement. This can be proved by Kolmogorov's maximal inequality through observing either

- $\bigcap_{M>0}\bigcup_{n,m>M}\{|S_n - S_m| > \epsilon\} = \bigcap_{M>0}\bigcup_{k=1}^{\infty}\{|S_{M+k} - S_M| > \epsilon\}$ or
- $\bigcup_{n,m>M}\{|S_n - S_m| > \epsilon\} \subseteq \left[\bigcup_{n>M}\{|S_n - S_M| > \epsilon/2\}\right] \cup \left[\bigcup_{m>M}\{|S_m - S_M| > \epsilon/2\}\right]$.

The remaining part of proof is easy. □

With the help of Kronecker's Lemma, we have the following corollary.

Corollary 2.1.3 (Kolmogorov's SLLN)**.** *Let $X_1, X_2, \cdots$ be independent r.v.s, each with finite mean and variance, and let $\{a_n\}$ be an increasing sequence of positive real numbers with $a_n \uparrow \infty$. If $\sum_{n=1}^{\infty} \frac{\operatorname{Var} X_n}{a_n^2} < \infty$, then*

$$\frac{1}{a_n}\sum_{k=1}^{n}(X_k - \mathrm{E}\,X_k) \to 0 \quad a.s.$$

Remark 2.1.4. There are two special cases of Kolmogorov's SLLN.

(i) If $\{X_n\}$ are independent r.v.s, each with finite mean m and finite variance σ^2, then $\frac{1}{n}\sum_{k=1}^{n} X_k \to m$ a.s.

(ii) If $\{X_n\}$ are independent and the fourth central moments are uniformly bounded, i.e. $\mathrm{E}\,|X_n - \mathrm{E}\,X_n|^4 < M$ for some $M > 0$. Then $\frac{1}{a_n}\sum_{k=1}^{n}(X_k - \mathrm{E}\,X_k) \to 0$ a.s. For by the Cauchy-Schwarz inequality,

$$\operatorname{Var} X_n = \mathrm{E}[(X_n - \mathrm{E}\,X_n)^2 \cdot 1] \le (\mathrm{E}\,|X_n - \mathrm{E}\,X_n|^4)^{1/2} \le M^{1/2}.$$

Then Kolmogorov's SLLN applies with $a_n = n$.

2.2 SLLN for i.i.d. Sequences

Theorem 2.2 (SLLN, i.i.d. case). *If $\{X_n\}$ be i.i.d. random variables. Then*

(i) If $\mathrm{E}\,|X_1| < \infty$*, then* $\frac{1}{n}\sum_{k=1}^n X_k \to \mathrm{E}\,X_1$ *a.s.*

(ii) If $\mathrm{E}\,|X_1| = \infty$*, then* $\limsup_n \frac{|S_n|}{n} = \infty$ *a.s.*

Proof. (i) Assume $\mathrm{E}\,|X_1| < \infty$. Let $Y_n = X_n \mathbf{1}_{|X_n| \le n}$. From Example 0.1.2, $\{Y_n\}$ and $\{X_n\}$ are equivalent. It suffices to show

(a) $\frac{1}{n}\sum_{k=1}^n \mathrm{E}\,Y_k \to \mathrm{E}\,X_1$ and
(b) $\frac{1}{n}\sum_{k=1}^n (Y_k - \mathrm{E}\,Y_k) \to 0$ a.s.

since these would imply $\frac{1}{n}\sum_{k=1}^n Y_k \to \mathrm{E}\,X_1$ a.s. which in turn implies $\frac{1}{n}\sum_{k=1}^n X_k \to \mathrm{E}\,X_1$ a.s. Part (a) is immediate as $\mathrm{E}\,Y_n = \mathrm{E}\,X_1 \mathbf{1}_{|X_1| \le n} \to \mathrm{E}\,X_1$ by DCT. For part (b), it is enough to check $\sum_{n=1}^\infty \frac{\mathrm{E}\,|Y_n|^2}{n^2} < \infty$ by variance criterion. Now

$$\begin{aligned}
\sum_{n=1}^\infty \frac{\mathrm{E}\,|Y_n|^2}{n^2} &= \sum_{n=1}^\infty \frac{1}{n^2} E X_n^2 \mathbf{1}_{\{|X_n| \le n\}} \\
&= \sum_{n=1}^\infty \sum_{k=1}^n \frac{1}{n^2} E\,|X_1|^2\, \mathbf{1}_{\{k-1 < |X_1| \le k\}} \\
&= \sum_{k=1}^\infty \sum_{n=k}^\infty \frac{1}{n^2} E X_1^2 I_{\{k-1 < |X_1| \le k\}} \\
&= \sum_{k=1}^\infty \left[E\left(X_1^2 I_{\{k-1 < |X_1| \le k\}} \right) \left(\sum_{n=k}^\infty \frac{1}{n^2} \right) \right] \\
&\le \sum_{k=1}^\infty \left[k E\left(|X_1|\, I_{\{k-1 < |X_1| \le k\}} \right) \left(\frac{C}{k} \right) \right] \\
&\le C E\,|X_1| < \infty,
\end{aligned}$$

where we used the elementary estimate $\sum_{n=k}^\infty \frac{1}{n^2} \le C/k$ for some $C > 0$ and all $k \ge 1$.

(ii) Assume $\mathrm{E}\,|X_1| = \infty$. Then for all $A > 0$, $\mathrm{E}\,\frac{|X_1|}{A} < \infty$ iff $\sum_{n=1}^\infty \mathrm{P}(|X_n| > An) = \infty$. By Borel-Cantelli Lemma (or equivalently, Borel 0-1 law), $|X_n| > An$ infinitely often a.s. This would lead to $|S_n/n| > A/2$ infinitely often a.s. since

$$|S_n| + |S_{n-1}| \ge |S_n - S_{n-1}| = |X_n| > An$$

implies $|S_n/n| > A/2$ or $|S_{n-1}/(n-1)| > A/2$. Hence $\limsup_n \frac{|S_n|}{n} = \infty$ a.s.

$\square$

2.3 SLLN for Independent Sequences, Second Look

Consider $\{X_n\}$ with $\mathrm{E}\,X_n \equiv m$. To conclude $\frac{1}{n}\sum_{k=1}^n X_k \to m$ a.s.,

- for SLLN for independent sequences, we may need $\mathrm{Var}\,X_k = \sigma^2$, as pointed out by Remark 2.1.4;

- for SLLN for i.i.d. sequences, we need "identically distributed" and without further conditions on moments.

Is there exists a sharper SLLN theorem for independent sequences but without second-order condition?

In this subsection, we can reduce the second-order condition to $1+p$ condition, for any $p>0$, which is very close to the first-order condition.

Technique: Kolmogorov's three series

Theorem 2.3 (Kolmogorov's three series). *Let $X_1, X_2, \dots$ be independent r.v.s. Let $Y_n \overset{def}{=} X_n \mathbf{1}_{|X_n| \le A}$.*

Then $\sum_{n=1}^{\infty} X_k < \infty$ a.s. iff for some $A > 0$, the following three series converge:

(a) $\sum_{n=1}^{\infty} \mathrm{P}(|X_n| > A) < \infty$,

(b) $\sum_{n=1}^{\infty} \mathrm{E}\, Y_n < \infty$ *and*

(c) $\sum_{n=1}^{\infty} \mathrm{Var}\, Y_n < \infty$.

Proof. We just present the proof of sufficiency, as we only need this part. For the second part, see [1, Note 4, Theorem 8.2.5, p. 135].

Condition (a) implies $\{Y_n\}$ and $\{X_n\}$ are equivalent. The variance criterion (c) for Y_n implies $\sum_{n=1}^{\infty}(Y_n - \mathrm{E}\, Y_n) < \infty$ a.s. Hence $\sum_{n=1}^{\infty} Y_n$ converges a.s. by condition (b), which in turn implies $\sum_{n=1}^{\infty} X_n$ converges a.s. □

Generalization of Variance Criterion

Theorem 2.4 (generalized SLLN for independent sequences). *Let $\{X_n\}$ are independent r.v.s and $0 < a_n \uparrow \infty$. Assume that*

$$\sum_{n=1}^{\infty} \frac{\mathrm{E}\,|X_n|^r}{a_n^r} < \infty, \quad 1 \le r \le 2.$$

Then we have $\frac{1}{a_n}\sum_{k=1}^{n}(X_k - \mathrm{E}\, X_k) \to 0$ a.s.

Proof. Check the conditions in Kolmogorov's three series Theorem and apply Kronecker's Lemma. □

Remark 2.4.1. As a special case, let $1 \le r \le 2$ assume $\mathrm{E}\,|X_n|^r \equiv m_r < \infty$ and $\mathrm{E}\, X_n \equiv \mu$. Then $\sum_{n=1}^{\infty} \frac{\mathrm{E}\,|X_n|^r}{a_n^r} < \infty$ iff $\sum_{n=1}^{\infty} \frac{1}{a_n^r} < \infty$. If a_n is of the form n^x, then the series converges iff $a_n = n^{(1+p)/r}$ for some $p > 0$. Apply this theorem,

$$\frac{1}{n^{(1+p)/r}} \sum_{k=1}^{n}(X_k - \mathrm{E}\, X_k) \to 0 \quad a.s.$$

Equivalently, $\frac{1}{n}\sum_{k=1}^{n} X_k - \mu = o(n^{\frac{1+p}{r}-1})$. As $n^{\frac{1+p}{r}-1} \to n^{\frac{1}{r}-1}$,

- when $r > 1$, the convergence rate can be n to the negative power as $\frac{1}{r} - 1 < 0$;
- when $r = 1$, the convergence rate can only be n to a positive power.

Hence the existence of $\mathrm{E}\,|X_k|^r \equiv m_r$ would guarantee the convergence of $\frac{1}{n}\sum_{k=1}^{n} X_k$ to μ a.s.

Remark 2.4.2. There are also theorems for $0 < r < 1$ and $r > 2$, see [1, Corollary 8.3.3, Assignment 2.5], respectively. Their proofs are all based on Kolmogorov's three series theorem.

References

[1] Zhu Ke. *Research Methods in Statistics, Lecture Notes.* The University of Hong Kong, not published, 2023.