1 Borel-Cantelli Lemma and Its Application

This section covers Borel-Cantelli Lemma, Borel 0-1 Law and some applications of the 0-1 law.

Lemma 1.1 (Borel-Cantelli)**.** *Let $\{A_n\}$ be a sequence of events on a probability space $(\Omega, \mathscr{A}, \mathrm{P})$* [1].

i. If $\sum_{n=1}^{\infty} \mathrm{P}(A_n) < \infty$, then $\mathrm{P}(A_n, i.o.) = 0$.

ii. Assume further that $\{A_n\}$ are independent. If $\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \infty$, then $\mathrm{P}(A_n, i.o.) = 1$.

Proof.

Proof of i. This is easy since

$$\begin{aligned}\mathrm{P}(A_n, i.o.) &= \mathrm{P}\left(\limsup_n A_n\right) = \mathrm{P}\left(\lim_k \bigcup_{n\geq k} A_n\right)\\ &= \lim_k \mathrm{P}\left(\bigcup_{n\geq k} A_n\right) \leq \lim_n \sum_{n=k}^{\infty} \mathrm{P}(A_m) = 0,\end{aligned}$$

where we have used the upper continuity of probability (or finite) measure.

Proof of ii. Also not hard by using the elementary inequality

$$1 - x \leq e^{-x}, \quad x \in \mathbb{R}. \tag{1}$$

Then

$$\begin{aligned}\mathrm{P}(A_n, i.o.) = \lim_k \mathrm{P}\left(\bigcup_{n\geq k} A_n\right) &= 1 - \lim_k \lim_N \mathrm{P}\left(\bigcap_{n=k}^{N} A_n^c\right)\\ &\geq 1 - \lim_k \lim_N \prod_{n=k}^{N}(1 - P(A_n)) = 1 - \exp\left(-\lim_k \sum_{n=k}^{\infty} P(A_n)\right) = 1,\end{aligned}$$

where we used (1) to overcome the difficulty of bounding products of the form $\prod_i (1 - a_i)$.

$\square$

Remark 1.2 (Comments on the condition of Lemma 1.1(ii)). If $\{A_n\}$ are strongly depend, then the result of Borel-Cantelli Lemma (ii) cannot hold. For example, take $A_n = A$. However, the independent assumption can be reduced to pairwise sense, [1, Theorem 4.2.5.].

We can understand Borel-Cantelli Lemma as the following "0-1 law".

Corollary 1.3 (Borel 0-1 Law)**.** *Let $\{A_n\}$ be pairwise independent events. Then $\mathrm{P}(A_n, i.o.) = 0$ iff $\sum_n \mathrm{P}(A_n) < \infty$; $\mathrm{P}(A_n, i.o.) = 1$ iff $\sum_n \mathrm{P}(A_n) = \infty$. The probability of $\{A_n, i.o.\}$ is either 0 or 1.*

In particular, if $A_n \to A$, then $\mathrm{P}(A)$ equals either 0 or 1.

[1] We may omit the probability space in the following.

Proof. Suppose $\mathrm{P}(A_n, i.o.) = 0$. Since $\sum_{n=1}^{N} \mathrm{P}(A_n)$ is an positive sequence, we must have either it converge or diverge. It must converge otherwise it is a contradiction. $\square$

Under the condition of pairwise independence, convergence in probability fast enough is equivalent to almost-sure convergence. We state this as a corollary of Borel 0-1 law to emphesis its condition.

Corollary 1.4. *Let $\{X_n\}$ be pairwise independent.*
Then $X_n \to 0$ a.s. iff $\sum_n \mathrm{P}(|X_n| \geq \epsilon) < \infty$ for all $\epsilon > 0$.

Proof. "$\sum_n \mathrm{P}(|X_n| \geq \epsilon) < \infty$ for all $\epsilon > 0$" iff "$\mathrm{P}(\{|X_n| \geq \epsilon, i.o.\}) = 0$ for all $\epsilon > 0$" iff "$|X_n| < \epsilon$ utimately a.s. for all $\epsilon > 0$" iff "$X_n \to 0$ a.s.". $\square$

Under the condition of pairwise independence, finiteness of r-moment is equivalent to growth less than order $n^{1/r}$, $r > 0$. We again state this as a corollary of Borel 0-1 law to emphesis its condition.

Corollary 1.5. *Let $\{X, X_n\}$ be pairwise independent and identially distributed.*
Then for $r > 0$, $\mathrm{E}\,|X|^r < \infty$ iff $X_n = o(n^{1/r})$ a.s.

Proof. We need a result to equivalently characterize $\mathrm{E}\,|X| < \infty$ in general situation first.

Lemma 1.6. *Let X be a r.v. $\mathrm{E}\,|X| < \infty$ iff $\sum_n \mathrm{P}(|X| > n) < \infty$.*

Proof of Lemma. Let $n \in \mathbb{N}$. For $x \in [n-1, n]$, $\mathrm{P}(|X| \geq n) \leq \mathrm{P}(|X| \geq x) \leq \mathrm{P}(|X| \geq n-1)$, so that

$$\mathrm{P}(|X| \geq n) \leq \int_{n-1}^{n} \mathrm{P}(|X| \geq x)\mathrm{d}x \leq \mathrm{P}(|X| \geq n-1).$$

Take summation on n,

$$\sum_{n\geq 1} \mathrm{P}(|X| \geq n) \leq \int_{0}^{\infty} \mathrm{P}(|X| \geq x)\mathrm{d}x \leq \sum_{n\geq 1} \mathrm{P}(|X| \geq n-1) = 1 + \sum_{n\geq 1} \mathrm{P}(|X| \geq n),$$

The result follows by the well-known result deduced by Fubini's Theorem. $\square$

It is enough to show the case $r = 1$. "$\mathrm{E}\,|X| < \infty$" iff "$\sum_n \mathrm{P}(|X| \geq \epsilon n) < \infty$" by Lemma iff "$\sum_n \mathrm{P}(|X_n|/n \geq \epsilon) < \infty$" by the identification of distribution "iff $|X_n|/n \to 0$ a.s." by the equivalence between convergence in probability fast enough and almost-sure convergence. $\square$

References

[1] K.L. Chung. *A Course in Probability Theory.* Elsevier Science, 2001.