Notes on Elementary Analysis

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Part I
Elementary Analysis

1 Integrate on Rational Functions

Functions of the form $R(x) = \frac{P(x)}{Q(x)}$ is called *rational functions*, where $P(x)$ and $Q(x)$ are polynomials. If $\deg(P(x)) < \deg(Q(x))$, then it is called a *proper fraction*; otherwise called *improper fraction*. We can always change an improper fraction into a polynomial plus a proper faction.

Example 1.1.

$$\frac{x^5}{1-x^2}$$

can be written as

$$\frac{x^5 - x^3 + x^3}{1-x^2} = \frac{x^3(x^2-1)+x^3}{1-x^2} = -x^3 + \frac{x^3 - x + x}{1-x^2} = -x^3 - x + \frac{x}{1-x^2}.$$

Therefore, we can only analyze on the integral of proper fraction.

Theorem 1.2 (decomposition). Assume $R(x) = \frac{P(x)}{Q(x)}$ is a proper fraction, where $Q(x) = (x-a_1)^{\alpha_1} \cdots (x-a_n)^{\alpha_n}(x^2+b_1x+c_1)^{\beta_1} \cdots (x^2+b_mx+c_m)^{\beta_m}$, where $\{a_i\}, \{b_i\}, \{c_i\} \subseteq \mathbb{R}$ and $\Delta_i = b_i^2 - 4c_i < 0$; also $\{\alpha_i\}, \{\beta_i\} \subseteq \mathbb{Z}_+$. Then $R(x)$ can be decomposed to

$$\begin{aligned}
R(x) = &\frac{A_{1\alpha_1}}{(x-a_1)^{\alpha_1}} + \cdots \frac{A_{11}}{x-a_1} \\
&+ \dots \\
&+ \frac{A_{n\alpha_n}}{(x-a_n)^{\alpha_n}} + \cdots \frac{A_{n1}}{x-a_1} \\
&+ \frac{B_{1\beta_1}x + C_{1\beta_1}}{(x^2+b_1x+c_1)^{\beta_1}} + \cdots + \frac{B_{11}x + C_{11}}{x^2+b_1x+c_1} \\
&+ \dots \\
&+ \frac{B_{m\beta_m}x + C_{m\beta_m}}{(x^2+b_mx+c_m)^{\beta_m}} + \cdots + \frac{B_{m1}x + C_{m1}}{x^2+b_mx+c_m},
\end{aligned}$$

where $\{A_{ij}\}, \{B_{ij}\} \subseteq \mathbb{R}$ and the coefficients are unique.

Proof. Find the proof in Complex analysis. □

The theorem told us we can only consider the integral of the form

$$\frac{A}{(x-a)^k} \quad \text{and} \quad \frac{Bx+C}{(x^2+bx+c)^l},$$

where $b^2 - 4c < 0$.

Recall that

$$\int \frac{\mathrm{d}x}{x-a} = \ln|x-a| + c$$

and

$$\int \frac{\mathrm{d}x}{(x-a)^k} = \frac{(x-a)^{1-k}}{1-k} + c$$

for $k \geq 2$. Therefore, we only need to investigate

$$\int \frac{Bx+C}{(x^2+bx+c)^l}\mathrm{d}x$$

where $b^2 - 4c < 0$ and $l \in \mathbb{Z}_+$.

We have

$$x^2 + bx + c = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}.$$

Let $a^2 = c - \frac{b^2}{4}$ and $u = x + \frac{b}{2}$. Then

$$\int \frac{Bx+C}{(x^2+bx+c)^l}\mathrm{d}x = B\int \frac{u}{(a^2+u^2)^l}\mathrm{d}u + (C - \frac{B \cdot b}{2})\int \frac{\mathrm{d}u}{(a^2+u^2)^l}.$$

When u in the nominator, the integral is easy, as

$$\int \frac{u}{a^2+u^2}\mathrm{d}u = \frac{1}{2}\ln(a^2+u^2) + c;$$

and for $l \geq 2$,

$$\int \frac{u}{(a^2+u^2)^l}\mathrm{d}u = \frac{1}{2(1-k)}(a^2+u^2)^{1-l} + c.$$

It remains the final step: calculate

$$I_l \triangleq \int \frac{\mathrm{d}u}{(a^2+u^2)^l},$$

for $l \in \mathbb{Z}_+$. To get the recurrence relation, use the method of integral by parts, then

$$\begin{aligned} I_l =& \frac{u}{(a^2+u^2)^l} + 2l\int \frac{u^2}{(a^2+u^2)^{l+1}}\mathrm{d}u \\ =& \frac{u}{(a^2+u^2)^l} + 2l\int \frac{a^2+u^2-a^2}{(a^2+u^2)^{l+1}}\mathrm{d}u \\ =& \frac{u}{(a^2+u^2)^l} + 2lI_l - 2la^2I_{l+1}, \end{aligned}$$

namely,

$$I_{l+1} = \frac{1}{2la^2}\frac{u}{(a^2+u^2)^l} + \frac{2l-1}{2la^2}I_l. \tag{1.1}$$

We then use this recurrence relation to calculate I_l, for $l \in \mathbb{Z}_+$. Recall that

$$I_1 = \int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + c.$$

For convenience, I list the following relation that we commonly use:

$$I_2 = \frac{1}{2a^2}\left(\frac{u}{a^2+u^2} + I_1\right); \tag{1.2}$$

$$I_3 = \frac{1}{4a^2}\left(\frac{u}{(a^2+u^2)^2} + 3I_2\right). \tag{1.3}$$

Part II
Functional Analysis

2 Separability

Definition 2.1 (separable). A metric space (X, d) is *separable* if it contains a countable dense set.

Lemma 2.2. If (X, d) is separable and $Y \subset X$, then (Y, d) is also separable.

Proof. Suppose $\{x_n\}$ is dense in X. Construct the separable set A for Y as following: for each $n, k \in \mathbb{N}$, if $B(x_n, 1/k) \cap Y \neq \emptyset$, then choose one point from $B(x_n, 1/k) \cap Y$ to A. □

It is a useful strategy to choose $1/k$ and consider a ball $B(x, 1/k)$ for a countable sequence. The proof of the following proposition is an example.

Proposition 2.3. Any compact metric space (X, d) is separable. Furthermore, in any compact metric space there exists a countable subset $(x_j)_{j=1}^{\infty}$ with the following property: for any $\epsilon > 0$, there is an $M(\epsilon)$ such that for every $x \in X$ we have $d(x_j, x) < \epsilon$ for some $1 \le j \le M(\epsilon)$.

Proof. Consider covering collection of radius $1/n$. □

To show a space is *not separable*, we should show that there is no dense subset. A common strategy is the following: construct an uncountable set B in which two distinct elements x, y must differ by a constant, i.e. $\|x - y\| \ge c$. Then, any dense set A must contain an uncountable number of elements. Indeed, for any distinct elements x, y in B, there exists $x', y' \in A$ such that $\|x - x'\| < 1/3$ and $\|y - y'\| < 1/3$. Then

$$\|x' - y'\| = \|x' - x + x - y + y - y'\| \le \|x - y\| - \|x' - x\| - \|y - y'\| > \frac{1}{3},$$

which shows x', y' are distinct. There exists a injectivity from B to A. A must be uncountable.

The proof of the non-separability of l^∞ and L^∞ uses the strategy.

All separable Hilbert spaces are isometrically isomorphic to l^2, so that in some sense l^2 is the only (infinite dimensional) separable Hilbert space.

Proposition 2.4. An infinite-dimensional Hilbert space is separable if and only if it has a countable orthonormal basis.

Proof. If a Hilbert space has a countable basis, then the conclusion is trivial. If a Hilbert space is separable, then consider Gram-Schmidt process. □

Theorem 2.5. Any infinite dimensional separable Hilbert space H over $\mathbb{K}$ is isometrically isomorphic to $l^2(\mathbb{K})$, i.e. $H \equiv l^2(\mathbb{K})$.

Most Hilbert spaces that occur in applications are separable, but there are non-separable Hilbert spaces, such as l^∞ and L^∞.

3 Completeness

"Completeness arguments" usually follow similar lines:

1. use the definition of what it means for a sequence to be Cauchy to identify a possible limit;
2. show that the original sequence converges to this "possible limit" in the appropriate norm;
3. check that the "limit" lies in the correct space.

In step two, it is enough to show a subsequence of the Cauchy sequence converges to the limit. This is because of the following lemma.

Lemma 3.1. Let (X, d) be a metric space and (x_n) is a Cauchy sequence in X. If there exists a subsequence (x_{n_k}) converges to x, then (x_n) must converge to x.

Therefore, to consider the convergence of a Cauchy sequence, it is equivalent to consider the convergence of the subsequence.

An example follows in the proof of the following theorem.

Theorem 3.2. The space $\mathbb{K}^d$ [a] is complete (with its standard norm [b]).

[a] Be careful that $\mathbb{K}$ does not mean all field of numbers here. Instead, it means either $\mathbb{R}$ or $\mathbb{C}$.

[b] The standard norm of $\mathbb{K}^d$ is the Euclid norm, i.e. l^2 norm for all $d \in \mathbb{N}_+$. However, the standard norm for $l^p(\mathbb{K})$ is l^p norm for each $1 \le p \le \infty$.

Proof. If x^n is Cauchy sequence in $\mathbb{K}^d$, then each coordinate is Cauchy in $\mathbb{K}$, which also has a limit in $\mathbb{K}$ since $\mathbb{K}$ is complete. Let $x \in \mathbb{K}^d$ be the limit where x_j is the limit of x_j^n. Step 2 is check $x^n \to x$. Step 3 is check $x^n \in \mathbb{K}^d$ which is trivial. □

With this theorem, we can deduce that any finite dimensional space with any norm $\|\cdot\|$ is complete, since $(V, \|\cdot\|_E) \equiv (\mathbb{K}^n, \|\cdot\|_{l^2})$. Here "$\equiv$" means "isometrically isomorphic to" and

$$\|x\|_E \triangleq \left(\sum_{i=1}^{n} \alpha_i^2\right)^{1/2},$$

where $x = \sum_{i=1}^n \alpha_i e_i$ and (e_i) is the basis of E.

Corollary 3.3. Any finite-dimensional normed space $(V, \|\cdot\|)$ is complete.

Here is a different strategy to prove the completeness.

Lemma 3.4. If $(X, \|\cdot\|)$ is a Banach space and Y is a linear subspace of X, then $(Y, \|\cdot\|)$ is a Banach space if and only if Y is closed.

Proof. If Y is complete, then the Cauchy sequence in Y converges in Y. If $(y_n) \to y$, then (y_n) is also Cauchy, so $y \in Y$. Contrarily, if Y is closed, since the Cauchy sequence in Y is also a Cauchy sequence in X, it converges. Because of closeness, it converges in Y. □

The statement of the following lemma provides a useful test for completeness. If there exists a sequence that does not satisfy the condition, then the space must not be complete.

Lemma 3.5. If $(X, \|\cdot\|)$ is a normed space with the property that whenever $\sum_{j=1}^{\infty} \|x_j\| < \infty$, the sum $\sum_{j=1}^{\infty} x_j$ converges in X, then X is complete.

Proof. Suppose (y_j) is a Cauchy sequence in X. Inductively find n_k such that $n_{k+1} > n_k$ and $\|y_i - y_j\| < 2^{-k}$ for $i, j > n_k$. Then set $x_1 = y_{n_1}$, $x_j = y_{n_j} - y_{n_{j-1}}$. Use the assumption to show that $y_{n_j} \to y$, where $y = \sum_{j=1}^{\infty} x_j$. □

4 Properties Preserved under Isomorphism

To be more specific, the "isomorphism" we talking about here is not the "set-isomorphism". Instead, it requires more: for $T : X \to Y$, where X, Y are normed spaces, T should have the following properties

1. bijectivity;

2. linearity;

3. norm preserving: there exists c_1, c_2 such that $c_1 \|x\| \leq \|Tx\| \leq c_2 \|x\|$.

Note that the third property ensures the injectivity, so the procedure to check isomorphism is first check the linearity, then norm preserving, finally surjective.

Note that T is automatically continuous by property item 2 and 3.

Proposition 4.1. Assume $T : X \to Y$ is a normed-space isomorphism, then T is a continuous map.

Proposition 4.2. If $(X, \|\cdot\|_X) \backsimeq (Y, \|\cdot\|_Y)$, then X is separable if and only if Y is separable.

Proof. Assume $X = \overline{\{x_j\}}$, show that $Y = \overline{\{Tx_j\}}$. □

Proposition 4.3. If $(X, \|\cdot\|_X) \backsimeq (Y, \|\cdot\|_Y)$, then $(X, \|\cdot\|_X)$ is complete if and only if $(Y, \|\cdot\|_Y)$ is complete.

Proof. If (x_n) is Cauchy in X, then (Tx_n) is Cauchy in Y. Assume Y is complete, then $Tx_n \to y \triangleq Tx$. Show that $x_n \to x$. □

5 The Contraction Mapping Theorem

In a complete normed space [1] $(X, \|\cdot\|)$, the Contraction Mapping Theorem, also known as Banach's Fixed Point Theorem, enables us to find a fixed point of any map that is a contraction.

Theorem 5.1 (Contraction Mapping Theorem). Let K be a non-empty closed subset of a complete normed space $(X, \|\cdot\|)$ and $f : K \to K$ a contraction, i.e. a map such that

$$\|f(x) - f(y)\| \leq \kappa \|x - y\|$$

for any $x, y \in K$ and some $\kappa < 1$. Then f has a unique fixed point in K, i.e. there exists a unique $x \in K$ such that $f(x) = x$.

Proof. Choose $x_0 \in K$ and set $x_{n+1} = f(x_n)$. Then note that

$$\|x_{j+1} - x_j\| \leq \kappa \|x_j - x_{j-1}\| \leq \cdots \leq \kappa^j \|x_1 - x_0\| .$$

Then use triangle inequality repeatedly, we have

$$\|x_k - x_j\| \leq \sum_{i=j}^{k-1} \|x_{i+1} - x_i\| \leq \frac{\kappa^j}{1-\kappa} \|x_1 - x_0\| .$$

It follows that (x_n) is a Cauchy sequence. By completeness, $x_n \to x$ and by closeness, $x \in K$. Then we have $x = f(x)$ by let $n \to \infty$, where the continuity of f is followed from the contraction. Such x is unique, since if $f(x) = x$ and $f(y) = y$, then $\|x - y\| = \|f(x) - f(y)\| \leq \kappa \|x - y\|$. Since $\kappa < 1$, this is not possible. □

Note that the conclusion of the theorem is no longer valid if we only have $\|f(x) - f(y)\| < \|x - y\|$ for any $x \neq y$, unless K is compact.

6 Compactness

There are two kinds of compactness.

Definition 6.1 (compact). A subset K of a metric space (X, d) is *compact* if any cover of K by open sets has a finite subcover.

Definition 6.2 (sequentially compact). If K is a subset of (X, d), then K is *sequentially compact* if any sequence in K has a subsequence that converges and whose limit lies in K.

Note that compactness implies close and bounded.

Lemma 6.3. If K is a compact subset of a metric space (X, d), then K is closed and bounded.

[1] The theorem also holds in any complete metric space (X, d) with the obvious changes

The opposite implication is only valid in finite dimensional space.

Theorem 6.4. A subset of a finite-dimensional normed space is compact if and only if it is closed and bounded.

Proof. The proof based on the fact that the result is true on $\mathbb{K}^d$ and construct a isometric isomorphism between two spaces. $\square$

A different characterization of finite-dimensional normed space based on compactness is the following theorem.

Theorem 6.5. A normed space X is finite-dimensional if and only if its closed unit ball is compact.

The proof based on Riesz's lemma.

Lemma 6.6 (Riesz's Lemma). Let $(X, \|\cdot\|)$ be a normed space and Y a proper closed subspace of X. Then there exists $x \in X$ with $\|x\| = 1$ such that $\|x - y\| \geq 1/2$ for every $y \in Y$.

Using the theorem, if in infinite dimensional space, a subset is bounded and closed implies compactness, then we can deduce that its closed unit ball is compact, which means the space is finite dimensional. Therefore, we have the following conclusion.

Corollary 6.7. The equivalence between compactness and closed-bounded is valid and only valid in finite dimensional normed space.

An easy example is the $(e^j)_j$ in l^p space. There is no subsequence of (e^j) converges, since no subsequence can be Cauchy. So the closed unit ball in $l^p(\mathbb{K})$ is not compact, which shows $l^p(\mathbb{K})$ is of infinite dimensional.

7 Fundamentals of Inner Product Space

Examples of the inner product spaces are:

1. $\mathbb{R}^n$ with $\langle x, y\rangle = \sum_{j=1}^n x_j \overline{y_j}$;
2. Sequence space $l^2(\mathbb{K})$ with $\langle x, y\rangle = \sum_{j=1}^\infty x_j \overline{y_j}$;
3. Functional space $L^2(\Omega)$ with $\langle f, g\rangle = \int_\Omega f(x)\overline{g(x)}\mathrm{d}x$.

Lemma 7.1 (Cauchy-Schwarz Inequality). Any inner product $\langle\cdot,\cdot\rangle$ in a vector space V satisfies

$$|\langle x, y\rangle| \leq \|x\| \|y\|, \tag{7.1}$$

for all $x, y \in V$, where $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$, which we called the *induced norm*.

Lemma 7.2 (Parallelogram Law). If V is an inner product space with induced norm $\|\cdot\|$, then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \tag{7.2}$$

for all $x, y \in V$.

Lemma 7.3 (Polarisation Identity). Let V be an inner product space with induced norm $\|\cdot\|$. Then if V is real,

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2, \tag{7.3}$$

while if V is complex,

$$4\langle x, y\rangle = \sum_{n=0}^{3} \mathrm{i}^n \|x + \mathrm{i}^n y\|^2. \tag{7.4}$$

Theorem 7.4 (Jordan-Neumann Theorem). If $\|\cdot\|$ is a norm on a real vector space V that satisfies Eq.(7.2), then the inner product defined by Eq.(7.3) is indeed an inner product.

8 Schauder Bases

Definition 8.1 (Schauder basis). A countable set $\{e_j\}$ is a *Schauder basis* for a normed space V if every $v \in V$ can be written uniquely as $v = \sum_{j=1}^{\infty} \alpha_j e_j$ for some $\{\alpha_j \in \mathbb{K}\}$.

Note that "uniquely" equivalently means independence in "countable sense", that is: if $\sum_{j=1}^{\infty} \alpha_j e_j = 0$ then $\alpha_j = 0$ for every j.

Therefore, a Schauder basis is a Hamel basis but a Hamel basis is not necessarily be a Schauder basis since (i) Schauder basis must be countable and (ii) the linearly independence on finite sum version is not enough. In other words, we now narrow the concept of basis from now on.

9 Orthonormal Sets

Lemma 9.1 (Generalized Pythagoras). If $\{e_1, \dots, e_n\}$ is an orthonormal set in an inner product space V, then for any $\{\alpha_j \in \mathbb{K}\}_{j=1}^{n}$,

$$\left\|\sum_{j=1}^{n} \alpha_j e_j\right\|^2 = \sum_{j=1}^{n} |\alpha_j|^2. \tag{9.1}$$

10 Bessel's Inequality and Parseval's Identity

Lemma 10.1 (Bessel's Inequality). Let V be an inner product space and $\{e_j\}_{j=1}^{\infty}$ an orthonormal set in V. Then for all $v \in V$,

$$\sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2 \le \|x\|^2 \tag{10.1}$$

and in particular the left hand side converges.

Proof. Note that $\|x - x_k\|^2 = \|x\|^2 - \|x_k\|^2$, where $\|x_k\| \triangleq \sum_{j=1}^{k} |\langle x, e_j \rangle|^2$. $\square$

Lemma 10.2 (Parseval's Identity). Let H be a Hilbert space and $\{e_j\}_{j=1}^{\infty}$ an orthornormal set in H. The series $\sum_{j=1}^{\infty} \alpha_j e_j$ converges if and only if $\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty$. And

$$\left\| \sum_{j=1}^{\infty} \alpha_j e_j \right\|^2 = \sum_{j=1}^{\infty} |\alpha_j|^2 . \tag{10.2}$$

Proof. Don't forget we are working with the Hilbert space, the proof includes the property of completeness. $\square$

Combining two lemmas, we have

Corollary 10.3. Let H be a Hilbert space and $\{e_j\}_{j=1}^{\infty}$ an orthonormal set in H. Then $\sum_{j=1}^{\infty} \langle x, e_j \rangle e_j$ converges for every $x \in H$.

Note that $\sum_{j=1}^{\infty} \langle x, e_j \rangle e_j$ converges but it not necessarily converges to x unless it is a orthonormal basis.

Proposition 10.4. Let $E = \{e_j\}_{j=1}^{\infty}$ be an orthonormal set in a Hilbert space H. Then the following statements are equivalent:

1. E is a basis for H;
2. for any x we have
$$x = \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j . \tag{10.3}$$
3. Parseval's identity holds:
$$\|x\|^2 = \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2 . \tag{10.4}$$
4. $\langle x, e_j \rangle = 0$ for all j implies that $x = 0$.
5. $\mathrm{clin}(E) = H$.

Recall that clin means the closed linear space of E, i.e. $\mathrm{clin}(E) = \overline{\mathrm{span}(E)}$. Compare Eq.(10.4) with Eq.(10.1), which shows the condition is also equivalent to the equality holds in the Bessel's inequality.

References