Notes on Elementary Analysis

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1 Integrate on Rational Functions

Functions of the form $R(x) = \frac{P(x)}{Q(x)}$ is called *rational functions*, where $P(x)$ and $Q(x)$ are polynomials. If $\deg(P(x)) < \deg(Q(x))$, then it is called a *proper fraction*; otherwise called *improper fraction*. We can always change an improper fraction into a polynomial plus a proper faction.

Example 1.1.

$$\frac{x^5}{1-x^2}$$

can be written as

$$\frac{x^5-x^3+x^3}{1-x^2} = \frac{x^3(x^2-1)+x^3}{1-x^2} = -x^3 + \frac{x^3-x+x}{1-x^2} = -x^3 - x + \frac{x}{1-x^2}.$$

Therefore, we can only analyze on the integral of proper fraction.

Theorem 1.2 (decomposition). Assume $R(x) = \frac{P(x)}{Q(x)}$ is a proper fraction, where $Q(x) = (x-a_1)^{\alpha_1} \cdots (x-a_n)^{\alpha_n}(x^2+b_1x+c_1)^{\beta_1} \cdots (x^2+b_mx+c_m)^{\beta_m}$, where $\{a_i\}, \{b_i\}, \{c_i\} \subseteq \mathbb{R}$ and $\Delta_i = b_i^2 - 4c_i < 0$; also $\{\alpha_i\}, \{\beta_i\} \subseteq \mathbb{Z}_+$. Then $R(x)$

can be decomposed to

$$
\begin{aligned}
R(x) = &\frac{A_{1\alpha_1}}{(x-a_1)^{\alpha_1}} + \cdots \frac{A_{11}}{x-a_1}\\
&+ \ldots\\
&+ \frac{A_{n\alpha_n}}{(x-a_n)^{\alpha_n}} + \cdots \frac{A_{n1}}{x-a_1}\\
&+ \frac{B_{1\beta_1}x + C_{1\beta_1}}{(x^2+b_1x+c_1)^{\beta_1}} + \cdots + \frac{B_{11}x+C_{11}}{x^2+b_1x+c_1}\\
&+ \ldots\\
&+ \frac{B_{m\beta_m}x + C_{m\beta_m}}{(x^2+b_mx+c_m)^{\beta_m}} + \cdots + \frac{B_{m1}x+C_{m1}}{x^2+b_mx+c_m},
\end{aligned}
$$

where $\{A_{ij}\}, \{B_{ij}\} \subseteq \mathbb{R}$ and the coefficients are unique.

Proof. Find the proof in Complex analysis. $\square$

The theorem told us we can only consider the integral of the form $\frac{A}{(x-a)^k}$ and $\frac{Bx+C}{(x^2+bx+c)^l}$, where $b^2 - 4c < 0$.

Recall that

$$\int \frac{\mathrm{d}x}{x-a} = \ln|x-a| + c$$

and

$$\int \frac{\mathrm{d}x}{(x-a)^k} = \frac{(x-a)^{1-k}}{1-k} + c$$

for $k \geq 2$. Therefore, we only need to investigate $\int \frac{Bx+C}{(x^2+bx+c)^l}\mathrm{d}x$ where $b^2 - 4c < 0$ and $l \in \mathbb{Z}_+$.

We have

$$x^2 + bx + c = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}.$$

Let $a^2 = c - \frac{b^2}{4}$ and $u = x + \frac{b}{2}$. Then

$$\int \frac{Bx+C}{(x^2+bx+c)^l}\mathrm{d}x = B\int \frac{u}{(a^2+u^2)^l}\mathrm{d}u + (C - \frac{B\cdot b}{2})\int \frac{\mathrm{d}u}{(a^2+u^2)^l}.$$

When u in the nominator, the integral is easy, as

$$\int \frac{u}{a^2+u^2}\mathrm{d}u = \frac{1}{2}\ln(a^2+u^2) + c;$$

and for $l \geq 2$,

$$\int \frac{u}{(a^2+u^2)^l}\mathrm{d}u = \frac{1}{2(1-k)}(a^2+u^2)^{1-l} + c.$$

It remains the final step: calculate

$$I_l \triangleq \int \frac{\mathrm{d}u}{(a^2+u^2)^l},$$

for $l \in \mathbb{Z}_+$. To get the recurrence relation, use the method of integral by parts, then

$$\begin{aligned} I_l =& \frac{u}{(a^2+u^2)^l} + 2l \int \frac{u^2}{(a^2+u^2)^{l+1}} \mathrm{d}u \\ =& \frac{u}{(a^2+u^2)^l} + 2l \int \frac{a^2+u^2-a^2}{(a^2+u^2)^{l+1}} \mathrm{d}u \\ =& \frac{u}{(a^2+u^2)^l} + 2lI_l - 2la^2 I_{l+1}, \end{aligned}$$

namely,

$$I_{l+1} = \frac{1}{2la^2} \frac{u}{(a^2+u^2)^l} + \frac{2l-1}{2la^2} I_l. \tag{1.1}$$

We then use this recurrence relation to calculate I_l, for $l \in \mathbb{Z}_+$. Recall that

$$I_1 = \int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + c.$$

For convenience, I list the following relation that we commonly use:

$$I_2 = \frac{1}{2a^2} \left(\frac{u}{a^2+u^2} + I_1 \right); \tag{1.2}$$

$$I_3 = \frac{1}{4a^2} \left(\frac{u}{(a^2+u^2)^2} + 3I_2 \right). \tag{1.3}$$

References