Graphic language for many-body physics

Now we try to translate the object in many-body physics into graphic object and rewrite it in graphic language.

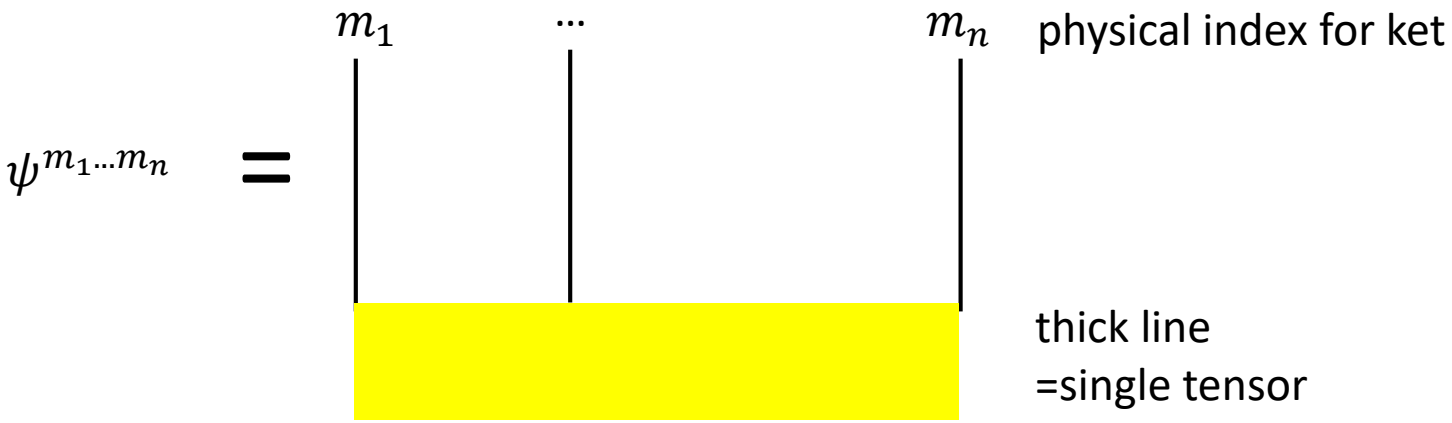
How to include the structure of entanglement?

First we discuss 1D system

A many-body ket is a (N, 0)-type tensor

$$\left|\psi\right\rangle = \sum_{\{m\}} \psi^{m_1 \ldots m_n} \underbrace{\left|m_1\right\rangle \otimes \ldots \otimes \left|m_n\right\rangle}_{\text{basis}}$$

The coefficient is just the amplitude on give state, and we can picture it as:



To here, we find that we just rediscover the language of quantum circuit!

https://www.bilibili.com/video/av38995130?from=search&seid=12285956102166412330

Inside the mirror space, there lives the bra:

$$\langle \phi | = \sum_{\{m'\}} \phi_{m_1' \ldots m_n'} \langle m_1' | \otimes \ldots \otimes \langle m_n' |$$

Overlap between two states

Consider two states in the same Hilbert space, so they are spanned by the same basis, we can calculate the overlap between these two states:

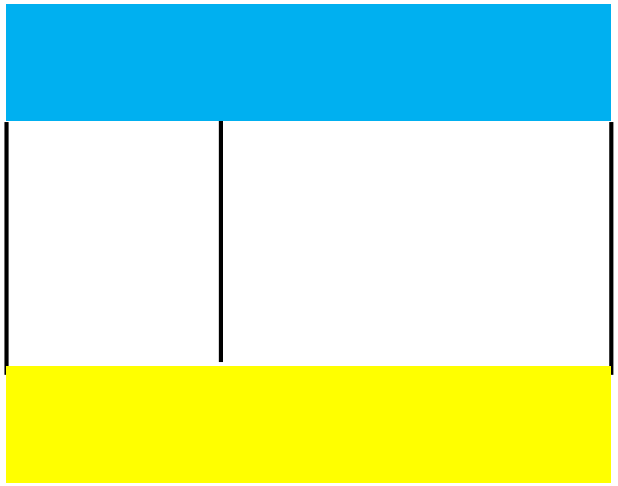
$$\langle\phi|\psi\rangle = \sum_{\{m\},\{m'\}} \phi_{m_1'\ldots m_n'} \psi^{m_1\ldots m_n} \underbrace{(\langle m_1'| \otimes \ldots \otimes \langle m_n'|)(|m_1\rangle \otimes \ldots \otimes |m_n\rangle)}_{\text{delta function, suppose our basis are orthonormal}}$$

Therefore, the only thing left are

$$\langle\phi|\psi\rangle = \sum_{\{m\}} \phi_{m_1\ldots m_n} \psi^{m_1\ldots m_n}$$

The overlap is just the contraction of two tensors, which gives a number, as is needed. In graphic language, we have:

$\langle\phi|\psi\rangle$ =



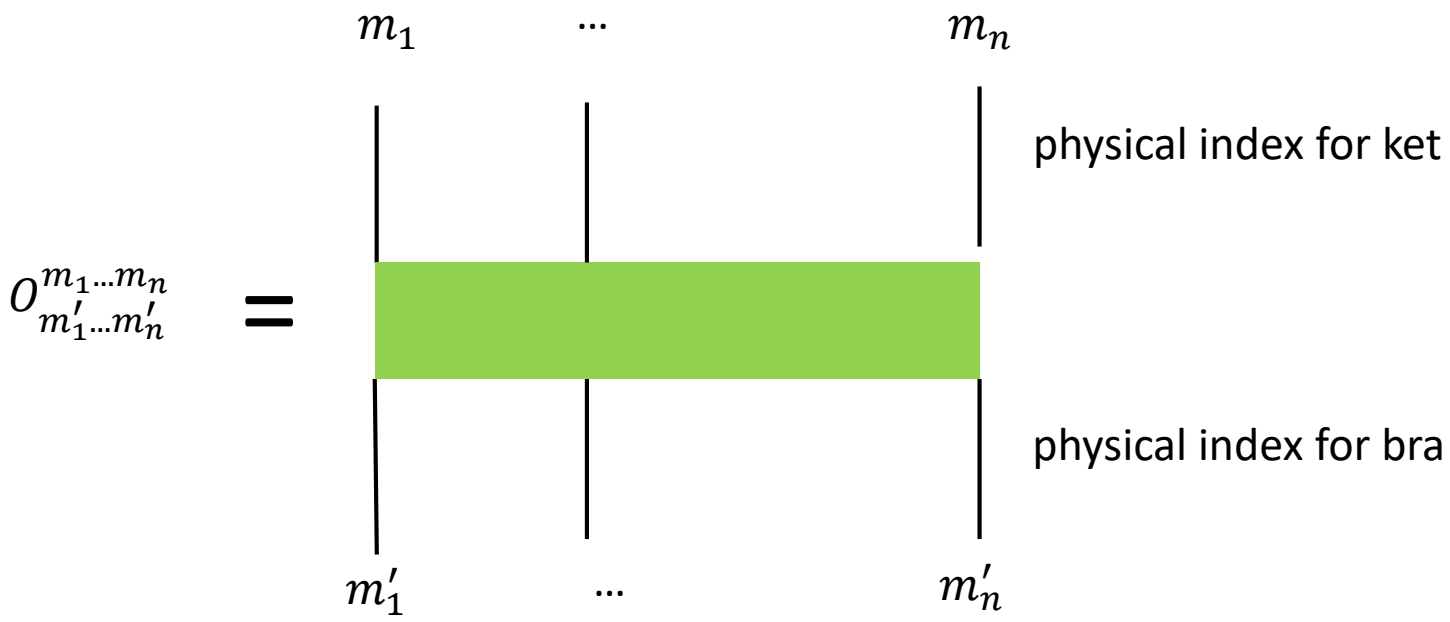
$\langle\phi|$

Each connected leg means summing over one physical index

$|\psi\rangle$

An operator is a (N, N)-type tensor

$$O = \sum_{\{m\},\{m'\}} O^{m_1 \ldots m_n}_{m_1' \ldots m_n'} (|m_1\rangle \otimes \ldots \otimes |m_n\rangle)(\langle m_1'| \otimes \ldots \otimes \langle m_n'|)$$



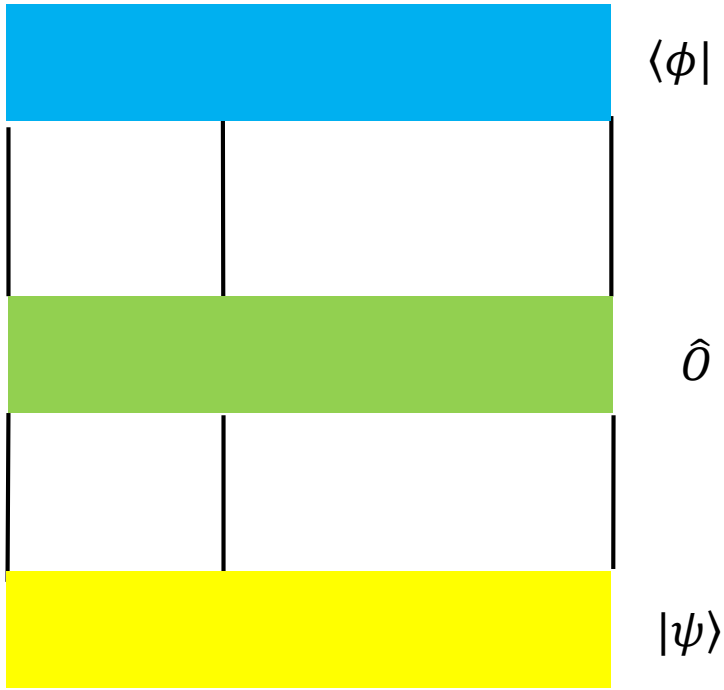
So far, there is nothing new, just expand the state and operator with basis.

Expectation value

$$\langle \phi | O | \psi \rangle = \sum_{\{m\},\{m'\}} \phi_{m_1 \ldots m_n} O^{m_1 \ldots m_n}_{m'_1 \ldots m'_n} \psi^{m'_1 \ldots m'_n}$$

A sandwich

$\langle \phi | \hat{O} | \psi \rangle$ =

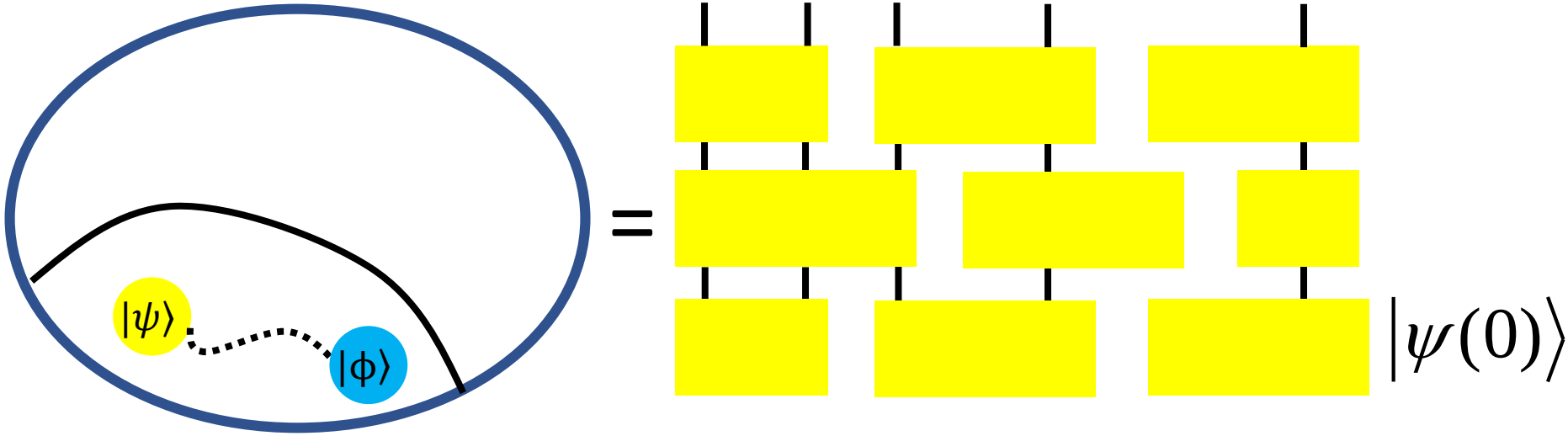


Each connected leg means summing over one physical index

Local Hamiltonian

All the above tensor is extremely high order and its really hard to treat.

If the system is local, i.e. the Hamiltonian of the system is made up of local operators (one-body, two-body…), then near a many-body state in the Hilbert space, is connected with other state by the product of local operators. This is the idea of quantum circuit means. Note the d.o.f. of quantum circuit is O(N).
If the G.S. is non-degenerate, then the states that are physical should be connected to the G.S. by quantum circuit, and these physical state should be of O(N). Therefore, the physical state lies at a corner of the Hilbert space.

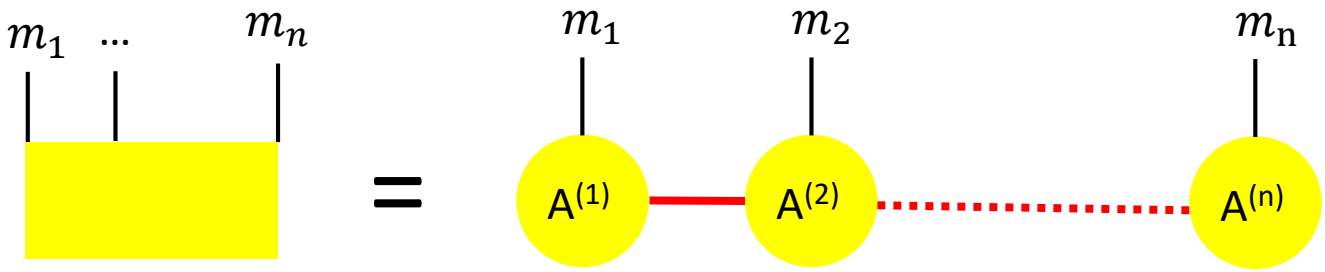


MPS ansatz

Basic tensor tells us that any tensor can be expressed by matrix product state

$$|\psi\rangle = \sum_{\{m\}} \psi^{m_1 \ldots m_n} |m_1\rangle \otimes \ldots \otimes |m_n\rangle$$

$$= \sum_{\{\mathrm{m}\},\{\mathrm{a}\}} A^{(1),\mathrm{m}_1}_{1 a_1} \ldots A^{(\mathrm{n}),\mathrm{m}_n}_{a_n 1} |m_1\rangle \otimes \ldots \otimes |m_n\rangle$$



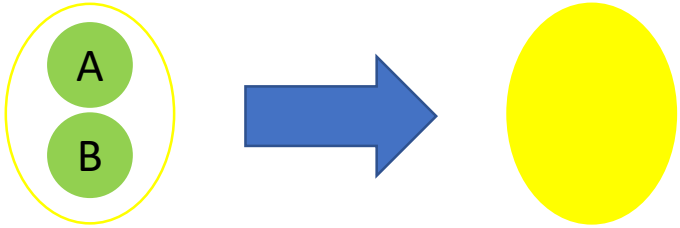
If {a} are not 1, then MPS is natural description of many-body ground state satisfying area law of entanglement entropy.

Here, we from a more physical root to derive MPS ansatz again:

Projection + maximally entanglement

Projector representation

We have seen how powerful projector representation is, here we will reverse what we have obtained.



$$\left|\uparrow\uparrow\right\rangle \qquad \left|1,1\right\rangle$$

$$\frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle+\left|\downarrow\uparrow\right\rangle\right) \qquad \left|1,0\right\rangle$$

$$\left|\downarrow\downarrow\right\rangle \qquad \left|1,-1\right\rangle$$

We can define the onsite projector on spin-triplet as:

$$P^{(1)}=\left|1,1\right\rangle\left\langle\uparrow_{1A}\uparrow_{1B}\right|+\left|1,0\right\rangle\frac{\left\langle\uparrow_{1A}\downarrow_{1B}\right|+\left\langle\downarrow_{1A}\uparrow_{1B}\right|}{\sqrt{2}}+\left|1,-1\right\rangle\left\langle\downarrow_{1A}\downarrow_{1B}\right|$$

Here (1) means the site.

Write it into matrix form, we have:

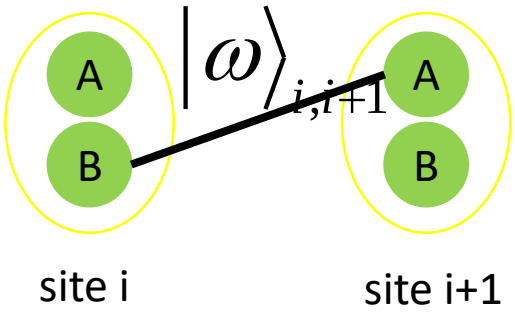
$$P^{(1)} = \sum_{\substack{i=1,\ldots,d \\ \alpha,\beta=1,\ldots,D}} A^{(1),\mathrm{i}}_{\alpha\beta} \left|i\right\rangle\left\langle\alpha\beta\right|$$

Here α and β are d.o.f. of auxiliary particle, as the effective spin-1/2 in spin-1 chain (AKLT model). The d.o.f. of α and β may be the same or not the same. D is called the bond dimension.

Each matrix $A^{(1)}$ has three indexes, one for physical index i with dimension d
and two for quasiparticle index α and β with dimension D.
We need to specify a given matrix for each physical d.o.f.

$$A^{(1),\mathrm{i}} = \begin{array}{c} \begin{matrix} |1A,\alpha 1\rangle & & |1A,\alpha D\rangle \end{matrix} \\ \begin{pmatrix} A^{(1),\mathrm{i}}_{11} & \cdots & A^{(1),\mathrm{i}}_{1d} \\ \cdots & \cdots & \cdots \\ A^{(1),\mathrm{i}}_{d1} & \cdots & A^{(1),\mathrm{i}}_{dd} \end{pmatrix} \end{array} \begin{matrix} |1B,\alpha 1\rangle \\ \\ |1B,\alpha D\rangle \end{matrix}$$

Now we code in maximally entanglement:



Since B of site i and A of site i+1 both has d.o.f. D, therefore, the maximally entangled state between these two are:

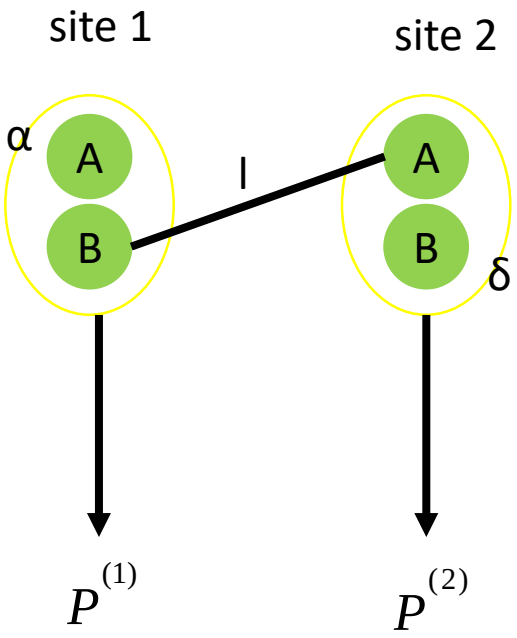
$$|\omega\rangle_{i,i+1} = \sum_{\mathrm{k},l=1,\ldots,D} \Sigma_{k,l}^{(i,i+1)} |k_{iB}; l_{i+1A}\rangle$$

So no matter where we set the boundary, the area law will be vary natural.

We can also write this into a matrix form:

$$\Sigma^{i,i+1} = \begin{array}{c} \begin{matrix} |iB,\alpha 1\rangle & & |iB,\alpha D\rangle \end{matrix} \\ \begin{pmatrix} \Sigma_{11}^{i,i+1} & \cdots & \Sigma_{1d}^{i,i+1} \\ \cdots & \cdots & \cdots \\ \Sigma_{d1}^{i,i+1} & \cdots & \Sigma_{dd}^{i,i+1} \end{pmatrix} \end{array} \begin{matrix} |i+1A,\alpha 1\rangle \\ \\ |i+1A,\alpha D\rangle \end{matrix}$$

Now we can consider the effect of projector on the maximally entangled state:



$$P^{(1)} \otimes P^{(2)} \left|\omega\right\rangle_{1,2} = \Big(\sum_{\substack{i=1,\ldots,d \\ \alpha,\beta=1,\ldots,D}} A^{(1),\mathrm{i}}_{\alpha\beta} \left|i\right\rangle_1 \left\langle \alpha_{1A}\beta_{1B}\right| \otimes \sum_{\substack{j=1,\ldots,d \\ \gamma,\delta=1,\ldots,D}} A^{(2),\mathrm{j}}_{\gamma\delta} \left|j\right\rangle_2 \left\langle \gamma_{2A}\delta_{2B}\right| \Big) \sum_{\mathrm{k},l=1,\ldots,D} \Sigma^{i,i+1}_{k,l} \left|k_{1B}; l_{2A}\right\rangle$$

According to our notation, the first k contract with β and the second k contract with γ.

So the index β and γ should be replaced by k, we have

$$P^{(1)} \otimes P^{(2)} \left|\omega\right\rangle_{1,2} = \sum_{\substack{i,j=1,\ldots,d \\ \alpha,\delta=1,\ldots,D}} \sum_{k,l} A^{(1),\mathrm{i}}_{\alpha k} \Sigma^{(1,2)}_{kl} A^{(2),\mathrm{j}}_{l\delta} \left|i\right\rangle_1 \left|j\right\rangle_2 \left\langle \alpha_{1A} \delta_{2B} \right|$$

$$= \sum_{\substack{i,j=1,\ldots,d \\ \alpha,\delta=1,\ldots,D}} (A^{(1),\mathrm{i}} \Sigma^{(1,2)} A^{(2),\mathrm{j}})_{\alpha\delta} \left|i\right\rangle_1 \left|j\right\rangle_2 \left\langle \alpha_{1A} \delta_{2B} \right|$$

what we want

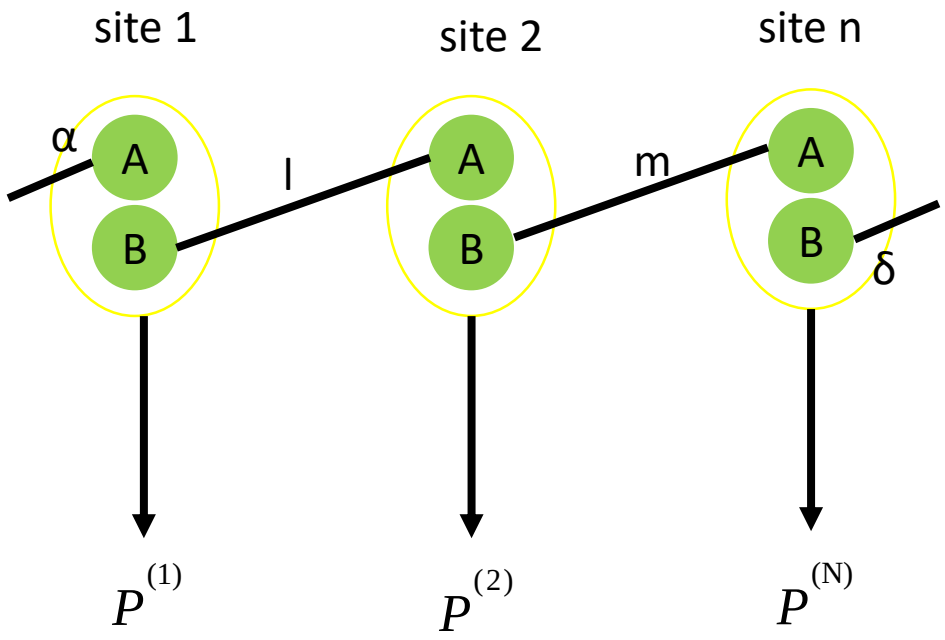
Since we have done nothing to the two end, they are left behind.

We can continue the above process, and If we have three sites:

$$P^{(1)} \otimes P^{(2)} \otimes P^{(3)} \left|\omega\right\rangle_{1,2} \left|\omega\right\rangle_{2,3} = \sum_{\substack{i,j,k=1,\ldots,d \\ \alpha,\delta=1,\ldots,D}} (A^{(1),\mathrm{i}} \Sigma^{(1,2)} A^{(2),\mathrm{j}} \Sigma^{(2,3)} A^{(3),\mathrm{k}})_{\alpha\delta} \left|i\right\rangle_1 \left|j\right\rangle_2 \left|k\right\rangle_3 \left\langle \alpha_{1A} \delta_{3B} \right|$$

The size grow bigger

We can iterate this process for 1D chain with n site,

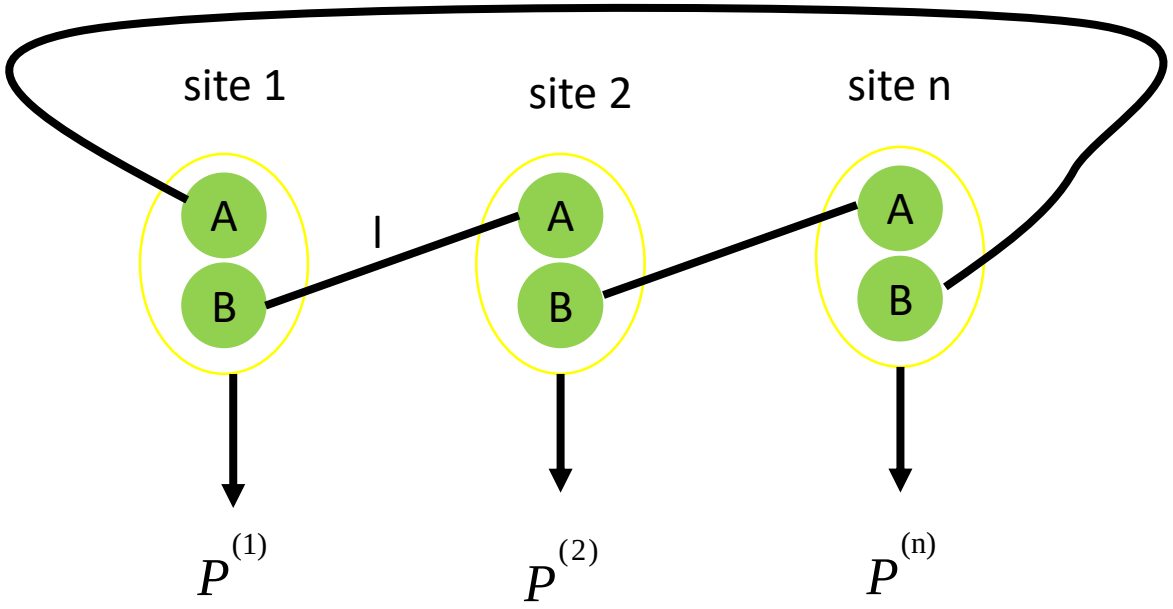


$$P^{(1)} \otimes P^{(2)} \otimes \ldots \otimes P^{(N)} \left|\omega\right\rangle_{1,2} \ldots \left|\omega\right\rangle_{N-1,N} = \sum_{\substack{i,j,\ldots,k=1,\ldots,d \\ \alpha,\delta=1,\ldots,D}} (A^{(1),i}\Sigma^{(1,2)}A^{(2),j}\ldots A^{(n),k})_{\alpha\delta} \boxed{\left|i\right\rangle_1 \left|j\right\rangle_2 \ldots \left|k\right\rangle_n} \left\langle \alpha_{1A}\delta_{nB}\right|$$

The basis of n sites:

$$\left|\psi\right\rangle = \sum_{\{m\}} \psi^{m_1 \ldots m_n} \left|m_1\right\rangle \otimes \ldots \otimes \left|m_n\right\rangle$$

PBC:



$$P^{(1)} \otimes P^{(2)} \otimes \ldots \otimes P^{(n)} \left|\omega\right\rangle_{1,2} \ldots \left|\omega\right\rangle_{n-1,n} \left|\omega\right\rangle_{n,1} = Tr(A^{(1),i}\Sigma^{(1,2)} A^{(2),j} \ldots A^{(n),k}\Sigma^{(n,1)}) \left|i\right\rangle_1 \left|j\right\rangle_2 \ldots \left|k\right\rangle_n$$

$$= Tr(A^{(1),i_1}\Sigma^{(1,2)} A^{(2),i_2} \ldots A^{(n),i_N}\Sigma^{(n,1)}) \left|i_1 i_2 \ldots i_n\right\rangle$$

The last line is used to make formula more compact.

If we absorb Σ into A:

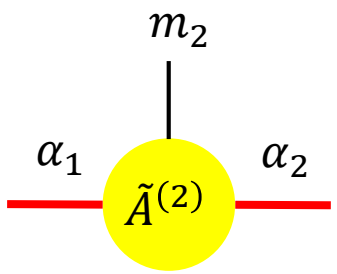
$$A^{(1),m_1} = A^{(1),i_1}\Sigma$$

We have the desired form

$$\psi^{m_1 \ldots m_n} = \sum_{\{i\}} tr(A^{(1),i_1} A^{(2),i_2} \ldots A^{(2),i_j} \ldots)$$

Introduce language of tensor network, represent a matrix with the following graph:

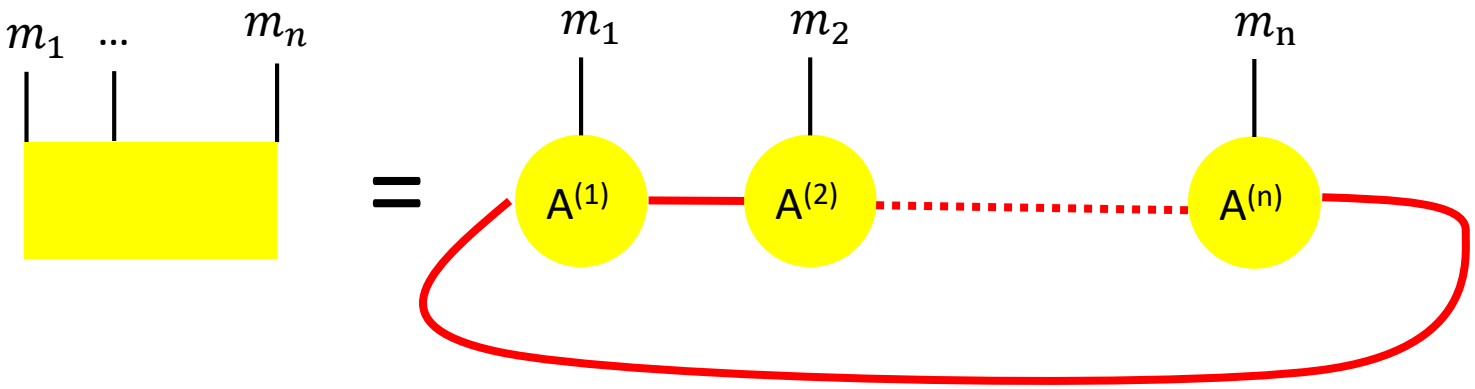
$\tilde{A}^{(2),m_2}_{\alpha_1\alpha_2} =$



Each line corresponds to an index of the tensor, where the top line corresponds to the physical Index: i=1,…,d and the horizontal lines-to the matrix indices α, which is called the bond dimension the bond dimension can not be the same to that of physical dimension.

For open boundary condition, we can express the above equation into the following way:

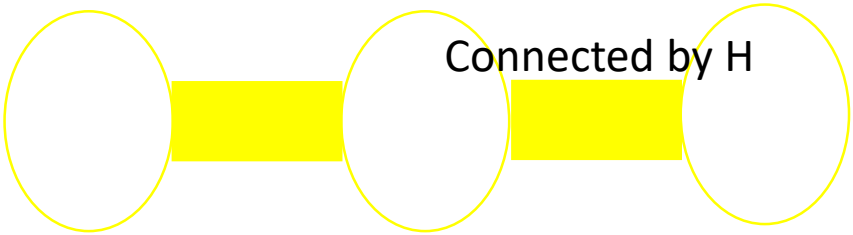
$$\psi^{m_1\ldots m_n} = \sum_{\{\mathrm{m}\}} tr(A^{(1),m_1} A^{(2),\mathrm{m}_2} \ldots A^{(2),\mathrm{m}_n})$$



Look back: what does matrix product state do?

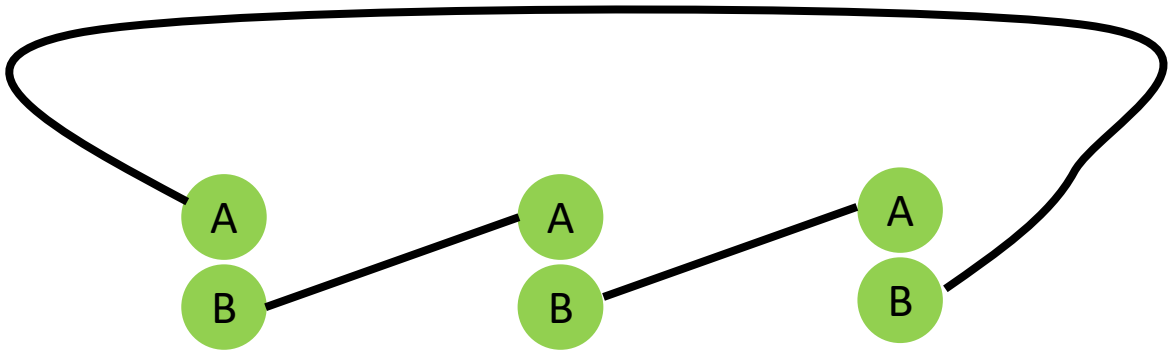
The answer is transforming the origin system to a auxiliary system:

State of
original system



Projection + Entanglement

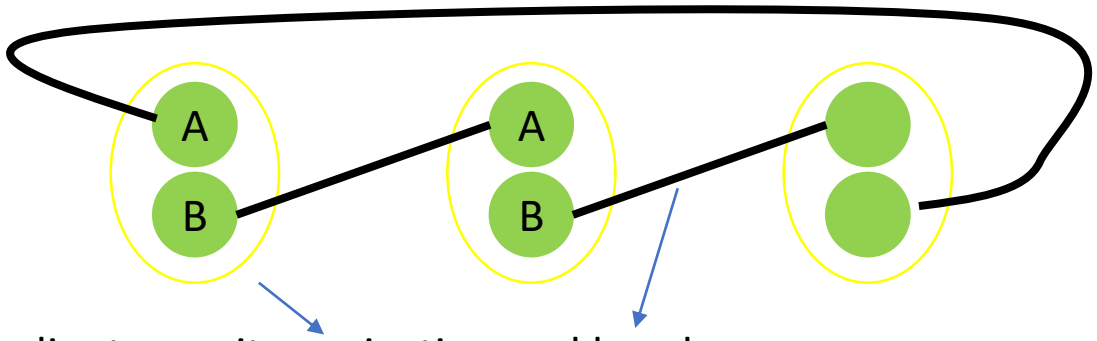
Matrix product
state



The entanglement takes place between the auxiliary particles.

Example: AKLT state with PBC

Since matrix product state is inspired by AKLT model, here we try to make the above process more clear for benchmark.



There are two ingredients: onsite projection and bond

First we treat onsite projection:

$$P^{(1)} = \sum_{i=1,.2,3,\alpha,\beta=1,2} A^{(1),i}_{\alpha\beta} |i\rangle\langle\alpha\beta|$$

$$= |1,1\rangle\langle\uparrow_{1A}\uparrow_{1B}| + |1,0\rangle \frac{\langle\uparrow_{1A}\downarrow_{1B}| + \langle\downarrow_{1A}\uparrow_{1B}|}{\sqrt{2}} + |1,-1\rangle\langle\downarrow_{1A}\downarrow_{1B}|$$

The only non-vanishing term is

$$A^{(1),1}_{\uparrow\uparrow} = 1, A^{(1),0}_{\uparrow\downarrow} = \frac{1}{\sqrt{2}}, A^{(1),0}_{\downarrow\uparrow} = -\frac{1}{\sqrt{2}}, A^{(1),-1}_{\downarrow\downarrow} = 1$$

We need three 2*2 matrix to store the information:

$$A^{(1),1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} |1A\uparrow\rangle \\ |1B\uparrow\rangle \\ |1B\downarrow\rangle \end{matrix} \begin{matrix} |1A\downarrow\rangle \end{matrix} \qquad A^{(1),0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^{(1),-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Second we treat the bond:

$$\left|\omega\right\rangle_{i,i+1} = \sum_{k,l=1,2} \Sigma_{kl}^{(i,i+1)} \left|k_{iB}; l_{i+1A}\right\rangle$$

Use the matrix to expand, we have

$$\Sigma^{(i,i+1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{matrix} |iB\uparrow\rangle \\ |i+1A\uparrow\rangle \\ |i+1A\downarrow\rangle \end{matrix} \begin{matrix} |iB\downarrow\rangle \end{matrix}$$

Obviously, $\Sigma_{i,i+1}$ must be clamped between $A^{(i),m1}$ and $A^{(i+1),m2}$ to ensure the consistency of basis.

So we have

$$P^{(1)} \otimes P^{(2)} \left|\omega\right\rangle_{1,2} = \sum_{\substack{i,j=1,\ldots,d \\ \alpha,\delta=1,\ldots,D}} (A^{(1),\mathrm{i}} \Sigma^{(1,2)} A^{(2),\mathrm{j}})_{\alpha\delta} \left|i\right\rangle_1 \left|j\right\rangle_2 \left\langle \alpha_{1A} \delta_{2B} \right|$$

If we have periodic boundary condition, the above formula goes like:

$$\psi^{m_1 \ldots m_n} = \sum_{\{\mathrm{m}_i = +1,0,-1\}} tr(A^{(1),\mathrm{m}_1} \Sigma^{(1,2)} A^{(2),\mathrm{m}_2} \Sigma^{(1,2)} \ldots A^{(2),\mathrm{m}_j} \ldots)$$

If we absorb Σ into A: $A^{(1),\mathrm{m}_1} = A^{(1),\mathrm{m}_1} \Sigma^{(1,2)}$

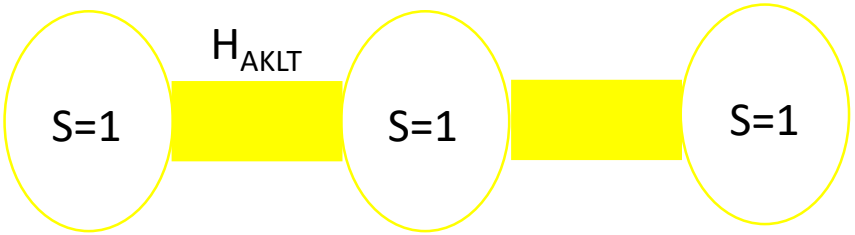
$$A^{(1),1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{matrix} |2A\uparrow\rangle \\ |2\mathrm{A}\downarrow\rangle \end{matrix} \qquad A^{(1),0} = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^{(1),-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$|1A\uparrow\rangle \quad |1\mathrm{A}\downarrow\rangle$$

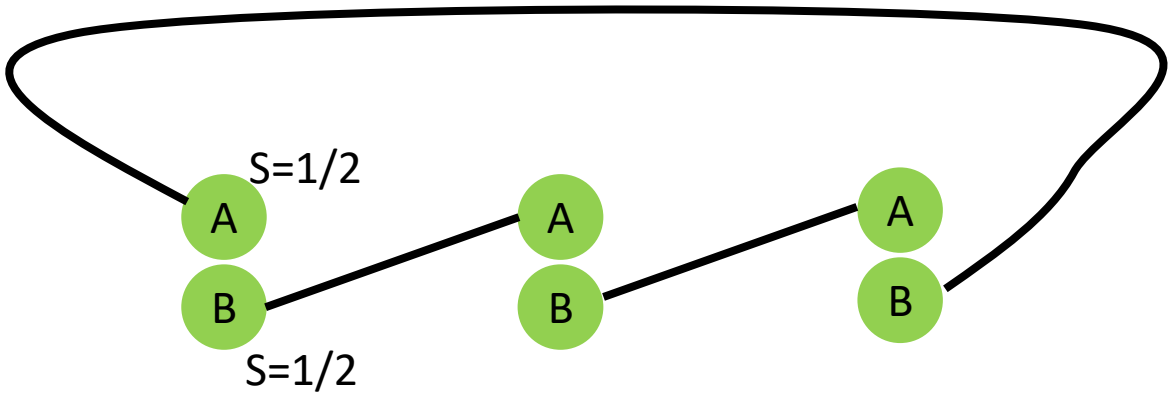
We have the desired form

$$\psi^{m_1 \ldots m_n} = \sum_{\{\mathrm{m}_i = +1,0,-1\}} tr(A^{(1),\mathrm{m}_1} A^{(2),\mathrm{m}_2} \ldots A^{(2),\mathrm{m}_n})$$

Goes back: what does AKLT do?



AKLT state



Example: Greenberger-Horne-Zeilinger state with PBC

$$\left|GHZ\right\rangle = \left|1.....1\right\rangle + \left|0.....0\right\rangle$$

Define projector:

$$P^{(1)} = \sum_{i=1,.2,\alpha,\beta=1,2} A_{\alpha\beta}^{(1),\mathrm{i}} \left|i\right\rangle\left\langle\alpha\beta\right|$$

Since the system has translational symmetry now, we have

$$A^{(\mathrm{i}),1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A^{(\mathrm{i}),0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

There is no entanglement between auxiliary particles

$$\sum^{(i,i+1)} = 1$$ 2*2 unitary matrix

$$\psi^{1\ldots1} = tr(A^{(i),1}) = tr(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\ldots) = 1$$

$$\psi^{0\ldots0} = tr(A^{(i),0}) = tr(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ldots) = 1$$

Other state vanishes

$$\psi^{0\ldots10\ldots0} = tr(A^{(1),0}\ldots A^{(i),1}A^{(i+1),0}\ldots A^{(n),0})$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ldots\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ldots\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= 0$$

This is just what we want.

Example: Greenberger-Horne-Zeilinger state with OBC

GHZ state which for N particles can be written as superposition of N zeros and N ones

$$\left|GHZ\right\rangle = \frac{1}{\sqrt{2}}\left(\underbrace{\left|1.....1\right\rangle}_{\text{n sites}} + \left|0.....0\right\rangle\right)$$

The physical d.o.f. is 2,

$$\psi^{m_1 \ldots m_n} = \sum_{\{m\}} \left(A^{(1),m_1} \Sigma^{(1,2)} A^{(2),m_2} \ldots A^{(n),m_n} \Sigma^{(n,1)} \right)_{\alpha\beta}$$

For given pair of (α, β), we can think the first matrix as row vector and and the last matrix as column vector

Since we have know the matrix for PBC, we only need to the row vector for the left end and column vector for the right end.

$$A^{(1),1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \end{pmatrix} \qquad A^{(1),0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$A^{(\mathrm{n}),1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad A^{(\mathrm{n}),0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The calculation gives

$$\psi^{1\ldots1} = A^{(1),1}A^{(\mathrm{i}),1}A^{(\mathrm{n}),1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\ldots\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\psi^{0\ldots0} = A^{(1),0}A^{(\mathrm{i}),0}A^{(\mathrm{n}),0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ldots\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\psi^{0\ldots10\ldots0} = A^{(1),0}\ldots A^{(\mathrm{i}),1}A^{(\mathrm{i}+1),0}\ldots A^{(\mathrm{n}),0} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ldots\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ldots\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

This is just what we want.

Bond and entanglement:
使用额外的自由度（bond d.o.f.）用来控制系统的纠缠状态，从而有效地将状态居于在整个Hilbert space的一个角落。而这就是物理状态可能的空间。

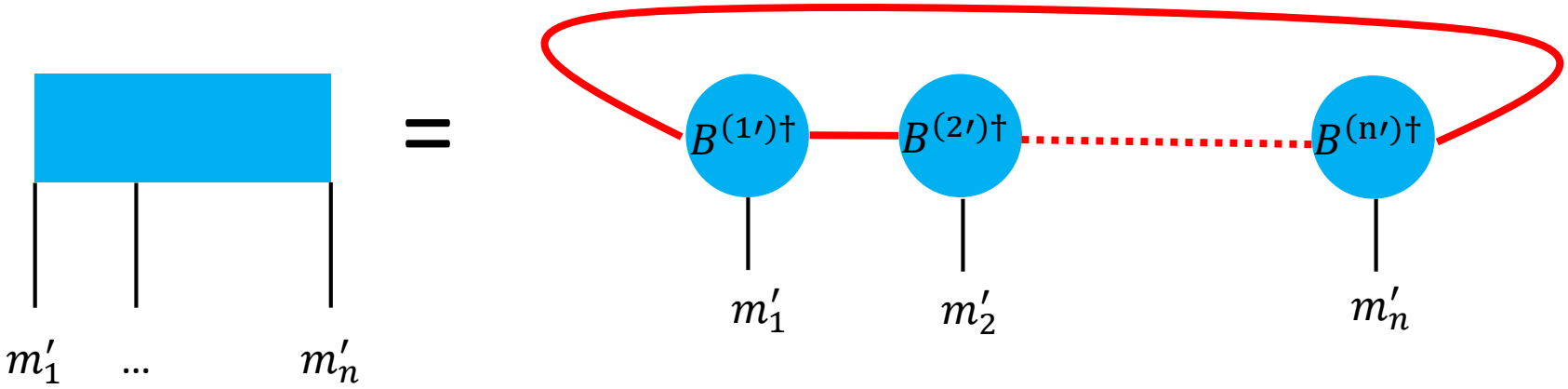
The meaning of bond.
Each leg means additional d.o.f., since each d.o.f. can be mapped to some physical quantum number, so leg can have actual physical meaning.
What if the leg is constructed by

More bond?

Now we can translate every thing into the language of matrix product state

bra

$$\langle\phi| = \sum_{\{m'\}} \mathrm{tr}\{(B^{(\mathrm{n}),m'_n})^\dagger \ldots (B^{(2),m'_2})^\dagger (B^{(1),m'_1})^\dagger\} \langle m'_1| \otimes \ldots \otimes \langle m'_n|$$

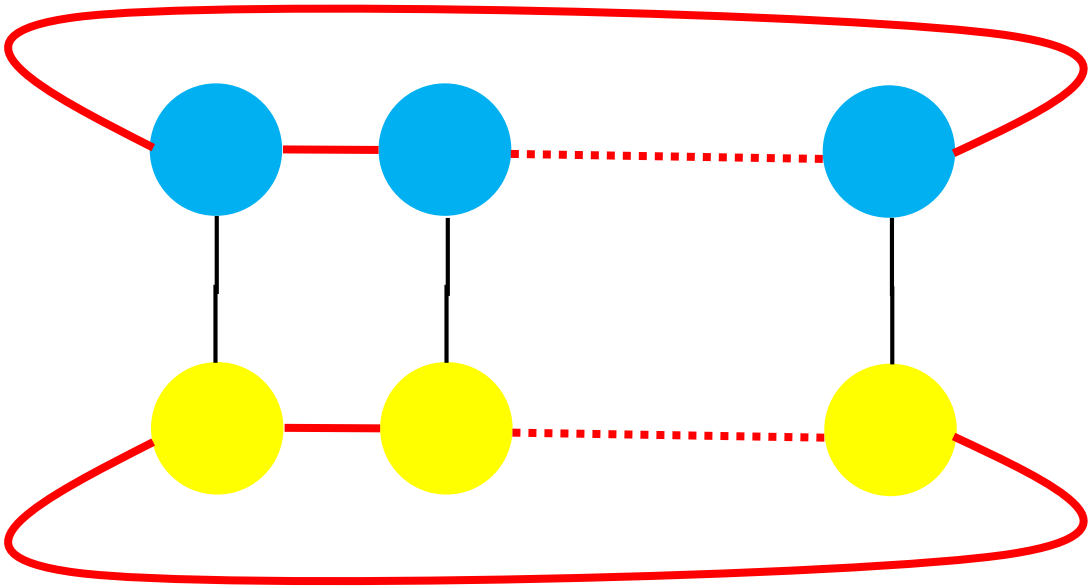


Overlap

$$\langle\phi|\psi\rangle = \sum_{\{m\}} \mathrm{tr}\{(\mathrm{B}^{(n),m_n'})^\dagger \ldots (\mathrm{B}^{(1),m_2'})^\dagger (\mathrm{B}^{(1),m_1'})^\dagger\}\, \mathrm{tr}\{A^{(1),m_1} A^{(2),m_2} \ldots A^{(n),m_n}\} \left\langle m_1' \right| \otimes \ldots \otimes \left\langle m_n' \right| \left| m_1 \right\rangle \otimes \ldots \otimes \left| m_n \right\rangle$$

$$= \sum_{\{m\}} \mathrm{tr}\{(\mathrm{B}^{(n),m_n})^\dagger \ldots (\mathrm{B}^{(1),m_2})^\dagger (\mathrm{B}^{(1),m_1})^\dagger\}\, \mathrm{tr}\{A^{(1),m_1} A^{(2),m_2} \ldots A^{(n),m_n}\}$$

=



remember that each connect leg means summation over one d.o.f.

Calculation of tr*tr

$$tr(\mathrm{A}) * \mathrm{tr}(\mathrm{B}) = \mathrm{tr}(\mathrm{A} \otimes \mathrm{B})$$

Proof:

$$tr(\mathrm{A}) * \mathrm{tr}(\mathrm{B}) = (\sum_i A_{ii}) * (\sum_j B_{jj}) = \sum_i (A_{ii} \sum_j B_{jj}) = \mathrm{tr}(\mathrm{A} \otimes \mathrm{B})$$

Therefore, we have

$$\langle \phi | \psi \rangle = \sum_{\{m\}} \mathrm{tr}\{[(\mathrm{B}^{(1),m_1})^\dagger \otimes A^{(1),m_1}][(\mathrm{B}^{(1),m_2})^\dagger \otimes A^{(2),m_2}]...[(\mathrm{B}^{(\mathrm{n}),m_n})^\dagger \otimes A^{(\mathrm{n}),m_n}]\}$$

For AKLT state, we have

$$A^{(1),1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{matrix} |1A\uparrow\rangle \\ |2A\downarrow\rangle \end{matrix} \qquad A^{(1),0} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^{(1),-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

We have the desired form

$$\psi^{m_1 \ldots m_n} = \sum_{\{m_i = +1,0,-1\}} tr(A^{(1),m_1} A^{(2),m_2} \ldots A^{(2),m_n})$$

Now we normalize the A matrix:

$$(A^{(1),1})^\dagger A^{(1),1} + (A^{(1),2})^\dagger A^{(1),2} + (A^{(1),3})^\dagger A^{(1),3}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4} I$$

The density-matrix renormalization group in the age of matrix product states

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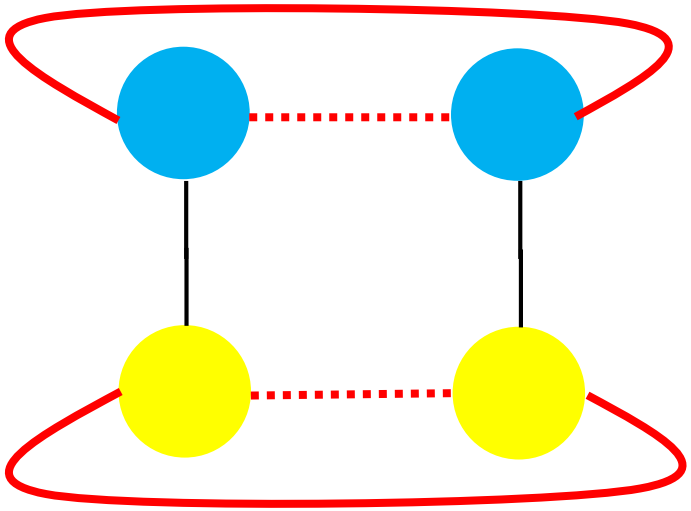
So we need to multiply by $\frac{2}{\sqrt{3}}$

$$A^{(1),1} = \sqrt{\frac{3}{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A^{(1),0} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad A^{(1),-1} = \sqrt{\frac{3}{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Then



=

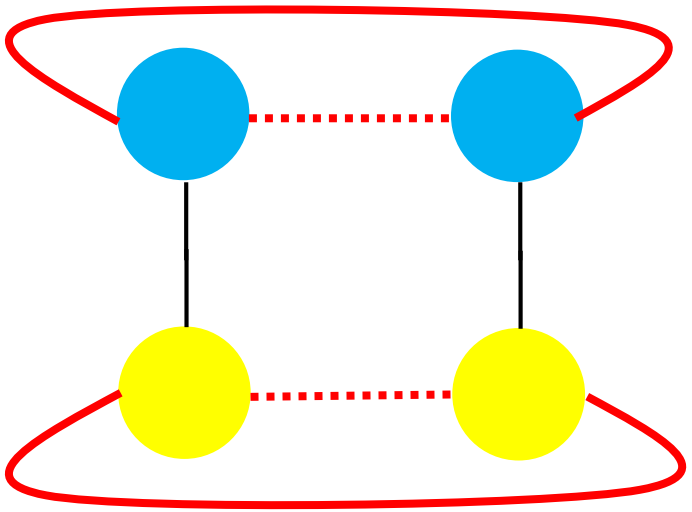
$$\sum_{\{m_i=+1,0,-1\}} tr\{[(A^{(1),m_1})^\dagger \otimes A^{(1),m_1}]...[(A^{(n),m_n})^\dagger \otimes A^{(n),m_n}]\}$$

Since

$$\sum_{\{m_i=+1,0,-1\}} [(A^{(1),m_1})^{\dagger} \otimes A^{(1),m_1}] = E = \begin{pmatrix} 1/4 & 0 & 0 & 1/2 \\ 0 & -1/4 & 0 & 0 \\ 0 & 0 & -1/4 & 0 \\ 1/2 & 0 & 0 & 1/4 \end{pmatrix}$$

And the eigenvalue of E is give by {1, -1/3, -1/3,-1/3}

Therefore,



$$= \quad 1+3(-\frac{1}{3})^n$$

To the thermodynamic limit, we have $\lim_{n\to\infty}(1+3(-\frac{1}{3})^n)=1$ just what we want!