Functional formalism

- A function that maps functions to numbers is called a functional;
- The argument of a functional $F[x(t)]$ is conventionally written in square brackets rather than parentheses;
- Just as an ordinary function $y(x)$ can be integrated over a set of points x, a functional $F[x(t)]$ can be integrated over a set of functions $x(t)$; the measure of such a functional integral is conventionally written with a script capital D;
- A functional can also be differentiated with respect to its argument (a function), and this functional derivative is denoted by $\delta F[x(t)]/\delta x(t)$;

Idea from classical mechanics

Lagrange equation of motion in classical mechanics

$$S = \int L(\mathrm{q}, \frac{dq}{dt})dt$$

Where the Lagrangian L depends on the general coordinate q and its velocity dq/dt. At the time of deriving equation of motion by the variational principle, q and dq/dt are independent as the function of t. Therefore, L is functional of q(t) and dq/dt.

Least action principle

$$\delta S = \int \delta L(\mathrm{q}, \frac{dq}{dt})dt = \int [\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}]dt = 0$$

$$0 = \int \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}\delta q)dt \Rightarrow \int \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\, dt = -\int \delta q \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}dt$$

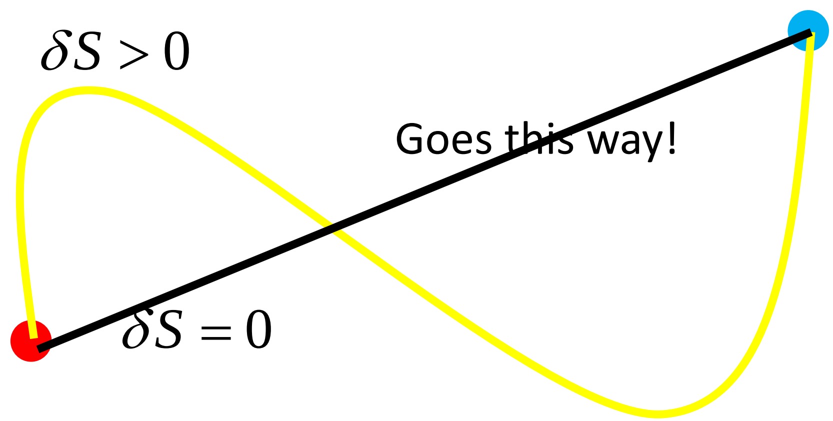
$$\delta S = \int \delta L(\mathrm{q}, \frac{dq}{dt})dt = \int [\frac{\partial L}{\partial q}\delta q - \delta q \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}]dt = 0 \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

In summary, least action principle exert a constraint, which will give an equation of motion:

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

两个时间求导在一起!
所以得到的方程是关于时间的二次导数

If we picture the above mechanism, we have:



Example：

$$L = -q^2 + \dot{q}^2$$

Harmonic oscillator

$$q + \ddot{q} = 0$$

Sin or cos oscillation

Time translational symmetry and Hamiltonian

The Lagrangian must be invariant under the infinitesimal time translation:

$$\mathrm{q(t)} \rightarrow \mathrm{q(t+}\,\varepsilon), \dot{q}(\mathrm{t}) \rightarrow \dot{q}(\mathrm{t+}\,\varepsilon)$$

$$L(\mathrm{q(t+}\,\varepsilon), \dot{q}(\mathrm{t+}\,\varepsilon)) - L(\mathrm{q(t)}, \dot{q}(\mathrm{t})) = 0$$

Since

$$\mathrm{q(t+}\,\varepsilon) = \underline{\mathrm{q(t)} + \dot{q}(\mathrm{t})\varepsilon}, \dot{q}(\mathrm{t+}\,\varepsilon) = \dot{q}(\mathrm{t}) + \dot{q}\,\dot{\varepsilon} + \ddot{q}\,\varepsilon$$

The constriction becomes:

$$L(\mathrm{q(t+}\,\varepsilon), \dot{q}(\mathrm{t+}\,\varepsilon)) = L(\mathrm{q(t)}, \dot{q}(\mathrm{t})) + \frac{\partial L}{\partial q}\dot{q}(\mathrm{t})\varepsilon + \frac{\partial L}{\partial \dot{q}}(\dot{q}\,\dot{\varepsilon} + \ddot{q}\,\varepsilon)$$

And the conservation equation becomes

$$0 = \frac{\partial L}{\partial q}\dot{q}(\mathrm{t})\varepsilon + \frac{\partial L}{\partial \dot{q}}(\dot{q}\,\dot{\varepsilon} + \ddot{q}\,\varepsilon)$$

$$\Rightarrow 0 = \frac{\partial L}{\partial q}\dot{q}(\mathrm{t})\varepsilon + \frac{\partial L}{\partial \dot{q}}\ddot{q}\,\varepsilon + \frac{\partial L}{\partial \dot{q}}\dot{q}\,\dot{\varepsilon} = \frac{\partial L}{\partial q}\dot{q}(\mathrm{t})\varepsilon + \frac{\partial L}{\partial \dot{q}}\ddot{q}\,\varepsilon - \varepsilon\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\dot{q}$$

$$\Rightarrow 0 = \frac{d}{dt}(L - \frac{\partial L}{\partial \dot{q}}\dot{q})$$

Which corresponds to the conservation of Hamiltonian:

$$H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L$$

The sign here is choose from physical meaning.

Example：

$$L = -q^2 + \dot{q}^2$$

Harmonic oscillator

$$H = q^2 + \dot{q}^2$$

Sum of kinetic and potential energy

Lagrange equation for fields

$$S = \int L(\psi, \partial_\mu \psi)\, \mathrm{d}\vec{x} dt$$

Least action principle

$$\frac{\partial L}{\partial \psi} = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) \quad \text{EL-equation}$$

The time and spatial derivative has the same contribution:

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\psi}} + \frac{\partial}{\partial x} \frac{\partial L}{\partial(\partial_x \psi)} + \frac{\partial}{\partial y} \frac{\partial L}{\partial(\partial_y \psi)} + \frac{\partial}{\partial z} \frac{\partial L}{\partial(\partial_z \psi)}$$

两个时间，空间求导在一起!
所以得到的方程是关于时间和空间的二次导数

Global gauge transformation and four-component current

$$\psi(x) \to e^{i\alpha}\psi(x) = \psi(x) + i\alpha\psi(x)$$

$$L(\mathrm{e}^{i\alpha}\,\psi(\mathrm{x}), \partial_\mu\,\mathrm{e}^{i\alpha}\,\psi(\mathrm{x})) - L(\psi(\mathrm{x}), \partial_\mu\psi(\mathrm{x})) = 0$$

The constriction:

$$L(\psi(\mathrm{x}) + \mathrm{i}\,\alpha\psi(\mathrm{x}), \partial_\mu(\psi(\mathrm{x}) + \mathrm{i}\,\alpha\psi(\mathrm{x}))) = L(\psi(\mathrm{x}), \partial_\mu\psi(\mathrm{x})) + \frac{\partial L}{\partial\psi(\mathrm{x})} i\alpha\psi(\mathrm{x}) + \frac{\partial L}{\partial(\partial_\mu\psi(\mathrm{x}))}\partial_\mu(i\alpha\psi(\mathrm{x}))$$

And the conservation equation becomes

$$0 = \frac{\partial L}{\partial\psi(\mathrm{x})} i\alpha\psi(\mathrm{x}) + \frac{\partial L}{\partial(\partial_\mu\psi(\mathrm{x}))}\partial_\mu(i\alpha\psi(\mathrm{x}))$$

$$\Rightarrow 0 = \underline{\frac{\partial L}{\partial\psi(\mathrm{x})} i\alpha\psi(\mathrm{x})} + \partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\psi(\mathrm{x}))} i\alpha\psi(\mathrm{x})\right] - \underline{i\alpha\psi(\mathrm{x})\partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\psi(\mathrm{x}))}\right]}$$

$$\Rightarrow 0 = \partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\psi(\mathrm{x}))} i\alpha\psi(\mathrm{x})\right]$$

EL-equation

Four-component current is conserved:

$$j^\mu = -i\frac{\partial L}{\partial(\partial_\mu \psi(\mathrm{x}))}\psi(\mathrm{x})$$

$$\partial_\mu j^\mu = 0$$

Integral over whole space give the conserved charge:

$$\int \frac{\partial j^0}{\partial t} d\vec{x} + \int (\underline{\vec{\nabla}\vec{J}}) d\vec{x} = 0$$

Surface integral vanishes

$$\Rightarrow \frac{\partial j^0}{\partial t} = 0$$

$$\Rightarrow j^0 = C$$

Time-space translational symmetry and energy-momentum tensor

$$\psi(x)\to\psi(x+t),\partial_\mu\psi(x)\to\partial_\mu\psi(x+t)$$

$$L(\psi(\mathrm{x+t}),\partial_\mu\psi(\mathrm{x+t}))-L(\psi(\mathrm{x}),\partial_\mu\psi(\mathrm{x}))=0$$

Here we need a little calculation since x and t here are four-component:

$$\psi(x+t)=\psi+(\partial_u\psi)t^\mu,\partial_\mu\psi(x+t)=\partial_\mu\psi+(\partial_\mu\partial_\nu\psi)t^\nu+\partial_\nu\psi\partial_\mu t^\nu$$

$$\frac{\partial L}{\partial\psi}(\partial_\nu\psi)t^\nu+\frac{\partial L}{\partial(\partial_\mu\psi)}[(\partial_\mu\partial_\nu\psi)t^\nu+(\partial_\nu\psi)(\partial_\mu t^\nu)]=0$$

$$\frac{\partial L}{\partial\psi}(\partial_\nu\psi)t^\nu+\frac{\partial L}{\partial(\partial_\mu\psi)}(\partial_\mu\partial_\nu\psi)t^\nu+\frac{\partial L}{\partial(\partial_\mu\psi)}(\partial_\nu\psi)(\partial_\mu t^\nu)=0$$

$$\frac{\partial L}{\partial\psi}(\partial_\nu\psi)t^\nu+\frac{\partial L}{\partial(\partial_\mu\psi)}(\partial_\mu\partial_\nu\psi)t^\nu+\underline{\partial_\mu[\frac{\partial L}{\partial(\partial_\mu\psi)}(\partial_\nu\psi)t^\nu]}-\partial_\mu[\frac{\partial L}{\partial(\partial_\mu\psi)}(\partial_\nu\psi)]t^\nu=0$$

Surface integral

$$\{\underline{\frac{\partial L}{\partial\psi}(\partial_\nu\psi)+\frac{\partial L}{\partial(\partial_\mu\psi)}(\partial_\mu\partial_\nu\psi)}-\partial_\mu[\frac{\partial L}{\partial(\partial_\mu\psi)}(\partial_\nu\psi)]\}t^\nu=0$$

From symmetry and its breaking in quantum field theory, by Takehisa Fujita

Evaluate the first two terms:

$$\delta L = \frac{\partial L}{\partial x^{\nu}} t^{\nu} = t^{\nu} \delta_{\nu}^{\mu} \partial_{\mu} L$$

$$\Rightarrow \partial_{\mu} L = \frac{\partial L}{\partial \psi} (\partial_{\mu} \psi) + \frac{\partial L}{\partial(\partial_{\nu} \psi)} (\partial_{\mu} \partial_{\nu} \psi)$$

The constraint becomes:

$$\{\partial_{\nu} L - \partial_{\mu} [\frac{\partial L}{\partial(\partial_{\mu} \psi)} (\partial_{\nu} \psi)]\} t^{\nu} = 0$$

Finally we have conserved energy-momentum tensor:

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_{\mu} \psi)} (\partial^{\nu} \psi) - \mathrm{g}^{\mu\nu} L$$

$$\Rightarrow \partial_{\mu} T^{\mu\nu} = 0$$

Integral over whole space gives the conserved energy:

$$T^{00} = \frac{\partial L}{\partial \dot{\psi}}\dot{\psi} - L = H$$

Which is just the Hamiltonian density.

Discrete symmetry

Specific symmetry and conserved quantity for a given field

In the above, we just mention the general symmetry a Lagrangian has. For a specific Lagrangian for a given field, some unequal symmetry exists. For example, the free Dirac field occupies the chiral symmetry. Here we list the Lagrangian for different fields:

	Lagrangian	Specific symmetry
Free boson field		
KG field		
Gauge field		
Free Dirac field		
QED field		
Chern-Simons field		

If a field theory satisfy Lorentz transformation, what’s the feature of such theory?

If a field theory satisfy gauge invariance, what’s the feature of such theory?

If a field theory is defined on a manifold rather than flat time-space, what’s the feature of such theory?

Classical topological field theory

Now quantized these theory

topological quantum field theory

Functional quantum mechanics

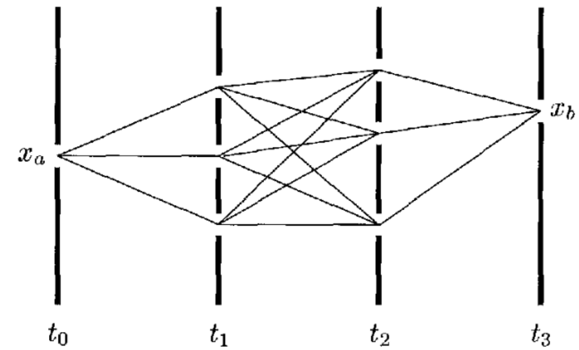


FIG. 2.1. The total amplitude is the sum of all amplitudes associated with the paths connecting x_a and x_b.

$$iG(\mathrm{x}_b, \mathrm{t}_b; \mathrm{x}_a, \mathrm{t}_a) = \langle x_b | \exp(-\mathrm{i}\, H(\mathrm{t}_b - \mathrm{t}_a)) | x_a \rangle$$

$$= \int dx_1 ... dx_{N-1} \prod_{j=1}^{N} iG(\mathrm{x}_j, \mathrm{t}_j; \mathrm{x}_{j-1}, \mathrm{t}_{j-1})$$

$$= A^N \int \prod_i dx_i \exp[\mathrm{i} \sum \Delta t L(\mathrm{t}_j, \frac{x_j + x_{j-1}}{2}, \frac{x_j - x_{j-1}}{\Delta t})]$$

Where $(t_0, x_0)=(t_a, x_a)$, $(t_N, x_N)=(t_b, x_b)$

Note that this is infinite dimensional integral, which is very hard to define accurately from mathematical view, which is an important thing for tensor network.

We always use the following label for simplicity:

$$A^N \prod_i dx_i = D[x(\mathrm{t})]$$

We can see this as a contracted form.

Therefore, we have:

$$iG(\mathrm{x}_b, \mathrm{t}_b; \mathrm{x}_a, \mathrm{t}_a) = \int D[\mathrm{x}(\mathrm{t})]\exp[\mathrm{i}\, S_M[\mathrm{x}(\mathrm{t})]]$$

$$S_M[\mathrm{x}(\mathrm{t})] = \int dtL(\mathrm{t}, \mathrm{x}, \dot{x})$$

Here M means Minkowski.

Note: the above picture is common known as path integral in quantum mechanics, but it is actually a kind of functional, therefore, in parallel with what's going on, we'd like to call it functional quantum mechanics here. Therefore, there are actually three kinds of views on quantum mechanics:

Heisenberg's view (matrix point of view)
Schrodinger's view (function point of view)
Feynman's view (functional point of view)

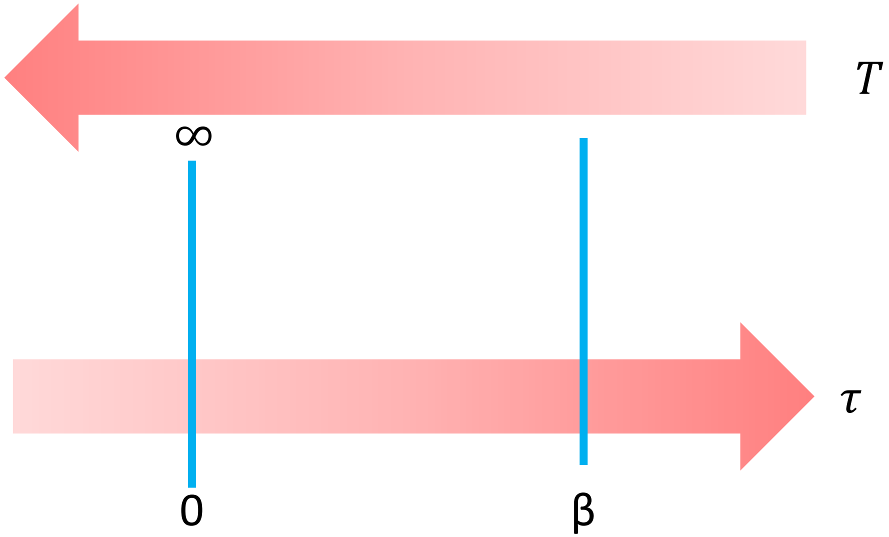
Duality between quantum mechanics and statistical mechanics

Partition function of one-particle system is given by:

$$Z(\beta) = Tr(\exp(-\beta \mathrm{H}))$$

Treat β as a variable τ, and τ has start point o and ends at β:

$$\tau = \frac{1}{k_B T}\begin{cases} = 0, T \to \infty \\ = \beta, T = \dfrac{1}{k_B \tau} \end{cases}$$



Compare the two systems:

quantum mechanics

statistical mechanics

$$U(t_b - t_a) = \exp(-i\,H(t_b - t_a))$$

$$Z(\beta) = Tr(\exp(-H(\beta - 0)))$$

$$t = -i\tau$$

$$\tau$$

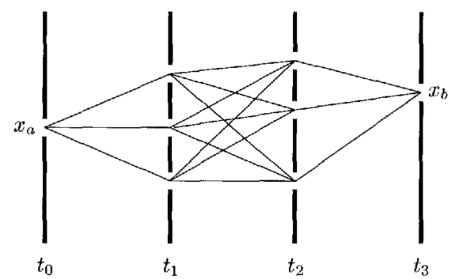
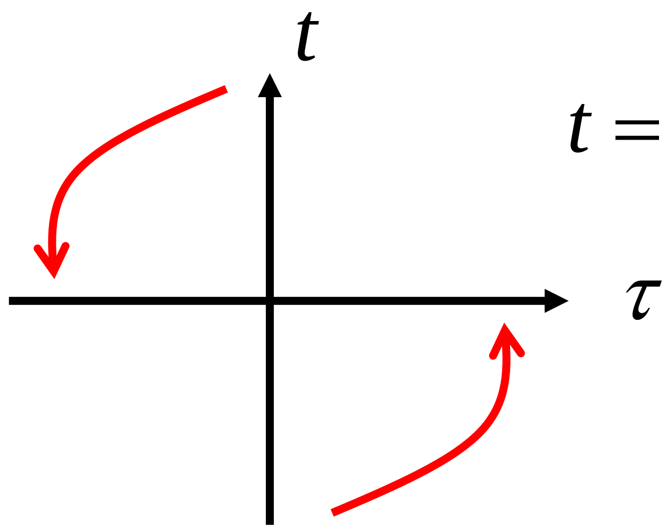


FIG. 2.1. The total amplitude is the sum of all amplitudes associated with the paths connecting x_a and x_b.

We apply functional method of quantum mechanics to treat problem in statistical mechanics.

Wick rotation



$$t = -i\tau$$

The metric between two spaces goes like:

$$(ds^2)_M = t^2 - \vec{r}^{\,2} \rightarrow (-\mathrm{i}\,\tau)^2 - \vec{r}^{\,2} = -(\tau^2 + \vec{r}^{\,2}) = -(ds^2)_E$$

The action between two spaces goes like:

$$iS_M[\mathrm{x(t)}] \rightarrow -S_E[\mathrm{x}(\tau)]$$

$$Z = \int D[\mathrm{x(t)}]\exp\{\mathrm{i}\,S_M[\mathrm{x(t)}]\} \rightarrow Z = \int D[\mathrm{x(t)}]\exp\{-S_E[\mathrm{x(t)}]\}$$

From one d.o.f. to infinite d.o.f.

The representation of quantum mechanics in functional formalism can be translated to field theory, with the following analogy. Here we concern a real scalar field ϕ(x).

$$i \leftrightarrow \vec{\mathrm{x}}$$

$$x_i(\mathrm{t}) \leftrightarrow \phi(\vec{\mathrm{x}}, \mathrm{t})$$

$$\prod_i dx_i(\mathrm{t}) \leftrightarrow \prod_{\vec{\mathrm{x}},t} d\phi(\vec{\mathrm{x}}, \mathrm{t})$$

$$S = \int dtL \leftrightarrow S = \int dtd\vec{\mathrm{x}}L$$

And the propagator now is:

$$\langle \phi_b(\mathrm{x}) | \exp(-i\mathrm{H\,T}) | \phi_a(\mathrm{x}) \rangle = \int D[\phi] \exp[\mathrm{i} \int_0^T dxL(\phi, \partial_\mu \phi)]$$

The integration measure Dϕ again involves an awkward constant, which we will not write explicitly.

Statistical field theory

Peskin, 9.3

Gauge field theory

From chapter 15 of An introduction to quantum field theory by Peskin, M. E.; Schroeder, D. V.

Here is the location of gauge quantum field in QFT:

In the case of classical field theories, $\phi(x)$ usually is an element of a finite dimensional real or complex manifold, which in many cases is a linear space. Prominent examples are:

- the real scalar field $\phi(x) \in R$
- the complex scalar field $\phi(x) \in C$
- the n-vector field $\vec{\phi}(x) \in R^n$
- the photon field $A_\mu(x) \in R$, where μ=0, 1, 2, 3 is a Lorentz index
- the Yang-Mills field $A_\mu(x) = \sum_b A_\mu^b(x) T_b$, whose components are elements of the Lie algebra of a compact Lie group with generators T_b

Abelian gauge quantum field:

Consider a system which is invariant under **local** gauge transformation (phase rotation):

$$\varphi(\mathrm{x}) \to \exp(\mathrm{i}\,\alpha(\mathrm{x}))\varphi(\mathrm{x})$$

Now we need to construct terms which are invariant under gauge transformation.

Obviously, the mass term is gauge invariant:

$$m\bar{\varphi}\varphi \to m\bar{\varphi}\varphi$$

The derivative is not gauge invariant:

$$\partial_\mu \varphi(\mathrm{x}) \not\to \exp(\mathrm{i}\,\alpha(\mathrm{x}))\partial_\mu \varphi(\mathrm{x})$$

This is easily seen by an explicit expansion: local phase can’t cross derivative

$$n^\mu \partial_\mu \varphi(\mathrm{x}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\varphi(\mathrm{x}+\varepsilon n) - \varphi(\mathrm{x})]$$

Since

$$\varphi(\mathrm{x}+\varepsilon \mathrm{n}) \to \exp(\mathrm{i}\,\alpha(\mathrm{x}+\varepsilon \mathrm{n}))\varphi(\mathrm{x}+\varepsilon \mathrm{n})$$

$$\varphi(\mathrm{x}) \to \exp(\mathrm{i}\,\alpha(\mathrm{x}))\varphi(\mathrm{x})$$

The two point follows different gauge transformation, that’s what’s the “**local**” actual means.

Now we add something to make the derivative gauge invariant:

We introduce a comparator which transfers point x to point $x + \varepsilon\hat{n}$, so we can do the minus at the same point:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\varphi(\mathrm{x}+\varepsilon n) - U(\mathrm{x}+\varepsilon n, x)\varphi(\mathrm{x})]$$

And the only requirement is that the comparator transforms like follows:

$$U(y,x) \to \exp(\mathrm{i}\,\alpha(\mathrm{y}))U(y,x)\exp(-\mathrm{i}\,\alpha(\mathrm{x}))$$

It's obvious that

- U(y, x) is a structure of the manifold, since it depends on only the two points on the manifold;
- U(y, x) form a Lie group, here since α(x) is just function, so the group is U(1), i.e. QED;
- The generator of the group lives the gauge boson, photon for U(1);
- As we shall see below, when the group is non-abelian, like SU(2) for Yang-Mills theory where , and SU(3) for .

Now we study the representation of U

Consider infinitesimal translation

$$U(\mathrm{x}+\varepsilon n, x) = \exp[-\mathrm{ie}\,\varepsilon n^{\mu} A_{\mu}(\mathrm{x}+\frac{\varepsilon}{2} n)]$$

Expansion gives

$$U(\mathrm{x}+\varepsilon n, x) = 1 - ie\varepsilon n^{\mu} A_{\mu}(\mathrm{x}) + \mathrm{O}(\varepsilon^2)$$

where e is the coupling constant, $A_{\mu}(x)$ is the generator field.

Now we back to the special derivative, we give it a name covariant derivative D:

$$n^{\mu} D_{\mu}\varphi(\mathrm{x}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\varphi(\mathrm{x}+\varepsilon n) - U(\mathrm{x}+\varepsilon n, x)\varphi(\mathrm{x})]$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[\varphi(\mathrm{x}+\varepsilon n) - \varphi(\mathrm{x}) + ig\varepsilon n^{\mu} A_{\mu}(\mathrm{x})\varphi(\mathrm{x})]$$

$$= n^{\mu}[\partial_{\mu} + igA_{\mu}(\mathrm{x})]\varphi(\mathrm{x})$$

Thus we have

$$D_{\mu} = \partial_{\mu} + ieA_{\mu}(\mathrm{x})$$

Such a relationship is quite uncommon, since two differential gives a field!

When field goes a local phase rotation, the generator field must also experience a change:

$$U(\mathrm{x}+\varepsilon n,x)\rightarrow \exp[\mathrm{i}\,\alpha(\mathrm{x}+\varepsilon n)]U(\mathrm{x}+\varepsilon n,x)\exp[-\mathrm{i}\,\alpha(\mathrm{x})]$$

$$1-ie\varepsilon n^{\mu}A_{\mu}(\mathrm{x})\rightarrow(1+\mathrm{i}\,\alpha(\mathrm{x}+\varepsilon n))(1-ie\varepsilon n^{\mu}A_{\mu}(\mathrm{x}))(1-\mathrm{i}\,\alpha(\mathrm{x}))$$

$$1-ie\varepsilon n^{\mu}A_{\mu}(\mathrm{x})\rightarrow 1-ie\varepsilon n^{\mu}A_{\mu}(\mathrm{x})+\mathrm{i}\,\alpha(\mathrm{x}+\varepsilon n)-\mathrm{i}\,\alpha(\mathrm{x})$$

$$1-ie\varepsilon n^{\mu}A_{\mu}(\mathrm{x})\rightarrow 1-ie\varepsilon n^{\mu}A_{\mu}(\mathrm{x})+i\varepsilon n^{\mu}\partial_{\mu}\alpha(\mathrm{x})$$

$$A_{\mu}(\mathrm{x})\rightarrow A_{\mu}(\mathrm{x})-\frac{1}{e}\partial_{\mu}\alpha(\mathrm{x})$$

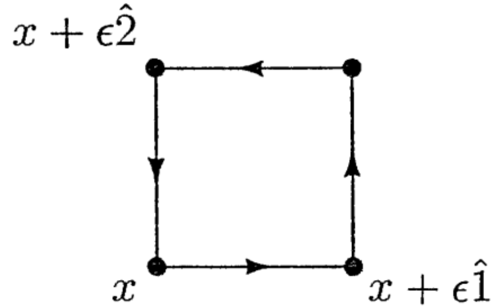
So we can show that the covariant derivative actually goes like the way we want, that’s why we give it the name “covariant”:

$$D_{\mu}\varphi(\mathrm{x})=[\partial_{\mu}+ieA_{\mu}(\mathrm{x})]\varphi(\mathrm{x})\rightarrow[\partial_{\mu}+ie\underbrace{(A_{\mu}(\mathrm{x})-\frac{1}{e}\alpha(\mathrm{x}))}_{\text{how generator changes}}]\underbrace{\exp(\mathrm{i}\,\alpha(\mathrm{x}))\varphi(\mathrm{x})}_{\text{how field changes}}$$

It’s obviously we have:

$$D_{\mu}\varphi(\mathrm{x})\rightarrow\exp(\mathrm{i}\,\alpha(\mathrm{x}))D_{\mu}\varphi(\mathrm{x})$$

Now we focus our new structure U:



Wilson's loop

We consider the above loop, at first sight, it seems that we do nothing, maybe product what we got by identity:

$$U_\lambda = U(\mathrm{x}, x+\varepsilon\hat{2})U(\mathrm{x}+\varepsilon\hat{2}, x+\varepsilon\hat{1}+\varepsilon\hat{2})U(\mathrm{x}+\varepsilon\hat{1}+\varepsilon\hat{2}, x+\varepsilon\hat{1})U(\mathrm{x}+\varepsilon\hat{1}, x)$$

Obviously, the loop product U(λ) is not gauge invariant

$$U_\lambda \to \exp[\mathrm{i}\,\alpha(\mathrm{x})]U_\lambda \exp[-\mathrm{i}\,\alpha(\mathrm{x})]$$

But the trace is gauge invariant:

$$W_\lambda \to W_\lambda, W_\lambda = tr(\mathrm{U}_\lambda)$$

Infinitesimal expansion gives:

$$U_\lambda = \lim_{\varepsilon\to 0} U(\mathrm{x}, x+\varepsilon\hat{2})U(\mathrm{x}+\varepsilon\hat{2}, x+\varepsilon\hat{1}+\varepsilon\hat{2})U(\mathrm{x}+\varepsilon\hat{1}+\varepsilon\hat{2}, x+\varepsilon\hat{1})U(\mathrm{x}+\varepsilon\hat{1}, x)$$

$$= 1 - \mathrm{i}\,\varepsilon^2\, \mathrm{e}[\partial_1 \mathrm{A}_2(\mathrm{x}) - \partial_2 \mathrm{A}_1(\mathrm{x})]$$

So we obtain the following gauge invariant quantity:

$$W_\lambda = \frac{1}{4}(F_{\mu\nu})^2, F_{\mu\nu} = \partial_\mu \mathrm{A}_\nu(\mathrm{x}) - \partial_\nu \mathrm{A}_\mu(\mathrm{x})$$

A world of two fields:

Here we take another point of view, which may be easy to adopt for CMP.
The generator is the gauge field and field is the matter field, i.e. the world is comprised by mater field and gauge field. The matter field is external and the gauge field is internal.

The QED Lagrangian is

$$\mathrm{L}_{QED} = \bar{\varphi}(\mathrm{i}\not{D})\varphi - \frac{1}{4}(F_{\mu\nu})^2 - m\bar{\varphi}\varphi$$

kinetic term of matter field + interaction between matter field and gauge field

kinetic term of gauge field

mass term of matter field

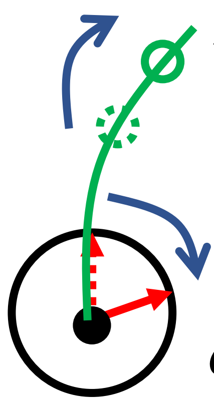
Now we try to picture what we get:

At each point, there is a matter field and gauge field. The gauge field determine how the matter field is mixed, so we can think matter field and gauge field as a embedded space.

In the following, we will use a black dot to represent space-time point:

● space-time point

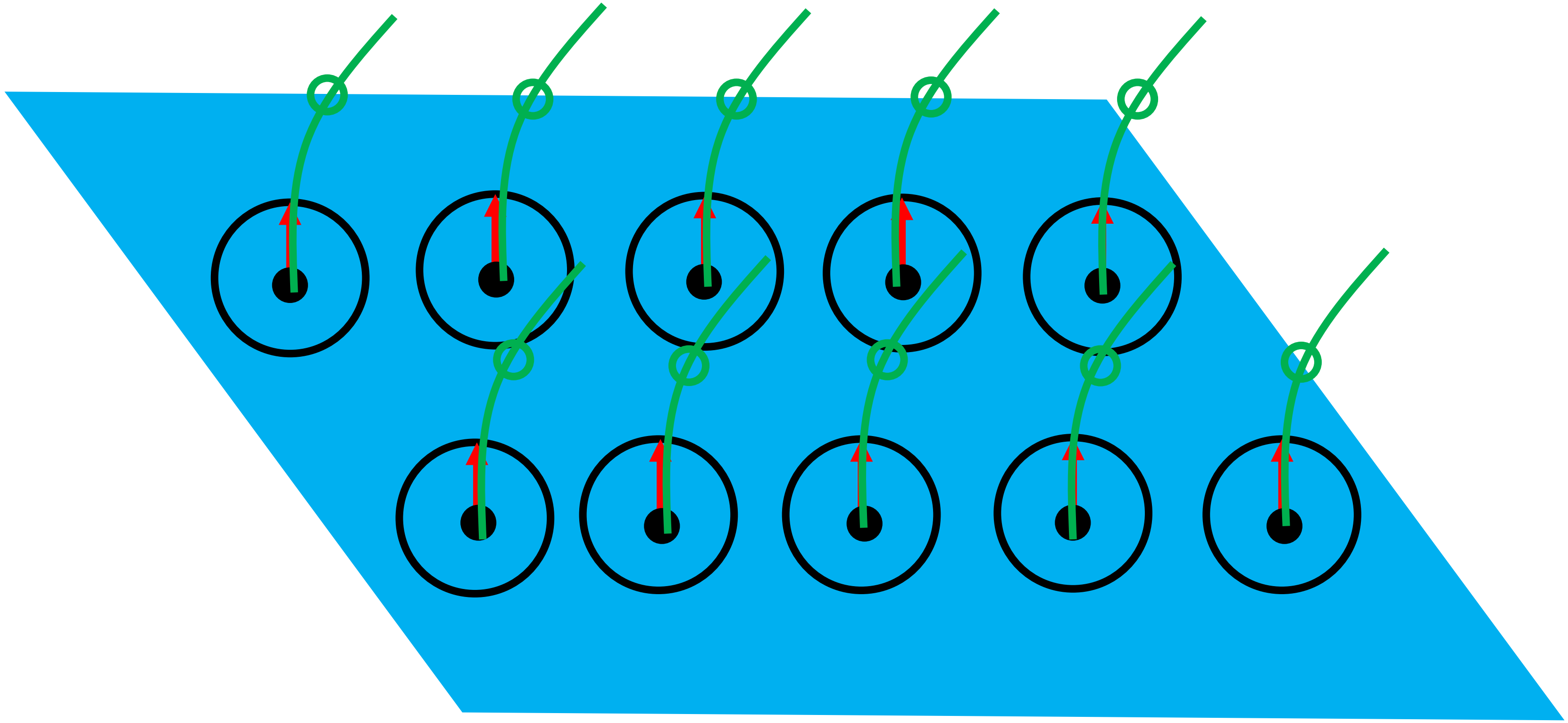
For U(1), the manifold can be represented by a circle, so we can use a circle to represent state of particle, and a red vector to show the relative phase;
The gauge field is represented by a fiber starting from a space-time point and a green circle stuck to the fiber silde up and down; Note that the magnitude of slide is determined by the e;
Then the gauge transformation is just the linkage between the rotation of red vector and the move of green circle:



$$A_\mu(\mathrm{x}) \to A_\mu(\mathrm{x}) - \frac{1}{e}\partial_\mu \alpha(\mathrm{x})$$

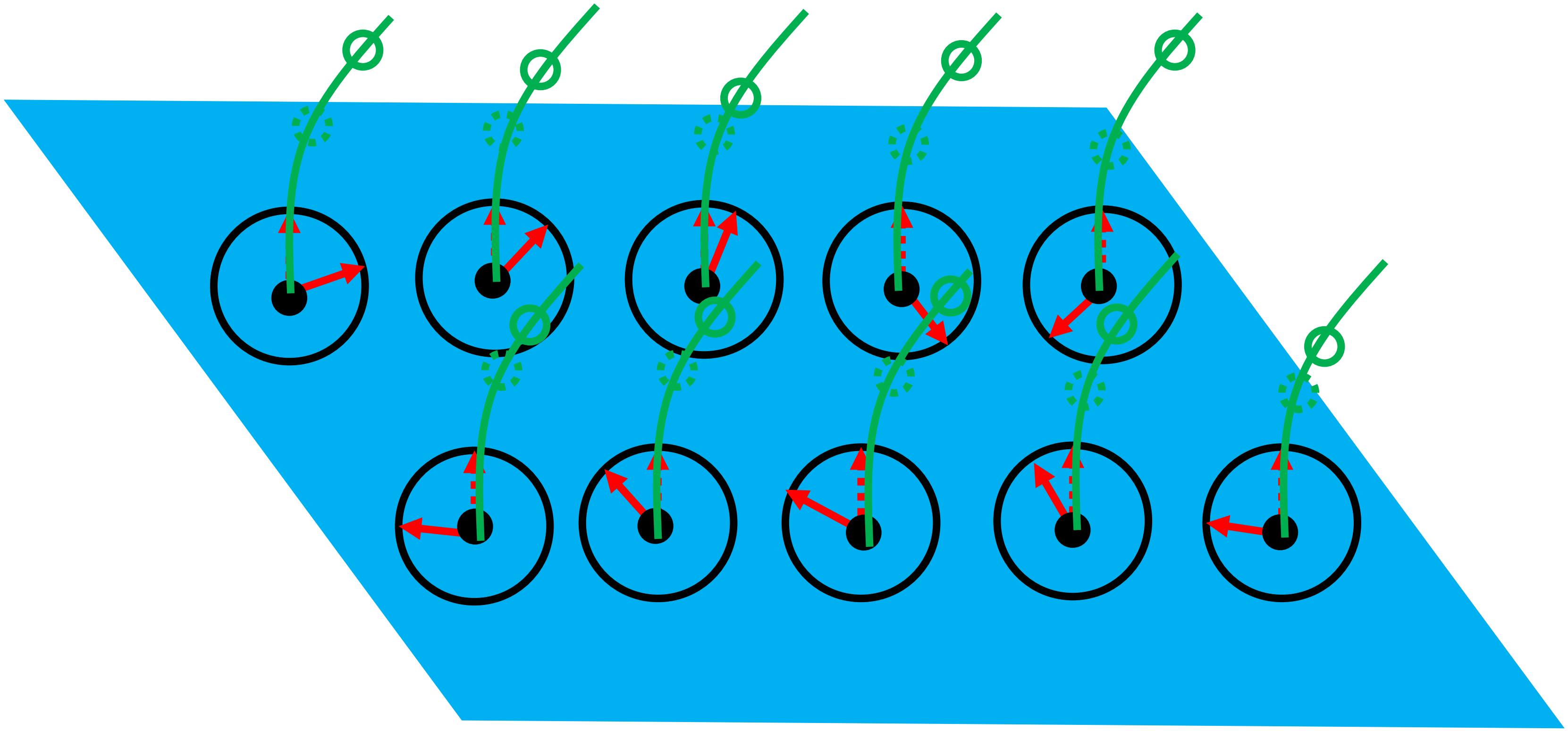
$$\varphi(\mathrm{x}) \to \exp(\mathrm{i}\,\alpha(\mathrm{x}))\varphi(\mathrm{x})$$

So without the effect of gauge field, the red arrow at each point points to the same direction and the yellow dot are at the same place of the fibers:



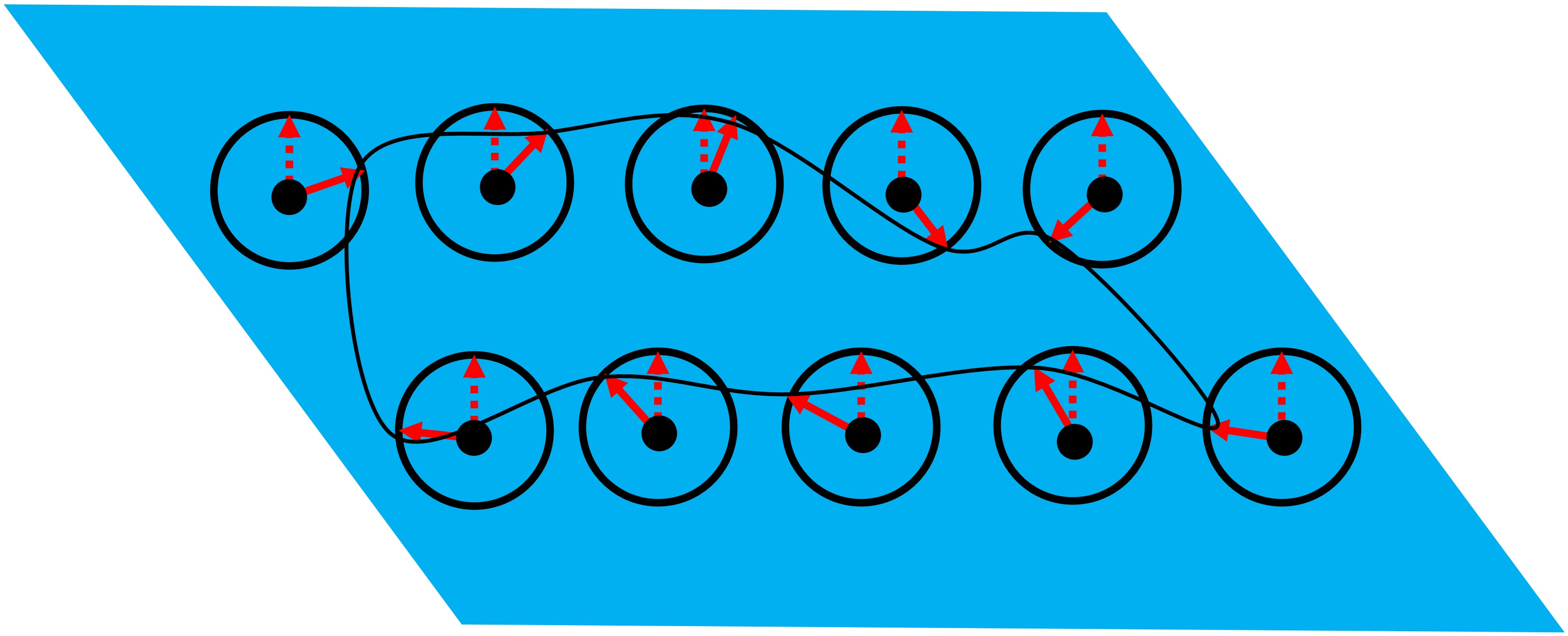
We can see the above picture is trivial since everything is the same.

Now if there exist gauge field, since the gauge field differs point by point, the place of green circles on the fibers vary which cause the angle of the red arrow at each point now is different:



We can see the above picture is more nontrivial since difference comes in.

To see the nontrivial about the picture, now we consider a electron moves along a close path:



Obviously, the phase of electron will rotate, and the angle must be integer of 2π. This is what Aharonov-Bohm effect say.

Gauge group

Now we use group theory for simplicity.

Gauge transformation is just an element in gauge group:

$$\varphi(\mathrm{x}) \to G(\mathrm{x})\varphi(\mathrm{x})$$

$$U(y,x) \to G(\mathrm{y})U(y,x)G(\mathrm{x})^{-1}$$

When G is expanded in basis, such as $\exp(i\alpha(x))$ in U(1),

The U can also be expanded in basis and generator of the group

$$U(x+\varepsilon\hat{n},x) = 1 - ie\varepsilon\hat{n}^{\mu}A_{\mu}(x) + O(\varepsilon^2)$$

1: the scalar product such as

$$\varphi^{\dagger}(y)U(y,x)\varphi(\mathrm{x}) \to \varphi^{\dagger}(y)G^{-1}(\mathrm{y})G(\mathrm{y})U(y,x)G^{-1}(\mathrm{x})G(\mathrm{x})\varphi(\mathrm{x})$$

$$= \varphi^{\dagger}(y)U(y,x)\varphi(\mathrm{x})$$

is then gauge invariant.

This gives the term $\bar{\varphi}(i\not{D})\varphi$ in the QED lagrangian

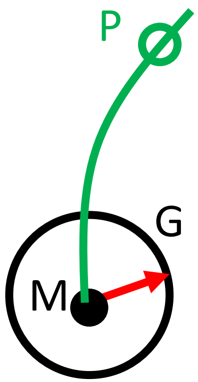
2, the Wilson's loop is gauge invariant: $W_{\lambda} \to W_{\lambda}$

This gives the term $-\frac{1}{4}\left(F_{\mu\nu}\right)^2$ in the QED lagrangian

Now we rewrite the theory in fiber bundle.

Let M be a manifold of dimension n and G a compact Lie group. Let P be a principal G-bundle over M. This forms the basic date of a gauge theory.

M	space-time
G	gauge group
P	gauge field



In a local trivialization of P (called a gauge) we express the connection through its connection 1-form A which takes values in the Lie algebra g of G.

The covariant derivative of A in this local trivialization is the curvature 2-form

$$F \equiv d_A A = dA + [\mathrm{A} \wedge \mathrm{A}]$$

again with values in g. Here the second term means that we take the exterior product on forms and the Lie bracket on Lie algebra elements.

From chapter 1 of **Discrete gauge theory: from lattices to TQFT** by Robert Oeckl

Non-abelian gauge quantum field: SU(2) and Yang-Mills theory

Consider the following generalization of gauge group. Instead of a single fermion field, we start with a doublet of Dirac fields:

$$\varphi = \begin{pmatrix} \varphi_1(\mathrm{x}) \\ \varphi_2(\mathrm{x}) \end{pmatrix}$$

which transform to one another under abstract 3D rotations as two-component spinor:

$$\varphi \to G(\mathrm{x})\varphi = \exp[\mathrm{i}\,\alpha^i(\mathrm{x})\frac{\sigma^i}{2}]\varphi$$

To construct a Lagrangian that is invariant under this new group of transformations, we must again define a covariant derivative that transforms in a simple way. Again we introduce a comparator. But since matter field now is a two-component object, the comparator must be a 2*2 matrix. The transformation law for U(y, x) now is

$$U(\mathrm{y},\mathrm{x}) \to G(\mathrm{y})U(\mathrm{y},\mathrm{x})\mathrm{G}^{\dagger}(\mathrm{x})$$

Knowledge of Lie group gives infinitesimal expansion:

$$U(\mathrm{x}+\varepsilon n, x) = 1 + ig\varepsilon n^{\mu} A^i_{\mu}(\mathrm{x})\frac{\sigma^i}{2} + \mathrm{O}(\varepsilon^2)$$

Then

$$D_{\mu} = \partial_{\mu} - igA_{\mu}^{i}(\mathrm{x})\frac{\sigma^{i}}{2}$$

$$F_{\mu\nu}^{i} = \partial_{\mu}A_{\nu}^{i} - \partial_{\nu}A_{\mu}^{i} + g\varepsilon^{ijk}A_{\mu}^{j}A_{\nu}^{k}$$

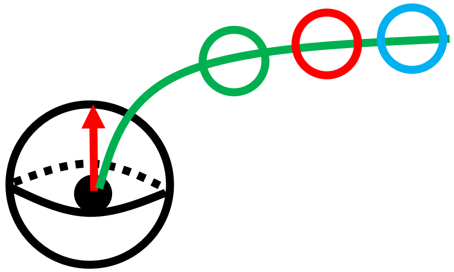
where ε^{ijk} is the structure constant and is determined by the commutation relations of Pauli matrices:

$$[\frac{\sigma^{i}}{2},\frac{\sigma^{j}}{2}]_{-} = i\varepsilon^{ijk}\frac{\sigma^{k}}{2}$$

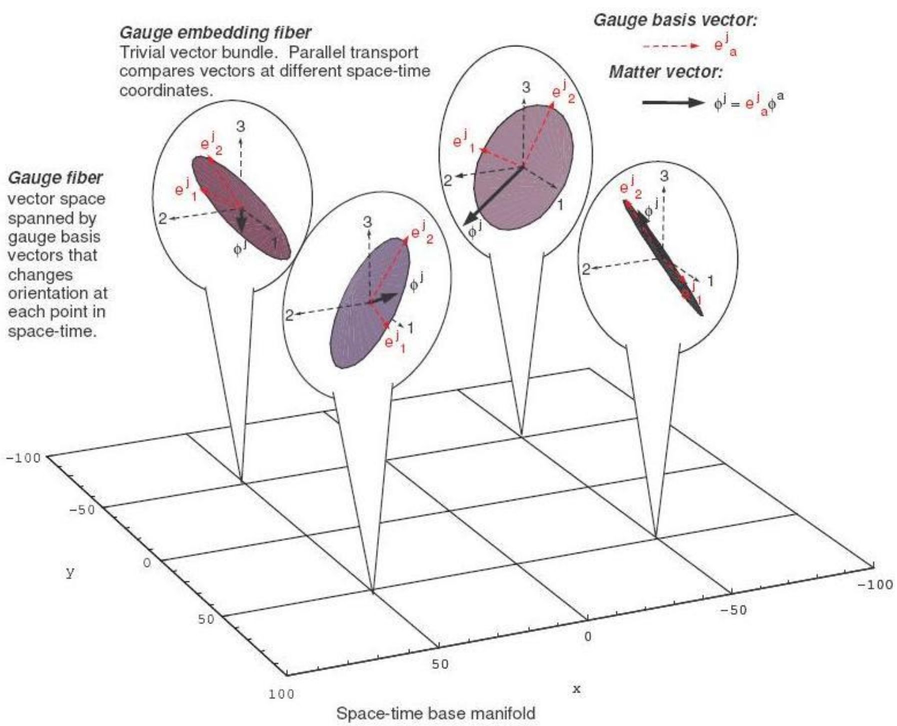
And the Lagrangian is:

$$\mathrm{L}_{QED} = \overline{\varphi}(\mathrm{i}\not{D})\varphi - \frac{1}{4}(F_{\mu\nu}^{i})^{2} - m\overline{\varphi}\varphi$$

For SU(2), the matter field is a sphere and the gauge field is a fiber, now with three circles stuck to the fiber rather than one:



And the story goes as well. For large group, it’s hard to imagine the high-dimensional sphere.



The representation of the Lie group gives the gauge bosons.

This picture is from:
https://www-thphys.physics.ox.ac.uk/people/MarioSerna/GaugeGeometryResearch/index.html

Non-abelian gauge quantum field: SU(3) and quantum chromodynamics

Relativistic quantum field theory and particles

Quantized

Dynamical variables (fields) Particles

Scalar fields Scalar particles

(Spin=0, boson) example: Higgs, phonons, etc

Spinor fields Fermion particles

(Spin=1/2, fermion) example: leptons
quarks

Gauge fields Gauge particles

(Spin=1, boson) example: photons (**U(1)**),
W and Z boson (**SU(2)**),
gluons (**SU(3)**)

Standard model