Projector representation

See Chapter 3 of http://users.ox.ac.uk/~phys1116/book.html

Projection operator

- A projector in general is an operator that squares to itself:

$$P^2 = P$$

- The eigenvalues of projectors can be only 0 or 1;

$$P|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow \lambda(\lambda - 1) = 0$$

- The projector can be expanded by subspace;

$$P^{(\mathrm{i})} = \sum_j |i, j\rangle\langle i, j|$$

- The projectors into different subspaces are orthogonal;

$$P^{(\mathrm{i})} P^{(\mathrm{j})} = \sum_{m,n} |i, \mathrm{m}\rangle\langle i, \mathrm{m}|\, j, n\rangle\langle j, n| = 0$$

- All the projector should be 1:

$$\sum_i P^{(\mathrm{i})} = \sum_i |i\rangle\langle i| = 1$$

Projector representation

What if the Hamiltonian can be written as sum of projectors with positive coefficients?

Any Hamiltonian that can be written in such a form necessarily has non-negative eigenvalues.

Proof:

Consider an eigenstate |E> of $\hat{H} = \sum_i \alpha_i \hat{P}^{(i)}$ where all the α_i are non-negative. Then:

$$E = \langle E | H | E \rangle = \sum_i \alpha_i \langle E | P^{(\mathrm{i})} | E \rangle$$

Consider now just one of the terms in this sum and expand |E> in a complete set of eigenstates of this $\hat{P}^{(i)}$, i.e.

$$|E\rangle = \sum_j \beta_{ij} |\mathrm{i};\, j\rangle, P^{(\mathrm{i})} |\mathrm{i};\, j\rangle = \lambda_{i,j} |\mathrm{i};\, j\rangle$$

where the eigenvalues $\lambda_{i,j}$ of a given $\hat{P}^{(i)}$ must be zero or one, and so non-negative. Therefore,

$$E = \langle E | H | E \rangle = \sum_i \alpha_i \lambda_{i,j} \left\| \beta_{i,j} \right\|^2 \geq 0$$

The condition to get E=0 is that for each $\hat{P}^{(i)}$, we have $\hat{P}^{(i)}|E\rangle$=0. Such a state is a ground state, and the Hamiltonians which have such a ground state are called frustration free in the quantum information world.

Single spin

$$J^2 \left| j,m \right\rangle = j(\mathrm{j}+1) \left| j,m \right\rangle$$

$$J_z \left| j,m \right\rangle = m \left| j,m \right\rangle \qquad \text{m=-j,-j+1,…,j-1,j}$$

A particle of spin j has a local Hilbert space $Hx=C^d$.

Casimir operator

The Casimir operator for a representation of SU(2) is the element $\hat{J}^2$.

In the case of an irreducible representation of SU(2), Schur's Lemma indicates that $\hat{J}^2$=cI for some constant c. For the irrep of dimension 2j+1, the constant c=j(j+1);

In the case of a reducible representation, $\hat{J}^2 = \sum_j j(j+1)\hat{P}^{(j)}$ where each j corresponds to an irrep in the decomposition of the representation, and $\hat{P}^{(j)}$ is the projection onto that subspace.

Many spin: the simplest tensor product we've ever met.

Each individual particle in a quantum spin chain is modeled by an irreducible representation of SU(2). To model the entire chain, we tensor neighboring representations together.
For example, let $D^{(j)}$ be the (2j+1)-irreducible representation of SU(2) describing a particle of spin j. If we wish to make a quantum spin chain of two consecutive particles with respective spins, j1, j2, the SU(2) of the representation of the chain is:

$$D^{(j_1)} \otimes D^{(j_2)}$$

We can decompose a tensor of two spin representations into irreducibles using the CG coefficients:

$$D^{(j_1)} \otimes D^{(j_2)} \cong D^{\|j_1 - j_2\|} \oplus D^{\|j_1 - j_2\|+1} \oplus \ldots \oplus D^{\|j_1 + j_2\|}$$

Symmetry breaking

An important set of operators is the total spin:

$$\vec{S} = \sum_{i=1}^{N} \vec{S}_i$$

With its three components:

$$S^{\alpha} = \sum_{i=1}^{N} S_i^{\alpha}$$

If the Hamiltonian commutes with these then the full system is SU(2) symmetric.

Now we prove the total spin commute with any “dot product”:

$$[\vec{S}, \vec{S}_i \vec{S}_j]_- = 0$$

Check:

$$[S^x, \vec{S}_i \vec{S}_j]_- = [S_i^x, S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z]_- + [S_j^x, S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z]_-$$

$$= S_i^z S_j^y - S_i^y S_j^z + S_i^y S_j^z - S_i^z S_j^y = 0$$

Therefore, the simplest example of SU(2)-symmetric spin Hamiltonian is the nearest neighbor Heisenberg model, where

$$H = -J \sum_{<i,j>} \vec{S}_i \vec{S}_j$$

Spin-1/2

Since the spin-1/2 operators can be written in term of Pauli matrices, the spin-1/2 Heisenberg Hamiltonian on a chain in terms of raising and lowering operators is then:

$$H = -J \sum_i \underbrace{\frac{1}{2}(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+)}_{\text{off-diagonal term}} + \underbrace{\frac{1}{4}\sigma_i^z \sigma_{i+1}^z}_{\text{diagonal term}}$$

- J>0, ferromagnetic case, where the diagonal terms in the Hamiltonian favors spin parallel;
- J<0, antiferromagnetic case, where diagonal terms in Hamiltonian favors spin antiparallel;

SU(2)

The SU(2) restricts the coupling constants of off-diagonal term and diagonal term to be 2:1. If the coefficient derivates from this ratio, then SU(2) symmetry is broken into lower symmetry.

U(1)*Z2

If we keep the coupling constants of the two off-diagonal terms the same, while change the ratio from 2:1, then we have the XXZ model:

$$H = -\sum_i J_\perp(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) + J_z\sigma_i^z\sigma_{i+1}^z$$

Obviously, there is a U(1) related to x and y
and a Z2 related to a mirror spin flipping.

Z2

We can continue break the symmetry, then we have the XYZ model:

$$H = -\sum_i J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z$$

Obviously, there is a Z2 related to a mirror spin flipping.

Warm up: add two spin-1/2

$$\vec{S}_1\vec{S}_2 = \frac{1}{2}(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + \frac{1}{4}\sigma_1^z\sigma_2^z$$

The basis is product state:

$$\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$$

So the matrix is:

$\lvert\uparrow\uparrow\rangle$	$\lvert\uparrow\downarrow\rangle$	$\lvert\downarrow\uparrow\rangle$	$\lvert\downarrow\downarrow\rangle$	
$1/4$	0	0	0	$\lvert\uparrow\uparrow\rangle$
0	$-1/4$	$1/2$	0	$\lvert\uparrow\downarrow\rangle$
0	$1/2$	$-1/4$	0	$\lvert\downarrow\uparrow\rangle$
0	0	0	$1/4$	$\lvert\downarrow\downarrow\rangle$

Exactly diagonalize it:

$$|1,1\rangle = |\uparrow\uparrow\rangle, E = \frac{1}{4}, S^z = 1$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), E = \frac{1}{4}, S^z = 0$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle, E = \frac{1}{4}, S^z = -1$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), E = -\frac{3}{4}, S^z = 0$$

So we have decompose the tensor product:

$$D^0 \oplus D^1 \cong D^{1/2} \otimes D^{1/2}$$

Now we are going to determine $P^{(0)}$ and $P^{(1)}$ in terms of $\hat{\vec{S}}$.

The Casimir operator for a representation of SU(2) is $\hat{\vec{S}}^2$,

In the case of a reducible representation, $\hat{\vec{S}}^2 = \sum_S S(S+1)\hat{P}^{(S)}$ where each j corresponds to an irrep in the decomposition of the representation, and $\hat{P}^{(S)}$ is the projection onto that subspace.

Therefore, we have

$$\vec{S}^{\,2} = (\vec{S}_1 + \vec{S}_2)^2 = 2P^{(1)} + 0P^{(0)}$$

i.e.

$$P^{(1)} = (\vec{S}_1 + \vec{S}_2)^2 = \frac{1}{2}(\vec{S}_1^{\,2} + \vec{S}_2^{\,2}) + \vec{S}_1\vec{S}_2$$

Since $\hat{\vec{S}}_i^2 = \frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}$

We have

$$P^{(1)} = \frac{1}{2}(\vec{S}_1 + \vec{S}_2)^2 = \frac{3}{4} + \vec{S}_1\vec{S}_2$$

And the sum of all projectors is 1, so we have

$$P^{(0)} = -\frac{1}{2}[(\vec{S}_1 + \vec{S}_2)^2 - 1] = \frac{1}{4} - \vec{S}_1\vec{S}_2$$

Now we check this results:

$$P^{(1)}\left|1,1\right\rangle = (\frac{3}{4} + \vec{S}_1\vec{S}_2)\left|1,1\right\rangle = 1\left|1,1\right\rangle$$

$$P^{(1)}\left|0,0\right\rangle = (\frac{3}{4} + \vec{S}_1\vec{S}_2)\left|0,0\right\rangle = 0\left|0,0\right\rangle$$

$$P^{(0)}\left|1,1\right\rangle = (\frac{1}{4} - \vec{S}_1\vec{S}_2)\left|1,1\right\rangle = 0\left|1,1\right\rangle$$

$$P^{(0)}\left|0,0\right\rangle = (\frac{1}{4} - \vec{S}_1\vec{S}_2)\left|0,0\right\rangle = 1\left|0,0\right\rangle$$

This is just what we want.

Physical interpretation

$$H = -J\vec{S}_1\vec{S}_2 = J\vec{P}^{(0)} - \frac{1}{4}J = -J\vec{P}^{(1)} + \frac{3}{4}J$$

Case 1 Case 2

Since we need to make sure the coefficients of projector are non-negative, there are two case according to sign of J.

Case 1: when J>0, to get the frustrated free (i.e. the ground state), we need to satisfy

$$\hat{\vec{P}}^{(0)}|\psi\rangle = 0$$

as we have described above, the state is that the pair of spins are the same. therefore, we obtained ferromagnetism as the ground state.

Case 2: when J<0, we need to find state which satisfies $\hat{\vec{P}}^{(1)}|\psi\rangle = 0, \forall i.$

the state is that the pair of spins are antiparallel. However, when there is a chain, there is no state annihilated by all the projectors. This is simple to see by checking explicitly just for three spins in a row. The antiferromagnetic Heisenberg Hamiltonian is therefore not frustration free, this is the difference between FM and AFM.

Valence bond

From now on, we call the singlet pair state “a valence bond”:

And introduce a condensed notation to describe the state:

$$[1,2] \equiv \frac{1}{\sqrt{2}}\left(\left|\uparrow_1\downarrow_2\right\rangle - \left|\downarrow_1\uparrow_2\right\rangle\right)$$

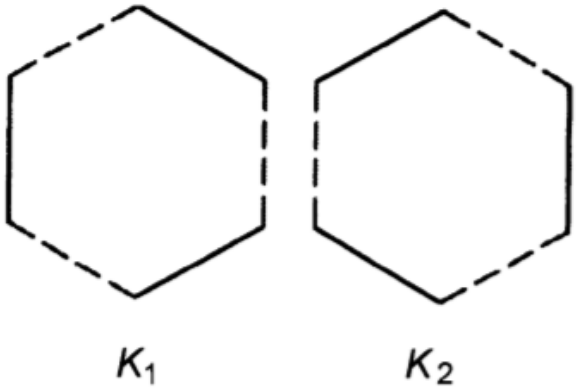
And the projector

$$P^{(1)}[1,2] = 0[1,2]$$

$$P^{(0)}[1,2] = 1[1,2]$$

Valence bond cluster

Benzene, there are 6 spin ½ states, the two degenerated states are K1 and K2



Either of the G.S. breaks the translation symmetry if we see benzene as a PBC system. If we see it as a cluster, either G.S. breaks the rotation symmetry. So they are symmetry breaking phase.

The non-symmetry breaking phase is in the form of a linear combination:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|K_1\rangle + |K_2\rangle)$$

The above picture is quite similar to bond resonating picture. Does the analogy is true for 1D spin system?

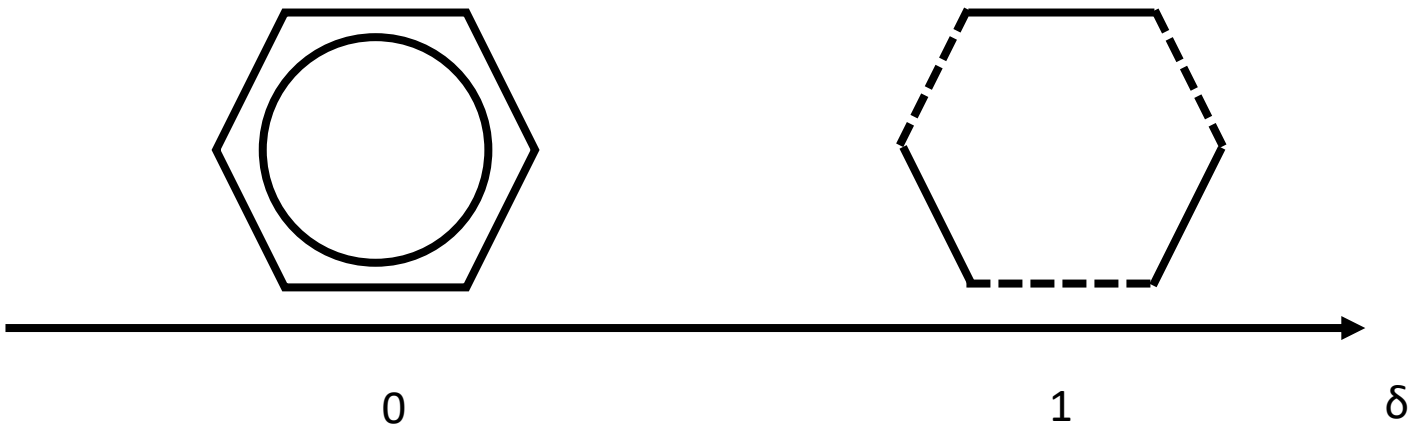
Excited state

Choose one symmetry breaking phase

We can couple into lattice distortion, just as the case in CDW.

$$H = \sum_i J(1+\delta)\vec{S}_{2i-1}\vec{S}_{2i} + J(1-\delta)\vec{S}_{2i}\vec{S}_{2i+1}, J > 0$$

Therefore, the original two states now split.



What happens with longer exchange coupling?

Here we still consider the case of benzene, in the case we described above, we only consider the nearest neighbor exchange coupling. If more exchange couplings are considered, we have

$$C_6^3 = 12$$

possible products of valence bonds. Here we plot five of them:

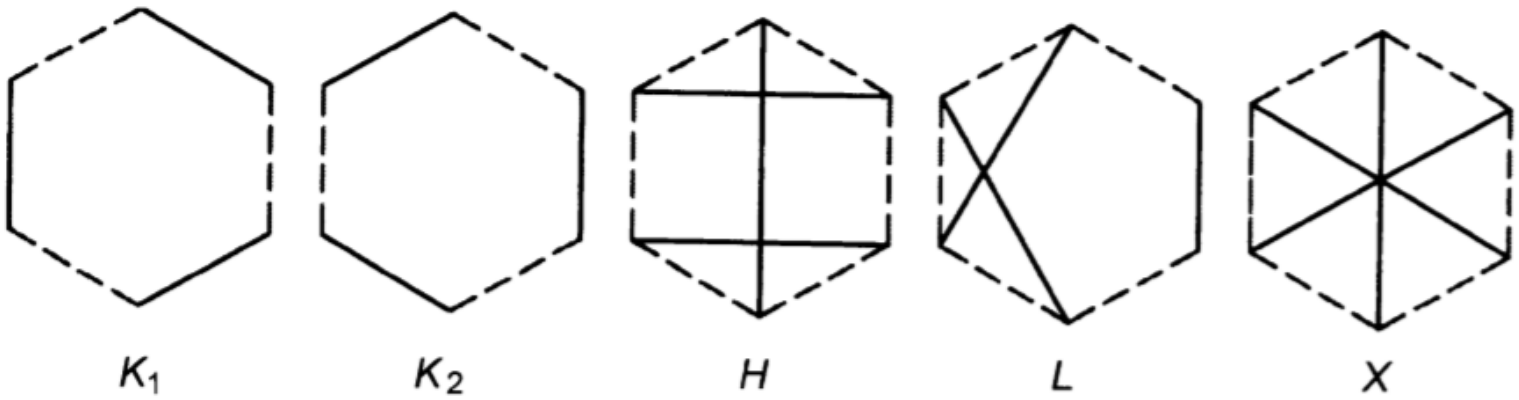


Fig. 10.5. Five possible products of valence bonds for the benzene structure. Only the first two, the Kekule states, are physical

See Chapter 10.3, book magnetism and superconductivity by Levy, L.-P.

Valence bond solid

Here we goes from cluster to solid. From cluster discussion, we know any local valence bond cluster state breaks rotational symmetry. To restore the symmetry, we turn local into global by summing up all the local terms and give equal weight. Here is the same idea, first we pick up a local case, then summing all things up. Now take spin ½ chains as example.

$$H = -J\sum_i \vec{S}_i \vec{S}_{i+1} = -J\sum_i \vec{P}^{(1)}_{i,i+1} + \frac{3}{4}NJ$$

Suppose we have J<0 for the Heisenberg Hamiltonian:

The singlet state we find satisfies

$$\vec{P}^{(1)}_{i,i+1}\left|\psi\right\rangle = 0\left|\psi\right\rangle, \forall i$$

Since

$$P^{(1)}_{i,i+1}[\mathrm{i},\mathrm{i}+1] = 0[\mathrm{i},\mathrm{i}+1]$$

Therefore,

$$\left|\psi\right\rangle = \sum_{\{\mathrm{i}\}} a(\mathrm{i}_1 \ldots \mathrm{i}_n)[\mathrm{i}_1,\mathrm{i}_2]\ldots[\mathrm{i}_{n-1},\mathrm{i}_n]$$

Here a means a particular covering of the entire lattice by the pairs $[i_1, i_2]\ldots[i_{n-1}, i_n]$.

For PBC in 1D, there are at least two degenerated singlet states.

Majumdar-Ghosh model

Here we consider spin-1/2 chain and with next nearest neighbor exchange coupling:

$$H = \sum_i (\vec{S}_i \vec{S}_{i+1} + \frac{1}{2}\vec{S}_i \vec{S}_{i+2}) = \frac{1}{2}\sum_i (\vec{S}_i \vec{S}_{i+1} + \vec{S}_{i+1}\vec{S}_{i+2} + \vec{S}_i \vec{S}_{i+2})$$

We have

$$\vec{S} = \vec{S}_i + \vec{S}_{i+1} + \vec{S}_{i+2}$$

The Casimir operator gives

$$\vec{S}^2 = \frac{3}{4}P^{(\frac{1}{2})} + \frac{15}{4}P^{(\frac{3}{2})}$$

On the other hand, we have

$$1 = P^{(\frac{1}{2})} + P^{(\frac{3}{2})}$$

Therefore, we have

$$P^{(\frac{1}{2})} = -\frac{1}{3}(\vec{S}^2 - \frac{15}{4})$$

$$P^{(\frac{3}{2})} = \frac{1}{3}(\vec{S}^2 - \frac{3}{4})$$

Here we give another way to write $\hat{P}^{\left(\frac{1}{2}\right)}$,$\hat{P}^{\left(\frac{3}{2}\right)}$, which is easy to generalize:

First we make the following table

	$\vec{S}^2$	$P^{(\frac{1}{2})}$	$P^{(\frac{3}{2})}$
$\left\|\frac{3}{2},i\right\rangle$	$\frac{15}{4}$	0	1
$\left\|\frac{1}{2},i\right\rangle$	$\frac{3}{4}$	1	0

From the table, we can construct the following polynomial:

$$(\vec{S}^2-\frac{15}{4})(\vec{S}^2-\frac{3}{4})$$

Since $\hat{P}^{\left(\frac{3}{2}\right)}$ project to state with $\hat{\vec{S}}^2=\frac{15}{4}$, we can write it as

$$P^{(\frac{3}{2})}=\frac{1}{3}(\vec{S}^2-\frac{3}{4})$$

And

$$P^{(\frac{1}{2})}=-\frac{1}{3}(\vec{S}^2-\frac{15}{4})$$

It's easy to check that

$$1 = P^{(\frac{1}{2})} + P^{(\frac{3}{2})}$$

$$\vec{S}^2 = \frac{3}{4} P^{(\frac{1}{2})} + \frac{15}{4} P^{(\frac{3}{2})}$$

Therefore:

$$P^{(\frac{3}{2})}_{i,i+1,i+2} = \frac{1}{3}(\vec{S}^2 - \frac{3}{4}) = \frac{1}{3}\vec{S}^2 - \frac{1}{4} = \frac{1}{3}(\vec{S}_i + \vec{S}_{i+1} + \vec{S}_{i+2})^2 - \frac{1}{4}$$

$$= \frac{2}{3}(\vec{S}_i \vec{S}_{i+1} + \vec{S}_{i+1}\vec{S}_{i+2} + \vec{S}_i \vec{S}_{i+2}) + \frac{1}{2}$$

Therefore, we find that the Majumdar-Ghosh Hamiltonian is also the direct sum of projectors with positive coefficients:

$$H = \frac{2}{3}\sum_i P^{(\frac{3}{2})}_{i,i+1,i+2}$$

with a constant ignored

The reverse is also true, the Majumdar-Ghosh Hamiltonian is so constructed to make sure that is has the above form.

Ground state

Now we need to know the state spanned by $\hat{P}^{\left(\frac{3}{2}\right)}_{i,i+1,i+2}$, this can be done by inspection rather than by exact diagonalization.

First consider two of the three spins, they can from spin-1 or spin-0 state.
Then consider the third spin. The spin-3/2 state for three spin can be found only by product a third spin with the spin-1 state of the pair. Since this product can be done in any order, this means that any two of the three consecutive spins in the MG chain must have spin 1 for the three to have total spin 3/2. Thus if any two of the three spins are in a singlet, $\hat{P}^{\left(\frac{3}{2}\right)}_{i,i+1,i+2}$ must annihilate the state.

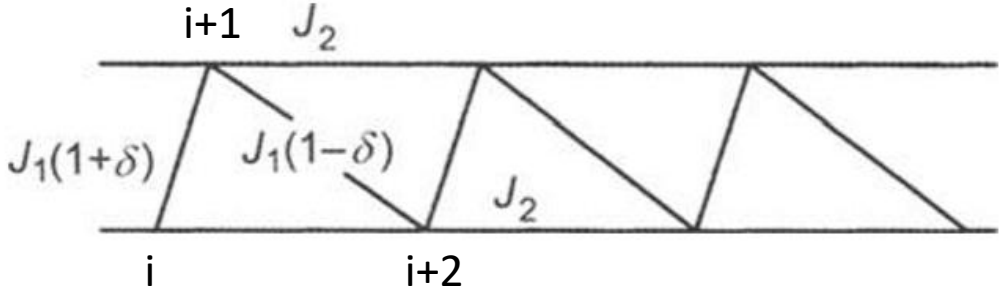
Therefore, the state

$$\left|\psi\right\rangle = \sum_{\{\mathrm{i}\}} a(\mathrm{i}_1 \ldots \mathrm{i}_n)[\mathrm{i}_1, \mathrm{i}_2]\ldots[\mathrm{i}_{n-1}, \mathrm{i}_n]$$

For PBC, there are at least two degenerated singlet states.

Since a singlet state is called a valence bond, so such ground states are called valence-bond solids.

To make a picture, we consider a spin ladder

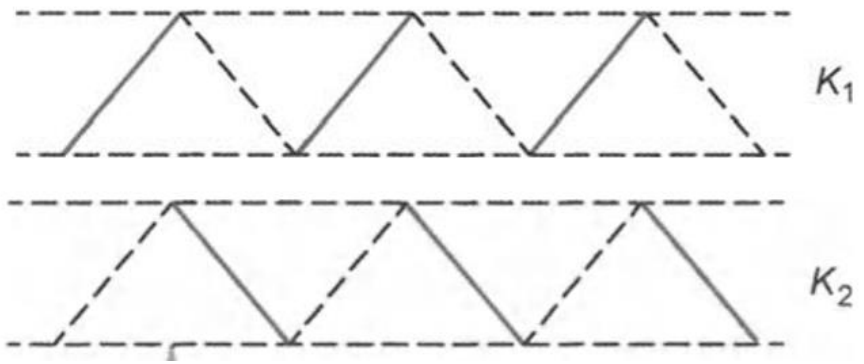


sublattice b

Here we have δ=0.

sublattice a

The two ground state is pictured just like the two Kekule states:

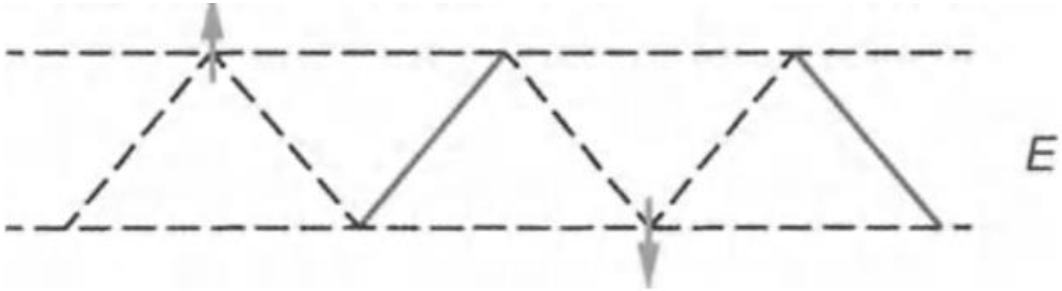


And the linear combination (the true G.S.) restore the symmetry:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|K_1\rangle + |K_2\rangle)$$

In this state, valence bonds have minimal length and hence correlations between spins are very short range, not exceeding one lattice spacing. The valence bonds are ordered but spins are disordered, as in a liquid state, and we speak of a spin liquid.

Excited state



Excited states are made up of two free spins behaving like two defects, which separate a |K1> state from a |K2> state.

The MG model and isotropic Heisenberg model belong to difference universal class;
The MG model and Haldane's liquid state belong to same universal class.

See Chapter 10.3, book magnetism and superconductivity by Levy, L.-P.

Finally we goes to the AKLT model:

$$H = \sum_i [\frac{1}{3} + \frac{1}{2}\vec{S}_i \vec{S}_{i+1} + \frac{1}{6}(\vec{S}_i \vec{S}_{i+1})^2] \quad \text{s is 1}$$

We have

$$\vec{S} = \vec{S}_i + \vec{S}_{i+1}$$

The Casimir operator gives

$$\vec{S}^2 = 0P^{(0)} + 2P^{(1)} + 6P^{(2)}$$

https://physics.stackexchange.com/questions/286601/detailed-derivation-and-explanation-of-the-aklt-hamiltonian

http://galileo.phys.virginia.edu/~pf7a/msm3.pdf

https://www.math.ucdavis.edu/~bxn/AKLT_Model-2.pdf

To explicitly write $\hat{P}^{(0)}$, $\hat{P}^{(1)}$, $\hat{P}^{(2)}$, we make the following table

	$\vec{S}^2$	$P^{(0)}$	$P^{(1)}$	$P^{(2)}$
$\lvert 2,i\rangle$	6	0	0	1
$\lvert 1,i\rangle$	2	0	1	0
$\lvert 0,i\rangle$	0	1	0	0

From the table, we can construct the following polynomial:

$$(\vec{S}^2-6)(\vec{S}^2-2)(\vec{S}^2-0)$$

Since $\hat{P}^{(2)}$ project to state with $\hat{\vec{S}}^2=6$, we can write it as

$$P^{(2)}=\frac{1}{24}(\vec{S}^2-2)(\vec{S}^2-0)$$

And $$P^{(1)}=-\frac{1}{8}(\vec{S}^2-6)(\vec{S}^2-0)$$

$$P^{(0)}=\frac{1}{12}(\vec{S}^2-6)(\vec{S}^2-2)$$

Now we check

$$P^{(2)} = \frac{1}{24}(\vec{S}^2 - 2)(\vec{S}^2 - 0)$$

$$P^{(1)} = -\frac{1}{8}(\vec{S}^2 - 6)(\vec{S}^2 - 0)$$

$$P^{(0)} = \frac{1}{12}(\vec{S}^2 - 6)(\vec{S}^2 - 2)$$

$$P^{(0)} + P^{(1)} + P^{(2)} = (\frac{1}{24} - \frac{1}{8} + \frac{1}{12})\vec{S}^4 + (-\frac{1}{12} + \frac{3}{4} - \frac{2}{3})\vec{S}^2 + 1 = 1$$

$$0P^{(0)} + 2P^{(1)} + 6P^{(2)} = (\frac{6}{24} - \frac{2}{8} + \frac{0}{12})\vec{S}^4 + (-\frac{6}{12} + \frac{6}{4} - \frac{0}{3})\vec{S}^2 = \vec{S}^2$$

Now we expand these projectors:

$$\vec{S}^{\,2} = (\vec{S}_i + \vec{S}_{i+1})^2 = \vec{S}_i^{\,2} + \vec{S}_{i+1}^{\,2} + 2\vec{S}_i \vec{S}_{i+1} = 4 + 2\vec{S}_i \vec{S}_{i+1}$$

$$\vec{S}^{\,4} = (4 + 2\vec{S}_i \vec{S}_{i+1})^2 = 16 + 16\vec{S}_i \vec{S}_{i+1} + 4(\vec{S}_i \vec{S}_{i+1})^2$$

$$P^{(2)}_{i,i+1} = \frac{1}{24}(\vec{S}^{\,4} - 2\vec{S}^{\,2}) = \frac{1}{3} + \frac{1}{2}\vec{S}_i \vec{S}_{i+1} + \frac{1}{6}(\vec{S}_i \vec{S}_{i+1})^2$$

Therefore, we find that the AKLT Hamiltonian is also the direct sum of projectors with positive coefficients:

$$H = \sum_i P^{(2)}_{i,i+1}$$

The reverse is also true, the AKLT Hamiltonian is so constructed to make sure that is has the above form.

G.S.

To see the G.S., here we use some trick.

Now we split one spin-1 into two spin-1/2,

If we abandon the spin singlet state,

$$|1,1\rangle = |\uparrow\uparrow\rangle, E = \frac{1}{4}, S^z = 1$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), E = \frac{1}{4}, S^z = 0$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle, E = \frac{1}{4}, S^z = -1$$

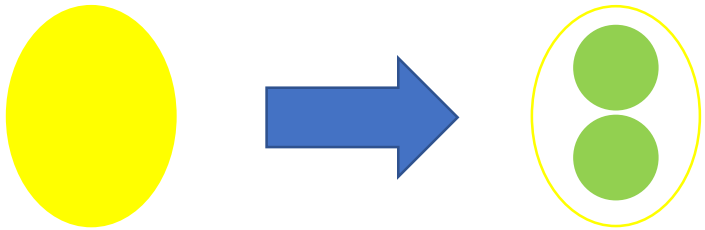
And we write as:

$$\{1,2\}_{+1} = |\uparrow_1\uparrow_2\rangle$$

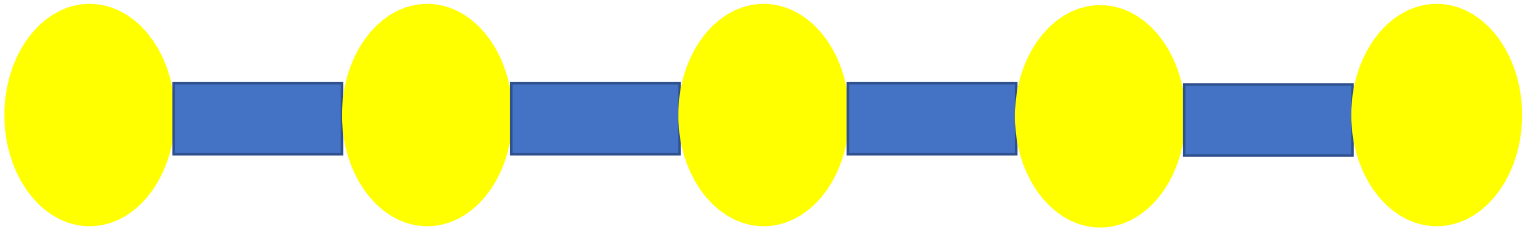
$$\{1,2\}_0 = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle)$$

$$\{1,2\}_{-1} = |\downarrow_1\downarrow_2\rangle$$

The total d.o.f. will still be 3^N.

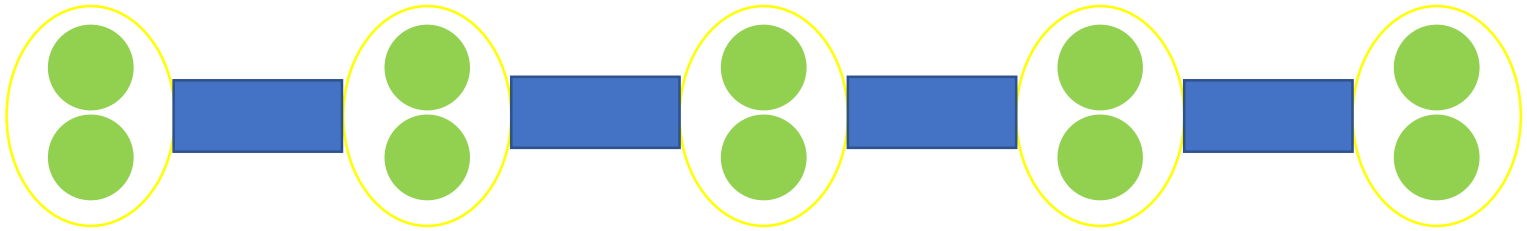


Our original chain is



spin-1 chain

Our new chain is

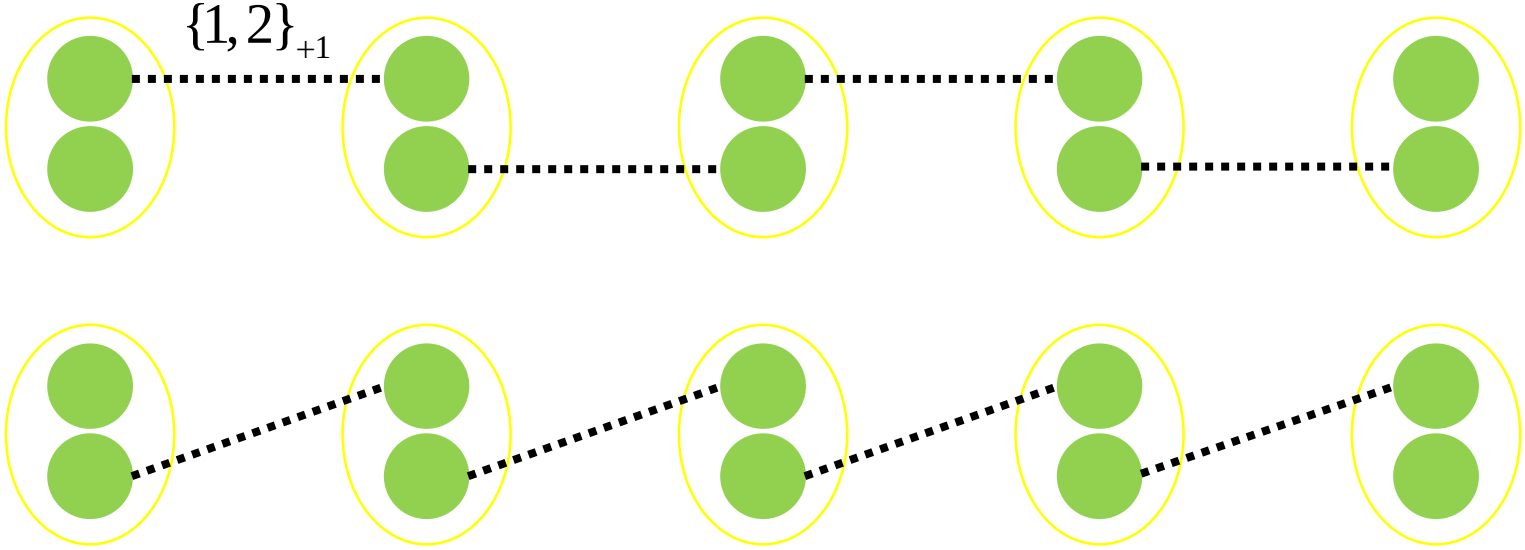


Each bond on the spin-1 chain has four spin-1/2 particles associated with it.

The Hilbert space for the spin-1 chain is therefore has a basis:

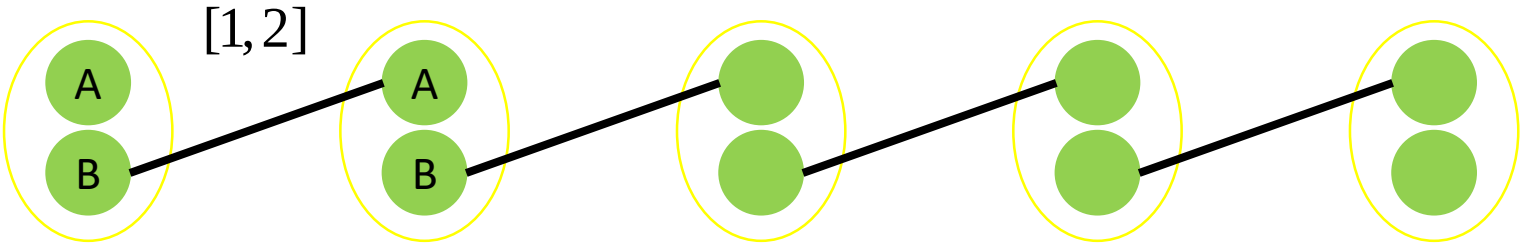
$$\{\mathrm{i}_1^1, \mathrm{i}_1^2\}_\alpha \{\mathrm{i}_2^1, \mathrm{i}_2^2\}_\beta \cdots$$

The key observation is that any pair of these four particles must be in a triplet state for the operator $\hat{P}^{(2)}_{i,i+1}$ to be non-vanishing:



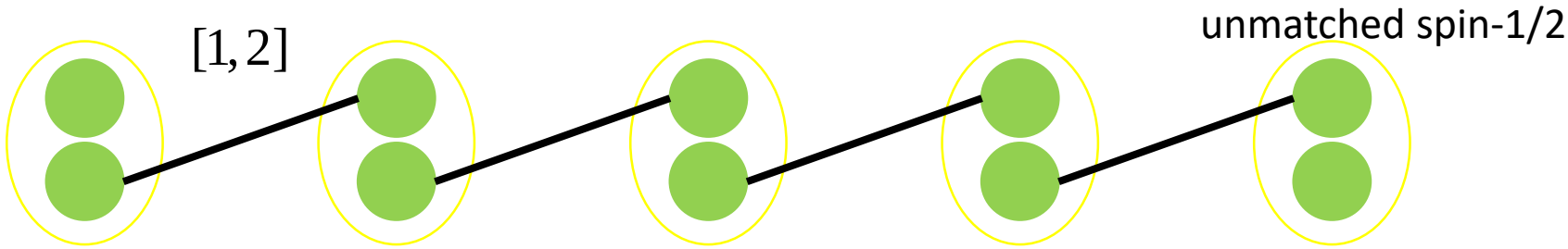
Since the two spin in a site is symmetric, therefore, the above two cases are the same.

The AKLT Hamiltonian therefore annihilates any state where any two of the four are in spin-singlet state.



Here we use that B of site i is connected to A of site i+1.

Excited state



unmatched spin-1/2

Excited states are made up of two unmatched spin-1/2 at each end.

Generalized trick:

The trick in AKLT model can be generalized: a higher spin can be always obtained by addition of lower spins.

See SOC from the perspective of entanglement

I find this topic is very important and decide to make it a project!

Code renormalization group with projector representation