Tensor and entanglement

Bipartite system $H_A, \{|i\rangle_A\}$

$$H_B, \{|j\rangle_B\}$$

The most general state in $H_A \otimes H_B$ is of the form

$$|\psi\rangle_{AB} = \sum_{i,j} C_{ij} |i\rangle_A \otimes |j\rangle_B$$

We can treat $|i\rangle_A \otimes |j\rangle_B$ as the product of two vector, so the two body wave function can be seen as (2, 0)-type tensor, i.e. we can treat many-body wave function as tensor.

There are two kinds of states:

1, Separable:

If there exist vectors so that

$$C_{ij} = [C_i^A]*[C_j^B], |\psi\rangle_A = \sum_i C_i^A |i\rangle_A, |\psi\rangle_B = \sum_j C_j^B |j\rangle_B$$

2, Inseparable:

Other state in the Hilbert space is called the entangled state.

A simple criterion of entanglement for two qubits

Here we consider both system is two-level (qubit),

$$\{|i\rangle_1, |i\rangle_2\}, \{|j\rangle_1, |j\rangle_2\}$$

Any state in the bipartite system is expressed by:

$$|\psi\rangle = \mathrm{a}_{11}|i\rangle_1|j\rangle_1 + \mathrm{a}_{12}|i\rangle_1|j\rangle_2 + \mathrm{a}_{21}|i\rangle_2|j\rangle_1 + \mathrm{a}_{22}|i\rangle_2|j\rangle_2$$

Any separable state is:

$$|\psi_s\rangle = (\mathrm{a}_1|i\rangle_1 + \mathrm{a}_2|i\rangle_2)(\mathrm{b}_1|j\rangle_1 + \mathrm{b}_2|j\rangle_2)$$

$|\psi\rangle$ is separable state if and only if:

$$\mathrm{a}_{11} = a_1 b_1$$

$$\mathrm{a}_{12} = a_1 b_2$$

$$\mathrm{a}_{21} = a_2 b_1$$

$$\mathrm{a}_{22} = a_2 b_2$$

Combining theses four expressions leaves us with a consistency condition:

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

If we use the matrix language to describe the 2-body wave function (tensor):

$$|\psi\rangle = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

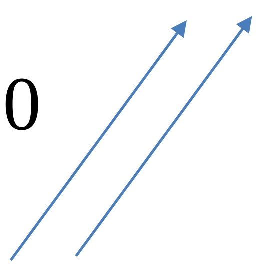
The condition will be

$$\det(|\psi\rangle) = 0$$

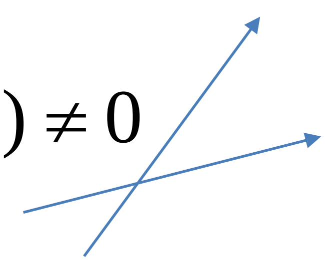
The reverse is also true, therefore, the $\det(|\psi\rangle) \neq 0$ is the necessary and sufficient condition.

We know from linear algebra that $\det(|\psi\rangle) = 0$ means that linear dependence of two equations, and $\det(|\psi\rangle) \neq 0$ means the linear independence of two equations.

$$\det(|\psi\rangle) = 0$$



$$\det(|\psi\rangle) \neq 0$$



Here we introduce another way to identify entangled state:

For separable state, we know we can may rearrange the basis to give new basis:

$$C_{ij} = [C_i^A]*[C_j^B], |\psi\rangle_A = \sum_i C_i^A |i\rangle_A, |\psi\rangle_B = \sum_j C_j^B | j\rangle_B$$

Actually we can do this for any state and gives

$$|u\rangle_A = U|i\rangle_A, |v\rangle_B = V| j\rangle_B$$

This gives the idea of Schmidt decomposition.

SVD and Schmidt decomposition

More generally, if state is

$$|\psi\rangle_{AB} = \sum_{i,j} C_{ij} |i\rangle_A \otimes | j\rangle_B$$

Then we can always do SVD decomposition to Cij

$$C = U\Sigma V^{\dagger}$$

$$|\psi\rangle_{AB} = \sum_{i,j} C_{ij} |i\rangle_A \otimes | j\rangle_B = \sum_{i,j}\sum_{k} U_{ik}\Sigma_{kk}(V^{\dagger})_{kj} |i\rangle_A \otimes | j\rangle_B$$

$$= \sum_{k} \Sigma_{kk} (\sum_{i} |i\rangle_A U_{ik}) \otimes (\sum_{j} (V^{\dagger})_{kj} | j\rangle_B)$$

$$= \sum_{k} \sigma_k |u_k\rangle_A \otimes |v_k\rangle_B$$

where we have defined the new basis

$$|u_k\rangle_A = \sum_{i} |i\rangle_A U_{ik}$$

$$|v_k\rangle_B = \sum_{j} (V^{\dagger})_{kj} | j\rangle_B$$

SVD tell us according to which for every $|\psi\rangle_{AB}$ there exist bases $|u_i\rangle_A, |v_j\rangle_B$ such that

$$|\psi\rangle_{AB} = \sum_{i=1}^{n} \underline{\sigma_i} |u_i\rangle_A \otimes |v_i\rangle_B$$

Schmidt coefficient

where n=min(dimH$_a$, dimH$_b$) and $1 = \sum_{i=1}^{n} \sigma_i^2$

Consider a pure ensemble and we can separate total system into two parts:

$$\rho_T = |\psi\rangle_{AB\,AB}\langle\psi|$$

We “trace out” system B to obtain the reduced density matrix on A:

Trace means annihilation all the d.o.f. of one subsystem.

$$\rho_A = tr_B(\rho_T) = \sum_j {}_B\langle v_j|\psi\rangle_{AB\,AB}\langle\psi|v_j\rangle_B = \sum_i \sigma_i^2 |u_i\rangle_{AA}\langle u_i|$$

i.e. the Schmidt coefficients are the square roots of the eigenvalues of subsystem’s density matrix.

Condition for entanglement for bipartite system

It's clear Schmidt decomposition is useful in characterizing entangled state:

- if and only if one Schmidt coefficient is 1 and the others are all zero, then the state is separable;
- if more than one Schmidt coefficients are non-zero, then the state is entangled.

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Entanglement in many-body systems

https://www.quantiki.org/wiki/bipartite-states-and-schmidt-decomposition

Example

First we consider an simple example

$$|\psi\rangle_{AB} = \frac{1}{2}(|+;+\rangle + |+;-\rangle + |-;+\rangle + |-;-\rangle)$$

A simple calculation tells us

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

Now we use Schmidt decomposition to do this:

We can write this as

$$|\psi\rangle_{AB} = \sum_{i,j} A_{ij} |i; j\rangle$$

With

$$A_{++} = A_{+-} = A_{-+} = A_{--} = \frac{1}{2}$$

So the matrix we have is

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

If we do single value decomposition, we will have

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{V^\dagger}$$

Since

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the only nonzero term is 11

Therefore, we only need to concern about the first column of U, V+

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So we have

$$|u\rangle_A = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), |v\rangle_B = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

Second we consider the simplest entangled state

$$\left|\psi\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|+;-\right\rangle - \left|-;+\right\rangle\right)$$

The matrix we have

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If we do single value decomposition, we will have

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we need to concern about both 11 and 22 matrix element

$$u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \sigma_1 = \frac{1}{\sqrt{2}}$$

$$u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \sigma_2 = \frac{1}{\sqrt{2}}$$

So we have just

$$\left|\psi\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|+;-\right\rangle - \left|-;+\right\rangle\right)$$ so it's an entangled state

Algebraic view of entangled state

von Neumann entropy

Given $\hat{\rho}$ we have

$$S(\rho) = -tr(\rho \ln \rho)$$

The negative sign here is to make sure S is positive, ln here is to make sure the law of large numbers.

If the density matrix is diagonalized:

$$\rho = diag[\rho_{ii}]$$

we have

$$S(\rho) = -\sum_i \rho_{ii} \ln \rho_{ii} = \begin{cases} 0, \text{ for a pure ensemble} \\ \ln N, \text{ for a maximally random ensemble} \end{cases}$$

Entanglement entropy

Here we explain why entanglement brings entropy.

Suppose we have two separated subsystems: A and B. Subsystem A has entropy S_A, and subsystem has entropy S_B. The whole entropy of A+B is the addition of $S_A + S_B$, this is because the probability of A is totally isolated from B, for example, if we measure the spin in A and obtain a randomly series, then we measure the spin in B will obtain randomly series. In other words, the correlation between A and B is 0.

Now we think what entanglement does for the subsystems. If the measure the spin in A, the result is a random series, and the result of B is also a random series, however, if we know the series of A, then we know the series of B. In other words, the correlation between A and B is no longer 0. Therefore, the whole entropy of A+B must be smaller than the addition of $S_A + S_B$

We can regard entanglement as hook which tie the two systems together.

Consider a pure ensemble and we can separate total system into two parts:

$$\rho_T = |\psi\rangle_{AB\,AB}\langle\psi|$$

And induced density matrix

$$\rho_A = tr_B(\rho_T) = \sum_j {}_B\langle v_j|\psi\rangle_{AB\,AB}\langle\psi|v_j\rangle_B$$

The entanglement entropy is the measure of von Neumann entropy on this subsystem:

$$S = -tr(\rho_A \ln(\rho_A))$$

Now we show that the entanglement entropy is directly related to Schmidt coefficients:

$$\rho_A = \sum_j {}_B\langle v_j|\psi\rangle_{AB\,AB}\langle\psi|v_j\rangle_B = \sum_i \sigma_i^2 |u_i\rangle_{A\,A}\langle u_i|$$

The above equation means that it is a mixed state. Such a density matrix is already diagonalized, and we have

$$S = -tr(\rho_A \ln(\rho_A)) = -\sum_i \sigma_i^2 \ln \sigma_i^2$$

in principle, once we work out the SVD, we will know entanglement entropy.

For separable state, suppose C1=1, other Ci are all zero:

$$S = 0$$

For entangled state

$$S = -\sum_i \sigma_i^2 \ln \sigma_i^2 > 0$$

There is an equality which gives the maximally entangled state:

$$\sigma_i^2 = \frac{1}{N}, S = -\sum_i \frac{1}{N} \ln \frac{1}{N} = \ln N$$

Note the entanglement entropy is symmetry for either subsystem:

$$\rho_B = \sum_j {}_A\langle u_j | \psi \rangle_{AB\,AB} \langle \psi | u_j \rangle_A = \sum_i \sigma_i^2 | v_i \rangle_{B\,B} \langle v_i |$$

$$S = -tr(\rho_B \ln(\rho_B)) = -\sum_i \sigma_i^2 \ln \sigma_i^2$$

So it doesn’t matter which subsystem we choose. The above definition is safe.

Now we take one example:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|+;-\rangle - |-;+\rangle)$$

The density matrix of subsystem A:

$$\rho_A = tr_B(\rho_T) = \frac{1}{2}{}_B\langle + |(|+;-\rangle - |-;+\rangle)(\langle -;+| - \langle +;-|)|+\rangle_B$$

$$+\frac{1}{2}{}_B\langle - |(|+;-\rangle - |-;+\rangle)(\langle -;+| - \langle +;-|)|-\rangle_B$$

$$= \frac{1}{2}(-|-\rangle)(-\langle -|) + \frac{1}{2}(|+\rangle)(\langle +|) = \frac{1}{2}(|-\rangle\langle -| + |+\rangle\langle +|)$$

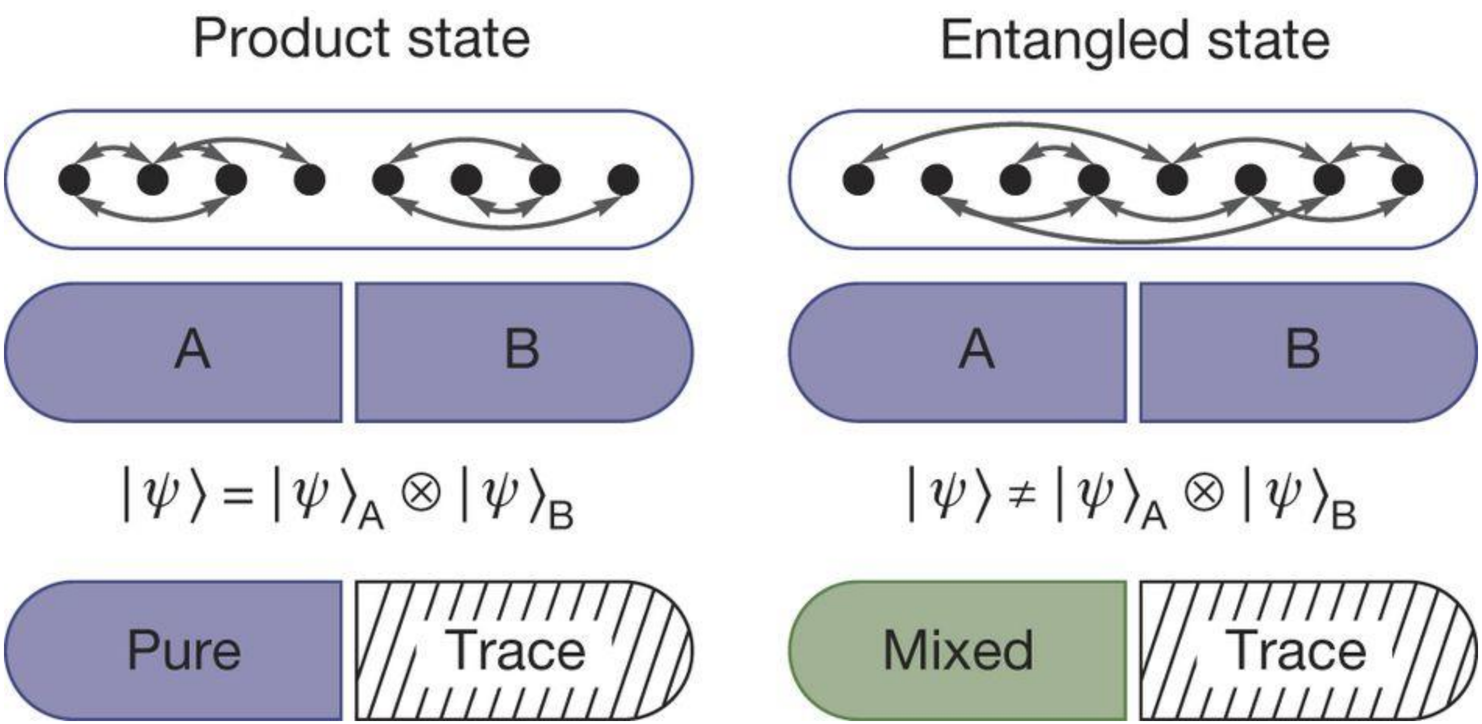
Such density matrix has already been diagonalized.

The entanglement entropy:

$$S = -tr(\rho_A \ln \rho_A) = -tr(\begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \ln\frac{1}{2} & \\ & \ln\frac{1}{2} \end{pmatrix}) = \ln 2$$

Here we quote a sentence to show the relationship between mixed state and entanglement:

"A mixed state reveals our ignorance about the quantum state of a system and can arise in various ways. To illustrate this we now consider two qubits in the entangled state above. where A plays the role of a system and B the role of an environment. Assume you cannot access qubit B at all. Mathematically, this corresponds to wiping out the information from B by performing trace over B. Ignoring part of a system leads in general to a mixed state. "



from Pachos, J. K. Introduction to topological quantum computation

What does $S = \ln 2$ mean?

Now we try to cut the entanglement off:

We found that there left two end points, one on each side. So it imply that

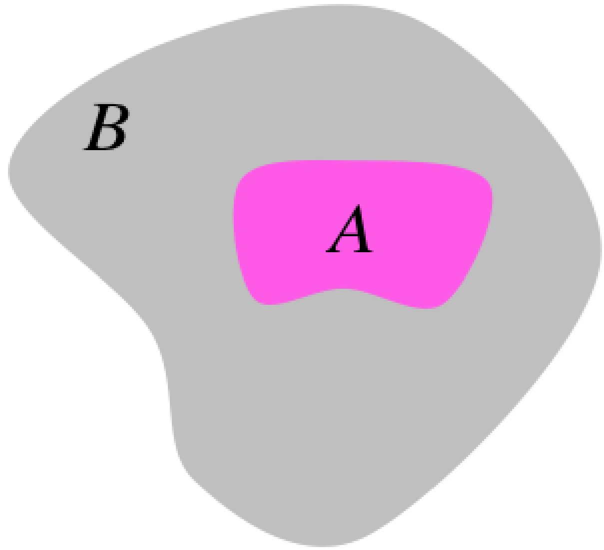
$$S \propto \partial$$

entanglement entropy measurement of boundary

This is called the area law.

Area law

The entanglement of gapped ground state satisfies the area law.



$$S \sim \partial A$$

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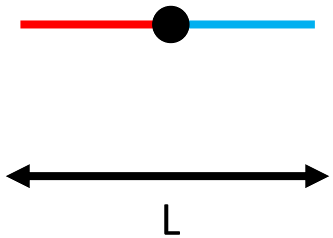
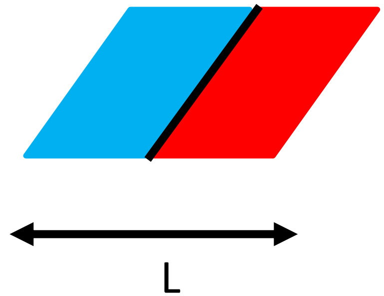
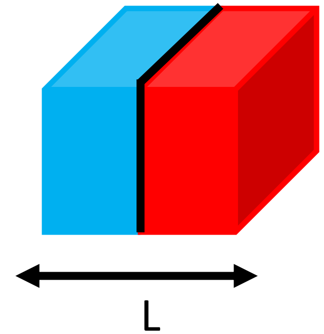
***Colloquium*: Area laws for the entanglement entropy**

Estimation of







$$S \propto \ln 2 \qquad S \propto L \qquad S \propto L^2$$

$$S = -tr(\rho_A \ln(\rho_A))$$

$$\rho_A \propto 2 \qquad \rho_A \propto 2^L \qquad \rho_A \propto 2^{L^2}$$

Zero entanglement

Product state

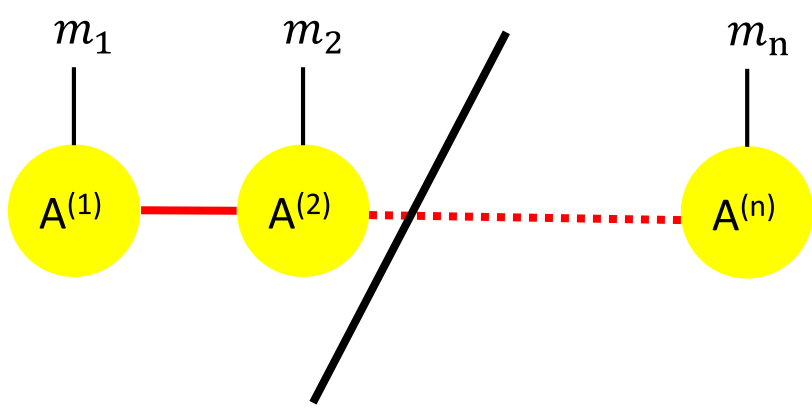
Mean field theory

Mean field theory

Matrix product state

High entanglement everywhere

Since matrix product state originate from SVD, when we cut at arbitrary auxiliary d.o.f., the entanglement is maximal.



According to the area low, the MPS is a natural representation of many-body ground state.

Entanglement Hamiltonian

Suppose we have made the Schmidt decomposition, but write the Schmidt coefficient into the exponential form:

$$\left|\psi\right\rangle_{AB} = \sum_{i=1}^{n} \frac{1}{\sqrt{Z}} \exp(-\frac{1}{2}\xi_i)\left|u_i\right\rangle_A \otimes \left|v_i\right\rangle_B$$

here ξ_i must be positive value and the normalized condition gives $Z = \sum_{i=1}^{n} \exp(-\xi_i)$

Define

$$p_{\xi_i} = [\frac{1}{\sqrt{Z}} \exp(-\frac{1}{2}\xi_i)]^2 = \frac{1}{Z} \exp(-\xi_i)$$

which is just the coefficient of the reduced density matrix

$$\rho_A = \sum_{i=1}^{n} p_{\xi_i} \left|u_i\right\rangle_{AA}\left\langle u_i\right|$$

And the entanglement entropy is

$$S = -tr(\rho_A \ln(\rho_A)) = -\sum_i p_{\xi_i} \ln p_{\xi_i}$$

The reason why we write the Schmidt coefficient in the above form is that we want analogy our system into a thermo system:

$$\rho = \frac{1}{Z}\exp(-\beta H)$$

If Hamiltonian is already diagonalized with {e1, e2, …}

$$H = \begin{pmatrix} \varepsilon_1 & & \\ & \cdots & \\ & & \varepsilon_n \end{pmatrix}$$

$$\rho = \frac{1}{Z}\exp(-\beta H) = \frac{1}{Z}\begin{pmatrix} \exp(-\beta\varepsilon_1) & & \\ & \cdots & \\ & & \exp(-\beta\varepsilon_n) \end{pmatrix}$$

$$\rho_i = \frac{1}{Z}\exp(-\beta\varepsilon_i)$$

And the entropy is given by

$$S = -\sum_i \rho_i \ln\rho_i$$

Duality map

Entanglement

?

?

$$\rho_A = \sum_{i=1}^{n} p_{\xi_i} |u_i\rangle_{AA}\langle u_i|$$

$$S = -\sum_i p_{\xi_i} \ln p_{\xi_i}$$

Thermodynamic system

$$H$$

$$\rho = \frac{1}{Z}\exp(-\beta H)$$

$$\rho = \frac{1}{Z}\begin{pmatrix} \exp(-\beta\varepsilon_1) & & \\ & \cdots & \\ & & \exp(-\beta\varepsilon_n) \end{pmatrix}$$

$$S = -\sum_i \rho_i \ln \rho_i$$

Suppose we can define an entanglement Hamiltonian $\hat{H}_A^e$ to fulfill the blank

Entanglement

$$\mathrm{H}_A^e$$

$$\rho_A = \frac{1}{Z_A^e}\exp(-\mathrm{H}_A^e)$$

$$\rho_A = \sum_{i=1}^{n} p_{\xi_i} |u_i\rangle_{AA}\langle u_i|$$

$$S = -\sum_i p_{\xi_i} \ln p_{\xi_i}$$

Thermodynamic system

$$H$$

$$\rho = \frac{1}{Z}\exp(-\beta H)$$

$$\rho = \frac{1}{Z}\begin{pmatrix} \exp(-\beta\varepsilon_1) & & \\ & \cdots & \\ & & \exp(-\beta\varepsilon_n) \end{pmatrix}$$

$$S = -\sum_i \rho_i \ln \rho_i$$

In other words, we can rewritten the reduced density matrix $\hat{\rho}_A$ in the form:

$$\rho_A = \frac{1}{Z_A^e} \exp(-\mathrm{H}_A^e)$$

so that the entanglement entropy is equivalent to the thermodynamic entropy of a system described by a Hermitian "Hamiltonian" H at "temperature" T=1 (β=1/kT=1)

Now we include "temperature" T:

$$\rho_A(\beta) = \exp(-\beta \mathrm{H})$$

And the corresponding entanglement entropy is given by:

$$S(\beta)$$

With initial condition S(1) is already known.

PRL 101, 010504 (2008)

Entanglement and manifold

Entanglement can distort the spacetime, change the metric.
Can entanglement change the genus of manifold, which makes the topological phase transition?

Geometric view of entangled state   Role of basis

Bell state

Bell states are a set of entangled basis vectors.

From the following Bell state:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|+;+\rangle + |-;-\rangle)$$

We can generate the following Bell states:

$$|\psi_i\rangle = (1 \otimes \sigma_i)|\psi_0\rangle$$

Write explicitly, we have

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|+;-\rangle + |-;+\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|+;-\rangle - |-;+\rangle)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|+;+\rangle |-;-\rangle)$$

Note that $|\psi_2\rangle$ is the AFM state

Orthogonality

$$\langle \psi_i | \psi_j \rangle = \langle \psi_0 | (1 \otimes \sigma_i)(1 \otimes \sigma_j) | \psi_0 \rangle$$

$$= \langle \psi_0 | 1 \otimes \sigma_i \sigma_j | \psi_0 \rangle$$

$$= \langle \psi_0 | 1 \otimes (1\delta_{ij} + i\varepsilon_{ijk}\sigma_k) | \psi_0 \rangle$$

$$= \delta_{ij} + i\varepsilon_{ijk} \langle \psi_0 | \psi_k \rangle = \delta_{ij}$$

Entanglement

$$|\psi_0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad |\psi_1\rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad |\psi_2\rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad |\psi_3\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Problem: after unitary operation, can a separable state becomes an entangled one?

Convex set

For vector spaces of dimensions different than two, the conditions for entanglement state is linked with the idea of convex set. To understand why convex enters, here we consider the case above.

Cut things into half