Basic Tensor

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不变

https://www.bilibili.com/video/av37599250/?p=17

From vector to tensor:

Vector and contravariant:

$$\vec{V} = \sum_i \vec{e}_i V^i$$

Following Einstein summation rule, we have

$$\vec{V} = \vec{e}_i V^i$$

Written in explicit:

$$\vec{V} = \left[\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \right] \begin{bmatrix} V^1 \\ V^2 \\ \cdots \\ V^n \end{bmatrix}$$

To obtain the following form,

$$\begin{bmatrix} V^1 \\ V^1 \\ \ldots \\ V^n \end{bmatrix} = B \begin{bmatrix} V^1 \\ V^1 \\ \ldots \\ V^n \end{bmatrix}$$

B for backward

Written in explicit,

$$V^i = B^i_j V^j$$

Note the order here

We define the transformation rule of basis

$$\left[\tilde{\vec{e}}_1, \tilde{\vec{e}}_2, \ldots, \tilde{\vec{e}}_n\right] = \left[\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\right] F$$

F for forward

Written in explicit,

$$\tilde{\vec{e}}_i = \vec{e}_j F_i^j$$

Note the order here

Since F and B are contra to each other, this is the reason of contravariant

The invariant of vector gives

$$\vec{V} = \left[\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\right] \begin{bmatrix} V^1 \\ V^1 \\ \ldots \\ V^n \end{bmatrix} = \left[\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\right] FB \begin{bmatrix} V^1 \\ V^1 \\ \ldots \\ V^n \end{bmatrix} = \left[\tilde{\vec{e}}_1, \tilde{\vec{e}}_2, \ldots, \tilde{\vec{e}}_n\right] \begin{bmatrix} V^1 \\ V^1 \\ \ldots \\ V^n \end{bmatrix}, FB = 1$$

Covector

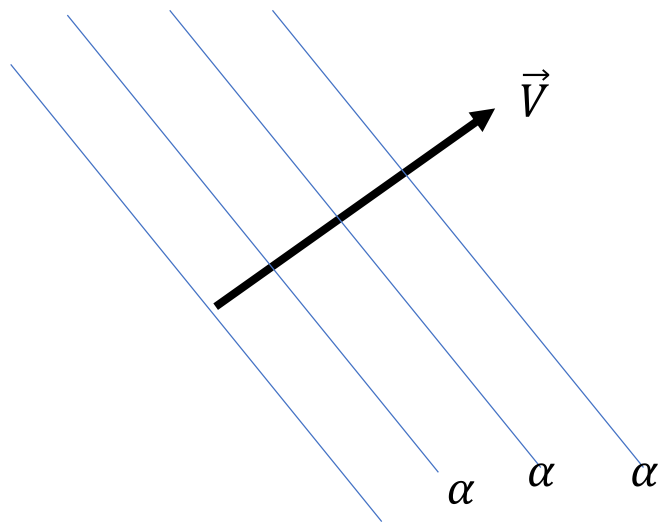
Map a vector to a scaler:

$$\alpha : V \to R$$

Geometric illustration:

Consider the fact that the inner product of two vectors is a scalar,

If we regard vector as arrow, then covector is stack, which is living in the mirror world (dual vector space)



This is because:

$$\alpha\overrightarrow{V} = aV_1 + bV_2$$

is a set of parallel lines (stack)

Define dual vector basis

$$\vec{\varepsilon}^{\,1}, \vec{\varepsilon}^{\,2}, \ldots, \vec{\varepsilon}^{\,n}$$

Which satisfies:

$$\vec{\varepsilon}^{\,i}\vec{e}_j = \delta^i_j$$

So we can express any covector as:

$$\overrightarrow{W} = \vec{\varepsilon}^{\,i} W_i$$

It’s easy to show

$$\vec{\varepsilon}^{\,i} = \sum_j \vec{\varepsilon}^{\,j} B^j_i$$

$$W_i = \sum_j W_j F_i^{\,j}$$

Summary here:

$$\vec{\tilde{e}}_i = \sum_j \vec{e}_j F_i^{\,j}$$

covarient

$$\vec{\tilde{\varepsilon}}^{\,i} = \sum_j B^i_j \vec{\varepsilon}^{\,j}$$

contravarient

把指标放在下标表示做transformation的时候使用F，即表示协变covariant

把指标放在上标表示做transformation的时候使用B，即表示逆变cotravariant

$$\tilde{V}^i = \sum_j B^i_j V^j$$

contravarient

$$\tilde{W}_i = \sum_j W_j F_i^{\,j}$$

covarient

把指标放在上标表示做transformation的时候使用B，即表示逆变cotravariant

把指标放在下标表示做transformation的时候使用F，即表示协变covariant

Matric tensor

Now we define length.

First, we define metric tensor, so we can separate the dependence of basis apart:

$$g_{ij} = \vec{e}_i \vec{e}_j$$

Written in explicit:

$$g = \begin{bmatrix} \vec{e}_1 \vec{e}_1 & \vec{e}_1 \vec{e}_2 & \dots & \vec{e}_1 \vec{e}_n \\ \vec{e}_2 \vec{e}_1 & \vec{e}_2 \vec{e}_2 & \dots & \vec{e}_2 \vec{e}_n \\ \dots & & & \\ \vec{e}_m \vec{e}_1 & \vec{e}_m \vec{e}_2 & \dots & \vec{e}_m \vec{e}_n \end{bmatrix}$$

And the length can be written as:

$$\left\| \vec{V} \right\| = \sum_{ij} V^i V^j \vec{e}_i \vec{e}_j = \sum_{ij} V^i V^j g_{ij}$$

And the metric tensor follows the transformation rule:

$$\tilde{g}_{ij} = \vec{\tilde{e}}_i \vec{\tilde{e}}_j = \vec{e}_k F_i^k \vec{e}_m F_j^m = \vec{e}_k \vec{e}_m F_i^k F_j^m$$

Metric tensor as a map:

$$g : \mathrm{V} \times \mathrm{V} \rightarrow R$$

Metric tensor maps two vectors into a scalar:

$$g(\vec{v}, \vec{w}) = v^i w^j g_{ij}$$

Multilinear maps

Let V1, V2, …, VN be vector spaces over the field R, a map

$$\mathrm{T}: \mathrm{V}_1 \times \mathrm{V}_2 \times \ldots \times \mathrm{V}_N \rightarrow R$$

$$\mathrm{T}(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n) = \mathrm{a}$$

is said to be multilinear if it is linear in each argument separately:

$$\mathrm{T}(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{av}_i + \mathrm{bv}_i^{'}, \ldots, \mathrm{v}_n)$$

$$= a\,\mathrm{T}(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_i, \ldots, \mathrm{v}_n) + b\,\mathrm{T}(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_i^{'}, \ldots, \mathrm{v}_n)$$

Multilinear maps can be added and multiplied by scalars in the usual fashion

$$(\mathrm{aT} + \mathrm{bS})(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n)$$

$$= \mathrm{aT}(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n) + \mathrm{bS}(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n)$$

Therefore, multilinear map forms a vector space, denoted by

$$\mathrm{V}_1^* \times \mathrm{V}_2^* \times \ldots \times \mathrm{V}_N^*$$

我们总是需要covector去吃掉vector

called the tensor product of the dual spaces V*1, V*2, …, V*N

Tensor spaces of type (r, s)

Let V be a vector space of dimension n over the field R. Setting

$$V_1 = V_2 = \ldots = V_r = V^*, V_{r+1} = V_{r+2} = \ldots = V_{r+s} = V$$

We refer to any multilinear map

$$T : \underbrace{V^* \times V^* \times \ldots \times V^*}_{\substack{r \\ \text{contravariant degree}}} \times \underbrace{V \times \ldots \times V}_{\substack{s \\ \text{covariant degree}}} \to R$$

as a tensor of type (r, s) on V.

因为T可以吃掉r个covector以及s个vector,所以T的基可以写成

$$V^{(r,s)} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s}$$

Basis representation

$$\mathrm{V} = \{\vec{e}_1, \ldots, \vec{e}_n\}$$

$$V^* = \{\vec{\varepsilon}^{\,1}, \ldots, \vec{\varepsilon}^{\,n}\}$$

where the dual basis is defined by

$$\vec{\varepsilon}^{\,i}(\vec{e}_j) = \delta^i_j$$

And the basis for (r, s)-tensor is

$$V^{(\mathrm{r,s})} = \{\vec{e}_{i_1} \otimes \vec{e}_{i_2} \otimes \ldots \otimes \vec{e}_{i_r} \otimes \vec{\varepsilon}^{\,j_1} \otimes \ldots \otimes \vec{\varepsilon}^{\,j_s}\}$$

And any tensor can be expanded as

$$T^{(\mathrm{r,s})} = T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \cdots j_s} \vec{e}_{i_1} \otimes \vec{e}_{i_2} \otimes \ldots \otimes \vec{e}_{i_r} \otimes \vec{\varepsilon}^{\,j_1} \otimes \ldots \otimes \vec{\varepsilon}^{\,j_s}$$

And the coefficient is:

$$T^{i_1 i_2 \ldots i_r}_{j_1 j_2 \cdots j_s} = T^{(\mathrm{r,s})}(\vec{\varepsilon}^{\,j_1}, \ldots, \vec{\varepsilon}^{\,j_r}, \vec{e}_{i_1}, \ldots, \vec{e}_{i_r})$$

Example: (0, 2)-tensor

$$V^{(0,2)} = \mathrm{V}^* \otimes \mathrm{V}^* \qquad \omega \otimes \rho$$

And u, v are two vector, so we have

$$\omega \otimes \rho(\mathrm{u}, \mathrm{v}) = \omega(\mathrm{u})\rho(\mathrm{v})$$

Into basis

$$\varepsilon^i \otimes \varepsilon^j$$

And we can represent a (0, 2)-tensor$\omega \otimes \rho$ as

$$\omega \otimes \rho = \omega_i \varepsilon^i \otimes \rho_j \varepsilon^j = \omega_i \rho_j \varepsilon^i \otimes \varepsilon^j$$

$$T = T_{ij} \varepsilon^i \otimes \varepsilon^j$$

So we have

$$T(\mathrm{u}, \mathrm{v}) = T_{ij} u^i v^j$$

We can express T as a matrix

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$$

For example, the metric tensor is a (2,0)-type tensor:

$$g=\{g|g_{ij}=\vec{e}_i\vec{e}_j\}$$

So it gives the expected length:

$$g(\vec{v},\vec{w}) = g_{ij}v^i w^j$$

For example, the metric tensor is a (0,2)-type tensor:

$$G=\{G|G^{ij}=\vec{\varepsilon}^i\vec{\varepsilon}^j\}$$

For example, the linear map is a (1,1)-type tensor:

$$L = L^i_j \vec{e}_i \vec{\varepsilon}^j$$

So the linear map gives the expected transformation rule:

$$L(\vec{v}) = L^i_j \vec{e}_i \vec{\varepsilon}^j \vec{v} = L^i_j \vec{e}_i v^j = w^i \vec{e}_i = \vec{w}$$

它使用一个covector收缩掉一个vector，剩下一个vector

Written in matrix form, we have

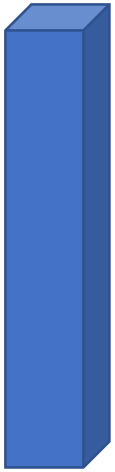
$$L = \begin{bmatrix} L^1_1 & L^1_2 & \cdots & L^1_m \\ L^2_1 & L^2_2 & \cdots & L^2_m \\ \cdots & \cdots & \cdots & \cdots \\ L^n_1 & L^n_2 & \cdots & L^n_m \end{bmatrix}$$

How to picture a tensor with cubic?

Tensor $\Longleftrightarrow$ Number array $\Longleftrightarrow$ Cubic

(1, 0)

$$\begin{bmatrix} V^1 \\ V^2 \\ \cdots \\ V^n \end{bmatrix}$$

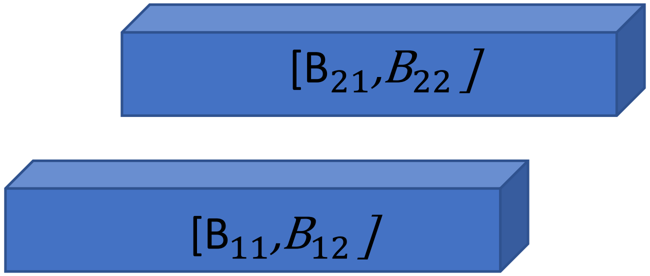


(0, 1)

$$\begin{bmatrix} W_1 & W_2 & \cdots & W_n \end{bmatrix}$$

(0, 2)

$$B=[[B_{11},B_{12}],[B_{21},B_{22}]]$$



(1, 1)

$$\begin{bmatrix} L_1^1 & L_2^1 & \cdots & L_m^1 \\ L_1^2 & L_2^2 & \cdots & L_m^2 \\ \cdots & \cdots & \cdots & \cdots \\ L_1^n & L_2^n & \cdots & L_m^n \end{bmatrix}$$

一个立方体最多可视化三阶张量，更高阶张量无法可视化。

Tensor equation

如果我们把所有的方程都写成张量方程的形式，我们知道不管在什么样的坐标系下，方程的形式都是一样的。这个就是张量方程的好处。

Tensor and quantum mechanics

量子力学里面的算符都是线性映射，都可以用张量进行表示。其中Dirac标记里面的bra和ket分别对应对偶空间和空间，所以量子力学是具有张量结构的。
因为单体问题是（0+1）维的问题，所以我们只需要矩阵就可以描述问题;
但是多体问题是（3+1）维的问题，所以矩阵已经不足够描述问题了，必须启用张量。