Integrated 1D system

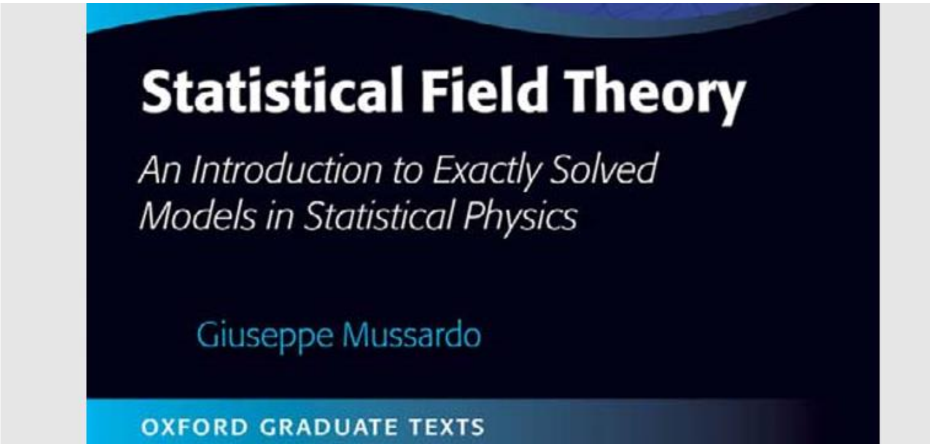
It provides:
1, a hydrogen-like unit understanding about physical system;
2, a analytic benchmark for numerical simulations;

Here we work out the following 1D systems which will be frequently confronted.

Betha ansatz about 1D quantum spin model

Kiteav model

- Toric-code model: only gauge field
- Transverse field Ising model: only matter field
- Local invariant Z2: matter field + gauge field
- Higgs theory


Statistical Field Theory
An Introduction to Exactly Solved
Models in Statistical Physics
Giuseppe Mussardo
OXFORD GRADUATE TEXTS

1, Toric code model and Z_2 gauge theory

Consider the string has two d.o.f.,

$$\{|\uparrow\rangle, |\downarrow\rangle\}$$

Here we use two colors to label the eigenstates of τ^z :

$|\uparrow\rangle$

$|\downarrow\rangle$

https://www.bilibili.com/video/av39845440/?spm_id_from=333.788.videocard.4

The gauge field can flip them, through the three Pauli matrix, we only need two, here we choose $\{\tau^z, \tau^x\}$

The basic math is

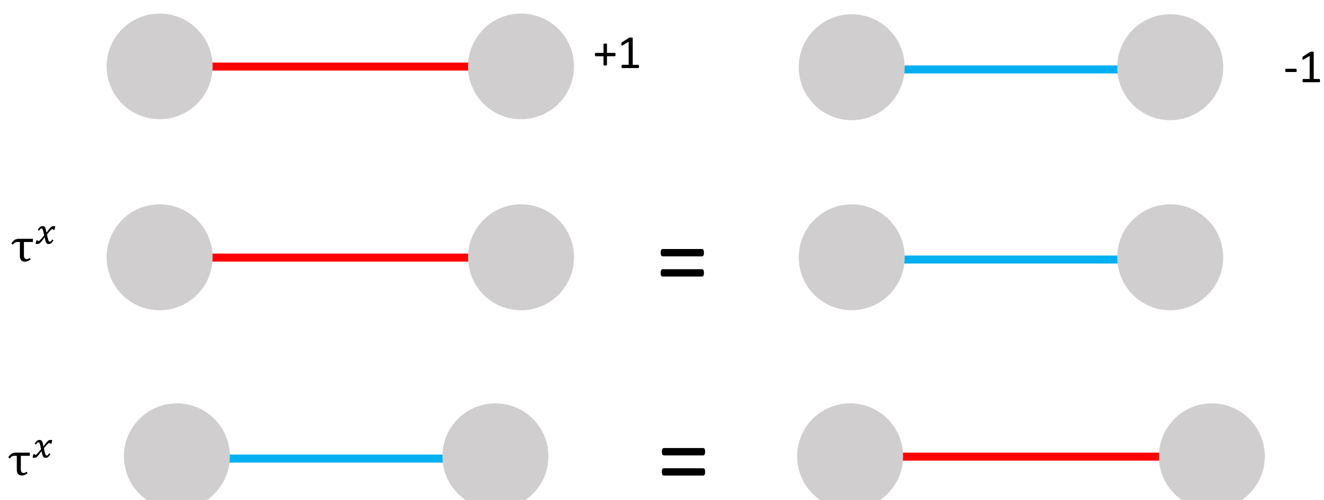
$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\tau^z\begin{pmatrix}1\\0\end{pmatrix} = +1\begin{pmatrix}1\\0\end{pmatrix}, \sigma^z\begin{pmatrix}0\\1\end{pmatrix} = -1\begin{pmatrix}1\\0\end{pmatrix}$$

mass term, Z_2 term

$$\tau^x\begin{pmatrix}1\\0\end{pmatrix} = +1\begin{pmatrix}0\\1\end{pmatrix}, \sigma^x\begin{pmatrix}0\\1\end{pmatrix} = +1\begin{pmatrix}1\\0\end{pmatrix}$$

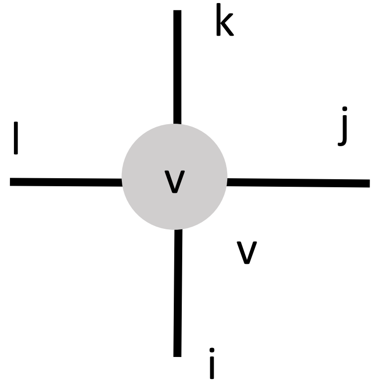
kinetic term,
annihilate one string
and create the opposite string



Now we consider the following operator:

1, mass term

Here we consider a vertex operator which is local



Here we consider a vertex term

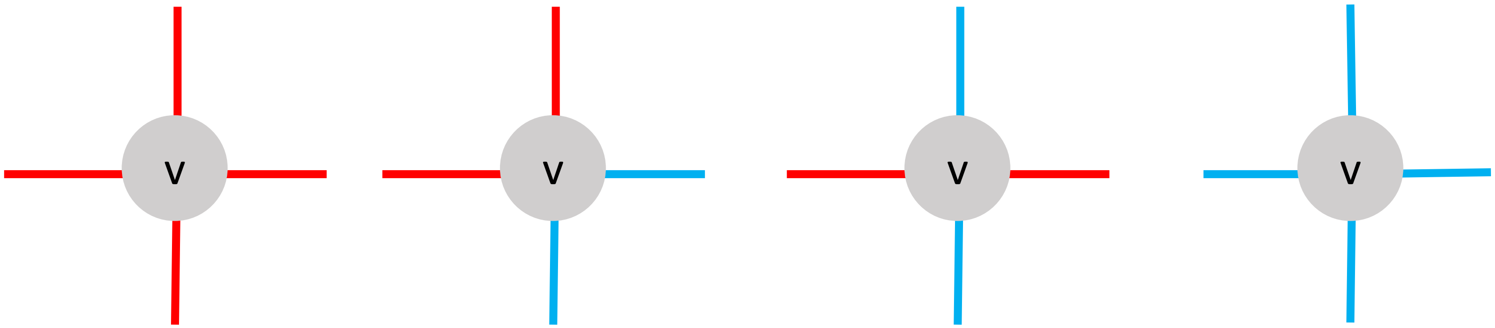
$$\underline{M_{glue} = 4\sigma a} \Rightarrow -A_v = -\prod_{i \in v} \tau_i^z = -\tau_i^z \tau_j^z \tau_k^z \tau_l^z$$

term from lattice QCD

Here we use i belong to v to denote the strings touching the vertex v

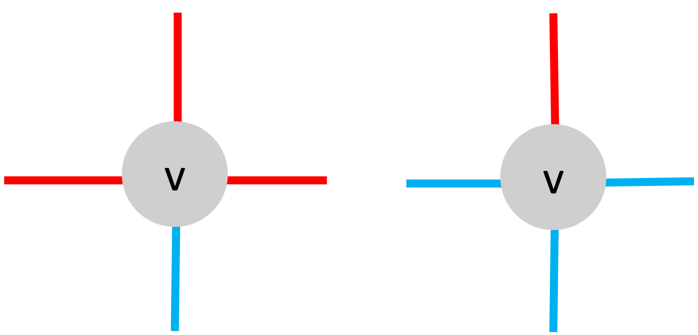
Highly degenerated G.S.

Since changing the color of even number of string will give the same energy, we find that the G.S. is highly degenerated for a site:



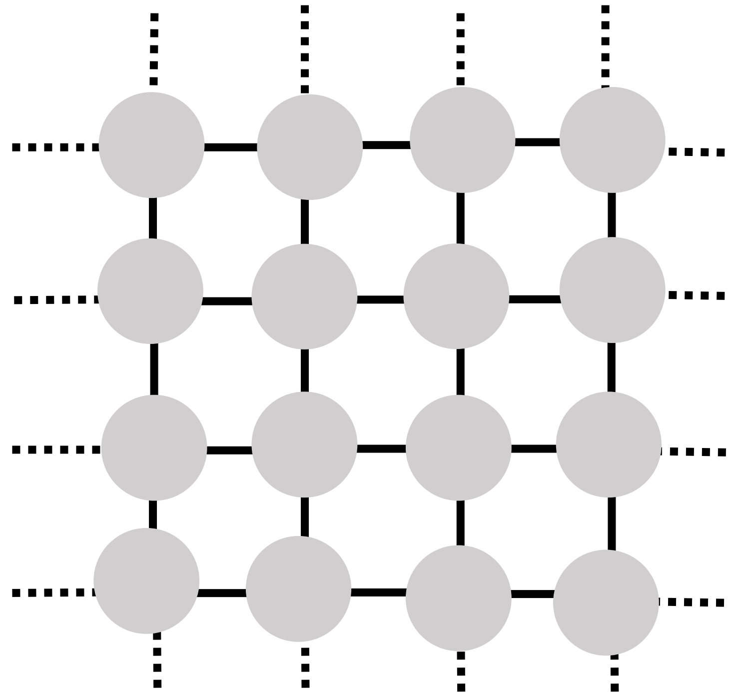
Even blue, even red

And the exited state is



Odd blue, odd red

Now we consider a lattice in flat spacetime



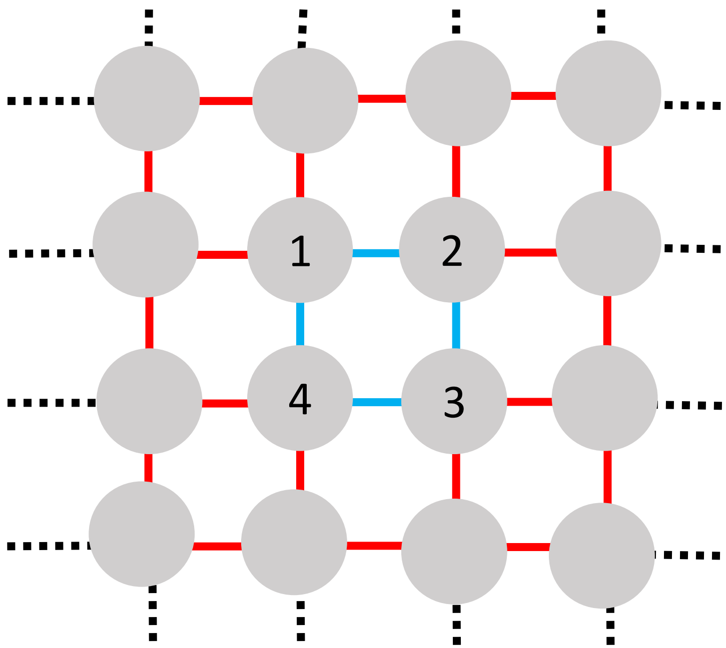
The mass term adding all the vertex term up:

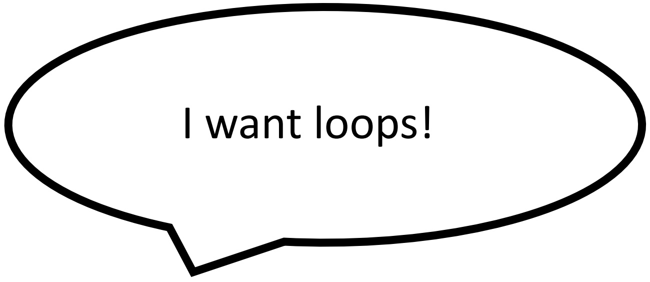
$$H_{mass} = -\sum_{v} A_v$$

Commutation between different vertex term:

$$[A_v, A_{v'}]_- = [\prod_{i \in v} \sigma_i^z, \prod_{i \in v'} \sigma_i^z]_- = [\sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z, \sigma_{i'}^z \sigma_{j'}^z \sigma_{k'}^z \sigma_{l'}^z]_- = 0$$

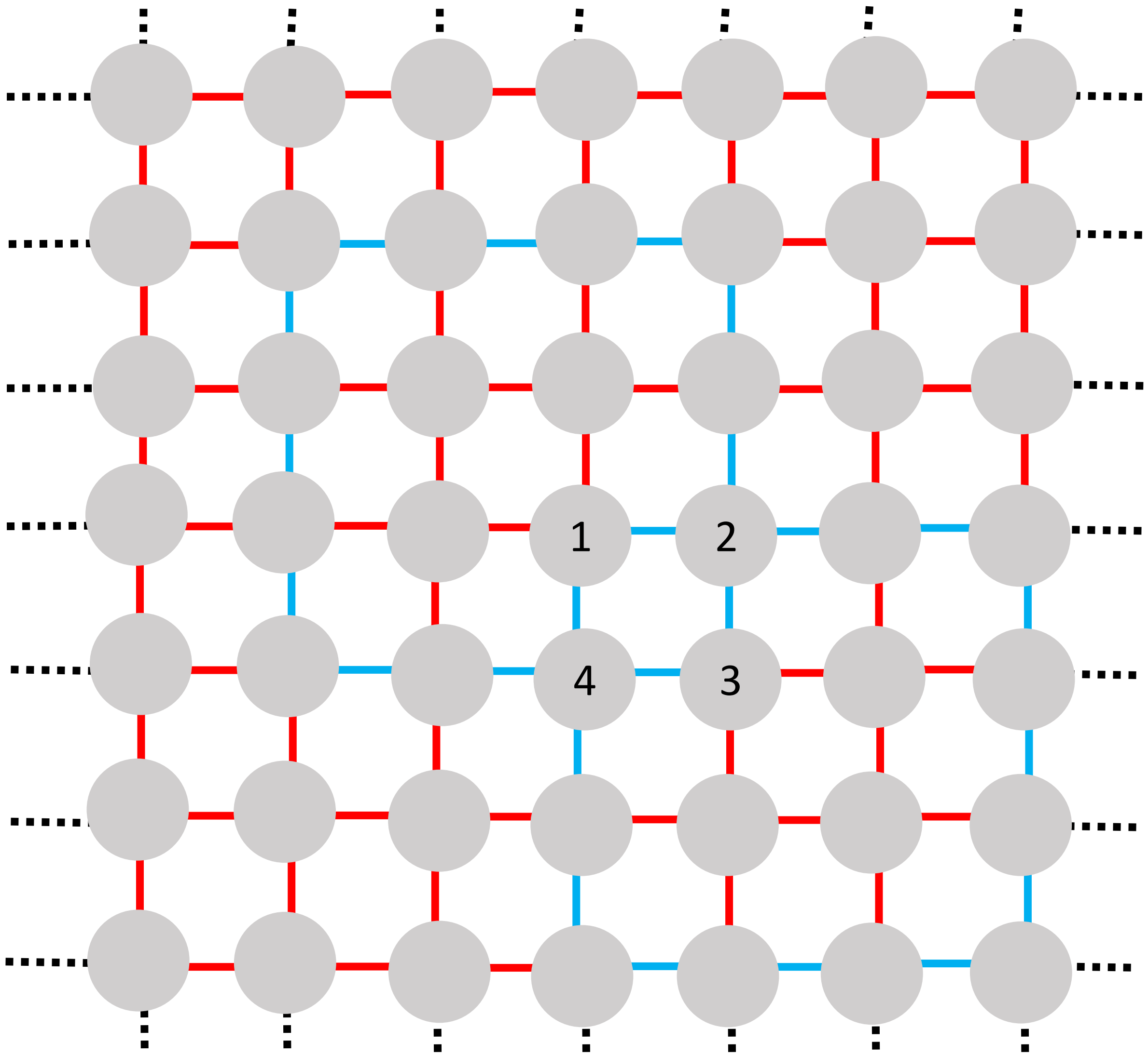
Since each vertex term is commute, the G.S. of mass term should have “even red and even blue” at each vertex, in other words, the G.S. is formed of loops:





$$H_{mass} = -\sum_v A_v$$

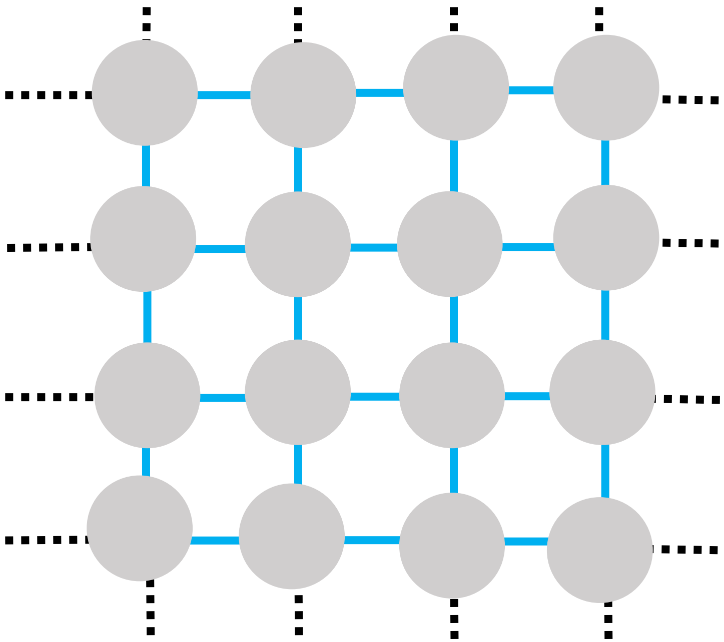
Continuous deformation of the loops:



We can turn the plaquette formed by 1234 from blue to red without changing the G.S., so the loop doesn't self-cross. i.e. the loop is simple loop.

Each loop state is topological equivalent

We can continue deform the G.S. until all the sites becomes blue:

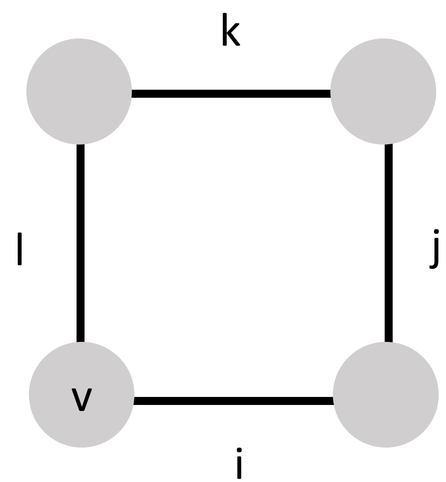


Each step is equivalent, so the all red state is equivalent to all blue state.
Therefore, each loop state is topological equivalent and a linear combination is also the ground state.

Now we consider the following operator:

2, kinetic term

The world of mass term is boring, now we add some term that can create the map between different loop state.



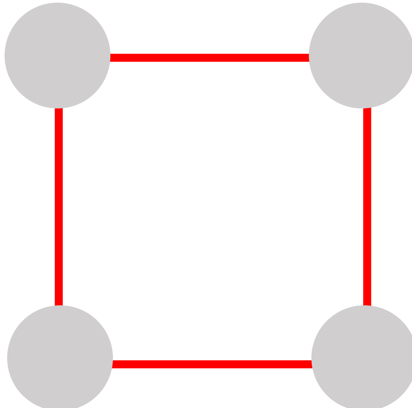
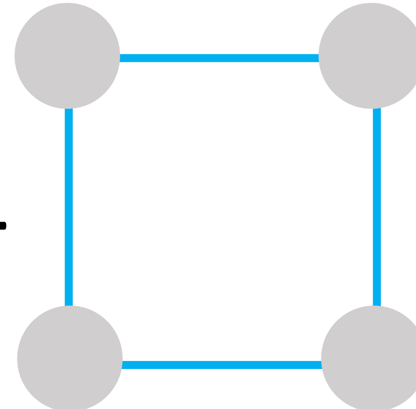
Here we consider a plaquette term

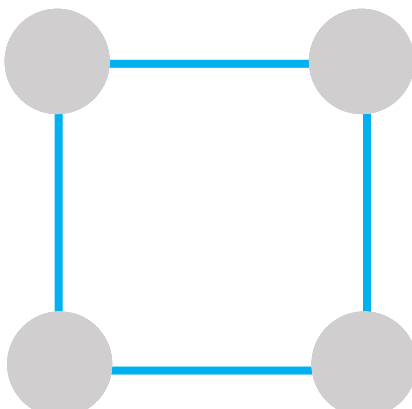
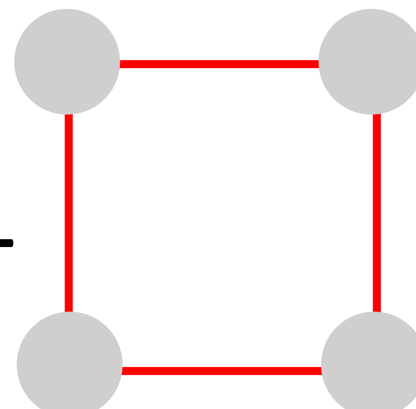
$$-B_p = -\prod_{i \in p} \tau_i^x = -\tau_i^x \tau_j^x \tau_k^x \tau_l^x$$

Here we use i belong to v to denote the strings surrounding the plaquette p.

Note that this is a local Z2 operator!

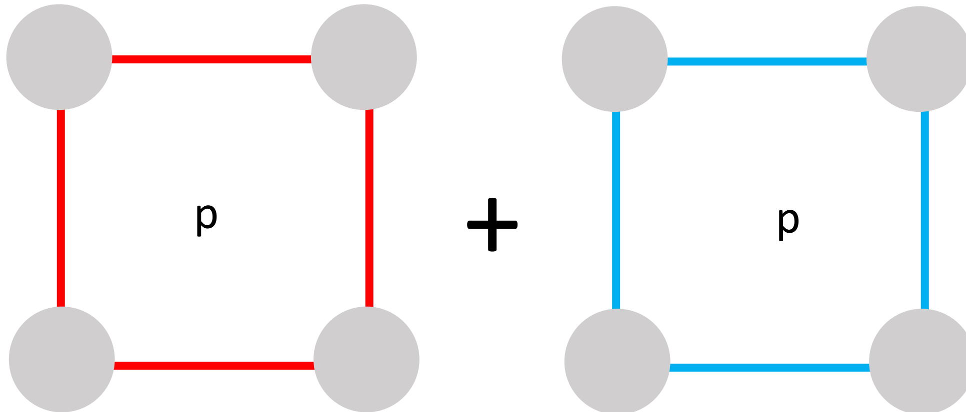
Each plaquette term map one loop state into another loop state

$-\hat{B}_p$  $= -$ 

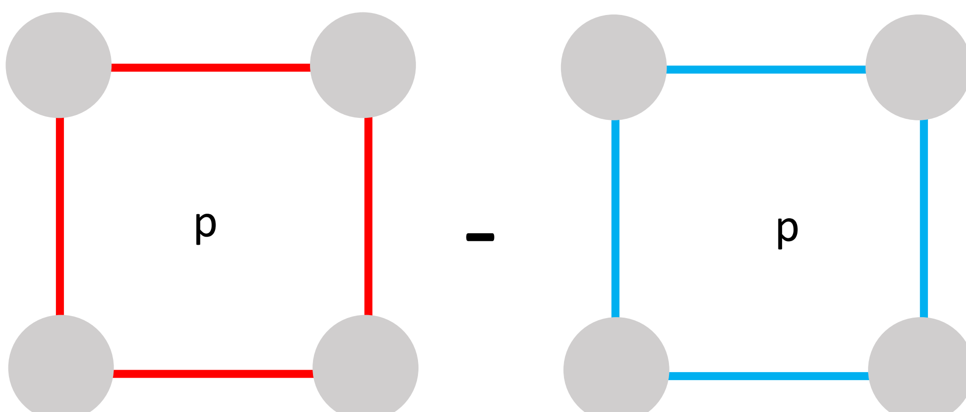
$-\hat{B}_p$  $= -$ 

The local kinetic term map a red plaquette into a blue one
and a blue plaquette to a red one.

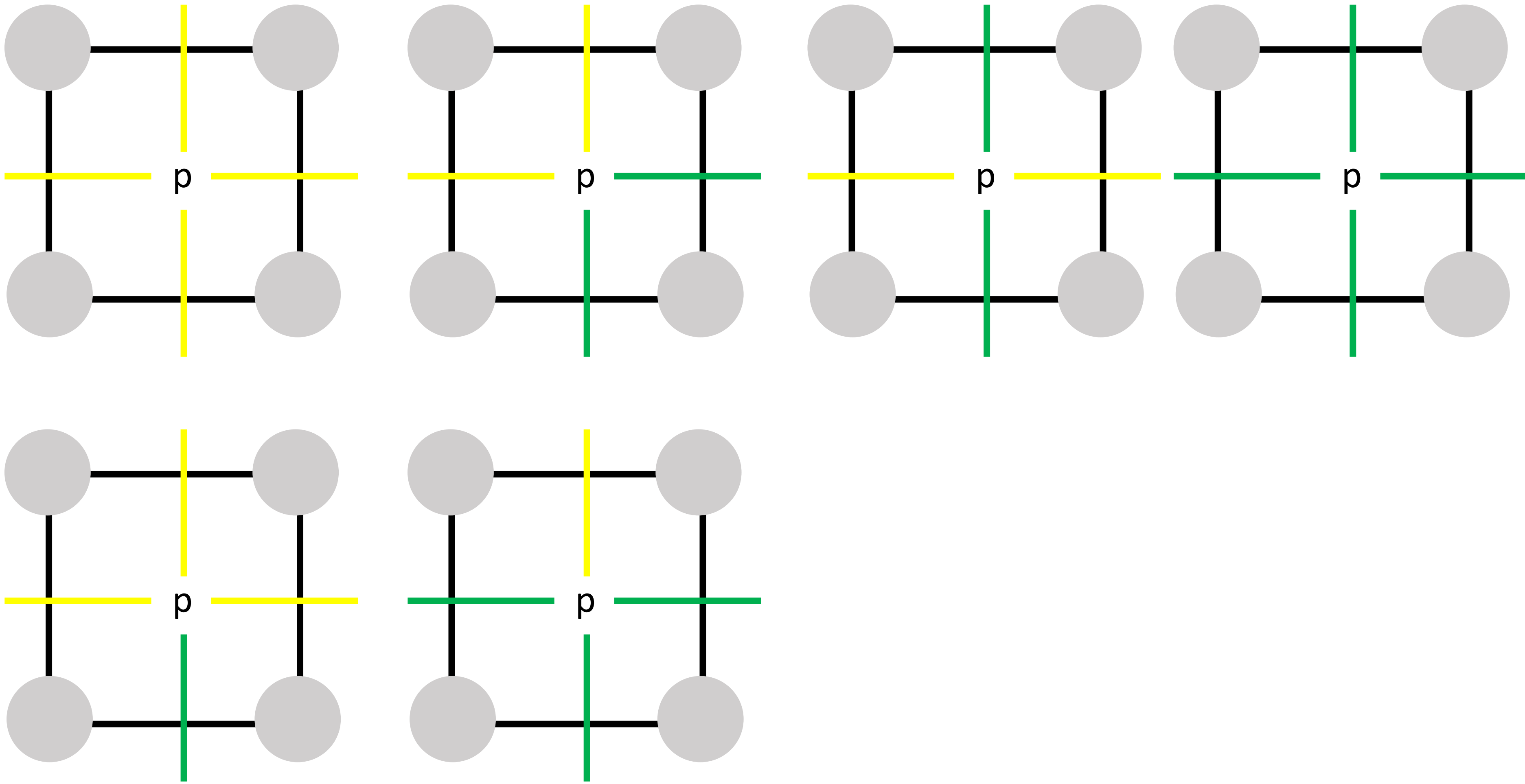
So we have the G.S. of plaquette term



And the excited state:

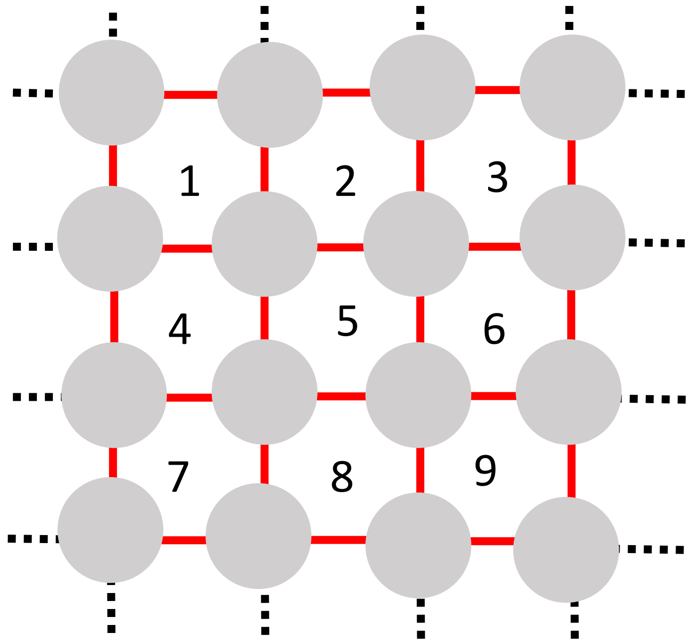


For convenience, here we use another way of representation



Actually, here we are in the basis of τ^x.

Now we consider a lattice in flat spacetime



The Hamiltonian is adding all the local operators up:

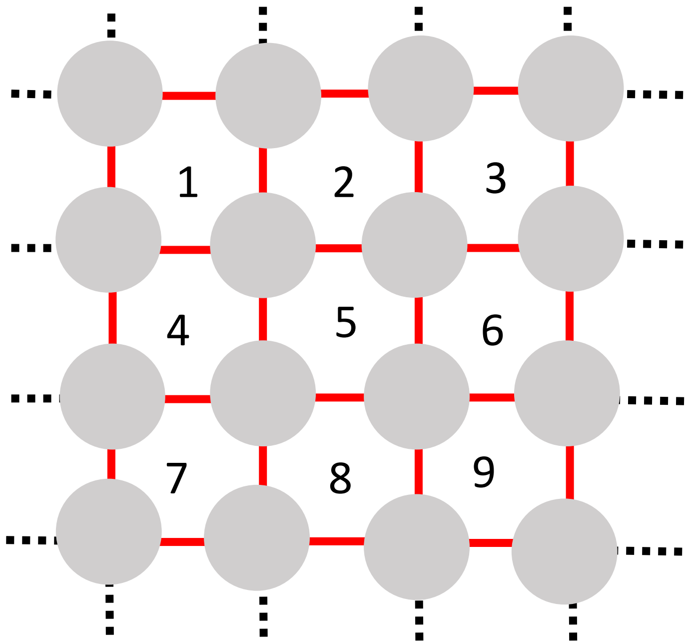
$$H_{kinetic} = -\sum_{p} B_p$$

Commutation between different plaquette:

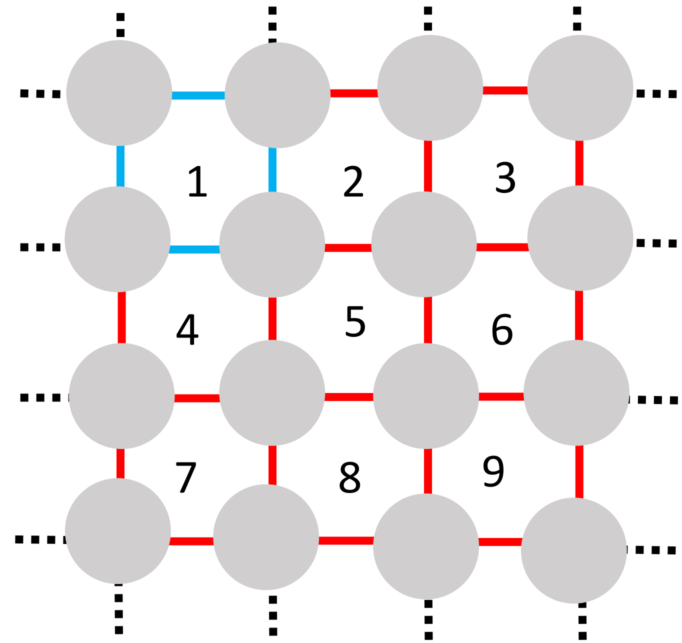
$$[B_p, B_{p'}]_- = [\prod_{i \in p} \sigma_i^x, \prod_{i \in p'} \sigma_i^x]_- = [\sigma_i^x \sigma_j^x \sigma_k^x \sigma_l^x, \sigma_{i'}^x \sigma_{j'}^x \sigma_{k'}^x \sigma_{l'}^x]_- = 0$$

First we think about the effect of kinetic term on the no loop G.S.:

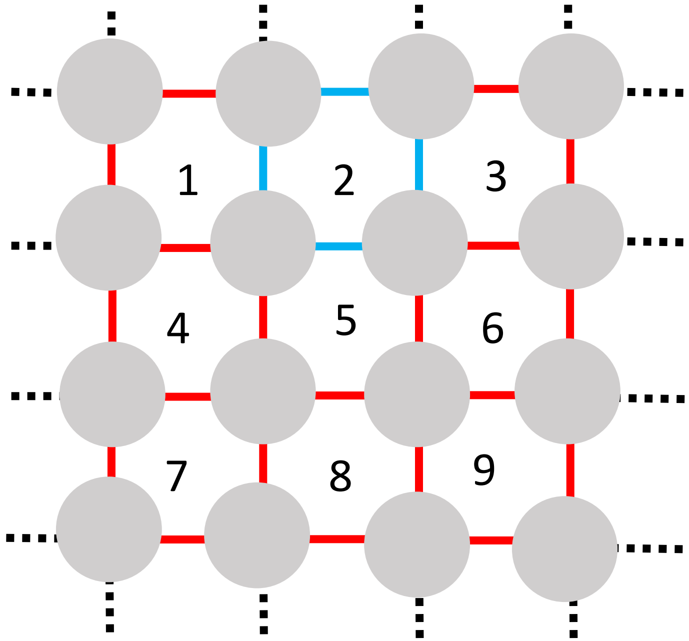
$$-\sum_{p} B_p$$



$=$



$+$

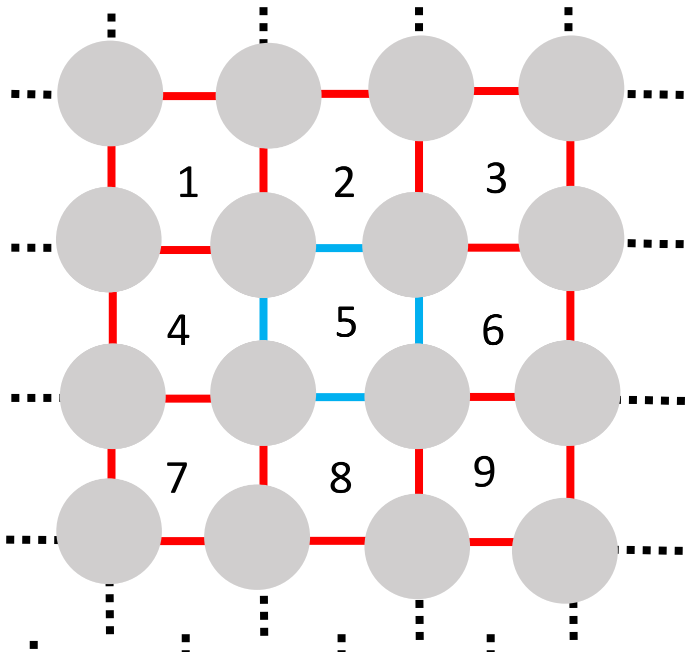


$+\ldots$

The one-loop state is constructed!

Second we think about the effect of kinetic term on the one loop G.S.:

$$-\sum_p B_p$$



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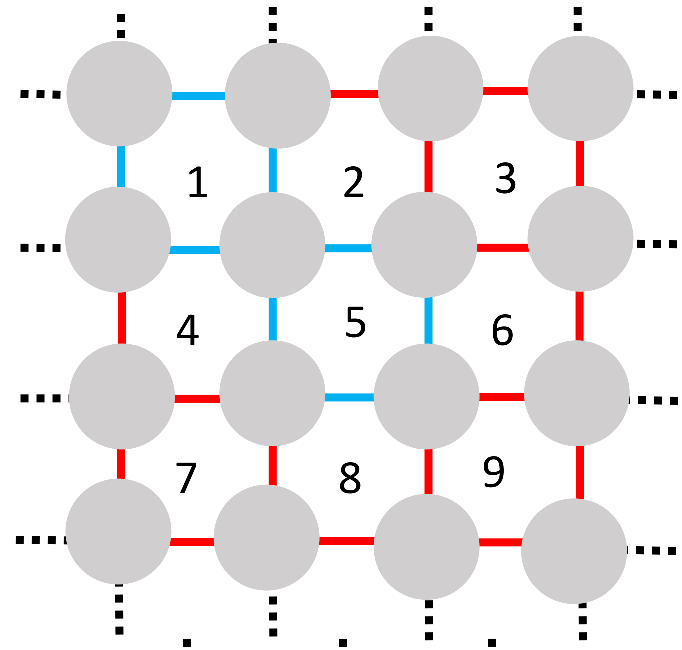

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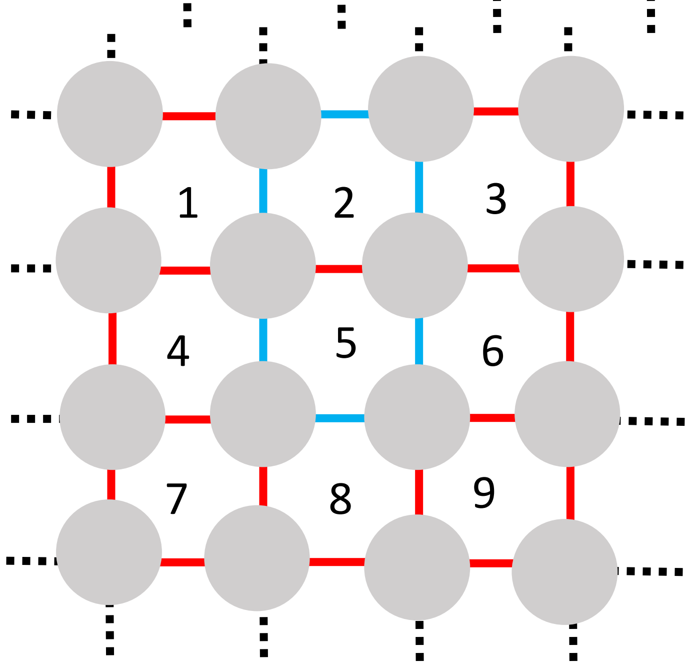

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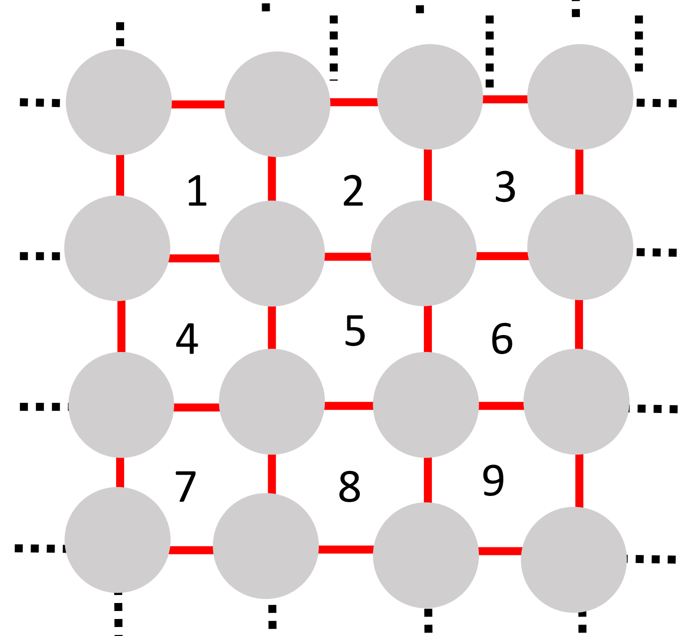
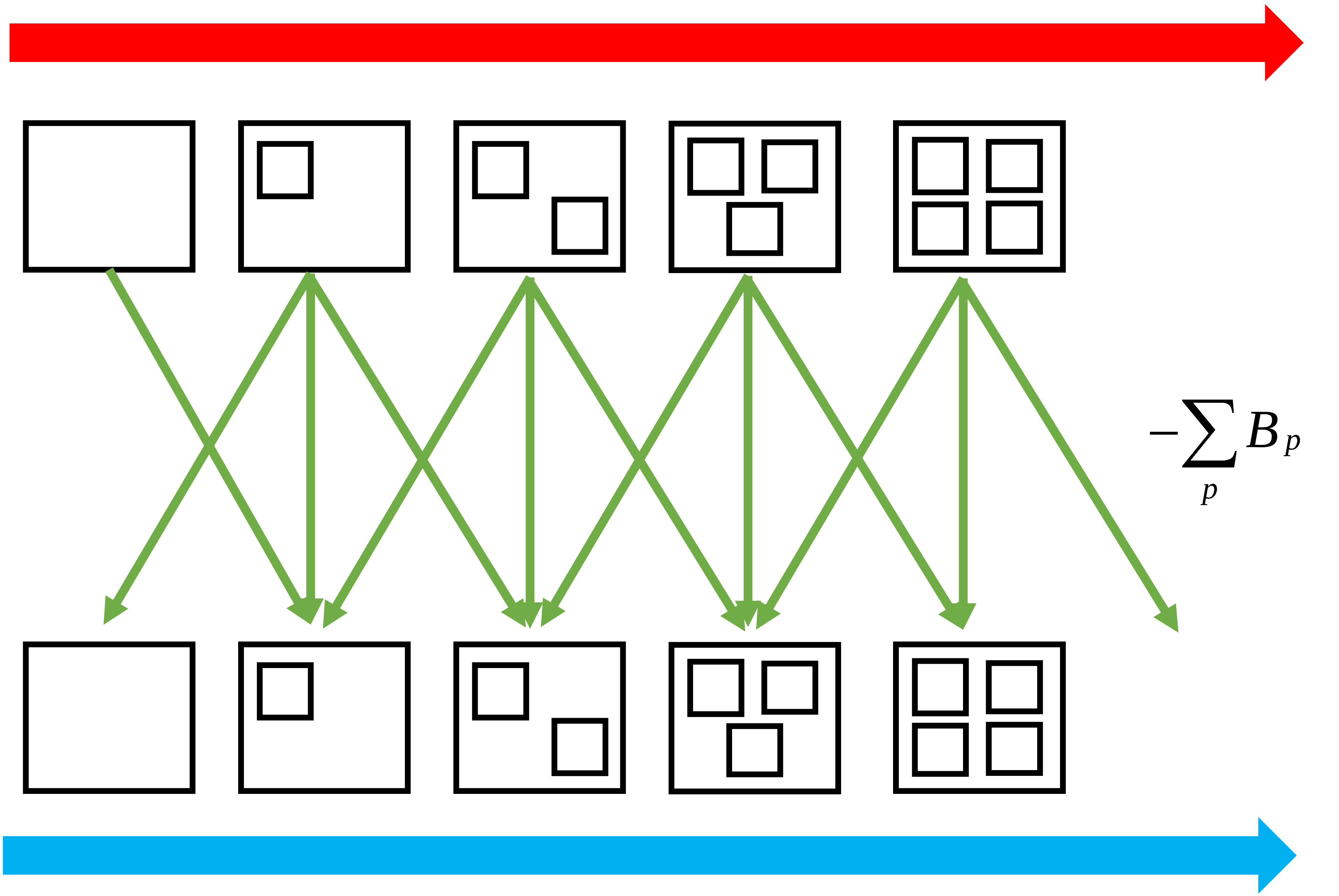


+...

The two-loop state is constructed! It also construct one-loop state and no loop state.

Picture



Therefore, the G.S. of kinetic term is the superposition of all loop states:

$$\sum_N |N - loop\rangle$$

To show this, we have:

$$H_{kinetic} \sum_N |N - loop\rangle = \sum_N |N - loop\rangle$$

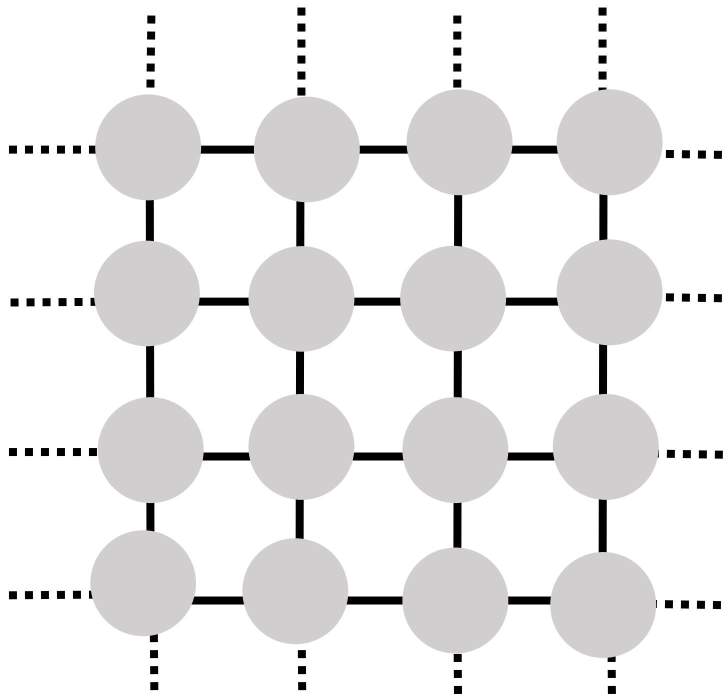
Since this is just a one-to-one correspondence.

I want loops to condensate!

$$H_{kinetic} = -\sum_p B_p$$

Physics on the lattice

Now we consider the lattice on flat plane



The Hamiltonian is adding the two terms up:

$$H = -\sum_{v} A_v - \sum_{p} B_p$$

Basic algebra:

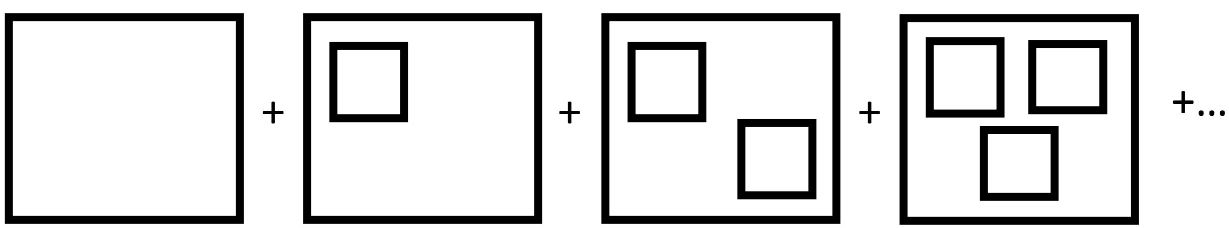
Each term are commute with each other:

$$[A_v, A_{v'}]_- = [B_p, B_{p'}]_- = [A_v, B_p]_-$$

The only terms that you might suspect not to commute are a plaquette term and a vertex term that share some bonds. Since a plaquette and a vertex share two bonds, they must commute with each other.

G.S.: condensate of loop state

According to above discussion, the G.S. wave-function must be the sum of all possible (i.e. ones that can be reached by applying the plaquette terms) loop configurations with equal weight. In other words, the G.S. is in a nonlocal state, or in a such a mess!

$\left|G.S.\right\rangle$ = 

Fractional excitation

First we consider the effect of τ^x:

$$[\tau_i^x, \tau_i^z \tau_j^z \tau_k^z \tau_l^z]_+ = 0 \Rightarrow [\tau_i^x, A_v]_+ = 0$$

Therefore, τ^x is anticommute with the vertex term where it lives in.

$$[\tau_i^x, \tau_i^x \tau_j^x \tau_k^x \tau_l^x]_- = 0 \Rightarrow [\tau_i^x, B_p]_- = 0$$

Therefore, τ^x is commute with the plaquette term where it lives in.

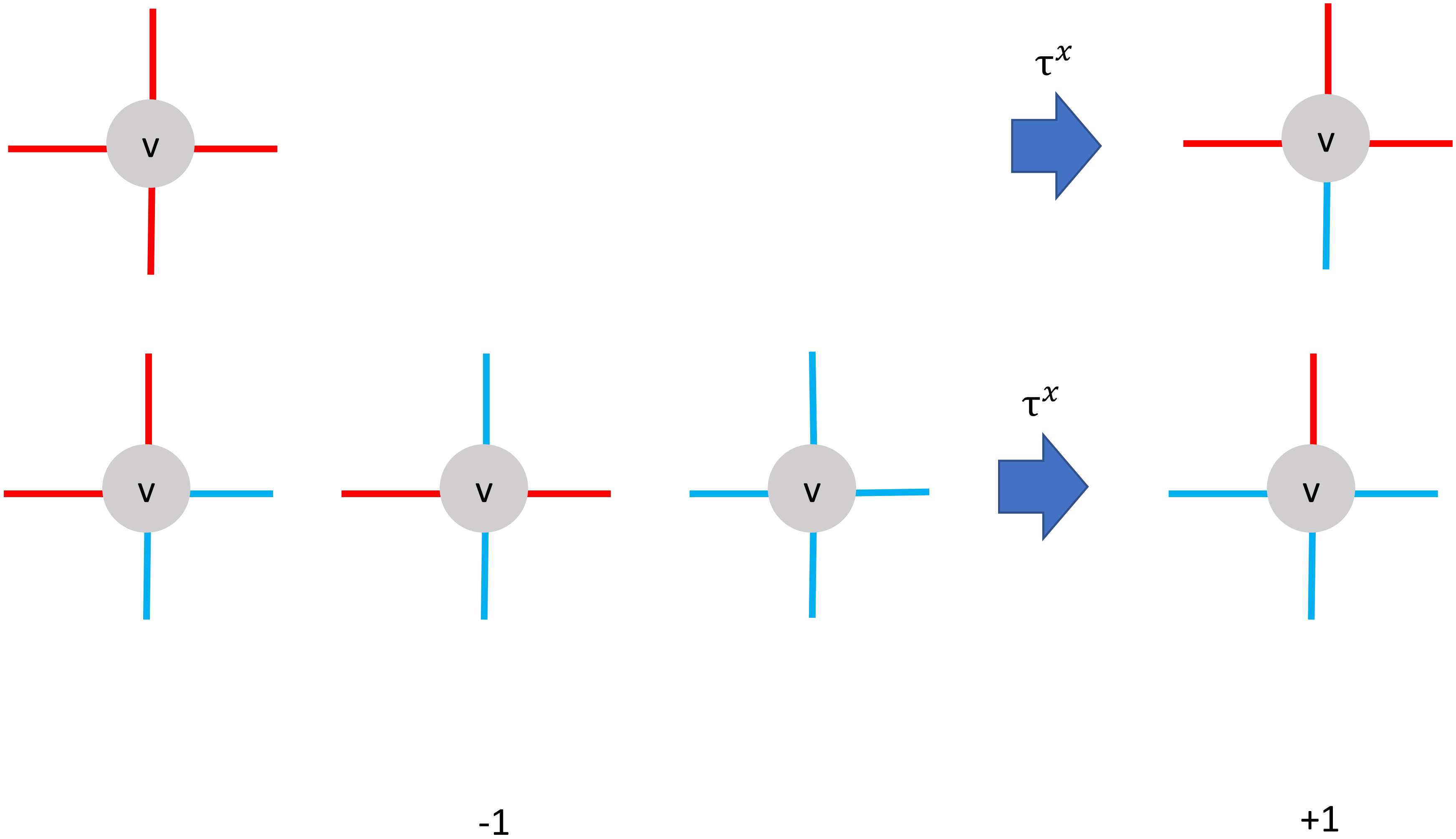
Suppose we have:

$$-A_v \left|\psi\right\rangle = -\left|\psi\right\rangle$$

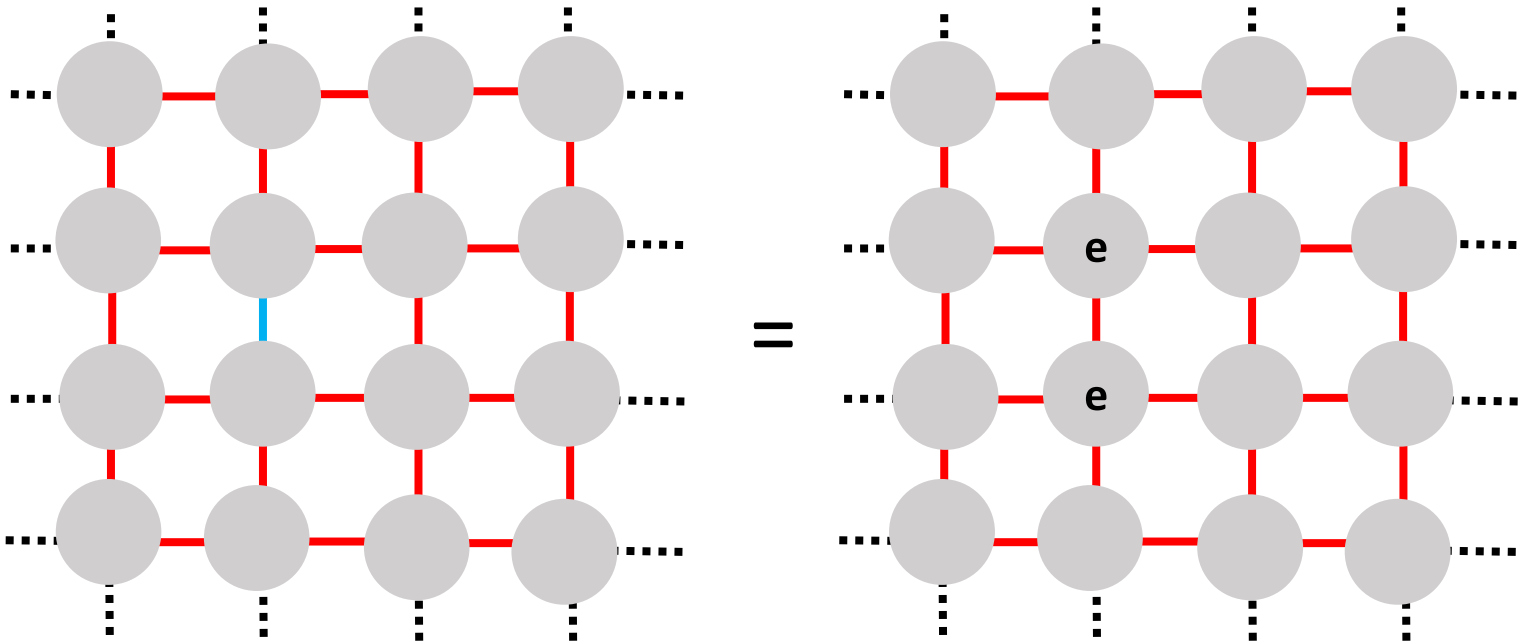
After the operation of the τ^x, the state will be on excited state:

$$-A_v(\tau_i^x \left|\psi\right\rangle) = \tau_i^x A_v \left|\psi\right\rangle = (\tau_i^x \left|\psi\right\rangle)$$

In picture, we have

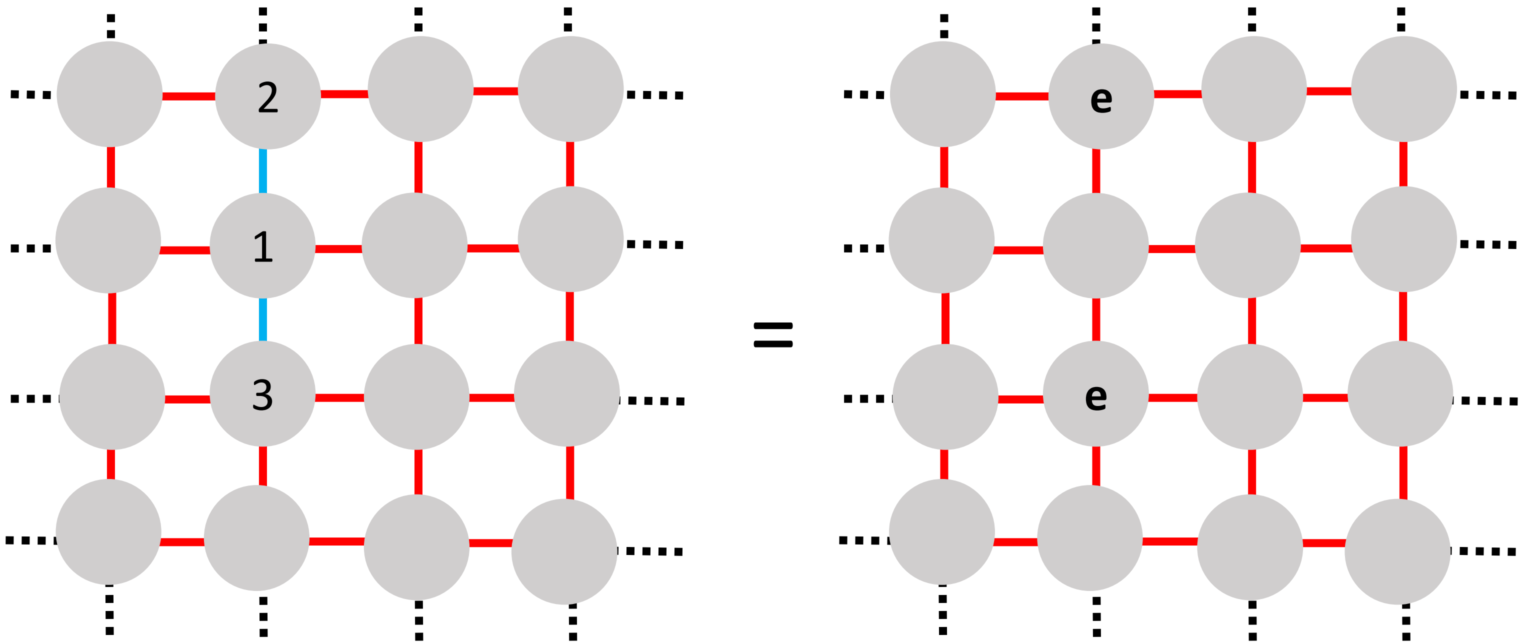


Since each sting is shared by two vertexes, τ^x creates pair excitations each time:



Here we use e to show the local excitation here.

Product of two adjoint τ^x



Now vertex 1 is in G.S., but 2 and 3 is now the excited state.

However, the loop excitation is the identity:

$$\prod_{i \in \square} \tau_i^x = 1$$

Second we consider the effect of τ^z:

$$[\tau_i^z, \tau_i^z \tau_j^z \tau_k^z \tau_l^z]_- = 0 \Rightarrow [\tau_i^z, A_v]_+ = 0$$

Therefore, τ^z is commute with the vertex term where it lives in.

$$[\tau_i^z, \tau_i^x \tau_j^x \tau_k^x \tau_l^x]_+ = 0 \Rightarrow [\tau_i^x, B_p]_+ = 0$$

Therefore, τ^z is anticommute with the plaquette term where it lives in.

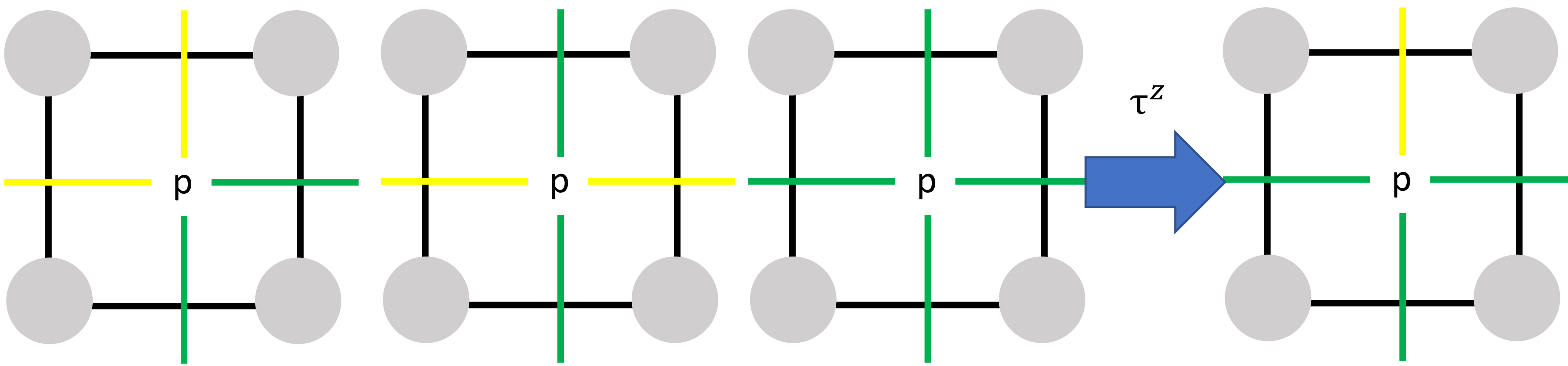
Suppose we have:

$$-B_p |\psi\rangle = -|\psi\rangle$$

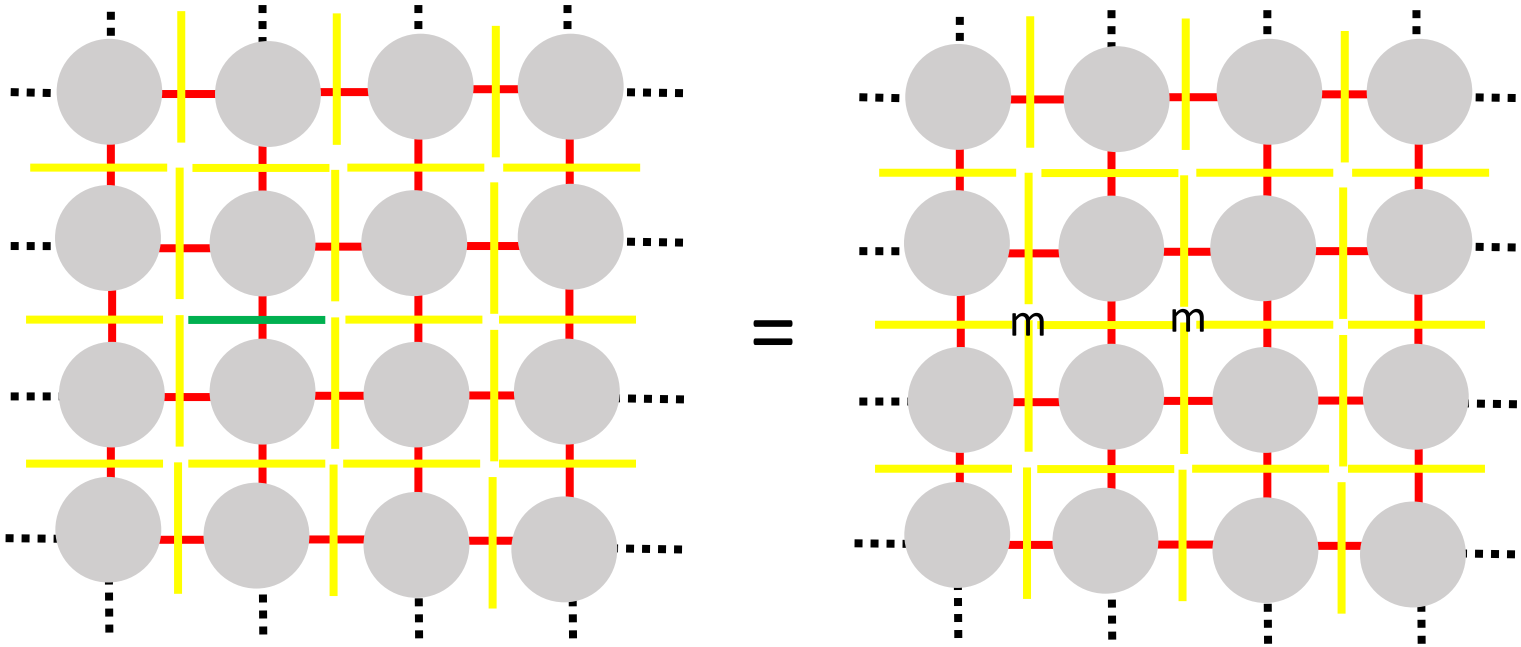
After the operation of the τ^z, the state will be on excited state:

$$-B_p(\tau_i^z |\psi\rangle) = \tau_i^z A_v |\psi\rangle = (\tau_i^z |\psi\rangle)$$

In picture, we have

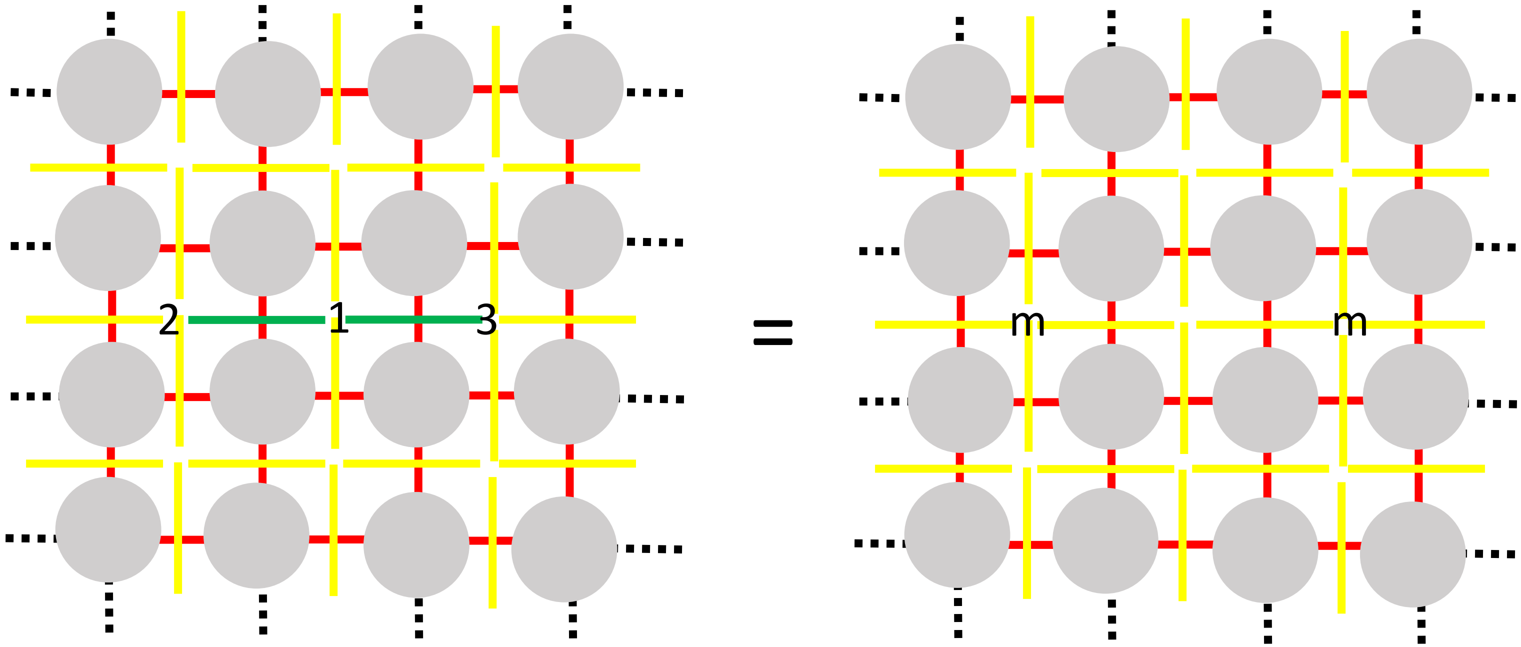


Since each string is shared by two plaquettes, τ^z also creates pair excitation



Here we use m to show the local excitation here.

Product of two adjoint τ^z:

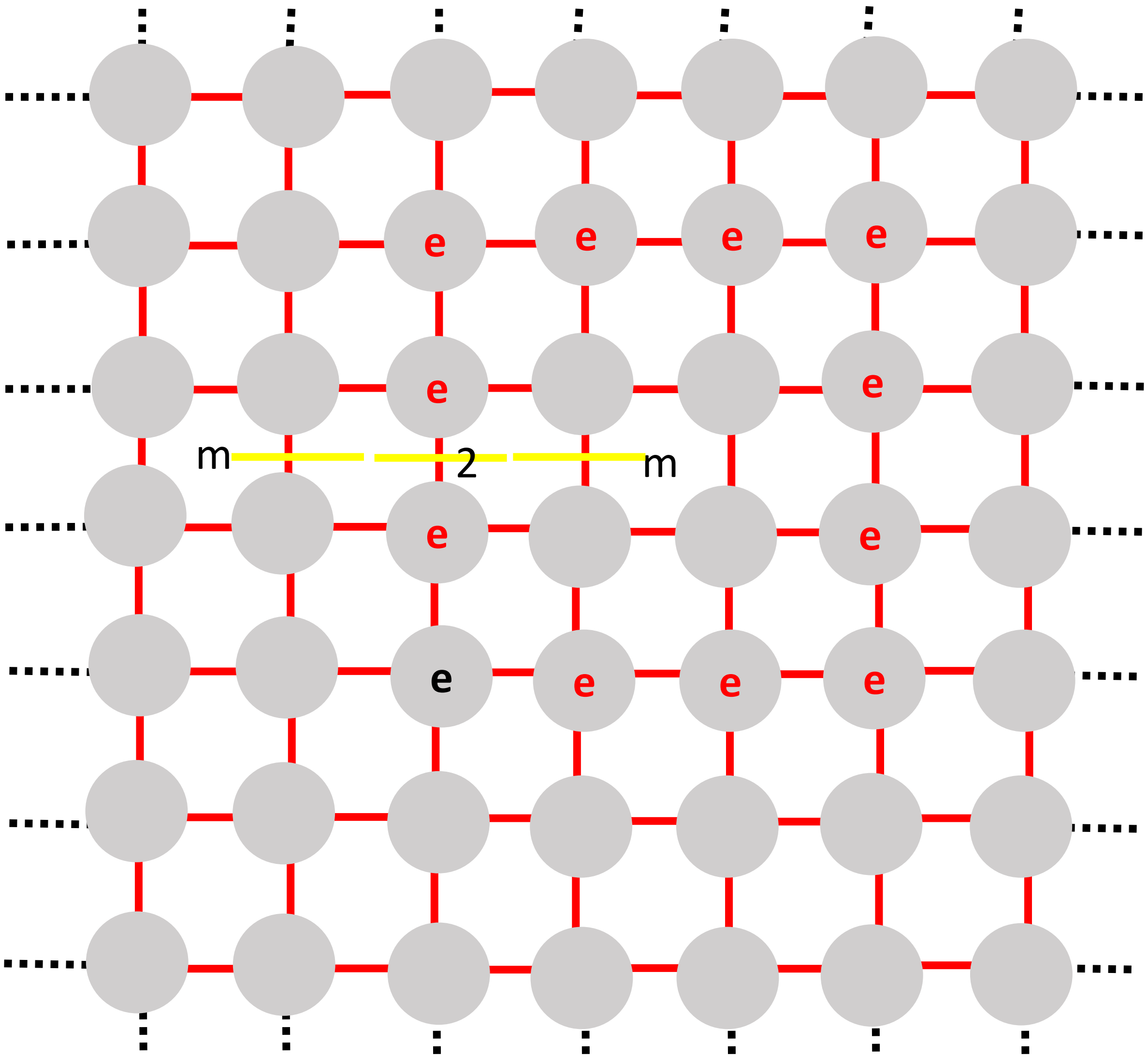


Now the plaquette 1 is in the G.S. while 2, 3 the excited state.

However, the loop excitation is the identity:

$$\prod_{i\in\square}\tau_i^z = 1$$

Lattice AB effect



Suppose the pair m excitation is created by the following path product operator:

$$A = \tau_3^z \tau_2^z \tau_1^z$$

And the pair e excitation is created by the following path product operator:

$$B = \ldots \tau_\beta^x \tau_\alpha^x \tau_2^x \ldots \tau_b^x \tau_a^x$$

Since path A and path B share only one string, here we have labelled it to be 2, we have

$$[A, \mathrm{B}]_+ = [\tau_3^z \tau_2^z \tau_1^z, \ldots \tau_\beta^x \tau_\alpha^x \tau_2^x \ldots \tau_b^x \tau_a^x]_+ = 0$$

Which means

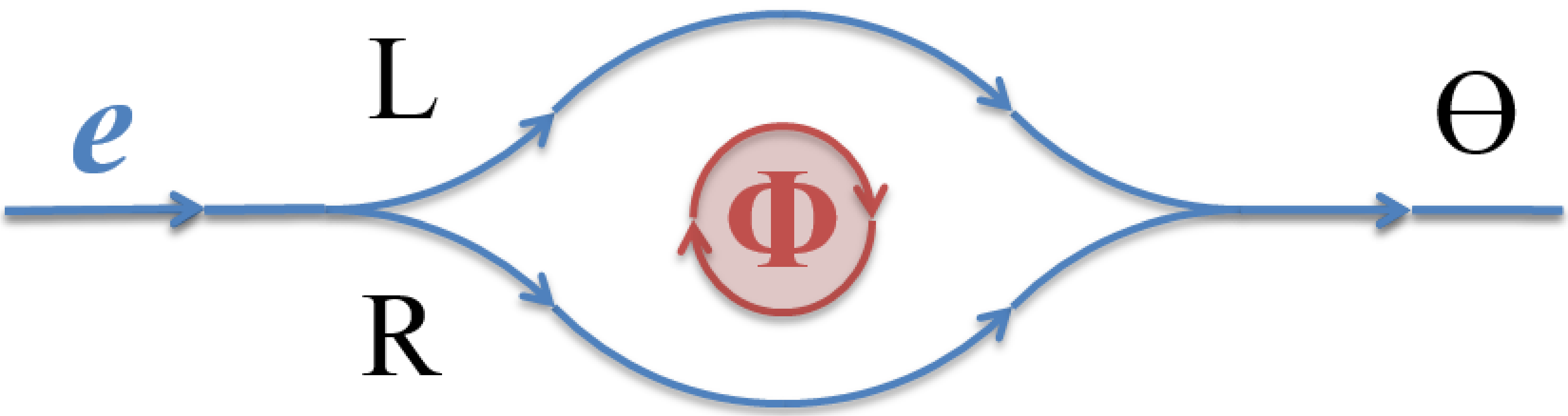
$$\mathrm{BA}\left|\Psi\right\rangle = -\mathrm{AB}\left|\Psi\right\rangle$$

左边是先产生m激发，后让e激发绕一圈得到的波函数

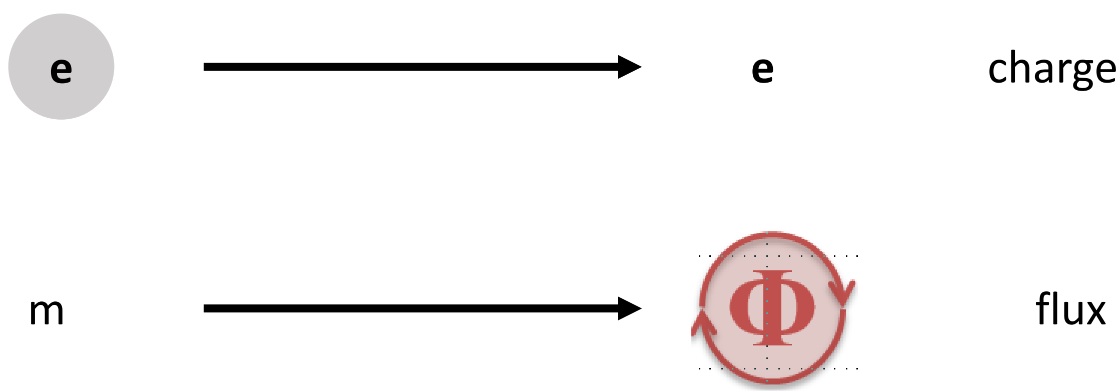
右边是先让e激发绕一圈，后产生m激发得到的波函数

这两个波函数之间差了一个π相位

根据AB效应，当电子绕矢量场转动的时候会积累一个相位，当矢量场不存在的时候此时电子不会积累一个相位:



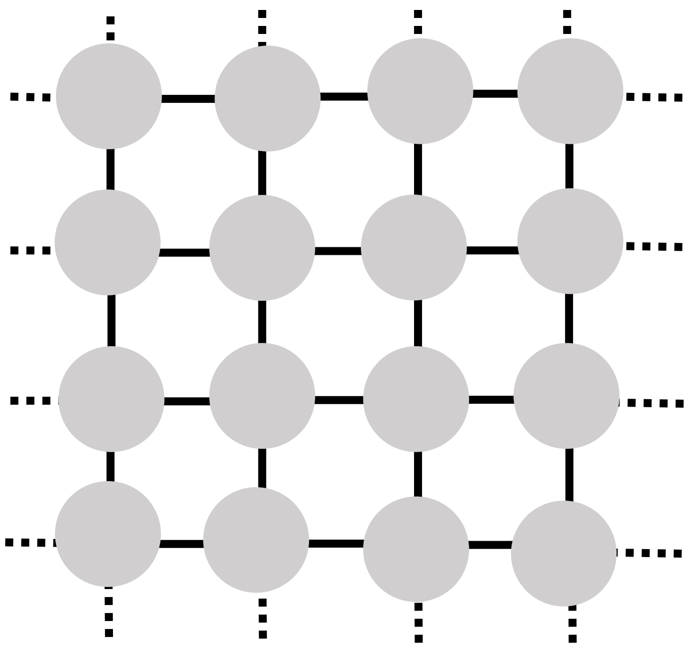
Compare the two picture, we know the correspondence:



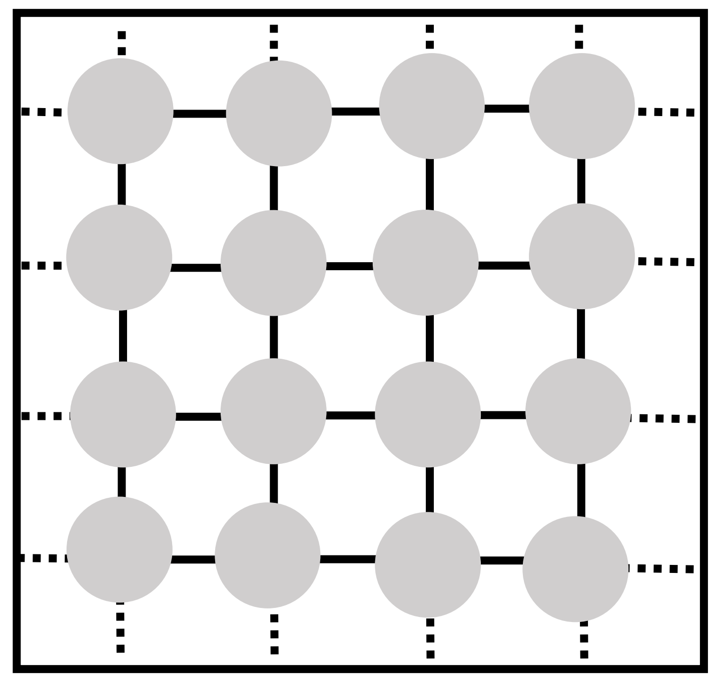
Combining classical topology

- as we discussed, we concern about loops on the manifold. And the homotopy describes the equivalence;
- we can replace the flatness space by either a sphere or a torus;
- sphere is simply connected, so all the loops can be shrink to a point, therefore, the G.S. is formed by vaccum;
- torus is compact, but not simply connected, therefore, we can have more G.S.

Since torus means periodic boundary condition, we will use the following to show PBC:

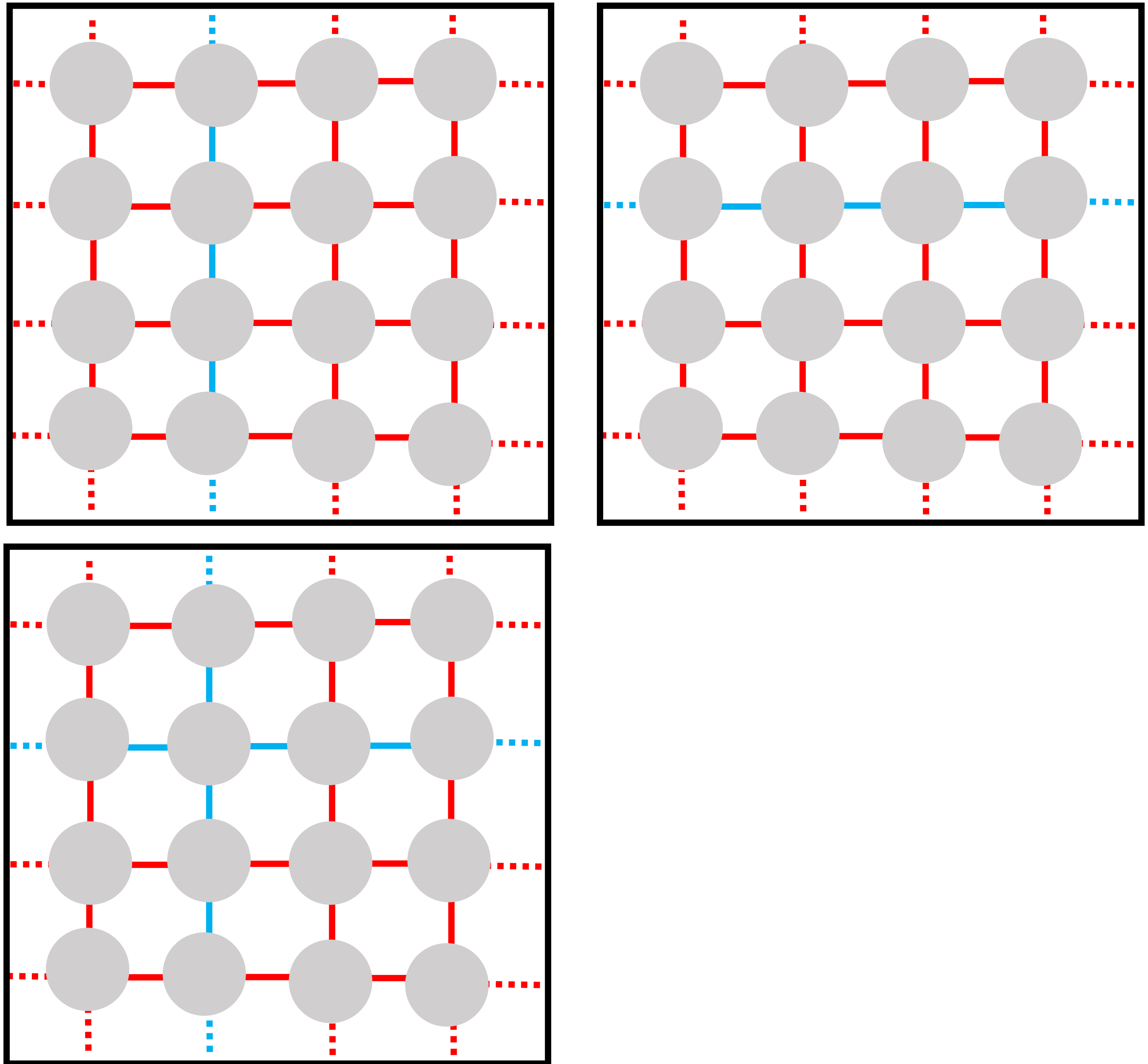


OBC

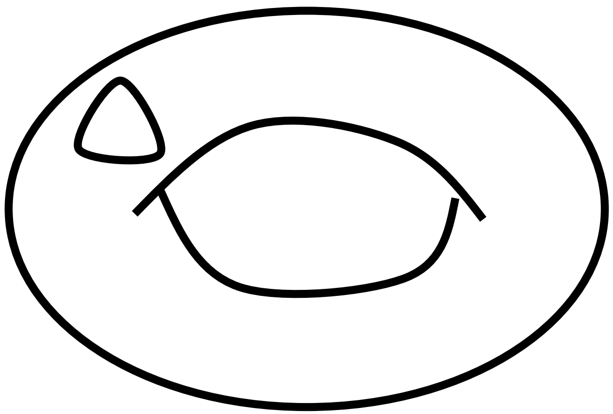


PBC

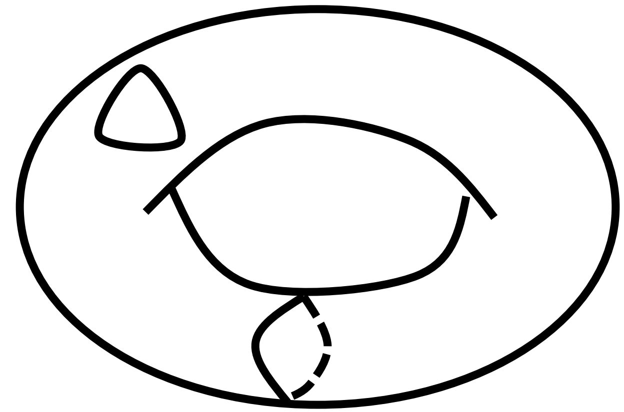
It’s clear that the following strings are topological different:



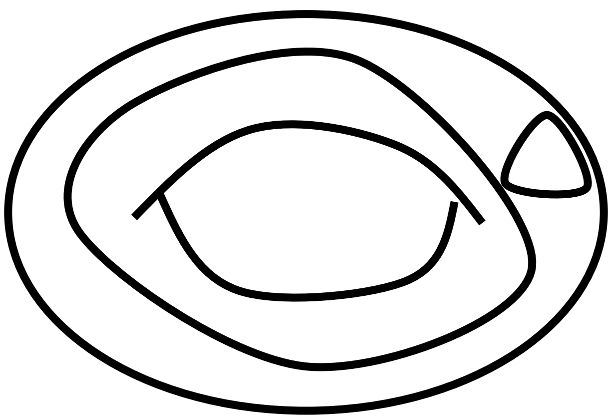
Combined with the G.S. of flatness, there are four different type of G.S.:



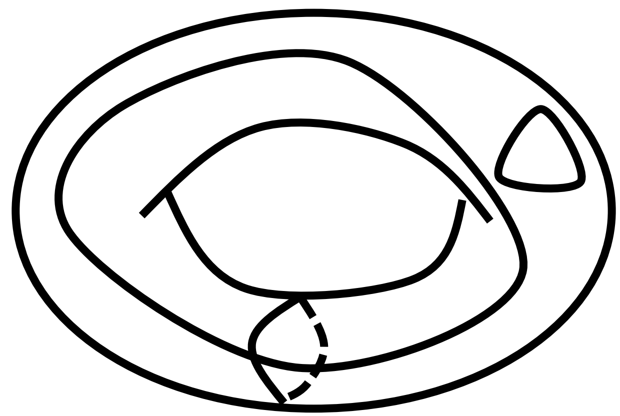
$\left|G.S.1\right\rangle$



$\left|G.S.2\right\rangle$



$\left|G.S.3\right\rangle$



$\left|G.S.4\right\rangle$

No local operator link to different type of G.S., so each type of G.S. is very robust. Until the perturbation is global.

The total degeneracy of G.S. is 4.

2, Transverse Ising model

$$H = -J\sum_{<i,j>}\sigma_i^x\sigma_j^x - \sum_i \sigma_i^z$$

Here we use σ matrix for matter field and τ matrix for gauge field as we used above.

Here we have the global symmetry:

$$U = \prod_i \sigma_i^z$$

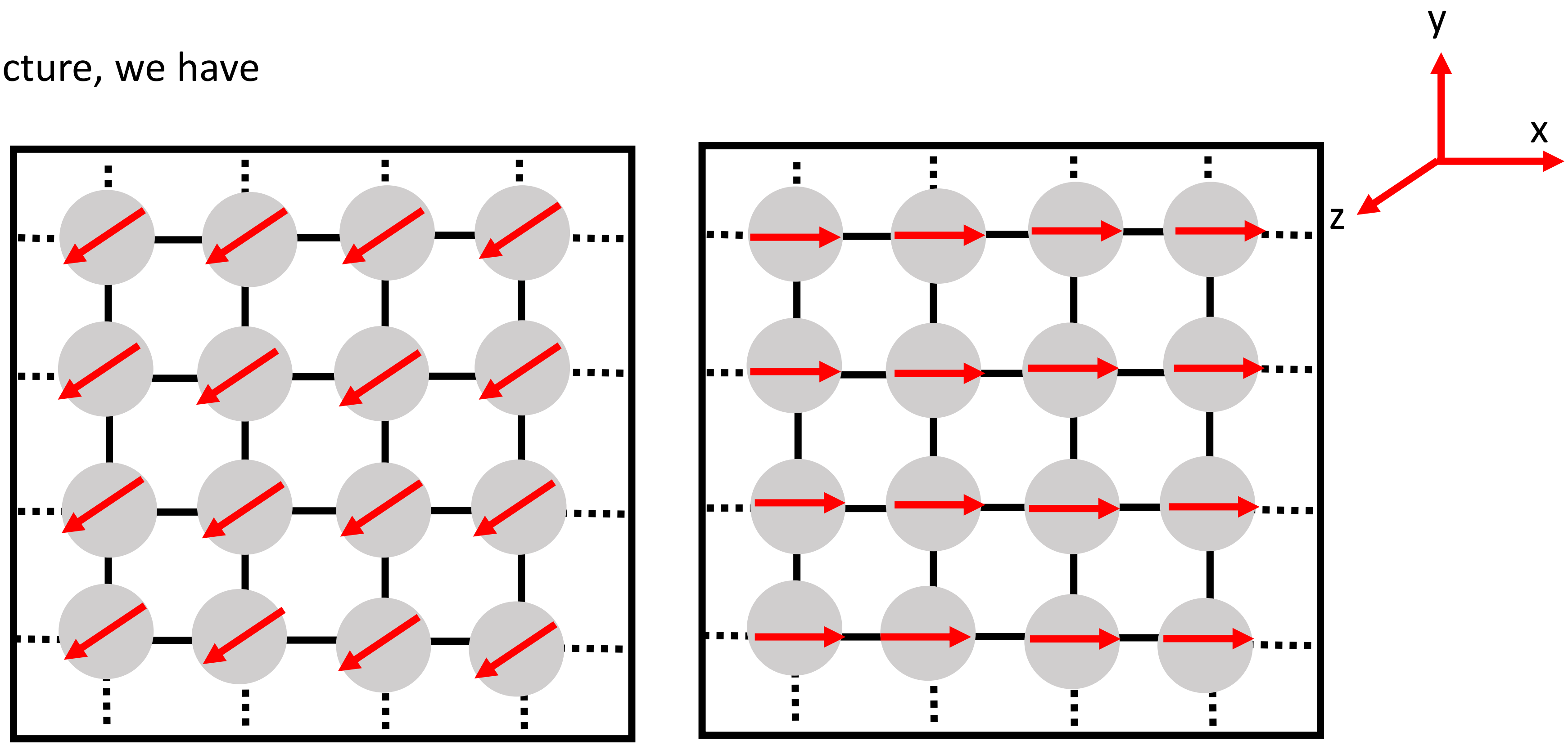
Here operation U is like a mirror reflection.

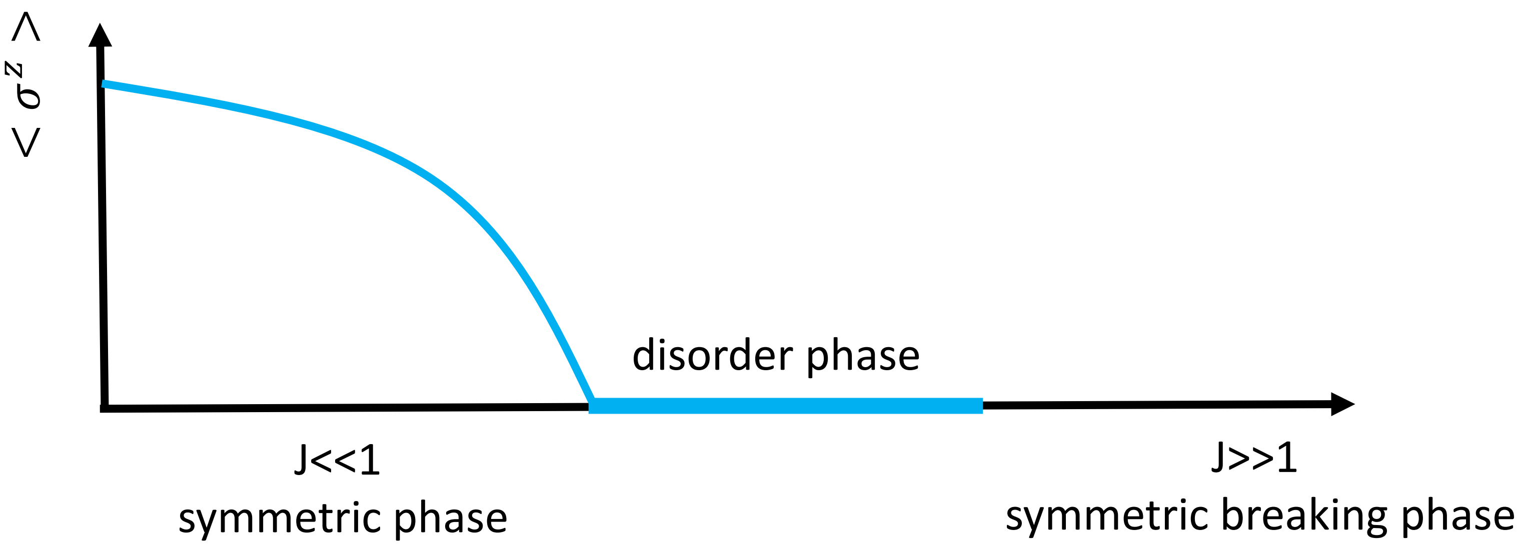
Obviously, we have

$$[H,U]_- = 0$$

If J<<1, we have the symmetric phase;
If J>>1, we have the symmetry breaking phase, which is a direct product state;

In picture, we have





3, Z2 gauge theory= transverse Ising model + gauge field

See the transverse Ising model as mass field on each vertex;
See the gauge field as gauge field on each string;
Now we are going to couple these two fields and derive the discrete lattice gauge theorywe haven't seen before.
The theory should be invariant under local gauge transformation.

The global symmetry is described by:

$$U = \prod_i \sigma_i^z$$

And the local symmetry is described by:

$$G_i = \sigma_i^z$$ local mirror z at each site

First, we need to guess the coupling between gauge field and matter field, from quantum field theory, we know:

If the matter field transforms as:

$$\sigma_i \to \sigma_i^z \sigma_i$$

The comparator transforms as:

$$U_{i,j} \to \sigma_j^z U_{i,j} \sigma_i^z$$

The coupling term should has the following form:

$$\sigma_j^x U_{i,j} \sigma_i^x$$

which is invariant under local gauge transformation

Proof:

$$\sigma_j^x U_{i,j} \sigma_i^x \to \sigma_j^x \sigma_j^z \sigma_j^z U_{i,j} \sigma_i^z \sigma_i^z \sigma_i^x = \sigma_j^x U_{i,j} \sigma_i^x$$

Since the comparator describes the d.o.f. of gauge field, the Uij above is just the gauge field living on the string:

Here we will use <ij> to label the string, rathe than i is past discussion.

So the coupling term is

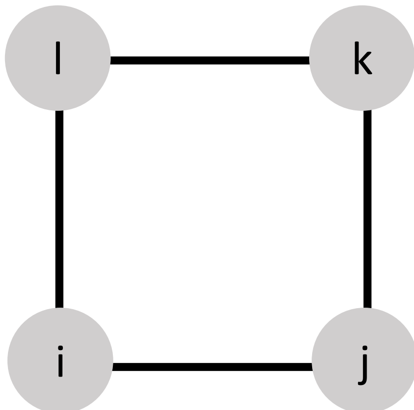
$$\sigma_j^x \tau_{<ij>}^x \sigma_i^x$$

Now we transfer to operator language, to keep the term commute with σ_i^z/σ_j^z, we should have

$$[\sigma_j^z, \tau_{<ij>}^x]_+ = [\tau_{<ij>}^x, \sigma_i^z]_+ = 0$$

which means that the gauge d.o.f. is effected by the local gauge transformation.

The introduction of coupling means the theory should contain of gauge field, it's clear that kinetic term is local gauge invariant, but not the mass term:



$$\sigma_i^z \tau_{<li>}^x \tau_{<kl>}^x \tau_{<jk>}^x \tau_{<ij>}^x \sigma_i^z$$

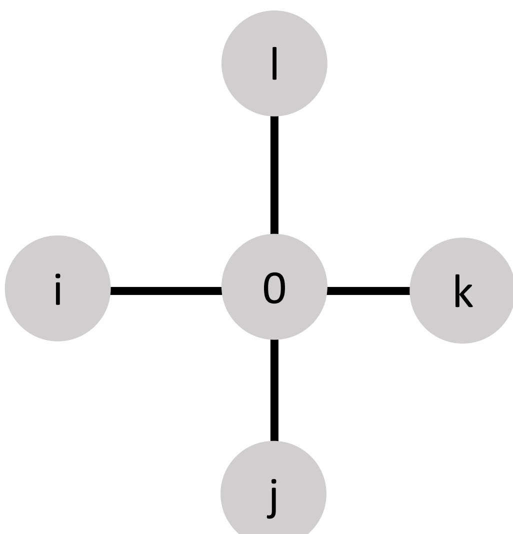
$$= -\sigma_i^z \tau_{<li>}^x \tau_{<kl>}^x \tau_{<jk>}^x \sigma_i^z \tau_{<ij>}^x$$

$$= -\sigma_i^z \tau_{<li>}^x \sigma_i^z \tau_{<kl>}^x \tau_{<jk>}^x \tau_{<ij>}^x$$

$$= - - \sigma_i^z \sigma_i^z \tau_{<li>}^x \tau_{<kl>}^x \tau_{<jk>}^x \tau_{<ij>}^x$$

$$= \tau_{<li>}^x \tau_{<kl>}^x \tau_{<jk>}^x \tau_{<ij>}^x$$

A vertex share two string with a plaquette.



$$\tau_{<0l>}^z \tau_{<0k>}^z \tau_{<0j>}^z \tau_{<0i>}^z \rightarrow$$

$$\sigma_l^z \tau_{<0l>}^z \sigma_0^z \sigma_k^z \tau_{<0k>}^z \sigma_0^z \sigma_j^z \tau_{<0j>}^z \sigma_0^z \sigma_i^z \tau_{<0i>}^z \sigma_0^z$$

$$\neq \tau_{<0l>}^z \tau_{<0k>}^z \tau_{<0j>}^z \tau_{<0i>}^z$$

Very an-natural here!

And term $\sum_i \sigma_i^z$ is locally gauge invariant

$$[\sum_i \sigma_i^z, \sigma_i^z]_- = 0$$

So the lattice Hamiltonian is:

Local
gauge invariant

$$H = -J \sum_{<i,j>} \sigma_i^x \tau_{<ij>}^x \sigma_j^x - \sum_i \sigma_i^z - \sum_p B_p$$

Global
gauge invariant

$$H = -J \sum_{<i,j>} \sigma_i^x \sigma_j^x - \sum_i \sigma_i^z$$

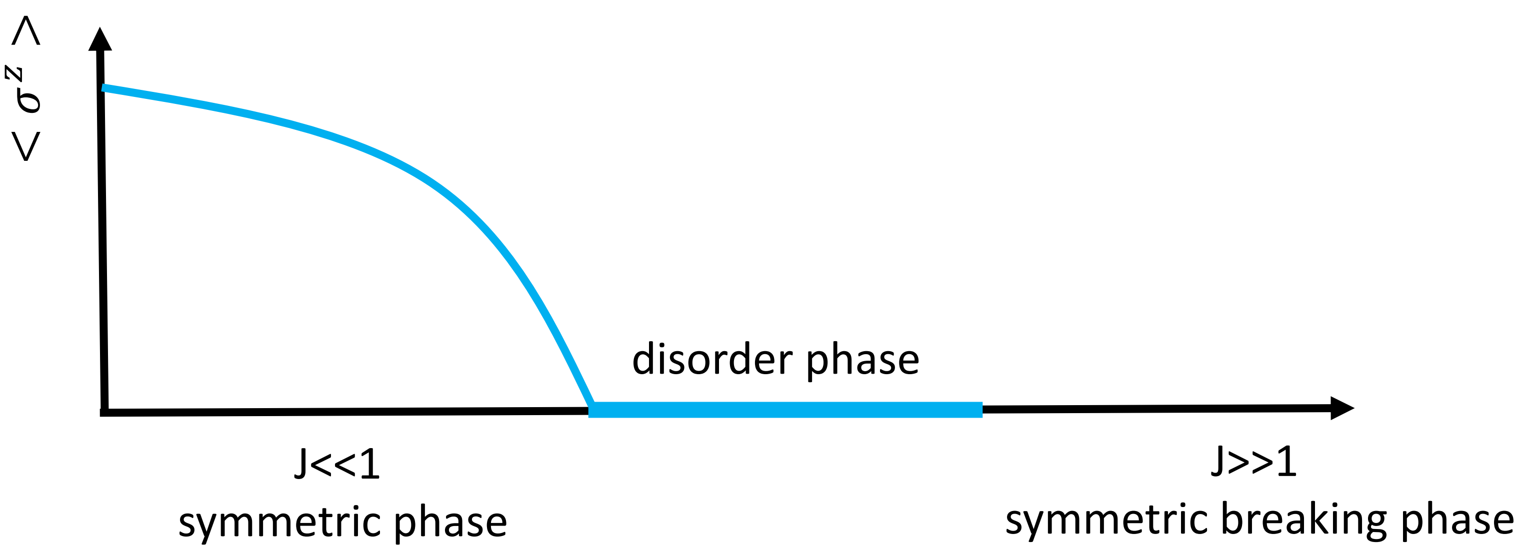
Weak coupling limit (J<<1)

See the vedio

Strong coupling limit

Medium regime

Phase diagram



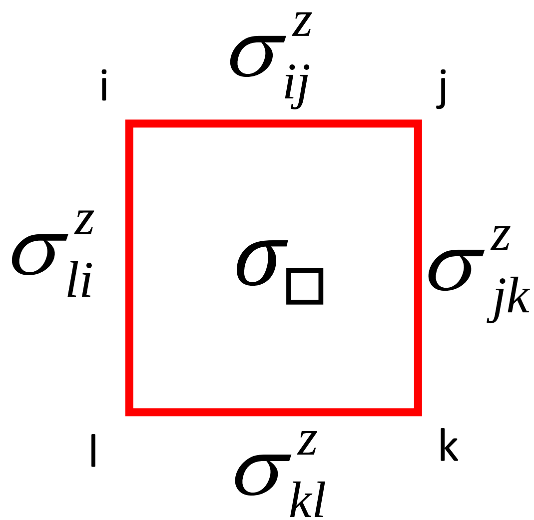
Kitaev toric code model

Encode with fermion

From chapter 9 of **field theories of condensed matter physics** by **Eduardo Fradkin**

Warm up: Ising lattice gauge theory

Here the system is formed by three parts: gauge term and matter-gauge interaction term:



The gauge field live on bond linking two sites, and the gauge term look like:

$$H_{gauge} = -K\sum_{\square}\sigma^z_\square, \sigma^z_\square = \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li}$$

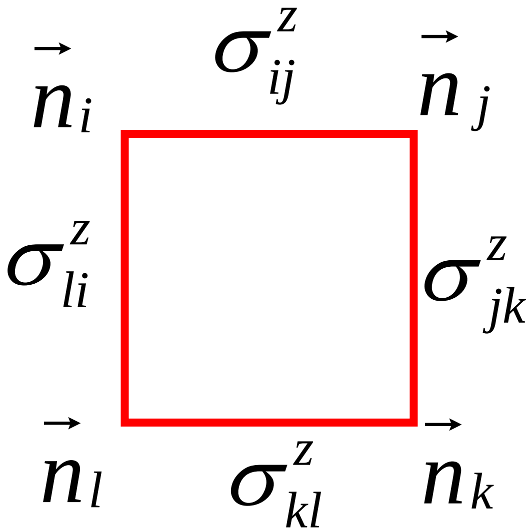
Where all σ=+1 or -1

PHYSICAL REVIEW X **6**, 041025 (2016)

Generalized Liquid Crystals: Giant Fluctuations and the Vestigial Chiral Order of *I*, *O*, and *T* Matter

Ke Liu (刘科 子竞),[1] Jaakko Nissinen,[1] Robert-Jan Slager,[1] Kai Wu,[2] and Jan Zaanen[1]

The matter term lives on lattice site, and the mass-gauge interaction is:



$$H_{Higgs} = -J \sum_{<ij>} \sigma^z_{ij} \vec{n}_i \vec{n}_j$$

So the full theory is:

$$H = H_{gauge} + H_{Higgs} = -K \sum_{\square} \sigma^z_{\square} - J \sum_{<ij>} \sigma^z_{ij} \vec{n}_i \vec{n}_j$$

The theory is controlled by two parameters (J, K).

- Symmetry analysis

  Gauge symmetry

- Limitation analysis

- Renormalization group analysis

  Fixed point→phase

- Generalization to higher group

$$U(1) \to \mathrm{Z}_N$$

- Tensor order parameter

Monte Carlo simulation

Between strong and weak coupling, Monte Carlo simulation provides a numerical method to study the medium region.