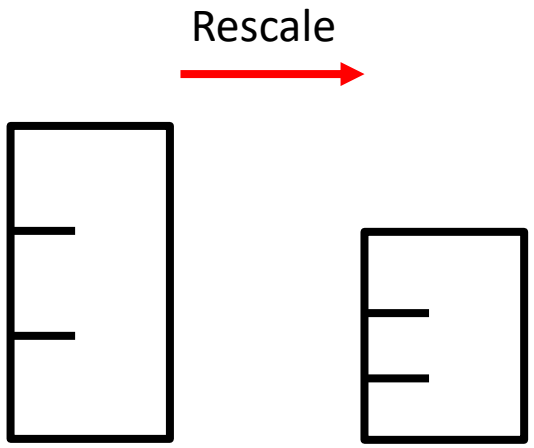
Renormalization Group

Scaling dimension in mathematics

First consider a parameter x and a function O, a scaling dimension of n is defined as:

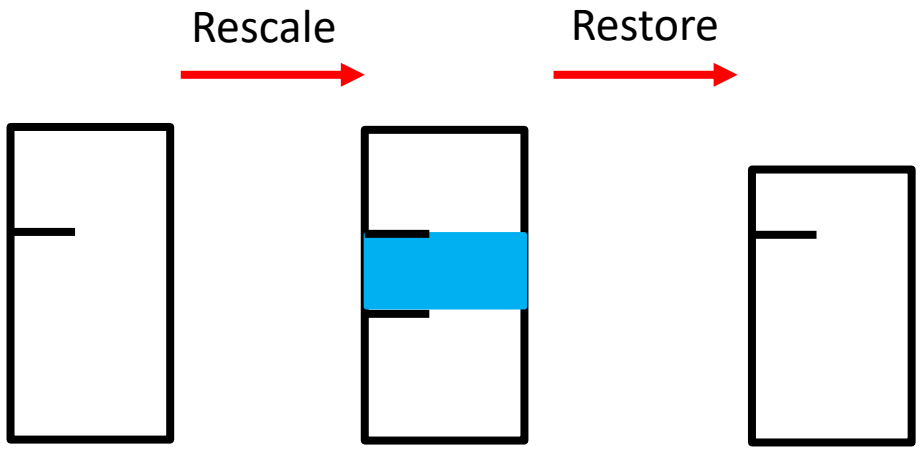
$$O(\lambda \mathrm{x}) = \lambda^{n} O(\mathrm{x})$$

n can be any number, spatial dimension for n=1, 2, 3 or fractional in fractal.



Rescale and restore in quantum field theory

Here we take two steps as follows:



Suppose we have action before renormalization:

$$Z=\int_0^{\Lambda}[\mathrm{D}\phi]\exp(-\int dxL(\phi,\partial_\mu\phi))$$

After integration of the high energy component, the rescale give

$$Z=\int_0^{\Lambda/\lambda}[\mathrm{D}\phi]\exp(-\int dxL_{eff}(\phi,\partial_\mu\phi))$$

Generally we can use perturbation theory to abandon the high order term (i.e. information is missed here, but very little) to get the effective Lagrangian:

$$L_{eff}(\phi,\partial_\mu\phi)$$

Now we want to goes back, so we restore the system by renormalize the field strength to keep the kinetic part the same, we can always have :

$$L_{eff}(\phi, \partial_\mu \phi) = \underline{L'(\phi', \partial'_\mu \phi')} + \Delta L + higer - order$$

这里L上的撇代表裸的L里面的系数都已经变了

Therefore, by ignore the high order contribution, a renormalization step is equivalent to:

$$L(\phi, \partial_\mu \phi) \to L'(\phi', \partial'_\mu \phi') + \Delta L$$

And we can list the transformation of each parameter:

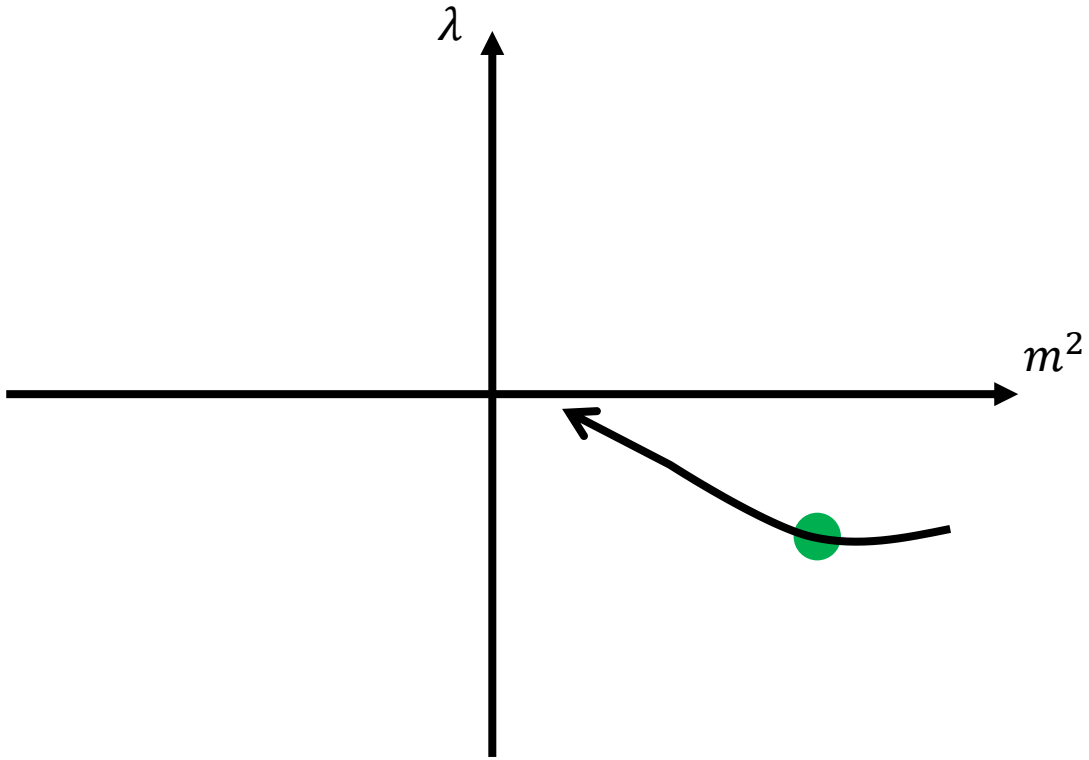
$$m^2 \to m'^2 = m^2 + \Delta m^2$$

$$\lambda \to \lambda' = \lambda + \Delta\lambda$$

...

Renormalization group flow

Now we give each parameter a d.o.f., so we have a high dimensional space:



Given a start point in the parameter space, then we can picture how it flows each step. until it reaches an end point, which is called fixed point.

Fixed point: scaling invariance

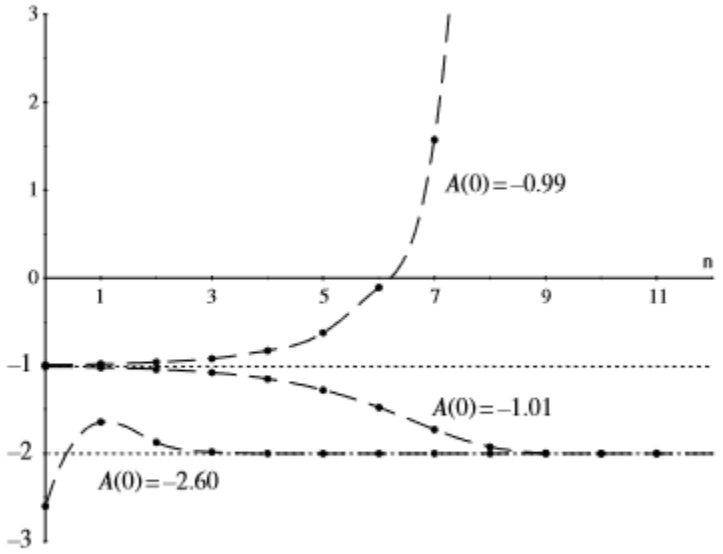
At the fixed point, the parameters don't change upon scaling:

$$x = f(x)$$

At such a point, the system has a new symmetry: scale invariance.

There maybe more than one solution to this equation, therefore, there may exist different fixed point. Some fixed point can never be reached, here we give an simple example:

$$X_{n+1} = (X_n + 4)X_n + 2$$



Repulsive fixed point

Attractive fixed point

FIG. 1.15 Illustration of recursion relations leading to fixed points.

Fixed point: stable phase for critical point

Now we give the physical meaning for the fixed point. Since at the fixed point, there is no scales left in the problem, therefore, a fixed-point can describes either

!) a system with a vanishing correlation length ξ and hence a divergent energy gap;

2) a system with a divergent correlation length ξ and hence a vanishing energy gap;

Fixed points with vanishing ξdescribe stable phases of matter,

Conversely, fixed points with divergent ξ describe systems at criticality, thus symbol of phase transition.

Here we can take ferromagnetic order for example:

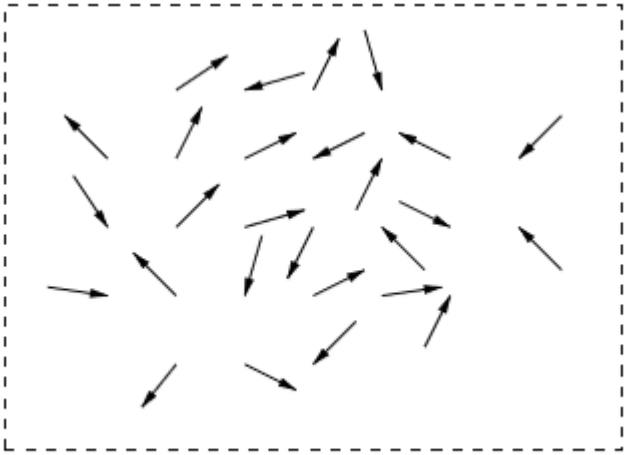


FIG. 1.13 Lattice spins randomly oriented at high temperatures.

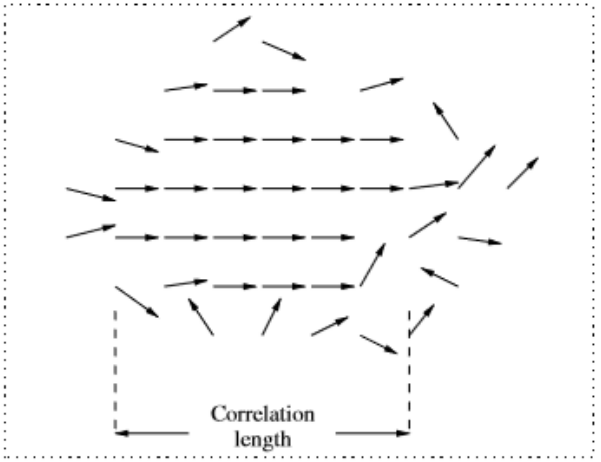


FIG. 1.14 Correlated groups of spins as the critical temperature is approached from above.

$$\xi = 0$$

$$\xi \to \infty$$

Four examples:

1: NRG treatment of 1D TB model

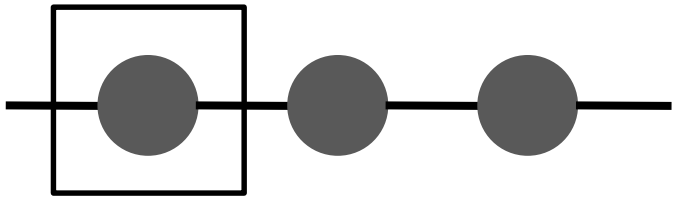
2: NRG treatment of 1D transverse Ising model

3: field theory treatment of superconductivity

4: 量子多体系统中的量子相变

E1: Numerical RG of 1D TB model

1D TB Here we treat it because we know the exact solution



Here the matrix in real space is

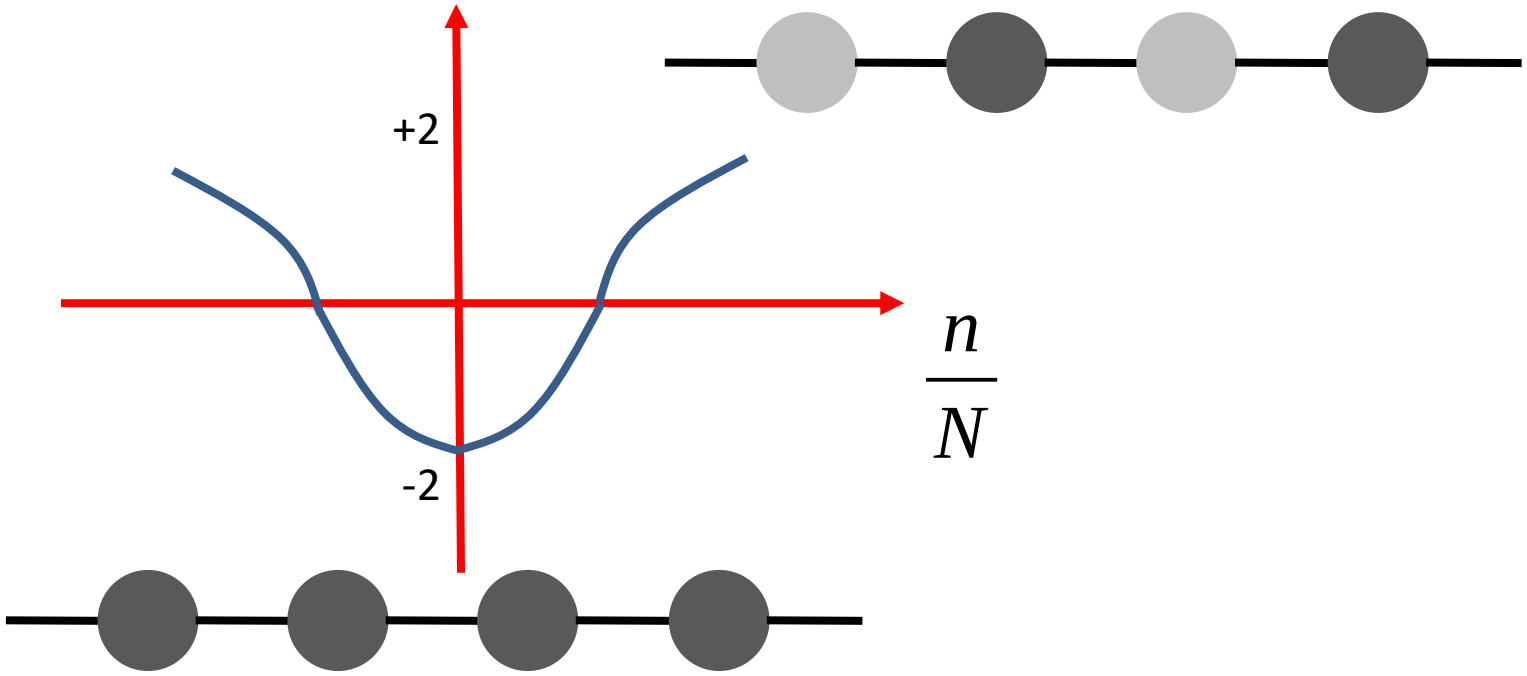
$$H_N = \begin{bmatrix} 0 & -1 & & \\ -1 & 0 & -1 & \\ & -1 & \ldots & -1 \\ & & -1 & 0 \end{bmatrix}_{N \times N}$$

Tridiagonal matrix

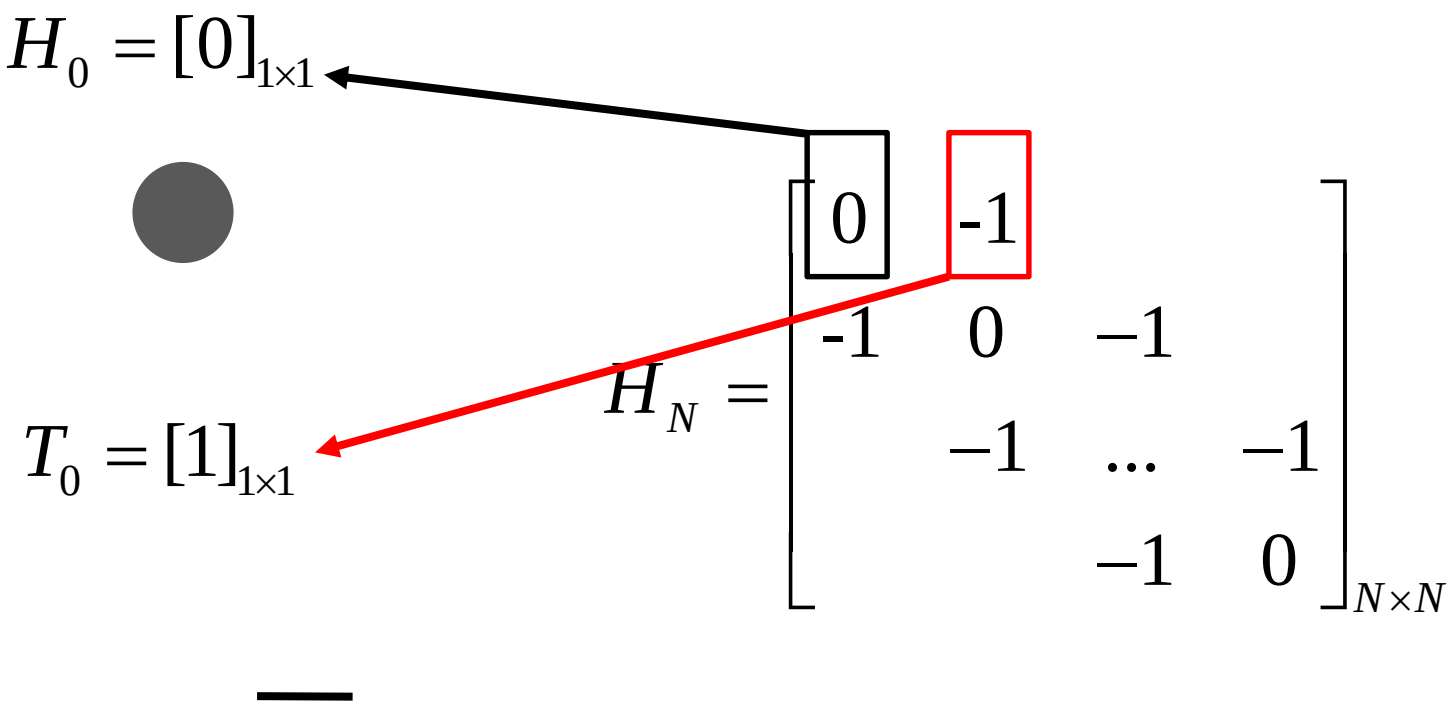
Exact solution

Although we can it effectively in real space, we can quickly get the answer in reciprocal space

$$E_n = -2\cos(\frac{n}{N}\pi) \qquad |\psi_n\rangle = \sum_t \exp(-\mathrm{i}\frac{n}{N}\mathrm{t})|s_t\rangle$$



Now we try to do NRG analysis

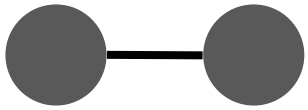


Here 0 is the start point, if we solve H0, the eigenvalue is 0.

Now we start to block the system

$$H_1 = \begin{bmatrix} H_0 & T_0 \\ T_0^T & H_0 \end{bmatrix}$$

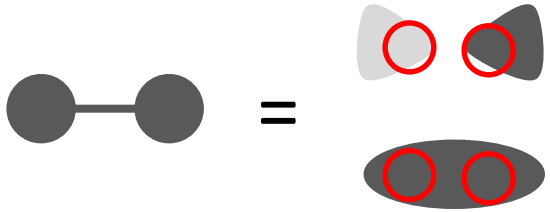
$$H_N = \begin{bmatrix} 0 & -1 & & & \\ -1 & 0 & -1 & & \\ & -1 & \ddots & \ddots & -1 \\ & & \ddots & \ddots & -1 \\ & & & -1 & 0 \end{bmatrix}_{N \times N}$$



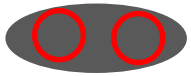
$$T_1 = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$$

Solve H1 by exact diagonalization

$$H_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

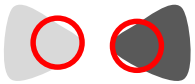


$$|u_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \mathrm{E}_1 = -1,$$



$$H_1|u_1\rangle = |u_1\rangle \mathrm{E}_1$$

$$|u_2\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \mathrm{E}_2 = 1,$$



$$H_2|u_2\rangle = |u_2\rangle \mathrm{E}_2$$

i.e.

$$H_1\begin{bmatrix}|u_1\rangle & |u_2\rangle\end{bmatrix}=\begin{bmatrix}|u_1\rangle & |u_2\rangle\end{bmatrix}\begin{bmatrix}E_1 & \\ & E_2\end{bmatrix}$$

Here we keep the lowest energy state

$$\begin{bmatrix}|u_1\rangle & \square\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \square \\ \frac{1}{\sqrt{2}} & \end{bmatrix}$$

Here we only concern on G.S.

And we can define the projection operator

$$P=|u_1\rangle=\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

And the truncated Hamiltonian and interaction is

$$H_1^{'} = P^{\dagger}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = [-1]_{1\times 1}$$

$$T_1^{'} = P^{\dagger}\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = [-\frac{1}{2}]_{1\times 1}$$

Here 0 is the start point, if we solve H1', the eigenvalue is -1.

Now we iterate by the following replacement:

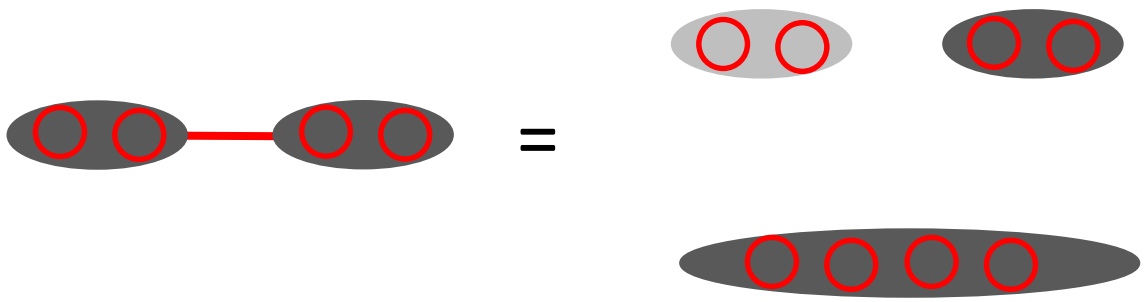
$$H_2 = \begin{bmatrix} H_1^{'} & T_1^{'} \\ T_1^{'T} & H_1^{'} \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 0 \\ T_1^{'} & 0 \end{bmatrix}$$

Solve H2 by exact diagonalization

$$H_2 = \begin{bmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix}$$

$$H_2 \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1.5 & \\ & -0.5 \end{bmatrix}$$



Truncation by defining the projection operator

$$P = |u_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

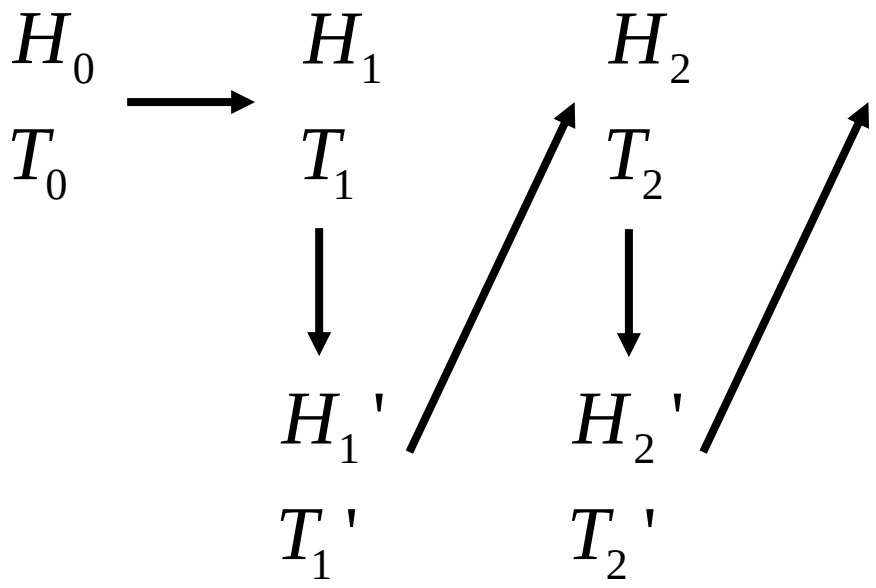
And the truncated Hamiltonian and interaction is

$$H_2^{'} = P^{\dagger} \begin{bmatrix} -1.5 & -1 \\ -1 & -0.5 \end{bmatrix} P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1.5 & -\frac{1}{2} \\ -\frac{1}{2} & -0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = [-1.5]_{1\times 1}$$

$$T_2^{'} = P^{\dagger} \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix} P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = [-\frac{1}{4}]_{1\times 1}$$

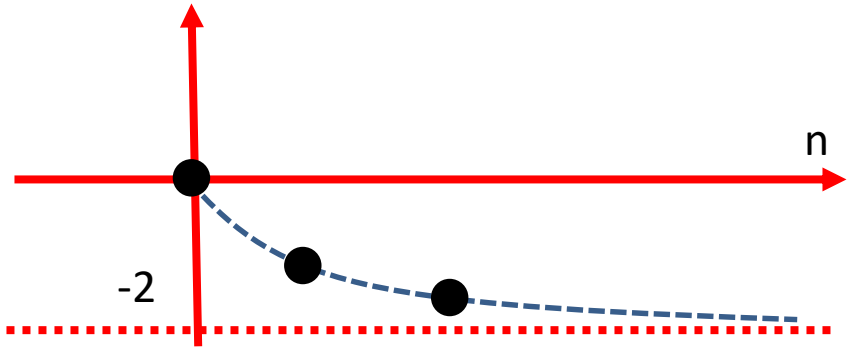
Here 0 is the start point, if we solve H2', the eigenvalue is -1.5.

Continue the above process, we have



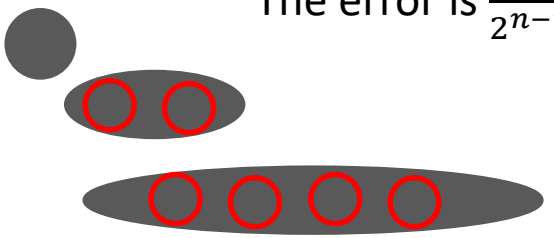
$$\cdots \quad a_0 = 0, a_1 = -1, a_2 = -1.5, \ldots$$

$$a_n = -2(1 - \frac{1}{2^n})$$



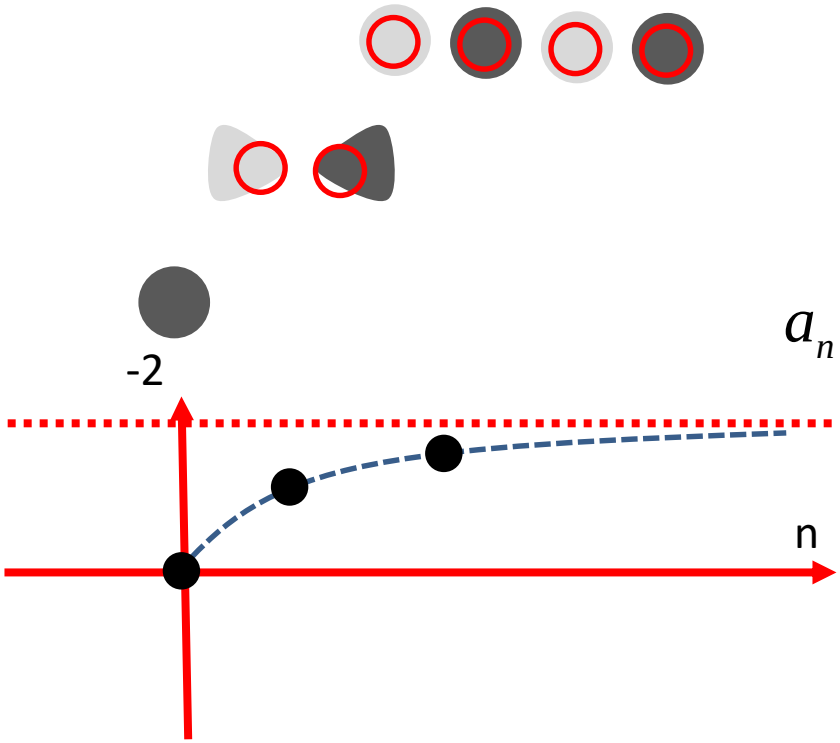
$$a_n = -2 + \frac{1}{2^{n-1}}$$

The error is $\frac{1}{2^{n-1}}$, quite slow



The process satisfy our physical picture.

What if we keep the higher state every time ?



$$a_n = -2 + \frac{1}{2^{n-1}}$$

The error is $\frac{1}{2^{n-1}}$, quite slow

The process satisfy our physical picture.

Write a code See NRG_1D.m in code/matlab/NRG_1D

```
H0=0;
T0=-1;

while 1
    %%%%Define block Hamiltonian
    H=[H0,T0;T0,H0];
    %%%%Define interaction between blocks
    T=[0,0;T0,0];

    %%%%Solve block Hamiltonian
    [v,d]=eig(H);
    d_val=real(eig(H));
    [d_sort,d_index]=sort(d_val);     %here v_val, d_sort, d_index are same size
    v_sort=v(:,d_index);              %sort eigenvector(column) by order of eigenvalue

    %%%%Normalize eigenvectors
    for t=1:2
        v_so(:,t)=v_sort(:,t)/norm(v_sort(:,t));
    end

    %%%%Define projector
    p=v_so(:,1);

    %%%%Truncate Hamiltonian
    H0=transpose(p)*H*p;

    %%%%Truncate Interaction
    T0=transpose(p)*T*p;

    if loop>M
       break;
    end
    loop=loop+1
end
```

After test, it is right.

Now we consider open boundary condition and set number of site equal to 8.

$$H_8 = \begin{bmatrix} & -1 & & & & & & \\ -1 & & -1 & & & & & \\ & -1 & & -1 & & & & \\ & & -1 & & -1 & & & \\ & & & -1 & & -1 & & \\ & & & & -1 & & -1 & \\ & & & & & -1 & & -1 \\ & & & & & & -1 & \end{bmatrix}$$

Exact diagonalization, we have

u =

-0.1612	-0.3030	-0.4082	0.4642	-0.4642	0.4082	-0.3030	-0.1612
-0.3030	-0.4642	-0.4082	0.1612	0.1612	-0.4082	0.4642	0.3030
-0.4082	-0.4082	0.0000	-0.4082	0.4082	-0.0000	-0.4082	-0.4082
-0.4642	-0.1612	0.4082	-0.3030	-0.3030	0.4082	0.1612	0.4642
-0.4642	0.1612	0.4082	0.3030	-0.3030	-0.4082	0.1612	-0.4642
-0.4082	0.4082	-0.0000	0.4082	0.4082	-0.0000	-0.4082	0.4082
-0.3030	0.4642	-0.4082	-0.1612	0.1612	0.4082	0.4642	-0.3030
-0.1612	0.3030	-0.4082	-0.4642	-0.4642	-0.4082	-0.3030	0.1612

More digits -1.879385241571817

v =

-1.8794	0	0	0	0	0	0	0
0	-1.5321	0	0	0	0	0	0
0	0	-1.0000	0	0	0	0	0
0	0	0	-0.3473	0	0	0	0
0	0	0	0	0.3473	0	0	0
0	0	0	0	0	1.0000	0	0
0	0	0	0	0	0	1.5321	0
0	0	0	0	0	0	0	1.8794

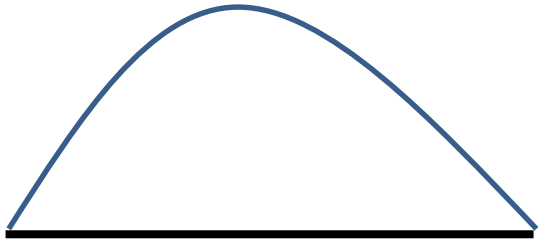
More digits

-1.879385241571817

If we use NRG,

$$a_n = -2 + \frac{1}{2^{n-1}} = -1.9921875$$

It can be seen that the difference is big in view of wavefunction:



Exact diagonalization

RG

E2: NRG treatment of 1D transverse Ising model

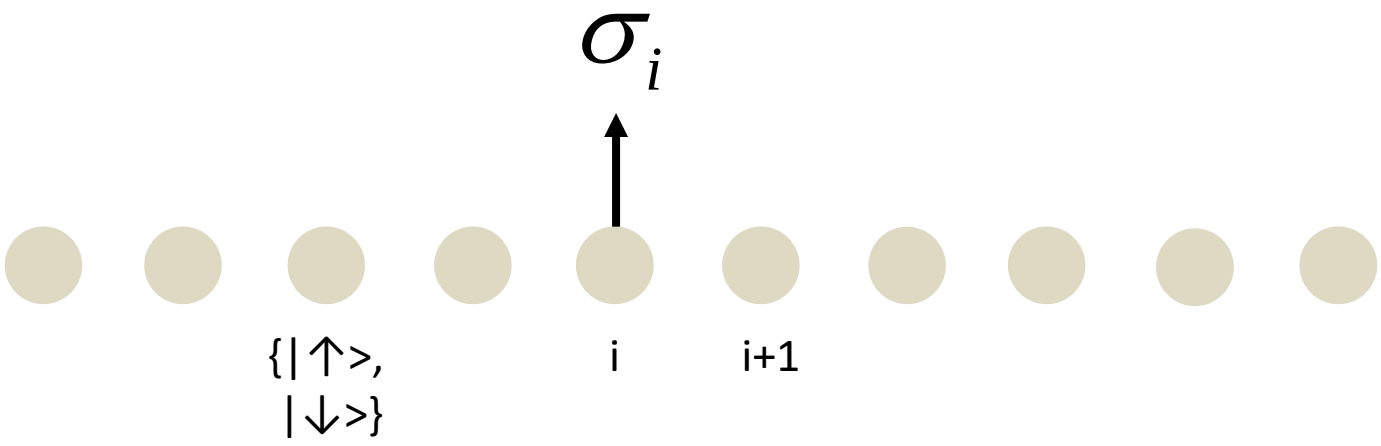
Exact treatment

Perturbation treatment

Now we consider another model, the exact solution can be obtained by Bethe ansatz, here we use perturbation theory for a general picture.

$$H = -J\sum_{i}^{N-1}\sigma_i^x\sigma_{i+1}^x - \Gamma\sum_{i}\sigma_i^z$$

Here we suppose N is even (bipartite)



From Chapter 2 of **quantum Ising phases and transitions in transverse Ising models** by Suzuki S.

Perturbative solution

$$H \to H = \Gamma \sum_i (1 - \sigma_i^z) - J \sum_i^{N-1} \sigma_i^x \sigma_{i+1}^x$$

perturbation term

Here we concern the gap of the system, i.e.

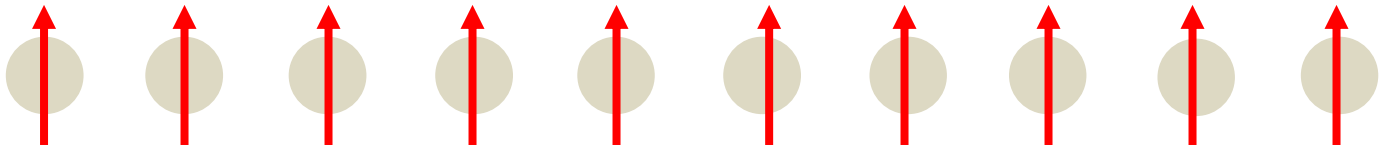
$$\Delta(\mathrm{J}) = E_1(\mathrm{J}) - E_0(\mathrm{J})$$

eigenvalue of 1 E.S. eigenvalue of the G.S.

$$H_0 = \Gamma \sum_i (1 - \sigma_i^z)$$

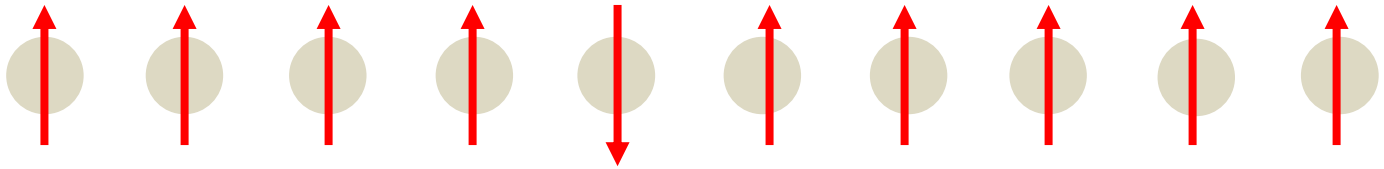
The G.S. is just product state and the eigen value is 0

$$|0\rangle = \prod_i |\uparrow_i\rangle, E_0 = 0$$

$|0\rangle =$ 

The 1 E.S. is N-fold:

$$|1\rangle = \sum_i \sigma_i^x |0\rangle, E_1 = +2$$

$|1\rangle =$ 

$$\Delta(0) = E_1(0) - E_0(0) = 2$$

Include first perturbation

$$(E_n(J))'_{J=0} = \langle n | H_1 | n \rangle$$

Set n=0, we have:

$$\langle 0 | -\sum_i^{N-1} \sigma_i^x \sigma_{i+1}^x | 0 \rangle = 0$$

Set n=1, we have:

$$\langle 1 | -\sum_i^{N-1} \sigma_i^x \sigma_{i+1}^x | 1 \rangle = -2$$

So we have

$$\Delta(\mathrm{J}) = E_1(\mathrm{J}) - E_0(\mathrm{J}) = 2(1-\mathrm{J})$$

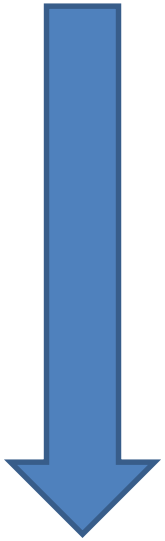
As J approaches from 0 to 1, the mass gap of the elementary excitations vanishes.

Duality

$$H = -J\sum_{i}^{N-1}\sigma_i^x\sigma_{i+1}^x - \Gamma\sum_i \sigma_i^z$$

$$\tau_i^z = \sigma_i^x\sigma_{i+1}^x$$

$$\tau_i^x = \prod_{k<i}\sigma_k^z$$



$$H = -\Gamma\sum_{i}^{N-1}\tau_i^x\tau_{i+1}^x - J\sum_i \tau_i^z = \frac{J}{\Gamma}\left(-\frac{\Gamma^2}{J}\sum_{i}^{N-1}\tau_i^x\tau_{i+1}^x - \Gamma\sum_i \tau_i^z\right)$$

Thus we have

$$H(\sigma, \mathrm{J}) = \frac{J}{\Gamma} H(\tau, \Gamma^2 / \mathrm{J})$$

Since both σ and τ operators satisfy the same algebra, the above symmetry implies one-to-one correspondence of the high and low temperature phases.

Here for the quantum model, it implies that each eigenvalue E of H satisfies the relation:

$$E(\mathrm{J}) = \frac{J}{\Gamma} E(\Gamma^2 / \mathrm{J})$$

- The above equation has important significance from the point of quantum phase transition in the model.
- For the quantum phase transition, at the critical point the mass gap vanishes and the correlation length ζ(J) (which is the inverse of the mass gap) diverges.
- For the present model, if the mass gap vanishes for some nonzero value

$$\mathrm{J}/\Gamma = 1$$

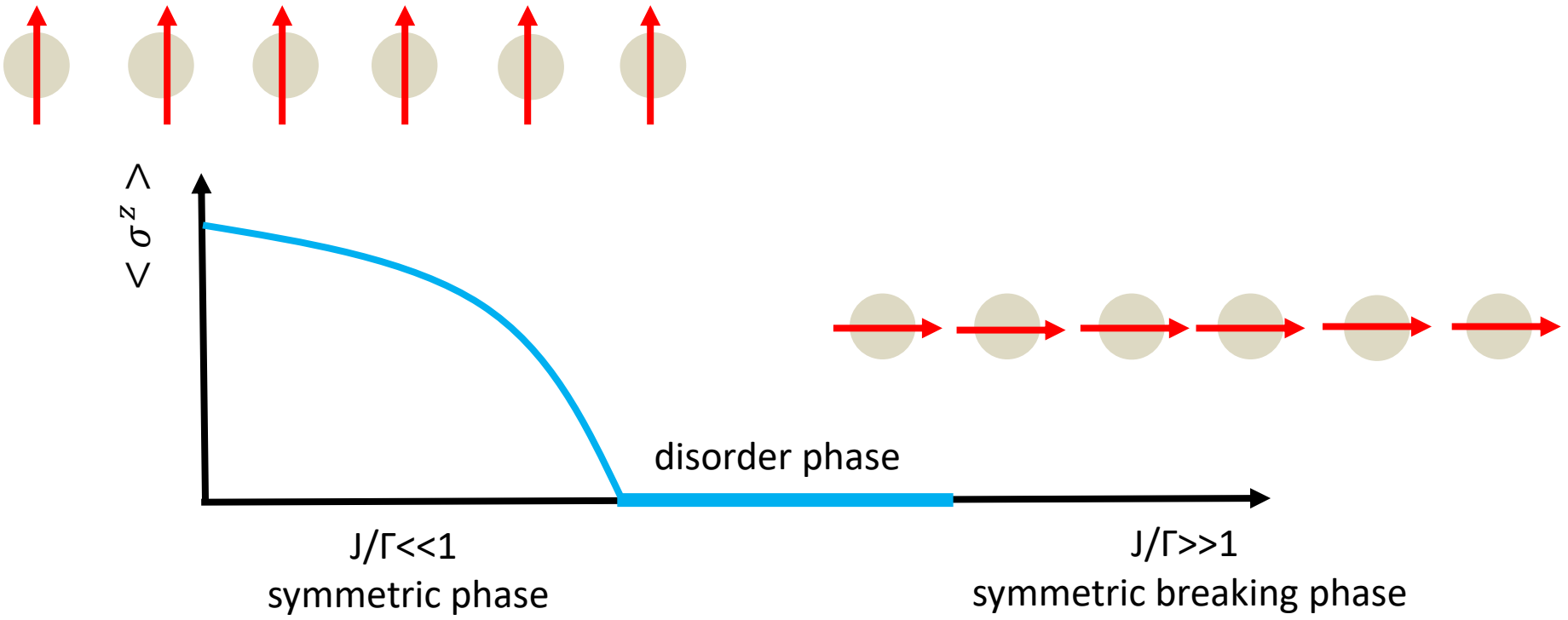
The duality is missing for higher dimensional models

Now back to the model, we can write the general form for mass gap as:

$$\Delta(\mathrm{J}) = 2\,\|1-\mathrm{J}\,\|$$

This suggests that the gap vanishes both from below and above the critical point J=1.
This also gives the estimate of correlation length through

$$\Delta(\mathrm{J}) \sim \|1-\mathrm{J}\,\|^{-\nu}, \nu = 1$$



Real space RG treatment :

We want to recombine the Hamiltonian into two terms:

first term is interblock and the second term is intrablock

$$H = H_{\text{int}\,raB} + H_{\text{int}\,erB}$$

Here we take two sites into a block

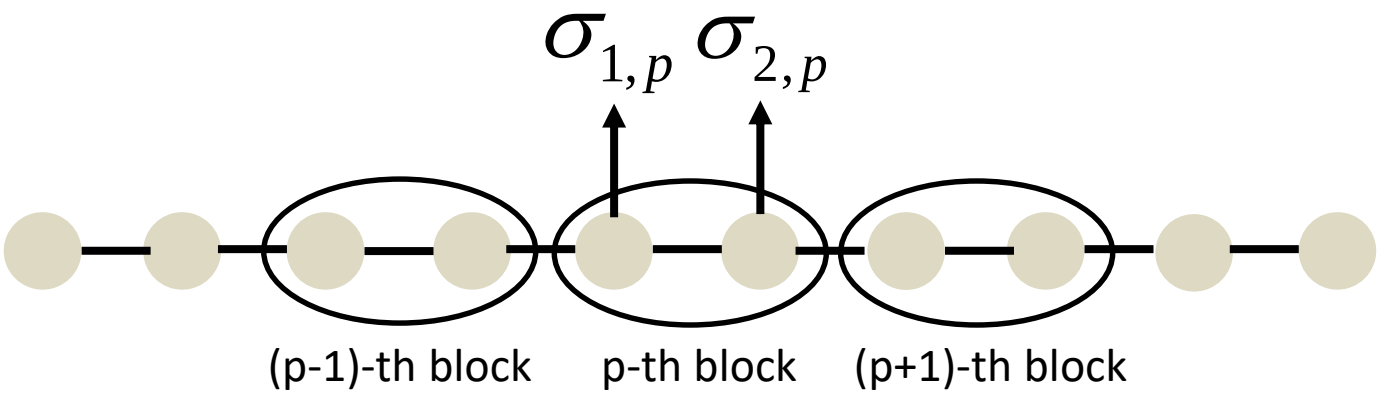
Every site now has two index rather than one:

$$\sigma_i \quad \Rightarrow \quad \sigma_{\text{i},p}$$

i from 1 to N

i from 1 to 2;
p from 1 to N/2

The total number of site doesn't change, what we do is just rearrange one index into two.



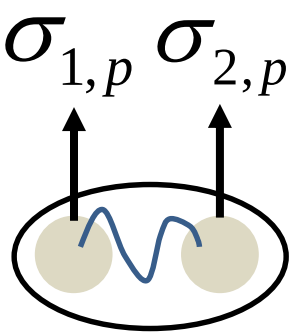
The intrablock term only depends on single block:

$$H_{\text{int}raB} = \sum_{p=1}^{N/2} \underline{H_p}$$

single block term

But the single block term is a self-consistent world, it must contain all the possible term inside the world:

$$H_p = -\Gamma\sigma_{1,p}^z - \Gamma\sigma_{2,p}^z - J\sigma_{1,p}^x\sigma_{2,p}^x$$



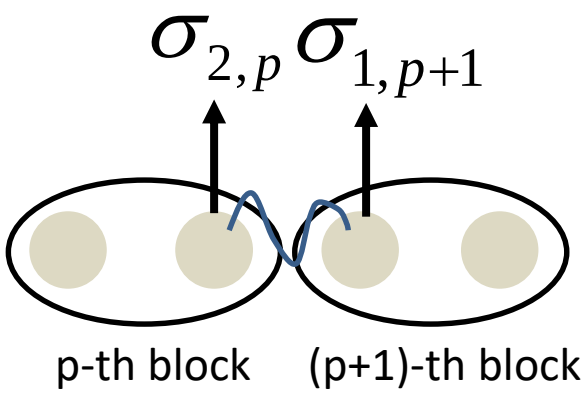
p-th block

The interblock term only depends on single block:

$$H_{interB} = \sum_{p=1}^{N/2-1} \underbrace{H_{p,p+1}}_{\text{adjoint block term}}$$

Now we need to link the adjoint worlds:

$$H_{p,p+1} = -J\sigma^{x}_{2,p}\sigma^{x}_{1,p+1}$$



First solve Hp by exact diagonalization:

$$H_p = -\Gamma\sigma^z_{1,p} - \Gamma\sigma^z_{2,p} - J\sigma^x_{1,p}\sigma^x_{2,p}$$

Use basis $\begin{bmatrix} |\uparrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \end{bmatrix}$

With $\langle\uparrow\uparrow|\sigma^z_{1,p}|\uparrow\uparrow\rangle = 1$

$\langle\uparrow\uparrow|\sigma^x_{1,p}\sigma^x_{2,p}|\downarrow\downarrow\rangle = 1$

$$H_p = \begin{pmatrix} -2\Gamma & -J & 0 & 0 \\ -J & +2\Gamma & 0 & 0 \\ 0 & 0 & 0 & -J \\ 0 & 0 & -J & 0 \end{pmatrix}$$

$$E_3 = +\sqrt{J^2 + 4\Gamma^2}, |3\rangle = \frac{1}{\sqrt{1+a^2}}(a|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$$

$$E_2 = +J, |2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$E_1 = -J, |1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$E_0 = -\sqrt{J^2 + 4\Gamma^2}, |0\rangle = \frac{1}{\sqrt{1+a^2}}(|\uparrow\uparrow\rangle + a|\downarrow\downarrow\rangle)$$

$$a = \frac{\sqrt{J^2 + 4\Gamma^2} - 2\Gamma}{J}$$

$$\frac{\sqrt{J^2 + 4\Gamma^2} - 2\Gamma}{J} * \frac{\sqrt{J^2 + 4\Gamma^2} + 2\Gamma}{J} = 1$$

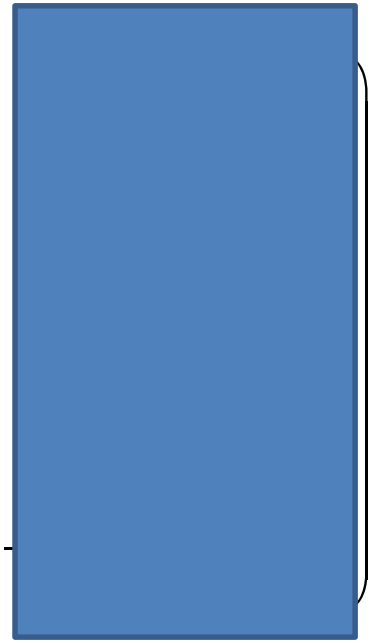
Note there is a big gap between state 2 and 3.

$$\begin{pmatrix} -2\Gamma & -J & 0 & 0 \\ -J & +2\Gamma & 0 & 0 \\ 0 & 0 & 0 & -J \\ 0 & 0 & -J & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & 0 & 0 & \frac{a}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} & 0 & 0 & -\frac{1}{\sqrt{1+a^2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & 0 & 0 & \frac{a}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} & 0 & 0 & -\frac{1}{\sqrt{1+a^2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\sqrt{J^2+4\Gamma^2} & 0 & 0 & 0 \\ 0 & -J & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & \sqrt{J^2+4\Gamma^2} \end{pmatrix}$$

Truncation by defining the projection operator:

$$\begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & 0 & \\ \frac{a}{\sqrt{1+a^2}} & 0 & \\ 0 & \frac{1}{\sqrt{2}} & \\ 0 & \frac{1}{\sqrt{2}} & \end{pmatrix}$$



Since there is a large gap between state 2 and state 3, we can approximate the Hamiltonian by cutting half of the d.o.f. off by keeping the G.S. and 1. E.S. only.

$$P = \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & 0 \\ \frac{a}{\sqrt{1+a^2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Truncate the block Hamiltonian

$$H_p^{'} = P^{\dagger} H_p P$$

$$= \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} & & \\ & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -2\Gamma & -J & 0 & 0 \\ -J & +2\Gamma & 0 & 0 \\ 0 & 0 & 0 & -J \\ 0 & 0 & -J & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & 0 \\ \frac{a}{\sqrt{1+a^2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{J^2+4\Gamma^2} & \\ & -J \end{pmatrix}$$

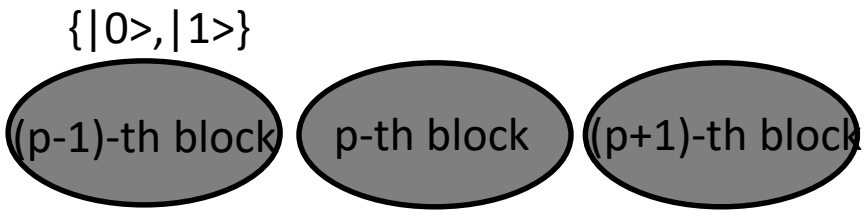
$$H_p \sim H_p^{'} = -\Gamma' \sigma_p^z + c' I_p, \Gamma' = \frac{E_1 - E_0}{2}, c' = \frac{E_1 + E_0}{2}$$

Now the intrablock term is just

$$H_{\mathrm{int}\,raB} = \sum_{p=1}^{N/2} H_p^{'}$$

$$H_p^{'} = -\Gamma' \sigma_p^z + c' I_p$$

which is equivalent to



Truncate the interaction connecting both blocks

The interblock coupling now should change, so that we can obtain running coupling constant:

$$H_{\mathrm{int}erB} = \sum_{p=1}^{N/2-1} H_{p,p+1}$$

$$H_{p,p+1} = -J\sigma^{x}_{2,p}\sigma^{x}_{1,p+1}$$

both need to be truncated

Written $\sigma_{2,p}^{x}$ into matrix form, we have

$$\sigma_{2,p}^{x} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} |\uparrow\uparrow\rangle_p \\ |\downarrow\downarrow\rangle_p \\ |\uparrow\downarrow\rangle_p \\ |\downarrow\uparrow\rangle_p \end{bmatrix}$$

$$\sigma_p^{x'} = \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} & & \\ & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & 0 \\ \frac{a}{\sqrt{1+a^2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{(1+a)}{\sqrt{2(1+a^2)}} \sigma_p^x$$

Written $\sigma^{x}_{1,p+1}$ into matrix form, we have

$$\sigma^{x}_{1,p+1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} |\uparrow\uparrow\rangle_{p+1} \\ |\downarrow\downarrow\rangle_{p+1} \\ |\uparrow\downarrow\rangle_{p+1} \\ |\downarrow\uparrow\rangle_{p+1} \end{bmatrix}$$

$$\sigma^{x'}_{p+1} = \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} & & \\ & & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & 0 \\ \frac{a}{\sqrt{1+a^2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{(1+a)}{\sqrt{2(1+a^2)}} \sigma^{x}_{p+1}$$

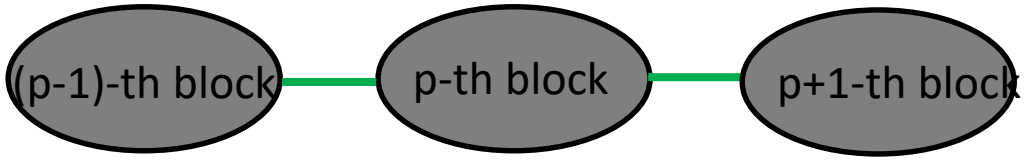
Therefore, we have

$$H_{p,p+1} = -J\sigma^x_{2,p}\sigma^x_{1,p+1} \rightarrow H^{'}_{p,p+1} = -J\frac{(a+1)^2}{2(1+a^2)}\sigma^x_p\sigma^x_{p+1}$$

So we have

$$H^{'}_{p,p+1} = -J'\sigma^x_p\sigma^x_{p+1}$$

$$J' = -J\frac{(a+1)^2}{2(1+a^2)}$$



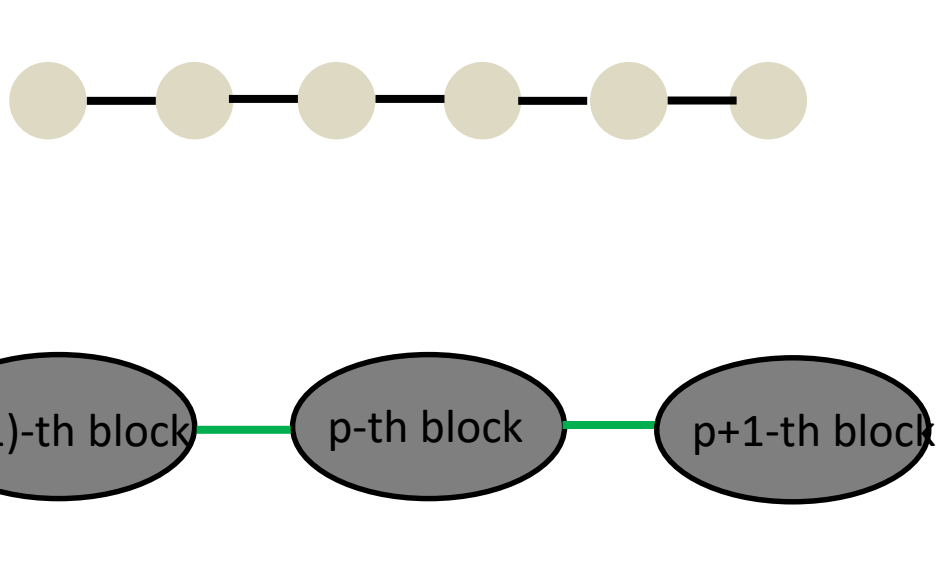
What we just do is a renormalization procedure

$$H = -J\sum_{i}^{N-1}\sigma_i^x\sigma_{i+1}^x - \Gamma\sum_i \sigma_i^z$$

$$J' = J\frac{(a+1)^2}{2(1+a^2)}$$

$$\Gamma' = \frac{E_1 - E_0}{2}$$

$$c' = \frac{E_1 + E_0}{2}$$



$$H' = -J'\sum_{p=1}^{N/2-1}\sigma_p^x\sigma_{p+1}^x - \Gamma'\sum_{p=1}^{N/2}\sigma_p^z + c'\sum_{p=1}^{N/2} I_p$$

We have restore the form of Hamiltonian.

Here we provide another way to calculate the interaction Hamiltonian

We consider the effect on two nearest block:

$$H_{p,p+1} = -J\sigma_{2,p}^{x}\sigma_{1,p+1}^{x}$$

The matrix element is

$\left\|0_p 0_{p+1}\right\rangle$	$\left\|0_p 1_{p+1}\right\rangle$	$\left\|1_p 0_{p+1}\right\rangle$	$\left\|1_p 1_{p+1}\right\rangle$
			$-J\frac{(a+1)^2}{2(1+a^2)}$ $\left\|0_p 0_{p+1}\right\rangle$
		$-J\frac{(a+1)^2}{2(1+a^2)}$	$\left\|0_p 1_{p+1}\right\rangle$
	$-J\frac{(a+1)^2}{2(1+a^2)}$		$\left\|1_p 0_{p+1}\right\rangle$
$-J\frac{(a+1)^2}{2(1+a^2)}$			$\left\|1_p 1_{p+1}\right\rangle$

Here we calculate a matrix element for example:

$$\left\langle 0_{p+1}0_p \left| \sigma^x_{2,p}\sigma^x_{1,p+1} \right| 1_p 1_{p+1} \right\rangle = \left\langle 0_p \left| \sigma^x_{2,p} \right| 1_p \right\rangle \left\langle 0_{p+1} \left| \sigma^x_{1,p+1} \right| 1_{p+1} \right\rangle$$

$$\left\langle 0_p \left| \sigma^x_{2,p} \right| 1_p \right\rangle = \frac{1}{\sqrt{2(1+a^2)}} (a \left\langle \downarrow\downarrow \right| + \left\langle \uparrow\uparrow \right|) \sigma^x_{2,p} (\left| \uparrow\downarrow \right\rangle + \left| \downarrow\uparrow \right\rangle) = \frac{1}{\sqrt{2(1+a^2)}} (a+1)$$

$$\left\langle 0_{p+1}0_p \left| \sigma^x_{2,p}\sigma^x_{1,p+1} \right| 1_p 1_{p+1} \right\rangle = \frac{(a+1)^2}{2(1+a^2)}$$

And the matrix can be written into the following form which resemble the origin form:

$$H'_{p,p+1} = -J' \sigma_p^x \sigma_{p+1}^x$$

$$\begin{array}{cccc} |0_p 0_{p+1}\rangle & |0_p 1_{p+1}\rangle & |1_p 0_{p+1}\rangle & |1_p 1_{p+1}\rangle \\ & & -J' & |0_p 0_{p+1}\rangle \\ & -J' & & |0_p 1_{p+1}\rangle \\ -J' & & & |1_p 0_{p+1}\rangle \\ & & & |1_p 1_{p+1}\rangle \end{array}$$

but with only the coupling constant renormalized:

$$J' = J \frac{(a+1)^2}{2(1+a^2)}$$

We can repeat the step further. Before doing this, we introduce an index n to label the steps,

$$H(\mathrm{n}) = -J(\mathrm{n}) \sum_{p=1}^{N/2^n-1} \sigma_p^x \sigma_{p+1}^x - \Gamma(\mathrm{n}) \sum_{p=1}^{N/2^n} \sigma_p^z + c(\mathrm{n}) \sum_{p=1}^{N/2^n} I_p$$

$$J(\mathrm{n}+1) = J \frac{(a(\mathrm{n})+1)^2}{2(1+a^2(\mathrm{n}))} = J(\mathrm{J},\Gamma)$$

$$\Gamma(\mathrm{n}+1) = \frac{E_1(\mathrm{n}) - E_0(\mathrm{n})}{2} = \Gamma(\mathrm{J},\Gamma)$$

$$c(\mathrm{n}+1) = \frac{E_1(n) + E_0(n)}{2} + 2c(\mathrm{n}) = \mathrm{c}(\mathrm{J},\Gamma)$$

And the initial system is

$$J(0) = J, \Gamma(0) = \Gamma, c(0) = 0$$

And the first step system is:

$$J(1) = J \frac{(a(0)+1)^2}{2(1+a^2(0))}$$

$$\Gamma(1) = \frac{E_1(0) - E_0(0)}{2} = \frac{-J + \sqrt{J^2 + 4\Gamma^2}}{2}$$

$$c(1) = \frac{E_1(0) + E_0(0)}{2} = \frac{-J - \sqrt{J^2 + 4\Gamma^2}}{2}$$

Fixed point

The property we concern is $\lambda = \frac{\Gamma}{J}$

The fixed point satisfy the equation:

$$\frac{\Gamma(\mathrm{n}+1)}{J(\mathrm{n}+1)} \equiv f(\lambda(\mathrm{n})) = \lambda(\mathrm{n})$$

For the first step, we have

$$f(\lambda(0)) = \frac{\Gamma(1)}{J(1)} = \frac{\frac{-J+\sqrt{J^2+4\Gamma^2}}{2}}{\frac{J(a(0)+1)^2}{2(1+a^2(0))}} = \frac{(1+a^2(0))(\sqrt{1+4\lambda^2(0)}-1)}{(a(0)+1)^2}$$

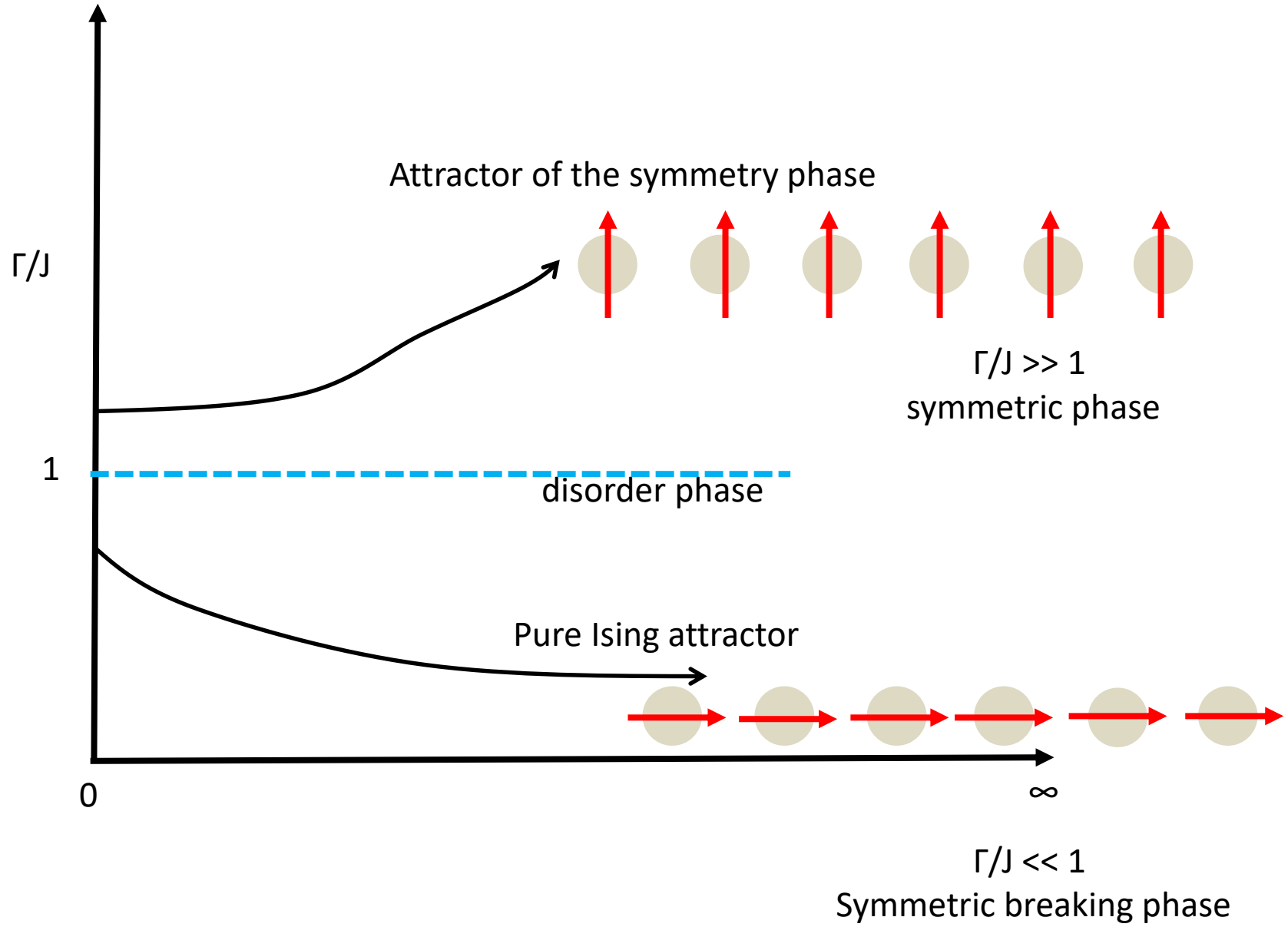
$$a(0) = \frac{\sqrt{J^2+4\Gamma^2}-2\Gamma}{J} = \sqrt{1+4\lambda^2(0)}-2\lambda(0)$$

Solving the above equation numerically, we can find the fixed point

$$\lambda^* = 1.227$$

The critical value of λ approaches the exact value, i.e. 1 as one considers larger and larger block size.

Phase diagram



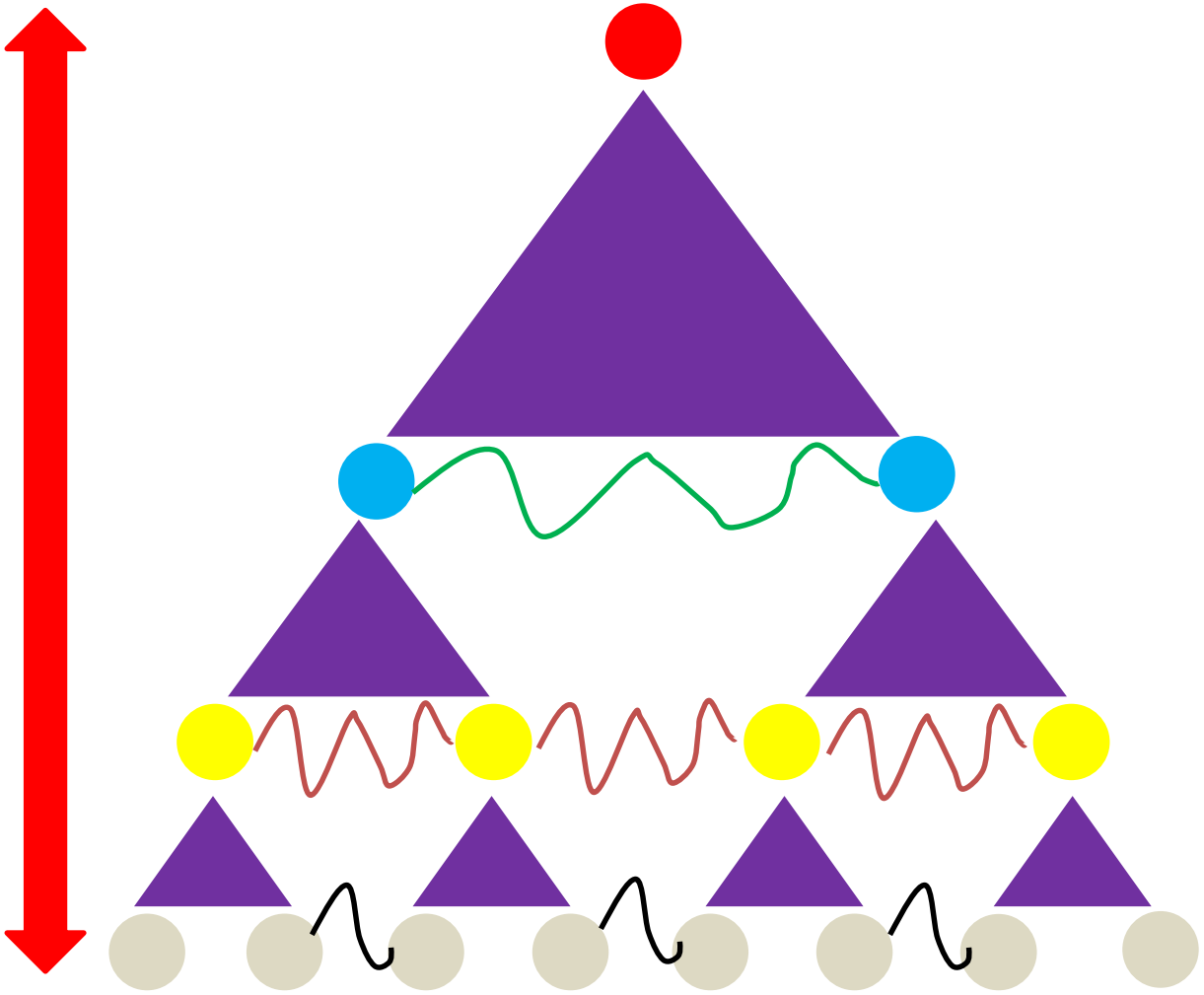
Summary of algorithm :

0. consider a system compose of N sites that can be studied in an exact numerical way. Build the Hamiltonian H_N: $\mathbb{C}^{d^N} \rightarrow \mathbb{C}^{d^N}$;
1. diagonalize H_N, finding its eigenvalues and eigenvectors $\hat{H}_N = \sum_{i=1}^{d^N} E_i |E_i\rangle\langle E_i|$ where the eigenvalues Ei are in increasing order. Consider the projector onto the lowest m eigenstates $\hat{P} = \sum_{i=1}^{m} |E_i\rangle\langle E_i|$ which project the Hilbert space on the subspace spanned by the first m low-energy laying eigenstates. Compute the projected Hamiltonian $\hat{\tilde{H}}_N = \hat{P}^\dagger \hat{H}_N \hat{P}$ as well as any other operator representation in the projected space, i.e. $\hat{\tilde{O}} = \hat{P}^\dagger \hat{O} \hat{P}$.
2. construct the Hamiltonian of a system of size 2N using the projected Hamiltonian $\hat{\tilde{H}}_N$ for each bipartition and the interaction among them, $\hat{H}_{2N} = \hat{\tilde{H}}_N \otimes 1 + 1 \otimes \hat{\tilde{H}}_N + \hat{\tilde{H}}_{\text{int}}$. The interaction Hamiltonian can be obtained as $\hat{\tilde{H}}_{\text{int}} = \hat{\tilde{A}}_N \otimes \hat{\tilde{B}}_N$ where $\hat{\tilde{A}}$ ($\hat{\tilde{B}}$) are the projected operator acting on each system bipartition $\hat{\tilde{A}} = \hat{P}^\dagger \hat{A} \hat{P} \left(\hat{\tilde{B}} = \hat{P}^\dagger \hat{B} \hat{P}\right)$
3. Repeat the step 1-2 until the desired system size is reached or convergence to the RG fixed point is achieved. Notice that, at each step of the algorithm, the dimension of described system is doubled while the dimension of the Hamiltonian representation is kept constant to m (which is used to keep the form of Hamiltonian the same).

Block-exact solve and truncation-restore Hamiltonian-running coupling constant

Picture

Macroscopic quantum phenomena (fixed point)



Microscopic quantum interaction

E3: Field theory treatment of superconductivity

RG in quantum field theory

Here we first review some basic knowledge.

See Chapter 13, book magnetism and superconductivity by Levy, L.-P.

Chern-Simmons theory in 1+2-D

The Lagrangian is

$$L_{CS} = \frac{k}{4\pi}\int d\vec{r}\varepsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda}$$

Here k is the coupling constant, μ, ν and λ run over t, x and y.

(A_x, A_y) is the vector potential

A_t is the scalar potential ϕ

$\varepsilon^{\mu\nu\lambda}$ is the rank-3 CL antisymmetric tensor

The action is

$$S_{CS} = \int dtL_{CS} = \frac{k}{4\pi}\int d\vec{r}dt\varepsilon^{\mu\nu\lambda}A_{\mu}\partial_{\nu}A_{\lambda}$$

http://www-personal.umich.edu/~sunkai/teaching/Fall_2013/chapter6.pdf

So the total term for gauge field is

$$\mathrm{L}_{gauge-field} = \int d\vec{r}[\frac{1}{4}(F_{\mu\nu})^2 + J_\mu A^\mu + C\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda]$$

See Chapter 7, book introduction to topological quantum computation by Jiannis, K. P.

Yong Tao's work

First he introduce formal field theory treatment of superconductivity, but not in 1+3-D, but in 0+3-D;
Second he introduce "Chern-Simmons term", but his three dimension is (x, y, z), not t, x, y, very wired;

Many undetermined thing about this paper

https://chuansongme.com/n/2834965

E4: 量子多体系统中的量子相变

我找到了此书，内容是关于重整化群在量子多体系统中的量子相变研究，
里面尽可能包含了做研究所用的计算细节，所以决定详读一下

量子相变严格的定义是在零温下（零温热力学涨落已经被严格压制，但是因为还有零点振动能等量子因素，所以可能会发生量子相变），由于Heisenberg不确定性原理，当系统的一个相互作用与系统其他部分的Hamiltonian不对易时，这一相互作用便会导致后者的本征态之间发生量子跃迁，也就是量子涨落。当量子涨落达到足够的强度以至于用来描述系统的主要物理过程的本征态发生了根本性的改变，这时候就是发生了量子相变。它是由量子涨落导致的。
尽管这一过程发生在零温，但是在许多有限温度的情况中，特别是系统在量子相变点附近的物理性质却主要地由零温下地相变行为决定，这使得零温下量子相变地研究显得尤为重要。

来自何培松，<<量子多体体系中的量子相变>>

Coherent representation

Coherent state is the eigenstate of annihilation operator.

$$a_\alpha \left| z_\alpha \right\rangle = z_\alpha \left| z_\alpha \right\rangle$$

where α labels the mode.

Why coherent representation?
Under coherent representation, a operator is transformed in a number (Grassmann number for fermion). When we treat path integral and loop diagram, we no longer need to treat the production of operators but numbers.

Bosonic coherent state

Bosonic coherent state is a semi-harmonic oscillator.

$$|z_\alpha\rangle = \exp[-\frac{1}{2}\|z_\alpha\|^2]\exp[z_\alpha a_\alpha^\dagger|0\rangle] \qquad [a_\alpha, a_\beta^\dagger]_- = \delta_{\alpha\beta}$$

Normalization factor: Gaussian function

Proof:

$$a_\alpha|z_\alpha\rangle = \exp[-\frac{1}{2}\|z_\alpha\|^2]a_\alpha \exp[z_\alpha a_\alpha^\dagger|0\rangle]$$

$$= \exp[-\frac{1}{2}\|z_\alpha\|^2]\sum_{n_\alpha}\frac{a_\alpha(z_\alpha a_\alpha^\dagger)^{n_\alpha}|0\rangle}{n_\alpha!}$$

$$\xrightarrow{|n_\alpha\rangle=\frac{(a_\alpha^\dagger)^{n_\alpha}}{\sqrt{n_\alpha!}}|0\rangle} \exp[-\frac{1}{2}\|z_\alpha\|^2]\sum_{n_\alpha}\frac{(z_\alpha)^{n_\alpha}a_\alpha|n_\alpha\rangle}{\sqrt{n_\alpha!}}$$

$$\xrightarrow{a_\alpha|n_\alpha\rangle=\sqrt{n_\alpha}|n_\alpha-1\rangle} \exp[-\frac{1}{2}\|z_\alpha\|^2]\sum_{n_\alpha}\frac{(z_\alpha)^{n_\alpha}\sqrt{n_\alpha}|n_\alpha-1\rangle}{\sqrt{n_\alpha!}}$$

$$= \exp[-\frac{1}{2}\|z_\alpha\|^2]\sum_{n_\alpha}\frac{(z_\alpha)^{n_\alpha}|n_\alpha-1\rangle}{\sqrt{(n_\alpha-1)!}} = z_\alpha \exp[-\frac{1}{2}\|z_\alpha\|^2]\sum_{n_\alpha}\frac{(z_\alpha)^{n_\alpha-1}|n_\alpha-1\rangle}{\sqrt{(n_\alpha-1)!}}$$

$$= z_\alpha|z_\alpha\rangle$$

So for more modes, we have

$$a_\alpha \left| z_1 \ldots z_\alpha \ldots \right\rangle = z_\alpha \left| z_1 \ldots z_\alpha \ldots \right\rangle$$

And

$$\left| z_1 \ldots z_\alpha \ldots \right\rangle = \exp[-\frac{1}{2}\sum_\alpha \left\| z_\alpha \right\|^2] \exp[\sum_\alpha z_\alpha a_\alpha^\dagger \left| 0 \right\rangle]$$

Fermion coherent state

$$\left|z_1 \ldots z_\alpha \ldots\right\rangle = \exp[-\frac{1}{2}\sum_\alpha \left\|z_\alpha\right\|^2]\exp[\sum_\alpha z_\alpha a_\alpha^\dagger \left|0\right\rangle]$$

Grassmann algebra

$$\left|z_1 \ldots z_\alpha \ldots\right\rangle = \exp[-\frac{1}{2}\sum_\alpha \xi_\alpha^* \xi_\alpha]\exp[\sum_\alpha \xi_\alpha a_\alpha^\dagger \left|0\right\rangle]$$

General form

Subject 1: 石墨烯的非朗道费米液体行为

Transform low-energy effective theory to quantum field theory

$$H_{\mathrm{TB},+1} = (-v_F)\sum_{\vec{q}} \varphi^{\dagger}_{+1,\vec{q}}(\sigma^x k_x + \sigma^y k_y)\varphi_{+1,\vec{q}}$$

$$H_{\mathrm{TB},-1} = (-v_F)\sum_{\vec{q}} \varphi^{\dagger}_{-1,\vec{q}}(-\sigma^x k_x + \sigma^y k_y)\varphi_{-1,\vec{q}}$$

Here +1 and -1 label the valley d.o.f. and $\hat{\varphi}$ is a spinor, where the d.o.f. is not a real spin, but pseudospin:

$$\varphi_{+1,\vec{q}} = \begin{pmatrix} a_{+1,\vec{q}} \\ \hat{b}_{+1,\vec{q}} \end{pmatrix}$$

Since $\vec{q} \to 0$, which correspond to a long-wave limit. And introduce replacement:

$$k_x \to i\partial_x, k_y \to i\partial_y$$

$$\sum_{\vec{q}} \to \int d\vec{r}$$

$$\varphi^{\dagger}_{+1,\vec{q}} \to \varphi^{\dagger}_{+1}(\vec{r}), \varphi_{+1,\vec{q}} \to \varphi_{+1}(\vec{r})$$

$$H_{\text{TB,+1}} = (-v_F)\sum_{\vec{q}} \varphi^{\dagger}_{+1,\vec{q}}(\sigma^x k_x + \sigma^y k_y)\varphi_{+1,\vec{q}}$$

$$\to (-iv_F)\int d\vec{r}\varphi^{\dagger}_{+1}(\vec{r})(\sigma^x\partial_x + \sigma^y\partial_y)\varphi_{+1}(\vec{r})$$

$$H_{\text{TB,-1}} = (-v_F)\sum_{\vec{q}\sigma} \varphi^{\dagger}_{-1,\vec{q}}(\sigma^x k_x - \sigma^y k_y)\varphi_{-1,\vec{q}}$$

$$\to (-iv_F)\int d\vec{r}\varphi^{\dagger}_{-1}(\vec{r})(\sigma^x\partial_x - \sigma^y\partial_y)\varphi_{-1}(\vec{r})$$

To get our full Hamiltonian, we need to add them up and now we label the pseudospin d.o.f. by σ and spin d.o.f. by s

And our Hamiltonian for pseudospin is

$$H = (-iv_F)\int d\vec{r}\{\varphi^{\dagger}_{+1}(\vec{r})(\sigma^x\partial_x + \sigma^y\partial_y)\varphi_{+1}(\vec{r}) + \varphi^{\dagger}_{-1}(\vec{r})(\sigma^x\partial_x - \sigma^y\partial_y)\varphi_{-1}(\vec{r})\}$$

And our Hamiltonian for pseudospin+spin is

$$H = (-iv_F)\int d\vec{r}\{\varphi^{\dagger}_{+1,\uparrow}(\vec{r})(\sigma^x\partial_x + \sigma^y\partial_y)\varphi_{+1,\uparrow}(\vec{r}) + \varphi^{\dagger}_{+1,\downarrow}(\vec{r})(\sigma^x\partial_x + \sigma^y\partial_y)\varphi_{+1,\downarrow}(\vec{r})$$

$$+\varphi^{\dagger}_{-1,\uparrow}(\vec{r})(\sigma^x\partial_x - \sigma^y\partial_y)\varphi_{-1,\uparrow}(\vec{r}) + \varphi^{\dagger}_{-1,\downarrow}(\vec{r})(\sigma^x\partial_x - \sigma^y\partial_y)\varphi_{-1,\downarrow}(\vec{r})\}$$

where there is no spin-orbit coupling term.

From Hamiltonian to Lagrangian

Equation of motion

Symmetry analysis

Into practice

After the derivation of Lagrangian, we can start from it can do renormalization group analysis