Perturbative or non-perturbative

Consider a action

$$\underbrace{S[\phi(\mathrm{x})}_{\text{Field}};\underbrace{\lambda]}_{\text{Coupling constant}}$$

First we have the free theory:

$$S_0[\phi(\mathrm{x});\lambda=0]$$

Then we expect we have

$$O(\lambda)=\underbrace{O_0}_{\text{Tree}}+\underbrace{O_1\lambda}_{\text{One loop}}+\underbrace{O_2\lambda^2}_{\text{Two loop}}+\dots$$

Ultraviolet divergence

All the loop diagram is divergent for we need to treat infinite k.

Cure ultraviolet divergence

- Regularization

Set a length scale below which process have no influence on the theory

- Renormalization

Absorb infinite to bare parameter, we suppose the bara parameter has no aforementioned restriction.

$$O(\lambda) = O_0^R + O_1^R \lambda + O_2^R \lambda^2 + \dots$$

Then each term is finite now. But the series may still be divergent!!!

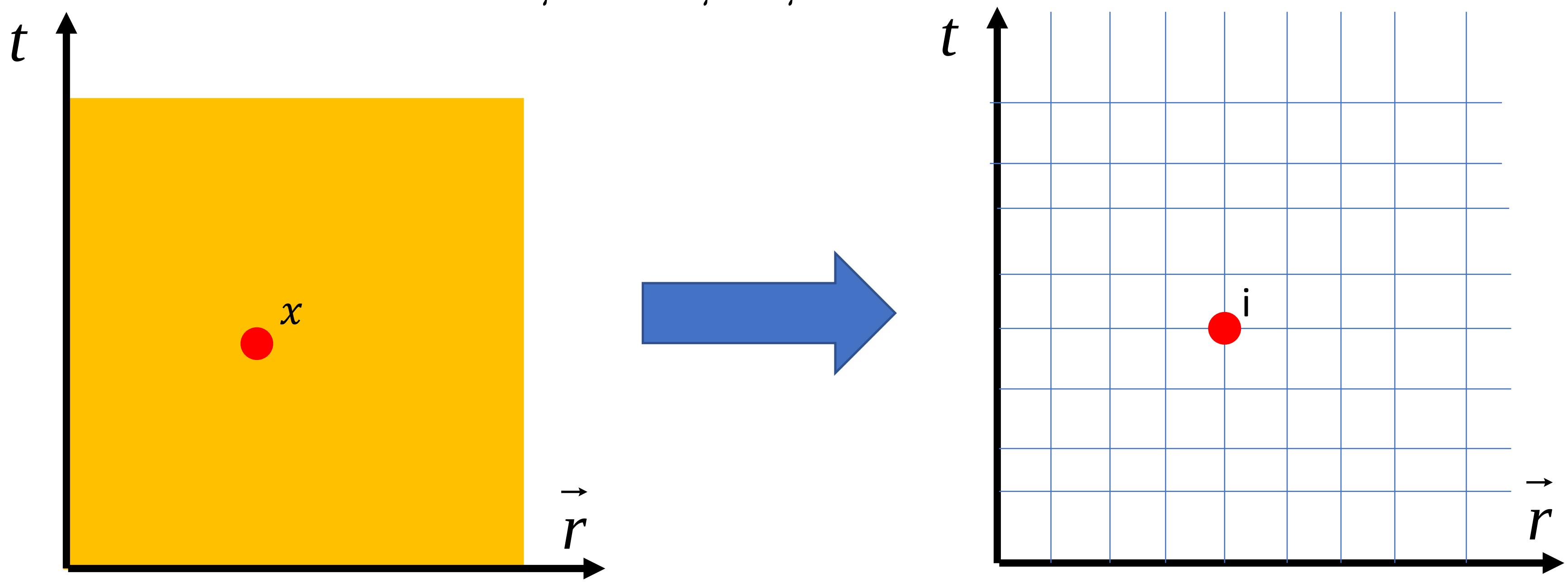
Lattice quantum field theory

Lattice discretization

As we try to show, the functional method is not well-defined. Now we try to discrete the theory. How is it defined?

In QM, the path integral representation can be derived as a limit of a discretization in time. As in QFT, the fields depend on the four Euclidean coordinates instead of a single time coordinate, we may now introduce a discretized space-time in form of a lattice, for example a hypercubic lattice, specified by

$$x_\mu = a n_\mu, n_\mu \in Z$$



The quantity a is called the lattice spacing for obvious reasons. It should be noted that the lattice spacing, being a dimensionful quantity, is not a parameter of the discretized theory, which could be inserted in a computer program for an evaluation of the path integral. The size of the lattice spacing in physical units is a derived quantity determined by the dynamics.

http://www.scholarpedia.org/article/Lattice_quantum_field_theory

The scalar field $\varphi(\vec{x}, t)$ is now defined on the lattice points only:

$$\varphi(\vec{\mathrm{x}}, \mathrm{t}) \rightarrow \varphi_i$$

Partial derivatives are replaced by finite differences:

$$\partial_\mu \varphi \rightarrow \Delta_\mu \varphi \equiv \frac{1}{a}(\varphi_{i+1\mu} - \varphi_i)$$

Space-time integrals by sums: $\int dx \rightarrow \sum_i a^4$

The measure Dφ:

$$\prod_x d\varphi(x) \leftrightarrow \sum_{\{\varphi_i\}}$$

The measure Dφ is ill-defined on QFT, but a well-defined object in lattice QFT, only a discrete set of variables has to be integrated. If the lattice is taken to be finite, one just has finite dimensional integrals.

The action

The partition function

Dual view of lattice

High energy	Condensed matter
Lattice is virtual tool (ultra-violet regulator)	Lattice is real (in real crystaks)
Lattice is usually 4-dimensional	Lattice is less than 4-dimensional
Simple hypercubic lattice is usually enough	Diverse lattice structure

Since the lattice spacing gives a natural cutoff for momentum, lattice QFT can cure the QFT-infinities. In this sense, defining QFT on a lattice is a better approximation than defining QFT on a smooth manifold.

An example: ϕ^4-theory

Here we show a action in the lattice form and the continuum form.

$$L(\varphi,\partial_\mu\varphi)=\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)-\frac{m_0}{2}\varphi^2-\frac{g_0}{4!}\varphi^4$$

$$S[\varphi]=\int dxL(\varphi,\partial_\mu\varphi)$$

$$Z=\int D[\varphi]\exp[\mathrm{i}S[\varphi]]$$

Now we discrete this theory

$$\partial_\mu\varphi\rightarrow\Delta_\mu\varphi\equiv\frac{1}{a}(\varphi_{i+1\mu}-\varphi_i)$$

$$(\partial_\mu\varphi)(\partial^\mu\varphi)\rightarrow(\Delta_\mu\varphi)^2$$

The theory goes like

$$L(\varphi,\partial_\mu\varphi)\rightarrow\frac{1}{2}(\Delta_\mu\varphi)-\frac{m_0}{2}\varphi_i^2-\frac{g_0}{4!}\varphi_i^4$$

$$S[\varphi]=\int dxL(\varphi,\partial_\mu\varphi)\rightarrow\sum_i a^4\{\frac{1}{2}(\Delta_\mu\varphi)-\frac{m_0}{2}\varphi_i^2-\frac{g_0}{4!}\varphi_i^4\}$$

$$Z=\int D[\varphi]\exp[\mathrm{i}S[\varphi]]\rightarrow\sum_{\{\varphi_i\}}\sum_i a^4\{\frac{1}{2}(\Delta_\mu\varphi)-\frac{m_0}{2}\varphi_i^2-\frac{g_0}{4!}\varphi_i^4\}$$

http://www.scholarpedia.org/article/Lattice_quantum_field_theory

Lattice gauge theory

From chapter 13 of **Gauge field theories: an introduction with applications** by Mike Guidy

Code gauge fields into space-time lattice

Following the book, we consider QCD, or the gauge group is SU(3).

gauge field theory

=

matter field

+

gauge field

First we consider matter field, which is quark and antiquark here:

- quark and antiquark are confined to the lattice sites;
- a quark or antiquark state can be specified by $|n, \alpha, i, \sigma\rangle$

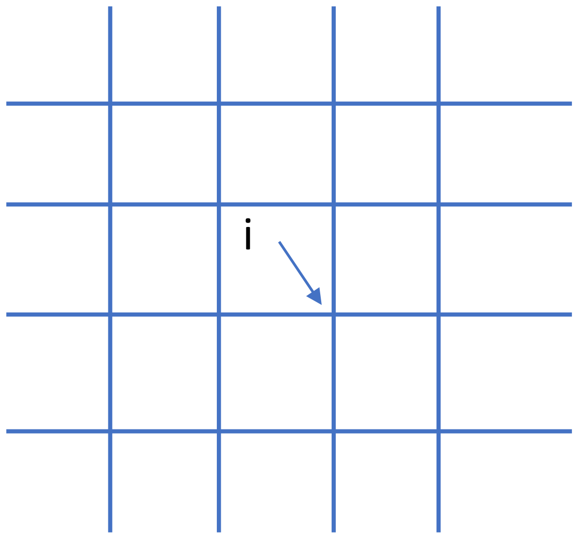
  where n is the site index,

  α is a flavor index (up, down, strange…),

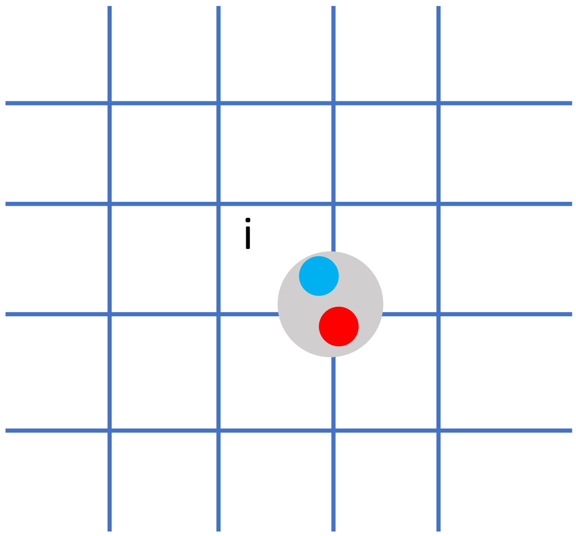
  i is a color index (red, blue, green),

  σ is a Dirac spin index
- multiple quark states can be formed by putting more than one quark on the lattice;
- there can be configuration with no quarks at all (quark vacuum);
- the # of quarks that can exist on a single lattice site is limited only by the Pauli principle;

  for example, if we restrict consideration to u, d and s flavors, as many as 18 quarks and 18 antiquarks could occupy a single lattice site.
- the Dirac operator $\bar{\psi}_{n\alpha i\sigma}$ and $\psi_{n\alpha i\sigma}$ that create and annihilate quarks are like those of a continuum theory, except they act only at the lattice sites.



quark vacuum

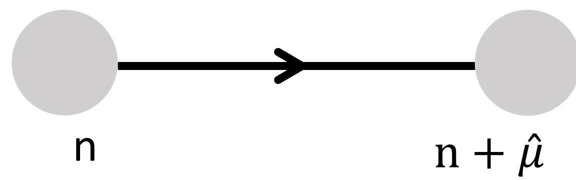


quark-antiquark at site i

Second we consider gauge fields.

Where does the gauge field lives?
Considering the fact that gauge fields are generators of "transport operators", it's natural to imagine gauge field to live between sites.

The path connecting adjacent sites on the lattice are called (directed) string bits.



String operators

Consider a cubic spacetime lattice of spacing a. The lattice string operator for an Abelian field $A_{n\mu}$ on the string bit between site n and $\mathrm{n} + \hat{\mu}$ is

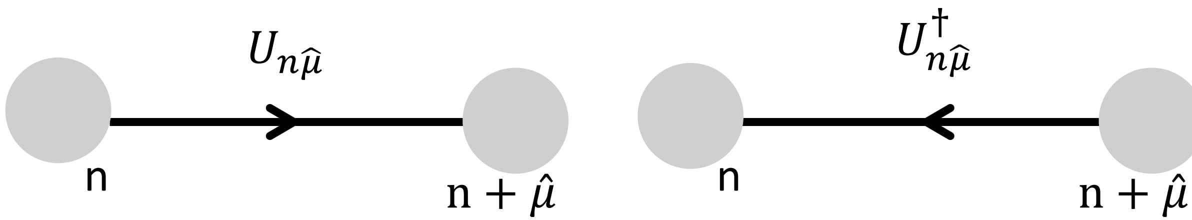
$$\mathrm{U}_{n\mu} \equiv U(n+\mu, n) = \exp(\mathrm{ig}\, a\, \mathrm{A}_{n\mu})$$

For a non-Abelian color field this generalizes to

$$\mathrm{U}_{n\mu} \equiv \exp(\mathrm{ig}\, a\, \mathrm{A}^{i}_{n\mu} \mathrm{F}_{i})$$

where Fi are the eight 3*3 Hermitian generators of SU(3) matrix.

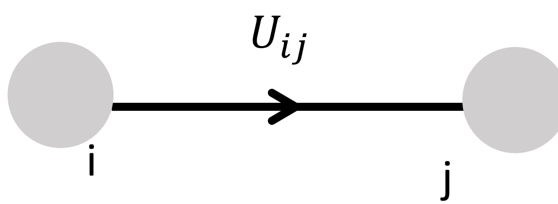
Therefore, $U_{n\hat{\mu}}$ is a unitary matrix with component $(U_{n\hat{\mu}})_{kl}$, where k is the anticolor index and l is the color index. Each component of this matrix is an operator that create or annihilate a string bit.
For example, $U_{n\hat{\mu}}$ creates a forward sting bit or annihilates a reverse string bit, and the Hermitian conjugate operator $U^{\dagger}_{n\hat{\mu}}$ annihilates a forward sting bit or creates a reverse string bit:



By unitary the string operator satisfy:

$$(\mathrm{U}_{n\mu})_{kl}(\mathrm{U}^{\dagger}_{n\mu})_{lq} = \delta_{kq}$$

To simplify expression, we will often drop the vector notation and just use U_{ij} to mean the string bit



In this notation, the two indices label adjacent sites of the lattice; they should not be confused with the color indices of the matrix U, which will not be displayed henceforth except where necessary. The notation below is understood to mean the inverse in the group sense:

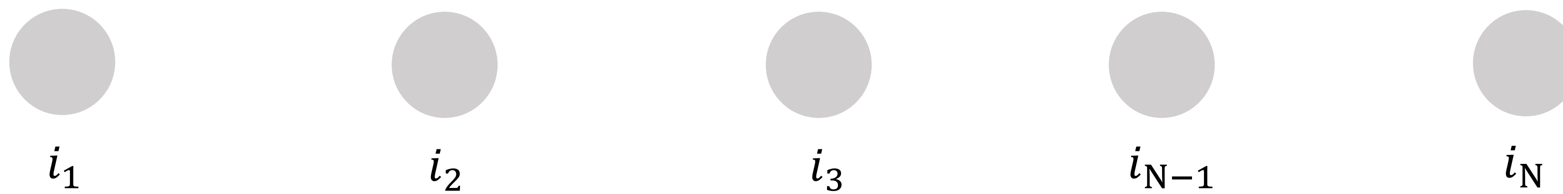
$$\mathrm{U}_{ji} = (\mathrm{U}_{ij})^{-1} = \mathrm{U}^{?}_{ij}$$

Create a string

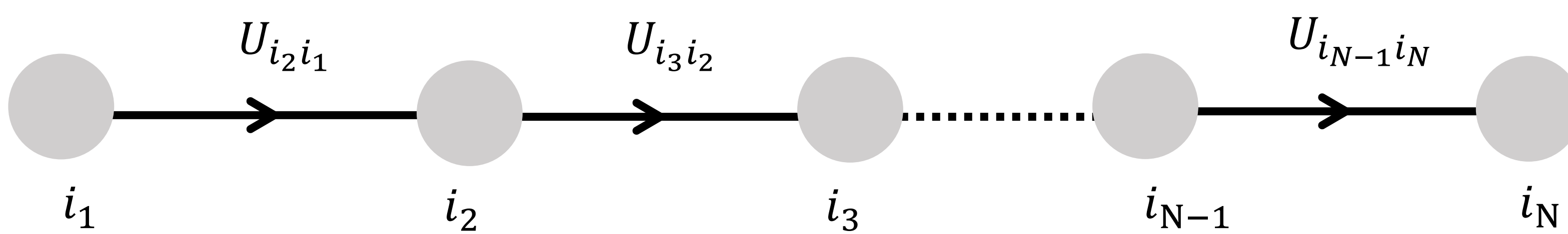
On the vacuum of both matter field and gauge field, we can create string state by:

$$U_\lambda = U_{i_N i_{N-1}} \ldots U_{i_3 i_2} U_{i_2 i_1}$$

Vacuum



String State



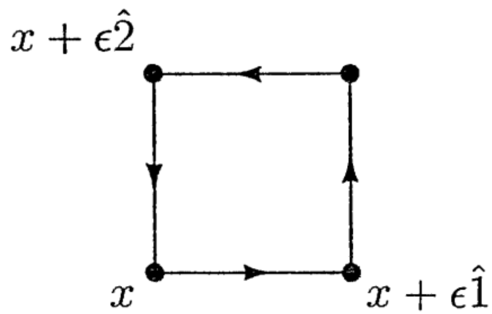
Obviously, such a state is not gauge invariant.

Create a plaquette

For a closed path c the trace of the transport operator

$$W(\mathrm{c}) = \mathrm{Tr}\, U_c = Tr \prod_{\substack{closed\\ path\ \mathrm{c}}} U_{ij}$$

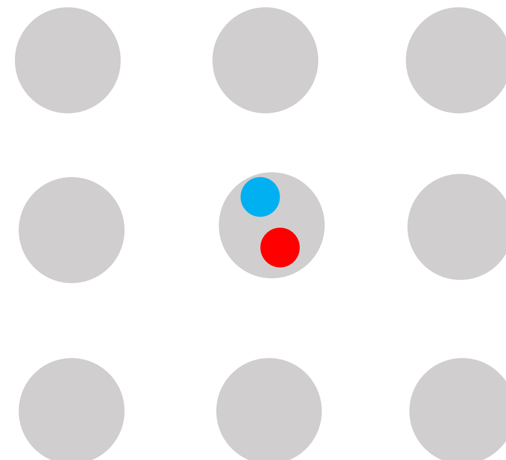
is called a Wilson loop, which a gauge invariant.



Gauge invariant states on lattice

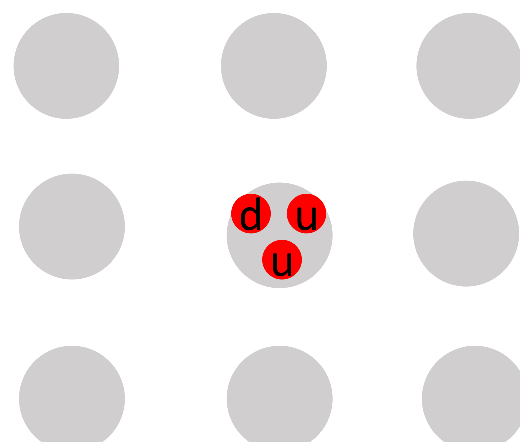
- mass term

1, Ground-state mesons, constructed from a quark and antiquark at the same lattice site;



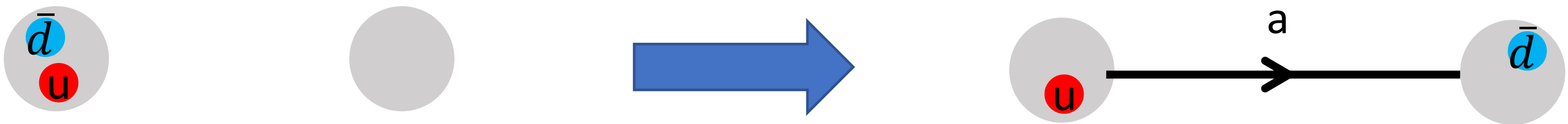
Only matter field,
no string bit, no gauge field.

2, Ground-state baryons, constructed from three quarks at the same lattice site;



Only matter field,
no string bit, no gauge field.

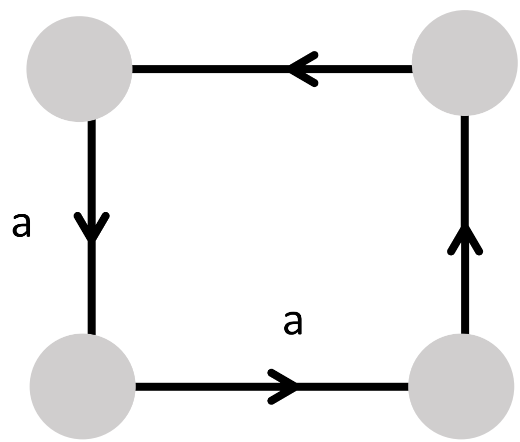
3, Excited meson and baryon states, constructed from quarks and string bits;



$$M_{ground} = m_u + m_d \qquad M_{excited} = 2m_u + \sigma a$$

Thus, for a string energy density σ and lattice spacing a the lowest excited state lies an energy σa above the G.S.

4, Glueballs, constructed from string bits arranged in closed loops;



$$M_{glue} = 4\sigma a$$

Gauge invariant states on lattice

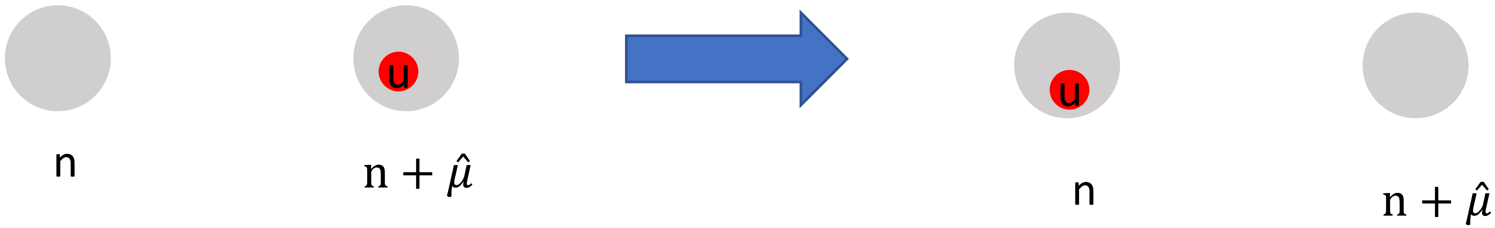
- kinetic term

  Kinetic energies are described by lattice operators that cause quarks or string bits to move on the lattice, by annihilation them at one spacetime site or string bit and creating them at another.

1, quark kinetic energy operator;

$$\Gamma_q = \overline{\psi}_n U_{n\mu} \psi_{n+\mu}$$

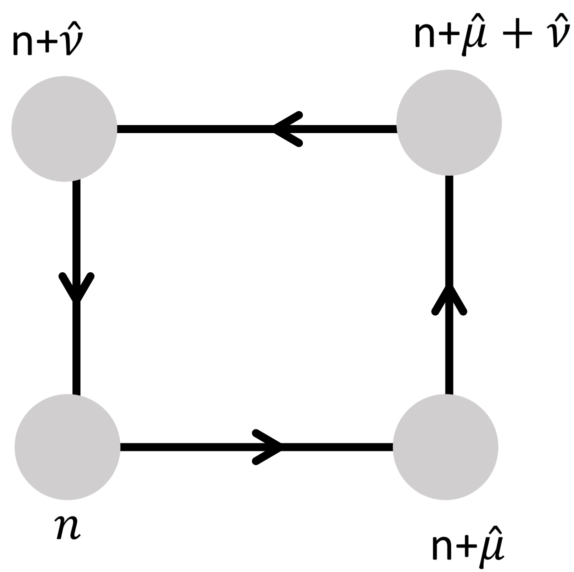
This operator can annihilate a quark at lattice site $\mathrm{n} + \hat{\mu}$ and create one at lattice site n:



2, quark kinetic energy operator;

In quark kinetic energy operator, we introduce the string bit $U_{n\hat{\mu}}$, now we need to have a term including $U_{n\hat{\mu}}$ to self-consistency:

$$\Gamma_g = Tr(U_{n\mu} U_{n+\mu,\nu} U^{\dagger}_{n+\nu,\mu} U^{\dagger}_{n\nu})$$



Now we have obtain all the building block for lattice gauge theory,
here we will go to study some models.

See **Lattice Gauge theories and Monte Carlo Simulations** by Claudio Rebbi

Lattice Higgs field theory

Lattice gauge theory
in the Hamiltonian form