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Herramientas para cálculos perturbativos en renormalización de Hamiltonianos

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Resumen

Los processos de renormalización son fundamentales para la comprensión de la teorías a distintas escalas, definiendo teorías efectivas que describen el comportamiento de los sistemas para las distintas energías. Sin embargo, los cálculos son no triviales, y se emplean la teoría de perturbaciones para obtener los resultados, representando los distintos términos de la teoría como diagramas que describen el proceso, similar a los diagramas de Feynman. Pero el número de diagramas aumenta exponencialmente con el orden de la perturbación, haciendo que el proceso de cálculo sea tedioso y propenso a errores.

El objetivo de este trabajo es desarrollar un programa que automatice el proceso de obtención de los diagramas asociados a un proceso determinado, hasta un orden dado, partiendo de unos diagramas base, llamado diagramas canónicos dados por la teoría. Siendo el programa capaz de descartar los diagramas que no contribuyen al proceso, detectar loops y añadir contratérminos a los diagramas para cancelar divergencias.

Analizando los diagramas obtenidos de ordenes inferiores, y comparando con resultados conocidos, se ha comprobado la veracidad del programa. Esto permite obtener los diagramas de ordenes superiores, y estudiar el comportamiento de la teoría a dichos ordenes, en las que se pueden observar fenómenos único de teorías no abelianas.

Abstract

The renormalization procedure is fundamental to understand the behavior of the theories at different scales, defining effective theories that describe the behaviour of the systems for different energy levels. However, the calculations are non-trivial, and perturbation theory is used to obtain the results, representing the different terms of the theory as diagrams that describe the process, similar to the Feynman diagrams. But the number of diagrams increases exponentially with the order of the perturbation, making the calculation process tedious and prone to errors.

The aim of this work is to develop a program that automates the process of obtaining the diagrams associated with a given process, up to a given order, starting from a set of base diagrams, called canonical diagrams given by the theory. The program is able to discard the diagrams that do not contribute to the process, detect loops and add counterterms to the diagrams to cancel divergences.

By analysing the diagrams obtained from lower orders, and comparing with known results, the correctness of the program has been verified. This allows to obtain the diagrams of higher orders, and study the behavior of the theory at those orders, where unique phenomena of non-abelian theories can be observed.

Contents

1 Introduction

When trying to build a theory that describes the particles and their interactions, it's fundamental that the theory is compatible with the 2 pillars of modern physics, the special relativity and quantum mechanics. The special relativity is a theory that's able to describe the behavior of particles at high energies, and quantum mechanics is a theory that describes the behavior of particles at small scales. The combination of these two theories is the basis of the quantum field theory, building a formalism that describes the particles as excitations of a field.

Other important aspect of the theory is that it should be able to describe the different phenomena that occur at different scales. In the context of QCD, the experiments performed at low energies, or high length scales, smear the interaction between the particles, and does not resolve the details of the structure inside the particles, observing strongly bound states. While at high energies, or low length scales, the hadrons are broken into their constituents, quarks and gluons, and the interaction between the particles is weak, forming the so-called asymptotic freedom. The theory should be able to describe the transition between these two regimes.

In the context of theoretical physics, the renormalization group procedure is a powerful tool to study the behavior of physical systems at different scales. In the case of quantum field theories, renormalization allows to study the system at different energy scales by introducing a scale parameter, and by changing this parameter, the "resolution" of the system is changed, allowing to focus from the smallest details (the short distance behavior) to the largest ones (the grand scale behavior).

We will adopt the Hamiltonian formalism to describe the dynamics of the system, working in the operator space, where the Hamiltonian operator governs the time evolution of the states.

Many widely used renormalization procedures tend to apply to Lagrangian dynamics. But in the framework of RGPEP, the renormalization group procedure is applied to Hamiltonian dynamics. The benefit of this approach is the ability obtain directly the solutions of the system (the spectrum of the theory, and the eigenstates of the Hamiltonian). RGPEP introduces an effective Hamiltonian, that describes the system at a given scale. This effective Hamiltonian is the solution to a differential equation, that describes the evolution of the effective Hamiltonian with respect to the scale parameter.

In general, obtaining the exact solution of the RGPEP equation is a non-trivial task, and a perturbative expansion of the effective Hamiltonian is used to obtain the solution. By identifying each order in the Hamiltonian expansion with a series of products of diagrams, the next order in the expansion can be obtained from the diagrams of previous orders. Typically, these diagrams were analyzed hand, but the number of diagrams increases exponentially with each order, making the process tedious and error-prone. The goal of this thesis is to develop a code that automates the process of obtaining the diagrams associated with a certain interaction for a given order.

This thesis is organized as follows. Section 2 describes the theoretical background needed to understand the renormalization group procedure for effective particles, and the Hamiltonian dynamics. Section 3 describes the case studied in this thesis, gluon self interactions, where holding the simplicity of the scalar case, some peculiarities of the QCD theory are present. The section 4 describes the connection between the diagrams and the objects are defined in the code, as well as the steps taken to obtain higher order diagrams, that are presented in the section 5. Finally, the section 6 presents the conclusions, future work and the improvement that can be done to the code.

2 Theoretical background

To understand the basis of the RGPEP, we need to understand the different concepts involved in the process, as well as the theories that the process is applied to.

2.1 Fock space

Intoduced by V.A. Fock in 1932 [2], the Fock space is a sum of a set of Hilbert spaces, each one corresponding to a different number of particles in the system. So in order to describe a variable number of particles in our system, the use of the Fock space is needed.

The Fock space is defined as the direct sum of tensor products of the single particle Hilbert space $\mathbb{H}$,

$$\mathbb{F} = \bigoplus_{n=0}^{\infty} \mathbb{H}^{\otimes n} = \mathbb{C} \oplus \mathbb{H} \oplus (\mathbb{H} \otimes \mathbb{H}) \oplus (\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}) \oplus \cdots , \tag{2.1}$$

where $\mathbb{C}$ is the complex scalar, corresponding to the states with no particles.

This way a general state in the Fock space can be expressed as,

$$|\Psi\rangle = |\Psi_0\rangle \oplus |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots = c\,|0\rangle + \sum_{i=1} c_i\,|\psi_i\rangle + \sum_{i,j=1} c_{ij}\,|\psi_i\psi_j\rangle + \cdots , \tag{2.2}$$

where $|\Psi_0\rangle$ is the vacuum state, $|\Psi_1\rangle$ is the one particle state, $|\Psi_2\rangle$ is the two particle state, and so on. The coefficients a_i are the amplitudes of the states, in general complex numbers.

In the case of QCD, a general state in the Fock space involves a superposition of all possible multiparticle states, build from quarks, antiquarks, and gluons, with the correct quantum numbers. For instance, a quarkonium state in QCD is given by:

$$|\Psi\rangle = c_1\,|q\bar{q}\rangle + c_2\,|q\bar{q}g\rangle + c_3\,|qqq\rangle + c_4\,|gg\rangle + \cdots , \tag{2.3}$$

2.2 Hamiltonian dynamics

In quantum field theory, 2 equivalent formulations of the dynamics can be used, the Lagrangian and the Hamiltonian formulations. Though the modeling of dynamics tends to start in the Lagrangian formulation, and then using the Legendre transformation to obtain the Hamiltonian formulation.

2.2.1 Canonical Hamiltonian

Consider a general field theory and a Lagrangian density $\mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))$[1], where $\phi_a(x)$ is a particular field of the system. To obtain the Hamiltonian formulation, we need to define the canonical momenta,

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a(x))}. \tag{2.4}$$

The Hamiltonian density, using the Legendre transformation,

$$\mathcal{H} = \sum_a \pi_a(x)\partial_0 \phi_a(x) - \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)). \tag{2.5}$$

In quantum mechanics, or quantum field theory, the Hamiltonian becomes an operator acting on a Hilbert space. This promotion of the Hamiltonian to an operator is done by replacing the canonical coordinates and momenta with the corresponding operators, and impose the canonical commutation relations,

$$[\phi_a(x), \pi_b(y)] = i\delta_{ab}\delta^3(x-y), \quad [\phi_a(x), \phi_b(y)] = 0, \quad [\pi_a(x), \pi_b(y)] = 0. \tag{2.6}$$

The field operators tend to be expressed in terms of creation, a_i and annihilation $a_i^\dagger$ operators, that act on the Fock space, creating and annihilating the mode i of the field. Mathematically, the creation and annihilation operators are the Fourier components of the field.

Any state component in the Fock space can be expressed as the action of a series of creation operators on the vacuum state, $|0\rangle$,

$$|\Psi_n\rangle = \sum_{i_1, i_2, \cdots, i_n} \frac{c_{i_1 i_2 \cdots i_n}}{\sqrt{n!}} a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle , \tag{2.7}$$

making the full state,

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{i_1, i_2, \cdots, i_n} c_{i_1 i_2 \cdots i_n} a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle . \tag{2.8}$$

The creation and annihilation operators satisfy some commutation or anticommutation relations, depending on the nature of the particles, bosons or fermions.

[1] where x indicate a 4-vector, with x^μ its components, and $\partial_\mu = \frac{\partial}{\partial x^\mu}$

For bosons, the creation and annihilation operators satisfy the following commutation relations,

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \quad \left[a_i, a_j\right] = 0, \quad \left[a_i^\dagger, a_j^\dagger\right] = 0. \tag{2.9}$$

For fermions, the creation and annihilation operators satisfy the following anticommutation relations,

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0. \tag{2.10}$$

This way, the creation and annihilation operators are the fundamental objects of the Fock space, and the Hamiltonian and the fields are expressed in terms of these operators.

When a combination of creation and annihilation operators is considered, normal-ordering is used when defining operators that contain both creation and annihilation operators and avoid the infinite vacuum contribution. The normal-ordering is defined as the process of rearranging the creation and annihilation operators in such a way that all the creation operators are to the left of all the annihilation operators.

The normal-ordering is denoted by the symbol $: \cdots :$, then for a fermionic case,

$$a_i a_j^\dagger = \{a_i, a_j^\dagger\} - a_j^\dagger a_i = \delta_{ij} - : a_j^\dagger a_i := \delta_{ij} - a_j^\dagger a_i. \tag{2.11}$$

2.2.2 Front form of Hamiltonian Dynamics

The quantization of the system is usually performed in the instant form of the dynamics, where the time is treated as the evolution parameter, and the rest of coordinates are treated as spatial coordinates. But other forms of dynamics can be considered.

The front form (FF) or light front of dynamics introduced by Dirac (1949) [1] offers a couple advantages to the instant form of dynamics typically considered in quantum field theory.

Considering a quantization hypersurface defined by the equation,

$$x^+ = \frac{1}{\sqrt{2}}(t + z) = \frac{1}{\sqrt{2}}(x^0 + x^3) = 0, \tag{2.12}$$

then the rest of coordinates will be defined as

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3), \quad x^\perp = (x^1, x^2), \tag{2.13}$$

In the FF quantization, the space-time coordinate x^+ is treated as the evolution parameter, similar to the time parameter in the instant form, and the rest of coordinates, x^- and $x^\perp$, are treated as spatial coordinates.

The Hamiltonian in FF quantization is obtained from the Lagrangian density using the Legendre transformation, similar to the procedure explained in the section 2.2.1, but with respect to the new coordinates.

Describing the system in the FF quantization, the vacuum state is "empty", since no particle creation are possible from $P^+ = 0$

2.3 RGPEP

The RGPEP, is a renormalization group procedure applied to the Hamiltonian formulation. By considering a series of unitary transformations, the RGPEP is able to construct a series of effective Hamiltonians $\mathcal{H}_s$, and the corresponding effective particles, by the use of effective particle operators (namely creation and annihilation operators) that differs from the canonical ones by the unitary transformation $\mathcal{U}_s$,

$$a_s = \mathcal{U}_s a_0 \mathcal{U}_s^\dagger, \tag{2.14}$$

labeled by the parameter s, and is associated with the renormalization group scale $\lambda = 1/s$. This parameter which has dimension of length, physically has the interpretation of the characteristic size of the effective particles.

Due to dimensional and notational reasons, it's convenient to consider the scale parameter $t = s^4$.

Then $s = 0$ correspond to the point-like or bare particles, and recovering the original Hamiltonian $\mathcal{H}_0(a_0)$.

The effective Hamiltonian is related to the regulated canonical one with counterterms by the condition,

$$\mathcal{H}_t(a_t) = \mathcal{H}_0(a_0) \tag{2.15}$$

Combining with the equation (2.14), and considering the parameter t instead of s, the condition becomes,

$$\mathcal{H}_t(a_0) = \mathcal{U}_t^\dagger \mathcal{H}_0(a_0) \mathcal{U}_t, \tag{2.16}$$

differentiating with respect of t,

$$\mathcal{H}_t'(a_0) \equiv \frac{d}{dt}\mathcal{H}_t(a_0) = \left[-\mathcal{U}_t^\dagger \mathcal{U}_t', \mathcal{H}_t(a_0)\right] = \left[\mathcal{G}_t(a_0), \mathcal{H}_t(a_0)\right], \tag{2.17}$$

where $\mathcal{G}_t$ is the RGPEP generator. We will consider the generator from Ref. [4]

$$\mathcal{G}_t = \left[\mathcal{H}_f, \mathcal{H}_{Pt}\right], \tag{2.18}$$

where $\mathcal{H}_f$, the free part of $\mathcal{H}_t$, and $\mathcal{H}_{Pt}$ is defined as function of the interacting term.

The free Hamiltonian $\mathcal{H}_f$ is the part of $\mathcal{H}_0(a_0)$ that does not depend on the coupling constants,

This way, the RGPEP equation have the form,

$$\mathcal{H}_t' = \left[\left[\mathcal{H}_f, \mathcal{H}_{Pt}\right], \mathcal{H}_t\right], \tag{2.19}$$

In general, the solution of the RGPEP equation is a non-trivial task, and a perturbative expansion of the effective Hamiltonian is used to obtain the solution. Expressing $\mathcal{H}_t$ as a power series of the coupling constant g,

$$\mathcal{H}_t = \sum_{n=0}^{\infty} g^n \mathcal{H}_{tn} = \mathcal{H}_0 + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2} + g^3\mathcal{H}_{t3} + g^4\mathcal{H}_{t4} + \cdots. \tag{2.20}$$

Plugging this expansion into the RGPEP equation, we obtain,

$$\begin{aligned} &\mathcal{H}_0' + g\mathcal{H}_{t1}' + g^2\mathcal{H}_{t2}' + \cdots = \\ &\quad = \left[\left[\mathcal{H}_0, \mathcal{H}_0 + g\mathcal{H}_{Pt1} + g^2\mathcal{H}_{Pt2} + \cdots\right], \mathcal{H}_0 + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2} + \cdots\right]. \end{aligned} \tag{2.21}$$

Order by order in g, the following differential equations are obtained,

$$\mathcal{H}_0' = 0, \tag{2.22}$$

$$g\mathcal{H}_{t1}' = \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], \mathcal{H}_0\right], \tag{2.23}$$

$$g^2\mathcal{H}_{t2}' = \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{Pt2}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], g\mathcal{H}_{t1}\right], \tag{2.24}$$

$$g^3\mathcal{H}_{t3}' = \left[\left[\mathcal{H}_0, g^3\mathcal{H}_{Pt3}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{Pt2}\right], g\mathcal{H}_{t1}\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], g^2\mathcal{H}_{t2}\right]. \tag{2.25}$$

$$\vdots$$

The order 0 term is solvable from a initial condition, and the solution will be a exponetial of the parameter t. The 1th order solution can be obtained from 0th order, and the 2nd order from the previous orders, and so on.

2.4 Regularization and Counterterms

In QCD, the bare Hamiltonian is ill-defined due to the presence of elements that contain divergences, ultraviolet (UV) divergences and infrared (IR) divergences. Th UV divergences are produced in processes where the large momentum transferes, or the high invariant mass difference between Fock states. While the IR divergences are produced in procceses where the particles are "soft", carring small longitudinal momentum fractions, $x_{p/P} = p^+/P^+$.

To deal with these divergences a regulating factor r is introduced in the interacting terms. These factors make the interacting terms rapidly tend to zero, if the change in the transverse momentum of any gluon exceeds a certain cutoff parameter Δ, or if the change in longitudinal momentum of any gluon is greater than a cutoff parameter δ.

The particle operators is multiplied by the regulating factor,

$$r_{\Delta\delta}(k^{\perp}, x) = r_{\Delta}(k^{\perp}) r_{\delta}(x)\theta(x). \tag{2.26}$$

The transverse regulator factor will be of the form,

$$r_{\Delta}(\mathcal{M}) = \exp\left(-\frac{\mathcal{M}^2}{\Delta^2}\right), \tag{2.27}$$

with $\mathcal{M}$ the invariant mass of the system, ensuring that UV processes are suppressed.

The longitudinal regulator factor must verify a similar condition, preventing terms of the form $1/x$ or $1/x^2$ to blow up as x approaches 0. The exact form of the longitudinal regulator factor is not important, as long as it verifies the condition, since it will be removed using the RGPEP.

Both cutoff parameters will disappear in the final result, since the theory can't depend on the cutoff parameters taken arbitrarily. This is done by taking the limit $\Delta \to \infty$ and $\delta \to 0$, recovering the original values taken by the momenta.

It's convenient to consider an abbreviated notation for the regulating factor, using the symbol $\tilde{r}_{P,p} = \tilde{r}_{\Delta\delta}(P, p)$, with,

$$\tilde{r}_{\Delta\delta}(P, p) = r_{\Delta\delta}(p^{\perp} - x_{p/P} P^{\perp}) r_{\Delta\delta}\left[P^{\perp} - p^{\perp} - \left(1 - x_{p/P}\right) P^{\perp}, 1 - x_{p/P}\right]. \tag{2.28}$$

The counterterms is an additional term added to the initial or bare Hamiltonian $\mathcal{H}_0$ to deal with the divergences due to loops, produced during the process. This way, ensure that the effective Hamiltonian $\mathcal{H}_t$ remains finite at all values of t.

The counterterms are defined in a way, such that the coefficients of products of creation and annihilation operators in the effective theory for gluons of size s become independent of the regularization parameter Δ when the regularization in dynamics of gluons of size zero is being removed. The rest of the unknown parts of the counterterms are adjusted to respect the symmetries of the theory, and must match the predictions of the theory with the experimental results.

2.5 Diagram representation

From the expression of the Hamiltonian, different terms can be separated and correlate to a diagram representation of the process. Each diagram will be composed of different elements,

- External legs, representation of the incoming and outgoing particles, it indicates the type of process that is being considered. Will have different colors/ types of lines depending on the type of particle.
- Internal legs, represent the virtual particles that are exchanged during the process, these particles are not observed in the final state. Similar to the external legs, will have different colors/types of lines depending on the type of particle.

- Vertices, points where interactions between particles occur, the number of vertices in a diagram indicates the order at which the diagram is contributing.

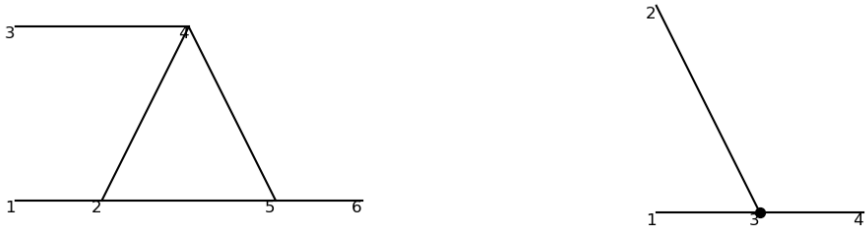
- Loops, closed paths in the diagrams, formed by the internal legs. It will indicate the presence of divergences in the process.

- Counterterms, additional terms added to the Hamiltonian to deal with the divergences produced during the process. These will be represented with a dot in the diagram.

Notice that the diagrams are similar to the Feynman diagrams, except for the presence of the counterterms. The main difference is the order that the processes occur, altering the order of the vertices in the diagram will produce a different process.

Other difference how the diagrams are read, in the context of RGPEP, the diagrams are read from the left to the right, meaning that the time evolution is from the left to the right.

As an example, consider the diagram in figure 1, this pair of diagrams are one of the contributions to the 3 gluon vertex, with the counterterm associated with it.



(a) Diagram with a 3 gluon loop.

(b) Counterterm of the diagram

Figure 1: A example of a possible diagram, a third order contribution to the process of 3 gluon vertex, 1 gluon incoming and 2 gluons outgoing. (Outputs of the programs.)

Following the previous definitions, the diagram represent a process where a gluon interacts in a third order process, forming a 3 gluon loop, and 2 gluons are produced in the final state. Due to the presence of a loop, the diagram is divergent, and a counterterm diagram is needed to deal with the divergence. This diagram is represented with a dot at the position of the loop, with the same external legs as the original diagram.

2.6 Order by order solutions.

The solution of the differential equations (2.23) to (2.25) and so on, without considering the multiplicative factors, and relative phases between the terms, can be expressed as,

$$\mathcal{H}_{t1} = \mathcal{H}_{01} \tag{2.29}$$
$$\mathcal{H}_{t2} = \mathcal{G}_{02} + \mathcal{H}_{01}\mathcal{H}_{01} \tag{2.30}$$
$$\mathcal{H}_{t3} = \mathcal{G}_{03} + \mathcal{H}_{01}\mathcal{H}_{01}\mathcal{H}_{01} + (\mathcal{G}_{02}\mathcal{H}_{01} + \mathcal{G}_{02}\mathcal{H}_{01}) \tag{2.31}$$
$$\mathcal{H}_{t4} = \mathcal{G}_{04} + \mathcal{H}_{01}\mathcal{H}_{01}\mathcal{H}_{01}\mathcal{H}_{01} + (\mathcal{G}_{02}\mathcal{H}_{01}\mathcal{H}_{01} + \mathcal{H}_{01}\mathcal{G}_{02}\mathcal{H}_{01} + \mathcal{H}_{01}\mathcal{H}_{01}\mathcal{G}_{02}) + (\mathcal{G}_{03}\mathcal{H}_{01} + \mathcal{H}_{01}\mathcal{G}_{03} + \mathcal{G}_{02}\mathcal{G}_{02})\,. \tag{2.32}$$
$$\vdots$$

This is an oversimplification of the solution, but it gives an idea of how the different solutions for each order are obtained.

3 Case study: Gluons self interactions

3.1 Canonical Hamiltonian

The Lagrangian density for the gluon fields is given by,

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}, \tag{3.1}$$

where $F^{\mu\nu}$ is the field strength tensor, defined as,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu], \tag{3.2}$$

and $A^\mu = A^{a\mu}t^a$, t^a are the generators of the gauge group, and g is the coupling constant. Verifying the following relations,

$$[t^a, t^b] = if^{abc}t^c, \quad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{3.3}$$

We will be working in the gauge $A^+ = 0$, where the Lagrange equations constrain the component A^- to become

$$A^- = \frac{1}{\partial^+}2\partial^\perp A^\perp - \frac{2}{\partial^{+2}}ig\left[\partial^+ A^\perp, A^\perp\right]. \tag{3.4}$$

In this way, the only degree of freedom left is the transverse component $A^\perp$.

As for the associated energy-momentum tensor,

$$\mathcal{T}^{\mu\nu} = -F^{a\mu\alpha}\partial^\nu A^a_\alpha + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{3.5}$$

The Hamiltonian in FF quantization is obtained from integrating the component $\mathcal{T}^{+-}$ of the energy-momentum tensor, over the hyperplane $x^+ = 0$.

By working in the gauge $A^+ = 0$, the Hamiltonian density can be expressed as the sum of 4 terms, as denoted in [3]

$$\mathcal{T}^{+-} = \mathcal{H}_{A^2} + \mathcal{H}_{A^3} + \mathcal{H}_{A^4} + \mathcal{H}_{[\partial AA]^2}, \tag{3.6}$$

with each of the terms,

$$\mathcal{H}_{A^2} = -\frac{1}{2} A^{\perp a} (\partial^\perp)^2 A^{\perp a}, \tag{3.7}$$

$$\mathcal{H}_{A^3} = g i \partial_\alpha A^a_\beta \left[A^\alpha, A^\beta \right]^a, \tag{3.8}$$

$$\mathcal{H}_{A^4} = -\frac{1}{4} g^2 \left[A_\alpha, A_\beta \right]^a \left[A^\alpha, A^\beta \right]^a, \tag{3.9}$$

$$\mathcal{H}_{[\partial AA]^2} = -\frac{1}{2} g^2 \left[i\partial^+ A^\perp, A^\perp \right]^a \frac{1}{(i\partial^+)^2} \left[i\partial^+ A^\perp, A^\perp \right]^a. \tag{3.10}$$

Replacing A^μ with the operator $\hat{A}^\mu(x)$, defined by its Fourier composition on the plane $x^+ = 0$,

$$\hat{A}^\mu(x) = \sum_{\sigma c} \int [k] \left[t^c \epsilon^\mu_{k\sigma} a_{k\sigma c} e^{-ikx} + t^c \epsilon^{\mu *}_{k\sigma} a^\dagger_{k\sigma c} e^{ikx} \right]_{x^+=0}, \tag{3.11}$$

where $[k] = \theta(k^+) dk^+ d^2k^\perp / (16\pi^3 k^+)$, $\epsilon^\mu_{k\sigma}$ are the polarization vectors, t^c are the generators of the gauge group, and $a^\dagger_{k\sigma c}$, $a_{k\sigma c}$ are the creation and annihilation operators (particle operators), respectively.

Substituting this expression into each term of the Hamiltonian densities, integrating over space coordinates and taking into account the completeness and orthonormality of the polarization vectors, we obtain the following expression for the different terms of the Hamiltonian [5],

$$H_{A^2} = \sum_{\sigma c} \int [k] \frac{k^{\perp 2}}{k^+} a^\dagger_{k\sigma c} a_{k\sigma c}, \tag{3.12}$$

$$H_{A^3} = \sum_{123} \int [123] \not{\delta}(p^\dagger - p) \tilde{r}_{\Delta\delta}(3,1) \left[g Y_{123} a^\dagger_1 a^\dagger_2 a_3 + g Y^*_{123} a^\dagger_3 a_2 a_1 \right], \tag{3.13}$$

$$H_{A^4} = \sum_{1234} \int [1234] \not{\delta}(p^\dagger - p) \frac{g^2}{4} \left[\left(\Xi_{A^4 1234} a^\dagger_1 a^\dagger_2 a^\dagger_3 a_4 + h.c. \right) + X_{A^4 1234} a^\dagger_1 a^\dagger_2 a_3 a_4 \right], \tag{3.14}$$

$$H_{[\partial AA]^2} = \sum_{1234} \int [1234] \not{\delta}(p^\dagger - p) g^2 \left[\left(\Xi_{[\partial AA]^2 1234} a^\dagger_1 a^\dagger_2 a^\dagger_3 a_4 + h.c. \right) + X_{[\partial AA]^2 1234} a^\dagger_1 a^\dagger_2 a_3 a_4 \right]. \tag{3.15}$$

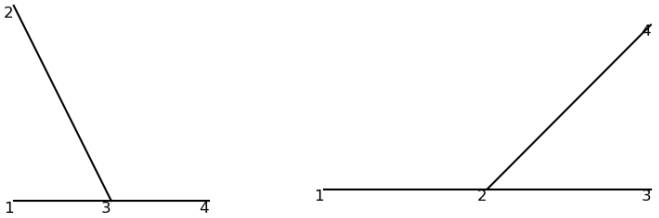
3.2 Canonical diagrams

The canonical diagrams are the diagrams that are obtained from the canonical Hamiltonian, and are the starting point to obtain the diagrams of higher order.

Following the creation and annihilation operators in the different terms of the Hamiltonian, we can distinguish the different types of process that are being considered.

Staring from the term H_{A^2} in equation (3.12), we can see that this a particle of certain momentum k is annihilated and created, producing the same particle. This corresponds to the kinetic term of the Hamiltonian, and the diagram will be a line with the same color/type of particle in both ends. Being an order 0 term that will not contribute to the perturbative expansion.

The term H_{A^3} in equation (3.13) is a 3 gluon vertex, where either 1 gluon is annihilated and 2 gluons are created, or 2 gluons are annihilated, and 1 gluon is created. This process can be represented by a diagram with 3 external legs, and 1 vertex. Being a order 1 term, with the corresponding diagrams shown in figure 2.



(a) (b)

Figure 2: Canonical diagrams of order 1

The term H_{A^4} in equation (3.14) is a 4 gluon vertex, where either 1 gluon is annihilated, and 3 gluons are created, 3 gluons are annihilated, and 1 gluon is created, or 2 gluons are annihilated, and 2 gluons are created. But we will not pay special attention to this term, since all the possible diagrams can be created from the diagrams of order 1.

The really interesting term is the term $H_{[\partial AA]^2}$ in equation (3.15), similar to the term H_{A^4}, this term is a 4 gluon vertex, but the concept of instantaneous interaction is introduced. This allows to consider processes where a gluon is annihilated and 3 gluons are created, to a combination of 3 gluon vertices.

4 Code implementation

The code is implemented in Python, but the general method can be applied to any programming language. The program is designed to be modular, and applicable to other theories, by considering some minor modifications and changing the canonical diagrams. These modifications are not implemented yet, and remain to be tested.

4.1 Diagrams definition

The diagrams are defined by 2 arrays,

- Points: arrays of dimension $N \times 2$, where N is the number of points in the diagram, each point is defined by its coordinates (x, y).

- Paths: arrays of dimension $M \times N' \times 2$, where M is the number of different types of particles to consider in the theory, N' is the number of paths for each types of particle in the diagram, and 2 indicate the points to connect.

This way of defining the diagrams is analogous to the way of defining the undirected graphs in the graph theory, where the points are the vertices and the paths are the edges.

In the case of gluon interactions, although only 1 type of particle is present, the instantaneous interactions have to be considered. This is done by defining this interaction as a new type of virtual particle in the program.

4.2 Order by order procedure

To calculate the diagrams of higher order, the code follows the order by order procedure described in the section 2. Having only the canonical diagrams, the program aims to obtain all the possible diagrams of an order that contribute to a certain process, discarding the other diagrams.

This process produces a huge amount of diagrams, many of them equivalent. The program detects these repeated diagrams, and adds their contribution, thus reducing the number of diagrams that needs to be used to calculate the next order. This procedure is the most time-consuming process of the program, needing a search algorithm to find the equivalent diagrams, meaning that as the order increases, the number of diagrams increases exponentially, and the time to find the equivalent diagrams increases exponentially too.

Although the process of finding the equivalent diagrams is time-consuming, it is fundamental to reduce the global computational time. Since the number of diagrams tend to decrease by 1 to 2 orders of magnitude, depending on the order and the number of particles in the process. This way reduces the time needed to calculate the diagrams of the next order, so at the grand scale, the time needed to calculate the diagrams of till a certain order is reduced by performing the search algorithm inbetewen the orders.

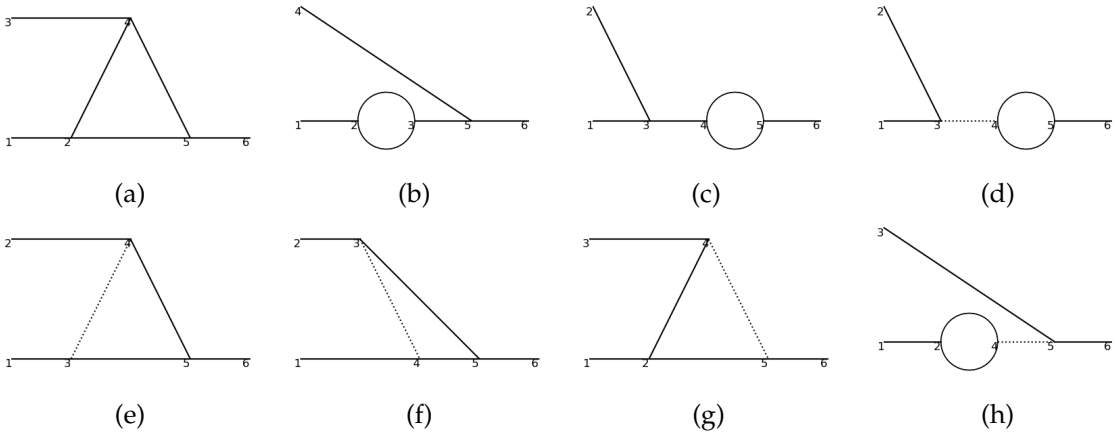
Other important aspect of the RGPEP are the counterterms, that are added to cancel the divergences produced in the process. The program is able to detect the 2 loops and 3 loops in the diagrams of a certain order, and add new diagrams to cancel the divergences. As shown in figure 1.

5 Diagrams obtained

5.1 Order 3

Considering the 3 gluon vertex, the process of 1 gluon going to 2 gluons, up till order 3, the diagrams obtained from the program are shown in the figures 3 and 4.

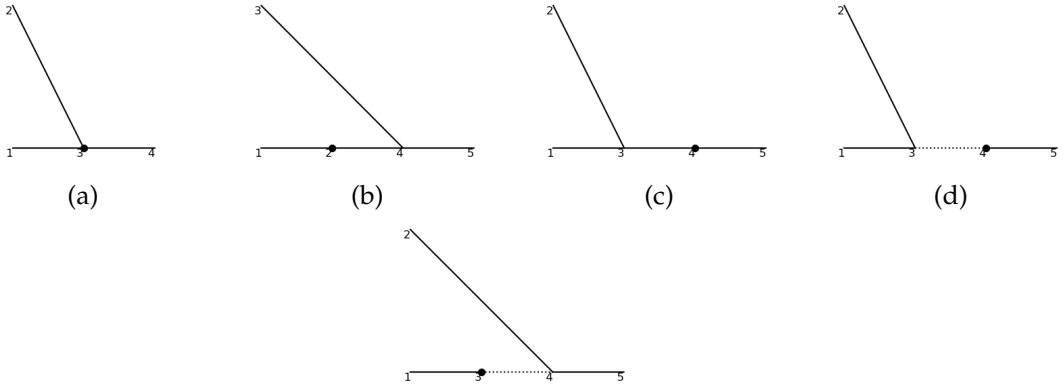
In the figure 3 we can see the diagrams obtained for the process of 1 gluon going to 2 gluons, with the corresponding counterterms in figure 4.



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 3: Diagrams of third order, contributing to the 3 gluon vertex.



(a) (b) (c) (d)

(e)

Figure 4: Counterterms to the diagrams of third order in figure 3

The diagrams 3e and 3f are really the same diagram, due to instant process being instantaneous. For any other particle, these 2 diagrams would be different, due to the importance of the order in the interactions so the program is kept to consider them as different, to not lose generality.

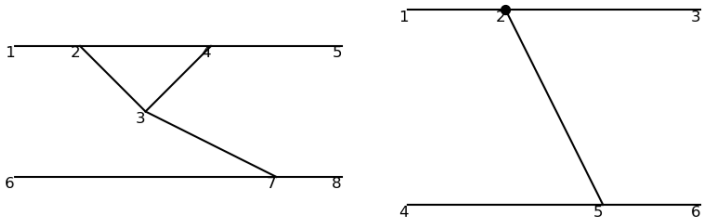
Other artifact of the program occur for the diagrams 4d and 4e, which are counterterms added to cancel the divergences in the canonical diagrams.

Comparing with the 3rd order contribution diagrams in [5], the same diagrams for the 3 gluon vertex are obtained, proving the validity of the program to reproduce the same results.

5.2 Order 4

Considering glueball states, the process of 2 gluons going to 2 gluons, the program produces about 100 diagrams, with the corresponding counterterms. All the diagrams are not shown here, since they are too many to be included in this document. But let's discuss some of the most interesting diagrams.

Starting at order 4, the distinct non-abelian nature of the QCD is present, since



(a) (b)

Figure 5: Diagrams of fourth order, contributing to the 2 gluon vertex.

the gluons interact with themselves, creating diagrams not possible in abelian theories, such as QED. One of such processes is shown in the figure 5, where the

5.3 Higher orders

At higher orders, the diagrams become more complex, and the number of diagrams scale exponentially. Is here where the power of the program is shown.

Order	Unique diagrams	Computational time (s)
2	16	0.150
3	76	0.198
4	612	1.166
5	5871	10.935
6	65000	157.121

Table 1: Number of diagrams obtained by the program at each order, and the Computational time taken to calculate the diagrams.

6 Conclusions and future work

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