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TRABAJO FIN DE GRADO

Herramientas para cálculos perturbativos en renormalización de Hamiltonianos

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Resumen

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Abstract

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Contents

1 Introduction

When trying to build a theory that describes the particles and their interactions, it's fundamental that the theory is compatible with the 2 pillars of modern physics, the special relativity and quantum mechanics. The special relativity is a theory that's able to describe the behavior of particles at high energies, and quantum mechanics is a theory that describes the behavior of particles at small scales. The combination of these two theories is the basis of the quantum field theory, building a formalism that describes the particles as excitations of a field.

Other important aspect of the theory is that it should be able to describe the different phenomena that occur at different scales. In the context of QCD, the experiments performed at low energies, or high length scales, smear the interaction between the particles, and does not resolve the details of the structure inside the particles, observing strongly bound states. While at high energies, or low length scales, the hadrons are broken into their constituents, quarks and gluons, and the interaction between the particles is weak, forming the so-called asymptotic freedom. The theory should be able to describe the transition between these two regimes.

In the context of theoretical physics, the renormalization group procedure is a powerful tool to study the behavior of physical systems at different scales. In the case of quantum field theories, renormalization allows to study the system at different energy scales by introducing a scale parameter, and by changing this parameter, the "resolution" of the system is changed, allowing to focus from the smallest details (the short distance behavior) to the largest ones (the grand scale behavior).

We will adopt the Hamiltonian formalism to describe the dynamics of the system, working in the operator space, where the Hamiltonian operator governs the time evolution of the states.

Many widely used renormalization procedures tend to apply to Lagrangian dynamics. But in the framework of RGPEP, the renormalization group procedure is applied to Hamiltonian dynamics. The benefit of this approach is the ability obtain directly the solutions of the system (the spectrum of the theory, and the eigenstates of the Hamiltonian). RGPEP introduces an effective Hamiltonian, that describes the system at a given scale. This effective Hamiltonian is the solution to a differential equation, that describes the evolution of the effective Hamiltonian with respect to the scale parameter.

In general, obtaining the exact solution of the RGPEP equation is a non-trivial task, and a perturbative expansion of the effective Hamiltonian is used to obtain the solution. By identifying each order in the Hamiltonian expansion with a series of products of diagrams, the next order in the expansion can be obtained from the diagrams of previous orders. Typically, these diagrams were analyzed hand, but the number of diagrams increases exponentially with each order, making the process tedious and error-prone. The goal of this thesis is to develop a code that automates the process of obtaining the diagrams associated with a certain interaction for a given order.

This thesis is organized as follows. Section 2 describes the theoretical background needed to understand the renormalization group procedure for effective particles, and the Hamiltonian dynamics. Section 3 describes the case studied in this thesis, gluon self interactions, where holding the simplicity of the scalar case, some peculiarities of the QCD theory are present. The section 4 describes the connection between the diagrams and the objects are defined in the code, as well as the steps taken to obtain higher order diagrams, that are presented in the section 5. Finally, the section 6 presents the conclusions, future work and the improvement that can be done to the code.

2 Theoretical background

To understand the basis of the RGPEP, we need to understand the different concepts involved in the process, as well as the theories that the process is applied to.

2.1 Hamiltonian dynamics

In quantum field theory, 2 equivalent formulations of the dynamics can be used, the Lagrangian and the Hamiltonian formulations. Though the modeling of dynamics tends to start in the Lagrangian formulation, and then using the Legendre transformation to obtain the Hamiltonian formulation.

2.1.1 Canonical Hamiltonian

Consider a general field theory and a Lagrangian density $\mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))$[1], where $\phi_a(x)$ is a particular field of the system. To obtain the Hamiltonian formulation, we need to define the canonical momenta,

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a(x))}. \tag{2.1}$$

The Hamiltonian density, using the Legendre transformation,

$$\mathcal{H} = \sum_a \pi_a(x)\partial_0\phi_a(x) - \mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x)). \tag{2.2}$$

In quantum mechanics, or quantum field theory, the Hamiltonian becomes an operator acting on a Hilbert space. This promotion of the Hamiltonian to an operator is done by replacing the canonical coordinates and momenta with the corresponding operators, and impose the canonical commutation relations,

$$[\phi_a(x), \pi_b(y)] = i\delta_{ab}\delta^3(x-y), \quad [\phi_a(x), \phi_b(y)] = 0, \quad [\pi_a(x), \pi_b(y)] = 0. \tag{2.3}$$

[1] where x indicate a 4-vector, with x^μ its components, and $\partial_\mu = \frac{\partial}{\partial x^\mu}$

The field operators tend to be expressed in terms of creation and annihilation operators, that act on the Fock space (see section 2.2).

The quantization of the system is usually performed in the instant form of the dynamics, where the time is treated as the evolution parameter, and the rest of coordinates are treated as spatial coordinates.

2.1.2 Front form of Hamiltonian Dynamics

The front form (FF) or light front of dynamics introduced by Dirac (1949) [1] offers a couple advantages to the instant form of dynamics typically considered in quantum field theory.

Considering a quantization hypersurface defined by the equation,

$$x^+ = \frac{1}{\sqrt{2}}(t+z) = \frac{1}{\sqrt{2}}(x^0 + x^3) = 0, \tag{2.4}$$

then the rest of coordinates will be defined as

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3), \quad x^\perp = (x^1, x^2), \tag{2.5}$$

In the FF quantization, the space-time coordinate x^+ is treated as the evolution parameter, similar to the time parameter in the instant form, and the rest of coordinates, x^- and $x^\perp$, are treated as spatial coordinates.

The Hamiltonian in FF quantization is obtained from the Lagrangian density using the Legendre transformation, similar to the procedure explained in the section 2.1.1, but with respect to the new coordinates.

2.2 Fock space

Intoduced by V.A. Fock in 1932 [2], the Fock space is a sum of a set of Hilbert spaces, each one corresponding to a different number of particles in the system. So in order to describe a variable number of particles in our system, the use of the Fock space is needed.

The Fock space is defined as the direct sum of tensor products of the single particle Hilbert space $\mathbb{H}$,

$$\mathbb{F} = \bigoplus_{n=0}^{\infty} \mathbb{H}^{\otimes n} = \mathbb{C} \oplus \mathbb{H} \oplus (\mathbb{H} \otimes \mathbb{H}) \oplus (\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}) \oplus \cdots, \tag{2.6}$$

where $\mathbb{C}$ is the complex scalar, corresponding to the states with no particles.

This way a general state in the Fock space can be expressed as,

$$|\Psi\rangle = |\Psi_0\rangle \oplus |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots = a\,|0\rangle + \sum_{i=1} a_i\,|\psi_i\rangle + \sum_{i,j=1} a_{ij}\,|\psi_i\psi_j\rangle + \cdots, \tag{2.7}$$

where $|\Psi_0\rangle$ is the vacuum state, $|\Psi_1\rangle$ is the one particle state, $|\Psi_2\rangle$ is the two particle state, and so on. The coefficients a_i are the amplitudes of the states, in general complex numbers.

In the case of QCD, a general state in the Fock space involves a superposition of all possible multiparticle states, build from quarks, antiquarks, and gluons, with the correct quantum numbers. For instance, a quarkonium state in QCD is given by:

$$|\Psi\rangle = c_1 |q\bar{q}\rangle + c_2 |q\bar{q}g\rangle + c_3 |qqq\rangle + c_4 |gg\rangle + \cdots , \tag{2.8}$$

Describing the system in the FF quantization, the vacuum state is "empty", since no particle creation are possible from $P^+ = 0$

2.3 RGPEP

The RGPEP, is a renormalization group procedure applied to the Hamiltonian formulation. By considering a series of unitary transformations, the RGPEP is able to construct a series of effective Hamiltonians $\mathcal{H}_s$, and the corresponding effective particles, by the use of effective particle operators (namely creation and annihilation operators) that differs from the canonical ones by the unitary transformation $\mathcal{U}_s$,

$$a_s = \mathcal{U}_s a_0 \mathcal{U}_s^\dagger, \tag{2.9}$$

labeled by the parameter s, and is associated with the renormalization group scale $\lambda = 1/s$. This parameter which has dimension of length, physically has the interpretation of the characteristic size of the effective particles.

Due to dimensional and notational reasons, it's convenient to consider the scale parameter $t = s^4$.

The effective Hamiltonian is related to the regulated canonical one with counterterms by the condition,

$$\mathcal{H}_t(a_t) = \mathcal{H}_0(a_0) \tag{2.10}$$

Then $s = 0$ correspond to the point-like or bare particles, and recovering the original

combining with the equation (2.9), and considering the parameter t instead of s, the condition becomes,

$$\mathcal{H}_t(a_0) = \mathcal{U}_t^\dagger \mathcal{H}_0(a_0) \mathcal{U}_t, \tag{2.11}$$

differentiating with respect of t,

$$\mathcal{H}_t'(a_0) \equiv \frac{d}{dt}\mathcal{H}_t(a_0) = \left[-\mathcal{U}_t^\dagger \mathcal{U}_t', \mathcal{H}_t(a_0)\right] = \left[\mathcal{G}_t(a_0), \mathcal{H}_t(a_0)\right], \tag{2.12}$$

where $\mathcal{G}_t$ is the RGPEP generator. This generator is defined as

We will consider the generator from Ref. [3]

$$\mathcal{G}_t = \left[\mathcal{H}_f, \mathcal{H}_{Pt}\right], \tag{2.13}$$

where $\mathcal{H}_f$, the free part of $\mathcal{H}_t$, and $\mathcal{H}_{Pt}$ is defined as function of the interacting term.

This way, the RGPEP equation have the form,

$$\mathcal{H}_t' = \left[\left[\mathcal{H}_f, \mathcal{H}_{Pt}\right], \mathcal{H}_t\right], \tag{2.14}$$

where $\mathcal{H}_t$ is the Hamiltonian interested in solving.

The free Hamiltonian $\mathcal{H}_f$ is the part of $\mathcal{H}_0(a_0)$ that does not depend on the coupling constants,

The Hamiltonian $\mathcal{H}_t$ can be expressed as a series of powers of the coupling constant g,

$$\mathcal{H}_t = \sum_{n=0}^{\infty} g^n \mathcal{H}_{tn} = \mathcal{H}_0 + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2} + g^3\mathcal{H}_{t3} + g^4\mathcal{H}_{t4} + \cdots. \tag{2.15}$$

Plugging this expression into the RGPEP equation, we obtain,

$$\begin{aligned} &\mathcal{H}_0' + g\mathcal{H}_{t1}' + g^2\mathcal{H}_{t2}' + \cdots = \\ &\quad = \left[\left[\mathcal{H}_0, \mathcal{H}_0 + g\mathcal{H}_{Pt1} + g^2\mathcal{H}_{Pt2} + \cdots\right], \mathcal{H}_0 + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2} + \cdots\right]. \end{aligned} \tag{2.16}$$

Order by order in g, the following differential equations are obtained,

$$\mathcal{H}_0' = 0, \tag{2.17}$$

$$g\mathcal{H}_{t1}' = \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], \mathcal{H}_0\right], \tag{2.18}$$

$$g^2\mathcal{H}_{t2}' = \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{Pt2}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], g\mathcal{H}_{t1}\right], \tag{2.19}$$

$$g^3\mathcal{H}_{t3}' = \left[\left[\mathcal{H}_0, g^3\mathcal{H}_{Pt3}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{Pt2}\right], g\mathcal{H}_{t1}\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], g^2\mathcal{H}_{t2}\right]. \tag{2.20}$$

$$\vdots$$

The order 0 term is solvable from a initial condition, and the solution will be a exponetial of the parameter t. The 1th order solution can be obtained from 0th order, and the 2nd order from the previous orders, and so on.

2.4 Regularization and Counterterms

In QCD, the bare Hamiltonian is ill-defined due to the presence of elements that contain divergences, ultraviolet (UV) divergences and infrared (IR) divergences. Th UV divergences are produced in processes where the large momentum transferes, or the high invariant mass difference between Fock states. While the IR divergences are produced in procceses where the particles are "soft", carring small longitudinal momentum fractions, $x_{p/P} = p^+/P^+$.

To deal with these divergences a regulating factor r is introduced in the interacting terms. These factors make the interacting terms rapidly tend to zero, if the change in the transverse momentum of any gluon exceeds a certain cutoff parameter Δ, or if the change in longitudinal momentum of any gluon is greater than a cutoff parameter δ.

The particle operators is multiplied by the regulating factor,

$$r_{\Delta\delta}(k^{\perp}, x) = r_{\Delta}(k^{\perp}) r_{\delta}(x)\theta(x). \tag{2.21}$$

The transverse regulator factor will be of the form,

$$r_{\Delta}(\mathcal{M}) = \exp\left(-\frac{\mathcal{M}^2}{\Delta^2}\right), \tag{2.22}$$

with $\mathcal{M}$ the invariant mass of the system, ensuring that UV processes are suppressed.

The longitudinal regulator factor must verify a similar condition, preventing terms of the form $1/x$ or $1/x^2$ to blow up as x approaches 0. The exact form of the longitudinal regulator factor is not important, as long as it verifies the condition, since it will be removed using the RGPEP.

Both cutoff parameters

The counterterms is an additional term added to the initial or bare Hamiltonian $\mathcal{H}_0$ to deal with the divergences due to loops, produced during the process. This way, ensure that the effective Hamiltonian $\mathcal{H}_t$ remains finite at all values of t.

2.5 Diagram representation

From the expression of the Hamiltonian, different terms can be separated and correlate to a diagram representation of the process, similar to the Feynman diagrams.

3 Case study: Gluons self interactions

3.1 Canonical diagrams

The Lagrangian density for the gluon fields is given by,

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr} F^{\mu\nu} F_{\mu\nu}, \tag{3.1}$$

where $F^{\mu\nu}$ is the field strength tensor, defined as,

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + ig[A^{\mu}, A^{\nu}], \tag{3.2}$$

and $A^{\mu} = A^{a\mu} t^a$, t^a are the generators of the gauge group, and g is the coupling constant. Verifying the following relations,

$$[t^a, t^b] = if^{abc}t^c, \quad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{3.3}$$

We will be working in the gauge $A^+ = 0$, where the Lagrange equations constrain the component A^- to become

$$A^- = \frac{1}{\partial^+} 2\partial^\perp A^\perp - \frac{2}{\partial^{+2}} ig \left[\partial^+ A^\perp, A^\perp\right]. \tag{3.4}$$

In this way, the only degree of freedom left is the transverse component $A^\perp$.

As for the associated energy-momentum tensor,

$$\mathcal{T}^{\mu\nu} = -F^{a\mu\alpha}\partial^\nu A^a_\alpha + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{3.5}$$

The Hamiltonian in FF quantization is obtained from integrating the component $\mathcal{T}^{+-}$ of the energy-momentum tensor, over the hyperplane $x^+ = 0$.

By working in the gauge $A^+ = 0$, the Hamiltonian density can be expressed as the sum of 4 terms, as denoted in [4]

$$\mathcal{T}^{+-} = \mathcal{H}_{A^2} + \mathcal{H}_{A^3} + \mathcal{H}_{A^4} + \mathcal{H}_{[\partial AA]^2}, \tag{3.6}$$

with each of the terms,

$$\mathcal{H}_{A^2} = -\frac{1}{2}A^{\perp a}(\partial^\perp)^2 A^{\perp a}, \tag{3.7}$$

$$\mathcal{H}_{A^3} = gi\partial_\alpha A^a_\beta \left[A^\alpha, A^\beta\right]^a, \tag{3.8}$$

$$\mathcal{H}_{A^4} = -\frac{1}{4}g^2 \left[A_\alpha, A_\beta\right]^a \left[A^\alpha, A^\beta\right]^a, \tag{3.9}$$

$$\mathcal{H}_{[\partial AA]^2} = -\frac{1}{2}g^2 \left[i\partial^+ A^\perp, A^\perp\right]^a \frac{1}{(i\partial^+)^2} \left[i\partial^+ A^\perp, A^\perp\right]^a. \tag{3.10}$$

Replacing A^μ with the operator $\hat{A}^\mu(x)$, defined by its Fourier composition on the plane $x^+ = 0$,

$$\hat{A}^\mu(x) = \sum_{\sigma c} \int [k] \left[t^c \epsilon^\mu_{k\sigma} a_{k\sigma c} e^{-ikx} + t^c \epsilon^{\mu *}_{k\sigma} a^\dagger_{k\sigma c} e^{ikx}\right]_{x^+=0}, \tag{3.11}$$

where $[k] = \theta(k^+)dk^+ d^2k^\perp/(16\pi^3 k^+)$, $\epsilon^\mu_{k\sigma}$ are the polarization vectors, t^c are the generators of the gauge group, and $a^\dagger_{k\sigma c}$, $a_{k\sigma c}$ are the creation and annihilation operators (particle operators), respectively.

Substituting this expression into each term of the Hamiltonian densities, integrating over space coordinates and taking into account the completeness and orthonormality of the polarization vectors, we obtain the following expression for the different terms of the Hamiltonian [5],

$$H_{A^2} = \sum_{\sigma c} \int [k] \frac{k^{\perp 2}}{k^+} a^\dagger_{k\sigma c} a_{k\sigma c}, \tag{3.12}$$

$$H_{A^3} = \sum_{123} \int [123] \not\delta(p^\dagger - p) \tilde{r}_{\Delta\delta}(3,1) \left[g Y_{123} a^\dagger_1 a^\dagger_2 a_3 + g Y^*_{123} a^\dagger_3 a_2 a_1 \right], \tag{3.13}$$

$$H_{A^4} = \sum_{1234} \int [1234] \not\delta(p^\dagger - p) \frac{g^2}{4} \left[\left(\Xi_{A^4 1234} a^\dagger_1 a^\dagger_2 a^\dagger_3 a_4 + h.c. \right) + X_{A^4 1234} a^\dagger_1 a^\dagger_2 a_3 a_4 \right], \tag{3.14}$$

$$H_{[\partial AA]^2} = \sum_{1234} \int [1234] \not\delta(p^\dagger - p) g^2 \left[\left(\Xi_{[\partial AA]^2 1234} a^\dagger_1 a^\dagger_2 a^\dagger_3 a_4 + h.c. \right) + X_{[\partial AA]^2 1234} a^\dagger_1 a^\dagger_2 a_3 a_4 \right]. \tag{3.15}$$

4 Code implementation

The code is implemented in Python, but the general method can be applied to any programming language. The program is designed to be modular, and applicable to other theories, by considering some minor modifications and changing the canonical diagrams. These modifications are not implemented yet, and remain to be tested.

4.1 Diagrams definition

The diagrams are defined by 2 arrays,

- Points: arrays of dimension $N \times 2$, where N is the number of points in the diagram, each point is defined by its coordinates (x, y).
- Paths: arrays of dimension $M \times N' \times 2$, where M is the number of different types of particles to consider in the theory, N' is the number of paths for each types of particle in the diagram, and 2 indicate the points to connect.

This way of defining the diagrams is analogous to the way of defining the undirected graphs in the graph theory, where the points are the vertices and the paths are the edges.

In the case of gluon interactions, although only 1 type of particle is present, the instantaneous interactions have to be considered. This is done by defining this interaction as a new type of virtual particle in the program.

4.2 Order by order procedure

To calculate the diagrams of higher order, the code follows the order by order procedure described in the section 2. Having only the canonical diagrams, the program aims to obtain all the possible diagrams of an order that contribute to a certain process, discarding the other diagrams.

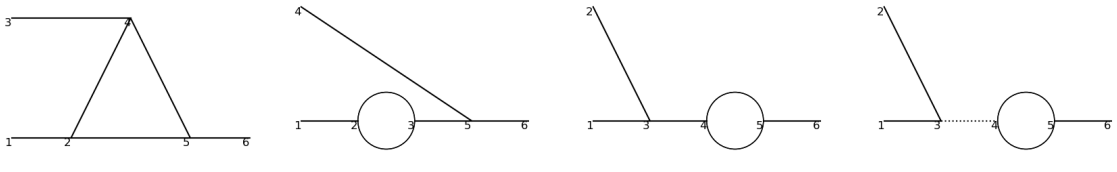
This process produces a huge amount of diagrams, many of them equivalent. The program detects these repeated diagrams, and adds their contribution, thus reducing the number of diagrams that needs to be used to calculate the next order. This procedure is fundamental to reduce the computational time.

Other important aspect of the RGPEP are the counterterms, that are added to cancel the divergences produced in the process. The program is able to detect the 2 loops and 3 loops in the diagrams of a certain order, and add new diagrams to cancel the divergences.

5 Diagrams obtained

5.1 Order 3

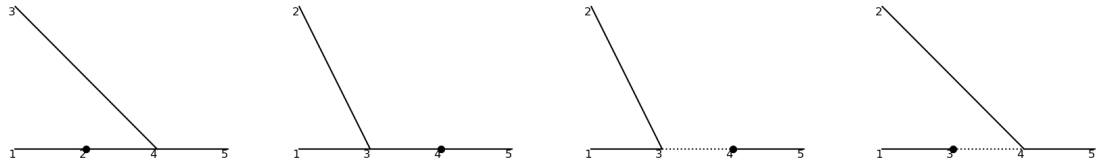
Considering the 3 gluon vertex.



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 1: Diagrams of third order, contributing to the 3 gluon vertex.



(a) (b) (c) (d)

Figure 2: Counterterms to the diagrams of third order in figure 1

In the figure 1 we can see the diagrams obtained for the process of 1 gluon going to 2 gluons, with the corresponding counterterms in figure 2.

The diagrams 1e and 1f are really the same diagram, due to instant process being instantaneous. For any other particle, these 2 diagrams would be different, due to the importance of the order in the interactions so the program is kept to consider them as different, to not lose generality.

Comparing with the 3rd order contribution diagrams in [5], the same diagrams

for the 3 gluon vertex are obtained, proving the validity of the program to reproduce the same results.

5.2 Order 4

Starting at order 4, the distinct non-abelian nature of the QCD is present, since the gluons interact with themselves, creating diagrams not possible in abelian theories, such as QED.

5.3 Higher orders

At higher orders, the diagrams become more complex, and the number of diagrams scale exponentially. Is here where the power of the program is shown.

6 Conclusions and future work

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