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Herramientas para cálculos perturbativos en renormalización de Hamiltonianos

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Resumen

Los processos de renormalización son fundamentales para la comprensión de la teorías a distintas escalas, definiendo teorías efectivas que describen el comportamiento de los sistemas para las distintas energías. Sin embargo, los cálculos son no triviales, y se emplean la teoría de perturbaciones para obtener los resultados, representando los distintos términos de la teoría como diagramas que describen el proceso, similar a los diagramas de Feynman. Pero el número de diagramas aumenta exponencialmente con el orden de la perturbación, haciendo que el proceso de cálculo sea tedioso y propenso a errores.

El objetivo de este trabajo es desarrollar un programa que automatice el proceso de obtención de los diagramas asociados a un proceso determinado, hasta un orden dado, partiendo de unos diagramas base, llamado diagramas canónicos dados por la teoría. Siendo el programa capaz de descartar los diagramas que no contribuyen al proceso, detectar loops y añadir contratérminos a los diagramas para cancelar divergencias.

Analizando los diagramas obtenidos de ordenes inferiores, y comparando con resultados conocidos, se ha comprobado la veracidad del programa. Esto permite obtener los diagramas de ordenes superiores, y estudiar el comportamiento de la teoría a dichos ordenes, en las que se pueden observar fenómenos único de teorías no abelianas.

Abstract

The renormalization procedure is fundamental to understand the behavior of the theories at different scales, defining effective theories that describe the behaviour of the systems for different energy levels. However, the calculations are non-trivial, and perturbation theory is used to obtain the results, representing the different terms of the theory as diagrams that describe the process, similar to the Feynman diagrams. But the number of diagrams increases exponentially with the order of the perturbation, making the calculation process tedious and prone to errors.

The aim of this work is to develop a program that automates the process of obtaining the diagrams associated with a given process, up to a given order, starting from a set of base diagrams, called canonical diagrams given by the theory. The program is able to discard the diagrams that do not contribute to the process, detect loops and add counterterms to the diagrams to cancel divergences.

By analysing the diagrams obtained from lower orders, and comparing with known results, the correctness of the program has been verified. This allows to obtain the diagrams of higher orders, and study the behavior of the theory at those orders, where unique phenomena of non-abelian theories can be observed.

Contents

1 Introduction

When trying to build a theory that describes the particles and their interactions, it's fundamental that the theory is compatible with the 2 pillars of modern physics, the special relativity and quantum mechanics. The special relativity is a theory that's able to describe the behavior of particles at high energies, and quantum mechanics is a theory that describes the behavior of particles at small scales. The combination of these two theories is the basis of the quantum field theory, building a formalism that describes the particles as excitations of a field.

Other important aspect of the theory is that it should be able to describe the different phenomena that occur at different scales. In the context of QCD, the experiments performed at low energies, or high length scales, smear the interaction between the particles, and does not resolve the details of the structure inside the particles, observing strongly bound states. While at high energies, or low length scales, the hadrons are broken into their constituents, quarks and gluons, and the interaction between the particles is weak, forming the so-called asymptotic freedom. The theory should be able to describe the transition between these two regimes.

In the context of theoretical physics, the renormalization group procedure is a powerful tool to study the behavior of physical systems at different scales. In the case of quantum field theories, renormalization allows to study the system at different energy scales by introducing a scale parameter, and by changing this parameter, the "resolution" of the system is changed, allowing to focus from the smallest details (the short distance behavior) to the largest ones (the grand scale behavior).

We will adopt the Hamiltonian formalism to describe the dynamics of the system, working in the operator space, where the Hamiltonian operator governs the time evolution of the states.

Many widely used renormalization procedures tend to apply to Lagrangian dynamics. But in the framework of RGPEP, the renormalization group procedure is applied to Hamiltonian dynamics. The benefit of this approach is the ability obtain directly the solutions of the system (the spectrum of the theory, and the eigenstates of the Hamiltonian). RGPEP introduces an effective Hamiltonian, that describes the system at a given scale. This effective Hamiltonian is the solution to a differential equation, that describes the evolution of the effective Hamiltonian with respect to the scale parameter.

In general, obtaining the exact solution of the RGPEP equation is a non-trivial task, and a perturbative expansion of the effective Hamiltonian is used to obtain the solution. By identifying each order in the Hamiltonian expansion with a series of products of diagrams, the next order in the expansion can be obtained from the diagrams of previous orders. Typically, these diagrams were analyzed hand, but the number of diagrams increases exponentially with each order, making the process tedious and error-prone. The goal of this thesis is to develop a code that automates the process of obtaining the diagrams associated with a certain interaction for a given order.

This thesis is organized as follows. Section 2 describes the theoretical background needed to understand the renormalization group procedure for effective particles, and the Hamiltonian dynamics. Section 3 describes the case studied in this thesis, gluon self interactions, where holding the simplicity of the scalar case, some peculiarities of the QCD theory are present. The section 4 describes the connection between the diagrams and the objects are defined in the code, as well as the steps taken to obtain higher order diagrams, that are presented in the section 5. Finally, the section 6 presents the conclusions, future work and the improvement that can be done to the code.

2 Theoretical background

To understand the basis of the RGPEP, we need to understand the different concepts involved in the process, as well as the theories that the process is applied to.

2.1 Fock space

Introduced by V.A. Fock in 1932 [2], the Fock space is a sum of different Hilbert spaces, each one corresponding to a different number of particles in the system. Allowing the description of quantum systems with a variable number of particles. It is essential in quantum field theory, where the number of particles is not conserved and processes such as particle creation and annihilation occur.

The Fock space is defined as the direct sum of tensor products of the single particle Hilbert space $\mathbb{H}$,

$$\mathbb{F} = \bigoplus_{n=0}^{\infty} \mathbb{H}^{\otimes n} = \mathbb{C} \oplus \mathbb{H} \oplus (\mathbb{H} \otimes \mathbb{H}) \oplus (\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}) \oplus \cdots , \tag{2.1}$$

where $\mathbb{C}$ is the complex scalar, corresponding to the states with no particles, and the terms $\mathbb{H}^{\otimes n}$ represents the Hilbert space for n-particle states.

This way a general state in the Fock space can be expressed as,

$$|\Psi\rangle = |\Psi_0\rangle \oplus |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots = c\,|0\rangle + \sum_{i=1} c_i\,|\psi_i\rangle + \sum_{i,j=1} c_{ij}\,|\psi_i \psi_j\rangle + \cdots , \tag{2.2}$$

where $|\Psi_0\rangle$ is the vacuum state, $|\Psi_1\rangle$ is the one particle state, $|\Psi_2\rangle$ is the two particle state, and so on. The coefficients c_i are the amplitudes of the states, in general complex numbers.

Fock space provides a natural framework for quantum field theories QCD, where, physical states are expressed as superposition of all allowable multiparticle configurations consistent with color confinement and other quantum numbers. For instance, a quarkonium state (a bound state of quark and antiquark) in QCD is given by:

$$|\Psi\rangle = c_1 |q\bar{q}\rangle + c_2 |q\bar{q}g\rangle + c_3 |qqq\rangle + c_4 |gg\rangle + \cdots, \tag{2.3}$$

2.2 Hamiltonian dynamics

In quantum field theory, 2 equivalent formulations of the dynamics can be used, the Lagrangian and the Hamiltonian formulations. Though the modeling of dynamics tends to start in the Lagrangian formulation, and then using the Legendre transformation to obtain the Hamiltonian formulation.

2.2.1 Canonical Hamiltonian

Consider a general field theory and a Lagrangian density $\mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))$[1], where $\phi_a(x)$ is a particular field of the system.

To formulate the theory in the Hamiltonian framework, we begin by defining the canonical conjugate momenta,

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a(x))}. \tag{2.4}$$

The Hamiltonian density, via the Legendre transformation,

$$\mathcal{H} = \sum_a \pi_a(x)\partial_0\phi_a(x) - \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)). \tag{2.5}$$

In quantum mechanics, or quantum field theory, the canonical variables $\phi_a(x)$ and $\pi_a(x)$ are promoted to operators acting on a Hilbert space. Quantization is achieved by imposing equal-time canonical commutation relations

$$[\phi_a(x), \pi_b(y)] = i\delta_{ab}\delta^3(x-y), \quad [\phi_a(x), \phi_b(y)] = 0, \quad [\pi_a(x), \pi_b(y)] = 0. \tag{2.6}$$

The field operators tend to be expressed in terms of creation, a_i and annihilation $a_i^\dagger$ operators, that act on the Fock space, creating and annihilating the mode i of the field. Mathematically, the creation and annihilation operators are the Fourier components of the field.

Any state component in the Fock space can be expressed as the action of a series of creation operators on the vacuum state, $|0\rangle$,

$$|\Psi_n\rangle = \sum_{i_1, i_2, \cdots, i_n} \frac{c_{i_1 i_2 \cdots i_n}}{\sqrt{n!}} a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle, \tag{2.7}$$

making the full state,

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{i_1, i_2, \cdots, i_n} c_{i_1 i_2 \cdots i_n} a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle. \tag{2.8}$$

[1] where x indicate a 4-vector, with x^μ its components, and $\partial_\mu = \frac{\partial}{\partial x^\mu}$

The creation and annihilation operators satisfy some commutation or anticommutation relations, depending on the nature of the particles, bosons or fermions.

For bosons, the creation and annihilation operators satisfy the following commutation relations,

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \quad \left[a_i, a_j\right] = 0, \quad \left[a_i^\dagger, a_j^\dagger\right] = 0. \tag{2.9}$$

For fermions, the creation and annihilation operators satisfy the following anticommutation relations,

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0. \tag{2.10}$$

This way, the creation and annihilation operators are the fundamental objects of the Fock space, and the Hamiltonian and the fields are expressed in terms of these operators.

When a combination of creation and annihilation operators is considered, normal-ordering is used when defining operators that contain both creation and annihilation operators and avoid the infinite vacuum contribution. The normal-ordering is defined as the process of rearranging the creation and annihilation operators in such a way that all the creation operators are to the left of all the annihilation operators.

The normal-ordering is denoted by the symbol $: \cdots :$, then for a fermionic case,

$$a_i a_j^\dagger = \{a_i, a_j^\dagger\} - a_j^\dagger a_i = \delta_{ij} - : a_j^\dagger a_i := \delta_{ij} - a_j^\dagger a_i. \tag{2.11}$$

2.2.2 Front form of Hamiltonian Dynamics

The quantization of relativistic systems is most commonly performed in the instant form (IF) of dynamics, where the ordinary time coordinate $x^0 = t$ serves as the evolution parameter, and spatial coordinates $\vec{x}$ define the hypersurface of equal time. However, alternative forms of dynamics are possible and were classified by Dirac in 1949 [1].

One such alternative is the front form (FF) of dynamics, also known as light-front quantization. It is defined by choosing a new set of coordinates where the evolution parameter is,

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3), \tag{2.12}$$

and the remaining coordinates are,

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3), \quad x^\perp = (x^1, x^2). \tag{2.13}$$

In this framework, quantization is performed on surfaces of constant x^+, treating it as the "light-front time", while $x^-, x^\perp$ play the role of spatial coordinates.

The corresponding momenta are defined as,

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3), \quad p^- = \frac{1}{\sqrt{2}}(p^0 - p^3), \quad p^\perp = (p^1, p^2). \tag{2.14}$$

The light-front Hamiltonian P^- is derived from the Lagrangian using a Legendre transformation with respect to x^+. It governs the evolution in x^+, analogous to the role of $H = P^0$ in instant-form quantization.

A key feature of FF quantization is the positivity condition $p^+ > 0$ for all physical particles. This kinematic constraint ensures that particle creation from the vacuum, which has total $P^+ = 0$, is forbidden due to momentum conservation. As a result, the vacuum in FF is trivial or "empty": it contains no virtual particles and cannot mix with multi-particle states.

This property simplifies the structure of the theory, particularly in the context of the RGPEP, where a clean separation between the vacuum and the particle spectrum is advantageous. FF quantization avoids the complexities associated with vacuum fluctuations that are typical in instant form dynamics.

2.3 RGPEP

The RGPEP, is a renormalization scheme applied within the Hamiltonian formalism of quantum field theory. By considering a series of unitary transformations applied to the canonical or bare Hamiltonian, the RGPEP is able to construct a series of effective Hamiltonians $\mathcal{H}_s$, each describing dynamics in terms of effective particles at a resolution scale set by the parameter s. This is associated with the renormalization group scale $\lambda = 1/s$, has dimension of length, and physically has the interpretation of the characteristic scale of the theory.

The corresponding effective particles, by the use of effective particle operators (namely creation and annihilation operators) that differs from the canonical ones by the same unitary transformation $\mathcal{U}_s$,

$$a_s = \mathcal{U}_s a_0 \mathcal{U}_s^\dagger. \tag{2.15}$$

Due to dimensional and notational reasons, it's convenient to consider the scale parameter $t = s^4$ instead.

Then $s = 0$ or equivalently $t = 0$, the theory describes point-like or bare particles, and recovering the original Hamiltonian $\mathcal{H}_0(a_0)$.

The effective Hamiltonian $\mathcal{H}_t$, written in terms of the effective particles operator a_s, is related to the regulated canonical Hamiltonian with counter-terms by the condition (section 2.4),

$$\mathcal{H}_t(a_t) = \mathcal{H}_0(a_0) \tag{2.16}$$

Using Eq. (2.15) and expressing all operators in terms of the original a_0, the Hamiltonian becomes,

$$\mathcal{H}_t(a_0) = \mathcal{U}_t^\dagger \mathcal{H}_0(a_0)\mathcal{U}_t, \tag{2.17}$$

differentiating with respect of t, one obtains the RGPEP differential equation,

$$\mathcal{H}_t'(a_0) \equiv \frac{d}{dt}\mathcal{H}_t(a_0) = \left[-\mathcal{U}_t^\dagger \mathcal{U}_t', \mathcal{H}_t(a_0)\right] = \left[\mathcal{G}_t(a_0), \mathcal{H}_t(a_0)\right], \tag{2.18}$$

where $\mathcal{G}_t$ is the generator of the RGPEP transformation.

Considering the generator from Ref. [4],

$$\mathcal{G}_t = \left[\mathcal{H}_f, \mathcal{H}_{Pt}\right], \tag{2.19}$$

where $\mathcal{H}_f$, the free part of $\mathcal{H}_t$ and the part of $\mathcal{H}_0(a_0)$ that does not depend on the coupling constants, while $\mathcal{H}_{Pt}$ is defined as function of the interacting term.

The resulting RGPEP equation have the form,

$$\mathcal{H}_t' = \left[\left[\mathcal{H}_f, \mathcal{H}_{Pt}\right], \mathcal{H}_t\right], \tag{2.20}$$

In general, the solution of the RGPEP equation is a non-trivial task, and a perturbative expansion of the effective Hamiltonian is used to obtain the solution. Expressing $\mathcal{H}_t$ as a power series of the coupling constant g,

$$\mathcal{H}_t = \sum_{n=0}^{\infty} g^n \mathcal{H}_{tn} = \mathcal{H}_0 + g\mathcal{H}_{t1} + g^2\mathcal{H}_{t2} + g^3\mathcal{H}_{t3} + g^4\mathcal{H}_{t4} + \cdots . \tag{2.21}$$

Substituting into the RGPEP equation, and collecting terms order by order in g, the following differential equations are obtained,

$$\mathcal{H}_0' = 0, \tag{2.22}$$

$$g\mathcal{H}_{t1}' = \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], \mathcal{H}_0\right], \tag{2.23}$$

$$g^2\mathcal{H}_{t2}' = \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{Pt2}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], g\mathcal{H}_{t1}\right], \tag{2.24}$$

$$g^3\mathcal{H}_{t3}' = \left[\left[\mathcal{H}_0, g^3\mathcal{H}_{Pt3}\right], \mathcal{H}_0\right] + \left[\left[\mathcal{H}_0, g^2\mathcal{H}_{Pt2}\right], g\mathcal{H}_{t1}\right] + \left[\left[\mathcal{H}_0, g\mathcal{H}_{Pt1}\right], g^2\mathcal{H}_{t2}\right]. \tag{2.25}$$

$$\vdots$$

The order 0 term is solvable from an initial condition, and the solution will be an exponential of the parameter t. The 1st order solution can be obtained from 0th order, and the 2nd order from the previous orders, and so on.

2.4 Regularization and Counterterms

In QCD, the bare Hamiltonian is ill-defined due to the presence of elements that contain divergences, ultraviolet (UV) divergences and infrared (IR) divergences. Th UV divergences are produced in processes where the large momentum transferes, or the high invariant mass difference between Fock states. While the IR divergences are produced in procceses where the particles are "soft", carring small longitudinal momentum fractions, $x_{p/P} = p^+/P^+$.

To deal with these divergences a regulating factor r is introduced in the interacting terms. These factors make the interacting terms rapidly tend to zero, if the change in the transverse momentum of any gluon exceeds a certain cutoff parameter Δ, or if the change in longitudinal momentum of any gluon is greater than a cutoff parameter δ.

The particle operators is multiplied by the regulating factor,

$$r_{\Delta\delta}(k^{\perp}, x) = r_{\Delta}(k^{\perp}) r_{\delta}(x) \theta(x). \tag{2.26}$$

The transverse regulator factor will be of the form,

$$r_{\Delta}(\mathcal{M}) = \exp\left(-\frac{\mathcal{M}^2}{\Delta^2}\right), \tag{2.27}$$

with $\mathcal{M}$ the invariant mass of the system, ensuring that UV processes are suppressed.

The longitudinal regulator factor must verify a similar condition, preventing terms of the form $1/x$ or $1/x^2$ to blow up as x approaches 0. The exact form of the longitudinal regulator factor is not important, as long as it verifies the condition, since it will be removed using the RGPEP.

Both cutoff parameters will disappear in the final result, since the theory can't depend on the cutoff parameters taken arbitrarily. This is done by taking the limit $\Delta \to \infty$ and $\delta \to 0$, recovering the original values taken by the momenta.

It's convenient to consider an abbreviated notation for the regulating factor, using the symbol $\tilde{r}_{P,p} = \tilde{r}_{\Delta\delta}(P, p)$, with,

$$\tilde{r}_{\Delta\delta}(P, p) = r_{\Delta\delta}(p^{\perp} - x_{p/P} P^{\perp}) r_{\Delta\delta}\left[P^{\perp} - p^{\perp} - \left(1 - x_{p/P}\right) P^{\perp}, 1 - x_{p/P}\right]. \tag{2.28}$$

The counterterms is an additional term added to the initial or bare Hamiltonian $\mathcal{H}_0$ to deal with the divergences due to loops, produced during the process. This way, ensure that the effective Hamiltonian $\mathcal{H}_t$ remains finite at all values of t.

The counterterms are defined in a way, such that the coefficients of products of creation and annihilation operators in the effective theory for gluons of size s become independent of the regularization parameter Δ when the regularization in dynamics of gluons of size zero is being removed. The rest of the unknown parts of the counterterms are adjusted to respect the symmetries of the theory, and must match the predictions of the theory with the experimental results.

2.5 Diagram representation

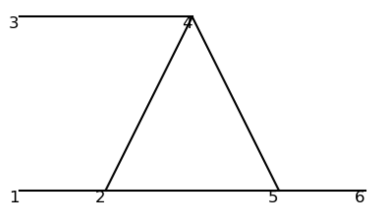
From the expression of the Hamiltonian, different terms can be separated and correlate to a diagram representation of the process. Each diagram will be composed of different elements,

- External legs, representation of the incoming and outgoing particles, it indicates the type of process that is being considered. Will have different colors/ types of lines depending on the type of particle.
- Internal legs, represent the virtual particles that are exchanged during the process, these particles are not observed in the final state. Similar to the external legs, will have different colors/types of lines depending on the type of particle.
- Vertices, points where interactions between particles occur, the number of vertices in a diagram indicates the order at which the diagram is contributing.
- Loops, closed paths in the diagrams, formed by the internal legs. It will indicate the presence of divergences in the process.
- Counterterms, additional terms added to the Hamiltonian to deal with the divergences produced during the process. These will be represented with a dot in the diagram.

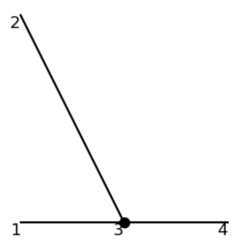
Notice that the diagrams are similar to the Feynman diagrams, except for the presence of the counterterms. The main difference is the order that the processes occur, altering the order of the vertices in the diagram will produce a different process.

Other difference how the diagrams are read, in the context of RGPEP, the diagrams are read from the left to the right, meaning that the time evolution is from the left to the right.

As an example, consider the diagram in figure 1, this pair of diagrams are one of the contributions to the 3 gluon vertex, with the counterterm associated with it.



(a) Diagram with a 3 gluon loop.



(b) Counterterm of the diagram

Figure 1: A example of a possible diagram, a third order contribution to the process of 3 gluon vertex, 1 gluon incoming and 2 gluons outgoing. (Outputs of the programs.)

Following the previous definitions, the diagram represent a process where a gluon interacts in a third order process, forming a 3 gluon loop, and 2 gluons are produced in the final state. Due to the presence of a loop, the diagram is divergent, and a counterterm diagram is needed to deal with the divergence. This diagram is represented with a dot at the position of the loop, with the same external legs as the original diagram.

2.6 Order by order solutions.

The solution of the differential equations (2.23) to (2.25) and so on, without considering the multiplicative factors, and relative phases between the terms, can be expressed as,

$$\begin{aligned}
\mathcal{H}_{t1} &= \mathcal{H}_{01} && (2.29)\\
\mathcal{H}_{t2} &= \mathcal{G}_{02} + \mathcal{H}_{01}\mathcal{H}_{01} && (2.30)\\
\mathcal{H}_{t3} &= \mathcal{G}_{03} + \mathcal{H}_{01}\mathcal{H}_{01}\mathcal{H}_{01} + (\mathcal{G}_{02}\mathcal{H}_{01} + \mathcal{H}_{01}\mathcal{G}_{02}) && (2.31)\\
\mathcal{H}_{t4} &= \mathcal{G}_{04} + \mathcal{H}_{01}\mathcal{H}_{01}\mathcal{H}_{01}\mathcal{H}_{01} + (\mathcal{G}_{02}\mathcal{H}_{01}\mathcal{H}_{01} + \mathcal{H}_{01}\mathcal{G}_{02}\mathcal{H}_{01} + \mathcal{H}_{01}\mathcal{H}_{01}\mathcal{G}_{02})\\
&\quad + (\mathcal{G}_{03}\mathcal{H}_{01} + \mathcal{H}_{01}\mathcal{G}_{03} + \mathcal{G}_{02}\mathcal{G}_{02})\,. && (2.32)\\
&\vdots
\end{aligned}$$

This is an oversimplification of the solution, but it gives an idea of how the different solutions for each order are obtained.

For instance, the diagrams for the first order is simply the canonical diagrams from $\mathcal{H}_{01}$. As for the second order, it contains the canonical diagrams from the second order $\mathcal{G}_{02}$, and the product of the first order diagrams. The third order contains the canonical diagrams from the third order $\mathcal{G}_{03}$, and the product of the first order, and second order canonical diagrams, as well as the product of exclusive first order diagrams. This way, we could rewrite the solution for each order in terms of the solution to the previous order,

$$\begin{aligned}
\mathcal{H}_{t1} &= \mathcal{H}_{01} = \mathcal{G}_{01}, && (2.33)\\
\mathcal{H}_{t2} &= \mathcal{G}_{02} + \mathcal{H}_{t1}\mathcal{G}_{01}, && (2.34)\\
\mathcal{H}_{t3} &= \mathcal{G}_{03} + \mathcal{H}_{t1}\mathcal{G}_{02} + \mathcal{H}_{t2}\mathcal{G}_{01}, && (2.35)\\
\mathcal{H}_{t4} &= \mathcal{G}_{04} + \mathcal{H}_{t2}\mathcal{G}_{02} + \mathcal{H}_{t1}\mathcal{G}_{03} + \mathcal{H}_{t3}\mathcal{G}_{01} && (2.36)\\
&\vdots
\end{aligned}$$

This way of writing the solution is useful to understand how the different orders are related to each other, and how an iterative process can be used to obtain the solution for each order.

3 Case study: Gluons self interactions

3.1 Canonical Hamiltonian

The Lagrangian density for the gluon fields is given by,

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu}, \tag{3.1}$$

where $F^{\mu\nu}$ is the field strength tensor, defined as,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu], \tag{3.2}$$

and $A^\mu = A^{a\mu}t^a$, t^a are the generators of the gauge group, and g is the coupling constant. Verifying the following relations,

$$[t^a, t^b] = if^{abc}t^c, \quad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{3.3}$$

We will be working in the gauge $A^+ = 0$, where the Lagrange equations constrain the component A^- to become

$$A^- = \frac{1}{\partial^+}2\partial^\perp A^\perp - \frac{2}{\partial^{+2}}ig\left[\partial^+ A^\perp, A^\perp\right]. \tag{3.4}$$

In this way, the only degree of freedom left is the transverse component $A^\perp$.

As for the associated energy-momentum tensor,

$$\mathcal{T}^{\mu\nu} = -F^{a\mu\alpha}\partial^\nu A^a_\alpha + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}. \tag{3.5}$$

The Hamiltonian in FF quantization is obtained from integrating the component $\mathcal{T}^{+-}$ of the energy-momentum tensor, over the hyperplane $x^+ = 0$.

By working in the gauge $A^+ = 0$, the Hamiltonian density can be expressed as the sum of 4 terms, as denoted in [3]

$$\mathcal{T}^{+-} = \mathcal{H}_{A^2} + \mathcal{H}_{A^3} + \mathcal{H}_{A^4} + \mathcal{H}_{[\partial AA]^2}, \tag{3.6}$$

with each of the terms,

$$\mathcal{H}_{A^2} = -\frac{1}{2}A^{\perp a}(\partial^\perp)^2 A^{\perp a}, \tag{3.7}$$

$$\mathcal{H}_{A^3} = gi\partial_\alpha A^a_\beta\left[A^\alpha, A^\beta\right]^a, \tag{3.8}$$

$$\mathcal{H}_{A^4} = -\frac{1}{4}g^2\left[A_\alpha, A_\beta\right]^a\left[A^\alpha, A^\beta\right]^a, \tag{3.9}$$

$$\mathcal{H}_{[\partial AA]^2} = -\frac{1}{2}g^2\left[i\partial^+ A^\perp, A^\perp\right]^a\frac{1}{(i\partial^+)^2}\left[i\partial^+ A^\perp, A^\perp\right]^a. \tag{3.10}$$

Replacing A^μ with the operator $\hat{A}^\mu(x)$, defined by its Fourier composition on the plane $x^+ = 0$,

$$\hat{A}^\mu(x) = \sum_{\sigma c} \int [k] \left[t^c \epsilon^\mu_{k\sigma} a_{k\sigma c} e^{-ikx} + t^c \epsilon^{\mu *}_{k\sigma} a^\dagger_{k\sigma c} e^{ikx} \right]_{x^+=0}, \tag{3.11}$$

where $[k] = \theta(k^+) dk^+ d^2k^\perp / (16\pi^3 k^+)$, $\epsilon^\mu_{k\sigma}$ are the polarization vectors, t^c are the generators of the gauge group, and $a^\dagger_{k\sigma c}$, $a_{k\sigma c}$ are the creation and annihilation operators (particle operators), respectively.

Substituting this expression into each term of the Hamiltonian densities, integrating over space coordinates and taking into account the completeness and orthonormality of the polarization vectors, we obtain the following expression for the different terms of the Hamiltonian [5],

$$H_{A^2} = \sum_{\sigma c} \int [k] \frac{k^{\perp 2}}{k^+} a^\dagger_{k\sigma c} a_{k\sigma c}, \tag{3.12}$$

$$H_{A^3} = \sum_{123} \int [123] \tilde{\delta}(p^\dagger - p) \tilde{r}_{\Delta\delta}(3,1) \left[g Y_{123} a^\dagger_1 a^\dagger_2 a_3 + g Y^*_{123} a^\dagger_3 a_2 a_1 \right], \tag{3.13}$$

$$H_{A^4} = \sum_{1234} \int [1234] \tilde{\delta}(p^\dagger - p) \frac{g^2}{4} \left[\left(\Xi_{A^4 1234} a^\dagger_1 a^\dagger_2 a^\dagger_3 a_4 + h.c. \right) + X_{A^4 1234} a^\dagger_1 a^\dagger_2 a_3 a_4 \right], \tag{3.14}$$

$$H_{[\partial A A]^2} = \sum_{1234} \int [1234] \tilde{\delta}(p^\dagger - p) g^2 \left[\left(\Xi_{[\partial A A]^2 1234} a^\dagger_1 a^\dagger_2 a^\dagger_3 a_4 + h.c. \right) + X_{[\partial A A]^2 1234} a^\dagger_1 a^\dagger_2 a_3 a_4 \right]. \tag{3.15}$$

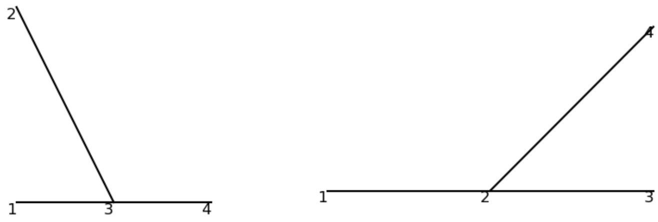
3.2 Canonical diagrams

The canonical diagrams are the diagrams that are obtained from the canonical Hamiltonian, and are the starting point to obtain the diagrams of higher order.

Following the creation and annihilation operators in the different terms of the Hamiltonian, we can distinguish the different types of process that are being considered.

Staring from the term H_{A^2} in equation (3.12), we can see that this a particle of certain momentum k is annihilated and created, producing the same particle. This corresponds to the kinetic term of the Hamiltonian, and the diagram will be a line with the same color/type of particle in both ends. Being an order 0 term that will not contribute to the perturbative expansion.

The term H_{A^3} in equation (3.13) is a 3 gluon vertex, where either 1 gluon is annihilated and 2 gluons are created, or 2 gluons are annihilated, and 1 gluon is created. This process can be represented by a diagram with 3 external legs, and 1 vertex. Being an order 1 term, with the corresponding diagrams shown in figure 2.



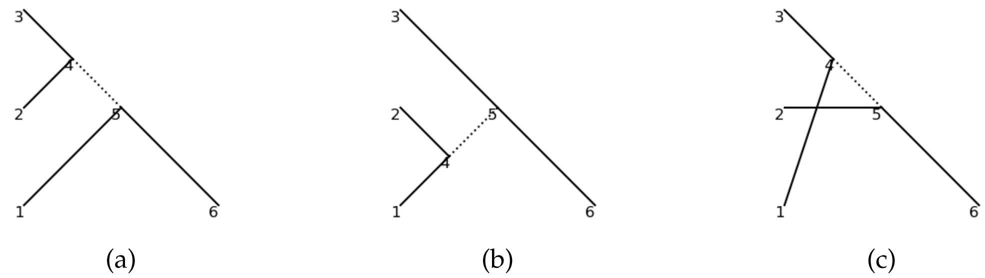
(a) (b)

Figure 2: Canonical diagrams of order 1

The term H_{A^4} in equation (3.14) is a 4 gluon vertex, where either 1 gluon is annihilated, and 3 gluons are created, 3 gluons are annihilated, and 1 gluon is created, or 2 gluons are annihilated, and 2 gluons are created. But since we are interested in building the diagrams at each order with diagrams of lower order, it's convenient to consider the 4 gluon vertex as a combination of two 3 gluon vertices. Meaning that for the process where one gluon goes to 3 gluons, we consider the process as, one gluon being annihilated, and 2 gluons are created, then one of the created gluons is annihilated, and 2 gluons are created, producing a total of 3 gluons in the final state. The same applies for the process where 3 gluons are annihilated. This way, these diagrams have 2 vertices, and thus a second order term, that can be obtained from the product of the canonical first order diagrams, so we will not consider them as a new canonical diagrams, but rather as a result that will be obtained during the perturbative expansion.

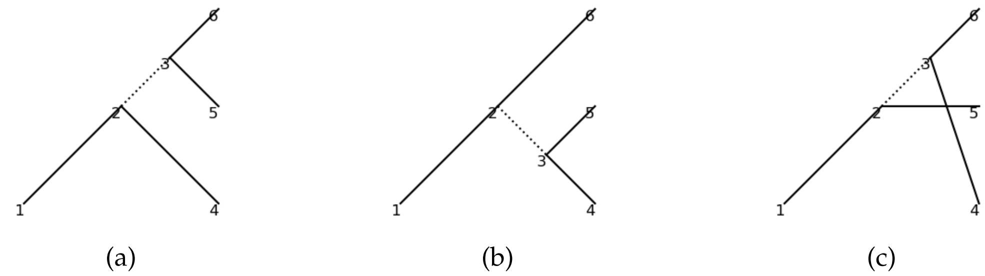
The fascinating term is $H_{[\partial AA]^2}$ in equation (3.15), similar to the term H_{A^4}, this term is a 4 gluon vertex, but the concept of instantaneous interaction is introduced. This allows to consider processes where a gluon is annihilated, and 3 gluons are created, to a pair of 3 gluon vertices, similar to H_{A^4}. The main difference is that the interaction is instantaneous, meaning that the gluon is annihilated, and the 3 gluons are created at the same time, without any time delay between them. This process is usually represented in by a perpendicular bar in the line of the gluon, indicating that the interaction is instantaneous, but due to program limitations, dotted lines are used in this work.

The different sum terms in the equation (3.15) indicate the different processes that can be considered, the first term $\Xi_{[\partial AA]^2 1234}$ indicate the process where 1 gluon is annihilated, and 3 gluons are created, as shown in figure 3. The second term $\Xi^*_{[\partial AA]^2 1234}$ indicate the hermitian conjugate of the first term, where 3 gluons are annihilated, and 1 gluon is created, as shown in figure 4. The third term $X_{[\partial AA]^2 1234}$ indicate the process where 2 gluons are annihilated, and 2 gluons are created, as shown in figure 5.



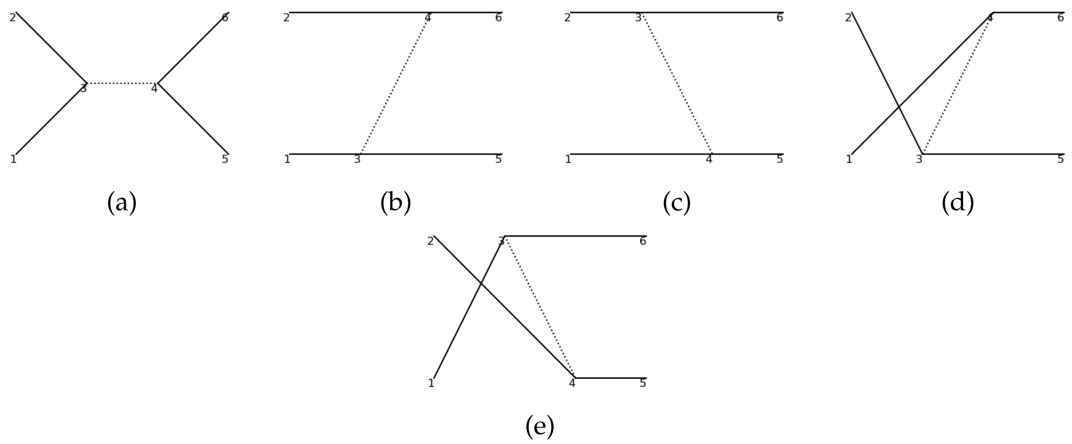
(a) (b) (c)

Figure 3: Canonical diagrams of order 2, for the term $\Xi_{[\partial AA]^2}$, where one gluon is annihilated, and 3 gluons are created.



(a) (b) (c)

Figure 4: Canonical diagrams of order 2, for the term hermitian conjugate of $\Xi_{[\partial AA]^2}$, where 3 gluons are annihilated, and 1 gluon is created.



(a) (b) (c) (d)

(e)

Figure 5: Canonical diagrams of order 2, for the term $X_{[\partial AA]^2}$, where 2 gluons are annihilated, and 2 gluons are created.

The combination of all these diagrams, are the canonical diagrams of order 2. Together with the diagrams of order 1, figure 2, they form the basis of the perturbative expansion of the Hamiltonian, and the starting point to obtain the diagrams of higher order.

As we can see, in the case of QCD, there are only canonical diagrams of order 1 and 2, then referring to the framework laid out in section 2.6, from order 3 and onwards, the diagrams only depend explicitly on the diagrams of previous order, as well as the order 1, and the order 2 canonical diagrams.

For example, in Eq. (2.36), the solution at order 4, $\mathcal{H}_{t4}$ consists of $\mathcal{H}_{t3}$ and the canonical diagrams, thus eliminating the term with $\mathcal{G}_{04}$ and $\mathcal{H}_{t1}\mathcal{G}_{03}$, since they are not present in the QCD theory. This way simplifying the number of diagrams that can be obtained at higher orders, and making the diagrams have the form,

$$\mathcal{H}_{t1} = \mathcal{H}_{01} = \mathcal{G}_{01}, \tag{3.16}$$
$$\mathcal{H}_{t2} = \mathcal{G}_{02} + \mathcal{H}_{t1}\mathcal{G}_{01}, \tag{3.17}$$
$$\mathcal{H}_{t3} = \mathcal{H}_{t1}\mathcal{G}_{02} + \mathcal{H}_{t2}\mathcal{G}_{01}, \tag{3.18}$$
$$\mathcal{H}_{t4} = \mathcal{H}_{t2}\mathcal{G}_{02} + \mathcal{H}_{t3}\mathcal{G}_{01} \tag{3.19}$$
$$\mathcal{H}_{t5} = \mathcal{H}_{t3}\mathcal{G}_{02} + \mathcal{H}_{t4}\mathcal{G}_{01} \tag{3.20}$$
$$\vdots$$
$$\mathcal{H}_{tn} = \mathcal{H}_{t(n-1)}\mathcal{G}_{02} + \mathcal{H}_{t(n-2)}\mathcal{G}_{01} \quad \text{for } n \geq 3. \tag{3.21}$$

Obtaining a simpler recursive relation to obtain the diagrams of higher order, that can be implemented in a program to obtain the diagrams of higher order.

4 Code implementation

The code is implemented in Python, but the general method can be applied to any programming language. The program is designed to be modular, and applicable to other theories, by considering some minor modifications and changing the canonical diagrams. These modifications are not implemented yet, and remain to be tested.

4.1 Diagrams definition

The diagrams are defined by 2 arrays,

- Points: arrays of dimension $N \times 2$, where N is the number of points in the diagram, each point is defined by its coordinates (x, y).
- Paths: arrays of dimension $M \times N' \times 2$, where M is the number of different types of particles to consider in the theory, N' is the number of paths for each types of particle in the diagram, and 2 indicate the points to connect.

This way of defining the diagrams is analogous to the way of defining the undirected graphs in the graph theory, where the points are the vertices and the paths are the edges.

In the case of gluon interactions, although only 1 type of particle is present, the instantaneous interactions have to be considered. This is done by defining this interaction as a new type of virtual particle in the program.

4.2 Order by order procedure

To calculate the diagrams of higher order, the code follows the order by order procedure described in the section 2. Having only the canonical diagrams, the program aims to obtain all the possible diagrams of an order that contribute to a certain process, discarding the other diagrams.

This process produces a huge amount of diagrams, many of them equivalent. The program detects these repeated diagrams, and adds their contribution, thus reducing the number of diagrams that needs to be used to calculate the next order. This procedure is the most time-consuming process of the program, needing a search algorithm to find the equivalent diagrams, meaning that as the order increases, the number of diagrams increases exponentially, and the time to find the equivalent diagrams increases exponentially too.

Although the process of finding the equivalent diagrams is time-consuming, it is fundamental to reduce the global computational time. Since the number of diagrams tend to decrease by 1 to 2 orders of magnitude, depending on the order and the number of particles in the process. This way reduces the time needed to calculate the diagrams of the next order, so at the grand scale, the time needed to calculate the diagrams of till a certain order is reduced by performing the search algorithm inbetewen the orders.

Other important aspect of the RGPEP are the counterterms, that are added to cancel the divergences produced in the process. The program is able to detect the 2 loops and 3 loops in the diagrams of a certain order, and add new diagrams to cancel the divergences. As shown in figure 1.

4.3 Applied to gluons self interactions

Although in this thesis the focus is on the gluons and its self interactions, and considering only one type of particle, there are tests that can be done to check the validity of the program for multiple types of particles. Thanks to the instantaneous interactions, the program can be adapted to consider this type of interactions as a new type of particles, and thus check for bugs or other issues that may arise when considering multiple types of particles.

This approach comes with its own problems. By considering an instantaneous process as a new type of particles, this process need to respect the time evolution of the system, meaning that in the eyes of the program, the instantaneous interaction is no longer instantaneous, but rather a process that happens in a certain time interval.

Nonetheless, this is a valid approach to consider the instantaneous interactions, since no possible diagrams will be lost, rather equivalent diagrams will be obtained, without the program being aware of it, and it rests on the user to identify these equivalences.

For lower orders, this approach works well, since the number of diagrams is manageable, but at higher orders, this is a problem that needs to be addressed in the future, and a general fix to the problem will be implemented in the program.

5 Diagrams obtained

In this section, we will present the diagrams obtained from the program. At each order, we will consider the types of processes that is the most relevant.

5.1 Order 3

Considering the 3 gluon vertex, the process of 1 gluon going to 2 gluons, up till order 3, the diagrams obtained from the program are shown in the figures 6 and 7.

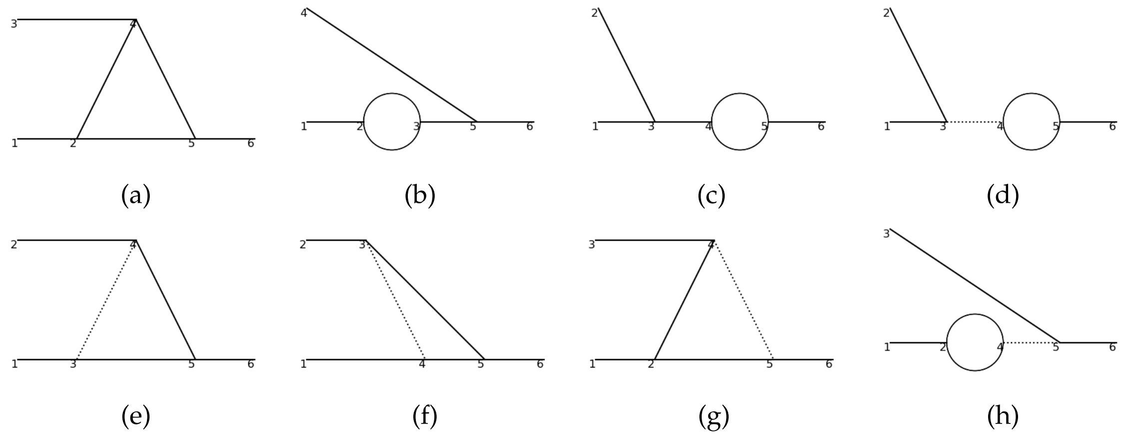


Figure 6: Diagrams of third order, contributing to the 3 gluon vertex.

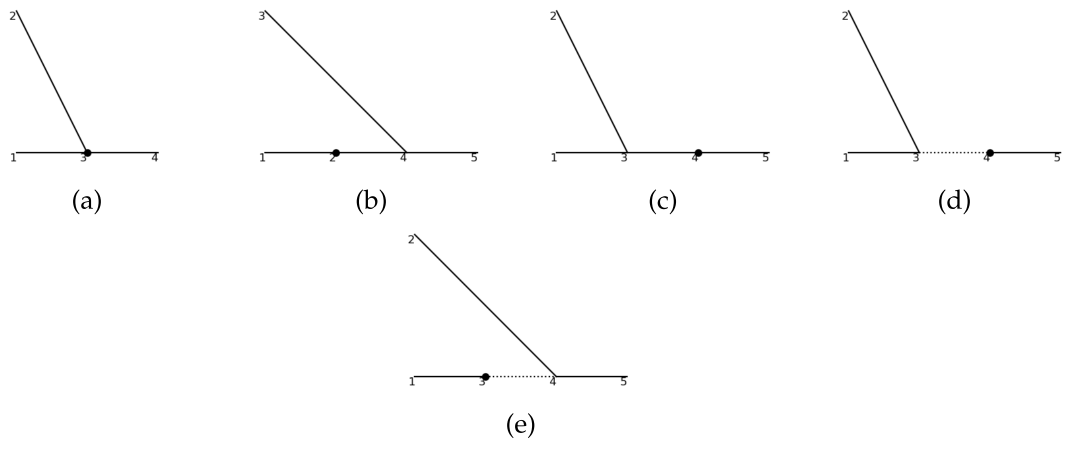


Figure 7: Counterterms to the diagrams of third order in figure 6

In the figure 6 we can see the diagrams obtained for the process of 1 gluon going to 2 gluons, with the corresponding counterterms in figure 7.

Depending on the types of particles in the process, we can deduce the origin of the different diagrams. With the once having dotted lines, and thus coming from the second order term canonical diagrams $H_{[\partial AA]^2}$, combined with a first order term canonical diagram H_{A^3}. While the diagrams without dotted lines come solely from the first order term canonical diagrams H_{A^3}

Referring to the problems mentioned in the section 4.3, the diagrams with dotted lines are instantaneous interactions, but the program considers them as a new type of particle, there are some artifacts that arise from this approach. For instance, the diagrams 6e and 6f are really the same diagram, due to instant process being instantaneous. For any other particle, these 2 diagrams would be different, due to the importance of the order in the interactions, so the program is kept to consider them as different, to not lose generality.

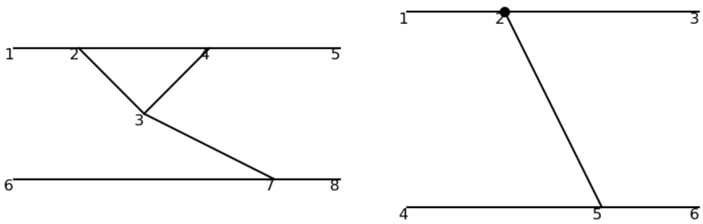
Other artifact of the program occur for the diagrams 7d and 7e, which are counterterms added to cancel the divergences in the canonical diagrams. These counterterms are already considered in the canonical diagrams, and thus not needed to be added again. But due to the way the program is implemented, it adds counterterms to all loop divergences in the diagrams.

Comparing with the 3rd order contribution diagrams in [5], the same diagrams for the 3 gluon vertex are obtained, proving the validity of the program to reproduce the same results.

5.2 Order 4

In a non abelian theory such as QCD, the gluons interact with themselves, thus making bound states formed by exclusively gluons, glueballs, possible.

Considering one of such glueball states, the process of 2 gluons going to 2 gluons, the program produces about 100 diagrams, with the corresponding counterterms. All the diagrams are not shown here, since they are too many to be included in this document. But let's discuss some of the most interesting diagrams.



(a) (b)

Figure 8: Diagrams of fourth order, contributing to the 2 gluon vertex.

Starting at order 4, the distinct non-abelian nature of the QCD is present, since the gluons interact with themselves, creating diagrams not possible in abelian theories, such as QED. One of such processes is shown in the figure 8, where 2 gluons interact with each other.

5.3 Higher orders

At higher orders, the diagrams become more complex, and the number of diagrams scale exponentially. Is here where the power of the program is shown.

Order	Unique diagrams	Computational time (s)
2	16	0.150
3	76	0.198
4	612	1.166
5	5871	10.935
6	65000	157.121

Table 1: Number of diagrams obtained by the program at each order, and the Computational time taken to calculate the diagrams.

6 Conclusions and future work

In this thesis, we have presented a program that implements the RGPEP to obtain the diagrams of the gluons self interactions, and the perturbative expansion of the Hamiltonian. The program is able to obtain the diagrams of higher order, starting from the canonical diagrams, and using the order by order procedure. The program is modular, and can be adapted to other theories, by changing the canonical diagrams and the types of particles considered. The program has been tested with the gluons self interactions, and has been able to obtain the diagrams of higher order, with the corresponding counterterms to cancel the divergences produced in the process. The program has been able to reproduce the diagrams obtained in the literature, proving its validity to obtain the diagrams of higher order in the perturbative expansion of the Hamiltonian.

The program is able to obtain the diagrams of higher order, but there are still some issues that need to be addressed, such as the artifacts produced by the instantaneous interactions, and the time taken to calculate the diagrams at higher orders.

The program is still in development, and there are many improvements that can be made to optimize the process of obtaining the diagrams. Other main issue to address is the counting of the diagrams, as the program is able to obtain the diagrams, but it does not output the symmetry factor correctly, due to the problems with the factors associated to the canonical diagrams, and the counterterms, as well as the method used to find the diagrams that prioritize ensuring that no diagrams are lost, rather than ensuring that the symmetry factor is correct.

As future work, we plan to fix the issues mentioned above, and to implement and test the program with other theories, and more types of particles, particularly considering the quarks and antiquarks, in the general case of QCD.

7 Acknowledgements

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