第十二讲：自适应有限元方法

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May 24, 2013

Outline

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The convergence and regularity

If the solution of the elliptic problem $Lu = f$ has the regularity $u \in H^2(\Omega)$, the finite element solution u_h has the optimal convergence

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}$$

For the domain with reentrant corners, the solution is no longer in $H^2(\Omega)$. So the classical finite element method fails to provide satisfactory result.

Harmonic functions in sector

We consider the harmonic functions in the sector $S_\omega = \{(r,\theta) : 0 < r < \infty, 0 \le \theta \le \omega\}$, where $0 < \omega < 2\pi$. We look for the solution of the form $u = r^\alpha \mu(\theta)$ for the Laplace equation $-\Delta u = 0$ in S_ω with boundary condtion $u = 0$ on $\Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{(r,\theta) : r > 0, \theta = 0\}, \Gamma_2 = \{(r,\theta) : r > 0, \theta = \omega\}$$

Let $u = r^\alpha \mu(\theta)$. In polar coordinates

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We have

$$\Delta = \alpha(\alpha-1) r^{\alpha-2}\mu(\theta) + \alpha r^{\alpha-2}\mu(\theta) + r^{\alpha-2}\mu''(\theta) = 0$$

Harmonic functions in sector(continued)

So we get

$$\mu''(\theta) + \alpha^2 \mu(\theta) = 0.$$

Therefore

$$\mu(\theta) = A\sin(\alpha\theta) + B\cos(\alpha\theta)$$

The boundary condition $\mu(0) = \mu(\omega) = 0$ yields that $\alpha = k\pi/\omega$ and $\mu(\theta) = A\sin(\frac{k\pi}{\omega}\theta)$. So boundary value problem $\Delta u = 0$ in S_ω, $u = 0$ on $\Gamma_1 \cup \Gamma_2$ has a solution

$$u = r^\alpha \sin(\alpha\theta), \alpha = \pi/\omega$$

Lemma u does not lie in $H^2(S_\omega \cap B_R)$ for any $R > 0$ if $\pi < \omega < 2\pi$

Harmonic functions in sector(continued)

$$\begin{aligned}\int_\Omega \left|\frac{\partial^2 u}{\partial r^2}\right|^2 dx &= \int_0^R \int_0^\omega |\alpha(\alpha-1)r^{\alpha-2}\sin(\alpha\theta)| r dr d\theta \\ &= \alpha^2(\alpha-1)^2 \int_0^\omega |\sin(\alpha\theta)|^2 d\theta \int_0^R r^{2\alpha-3} dr = cr^{2(\alpha-1)}|_0^R.\end{aligned}$$

Example Laplace equation on a L-shape domain with Dirichlet boundary condition so chosen that the true solution is $u = r^{2/3}\sin(2\theta/3)$ in polar coordinates.

Question What's the regularity of this u?

Outline

The domain and partition

Some configurations:

- Domain: Let Ω be a polyhedral domain;
- Mesh: $\mathcal{M}_h$ is a regular partition of Ω;
- FEM space: V_h be the standard H^1-conforming linear finite element space, $V_h^0 = V_h \cap H_0^1(\Omega)$;
- Geometry: The set of all interior sides of the mesh M_h is denoted as $\mathcal{B}_h$, for any $K \in \mathcal{M}_h$, let h_k be the diameter of K. For any $e \in \mathcal{B}_h$ with $e = K_1 \cup K_2$, let $\Omega_e = K_1 \cup K_2$ and let h_e be the diameter of e .

The Clément interpolation operator

The Classical Lagrange interpolation:

$$I_K v = \sum_{i=1}^{3} v(a_i)\phi_i.$$

Which needs $v \in C^0(\bar{K})$. The interpolation has error bound:

$$|u - I_K u|_{H^m(K)} \leq Ch^{2-m}|u|_{H^2(K)}, 0 \leq m \leq 2$$

where $C = const > 0$ independent of K and v.
Now we want to construct an interpolation operator for function in $L^1(\Omega)$, which has the same error bound with the classical Lagrange interpolant.

The Clément interpolation operator(continued)

Let $\{x_j\}_{j=1}^{J}$ be the set of nodes of the mesh $\mathcal{M}_h$, and $\{\phi_j\}_{j=1}^{J}$ be the set of nodal basis functions. For any x_j, define $S_j = \mathrm{supp}\phi_\mathrm{j}$, the star surround x_j, namely

$$S_j = \cup\{K \in \mathcal{M}_h, \mathrm{supp}\phi_\mathrm{j} \cap \mathrm{K} \neq \varnothing\}.$$

S_j has the following properties(Why ?):

- $\sharp\{K : K \in S_j\} \leq M$;
- $\sharp\{S_j : K \in S_j\} \leq M'$.

Then the macro-elements S_j can only assume a finite number of different configurations.

The Clément interpolation operator(continued)

Denote by $\Lambda = \{\hat{S}\}$ the set of reference configurations. The number of reference configurations $\sharp\Lambda$ depends only on the minimum angle of $\mathcal{M}_h$. For any S_j, let $\hat{S}_j$ be the corresponding reference configureation in Λ, there exists a continuous and invertible mapping F_j from $\hat{S}_j$ onto S_j such that

$$F_j|_{\hat{K}_i} \text{ is an affine mapping from } \hat{K}_i \text{ onto } K_i$$

for all $\hat{K}_i$ contained in $\hat{S}_j$.

The Clément interpolation operator(continued)

Define $\hat{R}_j : L^1(\hat{S}_j) \to P_1(\hat{S}_j)$ the L^2 projection operator by

$$\hat{R}_j\hat{\psi} \in P_1(\hat{S}_j) : \int_{\hat{S}_j} (\hat{R}_j\hat{\psi})\hat{v}_h dx = \int_{\hat{S}_j} \hat{\psi}\hat{v}_h dx$$

for any $\hat{\psi} \in L^1(\hat{S}_j)$. For any $\psi \in L^1(\Omega)$, denote by $\hat{\psi}_j = \psi \circ F_j$. Let $\{x_j\}_{j=1}^{J}$ be the set of interior nodes. The Clément interpolation operators Π_h and Π_h^0 are then defined by

$$\Pi_h : L^1(\Omega) \to V_h, \Pi_h\psi = \sum_{j=1}^{\bar{J}} (\hat{R}_j\hat{\psi}_j)(F_j^{-1}(x_j))\phi_j$$

$$\Pi_h^0 : L^1(\Omega) \to V_h^0, \Pi_h\psi = \sum_{j=1}^{J} (\hat{R}_j\hat{\psi}_j)(F_j^{-1}(x_j))\phi_j$$

The interpolation error

Theorem There exists constant C depending only on the minimum angle of $\mathcal{M}_h$ such that for any $\psi \in H_0^1(\Omega)$

$$\begin{aligned} \|\psi - \Pi_h^0 \psi\|_{L^2(K)} &\le Ch_K \|\nabla \psi\|_{L^2(\tilde{K})} \quad \forall K \in \mathcal{M}_h \qquad (1) \\ \|\psi - \Pi_h^0 \psi\|_{L^2(e)} &\le Ch_e^{1/2} \|\nabla \psi\|_{L^2(\tilde{e})} \quad \forall e \in \mathcal{B}_h \qquad (2) \\ \|\nabla \Pi_h \psi\|_{L^2(K)} &\le C \|\nabla \psi\|_{L^2(\tilde{K})} \quad \forall K \in \mathcal{M}_h \qquad (3) \end{aligned}$$

where $\tilde{K}$ is the unior of all elements in $\mathcal{M}_h$ having nonempty intersection with K, and $\tilde{e} = \tilde{K}_1 \cup \tilde{K}_2$ with $e = K_1 \cap K_2$.

The interpolation error(continued)

Proof The proof is divided into three steps.

1. (2) and (3) are direct consequences of (1).
2. Prove (1) for K is a interior element.
3. Prove (1) for K has a node on the boundary $\partial\Omega$

We will prove (1) for K is a interiori element and (3), (2).

The interpolation error(continued)

Proof of (3)
Let $\psi_K = \frac{1}{|K|}\int_K \psi dx$ be the average of ψ on K, then it follows frome the local inverse estimate and (1) that

$$
\begin{aligned}
\|\nabla \Pi_h^0 \psi\|_{L^2(K)} &= \|\nabla(\Pi_0\psi - \psi_K)\|_{L^2(K)} \le C h_K^{-1} \|\Pi_h^0 \psi - \psi_K\|_{L^2(K)} \\
&\le C h_K^{-1} (\|\Pi_h^0 \psi - \psi)\|_{L^2(K)} + \|\psi - \psi_K\|_{L^2(K)}) \\
&\le C \|\nabla \psi\|_{L^2(\tilde{K})}.
\end{aligned}
$$

We have used the Poincare-Friedichs inequality for the second term in the second line of above equations.

The interpolation error(continued)

Proof of (2)

By the scaled trace inequality, for $e \subset \partial K$ for some $K \in \mathcal{M}_h$,

$$
\begin{aligned}
\|\psi - \Pi_h^0 \psi\|_{L^2(e)} &\leq C(h_e^{-1/2}\|\psi - \Pi_h^0 \Psi\|_{L^2(K)} + h_e^{1/2}\|\nabla(\psi - \Pi_h^0 \psi)\|_{L^2(K)}) \\
&\leq Ch_e^{1/2}\|\nabla \psi\|_{L^2(\tilde{K})} \leq Ch^{1/2}\|\nabla \psi\|_{L^2(\tilde{e})}.
\end{aligned}
$$

The inequality in red colour is the so called scaled trace inequality, which can be proved by scaling argument(Try it).

The interpolation error(continued)

Proof of (1)

By Deny-Lions theorem and inverse estimate, for any $\hat{\psi} \in H^1(\hat{S}_j)$, we have

$$\|\hat{\psi} - \hat{R}_j\hat{\psi}\|_{L^2(\hat{S}_j)} \leq \inf_{\hat{p}\in P_1(\hat{S}_j)} \|\hat{\psi} - p\|_{L^2(\hat{S}_j)} \leq C\|\nabla\hat{\psi}\|_{L^2(\hat{S}_j)}$$

$$\begin{aligned} \|\nabla\hat{R}_j\hat{\psi}\|_{L^2(\hat{S}_j)} = \|\nabla\hat{R}_j(\hat{\psi} - \hat{\psi}_{\hat{S}_j})\|_{L^2(\hat{S}_j)} &\leq C\|\hat{R}_j(\hat{\psi} - \hat{\psi}_{\hat{S}_j})\|_{L^2(\hat{S}_j)} \\ &\leq C\|\hat{\psi} - \hat{\psi}_{\hat{S}_j}\|_{L^2(\hat{S}_j)} \leq C\|\nabla\hat{\psi}\|_{L^2(\hat{S}_j)} \end{aligned}$$

Denote by h_j the diameter of S_j. Since $\sum_{j=1}^{J}\phi_j = 1$, we have

The interpolation error(continued)

$$\begin{aligned}
\|\psi - \Pi_h\psi\|_{L^2(K)} &= \|\sum_{x_j\in K}(\psi - (\hat{R}_j\hat{\psi}_j)(F_j^{-1}(x_j))\phi_j\|_{L^2(K)} \\
&\leq C\sum_{x_j\in K}\|\psi - (\hat{R}_j\hat{\psi}_j)(F_j^{-1}(x_j))\|_{L^2(S_j)} \\
&\leq C\sum_{x_j\in K} h_j^{d/2}\|\hat{\psi}_j - (\hat{R}_j\hat{\psi}_j)(F_j^{-1}(x_j))\|_{L^2(\hat{S}_j)} \\
&\leq C\sum_{x_j\in K} h_j^{d/2}\big(\|\hat{\psi}_j - \hat{R}_j\hat{\psi}_j\|_{L^2(\hat{S}_j)} \\
&\quad +\|\hat{R}_j\hat{\psi}_j - (\hat{R}_j\hat{\psi}_j)(F_j^{-1}(x_j))\|_{L^2(\hat{S}_j)}\big) \\
&\leq C\sum_{x_j\in K} h_j^{d/2}\big(\|\hat{\psi}_j - \hat{R}_j\hat{\psi}_j\|_{L^2(\hat{S}_j)} + \|\nabla\hat{R}_j\hat{\psi}_j\|_{L^2(\hat{S}_j)}\big) \\
&\leq C\sum_{x_j\in K} h_j^{d/2}\|\nabla\hat{\psi}_j\|_{L^2(\hat{S}_j)} \leq Ch_K\|\nabla\phi\|_{L^2(\tilde{K})}
\end{aligned}$$

Outline

The variational problem and FEM problem

We consider the variational problem to find $u \in H_0^1(\Omega)$ such that

$$(a(x)\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega),$$

where $a(x)$ is assumed to be a piecewise constant function, $f \in L^2(\Omega)$, and Ω is not necessarily convex. Suppose that $a(x)$ is constant on each $K \in \mathcal{M}_h$. Let $u_h \in V_h^0$ be the finite element solution of the discrete problem

$$(a(x)\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_0^h. \tag{4}$$

The error indicator

For any $e \in \mathcal{B}_h$ with $e = K_1 \cap K_2$, we define the jump residual for u_h by

$$J_e = ([[a(x)\nabla u_h]] \cdot n)|_e = a(x)\nabla u_h|_{K_1} \cdot n_1 + a(x)\nabla u_h|_{K_2} \cdot n_2,$$

where n_i is the unit outer normal of ∂K_i restricted to e. Define $J_e = 0$ for any side $e \subset \partial\Omega$. For any $K \in \mathcal{M}_h$, define the indicator η_K by

$$\eta_K^2 := h_K^2 \|f\|_{L^2(K)}^2 + h_K \sum_{e \subset \partial K} \|J_e\|_{L^2(e)}^2$$

Upper bound

For any domain $G \subset \Omega$, let $|||\cdot|||_G = \|a^{1/2}\nabla \cdot\|_{L^2(G)}$. Note that $|||\cdot|||_\Omega$ is the energy norm in $H_0^1(\Omega)$.

Theorem(Upper bound) There exists a constant $C_1 > 0$ which depends only on the minimum angle of the mesh $\mathcal{M}_h$ and the minimum value of $a(x)$ such that

$$|||u - u_h|||_\Omega \leq C_1 \Big(\sum_{K \in \mathcal{M}_h} \eta_K^2 \Big)^{1/2}$$

Upper bound(continued)

Proof:

$$
\begin{aligned}
(a\nabla u, \nabla\phi) &= (f, \phi) \quad \forall\phi \in H_0^1(\Omega) \\
(a\nabla u_h, \nabla\phi_h) &= (f, \phi_h) \quad \forall\phi_h \in V_h^0.
\end{aligned}
$$

then

$$
\begin{aligned}
(a\nabla(u-u_h), \nabla\phi) &= (f,\phi) - (a\nabla u_h, \nabla\phi) \\
&= (f, \phi - \Pi_h^0\phi) - (a\nabla u_h, \nabla(\phi - \Pi_h^0\phi)) \\
&= (f.\phi - \Pi_h^0\phi) - \sum_{K\in\mathcal{M}_h} \int_{\partial K} a\nabla u_h \cdot n(\phi - \Pi_h^0\phi)ds \\
&= \sum_{K\in\mathcal{M}_h} \int_K f(\phi - \Pi_h^0\phi)dx - \sum_{e\in\mathcal{B}_h} J_e(\phi - \Pi_h^0\phi)ds
\end{aligned}
$$

Upper bound(continued)

Then we have

$$\begin{aligned}(a\nabla(u-u_h),\nabla\phi) &\le C\Big(\sum_{K\in\mathcal{M}_h} h_K^2\|f\|_{L^2(K)}^2\|\Big)^{1/2}\|\nabla\phi\|_{L^2(\Omega)} \\ &\quad + C\Big(\sum_{e\in\mathcal{B}_h} h_e\|J_e\|_{L^2(e)}^2\Big)^{1/2}\|\nabla\phi\|_{L^2(\Omega)} \\ &\le C\Big(\sum_{K\in\mathcal{M}_h}\eta_K^2\Big)^{1/2}\|\nabla\phi\|_{L^2(\Omega)}.\end{aligned}$$

The theorem follows by taking $\phi = u - u_h \in H_0^1(\Omega)$.

Local lower bound

Theorem(Local lower bound) There exists a constant $C_2 > 0$ which depends only on the minimum angle of the mesh and the maximum value of $a(x)$ such that any $K \in \mathcal{M}_h$

$$\eta_K^2 \leq C_2 |||u - u_h|||_{K^*}^2 + C_2 \sum_{K \subset K^*} h_K^2 \|f - f_K\|_{L(K)}^2.$$

where $f_K = \frac{1}{|K|} \int_K f dx$ and K^* is the union of all elements sharing at least on common side with K.

Local lower bound(continued)

Proof: The proof is devided into two steps

1. $h_k^2\|f\|_{L^2(K)}^2 \le C(\||u-u_h\||_\Omega^2 + h_K^2\|f-f_K\|_{L^2(K)}^2)$;
2. $h_e\|J_e\|_{L^2(e)}^2 \le C(\sum_{K\subset\Omega_e}\|h_K f\|_{L^2(K)}^2 + \||u-u_h\||_{\Omega_e}^2)$.

The key tools in the proof is the following bubble function:

- $\phi_K = (d+1)^{d+1}\lambda_1\lambda_2...\lambda_{d+1}$;
- $\phi_e = d^d\lambda_1...\lambda_d$.

Local lower bound(continued)

$$(a\nabla(u-u_h),\nabla\phi)=\sum_{K\in\mathcal{M}_h}\int_K f\phi dx-\sum_{e\in\mathcal{B}_h}\int_e J_e\phi dx.$$

1 , Choose constant α_K such that $\phi=\alpha_K\phi_K$ satisfies

$$\int_K f_K\phi dx = h_K^2\|f_K\|^2_{(L^2(K))}.$$

It is clear that

$$|\alpha_K|=\frac{h_K^2|f_K||K|}{\int_K\phi_K dx}\le Ch_K^{1-d}\|h_K f_K\|_{L^2(K)}$$

and thus

$$h_K^{-1}\|\phi\|_{L^2(K)},\|\nabla\phi\|_{L^2(K)}\le C|\alpha|h_K^{-1}|K|^{1/2}\le C\|h_K f_K\|_{L^2(K)}$$

Local lower bound(continued)

Now

$$\|h_K f\|_{L^2(K)}^2 \leq \|h_K f_K\|_{L^2(K)}^2 + \|h_K(f - f_K)\|_{L^2(K)}^2$$

and

$$\begin{aligned}
\|h_K f_K\|_{L^2(K)}^2 &= \int_K f\phi dx = \int_K (f_K - f)\phi dx + (a\nabla(u - u_h), \nabla\phi) \\
&\leq C\|h_K(f - f_K)\|_{L^2(K)}\|h_K^{-1}\phi\|_{L^2(K)} + C|||u - u_h|||_K\|\phi\|_{L^2(K)} \\
&\leq C\|h_K f_K\|_{L^2(K)}(|||u - u_h|||_K^2 + \|h_K(f - f_K)\|_{L^2(K)}^2)^{1/2}
\end{aligned}$$

Therefore,

$$\|h_K f\|_{L^2(K)}^2 \leq C(|||u - u_h|||_K^2 + \|h_K(f - f_K)\|_{L^2(K)}^2)$$

Local lower bound(continued)

2 , Denote by $\psi = \beta_e \psi_e$ the function satisfies

$$\int_e J_e \psi = h_K \|J_e\|_{L^2(e)}^2.$$

It is easy to check that

$$|\beta_e| \le C h_K |J_e| \le C h_K^{1-d/2} h_K^{1/2} \|J_e\|_{L^2(e)}$$

and thus

$$h_K^{-1} \|\psi\|_{L^2(\Omega_e)}, \|\nabla \psi\|_{L^2(\Omega_e)} \le |\beta_e| h_K^{-1} |\Omega_e|^{1/2} \le h_K^{1/2} \|J_e\|_{L^2(\Omega_e)}$$

Local lower bound(continued)

Note that $\psi \in H_0^1(\Omega)$, we have

$$\begin{aligned} h_K\|J_e\|^2_{L^2(e)} &= \int_e J_e\psi ds = \int_{\Omega_e} f\psi - \int_{\Omega_e} a\nabla(u-u_h)\nabla\psi dx \\ &\le Ch_K^{1/2}\|J_e\|_{L^2(e)}(\sum_{K\subset\Omega_e}\|h_K f\|^2_{L^2(K)} + |||u-u_h|||^2_{\Omega_e})^{1/2} \end{aligned}$$

Thie completes the proof with the help of estimate for $\|h_K f\|_{L^2(K)}$.

Local lower bound(continued)

The lower bound in the previous theorem implies that up to a high order quantity $(\sum_{K\subset K^*} h_K^2\|f-f_K\|^2_{L(K)})^{1/2}$, the local energy error $|||u-u_h|||_{K^*}$ is bounded from below by the error indicator η_K.

- Reliability: Lower bound of the a posteriori error;
- Efficiency: Upper bound of the a posteriori error.

Outline

Adaptive algorithm

Based on the local error indicators, the usual adaptive algorithm solving the variational problem (4) can be described as loops of the form:

$$\textbf{Solve} \rightarrow \textbf{Estimate} \rightarrow \textbf{Mark} \rightarrow \textbf{Refine}$$

The convergence property, which guarantees the iterative loop terminates in finite number of iterations starting from any given initial mesh, depends on the proper design of marking strategies.

Adaptive algorithm

The marking strategies:

1. The error equidistribution strategy: Give $\theta > 1$ and a tolerence TOL, mark all elements K such that
$$\eta_K \geq \theta \frac{TOL}{\sqrt{M}},$$
where M is the number of elements in $\mathcal{M}_h$.
2. The maximum strategy: Given $\theta \in (0,1)$, mark all elements K such that
$$\eta_K \geq \theta \max_{K' \in \mathcal{M}_h} \eta_{K'}$$
3. The Dorfler strategy: Given $\theta \in (0,1]$, mark elements in a subset $\hat{\mathcal{M}}_h$ such that
$$\eta_{\hat{\mathcal{M}}_h} \geq \theta \eta_{\mathcal{M}_h}$$

Question: How to implement the Dorfler strategy?

Adaptive algorithm

Given a triangulation $\mathcal{M}_H$ and a set of marked elements $\hat{\mathcal{M}}_H \subset \mathcal{M}_H$, the refinement of $\mathcal{M}_H$ usually consists of two steps: refining the marked elements and removing the hanging nodes. We make the following assumption on the first step:

Any marked simplex is subdivided into several subsimplices such that the measure of each subsimplex $\leq \frac{1}{m} \times$ *the measure of its father simplex.*

Here $m > 1$ is a fixed number. For example, in the case of one time bisection, $m = 2$.

Convergence analysis

Lemma Let $\mathcal{M}_h$ be a refinement of $\mathcal{M}_H$ such that $V_H \subset V_h$. Then the following ralation holds

$$|||u - u_h|||_\Omega^2 = |||u - u_H|||_\Omega^2 - |||u_h - u_H|||_\Omega^2$$

The proof is straightforward by the Galerkin orthogonality since $u_h - u_H \in V_h^0$:

$$\begin{aligned}
& (a\nabla(u - u_h), \nabla(u - u_h)) \\
= \; & (a\nabla(u - u_h), \nabla(u - u_H)) + (a\nabla(u - u_h), \nabla(u_H - u_h)) \\
= \; & (a\nabla(u - u_h), \nabla(u - u_H)) \\
= \; & (a\nabla(u - u_H), \nabla(u - u_H)) + (a\nabla(u_H - u_h), \nabla(u - u_H)) \\
= \; & |||u - u_H|||_\Omega^2 + (a\nabla(u_H - u_h), \nabla(u_h - u_H)) \\
& + (a\nabla(u_H - u_h), \nabla(u - u_h)) \\
= \; & |||u - u_H|||_\Omega^2 - (a\nabla(u_h - u_H), \nabla(u_h - u_H)).
\end{aligned}$$

Convergence analysis(continued)

Let

$$\tilde{\eta}_K^2 := \tilde{h}_K^2 \|f\|_{L(K)}^2 + \tilde{h}_K \sum_{e \subset \partial K} \|J_e\|_{L^2(e)}^2, \text{ where } \tilde{h}_K := |K|^{1/d}$$

It is clear that there exists positive constants c_1 and c_2 such that

$$c_2 \eta_K \leq \tilde{\eta}_K \leq c_1 \eta_K$$

Lemma Let $\hat{\mathcal{M}}_H \subset \mathcal{M}_H$ be the set of elements marked for refinement and let $\mathcal{M}_h$ be a refinement of $\mathcal{M}_H$ satisfying the assumption. Then there exists a constant C_3 depending only on the minimum angle of the meshes and the maximum value of $a(x)$ such that, for any $\delta > 0$

$$\tilde{\eta}_{\mathcal{M}_h}^2 \leq (1+\delta)(\tilde{\eta}_{\mathcal{M}_H}^2 - (1 - \frac{1}{m^{1/d}})\tilde{\eta}_{\hat{\mathcal{M}}_H}^2) + (1 + \frac{1}{\delta}) C_3 |||u_h - u_H|||_\Omega.$$

Convergence analysis(continued)

Proof: From the Young inequality with parameter δ

$$
\begin{aligned}
\tilde{\eta}^2_{\mathcal{M}_h} &= \sum_{K\in\mathcal{M}_h} \left(\tilde{h}_K^2\|f\|^2_{L^2(K)} + \tilde{h}_K \sum_{e\subset\partial K\cap\Omega} \|[[a\nabla(u_H + u_h - u_H)]]\cdot n)\|^2_{L(e)}\right) \\
&\leq \sum_{K\in\mathcal{M}_h} \left(\tilde{h}_K^2\|f\|^2_{L^2(K)} + (1+\delta)\tilde{h}_K \sum_{e\subset\partial K\cap\Omega} \|[[a\nabla(u_H)]]\cdot n)\|^2_{L(e)}\right) \\
&\quad +(1+\frac{1}{\delta}) \sum_{K\in\mathcal{M}_h} \tilde{h}_K \sum_{e\subset\partial K\cap\Omega} \|[[a\nabla(u_h - u_H)]]\cdot n)\|^2_{L(e)} \\
&:= I + II
\end{aligned}
$$

Convergence analysis(continued)

Note that $([[a\nabla u_H]]\cdot n)|_e = 0$ for any e in the interior of some element $K' \in \mathcal{M}_H$ and that $\tilde{h}_K = |K|^{1/d} \le \frac{1}{m^{1/d}}\tilde{H}_K$, for any $K \subset K' \in \hat{\mathcal{M}}_H$. We have

$$\begin{aligned}
I &\le (1+\delta)\sum_{K\subset K'\in\mathcal{M}_H\backslash\hat{\mathcal{M}}_H}(\tilde{h}_K^2\|f\|^2_{L^2(K)} + \tilde{h}_K\sum_{e\subset\partial K\cap\Omega}\|[[a\nabla(u_H)]]\cdot n)\|^2_{L(e)}) \\
&\quad +(1+\delta)\sum_{K\subset K'\in\hat{\mathcal{M}}_H}(\tilde{h}_K^2\|f\|^2_{L^2(K)} + \tilde{h}_K\sum_{e\subset\partial K\cap\Omega}\|[[a\nabla(u_H)]]\cdot n)\|^2_{L(e)}) \\
&\le (1+\delta)\sum_{K\subset K'\in\mathcal{M}_H\backslash\hat{\mathcal{M}}_H}(\tilde{H}_K^2\|f\|^2_{L^2(K)} + \tilde{H}_K\sum_{e\subset\partial K\cap\Omega}\|[[a\nabla(u_H)]]\cdot n)\|^2_{L(e)}) \\
&\quad +\frac{1+\delta}{m^{1/d}}\sum_{K\subset K'\in\hat{\mathcal{M}}_H}(\tilde{H}_K^2\|f\|^2_{L^2(K)} + \tilde{H}_K\sum_{e\subset\partial K\cap\Omega}\|[[a\nabla(u_H)]]\cdot n)\|^2_{L(e)}) \\
&= (1+\delta)\tilde{\eta}^2_{\mathcal{M}_H\backslash\hat{\mathcal{M}}_H} + \frac{1+\delta}{m^{1/d}}\tilde{\eta}^2_{\hat{\mathcal{M}}_H} = (1+\delta)(\tilde{\eta}^2_{\mathcal{M}_H} - (1-m^{-1/d})\tilde{\eta}^2_{\hat{\mathcal{M}}_H}).
\end{aligned}$$

Convergence analysis(continued)

Next we estimate II. For any $e \in \mathcal{B}_h$, denote by K_1 and K_2 the two elements having common side e. We have

$$
\begin{aligned}
II &\le C(1+\frac{1}{\delta})\sum_{e\in\mathcal{B}_h} h_e \|([[a\nabla(u_h-u_H)]]\cdot n)|_e\|^2_{L^2(e)} \\
&= C(1+\frac{1}{\delta})\sum_{e\in\mathcal{B}_h} h_e \|a\nabla(u_h-u_H)|_{K_1}\cdot n_1 + a\nabla(u_h-u_H)|_{K_2}\cdot n_2\|^2_{L^2(e)} \\
&\le C(1+\frac{1}{\delta})\sum_{e\in\mathcal{B}_h} h_e (\|a\nabla(u_h-u_H)\|^2_{L^2(K_1)} + \|a\nabla(u_h-u_H)\|^2_{L^2(K_2)}) \\
&\le C(1+\frac{1}{\delta})\sum_{e\in\mathcal{B}_h} h_e \|a\nabla(u_h-u_H)\|^2_{L^2(K_1\cup K_2)} \le (1+\frac{1}{\delta})C_3 \||u_h-u_H\|^2_{\Omega}.
\end{aligned}
$$

Convergence analysis(continued

Theorem(Convergence) Let $\theta \in (0,1]$, and let $\{\mathcal{M}_k, u_k\}_{k\geq 0}$ be the sequence of meshes and discrete solutions produced by the adaptive finite element algorithm based on the Dorfler marking strategy and the assumption. Suppose the family of meshes $\{\mathcal{M}_k\}$ is shape regular. Then there exist constants $\gamma > 0$, $C_0 > 0$, and $0 < \alpha < 1$, depending only on the shape-regularity of $\{\mathcal{M}_k\}$, m, and the marking parameter θ, such that

$$(|||u - u_k|||_\Omega^2 + \gamma \eta_{\mathcal{M}_k}^2)^{1/2} \leq C_0 \alpha^k.$$

Convergence analysis(continued)

Proof: We first show that there exist constant $\gamma_0 > 0$ and $0 < \alpha < 1$ such that

$$|||u - u_{k+1}|||_\Omega^2 + \gamma_0 \tilde{\eta}_{\mathcal{M}_{k+1}}^2 \leq (|||u - u_k|||_\Omega^2 + \gamma_0 \tilde{\eta}_{\mathcal{M}_k}^2). \tag{5}$$

For convenience, we use the notation

$$e_k = |||u - u_k|||_\Omega, \tilde{\eta}_k = \tilde{\eta}_{\mathcal{M}_k}, \lambda = 1 - \frac{1}{m^{1/d}}.$$

then by previous two lemmas and the Dorfler strategy, we have

$$\tilde{\eta}_{k+1}^2 \leq (1+\delta)(1-\lambda\theta^2)\tilde{\eta}_k^2 + (1+\frac{1}{\delta})(e_k^2 - e_{k+1}^2)$$

Convergence analysis(continued)

By the upper bound of the a posteriori error, we have

$$e_k^2 \le \tilde{C}_1 \tilde{\eta}_k^2 \text{ where } \tilde{C}_1 = C_1/c_2;$$

Let $\beta = (1+\frac{1}{\delta})C_3$. Then we have that, for $0 < \xi < 1$,

$$\begin{aligned} e_{k+1}^2 + \frac{1}{\beta}\tilde{\eta}_{k+1}^2 &\le e_k^2 + \frac{1}{\beta}(1-\lambda\theta^2)\tilde{\eta}_k^2 \\ &\le \xi e_k^2 + ((1-\xi)\tilde{C}_1 + \frac{1}{\beta}(1+\delta)(1-\lambda\theta^2))\tilde{\eta}_k^2 \\ &= \xi\big(e_k^2 + \frac{1}{\beta}(\beta\xi^{-1}(1-\xi)\tilde{C}_1 + \xi^{-1}(1+\delta)(1-\lambda\theta^2))\tilde{\eta}_k^2\big) \end{aligned}$$

Choose $\delta > 0$ such that $(1+\delta)(1-\lambda\theta^2) < 1$ and choose ξ such that $\beta\xi^{-1}(1-\xi)\tilde{C}_1 + \xi^{-1}(1+\delta)(1-\lambda\theta^2) = 1$ which amounts to take

$$\xi = \frac{(1+\delta)(1-\lambda\theta^2) + \beta\tilde{C}_1}{1+\beta\tilde{C}_1} < 1$$

This implies (5) holds with

$$\gamma_0 = \frac{1}{\beta} \text{ and } \alpha^2 = \xi.$$

To finish the proof, we note that $\tilde{\eta}_k \geq c_2 \eta_k$, and let

$$\gamma = \gamma_0 c_2^2 \text{ and } C_0 = (|||u - u_0|||_\Omega^2 + \gamma \eta_0^2)^{1/2}$$

This completes the proof.

Optimality of adaptive finite element method

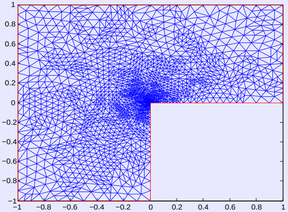
In two dimension cases, extensive numerical experiments strongly suggest that the adaptive finite element method based on a posteriori error estimates enjoys the remarkable property that the meshes and the associated numerical complexity are quasi-optimal in the following sense:

$$|||u - u_h||| \approx CN^{-1/2}(\text{ or } \log(|||u - u_h|||) \approx -\frac{1}{2}\log N + C)$$

where N is the number of elements of the underlying mesh. But the convergence give nothing about this optimality. And the optimality is still an open question.

An example

We consider the Laplace equation in a L-shape domain,



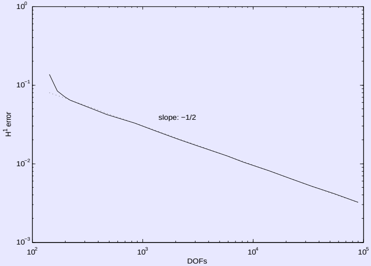
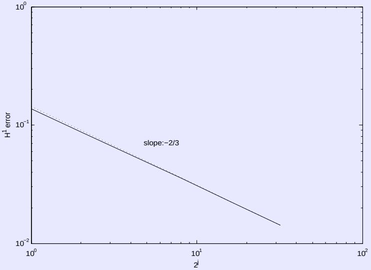


Figure: Left: mesh generated by the adaptive finite element method; Right: the error reduction with respect to the adaptive computation.

An example

If we solve this problem by standard finite element method, the error reduction with uniform refinement($2^j = h_0/h_j$) is shown in the following figure



It shows that the following error estimate is valid for linear finite element:

$$|||u - u_h||| \leq Ch^{2/3}$$

Some notes

- The Clément interpolation operator for non-smooth functions is introduced in 1975
- The adaptive finite element method based on a posteriori error estimates is originally provided by Babuska and Rheinboldt in 1978
- The upper bound of the a posteriori is proved by Babuska and Miller in 1987
- The lower bound of the a posteriori is proved by Verfurth in 1989
- The convergence of the adaptive finite element method is first considered by Dorfler in 1996, here I have adapt the convergence proof provided by Cascon et. al. in 2008.

Some references

- Regularity of PDE in nonsmooth domain:
  P. Grisvard, Elliptic Problems in nonsmooth domains, Pitman, London, 1985
- A posteriori error estimate:
  R. Verfurth, A review of a posteriori error estimation and adaptive mesh refinement technique, Teubner, 1996.
- Zhiming Chen and Haijun Wu, Selected topics in Finite Element Methods, Science press, 2010

Exercise

Let $\Omega = (0, 1)$. Derive a posteriori error estimate for the conforming linear finite element approximation to the two-point boundary value problem

$$-u'' = f \text{ in } \Omega,$$
$$u(0) = \alpha, u'(1) = \beta.$$