第十一讲：有限元方法简介

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Outline

1. Variational Formualation of Elliptic Problems

2. Finite Element Methods

3. Bounds for interpolation error

4. Convergence for second order elliptic problem

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1. Variational Formualation of Elliptic Problems
2. Finite Element Methods
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Variational Formulation of Elliptic Problems

We consider the following problem:

$$Lu = f \text{ in } \Omega, u = 0 \text{ on } \partial\Omega \tag{1}$$

Where Ω is a bounded open subset of $R^d (d = 1, 2, 3)$ and $u : \Omega \to \mathcal{R}$ is the unknown function. $f : \Omega \to R$ is a given function and L denotes the second order partial differential operator of the form:

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + c(x)u$$

Variational Formulation of Elliptic Problems

We shall assume the operator L is uniformly elliptic, that is, there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \text{ for a.e. } x \in \Omega \text{ and all } \xi \in R^d.$$

Remark: This is the key assumption for existence of a unique solution to the elliptic problem.

Some Sobolev space

Let Ω be an open set in R^d. We define $C_0^\infty(\Omega)$ to be the linear space of infinitely differential functions with compact support in Ω.

- $$L^1_{loc}(\Omega) = \{f : f \in L^1(K)\ \forall \text{ compact set } K \subset\subset \Omega\}$$
- $$L^p(\Omega) = \{u \in L^1_{loc}(\Omega) : \|u\|_{L^p(\Omega)} \leq \infty\}$$

  where $\|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{1/p}$.

Some Sobolev space

Weak derivatives

Assume $f \in L^1_{loc}(\Omega), 1 \le i \le d$, we say $g_i \in L^1_{loc}(\Omega)$ is the weak partial derivative of f with respect to x_i in Ω if

$$\int_\Omega f\frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i \phi dx,\ \forall \phi \in C_0^\infty(\Omega)$$

We write

$$\partial_{x_i} f = \frac{\partial f}{\partial x_i} = g_i, i = 1, ..., d, \nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_d})$$

For a multi-index $\alpha = (\alpha_1, ..., \alpha_d) \in N^d$ with length $|\alpha| = \alpha_1 + ... + \alpha_d$, $\partial^\alpha f \in L^1_{loc}(\Omega)$ is defined by

$$\int_\Omega \partial^\alpha f \phi dx = (-1)^{|\alpha|}\int_\Omega f \partial^\alpha \phi dx,$$

where $\partial^\alpha = \partial^{\alpha_1}_{x_1} ... \partial^{\alpha_d}_{x_d}$

Weak derivatives

Example

Let $d = 1$ and $\Omega = (-1, 1)$, $f(x) = 1 - |x|$. The weak derivative of f is

$$g = \begin{cases} 1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases}$$

But the weak derivative of g does not exist.

Some Sobolev Space

Sobolev Space: For a nonnegative integer k and a real $p \geq 1$, we define

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha p \in L^p(\Omega)^d \text{ for all } |\alpha| \leq k\}$$

$W^{k,p}$ is a Banach space with the norm:

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} (\sum_{|\alpha|\leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)})^{1/p} & 1 \leq p < \infty \\ \max_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)} & p = \infty. \end{cases}$$

$W_0^{k,p}(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. When p=2, we denote

$$H^k(\Omega) = W^{k,2}(\Omega), H_0^k(\Omega) = W_0^{k,2}(\Omega)$$

The space $H^k(\Omega)$ is a Hilbert space when equipped with the inner product

$$(u, v) = \sum_{|\alpha|\leq k} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Example

- Let $\Omega = (0, 1)$, and consider the function $u = x^{\alpha}$. We can easily verify that $u \in L^2(\Omega)$ if $\alpha > -1/2$, $u \in H^1(\Omega)$ if $\alpha > 1/2$, and $u \in H^k(\Omega)$ if $\alpha > k - 1/2$.
- Let $\Omega = \{x \in R^2 : |x| < 1/2\}$ and consider the function $f(x) = ln|ln|x||$. Then $f \in W^{1,p}(\Omega)$ for $p \leq 2$ but $f \notin L^{\infty}(\Omega)$.

This example shows that functions in $H^1(\Omega)$ are not necessarily continuous nor bounded.

Lipschitz Domain

A domain is referred to an open and connected set.

Lipschitz domain

We say that a domain Ω has a Lipschitz boundary $\partial\Omega$ if for each point $x \in \partial\Omega$ there exist $r > 0$ and a Lipschitz mapping $\phi : R^{d-1} \to R$ such that —upon rotating and relabeling the coordinate axes if necessarily—we have

$$\Omega \cap Q(x,r) = \{y : \phi(y_1, y_2, ..., y_{d-1}) < y_d\} \cap Q(x,r),$$

Where $Q(x,r) = \{y : |y_i - x_i| < r, i = 1, 2, ..., d|\}$ We call Ω a Lipschitz domain if it has a Lipschitz boundary.

Sobolev Imbedding Theorem

Let $\Omega \subset R^d$ be bounded Lipschitz domain and $1 \leq p \leq \infty$. Then

1. If $0 \leq k \leq d/p$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $L^q(\Omega)$ with $q = dp/(d-kp)$ and compactly imbedded in $L^{q'}(\Omega)$ for any $1 \leq q' < q$.
2. If $k = d/p$, the space $W^{k,p}(\Omega)$ is compactly imbedded in L^q for any $1 \leq q < \infty$.
3. If $0 \leq m < k - d/p < m+1$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $C^{m,\alpha}(\bar{\Omega})$ for $\alpha = k - d/p - m$, and compactly imbedded in $C^{m,\beta}(\bar{\Omega})$ for $0 \leq \beta < \alpha$.

Example: $H^1(\Omega)$ is continuously imbedded in $C^{0,1/2}(\bar{\Omega})$ for $d = 1$, in $L^q(\Omega)$, $1 \leq q < \infty$ for d=2, and in L^6 for d=3.

Poincare-Friedrichs Inequality

Let $\Omega \subset R^d$ be a bounded Lipschitz domain and $1 \leq p \leq \infty$. Then

$$\|u\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)} \forall u \in W_0^{1,p}(\Omega)$$
$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)} \forall u \in W^{1,p}(\Omega)$$

Where $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x) dx$.

The first inequality implies the equivalence between norm and semi-norm of space $W_0^{1,p}$.

Fractional Sobolev space

For two real numbers s, p with $p \geq 1$ and $s = k + \sigma$ where $\sigma \in [0, 1)$, we define $W^{s,p}(\Omega)$ when $p < \infty$ as the set of all functions $u \in W^{k,p}$ such that

$$\int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{d+\sigma p}} dxdy < +\infty, \forall |\alpha| = k|.$$

Likewise, when $p = \infty$, $W^{k,\infty}(\Omega)$ is the set of all functions $u \in W^{k,\infty}$ such that

$$\max_{|\alpha|=k} ess \sup_{x,y\in\Omega, x\neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x - y|^\sigma} < +\infty, \forall |\alpha| = k.$$

Norm:

$$\|u\|_{W^{s,p}(\Omega)} = \{\|u\|^p_{W^{k,p}(\Omega)} + \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{d+\sigma p}} dxdy\}^{1/p}$$

Soboble space on the boundary

Let Ω be a bounded Lipschitz domain in R^d with boundary Γ. Let s, p bw two real numbers with $0 \leq s < 1$ and $1 \leq p < \infty$. We define $W^{s,p}(\Gamma)$ as the set of all functions $u \in L^p(\Gamma)$ such that

$$\int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x - y|^{d-1+sp}} ds(x) ds(y) < \infty$$

$W^{s,p}(\Gamma)$ is a Banach space with the norm

$$\|u\|_{W^{s,p}(\Gamma)} = \{\|u\|^p_{L^p(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x - y|^{d-1+sp}} ds(x) ds(y)\}^{1/p}$$

Trace Theorem

Let Ω be a bounded Lipschitz domain with boundary Γ, $1 \leq p < \infty$, and $1/p < s \leq 1$

1. There exists a bouned linear mapping

$$\gamma_0 : W^{s,p}(\Omega) \to W^{s-1/p,p}(\Gamma);$$

2. For all $v \in C^1(\bar{\Omega})$ and $u \in W^{1,p}(\Omega)$,

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = -\int_\Omega \frac{\partial u}{\partial x_i} v dx + \int_\Gamma \gamma_0(u) v n_i ds,$$

3. $W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \gamma_0(u) = 0\}$.
4. γ_0 has a continuous right inverse, that is, there exists a constant C such that, $\forall g \in W^{s-1/p,p}(\Gamma)$, there exists $u_g \in W^{s,p}(\Omega)$ satisfying

$$\gamma_0(u_g) = g \text{ and } \|u_g\|_{W^{s,p}(\Omega)} \leq C\|g\|_{W^{s-1/p,p}(\Gamma)}$$

Variational formulation

Assuming $f \in L^2(\Omega)$ and the coefficients in operator L satisfies $a_{ij}, b_i, c \in L^\infty(\Omega), i, j = 1, 2, ..., d$. We multiply $Lu = f$ by $\phi \in C_0^\infty(\Omega)$, and integrate by part, to find

$$\int_\Omega (\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} \phi + cu\phi) dx = \int_\Omega f \phi dx$$

By the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, the above equation makes sense if $u \in H_0^1(\Omega)$, then define the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ as follows:

$$a(u, \phi) = \int_\Omega (\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} \phi + cu\phi) dx$$

Variational formulation(continued)

Variational formulation $u \in H_0^1(\Omega)$ is called a weak solution of the boundary value problem (1) if

$$a(u,\phi) = (f,\phi) \ \forall \phi \in H_0^1(\Omega) \tag{2}$$

Where $(\cdot,\cdot)$ denotes the inner product on $L^2(\Omega)$.

Question

- What is the variational formulation for the elliptic problem with inhomogeneous boundary condition?

$$Lu = f \text{ in } \Omega, \ \ u = g \text{ on } \partial\Omega$$

- if $f \in H^{-1}(\Omega)$, how to define the weak formulation?

Lax-Milgram Lemma

Assume that V is a real Hilbert space, with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Assume that $a : V \times V \to R$ is a bilinear form, for which there exist constants $\alpha, \beta > 0$ such that

$$|a(u,v)| \leq \beta\|u\|\|v\|, \ \forall u, v \in V \tag{3}$$
$$|a(u,u)| \geq \alpha\|u\|^2, \ \forall u \in V. \tag{4}$$

Let $f : V \to R$ be a bounded linear functional on V. Then there exists unique element $u \in V$ such that

$$a(u,v) = < f, v > \ \ \forall v \in V.$$

The bilinear form a is called V-elliptic(or V-coercive) if it satisfies (4).

Well-posedness and Regularity

Existance of a solution

If operator L is uniformly elliptic, $b_i = 0, i = 1, ..., d$, and $c(x) \geq 0$. Suppose $f \in H^{-1}(\Omega)$. Then the boundary value problem $Lu = f$ in Ω has a unique solution $u \in H_0^1(\Omega)$.

Regularity

Asssume that $a_{ij} \in C^1(\bar{\Omega}), b_i, c \in L^\infty(\Omega), i, j = 1, ..., d$, and $f \in L^2(\Omega)$. Suppose that $u \in H_0^1(\Omega)$ is the weak solution of $Lu = f$ in Ω. Assume that $\partial\Omega$ is smooth ($C^{1,1}$) or Ω is convex. Then $u \in H^2(\Omega)$ satifies the esitmate

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$$

Some references

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  R. Adams, Sobolev spaces, Academic press, 1975 (中文版，影印版)
- Regularity:
  D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order.
- PDEs:
  L. Evans, Partial differential equations, Vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, 1998

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Galerkin Method

Let V be a real Hilbert space with the norm $\|\cdot\|_V$ and inner product$(\cdot,\cdot)_V$. Assume that the bilinear form $a : V \times V \to R$ is bounded and V-coercive. Let $f : V \to R$ be a bounded linear functional on V. We consider the variational problem to find $u \in V$ such that

$$a(u,v) =< f, v > \ \ \forall v \in V. \tag{5}$$

Let V_h be a subspace of V which is finite dimensional, h stands for a discretization parameter. The Galerkin method of the variational problem is then to find $u_h \in V_h$ such that

$$a(u_h, v_h) =< f, v_h > \ \ \forall v_h \in V_h \tag{6}$$

Supposed that $\{\phi_1, ..., \phi_N\}$ is a basis for V_h. Then the Galerkin method is equivalent to

$$a(u_h, \phi_i) =< f, \phi_i >, i = 1, ..., N$$

Writing u_h in the form

$$u_h = \sum_{j=1}^{N} x_j \phi_j.$$

We reach the system of equations

$$\sum_{j=1}^{N} a(\phi_j, \phi_i) x_j =< f, \phi_i >, \ i = 1, ..., N,$$

which we can write in the matrix-vector form as

$$Ax = b$$

We remark that A is positive definite. The matrix A is called the stiffness matrix.

Céa Lemma

Suppose the blinear form $a(\cdot,\cdot)$ is bounded and coercive. Suppose u and u_h are the solution of the variational problem (5) and its Galerkin approximation, respectively. Then

$$\|u-u_h\|_V \leq \frac{\beta}{\alpha} \inf_{v_h \in V_h} \|u-v_h\|_V$$

According to the Céa Lemma, the arruracy of a numerical solution depends essentially on choosing function spaces which are capable of approximating the solution u well.

Galerkin Method(continued)

- **Rayleigh-Ritz method**: When the bilinear form $a : V \times V \to R$ is symmetric, then the variational problem (5) is equivalent to the minimization problem

$$\min_{v \in V} J(v), \;\; J(v) = \frac{1}{2} a(v, v) - < f, v >$$

  The Rayleight-Ritz method is then to solve the above problem by solving $u_h \in V_h$ as

$$\min_{v_h \in V_h} J(v_h)$$

- **Galerkin method**: The weak formulation (6) is solved for problems where the bilinear form is not necessarily symmetric. If the bilinear form is coercive and symmetric, then the term Ritz-Galerkin is often used.
- **Finite element method**: The finite element method can be regarded as a special kind of Galerkin method that uses piecewise polynomials to construct discrete approximating function spaces.

The construction of finite element spaces

For simplicity, we restrict our discussion primarily to the piecewise polynomial approximations over triangular(2d) or tetrahedral(3d) elements.

The finite element(cf. P. Ciarlet 1978)

A finite element is a triple $(K, \mathcal{P}, \mathcal{N})$ with the following properties:

- $K \subset R^d$ is a domain with piecewise smooth boundary(the element)
- $\mathcal{P}$ is a finite-dimensional space of function on K(the shape functions)
- $\mathcal{N} = \{N_1, N_2, ..., N_n\}$ is the base for $\mathcal{P}'$(the nodal variables or degrees of freedom)

Nodal basis

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element, and let $\{\phi_1, \phi_2, ..., \phi_n\}$ be the basis for $\mathcal{P}$ dual to $\mathcal{N}$, that is, $N_i(\phi_j) = \delta_{ij}$. It is called the nodal basis for $\mathcal{P}$.

It is clear that the following expansion hold for any $v \in \mathcal{P}$:

$$v = \sum_{i=1}^{n} N_i(v)\phi_i(x)$$

Linear element(1-d) - an example

Let $K = [0, 1]$, $\mathcal{P}$ is the set of linear polynomials and $\mathcal{N} = \{N_1, N_2\}$, where $N_1(v) = v(0), N_2(v) = v(1), \forall v \in \mathcal{P}$, then $(K, \mathcal{P}, \mathcal{N})$ is a finite element and the nodal basis consists of $\phi_1(x) = 1 - x$ and $\phi_2 = x$.

Linear element

Let K be simplex in R^d with vertices $A_i(i = 1, ..., d+1), \mathcal{P} = P_1$, and $\mathcal{N} = \{N_1, ..., N_{d+1}\}$, where $N_i v = v(A_i)$ for any $v \in \mathcal{P}$. Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element.

The nodal basis $\{\lambda_1(x), ..., \lambda_{d+1}\}$ of the linear element satisfies

$$\lambda_i(x) \text{ is linear and } \lambda_i(A_j) = \delta_{ij} \quad i, j = 1, ..., d+1. \tag{7}$$

Linear element(continued)

To construct the nodal basis, it is convenient to consider the associated barycentric coordinates defined as $(d+1)$-tuple $(\lambda_1, ..., \lambda_{d+1})$, where $\lambda_i(x)$ satisfies (7). Let α_i be the Cartesian coordinates for A_i, we have

$$\sum_{i=1}^{d+1} \lambda_i(x) = 1, \text{ and } x = \sum_{i=1}^{d+1} \alpha_i \lambda_i(x)$$

That is(for 2 d case)

$$\lambda_1 = \frac{1}{2S} \begin{vmatrix} x_1 & x_2 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}, \lambda_2 = \frac{1}{2S} \begin{vmatrix} a_1 & b_1 & 1 \\ x_1 & x_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}, \lambda_3 = \frac{1}{2S} \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ x_1 & x_2 & 1 \end{vmatrix}.$$

Where S is the area of simplex K and the coordinates of A_i is (a_i, b_i).

Lagrange Element

Local interpolant

Given a finite element $(K, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i, 1 \leq i \leq n\}$ be the basis dual to $\mathcal{N}$. If v is a function for which all $N_i \in \mathcal{N}, i = 1, 2, ..., N$ are defined, the we defined the local interpolant by

$$I_K v = \sum_{i=1}^{n} N_i(v)\phi_i$$

It is easy to see that I_K is linear and $I_K u = u$ for $u \in \mathcal{P}$.

Lagrange interpolant of linear elements

Let $(K, \mathcal{P}, \mathcal{N})$ be the linear finite element with nodal basis $\{\phi_i\}$. The Lagrange interpolant is defined as

$$(I_K v)(x) := \sum_{i=1}^{d+1} v(A_i)\phi_i(x)$$

Mesh

Triangular

A triangular(tetrahedral) mesh $\mathcal{M}_h$ is a partition of the domain Ω in $R^d, d = 2, 3$ into a finite collection of triangles(tetrahedral)$\{K_i\}$ satisfying the following the following conditions:

- $K_i \cap K_j = \varnothing$ for $i \neq j$;
- $\cup \bar{K}_i = \bar{\Omega}$;
- No vertex of any triangle (tetrahedral) lies in the interior of an edge(or a face) of another triangle(tetrahedral).

A triangular or tetrahedral mesh is called a *triangulation*, of simply a *mesh*.

Continuity

Let Ω be bounded domain in R^d and $\mathcal{M}_h = \{K_j\}_{j=1}^J$ be a partition of Ω, that is, $\cup K_i = \bar{\Omega}, K_i \cap K_j = \emptyset, i \neq j$. Assume that $\partial K_i (i = 1, ..., J)$ are Lipschitz. Let $k \geq 1$. The a piecewise infinite differentiable function $v : \bar{\Omega} \to R$ over the partition $\mathcal{M}_h$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\bar{\Omega})$.

Example: Conforming linear element

Let $(K, \mathcal{P}, \mathcal{N})$ be the linear element defined above. Since any piecewise linear function is continuous as long as it is continuous at the vertices, we can introduce

$$V_h = \{v : v|_K \in P_1 \ \forall K \in \mathcal{M}_h, v \text{ is continuous at the vertices of the element}$$

Then $V_h \subset H^1(\Omega)$, V_h is a H^1-conforming finite element space.

Computation of FEM

The computation of finite element methods can be devided into three steps:

- Construction of a mesh by partition Ω;
- Setting the stiffness matrix;
- Solution of the system of equations.

We consider

$$a(u, v) = \int_{\Omega} \sum_{k,l=1}^{d} a_{k,l}(x) \frac{\partial u}{\partial x_l} \frac{\partial v}{\partial x_k} dx$$

Computation of FEM

Let $\{\phi_j\}_{j=1}^{J}$ be a nodal basis of the linear finite space $V_h^0 = V_h \cap H_0^1(\Omega)$ so that $\phi_j(x_i) = \delta_{ij}, i, j = 1, ..., J$, where $\{x_i\}_{i=1}^{J}$ is the set of interior nodes of the mesh $\mathcal{M}_h$. Then

$$A_{ij} = a(\phi_j, \phi_i) = \sum_{K \in \mathcal{M}_h} \sum_{k,l}^{d} \int_K a_{k,l}(x) \frac{\partial \phi_j}{\partial x_l} \frac{\partial \phi_i}{\partial x_k} dx$$

In forming the sum, we need only take account of those triangles which overlap the support of both ϕ_i and ϕ_i. Note that $A_{ij} = 0$ if the x_i and x_j are not adjacent.

Remark: The stiffness matrix $A = (A_{ij})$ is sparse.

element stiffness matrix

On each element K, the nodal basis function reduces to one of the barcentric coordinate functions $\lambda_p, p = 1, 2, ..., d+1$. then we need only to evaluate the following $(d+1) \times (d+1)$ matrix

$$A_K : (A_K)_{p,q} = \sum_{k,l}^{d} \int_K a_{k,l}(x) \frac{\partial \lambda_q}{\partial x_l} \frac{\partial \lambda_p}{\partial x_k}$$

Denote by K_p the global index of the p-th vertex of the element K, then $\phi_{k_p}|K = \lambda_p$ and the global matrix may be assembled through the element striffness matrices as

$$A_{ij} = \sum_{\substack{K, p, q \\ K_p = i, K_q = j}} (A_K)_{pq}$$

Others

- In coding FEM, you can compute element stiffness matrix elementwise, then add them to the global matrix according the mapping relation between local indices(p) and global indices(K_p).
- For lower order term $\int_K \phi_i(x)\phi_j(x)dx$ and right hand side $\int_K f\phi_i dx$, or high order(≥ 2) elements, we need to deal with integrals about high order polynomial over K. Typically, we use Gaussian quadrature formula to compute the integral over K.
- To solve the linear system Ax=b, for postive definite A, the Preconditioned CG method is a suitable choice. If A is not positive definite, GMRES method is preferred.

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Interpolation in Sobolev spaces

We start from the following result which plays an important role in the error analysis of finite element methods.

Deny-Lions theorem

Let Ω be a bounded Lipschitz domain, For any $k \geq 0$, there exists a constant $C(\Omega)$ such that

$$\inf_{p \in P_k(\Omega)} \|v + p\|_{H^{k+1}(\Omega)} \leq C(\Omega)|v|_{H^{k+1}(\Omega)}, \ \forall v \in H^{k+1}(\Omega)$$

This theorem is also called equavilent norm theorem.

Scaling argument

Let Ω and $\hat{\Omega}$ be affine equivalent, i.e. there exists a affine mapping

$$F : \hat{\Omega} \to \Omega, \ F\hat{x} = B\hat{x} + b$$

with a nonsingular matrix B. If $v \in H^m(\Omega)$, then $\hat{v} = v \circ F \in H^m(\hat{\Omega})$, and there exists a constant $C = C(m, d)$ such that

$$|\hat{v}|_{H^m(\hat{\Omega})} \leq C\|B\|^m |detB|^{-1/2} |v|_{H^m(\Omega)}$$
$$|v|_{H^m(\Omega)} \leq C\|B^{-1}\|^m |detB|^{1/2} |\hat{v}|_{H^m(\hat{\Omega})}$$

Here $\|\cdot\|$ denotes the matrix norm associated with the Euclidean norm in R^d.

Scaling argument(contitued)

Let Ω and $\hat{\Omega}$ be affine equivalent with

$$F : \hat{x} \in \hat{\Omega} \to B\hat{x} + b \in \Omega$$

being an invertible affine mapping. Then the upper ounds

$$\|B\| \leq \frac{h}{\hat{\rho}}, \|B^{-1}\| \leq \frac{\hat{h}}{\rho}, (\frac{\rho}{\hat{h}})^d \leq |detB| \leq (\frac{\rho}{\hat{\rho}})^d$$

hold, where h=diam(Ω), $\hat{h}$=diam($\hat{\Omega}$), ρ and $\hat{\rho}$ are the maximum diameter of the ball contained in Ω and $\hat{\Omega}$, respectively.

Bounds for interpolation error

Theorem 11.1: Suppose $m - d/2 - l > 0$. Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ be a finite element satisfying

- $P_{m-1} \subset \hat{\mathcal{P}} \subset H^m(\hat{K})$;
- $\hat{\mathcal{N}} \subset C^l(\hat{K})'$.

Then for $0 \leq i \leq m$ and $\hat{v} \in H^m(\hat{K})$ we have

$$|\hat{v} - \hat{I}\hat{v}|_{H^i(\hat{K})} \leq C(m, d, \hat{K})|\hat{v}|_{H^m(\hat{K})}$$

where $\hat{I}$ is the local interpolation operator.

Bounds for interpolation error(continued)

Affine-Interpolant Equivalent

Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ be a finite element and $x = F(\hat{x}) = B\hat{x} + b$ be an affine map. Let $v = \hat{v} \circ F^{-1}$. The finite element $(K, \mathcal{P}, \mathcal{N})$ is affine-interpolant equivalent to $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ if

- $K = F(\hat{K})$;
- $P = \{p : \hat{p} \in \hat{\mathcal{P}}\}$;
- $\hat{Iv} = \hat{I}\hat{v}$.

Here $\hat{I}\hat{v}$ and Iv are the $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$-interpolant and $(K, \mathcal{P}, \mathcal{N})$-interpolant respectively.

Bounds for interpolation error(continued)

Shape regular

A family of meshes $\{M_h\}$ is called regular or shape regular provided there exists a numer $\kappa > 0$ such that each $K \in \mathcal{M}_h$ contains a ball of diameter ρ_K with

$$\rho_K \geq h_k/\kappa$$

Local error bound

Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ satisfy the condition of Theorem 11.1 and let $(K, \mathcal{P}, \mathcal{N})$ be affine-interpolant equivalent to $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$. Then for $0 \leq i \leq m$ and $v \in H^m(K)$, we have

$$|v - Iv|_{H^i(K)} \leq Ch_k^{m-i}|v|_{H^m(K)}$$

where C depends on $m, d, \hat{K}$ and h_k/ρ_k.

Bounds for interpolation error(continued)

Suppose $\{\mathcal{M}_h\}$ is a regular family of meshes of a polyhedral domain $\Omega \subset R^d$. Let $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ be a reference finite element satisfying the conditions for Theorem 11.1 for some l and m. For all $K \in \mathcal{M}_h$, suppose $(K, \mathcal{P}, \mathcal{N})$ is affine-interpolant equivalent to $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$. Then for $0 \le i \le m$, there exists a positive constant $C(\hat{K}, d, m, \kappa)$ such that

$$(\sum_{K \in \mathcal{M}_h} \|v - Iv\|^2_{H^i(K)})^{1/2} \le Ch^{m-i}|v|_{H^m(\Omega)}, \ h = \max_{K \in \mathcal{M}_h} h_K, \ \forall v \in H^m(\Omega).$$

Inverse estimate

Quasi-uniform

A family of meshes $\{\mathcal{M}_h\}$ is called quasi-uniform if there exists a constant ν such that

$$h/h_K \leq \nu \ \forall K \mathcal{M}_h,$$

where $h = \max_{K\in\mathcal{M}_h} h_K$.

Inverse estimate

Let $\{\mathcal{M}_h\}$ be a shape regular quasi-uniform family of triangulations of Ω and let X_h be a finite element space of piecewise polynomials of degree less than or equal to p. Then for $m \geq l \geq 0$, there exists a constant $C = C(p, \kappa, \nu, m)$ such that for any $v_h \in X_h$,

$$\Big(\sum_{K\in\mathcal{M}_h} |v_h|^2_{H^m(K)}\Big)^{1/2} \leq Ch^{l-m}\Big(\sum_{K\in\mathcal{M}_h} |v_h|^2_{H^l(K)}\Big)^{1/2}$$

Outline

1. Variational Formualation of Elliptic Problems
2. Finite Element Methods
3. Bounds for interpolation error
4. Convergence for second order elliptic problem

The energy error estimate

Let Ω be a polyhedral domain in R^d and $\{\mathcal{M}_h\}$ be regular family of triangulations of the domain. Let V_h be the piecewise linear conforming finite element space over $\mathcal{M}_h$. Denote $V_h^0 = V_h \cap H_0^1(\Omega)$. Let $u \in H_0^1(\Omega)$ be the weak solution of the variational problem

$$a(u, v) = < f, v > \quad \forall v \in H_0^1(\Omega),$$

and u_h be the corresponding finite element solution

$$a(u_h, v_h) = < f, v_h > \quad \forall v_h \in V_h^0.$$

We assume the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to R$ is bounded and $H_0^1(\Omega)$-elliptic:

$$|a(u, v)| \le \beta \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, a(u, u) \ge \alpha \|u\|^2_{H^1(\Omega)}$$

The H^1 error estimate

If the solution $u \in H_0^1(\Omega)$ has the regularity $u \in H^2(\Omega)$, then there exists a constant C independentof h such that

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}$$

.

Remark: If V_h is a continuous finite element space of piecewise polynomials of degree $\leq m$ and the weak solution $u \in H^{k+1}(\Omega)$,we have the following estimate:

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^{\min\{k,m\}}|u|_{H^{k+1}(\Omega)}$$

The H^1 convergence under weak regularity

If solution u only belongs to $H_0^1(\Omega)$, we have

$$\lim_{h\to 0} \|u - u_h\|_{H^1(\Omega)} = 0$$

Question: Prove this convergence results.

The L^2 error estimate

Nitshce-trick:
We introduce w to be the solution of the following variational problem:
Find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = (u - u_h, v) \ \forall v \in H_0^1(\Omega)$$

L^2-error estimate

Assume the solution $u \in H^2$ and the solution of above equation $w \in H^2(\Omega)$ satisfying

$$\|w\|_{H^2(\Omega)} \leq C\|u - u_h\|_{L^2(\Omega)}.$$

Then there exists a constant C independent of h such that

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)}$$

Some references

- P. Ciarlet, The finite element method for elliptic problems. North-Holland. (1978)
- S. Brenner and L. Scott, The mathematical theory of finite element methods, Springer-Verlag, 1994
- 许学军，王烈衡，有限元方法的数学基础。

Homework

1. Consider the elliptic problem with inhomogeneous boundary condtion:

$$Lu = f \text{ in } \Omega, u = g \text{ on } \partial\Omega$$

Assume that $g \in H^{3/2}(\partial\Omega)$ and the weak solution $u \in H^2$. Give the error estimates with H^1 norm when the linear H^1–conforming finite element is used.