第十五讲：电磁场有限元方法

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Outline

The Maxwell's equation

The Maxwell equations comprise four first-order partial differential equations linking the fundamental electromagnetic quantities, the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, the magnetic field $\mathbf{H}$, the electric flux density $\mathbf{D}$, the electric current density $\mathbf{J}$, and the space charge density ρ:

$$
\begin{aligned}
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \nabla \cdot \mathbf{D} = \rho, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \nabla \cdot \mathbf{B} = 0.
\end{aligned}
$$

They are usually supplemented by the following constitutive relations:

$$
\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H}.
$$

Maxwell equation in wave form

With the help of constitutive relations, we can get the following wave form of Maxwell equations:

$$\begin{aligned}
\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \nabla\times(\mu^{-1}\nabla\times\mathbf{E}) &= -\frac{\partial\mathbf{J}}{\partial t}, \nabla\cdot(\varepsilon\mathbf{E}) = \rho \\
\mu\frac{\partial^2\mathbf{H}}{\partial t^2} + \nabla\times(\varepsilon^{-1}\nabla\times\mathbf{H}) &= \nabla\times(\varepsilon^{-1}\mathbf{J}), \nabla\cdot(\mu\mathbf{H}) = 0.
\end{aligned}$$

Let $\mathbf{E}(x,t) = Re(\hat{\mathbf{E}}e^{-i\omega t}), \mathbf{H}(x,t) = Re(\hat{\mathbf{H}}e^{-i\omega t}), \mathbf{J}(x,t) = Re(\hat{\mathbf{J}}e^{-i\omega t})$, the time harmonic form is

$$\begin{aligned}
\nabla\times(\mu^{-1}\nabla\times\hat{\mathbf{E}}) - \varepsilon\omega^2\hat{\mathbf{E}} &= \imath\omega\hat{\mathbf{J}}, \nabla\cdot(\varepsilon\hat{\mathbf{E}}) = \rho \\
\nabla\times(\varepsilon^{-1}\nabla\times\hat{\mathbf{H}}) - \mu\omega^2\hat{\mathbf{H}} &= \nabla\times(\varepsilon^{-1}\hat{\mathbf{J}}), \nabla\cdot(\mu\hat{\mathbf{H}}) = 0.
\end{aligned}$$

The function space $H(curl;\Omega)$

Let Ω be a bounded domain in R^3 with a Lipschitz boundary Γ, we define

$$H(curl;\Omega) = \{v \in L^2(\Omega)^3 : \nabla \times v \in L^2(\Omega)^3\}$$

with the norm

$$\|v\|_{H(curl;\Omega)} = (\|v\|^2_{L^2(\Omega)} + \|\nabla \times v\|^2_{L^2(\Omega)})^{1/2}.$$

We define $H_0(curl;\Omega)$ to be the closure of $C_0^\infty(\Omega)^3$ in $H(curl;\Omega)$. $H(curl;\Omega)$ and $H_0(curl;\Omega)$ are Hilbert spaces.
Lemma: Let Ω be a bounded Lipschitz domain. Let $v \in H(curl;R^3)$ vanish outside Ω. Then $v \in H_0(curl;\Omega)$

Theorem: Let $D(\bar{\Omega})$ be the set of all functions $\phi|_{\Omega}$ with $\phi \in C_0^{\infty}(R^3)$. Then $D(\bar{\Omega})^3$ is dense in $H(curl;\Omega)$.

Theorem(trace): The mapping $\gamma_{\tau} : \mathbf{v} \to \mathbf{v} \times \mathbf{n}|_{\Gamma}$ defined on $D(\bar{\Omega})^3$ can be extended by continuity to a linear and continuous mapping from $H(curl;\Omega)$ to $H^{-1/2}(\Gamma)^3$. Moreover, the following Green formula holds

$$< \mathbf{v} \times \mathbf{n}, \mathbf{w} >_{\Gamma} = \int_{\Omega} \mathbf{v} \cdot \nabla \times \mathbf{w} dx - \int_{\Omega} \nabla \times \mathbf{v} \cdot \mathbf{w} dx \quad \forall \mathbf{w} \in H^1(\Omega)^3, \mathbf{v} \in H(cu$$

γ_{τ} is not a surjective mapping.

Lemma:

$$H_0(curl;\Omega) = \{\mathbf{v} \in H(curl;\Omega) : \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma\}$$

The function space $H(curl;\Omega)$(continued)

The following theorem is a generalization of the classical Stokes theorem:
Theorem: Let Ω be a simply connected Lipschitz domain. Then $\mathbf{u} \in L^2(\Omega)^3$ and $\nabla \times \mathbf{u} = 0$ if and only if there exists a function $\phi \in H^1(\Omega)/R$ such that $u = \nabla\phi$.

The function space $H(curl;\Omega)$(continued)

Theorem A vector field $\mathbf{v} \in L^2(\Omega)^3$ satisfies

$$\nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, < \mathbf{v} \cdot \mathbf{n}, 1 >_{\Gamma_i} = 0, 0 \leq i \leq p.$$

if and only if there is a vector potential $\mathbf{w} \in H^1(\Omega)^3$ such that

$$\mathbf{v} = \nabla \times \mathbf{w}.$$

Moreover, $\mathbf{w}$ may be chosen such that $\nabla \cdot \mathbf{w} = 0$ and the following estimate holds

$$\|\mathbf{w}\|_{H^1(\Omega)} \leq C\|\mathbf{v}\|_{L^2(\Omega)}.$$

The function space $H(curl;\Omega)$(continued)

Theorem(Helmholtz decomposition): Any vector field $\mathbf{v} \in L^2(\Omega)^3$ has the following orthogonal decomposition

$$\mathbf{v} = \nabla q + \nabla \times \mathbf{w},$$

where $q \in H^1(\Omega)/R$ is the unique solution of the following problem

$$(\nabla q, \nabla \phi) = (\mathbf{v}, \nabla \phi) \quad \forall \phi \in H^1(\Omega),$$

and $\mathbf{w} \in H^1(\Omega)^3$ satisfies $\nabla \cdot \mathbf{w} = 0$ in Ω, $\nabla \times \mathbf{w} \cdot \mathbf{n} = 0$ on Γ.

The function space $H(curl;\Omega)$(continued)

The embedding theorem for function spaces $X_N(\Omega)$ and $X_T(\Omega)$ are very useful.

$$X_N(\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3, \nabla \cdot \mathbf{v} \in L^2(\Omega), \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma\}$$
$$X_T(\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3, \nabla \cdot \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$$

Theorem(Embedding): If Ω is a C^1 or convex domain, $X_N(\Omega), X_T(\Omega)$ are continuously embedded into $H^1(\Omega)^3$

Outline

The curl conforming finite element

We only consider the lowest order Nédélec finite element space. The lowest order Nedelec finite element is a triple (K, P, N) with the following properties

- $K \subset R^3$ is a tetrahedron;
- $P = \{\mathbf{u} = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x}\ \forall \mathbf{a}_K, \mathbf{b}_K \in R^3\}$;
- $N = \{M_e : M_e(\mathbf{u}) = \int_e \mathbf{u} \cdot \mathbf{t} dl\ \forall edge\ e\ of\ K, \forall \mathbf{u} \in P\}$. $M_e(\mathbf{u})$ is called the moment of $\mathbf{u}$ on the edge e.

The curl conforming finite element(continued)

Lemma The nodal basis of the lowest order Nedelec element is $\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i, 1 \leq i < j \leq 4\}$. Here $\lambda_i, i = 1, 2, 3, 4$ are barycentric coordinate functions of the element K.

The curl conforming finite element(continued)

Let K be a tetrahedron with vertices A_i, $1 \leq i \leq 4$, Let $F_K : \hat{K} \to K$ be the affine transform from the reference element *hatK* to K:

$$x = F(\hat{x}) = B_K\hat{x} + b_K, \hat{x} \in \hat{K}, B_K \text{ is invertible.}$$

Notice that the normal and tangential vector $\mathbf{n}, \hat{\mathbf{n}}$ and $\mathbf{t}, \hat{\mathbf{t}}$ to the faces satisfy

$$\mathbf{n} \circ F_K = (B_K^{-1})^T\hat{\mathbf{n}}/|(B_K^{-1})^T\hat{\mathbf{n}}|, \mathbf{t} = B_K\hat{\mathbf{t}}/|B_K\hat{\mathbf{t}}|.$$

For any scaler function ϕ defined on K, we associate

$$\hat{\phi} = \phi \circ F_K, \text{ that is, } \hat{\phi} = \phi(B_K\hat{x} + b_K).$$

For any vector valued function $\mathbf{u}$ defined on K, we associate

$$\hat{\mathbf{u}} = B_K^T\mathbf{u} \circ F_K, \text{ that is, } \hat{\mathbf{u}}(\hat{x}) = B_K^T\mathbf{u}(B_K\hat{x} + b_K).$$

The curl conforming finite element(continued)

Denote by $\mathbf{u} = (u_1, u_2, u_3), \hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$. We introduce

$$C = (\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})_{i,j=1}^3 \text{ and } \hat{C} = (\frac{\partial \hat{u}_i}{\partial \hat{x}_j} - \frac{\partial \hat{u}_j}{\partial \hat{x}_i})_{i,j=1}^3.$$

Then we have

$$C \circ F_K = (B_K^{-1})\hat{C} B_K^{-1}.$$

In fact,

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_j} = \frac{\partial}{\partial \hat{x}_j}(\sum_k b_{ki}(u_k \circ F_K)) = \sum_{k,l} b_{ki} \frac{\partial u_k}{\partial x_l} b_{lj}$$

and

$$\frac{\partial \hat{u}_i}{\partial \hat{x}_j} - \frac{\partial \hat{u}_j}{\partial \hat{x}_i} = \sum_{k,l} b_{ki} \frac{\partial u_k}{\partial x_l} b_{lj} - \sum_{k,l} b_{kj} \frac{\partial u_k}{\partial x_l} b_{li} = \sum_{k,l} b_{ki}(\frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k}) b_{lj}$$

This yields

$$\hat{C}_{ij} = \sum_{k,l} b_{ki} C_{kl} b_{lj} \text{ and hence } \hat{C} = B_K^T C B^T$$

The curl conforming finite element(continued)

Lemma We have

(i) $\mathbf{u} \in P(K) \Leftrightarrow \hat{\mathbf{u}} \in \hat{P}(\hat{K})$;

(ii) $\nabla \times \mathbf{u} = 0 \Leftrightarrow \hat{\nabla} \times \hat{\mathbf{u}} = 0, \forall \mathbf{u} \in P(K)$;

(iii) $M_e(\mathbf{u}) = 0 \Leftrightarrow M_{\hat{e}}(\hat{\mathbf{u}}) = 0, \forall \mathbf{u} \in P(K)$;

(iv) Let $\mathbf{u} \in P(K)$ and F be a face of K, If $M_e(\mathbf{u}) = 0$ for any edge $e \subset \partial F$, then $\mathbf{u} \times \mathbf{n} = 0$ on F;

(v) If $\mathbf{u} \in P(K)$ and $M_e(\mathbf{u}) = 0$ for any edge e, then $\mathbf{u} = 0$ in K.

This lemma induces a natural interpolation operator on K.

Definition Let K be an arbitrary tetrahedron in R^3 and $\mathbf{u} \in W^{1,p}(K)^3$ for some $p > 2$. Its interpolant $\gamma_K \mathbf{u}$ is a unique polynomial in $P(K)$ that has the same moments as $\mathbf{u}$ on K. In other words, $M_e(\gamma_K \mathbf{u} - \mathbf{u}) = 0$

The curl conforming finite element(continued)

For any $p > 2$, the operator γ_K is continuous on the space

$$\{\mathbf{v} \in L^p(K)^3 : \nabla \times \mathbf{v} \in L^p(K)^3 \text{ and } \mathbf{v} \times \mathbf{n} \in L^p(\partial K)^3\}$$

The curl conforming finite element(continued)

Let Ω be a bounded polyhedron and $\mathcal{M}_h$ be a regular mesh of Ω. We set

$$X_h = \{\mathbf{u}_h \in H(curl;\Omega) : \mathbf{u}_h|_K \in P(K) \forall K \in \mathcal{M}_h\}.$$

For any function $\mathbf{u}$ whose moments are defined on all edges of the mesh $\mathcal{M}_h$, we define the interpolation operator γ_h by

$$\gamma_h \mathbf{u}|_K = \gamma_K \mathbf{u} \text{ on } K, \forall K \in \mathcal{M}_h$$

Theorem: Let $\mathbf{u} \in H^1(curl;\Omega)$, that is $\mathbf{u} \in H^1(\Omega)$ and $\nabla \times \mathbf{u} \in H^1(\Omega)^3$, we have

$$\|\mathbf{u} - \gamma_h \mathbf{u}\|_{H(curl;\Omega)} \leq Ch(|\mathbf{u}|_{H^1(\Omega)} + |\nabla \times \mathbf{u}|_{H^1(\Omega)}).$$

Outline

Weak formulation and the discrete problem

Let Ω be bounded polyhedral domain in R^3. We will consider the following porblem

$$\nabla \times (\alpha \nabla \times \mathbf{E}) + \beta \mathbf{E} = \mathbf{f} \text{ in } \Omega$$

with boundary condition

$$\mathbf{E} \times \mathbf{n} = 0.$$

We assume $\mathbf{f} \in L^2(\Omega)^3$, $\alpha, \beta \in L^\infty$ such that $\alpha \geq \alpha_0 > 0, \beta \geq \beta_0 > 0$. The variational problem is to find $\mathbf{E} \in H_0(curl; \Omega)$ such that

$$(\alpha \nabla \times \mathbf{E}, \nabla \times \mathbf{v}) + (\beta \mathbf{E}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \forall \mathbf{v} \in H_0(curl; \Omega) \tag{1}$$

Weak formulation and the discrete problem

There exists a unique solution for the variational problem according to Lax-Milgram theorem. Let $X_h^0 = X_h \cap H_0(curl;\Omega)$. Then the finite element approximation is to find $\mathbf{E}_h \in X_h^0$ such that

$$(\alpha\nabla \times \mathbf{E}_h, \nabla \times \mathbf{v}_h) + (\beta\mathbf{E}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \forall \mathbf{v}_h \in X_h^0. \tag{2}$$

This discrete problem has a unique solution since it is a conforming finite element approximation for the variational problem.

The convergence of finite element method

Theorem: Let Ω be a convex polyhedral domain in R^3, and $\alpha = \beta = 1$. Assume that the solution $\mathbf{E}$ of (1) satisfies $\mathbf{E} \in H^1(\Omega)^3, \nabla \times \mathbf{E} \in H^1(\Omega)^3$, then the following error estimates holds.

$$\|\mathbf{E} - \mathbf{E}_h\|_{H(curl;\Omega)} \leq Ch(|\mathbf{E}|_{H^1(\Omega)} + |\nabla \times \mathbf{E}|_{H^1(\Omega)})$$

Outline

BHHW interpolation operator

Theorem: There exists a linear projection $\Pi_h : H^1(\Omega) \cap H_0(curl;\Omega) \to X_h^0$ such that for all $\mathbf{v} \in H^1(\Omega)^3$

$$
\begin{aligned}
\|\Pi_h \mathbf{v}\|_{L^2(K)} \leq C(\|\mathbf{v}\|_{L^2(\tilde{K})} + h_K |\mathbf{v}|_{H^1(\tilde{K})}), \forall K \in \mathcal{M}_h,\\
\|\nabla \times \Pi_h \mathbf{v}\|_{L^2(K)} \leq C|\mathbf{v}|_{H^1(\tilde{K})}, \forall K \in \mathcal{M}_h,\\
\|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2(K)} \leq C h_k |\mathbf{v}|_{H^1(\tilde{K})}, \forall K \in \mathcal{M}_h,\\
\|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2(F)} \leq C h_F^{1/2} |\mathbf{v}|_{H^1 \tilde{F}}, \forall \text{ face } F \in \mathcal{F}_h
\end{aligned}
$$

where $\mathcal{F}_h$ is the set of all interior faces of the mesh $\mathcal{M}_h$, $\tilde{K}$ and $\tilde{F}$ are the union of the elements in $\mathcal{M}_h$ having having nonempty intersection with K and F, respectively.

BHHW interpolation operator

Proof. Let $\mathcal{E}_h$ be the set of all edges of the mesh $\mathcal{M}_h$. For any edge $e \in \mathcal{E}_h$, let $\mathbf{w}_e \in X_h$ be the associated canonical basis function of X_h, that is, $\{\mathbf{w}_e\}_{e\in\mathcal{E}_h}$ be the basis of X_h satisfying

$$\int_e \mathbf{w}_e \cdot \mathbf{t}_e dl = 1, \int_{e'} \mathbf{w}_e \cdot \mathbf{t}_{e'} dl = 0, \forall e, e' \in \mathcal{E}_h, e' \neq e.$$

On each face $F \in \mathcal{F}_h$ with edges $\{e_1, e_2, e_3\}$, we construct a dual basis $\{\mathbf{q}_i\}$ of $\{\mathbf{w}_i \times \mathbf{n}\}$ as follows

$$\int_F (\mathbf{w}_i \times \mathbf{n}) \cdot \mathbf{q}_j ds = \delta_{ij}, i, j = 1, 2, 3$$

We claim that

$$\|\mathbf{q}_i\|_{L^\infty(F)} \leq Ch_F^{-1}. \tag{3}$$

which implies that $\|\mathbf{q}_i\|_{L^2(F)} \leq C$

BHHW interpolation operator

Without loss of generality, we will prove that the claim(3) holds for $i = 1$. We first find $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ such that

$$\mathbf{q}_1 = \alpha_1 \mathbf{w}_1 \times \mathbf{n} + \alpha_2 \mathbf{w}_2 \times \mathbf{n} + \alpha_3 \mathbf{w}_3 \times \mathbf{n}, \int_F (\mathbf{w}_i \times \mathbf{n}) \cdot \mathbf{q}_1 ds = \delta_{i1}, i = 1, 2, 3$$

It is clear that α is the solution of the linear system

$$A_F \alpha = (1, 0, 0)^T, \text{ where } A_F = \left(\int_F (\mathbf{w}_i \times \mathbf{n}) \cdot (\mathbf{w}_j \times \mathbf{n}) ds \right)_{3\times 3}$$

We will show that A_F is invertible. Let F be the face F_{123} of a tetrahedron K with vertices $A_i, i = 1, 2, 3, 4$ and let e_1, e_2, e_3 be the edges A_2A_3, A_3A_1, A_1A_2. Then

$$\mathbf{w}_1 = \lambda_2 \nabla \lambda_3 - \lambda_3 \nabla \lambda_2, \mathbf{w}_2 = \lambda_3 \nabla \lambda_1 - \lambda_1 \nabla \lambda_3, \mathbf{w}_3 = \lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1.$$

Let $b_{ij} = (\nabla\lambda_i \times \mathbf{n}) \cdot (\nabla\lambda_j \times \mathbf{n})$. Since $\sum_{i=1}^4 \nabla\lambda_i = 0$ and $\nabla\lambda_4$ is perpendicular to the face F_{123}, we have

$$\sum_{j=1}^{3} b_{ij} = 0 \text{ and } b_{ij} = b_{ji}.$$

Therefore, A_F can be rewritten as

$$A_F = \frac{|F|}{12} \begin{pmatrix} 3b_{22} + 3b_{33} - b_{11} & -3b_{33} + b_{11} + b_{22} & -3b_{22} + b_{33} + b_{11} \\ -3b_{33} + b_{11} + b_{22} & 3b_{11} + 3b_{33} - b_{22} & -3b_{11} + b_{22} + b_{33} \\ -3b_{22} + b_{33} + b_{11} & -3b_{11} + b_{22} + b_{33} & 3b_{11} + 3b_{22} - b_{33} \end{pmatrix}$$

BHHW interpolation operator

It follows from $\nabla\lambda_1 \perp F_{234}$ that

$$|\nabla\lambda_1| = 1/\text{the height of K to the face} F_{234}.$$

which implies that

$$b_{11} = |\nabla\lambda_1 \times \mathbf{n}|^2 = \frac{|e_1|^2}{4|F|^2}.$$

Similarly,

$$b_{22} = \frac{|e_2|^2}{4|F|^2}, b_{33} = \frac{|e_3|^2}{4|F|^2}$$

Straightforward computation shows that

$$\det A_F = \frac{|e_1|^2 + |e_2|^2 + |e_3|^2}{576|F|} \geq c_0,$$

where c_0 is a positive constant that depends only on the minimum angle of the elements in the mesh. Thus A_F is invertible. Since $A_F = O(1)$, we have $A_F^{-1} = O(1)$ which implies $\alpha = O(1)$, that is, (3) holds.

Now for each $e \in \mathcal{E}_h$, we assign one of those faces with edge e and call it $F_e \in \mathcal{F}_h$. We have to comply with the restriction that for e on the boundary, F_e also on the boundary. Then we can define

$$\Pi_h \mathbf{v} = \sum_{e \in \mathcal{E}_h} \left(\int_{F_e} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{q}_e^{F_e} ds \right) \mathbf{w}_e.$$

This defines a projection. Obviously the boundary condition is respected. By the claim(3),

$$\left| \int_{F_e} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{q}_e^{F_e} ds \right| \leq \|\mathbf{v}\|_{L^2(F_e)} \|\mathbf{q}_e^{F_e}\|_{L^2(F_e)} \leq C \|\mathbf{v}\|_{L^2(F_e)}$$

BHHW interpolation operator

Let $K_e \in \mathcal{M}_h$ be an element with F_e as one of its face. By the scaled trace inequality, we have

$$\|\mathbf{v}\|^2_{L^2(F_e)} \leq C(h_e^{-1}\|\mathbf{v}\|^2_{L^2(K_e)} + h_e|\mathbf{v}|^2_{H^1(K_e)}).$$

Therefore

$$\begin{aligned} \|\Pi_h \mathbf{v}\|^2_{L^2(K)} &\leq Ch_K \sum_{e\in\mathcal{E}_h, e\subset\partial K} \Big|\int_{F_e} (\mathbf{v}\times\mathbf{n})\mathbf{q}_e^{F_e} ds\Big|^2 \\ &\leq Ch_K \sum_{e\in\mathcal{E}_h, e\subset\partial K} (h_e^{-1}\|\mathbf{v}\|^2_{L^2(K_e)} + h_e|\mathbf{v}|^2_{H^1(K_e)}) \\ &\leq C(\|\mathbf{v}\|^2_{L^2(\tilde{K})} + h_e^2|\mathbf{v}|^2_{H^1(\tilde{K})}) \end{aligned}$$

This proves the first estimate in the theorem.

BHHW interpolation operator

Since Π_h is a projection, we know that $\Pi_h \mathbf{c}_K = \mathbf{c}_K$ for any constant $\mathbf{c}_K$. Thus

$$\begin{aligned}
\|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2(K)} &= \inf_{\mathbf{c}_K} \|(\mathbf{v} + \mathbf{c}_K) - \Pi_h(\mathbf{v} + \mathbf{c}_K)\|_{L^2(K)} \\
&\leq C \inf_{\mathbf{c}_K} (\|\mathbf{v} + \mathbf{c}_K\|_{L^2(\tilde{K})} + h_K |\mathbf{v} + \mathbf{c}_K|_{H^1(\tilde{K})}) \\
&\leq C h_K |\mathbf{v}|_{H^1(\tilde{K})}.
\end{aligned}$$

where we have used the scaling argument and Deny-Lions theorem in the last inequality. This prove the third inequality. The last inequality can be proved similarly.

A regular decomposition

Lemma(Birman-Solomyak): Let Ω be a bounded Lipschitz domain. Then for any $\mathbf{v} \in H_0(curl;\Omega)$, there exists a $\psi \in H_0^1(\Omega)$ and a $\mathbf{v}_s \in H^1(\Omega)^3 \cap H_0(curl;\Omega)$ such that $\mathbf{v} = \mathbf{v}_s + \nabla\psi$ in Ω, and

$$\|\psi\|_{H^1(\Omega)} + \|\mathbf{v}_s\|_{H^1(\Omega)} \leq C\|\mathbf{v}\|_{H(curl;\Omega)},$$

where the constant C depends only on Ω.

A regular decomposition

Proof. Let $\mathcal{O}$ be a ball constaining Ω. We extend $\mathbf{v}$ by zero to the exterior of Ω and denote the extension by $\tilde{\mathbf{v}}$. Clearly $\tilde{\mathbf{v}} \in H_0(curl; \mathcal{O})$ with compact support in $\mathcal{O}$. By the previous theorem, there exists a $\mathbf{w} \in H^1(\mathcal{O})^3$ such that

$$\nabla \times \mathbf{w} = \nabla \times \tilde{\mathbf{v}}, \nabla \cdot \mathbf{w} = 0 \text{ in } \mathcal{O}$$

and

$$\|\mathbf{w}\|_{H^1(\mathcal{O})} \leq C\|\nabla \times \tilde{\mathbf{v}}\|_{L^2(\mathcal{O})} = C\|\nabla \times \mathbf{v}\|_{L^2(\Omega)}$$

A regular decomposition

Now since $\mathcal{O}$ is simply-connected, $\nabla \times (\mathbf{w} - \tilde{\mathbf{v}}) = 0$, there exists a $\phi \in H^1(\mathcal{O})/R$ such that $\tilde{\mathbf{v}} = \mathbf{w} + \nabla\phi$ in $\mathcal{O}$, and

$$\|\phi\|_{H^1(\mathcal{O})} \leq C|\phi|_{H^1(\mathcal{O})} \leq C\big(\|\tilde{\mathbf{v}}\|_{L^2(\mathcal{O})} + \|\mathbf{w}\|_{L^2(\mathcal{O})}\big) \leq C\|\mathbf{v}\|_{H(curl;\Omega)}$$
$$|\phi|_{H^2(\mathcal{O}\setminus\bar{\Omega})} \leq |\mathbf{w}|_{H^1(\mathcal{O})} \leq C\|\nabla \times \mathbf{v}\|_{L^2(\Omega)}$$

A regular decomposition

Since $\mathcal{O}\backslash\bar{\Omega}$ is a Lipschitz domain, by the extension theorem of Necas, there exists an extension of $\phi|_{\mathcal{O}\backslash\bar{\Omega}}$, denoted by $\tilde{\phi} \in H^2(R^3)$, such that

$$\tilde{\phi} = \phi \in \mathcal{O}\backslash\bar{\Omega}, \|\tilde{\phi}\|_{H^2(R^3)} \leq C\|\phi\|_{H^2(\mathcal{O}\backslash\bar{\Omega})} \leq C\|\mathbf{v}\|_{H(curl;\Omega)}.$$

This completes the proof by letting $\psi = \phi - \tilde{\phi} \in H_0^1(\Omega)$ and $\mathbf{v}_s = \mathbf{w} + \nabla\tilde{\phi}$. Remember that $\tilde{\mathbf{v}} = \mathbf{v} + \nabla\psi$ in $\mathcal{O}$ and $\mathbf{v}_s = \tilde{\mathbf{v}} = 0$ in $\mathcal{O}\backslash\bar{\Omega}$. Thus $\mathbf{v}_s \in H^1(\Omega)^3 \cap H_0(curl;\Omega)$.

A posteriori error estimate

Theorem: Let $\mathbf{E} \in H_0(curl;\Omega)$ and $\mathbf{E}_h \in X_h^0$ be respectively the solutions of (1) and (2). We have the following a posteriori error estimate

$$\|\mathbf{E}-\mathbf{E}_h\|_{H(curl;\Omega)} \leq C\Big(\sum_{K\in\mathcal{M}_h} \eta_K^2\Big)^{1/2}$$

where

$$\begin{aligned}\eta_k^2 &= h_K^2\|f-\nabla\times(\alpha\nabla\times\mathbf{E}_h)-\beta\mathbf{E}_h\|_{L^2(K)}^2 + h_K^2\|\nabla\cdot(\mathbf{f}-\beta\mathbf{E}_h)\|_{L^2(K)}^2 \\ &+ \sum_{F\subset\partial K}\big(h_F\|[[\mathbf{n}\times(\alpha\nabla\times\mathbf{E}_h)]]\|_{L^2F}^2 + h_F\|[[(\mathbf{f}-\beta\mathbf{E}_h)\cdot\mathbf{n}]]\|_{L^2(F)}^2\big)\end{aligned}$$

Proof.By (1) and (2) we know that

$$a(\mathbf{E}-\mathbf{E}_h,\mathbf{v}_h)=0,\ \forall\mathbf{v}_h\in X_h^0.$$

For any $\mathbf{v}\in H_0(curl;\Omega)$, by the regular decomposition theorem, there exists a $\psi\in H_0^1(\Omega)$ and a $\mathbf{v}_s\in H^1(\Omega)^3\cap H_0(curl;\Omega)$ such that $v=\nabla\psi+\mathbf{v}_s$,and

$$\|\psi\|_{H^1(\Omega)}+\|\mathbf{v}_s\|_{H^1(\Omega)}\le C\|\mathbf{v}\|_{H(curl;\Omega)}. \tag{4}$$

Let $r_h:H^1(\Omega)\to V_h^0$ be the Clément interpolant defined before, and define

$$\mathbf{v}_h=\nabla r_h\psi+\Pi_h\mathbf{v}_s\in X_h^0$$

By integrating by parts, we have

$$
\begin{aligned}
& a(\mathbf{E}-\mathbf{E}_h, \mathbf{v}-\mathbf{v}_h) \\
= \; & (\mathbf{f}, \mathbf{v}-\mathbf{v}_h) - (\alpha\nabla\times\mathbf{E}_h), \nabla\times(\mathbf{v}-\mathbf{v}_h)) - (\beta\mathbf{E}_h, \mathbf{v}-\mathbf{v}_h) \\
= \; & (\mathbf{f}, (\nabla\psi+\mathbf{v}_s) - (\nabla r_h\psi + \Pi_h\mathbf{v}_s)) - (\alpha\nabla\times\mathbf{E}_h, \nabla\times(\mathbf{v}_s-\Pi_h\mathbf{v}_s)) \\
& -(\beta\mathbf{E}_h, \mathbf{v}_s-\Pi_h\mathbf{v}_s) - (\beta\mathbf{E}_h, \nabla(\psi-r_h\psi)) \\
= \; & \sum_{K\in\mathcal{M}_h} (\mathbf{f}-\nabla\times(\alpha\nabla\times\mathbf{E}_h)-\beta\mathbf{E}_h, \mathbf{v}_s-\Pi_h\mathbf{v}_s)_K \\
& + \sum_{F\in\mathcal{F}_h} \int_F [[\mathbf{n}\times(\alpha\nabla\times\mathbf{E}_h)]]\cdot(\mathbf{v}_s-\Pi_h\mathbf{v}_s) \\
& - \sum_{K\in\mathcal{M}_h} (\nabla\cdot(\mathbf{f}-\beta\mathbf{E}_h), \psi-r_h\psi)_K + \sum_{F\in\mathcal{F}_h} \int_F [[(\mathbf{f}-\beta\mathbf{E_h})\cdot\mathbf{n}]](\psi-r_h\psi)
\end{aligned}
$$

A posteriori error estimate

Now with the help of BHHW interpolation operator and (4), we have

$$\begin{aligned} a(\mathbf{E}-\mathbf{E}_h,\mathbf{v}) &= a(\mathbf{E}-\mathbf{E}_h,\mathbf{v}-\mathbf{v}_h) \\ &\leq C\Big(\sum_{K\in\mathcal{M}_h}\eta_K^2\Big)^{1/2}\big(\|\mathbf{v}_s\|_{H^1(\Omega)}+|\psi|_{H^1(\Omega)}\big) \\ &\leq C\Big(\sum_{K\in\mathcal{M}_h}\eta_K^2\Big)^{1/2}\|\mathbf{v}\|_{H(curl;\Omega)}. \end{aligned}$$

The proof is completed by taking $\mathbf{v}=\mathbf{E}-\mathbf{E}_h$.

Outline

An Adaptive Finite Element Method for Wideband Impedance Extraction

The governing equation of eddy current field is magneto-quasi-static Maxwell's equations:

$$
\begin{aligned}
\nabla \times \mathbf{E} &= -i\omega\mathbf{B} \qquad (5)\\
\nabla \times \mathbf{H} &= \mathbf{J} + \mathbf{J}_s \qquad (6)
\end{aligned}
$$

$$\mathbf{E}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \mathbf{H}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty.$$

Where $\mathbf{B} = \mu\mathbf{H}, \mathbf{J} = \sigma\mathbf{E}$, μ and σ are magnetic permeability and electric conductivity respectively.

A-ϕ formulation

Dimensionless($\mathbf{x}' = \mathbf{x}/s$) version:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A} + is^2\sigma\mu\omega\mathbf{A} &= -s\sigma\mu\nabla\phi_0 + s^2\mu\mathbf{J}_s \quad \text{in } \Omega. && (7) \\ \mathbf{A} \times \mathbf{n} &= 0 \quad \text{on } \Gamma. && (8) \end{aligned}$$

Lemma The solution $\mathbf{A}$ of (7)-(8) is unique in Ω_c. Moreover, the eddy current $\mathbf{J} = \sigma\mathbf{E} = \sigma(-i\omega\mathbf{A} - s^{-1}\nabla\phi_0)$ depends only on the voltage U_j on the electrodes $S_j, j = 1, \cdots, N$, and is independent of the particular form of the function $\phi_0 \in H^1(\Omega)$ such that $\phi_0 = U_j$ on S_j, $j = 1, \cdots, N$.

Weak formula

Define the sesquilinear form

$$a(\mathbf{A}, \mathbf{G}) = (\nabla \times \mathbf{A}, \nabla \times \mathbf{G}) + is^2\omega\mu(\sigma\mathbf{A}, \mathbf{G})_{\Omega_c}.$$

The weak formulation of the problem (7)-(8) then reads as follows: Find $\mathbf{A} \in H_0(\mathbf{curl}; \Omega)$ such that

$$a(\mathbf{A}, \mathbf{G}) = -s\mu(\sigma\nabla\phi_0, \mathbf{G})_{\Omega_c} + s^2\mu(\mathbf{J}_s, \mathbf{G}), \qquad \forall \mathbf{G} \in H_0(\mathbf{curl}; \Omega). \tag{9}$$

Finite element approximation

Let $\mathcal{M}_h$ be a regular tetrahedral triangulation of Ω and $\mathcal{F}_h$ be the set of faces not lying on Γ.

The Nédélec lowest order edge element space $\mathbf{U}_h$ over $\mathcal{M}_h$ is:

$$
\begin{aligned}
\mathbf{U}_h := \quad & \{\mathbf{u} \in H(\mathbf{curl};\Omega) : \mathbf{u} \times \mathbf{n}|_\Gamma = \mathbf{0} \quad \text{and} \\
& \mathbf{u}|_T = \mathbf{a}_T + \mathbf{b}_T \times \mathbf{x} \quad \text{with} \quad \mathbf{a}_T,\ \mathbf{b}_T \in \mathbb{R}^3, \quad \forall T \in \mathcal{M}_h\}.
\end{aligned}
$$

The finite element approximation to (7)-(8) is: Find $\mathbf{A}_h \in \mathbf{U}_h$ such that

$$
a(\mathbf{A}_h, \mathbf{G}_h) = -s\mu(\sigma\nabla\phi_0, \mathbf{G}_h)_{\Omega_c} + s^2\mu(\mathbf{J}_s, \mathbf{G}_h), \quad \forall \mathbf{G}_h \in \mathbf{U}_h. \tag{10}
$$

The solution of the problem (10) is not unique.

A posteriori error analysis

Theorem: Let $\mathbf{A}$ be the solution of (9) and $\mathbf{A}_h$ be the solution of (10). There exists a constant C depending only on the minimum angle of the mesh $\mathcal{M}_h$ and the size of the domain Ω_{nc} such that

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{L^2(\Omega)} + \|\mathbf{A} - \mathbf{A}_h\|_{L^2(\Omega_c)} \leq C \min(1, \alpha)^{-1} \left(\sum_{T \in \mathcal{M}_h} \eta_T^2 \right)^{1/2},$$

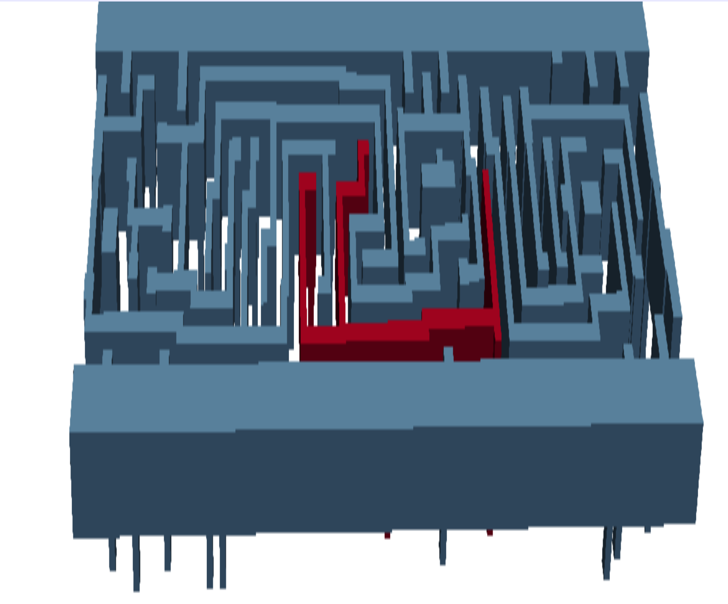
where $\alpha = \sqrt{s^2 \omega \sigma \mu}$.

Theorem(lowerbound) There exists a constant C depending only on the minimum angle of the mesh $\mathcal{M}_h$ such that for any $T \in \mathcal{M}_h$,

$$\eta_T \leq C \left(\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|_{L^2(\tilde{T})} + \alpha^2 \|\mathbf{A} - \mathbf{A}_h\|_{L^2(\Omega_c \cap \tilde{T})} + \sum_{T \subset \tilde{T}} h_T \|\mathbf{f} - Q_h \mathbf{f}\|_{L}\right.$$

where $\tilde{T}$ is the union of T and the adjacent elements of T, $\mathbf{f} = s^2 \mu (\mathbf{J}_s - s^{-1} \sigma \nabla \phi_0)$, and $Q_h : L^2(T) \to P_1(T)$ is the L^2 projection to the space of linear functions on T.

Numerical Example



Numerical example

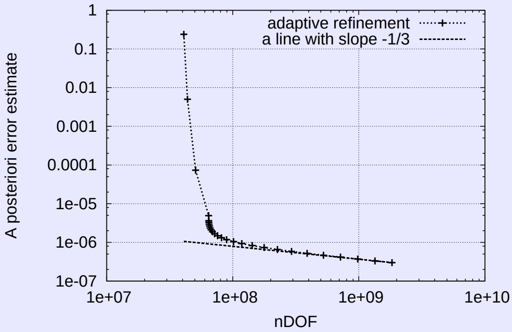


Figure: `the a posteriori error estimate` $(f = 1\text{GHz})$

Numerical example

CPUs	DOFs	Time(s)	Its	Efficiency for single step
32	50395098	402.9	8	100%
64	63486700	321.2	7	69.1%
128	130965380	359.2	8	72.8%
256	295872484	573.3	10	64.4%
512	522404008	555.5	12	70.4%
1024	1037479176	896.4	13	46.9%

Table: Parallel Scalability (Adder Circuit, frequence $f = 10GHz$, 1 OpenMP threads for each MPI process)

Some references

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