Mathematical Modeling of Behavior (2024-2025)

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Choice theory – supplement (correction) — September 17, 2024

Question 1: Cobb-Douglas demand function

In this exercise, we address the following question: given all the possible values of q_1 and q_2, which specific quantity of q_1 and quantity of q_2 does the consumer choose? The behavioral assumption is that the consumer wants to maximize her utility. What stops her from consuming an infinite number of goods? These goods have prices and the consumer has a limited budget (I) to spend on the goods.

Consumer behavior can be expressed as an optimization problem where the consumer selects the quantities q_1 and q_2 that maximize her utility $\widetilde{U}$ and are compatible with her available budget I:

$$\max_{q_1, q_2} \widetilde{U} = \theta_0 q_1^{\theta_1} q_2^{\theta_2} \tag{1}$$

subject to

$$p_1 q_1 + p_2 q_2 = I. \tag{2}$$

The optimal solution of this optimization problem verifies the necessary optimality conditions, based on the Lagrangian function:

$$L(q_1, q_2, \lambda; \theta) = \theta_0 q_1^{\theta_1} q_2^{\theta_2} - \lambda(p_1 q_1 + p_2 q_2 - I), \tag{3}$$

where λ is the Lagrange multiplier.

The Lagrangian somehow turns a constrained optimization problem (1)–(2) into an unconstrained optimization problem where the objective function is (3). In this way, the necessary optimality conditions for unconstrained optimization apply: the first derivatives are equal to zero. Here, the Lagrangian has three unknowns: q_1, q_2 and the Lagrange multiplier λ. Therefore,

$$\partial L/\partial q_1 = \theta_0 \theta_1 q_1^{\theta_1 - 1} q_2^{\theta_2} - \lambda p_1 = 0, \tag{4}$$

$$\partial L/\partial q_2 = \theta_0 \theta_2 q_1^{\theta_1} q_2^{\theta_2 - 1} - \lambda p_2 = 0, \tag{5}$$

$$\partial L/\partial \lambda = p_1 q_1 + p_2 q_2 - I = 0. \tag{6}$$

Multiplying (4) by q_1 and (5) by q_2, we have

$$\theta_0 \theta_1 q_1^{\theta_1} q_2^{\theta_2} - \lambda p_1 q_1 = 0, \tag{7}$$

$$\theta_0 \theta_2 q_1^{\theta_1} q_2^{\theta_2} - \lambda p_2 q_2 = 0. \tag{8}$$

Adding the two and using (6) we obtain

$$\lambda I = \theta_0 q_1^{\theta_1} q_2^{\theta_2}(\theta_1+\theta_2) \tag{9}$$

or, equivalently,

$$\theta_0 q_1^{\theta_1} q_2^{\theta_2} = \frac{\lambda I}{(\theta_1+\theta_2)}. \tag{10}$$

Using (10) in (7), we obtain

$$\frac{\lambda p_1 q_1}{\theta_1} = \frac{\lambda I}{(\theta_1+\theta_2)}. \tag{11}$$

Solving (11) for q_1, we obtain

$$q_1^* = \frac{\theta_1}{(\theta_1+\theta_2)}\frac{I}{p_1}. \tag{12}$$

Similarly, we obtain

$$q_2^* = \frac{\theta_2}{(\theta_1+\theta_2)}\frac{I}{p_2}. \tag{13}$$

Note that the constraints $q_1, q_2 \geq 0$ should also have been included in the optimization problem (1)–(2). As the parameters θ are positive, if the budget is non zero, the optimal quantities are positive, and these constraints are not active at the solution. Therefore, it was appropriate to ignore them. The Cobb-Douglas function has the property that the demand for a good is only dependent on its own price and independent of the price of any other good, which is a fairly restrictive assumption. The equations can also be solved for the third unknown, the Lagrange multiplier λ:

$$\lambda = \theta_0(\theta_1+\theta_2)\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{\theta_1}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^{\theta_2}\frac{I^{(\theta_1+\theta_2-1)}}{p_1^{\theta_1}p_2^{\theta_2}} \tag{14}$$

The parameter λ is not just a nuisance parameter but has a useful interpretation. Its value is the marginal utility of income, that is the increase in utility that results if income is increased by one unit. Equivalently, λ is equal to the marginal utility of good ℓ ($\partial\widetilde{U}/\partial q_\ell$) divided by the marginal cost of good ℓ (equal to p_ℓ in this example) for all goods, or

$$\lambda = \frac{\partial\widetilde{U}/\partial q_\ell}{p_\ell} \quad \text{for all goods } \ell. \tag{15}$$

The above equation is directly derived from (4) and (7), that can be written as

$$\partial L/\partial q_\ell = \partial\tilde{U}/\partial q_\ell - \lambda p_\ell = 0. \tag{16}$$

Equation (15) is often described as an optimality condition. Conceptually, at optimal consumption each good should yield the same marginal utility per monetary unit spent. At optimality, if given one extra unit of income to spend, the consumer is indifferent as to which good to purchase more. If the consumer is not indifferent, then she was not at optimality and should adjust her consumption bundle towards the preferred good. The optimality conditions can also be rearranged to state that the marginal rate of substitution of good i for good j is equal to the ratio of the marginal costs of good i relative to good j. For the two commodity case and linear budget constraint, this optimality condition is obtained by calculating the ratio of (15) for $\ell = 1$ and $\ell = 2$ as

$$\frac{\partial\widetilde{U}/\partial q_1}{\partial\widetilde{U}/\partial q_2} = \frac{p_1}{p_2}. \tag{17}$$

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Question 2: Derivation of a choice model

The CDF of the error terms is given by

$$F_\varepsilon(\varepsilon_i, \varepsilon_j) = e^{-e^{-\varepsilon_i}} e^{-e^{-\varepsilon_j}}. \tag{18}$$

We have

$$P(i|\{i,j\}) = \int_{\varepsilon=-\infty}^{+\infty} \frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon, V_i - V_j + \varepsilon) d\varepsilon. \tag{19}$$

From (18), we have

$$\frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon_i, \varepsilon_j) = e^{-e^{-\varepsilon_i}} e^{-e^{-\varepsilon_j}} e^{-\varepsilon_i}. \tag{20}$$

Therefore,

$$\frac{\partial F_\varepsilon}{\partial \varepsilon_i}(\varepsilon, V_i - V_j + \varepsilon) = e^{-e^{-\varepsilon}} e^{-e^{-(V_i - V_j + \varepsilon)}} e^{-\varepsilon} = e^{-e^{-\varepsilon}} e^{-Ke^{-\varepsilon}} e^{-\varepsilon} \tag{21}$$

where

$$K = \exp(-(V_i - V_j)). \tag{22}$$

Therefore,

$$P(i|\{i,j\}) = \int_{\varepsilon=-\infty}^{+\infty} e^{-e^{-\varepsilon}} e^{-Ke^{-\varepsilon}} e^{-\varepsilon} d\varepsilon. \tag{23}$$

Define

$$t = -e^{-\varepsilon}, \; dt = e^{-\varepsilon} d\varepsilon,$$

to obtain

$$P(i|\{i,j\}) = \int_{t=-\infty}^{0} e^{(1+K)t} dt = \frac{1}{1+K}. \tag{24}$$

Using (22), we obtain the simple expression:

$$P(i|\{i,j\}) = \frac{1}{1 + \exp(-(V_i - V_j))} = \frac{e^{V_i}}{e^{V_i} + e^{V_j}}. \tag{25}$$

This happens to be the binary logit model.