Quantum Analog

Ricardo Guerra, Nora Colunga
University of Texas Rio Grande Valley

10/01/2021

The purpose of this experiment is to make the complex idea of Quantum Mechanics a bit easier to conceptualize, by introducing a few topics and equations that surround the topic.

Introduction

In 1923 **Louis de Broglie** introduced the idea that electrons, like previously seen with light, can also behave as a wave and a particle. This led of course to a Nobel prize in 1929 establishing the foundation for the future of quantum mechanics.

$$\lambda = h/p \tag{1}$$

Where:

- λ is the wavelength
- h is *Planck's Constant*
- p is momentum

Which could also be written as.

$$\lambda = h/mv \tag{2}$$

This contribution to science was paramount, and led to Heisenberg principle and Schrodinger's equation. Waves exhibit similar behaviors, and because of de Broglie's contributions, we are able to make the analogous connection between sound waves and the waves that describe the probability of the location of an electron using the hydrogen wave equation.

What's in a Wave

What exactly is a wave? Commonly known as oscillations, to call something a wave is to describe a disturbance in matter. This includes sound waves, water waves, light waves, etc.

To describe a wave we use mathematics. Giving us the equation for the wave function:

$$y(x,t) = A\sin(kx - \omega t + \phi_0) \tag{3}$$

In Which:

- k is the wave number or the direction of wave propagation
- ω is the angular frequency in radians per second
- ϕ_0 is a phase factor

This equation can especially be used to represent a standing wave in a tube that will later be used in the first experiment.

In this experiment we're focusing on the amplitude of a wave. We'll be using equipment to produce a standing wave, a wave who's amplitude changes but doesn't move through time. A standing wave is produced by having two waves travel in opposite directions. It is also important to note that the wave must be orthonormal for the purposes of the two experiments. This means that, from negative infinity to positive infinity, the integral of the product of the wave equation and its complex conjugate is one.

Probability

In quantum mechanics due to electrons behaving as waves there is no certainty that we can find an

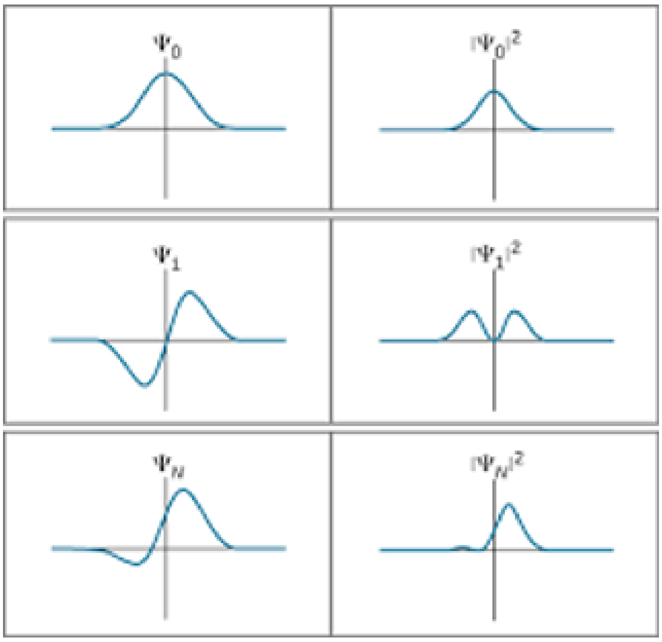

Figure 1: *Example Wave Functions*

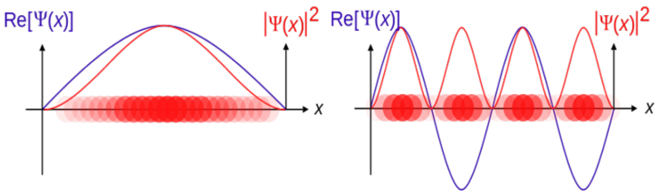

Figure 2: *Probability Distribution.*

electron at any given point. We can, however, use probability to determine the likelihood of finding it around the nucleus. *Figure 4* shows a visual representation of what probability would look like depending on the amplitudes of a wave.

As you can see $|\psi(x)|^2$ is how to find the probability of finding the electron at this position.

Schrodinger's Equation

This equation features an i for the imaginary part times *Planck's constant*. Rate of change, in respect to time of the *wave function*.

$$i\hbar\frac{d}{dt}\Psi(\vec{r},t) = \hat{H}\Psi(\vec{r},t) \tag{4}$$

The end product being the Hamiltonian operator which in this case is time independent. The full equation for an electron along with the potential *V(r)*:

$$\frac{-\hbar^2}{2m}\Delta\Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t) = i\hbar\frac{d}{dt}\Psi(\vec{r},t) \tag{5}$$

Where the first part is the *kinetic energy* of a system plus the *potential energy* of the system. As can be seen:

$$E\Psi = \hat{H}\Psi \tag{6}$$

This equation is particularly important because it uses a complex number to describe a real physical property.

Infinite Potential Well

The infinite potential well is a setup in quantum where in some arbitrary length, the potential is zero and anywhere else it is infinite as shown in Figure 4.

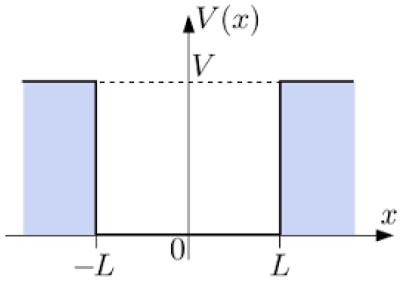

Figure 3: *Infinite Potential Well*

Because an infinite potential couldn't work in Schrodinger's equation, we are left with using *V(r)*=0 which gives us

$$\frac{-\hbar^2}{2m}\Delta\Psi(x,t) = i\hbar\frac{d}{dt}\Psi(x,t) \tag{7}$$

Equation (7) may have solutions that tend to be time dependent charge densities while also having time independent charge density. To find the time independent charge density, the following time-independent Schrodinger equation must be used.

$$\frac{-\hbar^2}{2m}\Delta\Psi(\vec{r}) + V(\vec{r})\Psi(\vec{r}) = E\Psi(\vec{r}) \tag{8}$$

And as previously mentioned, the potential between the barriers of the well is nonexistent, so the equation simplifies to

$$\frac{-\hbar^2}{2m}\Delta\Psi(x) = E\Psi(x) \tag{9}$$

The general solution to this differential equation is

$$\psi(x) = A\sin(kx+\alpha) \tag{10}$$

The ends of the tube are nodes because in the infinite potential well, the waves reach a type of resonance just like sound resonance in the tubes in the experiment. Therefore, $\psi(0) = 0$ and $\psi(L) = 0$, so $\alpha = 0$ and $k = n\pi/L$, where n is an integer.

From the probability section, we know that we must square the wave equation, set it equal to one and find the amplitude in this manner to make the wave equation orthonormal. After those calculations, we get $A = \sqrt{\frac{2}{L}}$.

Quantum Numbers

Quantum numbers are used to represent different aspects of the position of an electron at any given time. Since it was established that electrons have wave-like properties, and **Bohr's model is inaccurately practical**, we cannot be certain where it will be located. We can label these aspects in different ways:

- n describes the energy level (1, 2, 3,...)
- l describes the angular momentum (0, 1, 2,...)
- m_l describes orbital orientation ($-l$,...,l)
- m_s describes the magnetic spin ($-\frac{1}{2}, \frac{1}{2}$)

Hydrogen Wave Equation

The hydrogen wave equation is a solution for the time independent Schrodinger equation. In its simplified version, it is:

$$\Psi(r, \theta, \phi) = R_{n,l}(r) Y_{l,m}(\theta, \phi) \tag{11}$$

with $R_{n,l}(r)$ describes how far the electron being observed is around the proton and $Y_{l,m}(\theta, \phi)$ provides information about where the electron is around the proton. A fleshed out version of $R_{n,l}(r)$ is:

$$R_{n,l}(r) = \sqrt{(\frac{2}{na_0})^3 \frac{(n-l-1)!}{2n(n+l)!}} \exp\frac{-r}{na_0} (\frac{2r}{na_0})^l \mathcal{L}_{n-l-1}^{2l+1}(\frac{2r}{na_0}) \tag{12}$$

Breaking this all down, the n and l are the aforementioned quantum numbers, the energy levels and the angular momentum, respectively, of the electron being observed. The constant a_0 is the reduced Bohr radius and can be represented as:

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2} \approx 5.295 \times 10^{-11}\text{m} \tag{13}$$

and the $\mathcal{L}_a^b(r)$ are the associated Laguerre Polynomials:

$$\mathcal{L}_a^b(r) = \frac{r^{-b}}{a!} \exp(r) \frac{\mathrm{d}^a}{\mathrm{dr}^a} (\exp(r) r^{a+b}) \tag{14}$$

A fleshed out version of $Y_{l,m}(\theta, \phi)$ is:

$$Y_{l,m}(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} (-1)^m \exp(-im\phi) P_l^m(\cos\theta) \tag{15}$$

where P_l^m are the associated Laguerre Polynomials:

$$P_l^m(x) = \frac{(-1)^l}{2^l l!} (1-x^2)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{dx}^{l+m}} (x^2-1)^l \tag{16}$$

Because everything else are just constants, the spherical harmonics $Y_l^m(\theta, \phi)$ can be simplified to, as stated in Teachspin's manual,

$$Y_l^m(\theta, \phi) \propto P_l^m(cos\theta) e^{im\phi} \tag{17}$$

and the aforementioned m quantum number is merely the orientation of the wave and the speaker creates waves with cylindrical symmetry about the speaker axis. So $e^{im\phi}$ becomes 1 and $P_l^m(cos\theta)$ becomes

$$Y_l^0(\theta, \phi) \propto P_l^0(cos\theta). \tag{18}$$

Now, we are left to depend on only one angle, θ, simplifying the experiment to associating the spherical harmonics to the Legendre Polynomials.

Analogies

Because of the wave properties that both sound waves and topics in quantum mechanics exhibit, there are analogies between both the infinite potential well and the standing waves in a tube and the hydrogen atom with a spherical resonator. In quantum mechanics the wave-function isn't exactly a wave but a probability distribution as already mentioned. However, the distribution has a specific pattern and using sound waves, which are space dependant, we can see these distributions.

Analogy to a quantum mechanical particle in a box

Schrodinger's equation from equation (5) can serve to describe the particle in the box as previously mentioned which then simplifies to equation (7) because of the nonexistent potential within the barriers in the infinite potential well.

The sound in the tube has barriers; we assume that there is no transmission of sound energy past the ends of the tubes and that it's completely reflected. This phenomenon also occurs in the infinite potential well as previously explained. Sound energy and the particle in the box also have solutions to Schrodinger's equation.

In sound, the amplitude is the intensity while the amplitude is indicative of the probability of finding an electron at that point.

Experimental Design

The goal of this experiment is to help make the concept of quantum orbitals easier to comprehend. We will be using sound as a wave analog to conduct the experiment and gather measurements. By using sound waves as analogs to the wave-function we would be able to see that although the electron position could be random, these probabilities can be quantified into specific locations.

Standing Sound Waves in a Tube

A microphone will be providing a sound and our work will include using the frequency to match this wave and produce a standing wave. Using the concept of *'particle in a box'* we will keep a wave bouncing within the potential barriers which are used to represent orbitals. In the 1-dimensional model it should become readily apparent that our bouncing wave should be easier to match the sound made by the microphone.

Modeling a Hydrogen Atom with a Spherical Resonator

For the 3-dimensional model we will also use l to find resonance. Plotting the gathered data and if done correctly should produce the speed of sound for the first part of the experiment. Using the hydrogen 3d model we should be able to find the quantum numbers and orbitals for the second part for the experiment.

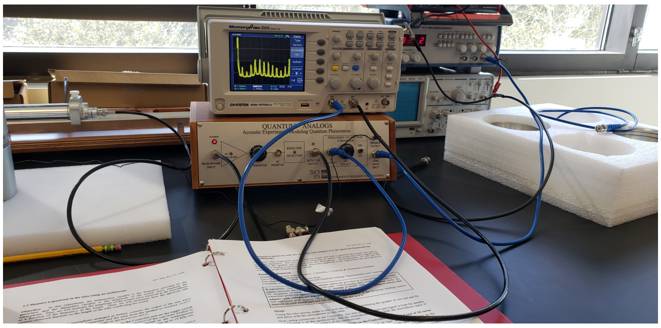

Figure 4: *Equipment Setup*

Equipment

For the experiment we will be using 2 models to represent analogy to a wave-function. The first model consists of 75mm hollow tubes to represent the atom in 1-dimension. The second part of the equipment is a 3-dimensional model of a hydrogen atom. We will be using a digital oscilloscope and a function generator to provide a frequency will be using to gather our data. This experiment requires us to use a Quantum analog box produced by *TeachSpin* which has the microphone speaker, a sine wave generator, and attenuating dial.

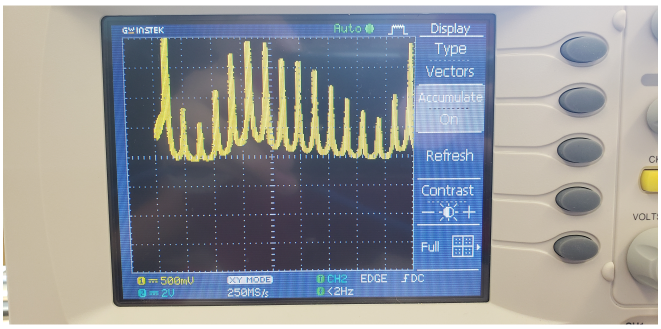

Figure 5: *Resonance*

[h]

Table 1: *1-Dimensional Data: 600mm*

	Frequency in Hz	
Resonance (n)	Theoretical	Experimental
1	290	288.8
2	573	572
3	859	854.1
4	1144	1147.8
5	1427	1522
6	1790	1786
7	2050	2040

Table 2: *1-Dimensional Data: 575mm*

	Frequency in Hz	
Resonance (n)	Theoretical	Experimental
1	326.67	327.2
2	653.3	643.8
3	980	978.7
4	1306.7	1314.6

Experimental Observations/Data

Experimental Analysis

1-Dimensional Data

The data gathered in *Table 1* shows each instance of a resonance which is the attributed as the value for n. Each frequency was recorded and compared against the theoretical value. Using the equation:

$$2L = n\frac{c}{f} = n\lambda \tag{19}$$

As can be seen all *experimental* values are well within the range of the theoretical values. The slight offset could be attributed to things such as the temperature of the tubes, pressure of the room, or noise within range. When the values were plotted which can be seen in *Figure 9* give us a slope which represents $\delta f/\delta n$. We can then get the slope and plug it in to $c = 2L * f$ and get approximately 357.8 m/s which is quite faster than the speed of light.

3-Dimensional Data

Using the makeshift 'Hydrogen' atom we were able to obtain data throughout a different set of frequencies. Resonance was obtained by sweeping the top half of the atom across various angles (α). Throughout the experiment at roughly around 90°it wasn't unusual for the oscillator to go wacky. This represented a change in amplitude. As demonstrated in *Figure 8.*

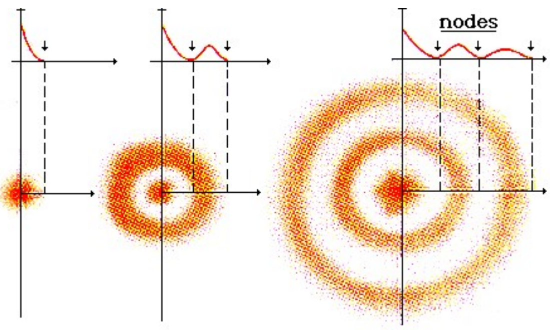

Figure 6: *Nodes*

These nodes are the potential barriers within each orbital. The likelihood of finding an electron between these nodes is higher compared to the cloud in the rest of the orbit. *Table 2* shows the measurements obtained of α translated to θ using the equation:

$$\theta = arccos(\frac{1}{2}cos\alpha - \frac{1}{2}) \tag{20}$$

Table 3: $3 - Dimensional Amplitude$

Amplitude (au)						
—	90°	105	120°	135	150°	180°
2315	0.3	0.1	-0.2	-0.6	-0.8	-1
3678	-0.5	-0.4	-0.1	0.1	0.5	1
4873	-0.2	0.3	0.5	0.5	-0.2	-1

This table also shows the frequency at which resonance was obtained. The data points when plotted show the sphere harmonics of the electron. If the gathered data is viable we should, once plotted, able to obtain our *quantum number l* for each of the resonances.

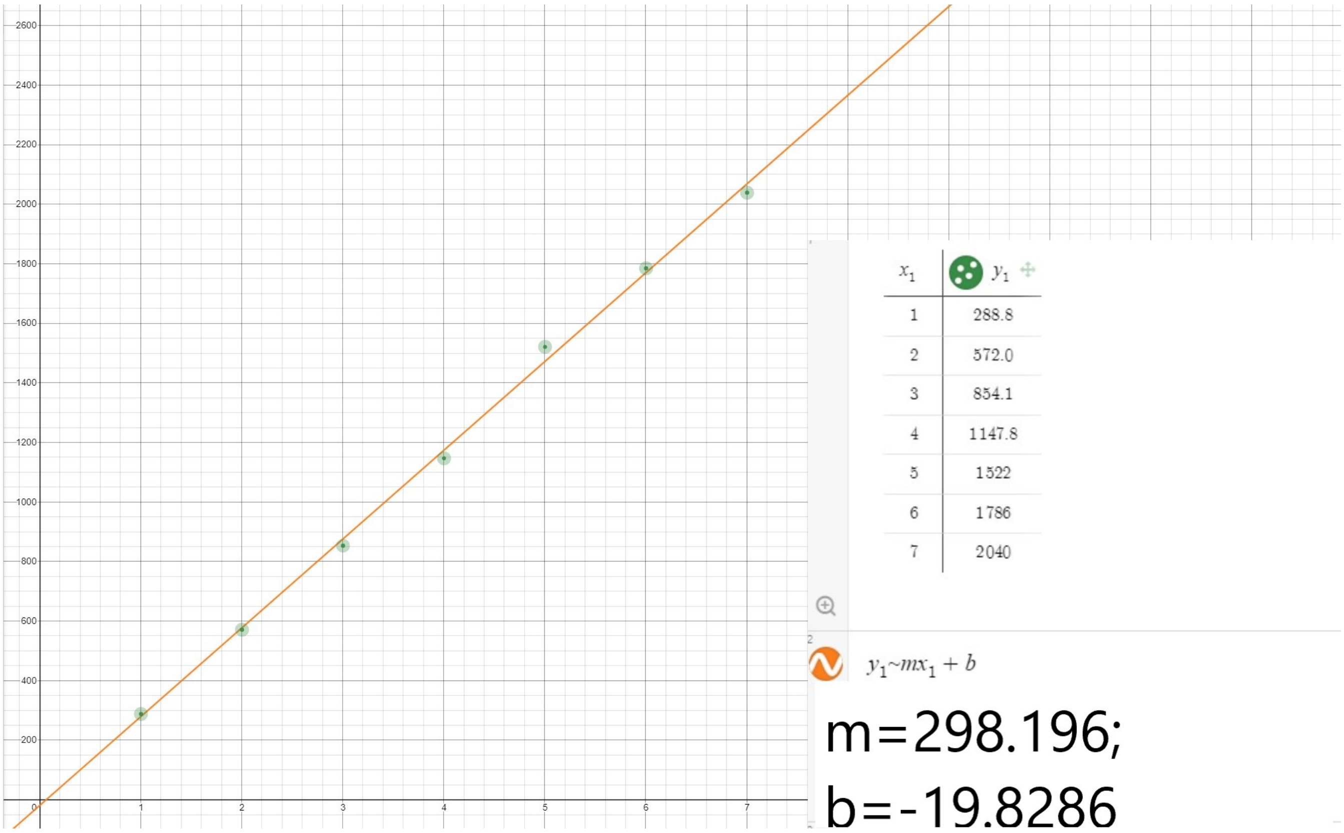

Figure 7: *1-Dimensional Plot Data*

Conclusion

For the one-dimensional experiment, it was evident that there were analogies between the infinite potential wall and the classical sound wave in the tube. The similarities mostly arise when you compare their waves' abilities to fit in Schrodinger's equation. Experiment-wise, the similarities can be seen from the ends of the tube being the "walls" of the infinite potential well. How the sound resonances represent the different resonances that can occur in it. As for the hydrogen model part of the experiment, the analogies are present in the sound resonances and their energies at those resonances with the energy levels of the electrons in their "electron cloud." This can be seen with the relationship between the amplitude, which more or less depicts the intensity of the sound, and the frequency in the polar graphs that are given in the previous pages. The plot points line up pretty consistently with the pre-established polar graphs of the angular momentum of the electron being observed. As can be seen using sound waves and controlling the angle at which it was transmitted we were able to find our quantum number as well as find that probability distributions behaving like wave patterns. This solidifies that although a wave-function is a measurement it does indeed have accurate space-time values. All in all, the connection between the wave-like properties of sound and the wave-like properties of phenomena in quantum mechanics are clearly present based on the two aforementioned experiments.

References

[1] Naiman, D., (2021). Sound Waves. *Mathematics of Music.* https://www.ams.jhu.edu/dan-mathofmusic/sound-waves/

[2] Electrons behaving as waves. *Institute of Physics.*, https://spark.iop.org/electrons-behaving-wavesgref

[3] Paul Reimer, *http://www.phy.anl.gov/mep/SeaQuest/*, 2014.

[4] Joseph C. Amato and Roger E. Williams, *AAPT Apparatus Competition*, 2008.

[5] R. Matzdorf. *Quantum Analogs-Acoustic Experiments Modeling Quantum Phenomena.* Universitaet Kassel