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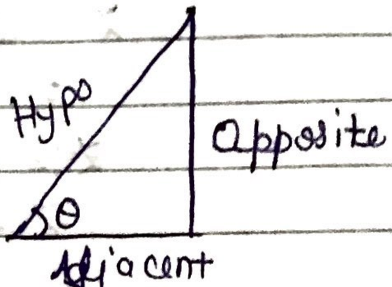
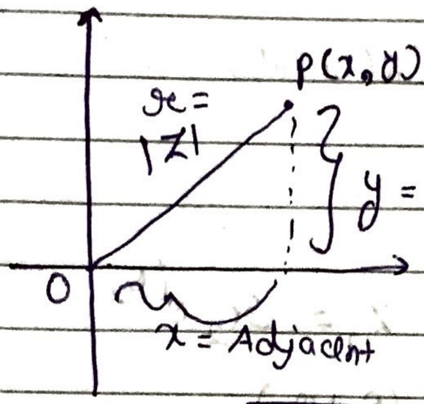
$\cos\theta = \dfrac{\cancel{\text{Adjust}}\ \text{Adjacent}}{\text{Hypotenuse}}$

$\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$





$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$r = |z| = \sqrt{x^2 + y^2}$

$\sin\theta = \dfrac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = \dfrac{x}{r} \Rightarrow x = r\cos\theta$

$z = x + iy$
$= r\cos\theta + i\sin\theta$

$\boxed{z = r(\cos\theta + i\sin\theta)} \rightarrow$ Euler's formula

* $e^{i\theta} = 1 + i\theta + \dfrac{(i\theta)^2}{2!} + \dfrac{(i\theta)^3}{3!} + \dfrac{(i\theta)^4}{4!} + \dfrac{(i\theta)^5}{5!} + \ldots$

$= 1 + i.\theta - \dfrac{\theta^2}{2!} - i\dfrac{\theta^3}{3!} + \dfrac{\theta^4}{4!} + i\dfrac{\theta^5}{5!} + \ldots$

$= \left(1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \ldots\right) + i\left(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \ldots\right)$

$e^{i\theta} = \cos\theta + i\sin\theta$

Teacher's Signature

$z = r\cos\theta + ri\sin\theta$
$= r(\cos\theta + i\sin\theta)$
$\cancel{z =}$
$z = re^{i\theta}$

Q $z = \sqrt{3} + i$, Calculate z^{61}

★ Some properties:-

(i) $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

Multiplication of two Complex numbers - $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ → Arguments get added up in multiplication of two complex numbers.

(ii) $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$

Division of two Complex Numbers - $= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Q (i) $z = 1 + i$
$r = \sqrt{1^2+1^2} = \sqrt{2}$
$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$(\sqrt{2}, \frac{\pi}{4})$
r, θ

(ii) $-1-i$
$r = \sqrt{2}$
$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \frac{\pi}{4}$

$(\sqrt{2}, \frac{\pi}{4})$
$r ; \theta$

Arg → Principle Argument
arg → general

But, $z = 1+i$ & $z = -1-i$ can not have same coordinates.

(iii) $z = 1-i$
$r = \sqrt{2}$
$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$
$\left(\sqrt{2}, -\frac{\pi}{4}\right)$

(iv) $z = -1+i$
$r = \sqrt{2}$
$\theta = -\frac{\pi}{4}$
$\left(\sqrt{2}, -\frac{\pi}{4}\right)$

Same case occurs in above two examples. To solve this issue, 'Argument' concept was introduced.

★ Argument :- → Argument → $\arg z = \text{Arg } z + 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$
→ Principal Argument

~~Principal argument → arg z~~

$$\text{Arg } z = \begin{cases} \tan^{-1} y/x & x \geq 0 \\ \tan^{-1} \frac{y}{x} + \pi & x < 0, y \geq 0 \\ \tan^{-1} y/x - \pi & x < 0, y < 0 \end{cases}$$

$-\pi < \arg z \leq \pi$

Remember technique:- if $x \to -ve$
(i) $y \to -ve \to$ Add '$-\pi$'
(ii) $y \to +ve \to$ Add 'π'

So, in $z = -1-i$
$\text{Arg } z = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$

& $z = -1 + i$

$$\text{Arg } z = \frac{-\pi}{4} + \pi = \frac{3\pi}{4}$$

Q (i) $\sqrt{3} + i$

$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(ii) $z = \sqrt{3} - i$

$\arg z = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$= -\frac{\pi}{6}$

(iii) $-\sqrt{3} - i$

$\arg z = \frac{\pi}{6} - \pi$

(iv) $-\sqrt{3} + i$

$\arg z = -\frac{\pi}{6} + \pi$

Properties :-

(i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(ii) $\text{Arg}(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2$ is not valid.

e.g. $z_1 = i$, $z_2 = -1$

$z_1 z_2 = -i$

$\text{Arg}(z_1 z_2) = -\frac{\pi}{2}$

$\text{Arg } z_1 = \text{Arg}(i) = \pi/2$

$\text{Arg } z_2 = \text{Arg}(-1) = 0 + \pi$

$\text{Arg } z_1 + \text{Arg } z_2 = \frac{3\pi}{2}$

(iii) $e^{i\theta} = \cos\theta + i \sin\theta$

$(e^{i\theta})^n = e^{in\theta}$

$= \cos n\theta + i \sin n\theta$

$(e^{i\theta})^n = (\cos\theta + i \sin\theta)^n$

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$\therefore \boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta} \rightarrow$ De-moivre Theorem

Q Prove De-Moiver's Theorem.

Ans By induction method,
For $n = 1$

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

For $n = 2$

$(\cos\theta + i\sin\theta)^2 = \cancel{\cos^2\theta + i}\cos\theta\cos\theta - \sin\theta\sin\theta$
$+ i\cos\theta\sin\theta + i\sin\theta\cos\theta$
$= \cos(\theta+\theta) + i\sin(\theta+\theta)$
$= \cos 2\theta + i\sin 2\theta$

Say, it is true for $n = k$
$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ — (1)

For $n = k+1$,
$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$
$= \cos k\theta\cos\theta - \sin k\theta\sin\theta$
$+ i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$
$= \cos(k+1)\theta + i\sin(k+1)\theta$

Teacher's Signature

Q. $z = (\sqrt{3}+i)^{61}$, Calculate z^{61}

Ans
$$= (2\, e^{\pi/6\, i})^{61}$$
$$= 2^{61}\, e^{10\pi i}\, e^{\pi/6\, i}$$
$$= 2^{61}\, e^{\pi/6\, i}$$
$$= 2^{61}\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$
$$= 2^{60}(\sqrt{3}+i)$$

Root of Complex Numbers:-

Let ω be n^{th} root of z

$$\boxed{\omega^n = z}$$

$$\omega = R(\cos\phi + i\sin\phi)$$
$$= R e^{i\phi}$$

$$z = r(\cos\theta + i\sin\theta)$$
$$= r e^{i\theta}$$

$$\omega^n = z \Rightarrow \omega = z^{1/n}$$

$$\Rightarrow R e^{i\phi} = (r e^{i\theta})^{1/n}$$

$$\Rightarrow R e^{i\phi} = \left[r e^{i(\theta + 2k\pi)}\right]^{1/n}$$
$$= r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$$

So, $\boxed{R = r^{1/n}}$

& $\boxed{\phi = \frac{\theta}{n} + \frac{2K\pi}{n}}$ → Significance of this term: It ~~produc~~ provides all roots of a given number for $K = 0, \pm1, \pm2 \ldots$

Q $(-1)^{1/3}$

Ans $z = -1 \Rightarrow z = e^{\pi i}$

$n = 3$

$\omega = z^{1/3} \Rightarrow \omega = e^{\pi i/3}$ (not adding $\frac{2K\pi}{n}$ term)

$\Rightarrow \omega = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3}$

↳ Clearly, only one root is visible when ignored $\frac{2K\pi}{n}$ term.

Now,

$\omega = \left(e^{\pi i + 2K\pi i}\right)^{1/3}$

$= e^{\pi i/3 + \frac{2K\pi i}{3}}$

$[K = 0, \pm1, \pm2, \pm3 \ldots]$

At $K=0$, $\omega_0 = e^{\pi i/3}$

At $K = 1$, $\omega_1 = e^{\pi i/3 + 2\pi/3\, i} = e^{\pi i}$

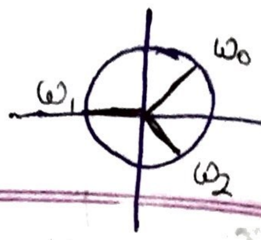
At $K = 2$, $\omega_2 = e^{\pi i/3 + 4\pi i/3} = e^{5\pi/3\, i}$

$\Bigg\}$ possible roots

At $K = 3$, $\omega_3 = e^{\pi i/3 + 6\pi/3\, i} = e^{\pi i/3} e^{2\pi i}$

$= e^{\pi i/3} = \omega_0$ → Repeat

~~$\omega_{-1} = e^{\pi/3 + 6\pi/3}$~~



$$\omega_{-1} = e^{\pi/3\, i - \frac{2\pi}{3} i}$$
$$= e^{-\pi i/3}$$
$$= e^{-\pi i/3 + 2\pi i}$$
$$= e^{5\pi/3\, i} = \omega_2 \rightarrow \text{Repeated}$$

<u>Note:-</u> All roots of a complex number occur at same angular distance (θ)

Root of Unity:-

$$(1)^{1/n} = (e^{0i})^{1/n} = \left(e^{0i + 2\pi k i}\right)^{1/n}$$

$$R = 1, \quad \phi = \frac{2k\pi}{n}$$

★ Sum & Multiplication of roots:-

$$Re^{i\phi} = [re^{i(\theta + 2k\pi)}]^{1/n} \qquad z = re^{i\theta}$$

$$\omega = \sqrt[n]{r}\; e^{i\theta/n}\; e^{\frac{2k\pi}{n} i}$$

$$\text{Let } d = e^{\frac{2\pi i}{n}}$$
$$d^k = e^{\frac{2k\pi i}{n}}$$

$$\omega_0 = \sqrt[n]{r}\; e^{i\theta/n}$$

$$\omega_1 = \sqrt[n]{r}\; e^{i\theta/n}\, e^{\frac{2\pi i}{n}} = \omega_0 d$$

$$\omega_2 = \sqrt[n]{r}\; e^{i\theta/n}\; e^{\left(\frac{2\pi i}{n}\right)^2}$$
$$= \omega_0 d^2$$

$\omega_3 = \omega_0 d^3$

Similarly,

last root $\rightarrow$ $\omega_{n-1} = \omega_0 d^{n-1}$

(i) Sum of roots :-

$= \omega_0 + \omega_1 + \omega_2 + \dots + \omega_{n-1}$

$= \omega_0 + \omega_0 d + \omega_0 d + \dots \omega_0 d^{n-1}$

$= \omega_0 (1 + d + d^2 + \dots d^{n-1})$

$= \omega_0 \left(\frac{1 - d^n}{1 - d} \right)$

$= \omega_0 \left(\frac{1-1}{1-d} \right)$ $\quad \left[d^n = \left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i} = 1 \right]$

$= \omega_0 \times 0 = 0$

(ii) Product of roots :-

$\cancel{\omega_0, \omega_1, \omega_2, \omega_3 \dots \omega_{n-1}}$

$= \omega_0 \omega_1 \omega_2 \omega_3 \dots \omega_{n-1}$

$= \omega_0 \omega_0 d \, \omega_0 d^2 \dots \omega_0 d^{n-1}$

$= \omega_0^n (d^{1+2+\dots n-1})$

$= \omega_0^n d^{\frac{n}{2}(n-1)}$

$= (\sqrt[n]{r} e^{i\theta/n})^n d^{\frac{n}{2}(n-1)}$

$= r e^{i\theta} \times e^{(n-1)\pi i}$

Teacher's Signature

$= z(-1)^{n-1}$ $\quad [\because (-1)^n = e^{n\pi i}]$

$= (-1)^{n-1} z$

Q Find all the roots of $z^3 = -8i$ & their representation graphically.

Ans $z^3 = -8i \Rightarrow z = (-8i)^{1/3}$

$= 2(-i)^{1/3}$

$\Rightarrow z = 2\left[e^{-\pi/2 i + 2k\pi i}\right]^{1/3}$

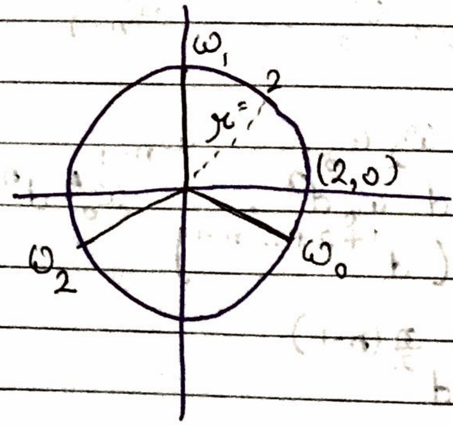
$= 2\left[e^{-\pi/6 i + \frac{2k\pi}{3} i}\right]$

$k = 0, 1, 2 \ldots$

For $k=0$, $\omega_0 = 2e^{-\pi/6 i} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

For $k=1$, $\omega_1 = 2e^{-\pi/6 i + \frac{2\pi}{3} i} = 2e^{\frac{3\pi}{6} i} = 2e^{\pi/2 i}$

For $k=2$, $\omega_2 = 2e^{-\pi/6 i + \frac{4\pi}{3} i} = 2e^{\frac{7\pi}{6} i}$



nbd :- all points in surrounding excluding centre point.

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Q Find all the roots & locate them graphically.

(a) $(-1+i)^{1/3}$ (b) $(-1-i)^{\frac{1}{4}}$

$\sqrt{2}\, e^{\frac{3\pi}{4} i}$

(c) $(-2\sqrt{3}-2i)^{1/4}$

Q (a) $z^5 = 32$ (b) $z^4 = (-2\sqrt{3}-2i)$

★ Somes definitions :-

(i) Neighbourhood (nbd) : A deleted nbd of a point z_0 is the set of all points z such that $|z-z_0| < \delta$, when δ is any positive real number.

$N_\delta(z_0) = \{z : |z-z_0| < \delta\}$ | Deleted nbd :- $\{z : 0 < |z-z_0| < \delta\}$

(ii) Interior point :- A point z_0 is called an interior point of a set S if we can find a δ- nbd of z_0 all of whose points belong to S.

[$\delta \rightarrow$ be any size]

(iii) Boundary :- If $\boxed{\text{every}}$ nbd of z_0 consists some point of S and some other tha[n] S.

(iv) Exterior point :- If there exists a nbd of z_0 which does not contain any point of S.

Teacher's Signature

connected

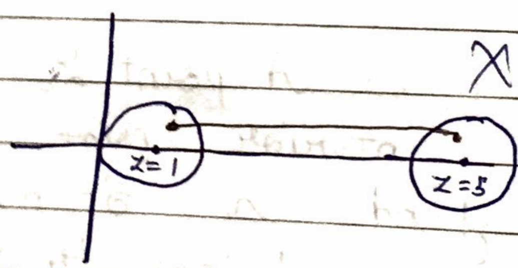
(v) Open set :- which consists only interior point.

(vi) Connected set :- If any two points of the set can be jointed by a path consisting of straight line segment all point of which one in S
[line segments has to be from set S]

(vii) Domain :- An open & connected set

Bounded set :- If there exists a constant m such that -
$|z| < m \quad \forall\ z \in S$

e.g. $S = \{z : |z-1| < 1 \text{ or } |z-5| < 1\}$



Not connected

★ Function :-

$f : A \to B$

Let A & B are two non-empty set then a rule which assigns each element of A to some element of B.

→ ~~unique~~ → ~~real number~~ system

→ some → Complex number system

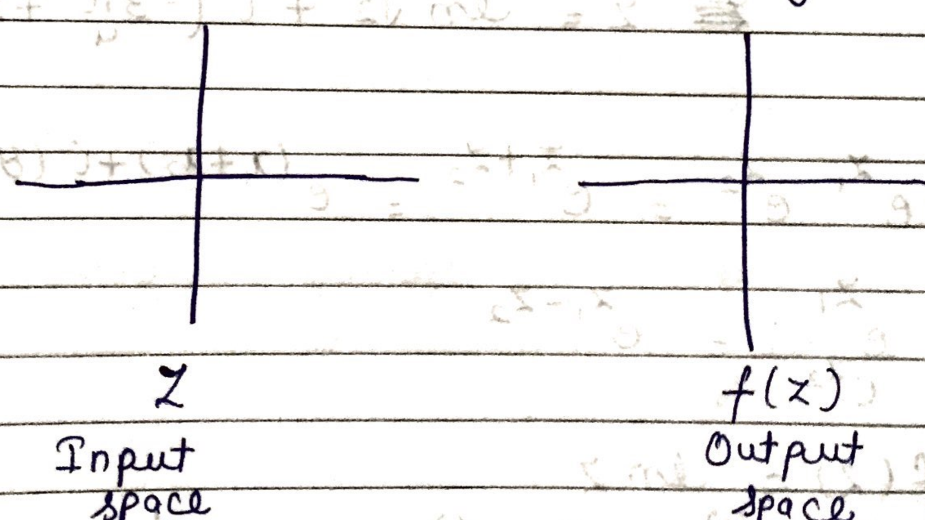
Function
↙ Single-Valued
↘ Multi-Valued
(Collection of single-valued functions)
↓
each single-valued function is called branch.

e.g. $$f(z) = \sqrt{r}\, e^{\frac{i\theta + 2k\pi i}{2}}$$

$$= \begin{cases} \sqrt{r}\, e^{i\theta/2} & ; k = 0 \\ \sqrt{r}\, e^{i\theta/2 + \pi i} & ; k = 1 \end{cases} \Bigg] \rightarrow \text{Branches}$$

<u>Note</u>:- To plot $f(z) = z+1$, input space & output space both contain 2 points each, so that has to be plot in 4-Dimensional space.

To plot this we use "Mapping Method".



Function

Single-Valued

Multi-Valued

(Collection of single-valued functions)

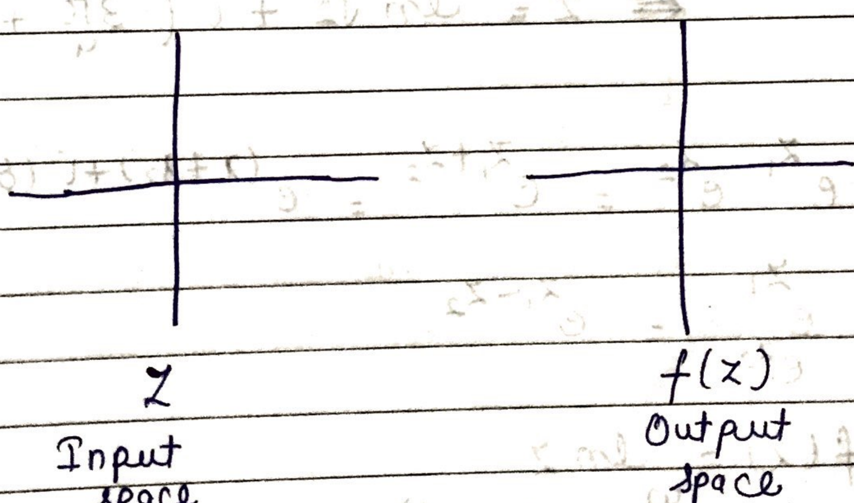
each single-valued function is called branch.

e.g. $f(z) = \sqrt{r}\, e^{\frac{i\theta + 2k\pi i}{2}}$

$$= \left\{ \begin{array}{ll} \sqrt{r}\, e^{i\theta/2} & ; k = 0 \\ \sqrt{r}\, e^{i\theta/2 + \pi i} & ; k = 1 \end{array} \right] \rightarrow \text{Branches}$$

<u>Note</u>:- To plot $f(z) = z+1$, input space & output space both contain 2 points each, so that has to be plot in 4-Dimensional space.

To plot this we use "Mapping Method."



Teacher's Signature

(a) Logarathemic function:-

We know,

$$-1 = e^{\pi i + 2k\pi i}$$

$$\therefore \log_e(-1) = \log_e(\cancel{1+}\,e^{\pi i + 2k\pi i})$$

$$= \pi i + 2k\pi i$$

(b) Exponential function:-

$$f(z) = e^z = re^{i\theta}$$

$$e^z = e^{\ln r} e^{i\theta}$$

$$e^z = e^{\ln r + i\theta}$$

$$\therefore z = \ln r + i\theta$$

e.g. $e^z = -1-i$. Find z

Ans

$$e^z = \sqrt{2}\, e^{-\frac{3\pi}{4} i}$$

$$e^x e^{iy} = e^{\ln\sqrt{2}}\, e^{-\frac{3\pi}{4} i}$$

$$z = \ln\sqrt{2} + i\left(-\frac{3\pi}{4} + 2k\pi\right)$$

(i) $e^{z_1} e^{z_2} = e^{z_1+z_2} = e^{(x_1+x_2)+i(y_1+y_2)}$

(ii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

\# $f(z) = \ln z$

$$e^w = z \Leftrightarrow w = \log z$$

To calculate z, we need to know 'w', which we know -

$$w = \ln r + i(\theta + 2k\pi)$$

$\log z \to$ ~~principal~~ General
$\text{Log } z \to$ Principal

Q Calculate $\log(-1-\sqrt{3}i)$

Ans Let $e^{\omega} = -1-\sqrt{3}i = 2e^{i\left(-\frac{2\pi}{3}+2k\pi\right)}$

$\Rightarrow \omega = \log_e\left(2e^{i\left(-\frac{2\pi}{3}+2k\pi\right)}\right) = \log(-1-\sqrt{3}i)$

$= \ln 2 + i\left(-\frac{2\pi}{3}+2k\pi\right)$

Principal value of $[\log(-1-\sqrt{3}i)]$

$\text{Log}(-1-\sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3}\right) = \cancel{\ln 2 + i\left(\frac{4\pi}{3}\right)}$

~~$\hookrightarrow$ Arg z is less than π which is not possible.~~

(c) Complex Exponent:-

When $z \neq 0$ & the exponent 'c' is any complex number the function z^c is defined by means of the eqn:-

$$\boxed{z^c = e^{c \log_e z}}$$

e.g. (i) $(i)^{-2i}$

$(i)^{-2i} = e^{\log_e (i)^{-2i}} = e^{-2i \log_e i}$ — (1)

Now, $\log i = \log|i| + i \arg(i)$
$= \log(1) + i\left(\frac{\pi}{2}+2k\pi\right)$

$= \left(\frac{\pi}{2}+2k\pi\right)i$

Into (1),
$(i)^{-2i} = e^{-2i\left(\frac{\pi}{2}+2k\pi\right)i}$
$= e^{(\pi+4k\pi)}$ $\quad [k = 0, \pm 1 \ldots]$

Teacher's Signature

(iii) $(2)^{\frac{1}{\pi}}$

$$(2)^{\frac{1}{\pi}} = e^{\log_e 2^{\frac{1}{\pi}}} = e^{\frac{1}{\pi}\ln 2}$$

$$= e^{\frac{1}{\pi}\{\ln 2 + i\, 2k\pi\}}$$

$$= e^{\frac{1}{\pi}\ln 2}\, e^{2ki}$$

Q What is principal value of $(-1)^{\frac{1}{\pi}}$ & $(-1-i)^{\frac{1}{\pi}}$

Ans

$$(-1)^{\frac{1}{\pi}} = e^{\frac{1}{\pi}\log_e(-1)}$$

$$= e^{\frac{1}{\pi}[\pi i]}$$

$$= e^{i} = \cos 1 + i \sin 1$$

Note:- As we know,

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$ but $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \neq \operatorname{Arg}(z_1 z_2)$

Similarly,

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$ but $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

(ii) $(z_1 z_2)^c = z_1^c z_2^c$ but $P.V.(z_1 z_2)^c \neq z_1^c z_2^c$

$\downarrow$

Principal Value

But the above properties hold true, when real part of z_1 & $z_2 > 0$.

Q $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = -1-i$

Check (i) $(z_1 z_2)^i = z_1^i z_2^i$

(ii) $(z_2 z_3)^i = z_2^i z_3^i$

Ans

(i) $z_1 z_2 = (1+i)(1-i) = 2$

$(z_1 z_2)^i = 2^i$

Q $e^{z} = 0$

$= e^{\log 0} = e^{(\ln|0| + i\,\mathrm{Arg}\,0)}$

[Since, Arg '0' is not defined

$\therefore$ $e^{z} = 0$ does not have any solution.

$z_1^{i} = (1+i)^{i} = e^{i \log_e (1+i)}$ $= \frac{1}{2} \ln 2$

$= e^{i\left[\frac{1}{2}\ln 2 + i\frac{\pi}{4}\right]}$

$= (\sqrt{2})^{i} e^{-\pi/4}$

$z_2^{i} = (1-i)^{i} = e^{i \log_e (1-i)}$

$= e^{i\left[\frac{1}{2}\ln 2 + i(-\pi/4)\right]}$

$= (\sqrt{2})^{i} e^{\pi/4}$

$\therefore (z_1 z_2)^{i} = z_1^{i} z_2^{i}$

(ii) $z_2 z_3 = (1-i)(-1-i)$

$= -2$

P.V. $(z_2 z_3)^{i} = (-2)^{i} = e^{i \log(-2)} = e^{i[\ln 2 + \pi i]}$

$= e^{i \ln 2} e^{-\pi}$

$(z_2)^{i} = (\sqrt{2})^{i} e^{\pi/4} = e^{i/2 \ln 2}$

$(z_3)^{i} = (-1-i)^{i} = e^{i \log(-1-i)}$

$= e^{i\left(\frac{1}{2}\ln 2 - \frac{3\pi}{4} i\right)}$

$= e^{i/2 \ln 2} e^{\frac{3\pi}{4}}$

To check $(z_2 z_3)^{i} = z_2^{i} z_3^{i}$

$e^{i \ln 2} e^{-\pi} = \left(e^{\frac{i}{2}\ln 2} e^{\frac{\pi}{4}}\right)\left(e^{\frac{i}{2}\ln 2} e^{\frac{3\pi}{4}}\right)$

$e^{i \ln 2} e^{-\pi} \neq e^{i \ln 2} e^{\pi}$

(d) Trignometric function:-

By Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\therefore \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

So, $$\cos(i\theta) = \frac{e^{i(i\theta)} + e^{-i(i\theta)}}{2} = \frac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta$$

Cosine hyperbolic function

$$\sin(i\theta) = \frac{e^{-\theta} - e^{\theta}}{2i} = i\left[\frac{e^{\theta} - e^{-\theta}}{2}\right]$$
$$= \sinh\theta$$

Note:- ~~Hyperb~~ Hyperbolic Sine & Cosine are both unbounded functions.

(i) $\sin z$ = $\sin(x+iy)$
Unbounded
$$= \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + \cos x \sinh y$$

(ii) $\cos z$ = $\cos(x+iy)$
Unbounded
$$= \cos x \cosh y - i \sin x \sinh y$$

e.g. $\log(i)^2 = 2\log i = 2\left(\frac{\pi}{2}i\right) = \pi i$

Or, $\log(i)^2 = \log i^2 = \log(-1) = 0 + \pi i$

Functions — Single-Valued → limit → Continuity → differentiability → Analytic
Functions — Multi-valued → only $K=0$ (principal) branch

Now,
$\log(i)^3 = 3\log i = 3\left(\frac{\pi}{2}i\right) = \frac{3\pi i}{2}$
Or, $\log(i^3) = \log(-i) = -\frac{\pi}{2}i$
} not same?

Q Find all the roots of the eq: $\log z = i\frac{\pi}{2}$

★ Limit:-
Let a function 'f' be defined at all points z in some deleted neighbourhood of z_0. The limit of $f(z)$, as z approaches to z_0, is a number (ω_0) given as-

$$\lim_{z \to z_0} f(z) = \omega_0 \quad — (1)$$

In other words,
for each ~~point~~ positive number ϵ there is a positive number δ such that
$|f(z) - \omega_0| < \epsilon$, whenever $0 < |z - z_0| < \delta$

This definition says that for each ϵ-neighbourhood, there is δ-nbd $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image ω_0 lying in the ϵ-nbd.
[$f(z)$ plane has 'ω' as point in $|f(z) - \omega_0| < \epsilon$
i.e. soln. of $|f(z) - \omega_0| < \epsilon$ in general is ω

Q 1. $\lim_{z \to 0} \frac{z}{|z_0|}$

Ans $= \lim_{x+iy \to 0} \frac{x+iy}{\sqrt{x^2+y^2}}$

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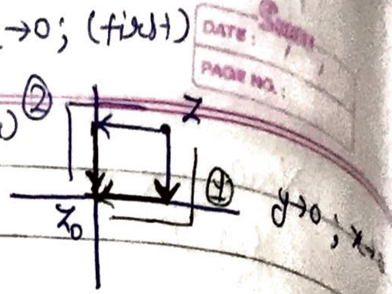
Iterative (repeated) limit :-



$$\lim_{x+iy\to 0} \frac{x+iy}{\sqrt{x^2+y^2}} =$$

First $x\to 0$, Later $y\to 0$ $\Rightarrow \lim_{x\to 0,\, y\to 0} \dfrac{x+iy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \dfrac{iy}{y} = i$

First $y\to 0$, Later $x\to 0$ $\Rightarrow \lim_{y\to 0;\, x\to 0} \dfrac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \dfrac{x}{x} = 1$

different paths, different limits $\therefore$ limit does not exist.

Q2. $\displaystyle\lim_{z\to 0} \frac{(\mathrm{Re}\, z - \mathrm{Im}\, z)^2}{|z|^2}$

Ans Let $y = mx$ be the path taken -

$$\lim_{x+iy} \frac{(x-y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x-mx)^2}{x^2(1+m^2)} = \frac{(1-m)^2}{1+m^2}$$

For $m = 0$ $\quad \lim \to 1$
$m = 1$ $\quad \lim \to 0$

diff. paths, limit diff.

So limit does not exist.

Note:- To prove non-existence of limit of a function we apply "Method of Contradiction":
(i) Iterative method
(ii) Path method

which to choose? depends on nature of problem

To prove existence of a limit :-

Q (i) $\lim_{z \to 1} \frac{i\bar{z}}{2} = \frac{i}{2}$ (Prove)

$f(z) = \frac{i\bar{z}}{2}$, $z_0 = 1$, $\omega_0 = \frac{i}{2}$

Let $0 < |z-1| < \delta$

$|f(z) - \omega_0| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{1}{2} |\bar{z} - 1| = \frac{\delta}{2}$ (let)

Q (ii) $\lim_{z \to i} z^2 = -1$

Ans $f(z) = z^2$, $\omega_0 = -1$, $z_0 = i$ ($f(z_0)$)

$0 < |z-i| < \delta$

$|f(z) - f(z_0)| = |z^2 - (-1)| = |(z-i)(z+i)|$

$= |(z-i)(z-i+2i)|$

$\leq |z-i| \{|z-i| + |2i|\} < \delta(\delta+2)$

$= \varepsilon$

(Triangular inequality)

Here, $\delta(\delta+2) = \varepsilon$

★ Continuity :- A function 'f' is continuous at a point z_0 if all three of the following conditions hold :

(i) $\lim_{z \to z_0} f(z)$ exist

(ii) $f(z_0)$ exist

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

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In other words,

For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

I

Q $f(z) = \cancel{\log z}\ \dfrac{i\bar{z}}{2}$ at $z = i$ (Check for continuity)

Ans $\lim\limits_{z \to i} \dfrac{i\bar{z}}{2} = \dfrac{1}{2}$

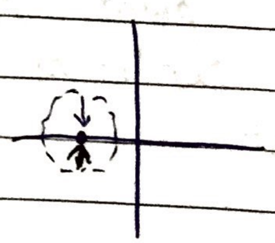
$f(z) = \dfrac{1}{2}$

Q (i) Find points of discontinuity for $f(z) = \log z$
(ii) Check whether $f(z)$ is continuous over negative real axis.

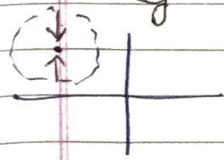
Ans $f(-1) = i\pi$

$$\lim_{z \to -1} f(z) = \lim_{z \to -1} [\ln|z| + i \operatorname{Arg} z]$$

$$\lim_{z \to -1} f(z) = \lim_{\substack{x \to -1 \\ y \to 0}} = \begin{cases} +\pi \ ; & y > 0, x < 0 \\ -\pi \ ; & y < 0, x < 0 \end{cases}$$



$\downarrow$
different paths
different limits
$\therefore$ limit does not exist



If ~~$\exists$~~ $z = x + iy$ where $x<0$, $y \neq 0$ limit exists, because there would be at least a circle of 'δ' radius possible for limit to exist.

★ **Differentiability:-** Let $f(z)$ be a single-valued function defined in a domain D, the function $f(z)$ is said to be differentiable at z_0 if-

$$\lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \text{ exist}$$

Let this limit be denoted by - $f'(z_0)$

~~At $z = z_0$~~ At $z - z_0 = \Delta z_0$

$$\Rightarrow \Delta z + z_0 = z$$

We get

$$f'(z_0) = \lim_{z \to 0} \left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right)$$

In other words,

For every $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that -

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \text{whenever}$$

$$0 < |z - z_0| < \delta$$

e.g. (i) $f(z) = z^2$

Sol:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z^2 + 2z\Delta z + (\Delta z)^2) - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

(ii) $f(z) = \bar{z}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\overline{z+\Delta z}) - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \Delta\bar{z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z}$$ → does this limit exist?

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z \to 0} \frac{x - iy}{x + iy}$$

For $y = mx$

$$= \lim_{x \to 0} \frac{x - xim}{x + imx}$$ → limit does not exist.

$$= \frac{1 - im}{1 + im}$$

Similarly,

$$\lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z} = \lim_{\Delta x + i\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Put $\Delta y = m\Delta x$

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x} = \frac{1-im}{1+im}$$

limit does not exist.

★ Cauchy - Riemann Equation :-

Let $f(z) = u(x,y) + iv(x,y)$ is defined and is continuous at a point $z = x+iy$ & differentiable at z, then at point z, first order partial derivative exists and satisfies the equation

C-R eqn $\rightarrow$ $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

e.g. $f(z) = \bar{z} = x - iy$

$f(z) = \overline{u+iv} = u - iv$

$u = x$, $v = -y$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

Since, C-R eqn does not satisfy, so, $f(z)$ is not differentiable.

Q $f(z) = |z|^2 = x^2+y^2$

Find all points of differentiability.

Ans $U = x^2+y^2$, $V=0$

$$\frac{\partial U}{\partial x} = 2x \quad , \quad \frac{\partial V}{\partial y} = 0 \quad — (1)$$

$$\frac{\partial U}{\partial y} = 2y \quad , \quad \frac{\partial V}{\partial x} = 0 \quad — (2)$$

(1) & (2) $\rightarrow$ Only at $z=0$, $f(z)$ is differentiable

Proof of Cauchy-Riemann Equation:-

$z = x+iy$, $\Delta z = \Delta x + i\Delta y$

$z+\Delta z = (x+\Delta x) + i(\Delta y + y)$

$f(z) = U(x,y) + i\,V(x,y)$

$f(z+\Delta z) = U(x+\Delta x, y+\Delta y) + i\,V(x+\Delta x, y+\Delta y)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{U(x+\Delta x, y+\Delta y) - U(x,y)}{\Delta z} + i\left(\frac{V(x+\Delta x, y+\Delta y) - V(x,y)}{\Delta z}\right)$$

By Iterative lim't Concept,

Take $\Delta y \to 0$ (first)

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$\therefore f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$

$+ i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$

$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad — ①$

Now, take $\Delta x \to 0$

$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y}$

$+ i \lim_{\Delta y \to 0} \frac{(v(x, y+\Delta y) - v(x,y))}{i \Delta y}$

$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad — ②$

As 'f' is differentiable --

① = ②

$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

So, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

& $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\Big] \to$ H.P.

Q
$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & ; z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Check C-R eq^n at $z = 0$

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Q. $f(z) = e^z$

$$e^z = e^x(\cos y + i\sin y)$$
$$= u + iv$$

$\therefore U = e^x \cos y \qquad V = e^x \sin y$

$\frac{\partial U}{\partial x} = U_x = e^x \cos y \quad ; \quad \frac{\partial U}{\partial y} = U_y = e^x(-\sin y)$

$\frac{\partial V}{\partial x} = V_x = e^x \sin y \quad ; \quad \frac{\partial V}{\partial y} = V_y = e^x \cos y$

$\because U_x = V_y \;\&\; U_y = -V_x$

$f(z) = U(x,y) + iV(x,y)$

As per C-R eq^n:- $U_x = V_y \;\&\; U_y = -V_x$

Similarly,

In polar form,

$f(z) = U(r,\theta) + iV(r,\theta)$

then -

$$U_r = \frac{1}{r}V_\theta \;\&\; V_r = -\frac{1}{r}U_\theta$$

* Analytic at z_0 :-

If $f(z)$ is differentiable at each point of some nbd of z_0 then we say $f(z)$ is analytic at z_0.

e.g. $f_1(z) = |z|^2$, $f_2(z) = \bar{z}$

$f_3(z) = z^2$

Check their differentiability at $z_0 = 0$

Ans

$f_1(z) = \|z\|^2 = x^2+y^2$	$f_2(z) = \bar{z} = x+i(-y)$	$f_3(z) = z^2 = x^2-y^2+i(2xy)$
$U = x^2+y^2$, $V=0$	$U = x$, $V = -y$	$U = x^2-y^2$, $V = 2xy$
$U_x = 2x$, $V_x = 0$	$U_x = 1$, $V_x = 0$	$U_x = 2x$, $V_x = 2y$
$U_y = 2y$, $V_y = 0$	$V_y = 0$, $V_y = -1$	$U_y = -2y$, $V_y = 2x$
C-R eqn holds only if $z=0$	does not satisfy C-R eqn	satisfies C-R eqn

Note:- For differentiability, C-R eqn must satisfy $\rightarrow$ Necessary condition

Q Check analyticity at $z=0$ of above functions.

Ans $f_1(z) \longrightarrow$ At $z=0$ differentiable, but in its surrounding not differentiable
$\hookrightarrow$ Not analytical

$f_2(z) \rightarrow$ Not analytical

$f_3(z) \rightarrow$ Analytic

Q Check if $f(z)$ is differentiable at $z=0$

$$f(z) = \begin{cases} \dfrac{\text{Im } z - \text{Re } z}{|z|} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$$

Ans Let's check for continuity,

$$\lim_{z\to 0} f(z) = \lim_{z \to 0} \frac{y-x}{\sqrt{x^2+y^2}}$$

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Take $y = mx$

$$\lim_{x \to 0} \frac{mx - x}{\sqrt{x^2 + m^2x^2}} = \frac{m-1}{\sqrt{1+m^2}}$$

$\downarrow$

Not continuous $\rightarrow \therefore$ not differentiable at $z=0$

Q $f(z) = \cancel{\log z}$ $\text{Log } z$

(i) Is it continuous on negative real axis.

(ii) Is it differentiable on negative real axis.

(iii) Is it differentiable other than negative axis.

Ans (i) We already know, $f(z)$ is discontinuous on negative real axis.

(ii) Discontinuity $\rightarrow$ Not differentiable on $-$ve real axis.

(iii) ~~f(z) = log z = e ln|z| + ...~~

$$f(z) = \text{Log } z = \ln|z| + i \, \text{Arg } z$$

$$= \frac{1}{2} \ln(x^2+y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

$$U = \frac{1}{2} \ln(x^2+y^2), \quad V = \tan^{-1}\left(\frac{y}{x}\right)$$

$$U_x = \frac{x}{x^2+y^2}, \quad V_y = \frac{x}{x^2+y^2}$$

$$U_y = \frac{y}{x^2+y^2}, \quad V_x = \frac{-y}{x^2+y^2}$$

It satisfies C-R eq^n

It is evident that C-R eq^n is only necessary condition because in above problem ~~contin~~ without checking continuity of eq^n, we may comment that $f(z) = \log z$ is differentiable everywhere, which isn't true

Q

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Ans $\quad U = \dfrac{x^3 - y^3}{x^2+y^2}, \quad V = \dfrac{x^3+y^3}{x^2+y^2}$

Note that $U_x(0,0)$ calculation by going by partial derivative $\because$ ~~approach would be wrong~~ as denominator would become '0' $(x^2+y^2=0)$

$$U_x(a,b) = \lim_{h \to 0} \frac{U(a+h, b) - U(a,b)}{h}$$

$$U_y(a,b) = \lim_{K \to 0} \frac{U(a, b+K) - U(a,b)}{K}$$

That is why limit approach is used to calculate U_x.

At $z = 0$

$$U_x(0,0) = \lim_{h \to 0} \frac{U(0+h, 0) - U(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

$$U_y(0,0) = \lim_{K \to 0} \frac{U(0, 0+K) - U(0,0)}{K}$$

$$= \frac{-K^3/K^2}{K} = -1$$

$$V_x(0,0) = \lim_{h \to 0} \frac{V(0+h,0) - V(0,0)}{h} = 1$$

$$V_y(0,0) = \lim_{k \to 0} \frac{V(0,0+k) - V(0,0)}{k} = 1$$

↳ satisfies C-R eqn

Now, let's take another approach to solve same problem →

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{x+iy \to 0} \frac{\frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} - 0}{x+iy}$$

$$= \lim_{x+iy \to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$$

Take $y = mx$

↳ not differentiable.

Again, In above problem C-R eqn satisfies still $f(z)$ is not differentiable.

★ Sufficient condition of differentiability :-

(i) Partial derivatives exist U_x, U_y, V_x, V_y
(ii) C-R eqn holds.
(iii) U_x, U_y, V_x, V_y are Continuous.

Real-valued fn → Any input
Output → Real

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Q Is $f(z) = e^{\bar{z}}$ differentiable at any point

Entire function:- If function is analytic everywhere in Z.

There is no co-relation b/w analyticity & entireness.

☆ Harmonic function:-

A "real-valued function" 'H' of two variable x & y is said to be harmonic in a given domain of xy-plane if throughout the domain it has continuous partial derivative of the first & second order and satisfies the equation-

$$H_{xx}(x,y) + H_{yy}(x,y) = 0 \rightarrow \text{Laplace eq}^n$$

$$[\nabla^2 H = 0]$$

Theorem:- If a function $f(z) = u + iv$ is analytic in a domain 'D', then its component function u & v are harmonic in D.

e.g. $f(z) = |z|^2$
$= x^2 + y^2 + i0$

$u = x^2 + y^2$, $v = 0$

$U_x = 2x$ $\qquad V_x = 0$
$U_{xx} = 2$ $\qquad V_{xx} = 0$

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$u_y = 2y$ $\qquad\qquad\qquad V_{yy} = 0$

$u_{yy} = 2$ $\qquad\qquad\qquad V_{xx} + V_{yy} = 0$

$U_{xx} + U_{yy} = 4 \neq 0$

Here, function itself is not analytic at z_0
$u, v \rightarrow$ not even harmonic.

(ii) $f(z) = \bar{z}$

$= x - iy$

$U = x$ $\qquad\qquad V = -y$

$U_{xx} = 0$ $\qquad\qquad V_{xx} = 0$

$U_{yy} = 0$ $\qquad\qquad V_{yy} = 0$

$U_{xx} + U_{yy} = 0$ $\qquad\qquad V_{xx} + V_{yy} = 0$

Here, u & v are harmonic
But function $f(z)$ is not analytic.

This Clearly, does not contradict our Theorem as theorem suggests $f(z)$ to be analytic which here isn't the case.

Q For what value of a & b given function is harmonic.

Ans

$f(z) = ax^2 + by^2$

$U = ax^2 + by^2$

$U_{xx} = 2a$, $U_{yy} = 2b$

$U_{xx} + U_{yy} = 0$

$2a + 2b = 0 \Rightarrow a = -b$

p.d. → partial derivative

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$\therefore U = x^2 - y^2$

Proof of Theorem:-

$f(z)$ is analytic $\Rightarrow$ u & v are harmonic

Given: $f(z)$ is analytic $[f(z) = u(x,y) + i\,v(x,y)]$

$\therefore U_x = V_y$ — ① $U_y = -V_x$ — ②

p.d. ① w.r.t. x p.d. ② w.r.t. y

$U_{xx} = V_{yx}$ $U_{yy} = -V_{xy}$

$U_{xx} + U_{yy} = V_{yx} - V_{xy} = 0$ [Since V_x & V_y are continuous]

Note:- If f_x & f_y are continuous, then-

$f_{xy} = f_{yx}$

Note:- Converse of Theorem is not true in general.

Q (i) Check whether given function is harmonic -

$U(x,y) = e^x \cos y.$

(ii) If yes, find conjugate harmonic of U.

(iii) Find $f(z) = U + iV$

(iv) Convert $f(z)$ in terms of z.

Ans (i) $U_{xx} = e^x \cos y$ & $U_{yy} = -e^x \cos y$

$\therefore U_{xx} + U_{yy} = e^x \cos y - e^x \cos y = 0$

↓

Harmonic

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(ii) $U_x = e^x \cos y$

Using C-R eq^n $(U_x = V_y)$

$\therefore V_y = e^x \cos y$

$V = e^x \sin y + \phi(x)$ — ①

$V = 2xy + x^2 + 5$

$\frac{\partial V}{\partial y} = 2x$

$\Downarrow$

$V = 2x + C$ ✗

So, C has to be fn of x.

$V_x = e^x \sin y + \phi'(x)$ — ②

Again, $U_y = -V_x$

$-V_x = -e^x \sin y$

$\Rightarrow V_x = e^x \sin y$ — ③

From ② & ③

$e^x \sin y = e^x \sin y + \phi'(x)$

$\Rightarrow \phi'(x) = 0$

$\phi(x) = C$

So, in ①

$V = e^x \sin y + C$

(iii) $f(z) = u + iv$

$= e^x \cos y + i e^x \sin y + iC$

$= e^x (\cos y + i e^x \sin y) + iC$

$= e^x e^{iy} + iC$

$= e^{x+iy} + iC$

$V = 2xy + x^2 + 5$

$\frac{\partial V}{\partial y} = 2x$

$\Downarrow$

$V = 2x + C$ ✗

So, C has to be fn of x.

Note:- Idea of such questions is to construct analytic function if one component function is provided to be harmonic.
[Hence use of C-R eqn]

(iv) $e^{x+iy} + iC = e^{z} + iC \quad (z = x+iy)$

'v' is conjugate harmonic of u.

Q Check harmonticity for $u = x^2 + y^2$

Ans

$u_x = 2x \qquad\qquad u_y = 2y$

$v_y = 2x \qquad\qquad -v_x = 2y$

$v = 2xy + \phi(x) \quad — (A) \qquad\qquad v_x = -2y \quad — (2)$

$v_x = 2y + \phi'(x) \quad — (1)$

From (1) & (2)

$2y + \phi'(x) = -2y$

$\phi'(x) = -4y$

$\phi(x) = -4xy + C \quad — (B)$

From (A) & (B)

$v = -2xy + C$

$\therefore f(z) = (x^2+y^2) - i\,2xy + iC$

$u = x^2 + y^2 \quad , v = -2xy$

$u_x = 2x \quad , v_y = -2x$

$u_y = 2y \quad , v_x = -2y$

→ does not follow C-R eqn

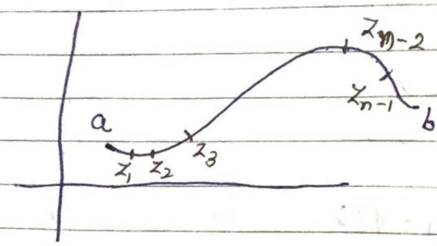
$\downarrow$

Hence not analytic

Integration (Complex No.)

★ Complex line integral :-

Let $f(z)$ be continuous at all points of a curve C, which we shall assume has a finite length



Subdivide 'C' into n points by means of points $z_1, z_2, \ldots z_{n-1}$ chosen arbitrarily and call $a = z_0$, $b = z_n$

On each Arc joining z_{k-1} to z_k, choose a point d_k. ~~From~~ Form the sum-

$$S_n = f(d_1)(z_1 - a) + f(d_2)(z_2 - z_1) + \ldots + f(d_{n-1})(z_{n-1} - z_{n-2})$$
$$+ f(d_n)(b - z_{n-1})$$

Let the number of subdivisions 'n' increases in such a way that the longest of the chord approaches to zero. Then, since $f(z)$ is continuous, sum approaches to $\int_a^b f(z)\,dz$ or $\int_C f(z)\,dz$

$\int f(z)\,dz$
$= \int (u+iv)(dx+idy)$
$= \int (u\,dx - v\,dy) + i(u\,dy + v\,dx)$

Let $\Delta z_k = z_k - z_{k-1}$

$$S_n = \sum_{k=1}^{n} f(d_k)\,\Delta z_k$$

This is called Complex integral or simply line integral of $f(z)$.

e.g. (i) $\int_0^{1+i} z^2\,dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{(1+i)^3}{3}$

(ii) $\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z \Big|_{-\pi i}^{\pi i}$

$= \sin \pi i - \sin(-\pi i)$
$= \sin \pi i + \sin \pi i$
$= 2\sin(\pi i)$
$= 2i \sinh \pi \quad [\because \sin ix = i\sinh x]$

★ Real line integral :-

Let $P(x,y)$ & $Q(x,y)$ be real functions of x and y continuous at all points of Curve 'C'. Then the real line integral of $P\,dx + Q\,dy$ along curve C can be defined as-

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

or

$$\int_C P\,dx + Q\,dy$$

If $x = \phi(t)$ & $y = \psi(t)$, $t_1 \leq t \leq t_2$

$$= \int_{t_1}^{t_2} P(\phi(t),\psi(t))\,\phi'(t)\,dt + Q(\phi(t),\psi(t))\,\psi'(t)\,dt$$

(To convert it into one variable 't')

★ **Arc:-** A set of points $z = (x, y)$ in the complex plane is said to be an arc if -

$$x = x(t), \quad y = y(t) \quad (a \le t \le b)$$

where $x(t)$ & $y(t)$ are continuous functions of real parameters 't'.

The arc 'c' is a simple Arc (or Jordan Arc) if it does not ~~intersect~~ cross itself.
In other words,

'C' is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$.
When C is simple except for the fact that $z(a) = z(b)$. We say that 'C' is simple closed curve. [a, b → end points]

Q Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along:

(i) The parabola $x = 2t$, $y = t^2 + 3$
(ii) Straight lines from $(0, 3)$ to $(2, 3)$ & then from $(2, 3)$ to $(2, 4)$
(iii) A straight line from $(0, 3)$ to $(2, 4)$

Ans (i) $x = 2t \Rightarrow dx = 2\,dt$

$y = t^2 + 3 \Rightarrow dy = 2t\,dt$

$0 \le t \le 1$

$$I = \int_0^1 \left[2(t^2+3) + 4t^2\right] 2\,dt + (6t - t^2 - 3)\,2t\,dt$$

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$$I = \int_0^1 (4t^2 + 12 + 8t^2)dt + (12t^2 - 2t^3 + 8t)dt$$

$$= \int_0^1 (-2t^3 + 24t^2 - 6t + 12)\,dt$$

$$= \left[-\frac{t^4}{2} + 8t^3 - 3t^2 + 12t\right]_0^1$$

$$= -\frac{1}{2} + 8 - 3 + 12 = 17 - \frac{1}{2} = \frac{33}{2}$$

(ii) $(0, 2)$ to $(2, 3)$

$y = 3$, $dy = 0$

$$\therefore I_1 = \int_0^2 (6 + x^2)\,dx + (3x - 3)\,0$$

$$= \left[6x + \frac{x^3}{3}\right]_0^2 = \frac{44}{3}$$

$(2, 3)$ to $(2, 4)$

$x = 2$, $dx = 0$

$$\therefore I_2 = \int_3^4 (2y + x^2)\,0 + (6 - y)\,dy$$

$$= \left[6y - \frac{y^2}{2}\right]_0^2 = \frac{5}{2}$$

$$I = I_1 + I_2 = \frac{103}{6}$$

(iii) For eq^n of line →

$$\frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - x_0}$$

$$\Rightarrow \frac{y - 3}{1} = \frac{x - 0}{2}$$

$$\Rightarrow 2y - 6 = x \quad \Rightarrow dx = 2dy$$

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then-

$$I = \int_3^4 (2y + (2y-6)^2] \, 2\,dy + [(3(2y-6) - y] \, dy$$

$$= \frac{97}{6}$$

We observe, with three different paths line integral comes out to be different.

Q (a) $\int_C \bar{z}\,dz$ and (b) $\int_C z\,dz$

where C is curve from $z=0$ to $z=4+2i$ given by -

(i) $z = t^2 + it$

(ii) The line from $z=0$ to $z=2i$ and then the line from $z=2i$ to $z=4+2i$

Ans

(a) $\int_C \bar{z}\,dz = \int_C (x-iy)\, d(x+iy)$

$$= \int_C (x\,dx + y\,dy) + i(x\,dy - y\,dx)$$

Along (i) path - $z = t^2 + it$

Here $x = t^2$, $y = t$

& $t \to 0$ to 2

$$I = \int_0^2 (t^2 - it)\, d(t^2 + it)$$

$$= \int_0^2 (t^2 - it)(2t + i)\,dt = 10 - \frac{8}{3}i$$

Along (ii) path -

$$I_1 = \int_0^2 (0\cdot 0 + y\,dy) + i\,(0\cdot dy - y\cdot 0)$$

$$= \left[\frac{y^2}{2}\right]_0^2 = 2$$

$$I_2 = \int_0^4 (x\,dx + \cancel{x}\cdot 0) + i\,(x\cdot 0 - 2\,dx)$$

$$= \int_0^4 x\,dx - 2i\,dx$$

$$= \left[\frac{x^2}{2} - 2xi\right]_0^4 = 8 - 8i$$

$$I = I_1 + I_2 = 10 - 8i$$

(b) $$I = \int_C z\,dz = \int_C (x+iy)(dx+i\,dy)$$

$$= \int_C (x\,dx - y\,dy) + i\,(y\,dx + x\,dy)$$

Along (i) path –

$z = t^2 + it$ $\quad z = 0$ to $z = 4+2i$

$t \to 0$ to 2

$$I = \int_0^2 (t^2 + it)\, d(t^2 + it)$$

$$= \int_0^2 (t^2 + it)(2t + i)\,dt$$

$$= \int_0^2 2t^3 - t + i\,(2t^2 + t^2)$$

$$= \left[\frac{t^4}{2} - \frac{t^2}{2} + i\left(\frac{2t^3}{3} + \frac{t^3}{3}\right)\right]_0^2$$

$$= 8 - 2 + 8i = 6 + 8i$$

Along (ii) path -

$$I_1 = \int_0^2 -y\,dy = -2$$

$$I_2 = \int_0^4 (2\,dx - 2\cdot 0) + i\,(2\cdot dx + x\cdot 0)$$

$$= 8 + 8i$$

$$I = I_1 + I_2 = 6 + 8i$$

	$f(z) = \bar{z}$	$f(z) = z$
Continuous	Yes	Yes
Differentiable	No	~~No~~ Yes
Analytic	No	Yes

Note:- In case of Analytic function, along any path chosen the line integral would be same. But In case of non-Analytic function, line integral along different paths would be different.

In simpler words, line integral depends on end points rather than the curve (path)

★ Smooth Curve :-

A curve $z(t)$ $[z : [a,b] \to C]$ ~~$a \le t \le b$~~ $a \le t \le b$ is smooth if the derivative $z'(t)$ is continuous on the closed interval $a \le t \le b$ and non-zero on the open ~~..~~ $a < t < b$.

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Contour or piecewise Smooth Curve :-

is an arc consisting of a finite number of smooth arc joined end to end.

Length of a Curve :-

$$z(t) = x(t) + i\,y(t)$$
$$z'(t) = x'(t) + i\,y'(t)$$

length of Curve

$$L = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

☆ ML - inequality :-

Let 'C' be a piecewise smooth curve $C : z(t) = x(t) + i\,y(t)$
$a \leq t \leq b$

then

$$\left| \int_C f(z)\,dz \right| \leq ML$$

where, L = length of the Curve
& $|f(z)| \leq M$ everywhere on C.
↳ Here $f(z)$ is bound on the Curve. not necessarily bounded in entire Complex plane

We know,

$$S_n = \sum_{k=1}^{n} f(\alpha_k)\,\Delta z_k$$

$$\Rightarrow |S_n| = \left| \sum_{k=1}^{n} f(\alpha_k)\,\Delta z_k \right|$$

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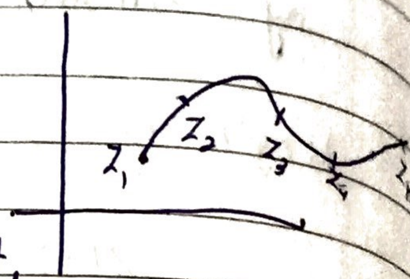
$$= \sum^{n} |\cancel{\ } f(\alpha_k)| \, |\Delta z_k|$$

$$\leq ML$$

Again,



From the definition of Contour integral. We get,

$$S_n = \sum_{k=1}^{n} f(\alpha_k) \Delta z_k.$$

$$|S_n| = \left| \sum_{k=1}^{n} f(\alpha_k) \Delta z_k \right|$$

$$\leq M \left| \sum_{k=1}^{n} \Delta z_k \right|$$

It is clear that $\sum_{k=1}^{n} |z_k - z_{k-1}|$ represent the ~~length~~ length of the chords whose end parts are $z_0, z_1 \ldots z_n$

Since a straight line path is the shortest distance b/w any two paths $|z_k - z_{k-1}|$ does not exceed the length of the arc joined to point z_{k-1} & z_k.

Thus $n \to \infty$ such that -

max. $|\Delta z_k| \to 0$, we have-

$$L^* \to L$$

We get.

$$\lim_{n\to\infty} |S_n| \le M \lim_{n\to\infty} \sum_{K=1}^{n} |x_K - x_{K-1}|$$

$$\left|\int_C f(z)\,dz\right| \le ML$$

~~Let C be the arc of the circle $|z|=2$~~

Q Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant, then-

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7} \rightarrow ML$$

Ans $|f(z)| = \left|\frac{z+4}{z^3-1}\right|$

$|f_1(z)| = |z+4| \le |z| + 4 = 2 + 4 = 6$

$|f_2(z)| = \frac{1}{z^3-1} \le \frac{1}{|z|^3-1} = \frac{1}{8-1} = 7$

$\therefore |f(z)| = \left|\frac{f_1(z)}{f_2(z)}\right| \le \frac{6}{7}$

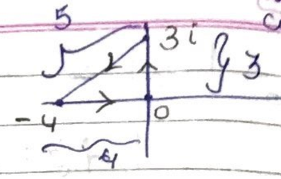
Q Show that if 'C' is the boundary of the triangle with vertices at the points 0, $3i$ & -4, oriented in the counter-clockwise direction, then-

$$\left|\int_C (e^z - \bar{z})\,dz\right| \le 60$$

Ans $f(z) = e^z - \bar{z}$

e^{z}

direction $\}$ counter-clockwise



Length of Curve (L)

$= 3+4+5 = 12$

$|f(z)| = |e^{z} - \bar{z}| \geq |e^{z}| - |\bar{z}|$ → This will give minimum value of function

$\therefore \quad |e^{z} - \bar{z}| \leq |e^{z}| + |\bar{z}|$

For maximum value of $|f(z)|$

$\Rightarrow |f(z)| \leq |e^{z}| + |\bar{z}| = |e^{z}| + |z|$

Maximum value of $|z| = 4$

$|e^{z}| = |e^{x+iy}|$

$= |e^{x}|\,|e^{iy}|$

$= |e^{x}|\,|\cos y + i\sin y|$

$= |e^{x}| \times 1 = |e^{x}|$

& Maximum value for $x = 0$

$\therefore \quad |e^{z}|_{max} = |e^{0}| = 1$

$\therefore \quad |f(z)| \leq 1 + 4 = 5$

$|f(z)| \leq 5$

Therefore, by ML - eqn:

$$\left| \int_{C} (e^{z} - \bar{z})\, dz \right| \leq M*L = 5 \times 12 = 60$$

$\cancel{\leq 5 \times 12}$

$\cancel{\leq 60}$

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Q. Find the length of Curve C: $z(t) = t^3 + it$, $0 \le t \le 1$

Ans. $z'(t) = 3t^2 + i$

Length of Curve (L) → $\int_a^b |z'(t)|\, dt$

$$L = \int_0^1 |3t^2 + i|\, dt$$

$$= \int_0^1 \sqrt{(3t^2)^2 + 1}\, dt = \int_0^1 \sqrt{9t^4 + 1}\, dt$$

Q. Find the upperbound for the absolute value of integral:

$$I = \int_L e^{(\bar{z})^2}\, dz \quad ; \quad C: |z| = 1$$

where C is traversed in the anti-clockwise direction.

Ans.

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Q Evaluate the integral $\int_C \frac{2z+3}{z} dz$ where
(a) Upper half of the circle $|z| = 2$ traversed in the clockwise direction.

Ans $|z| = 2 \Rightarrow z = 2e^{i\theta}$

$z'(\theta) = 2i\,e^{i\theta}$

We know when doing parameters exchange

$$I = \int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

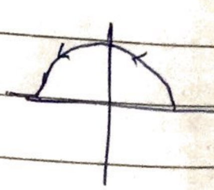
$$I = \int_C \frac{2z+3}{z} dz = \int_C \frac{2(2e^{i\theta})+3}{2e^{i\theta}} (2ie^{i\theta})\,d\theta$$

$$= i\int_\pi^0 (4e^{i\theta}+3)\,d\theta$$

limit from π to 0
as it's upper half of circle
& in clockwise direction

(b) The upper half of the circle $|z| = 2$ traversed in the anti-clockwise direction.

$$I = i\int_0^\pi (4e^{i\theta}+3)\,d\theta$$



Anti-clockwise → +ve
Clockwise → −ve

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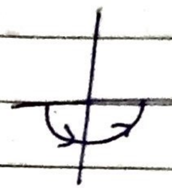
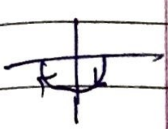
(c) Lowerhalf of the circle $|z| = 2$ in the counter-clockwise direction

$$I = i\int_{-\pi}^{0} (4e^{i\theta}+3)\,d\theta$$



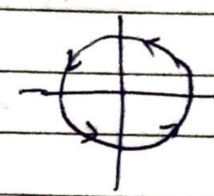
(d) Lowerhalf of the circle $|z| = 2$ in the clockwise direction



$$I = i\int_{0}^{-\pi} (4e^{i\theta}+3)\,d\theta = i\left[\frac{4e^{i\theta}}{i} + 3\theta\right]_{0}^{-\pi}$$

(e) Circle $|z| = 2$ in the anti-clockwise direction.

$$I = i\int_{0}^{2\pi} (4e^{i\theta}+3)\,d\theta =$$



$$= i\left[\frac{4e^{i\theta}}{i} + 3\theta\right]_{0}^{2\pi}$$

$$= 4e^{i2\pi} + 6\pi i = 6\pi i$$

Q. Evaluate the integral $\int_C z^2\,dz$ when C is circle $|z| = 2$ is traversed in anti-clockwise direction.

Ans. $|z| = 2 \Rightarrow z = 2e^{it}$ $\quad f(z) = z^2$

$z'(t) = 2ie^{it}$

$$I = \int_{0}^{2\pi} f(z)\,dz = i\int_{0}^{2\pi} [2e^{it}]^2\,[2ie^{it}]\,dt$$

$$= i2^3 \int_{0}^{2\pi} e^{(2+1)it}\,dt = \frac{8i}{3}\left[e^{3it}\right]_{0}^{2\pi}$$

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$$= \frac{8i}{3}[1-1] = 0$$

Ans

Let's repeat same problem for $f(z) = z$

$$I = \oint_0^{2\pi} [2e^{it}][2ie^{it}]\,dt$$

$$= i\,2^2 \int_0^{2\pi} e^{(1+1)it} = 4i[1-1] = 0$$

Similarly,

For $f(z) = z^n$

$$I = \int_C f(z)\,dz = 0 \qquad \text{for } n \neq -1$$

For $f(z) = \frac{1}{z}$

$$I = \int_0^{2\pi} \frac{1}{[2e^{it}]}\, i\,[2e^{it}]\,dt = i[t]_0^{2\pi}$$

$$= 2\pi i$$

So,

Note:- $$I = \int z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n = 0, \pm 1, \pm 3, \dots \end{cases}$$

~~Note~~:- Q Evaluate $I = \int_C z^n\,dz \qquad n = 0, \pm 1, \pm 2 \dots$

where $C: |z| = r$ is traversed in the counter-clockwise direction

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Ans $z(t) = r e^{it}$; $z'(t) = r i e^{it}$

$$I = \int_0^{2\pi} [r e^{it}]^n [i r e^{it}] \, dt$$

$$= i r^{n+1} \int_0^{2\pi} e^{(n+1)it} \, dt$$

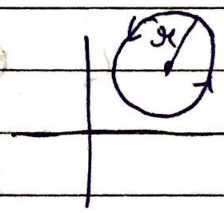
$$= \left[\frac{r^{n+1}}{n+1} \left[e^{(n+1)it} \right]_0^{2\pi} \right] \overset{=0}{(\text{if } n \neq -1)}$$

Q $I = \int_C (z - z_0)^n \, dz$; $C : |z - z_0| r$ anti-clockwise

Ans Take $z - z_0 = \omega$ $\quad \int_C \omega^n \, d\omega$; $C : |\omega| = r$
$dz = d\omega$

It is same problem only which we have calculated.

$|z - z_0| = r$
$\therefore z - z_0 = r e^{it}$
$z = z_0 + r e^{it}$



$dz = r i e^{it} \, dt$

$$I = \int_C (z - z_0)^n \, dz$$

$$= \int_0^{2\pi} [r e^{it}]^n \, r i e^{it} \, dt$$

$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt$$

$$= \frac{r^{n+1}}{n+1} \left[e^{i(n+1)t} \right]_0^{2\pi}$$

$= 0$ [when $n \neq -1$]

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When $n = -1$

$$I = \int_0^{2\pi} \frac{1}{r e^{it}} \, r i e^{it} \, dt$$

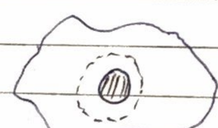
$$= \int_0^{2\pi} i \, dt = 2\pi i$$

<u>Note:-</u> Neither circle radius(r) nor its centre (z_0) location changes the value of integration

$$\int_C (z-z_0)^n \, dz \quad ; \; C : |z-z_0| = r$$

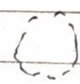
<u>A simply Connected domain:-</u>

In a Complex plane a domain (open & connected) that every simple closed curve in Domain encloses only point of domain.



D_1

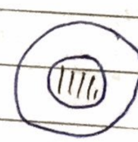
//// → region represents not included in domain

 → Contains points other than domain D_1 too.

~~not con~~

Thus, not simply connected Domain (D_1)

e.g. (i) $1 \le |z| \le 2$ → not simply connected



(ii) $|z| < 5$ → Simply Connected

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Multiply Connected Domain :-
Domain that is not simply connected is called Multiply Connected.

e.g. $1 < |z| < 7$

★ Green's Theorem :-
Let 'C' be a positively oriented, piecewise smooth, simple closed curve in a plane and let 'D' be a region bounded by 'C'.
If L & M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives, then-

$$\boxed{\oint_C L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy}$$

★ Cauchy's Theorem :-
Let $f(z)$ is analytic in a simply connected domain D & $f'(z)$ is continuous in D, then for every simple closed curve 'C' in D -

$$\oint_C f(z)\,dz = 0$$

Q1. $I = \oint \sin z\,dz$; $C : |z-1| = 5$

$= 0$ [As $\sin z$ is analytic in its domain]

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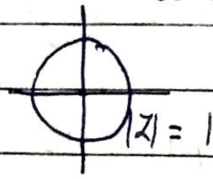
$|z - z_0|$

Q $I = \oint \frac{\cos z}{z+3} \, dz$; $C : |z| = 1$
oriented in anti-clockwise direction

$f(z) = \frac{\cos z}{z+3}$ → not defined ~~for~~ for $z = -3$

But $z = -3$ isn't in the domain of Curve



$\therefore I = 0$ (Cauchy's Theorem)

Q $I = \oint_C \frac{z^2}{z^2+4} \, dz$; $C : |z - i| = 1.5$

Ans $f(z) = \frac{z^2}{z^2+4} = \frac{z^2}{(z+2i)(z-2i)}$
↳ not defined for $z = 2i$ & $z = -2i$

In domain of 'C'
$z = 2i$ is coming there.

$\therefore I \neq 0$

Note:- In Cauchy's Theorem, simply connected domain is 'Necessary' condition & Analyticity is 'Sufficient' condition for $\int_C f(z)\, dz = 0$ to hold.

e.g. $f(z) = \frac{1}{z}$ $C : \frac{1}{2} < |z| < 1$
↳ In domain of 'C' function is analytic.

Still, $I \neq 0$

Therefore, simply connected domain is required.

Proof of Cauchy's Theorem:-

Let $f(z) = u + iv$

$dz = dx + idy$

Consider $f(z) \rightarrow$ Analytic (as per Cauchy's Theorem initial given condition)

$\therefore u, y \rightarrow$ ~~Analytic~~ partial derivatives exist & continuous

$$I = \oint_C f(z)\,dz = \oint_C (u + iv)(dx + idy)$$

$$= \oint (u\,dx - v\,dy) + i\int (u\,dy + v\,dx)$$

Apply Green's Theorem as u & v have continuous partial derivatives.

$$\int u\,dx + (-v)\,dy = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

using C-R eq^n [$u_x = v_y$ & $v_x = -u_y$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$$

Similarly,

$$\int u\,dy + v\,dx = \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$$

(Using C-R eq^n)

$\therefore$ $I = 0 + 0 = 0$ - Hence proved

$\sqrt{z} = \sqrt{re^{i\theta}}$

$= \sqrt{r}\, e^{i\theta/2}$

$= \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$ — ①

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

Replace θ by $\theta/2$

$\therefore \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

So, in ①

$\sqrt{z} = \sqrt{r}\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{r}\sqrt{\frac{1-\cos\theta}{2}}$

$= \sqrt{\frac{r+r\cos\theta}{2}} + i\sqrt{\frac{r-r\cos\theta}{2}}$

$= \sqrt{\frac{|z|+x}{2}} + i\,(\text{Sig}\,y)\sqrt{\frac{|z|-x}{2}}$

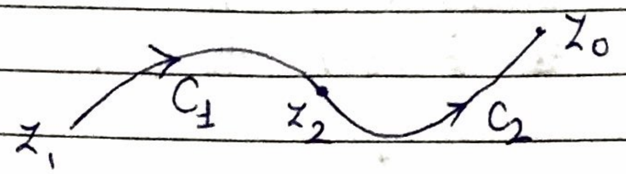
It is important to add signature (sign) of y in the expression as there is no 'y' involvement in expression

e.g. $\sqrt{z} = \sqrt{1-i}$; $\sqrt{z} = \sqrt{1+i}$

Both will have same solution if sign of 'y' is ignored.

Properties :-

1. **Partition of Path :-**



$C_1 : z_1 \to z_2$

$C_2 : z_2 \to z_0$

$C = C_1 + C_2$ $\quad$ $C : z_1 \to z_0$

then - $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

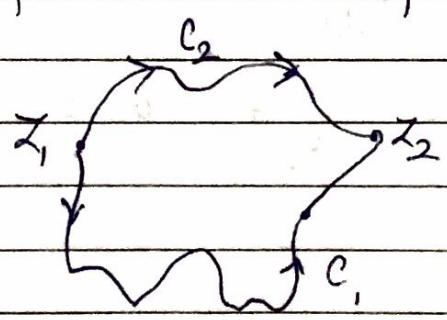
2. **Sense reversal :-**

$$\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz$$

3. **Independent of path :-**

If $f(z)$ is analytic in a simply ~~convert~~ connected domain D then the integral of $f(z)$ is independent of path in D.

Proof:



Let z_1 & z_2 be any point in D. Consider two path C_1 & C_2 in D from z_1 to z_2 ~~&~~ without further common points.

Denote path C_1 & C_2, where C_2^* is the path C_2

$C_2 \rightarrow z_1$ to z_2

$C_2^* \rightarrow z_2$ to z_1

with orientation reversed.

Integrating z_1 over C_1 to z_2 and over C_2^* back to z_1.

This ($C = C_1 + C_2^*$) becomes simple closed path in simply connected domain D then -

By Cauchy's Theorem -

$$\int_C f(z)\,dz = 0 \qquad -(1)$$

but
$$\int_C f(z)\,dz = \int_{C_1 + C_2^*} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz \qquad -(2)$$

From (1) & (2)

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

In case of Analytic function, this integral depends on end points but not on path.

Singular point → point where function

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Q Find the solution of following:

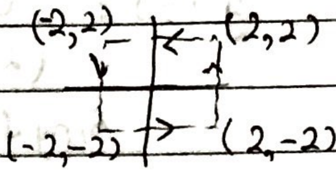
(i) $\int_{C_1} e^z \, dz$; $C_1 : |z| = 2$

Ans $e^z \to$ Analytic

$\therefore$ By Cauchy's Theorem

$I = 0$

(ii) $\int_{C_2} \frac{z^2}{z+8} \, dz$



C_2

Ans $f(z) = \frac{z^2}{z+8}$

$\downarrow$ Analytic in C_2

[Singularity $z = -8$]

So, By Cauchy's Theorem

$I = 0$

(iii) $\int \tan z \, dz$ $C_3 : |z| = 1$

Ans $\int \frac{\sin z}{\cos z} \, dz$

$f(z) = \frac{\sin z}{\cos z}$ → not differentiable at $z = \frac{\pi}{2}$ which is not in C_3

[Singularity $(2n+1)\frac{\pi}{2}$]

So, $f(z) \to$ Analytic in C_3

So, $I = 0$ (By Cauchy's Theorem)

(iv) $f_1(z) = e^{\bar{z}}$

↳ not differentiable anywhere (but continuous everywhere)

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(v) $f_2(z) = \bar{z}$

↳ ~~not~~ continuous
but not differentiable anywhere

(iv), (v) → No point is singlular point

(vi) $g_1(z) = \dfrac{e^z}{(z-1)(z-2)}$

Singular point → $z = 1, 2$

(vii) ~~$g_2(z) = \tan z$~~ $g_2(z) = \dfrac{1}{z}$

Singlular point $z = 0$

(viii) $\log z$

↳ differentiable everywhere except -ve real axis.

& ↳ all points on -ve real axis are singular points.

★ <u>Singularity</u>:-

A single valued function is said to have a singularity at a point if the function is not analytic at the point while every neighbourhood of that point contains at least one point at which function is analytic.

e.g. (i) $f(z) = \frac{1}{z}$ has singularity at $z=0$ as every nbd of $z=0$ contains ~~infinitely~~ some regular (analytic) point.

(ii) $f(z) = \bar{z}$ This function is nowhere differentiable. Hence no nbd contains any regular points. Thus it has no singularity.

There are two types of singularities:
(i) Isolated Singularity
(ii) Non-isolated Singularity

(i) Isolated Singularity: In every nbd of a particular singularity if there is ~~not~~ no other singularity, then such singularity is isolated.

e.g. (a)
$$f(z) = \begin{cases} z+1 & \text{for } z \neq 0 \\ 3 & \text{for } z = 0 \end{cases}$$

Singularity at $z=0$

(b) $f(z) = \frac{1}{z}$ for $z \neq 0$

Singularity at $z=0$

(c) $f(z) = \sin\left(\frac{1}{z}\right)$, $z \neq 0$

Singularity at $z=0$

(d) $f(z) = \frac{5}{z(z-1)^2(z-3)}$

↳ Singularity at $z = 0, 1, 3$

We can get at least one nbd of $z=0,1,3$ (seperately) where ~~function~~ there is no singularity.

In other words, if function is analytic in the deleted nbd of the singularity → Isolated singularity

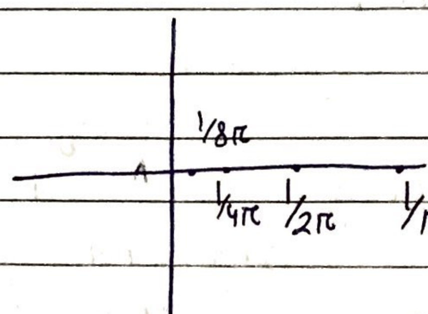
(ii) Non-isolated singularity: If singularity is not isolated, it is non-isolated.

e.g. (i) $g(z) = \text{Log } z$

(ii) $g(z) = \dfrac{1}{\sin\left(\frac{1}{z}\right)}$

Why? Why non-isolated?

$\sin\left(\frac{1}{z}\right) = 0 \Rightarrow z = \frac{1}{n\pi}, n \in \mathbb{Z}$



$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{4\pi}$ ---- → Isolated

$z = 0$ → Non-isolated $\left[\, |z| = r \quad r = \frac{1}{\pi\, 10000} \right.$ ↓ Another singularity in nbd of $z=0$ ∴ $z = 0$ → non-isolated

★ **Pole :-** If z_0 is an isolated singularity and we can find a positive integer 'n' such that -
$(n>0)$

$$\lim_{(z \to z_0)} (z-z_0)^n f(z) = A \neq 0 \text{ [finite Value]}$$

then $z = z_0$ is called pole & 'n' is its order.

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e.g. (i) $\dfrac{z^2}{(z-1)^2(z-3)}$

$\Rightarrow$ For $z_0 = 1$,

$$\lim_{z\to 1} (z-1)^n \frac{z^2}{(z-1)^2(z-3)}$$

For $n = 2$

$$\lim_{z\to 1} (z-1)^2 \frac{z^2}{(z-1)^2(z-3)} = -\frac{1}{2} \neq 0$$

Thus $z = 1$ is a pole of order 2.

$\Rightarrow$ For $z_0 = 3$ & $n = 1$

$$\lim_{z\to 3} (z-3) \frac{z^2}{(z-1)^2(z-3)} = \frac{9}{4} \neq 0$$

Thus, $z = 3$ is a pole of order 1

(ii) $f(z) = \tan z$; Is $z = \frac{\pi}{2}$ a pole?
And, if yes what's order?

Ans $\lim_{z\to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \dfrac{\sin z}{\cos z}$ [For $n = 1$]

Applying L'Hospital law-

$$= \lim_{z\to \frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)\cos z + 1 \times \sin z}{-\sin z} = -1 \neq 0$$

$z = \frac{\pi}{2} \rightarrow$ a pole of order of 1.

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★ Zero of function :-

$z = z_0$ is said to be zero of order m $(m>0)$ if

$$\lim_{z \to z_0} (z-z_0)^{-m} f(z) \neq 0$$ (finite non-zero Value)

e.g. (i) $f(z) = \tan z$

Zeroes → $n\pi$, $n \in Z$

Poles → $(2n+1)\frac{\pi}{2}$, $n \in Z$

(ii) $f(z) = \cot z$
$= \frac{\cos z}{\sin z}$

Poles → $n\pi$, $n \in Z$

Zeroes → $(2n+1)\frac{\pi}{2}$ $n \in Z$

Removable type discontinuity :-

Function is originally discontinuous at particular point, but by redefining the function we can make it continuous.

e.g. (i) $f(x) = \frac{\sin x}{x}$ ↳ discontinuous at $x=0$ [not defined]

But redefining

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$ ✓ continuous

(ii) $f(x) = \begin{cases} x+1 & x \neq 0 \\ 3 & x = 0 \end{cases}$ → not continuous

By redefining → $f(x) = \begin{cases} x+1, & x \neq 0 \\ 1, & x = 0 \end{cases}$

✓ continuous

e.g. (a) $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ [$\sin \frac{1}{x}$ → bounded $x \to 0$]

(b) $\lim_{z \to 0} z \sin \frac{1}{z} \neq 0$ [$\sin \frac{1}{z}$ → not bounded but $z \to 0$]

* Removable singularity :-

An isolated singularity or singular point z_0 is called a removable of $f(z)$ if $\lim_{z \to z_0} f(z)$ exist.

e.g. (i) $f(z) = \frac{\sin z}{z}$ has singularity at $z = 0$

$\lim_{z \to 0} \frac{\sin z}{z} = 1$ [Here singularity $z_0 = 0$ is removable]

(ii) $f(z) = \frac{z+2}{(z^2-4)\, z^2}$

$= \frac{z+2}{(z-2)(z+2)(z^2)}$

$z = -2$ is removable singularity

as –

$\lim_{z \to -2} \frac{z+2}{(z-2)(z+2)z^2} = \lim_{z \to -2} \frac{1}{(z-2)z^2}$

$= \frac{-1}{16}$

(c) $f(z) = z \sin \frac{1}{z}$ does not have removable singularity at $z=0$

[Reason:- $\sin \frac{1}{z} \to$ unbounded hence $\lim_{z\to 0} f(z)$ does not exist]

(d) $f(z) = e^{\frac{1}{z}}$

$$= 1 + \underset{\downarrow \infty}{\frac{1}{z}} + \underset{\downarrow \infty}{\frac{1}{2!\,z^2}} + \underset{\downarrow \infty}{\cdots} \cdots\cdots$$

For Each term in series expansion the lim approaches to ∞.

↳ not removable singularity at $z=0$

↳ $z=0$ is not even pole

↳ It is "essential singularity"

(e) $f(z) = \sin \frac{1}{z}$

↳ not removable singularity at $z=0$

↳ $z=0$ not even pole

↳ It is categorized under "Essential Singularity"

Essential singularities:-

An isolated singularity which is not pole or removable singularity.

Singularity at infinity:-

The type of singularity of $f(z)$ at $z=\infty$ is the same as that $f\left(\frac{1}{z}\right)$ at $z=0$.

e.g. $f(z) = z$ At $z=\infty$, $f(z) \to \infty$.

[Note that, $f(z)$ has to be analytic in z for singularity ...]

$f_1(z) = z^{1/3}$ $\quad$ $f_2(z) = z^{1/2}$

$f_1(z)$ → Triple Valued

$f_2(z)$ → Double Valued

(Their roots repeat as they move in circular fashion)

$f(z) = \log z$

$= \ln|z| + i(\text{Arg } z + 2k\pi)$

↓

roots move in linear fashion → Hence they don't get repeated

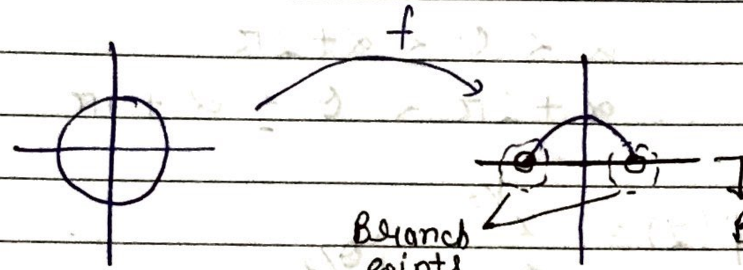
↓

Multi-Valued fn

For $f(z) = z^{1/2}$

$= \sqrt{r}\, e^{i\theta/2}$

Full circle completed →



Half Circle (Branch)

→ Half Circle completed

Branch points

Branch line

z-plane (Input) | w-plane (Output)

z-plane (Input) θ	w-plane (Output) $\frac{\theta}{2}$
$0°$	$0°$
$\frac{\pi}{2}$	$\frac{\pi}{4}$
π	$\frac{\pi}{2}$
2π	π

Branch points ↓ intersection with branch line.

Branch :-

A branch of a multi-valued function f is any single-Valued function 'F' that is analytic in some domain at each point z of which the value $f(z)$ one of value of $f(z)$.

e.g. (i) $f(z) = \text{Log } z$

$\text{Log } z = \ln|z| + i\theta \quad (r > 0, -\pi < \theta < \pi)$

is called principal branch.

(ii) $f(z) = z^{1/2}$

$$= \begin{cases} \sqrt{r}\, e^{i\theta/2} & -\pi < \theta \le \pi \\ \sqrt{r}\, e^{i\theta/2} & \pi < \theta \le 2\pi \end{cases} \quad \text{Branches}$$

~~Principal~~ branch →

$\alpha < \theta \le \alpha + 2\pi$

$\alpha + 2\pi < \theta \le \alpha + 4\pi$

(iii) $f(z) = z^{1/3}$

$= \sqrt[3]{r}\, e^{i\theta/3}$

Argument → $\theta/3$

$$\left.\begin{aligned} \alpha < \theta &\le \alpha + 2\pi \\ \alpha + 2\pi < \theta &\le \alpha + 4\pi \\ \alpha + 4\pi < \theta &\le \alpha + 6\pi \end{aligned}\right\} \text{Branches}$$

To calculate branch point →

e.g. (i) $f(z) = (z^2 + 2z - 3)^{1/2}$

Put $f(z) = 0$

$\Rightarrow z^2 + 2z - 3 = 0 \Rightarrow z = 1, -3$

Branch point

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(ii) $\log(z^2-1) = f(z)$

Put $z^2-1 = 0$

$\Rightarrow z = \pm 1$

(iii) $\log(z-2) = f(z)$

$z-2=0 \Rightarrow z=2 \rightarrow$ Branch point

Branch Cut :-

A branch cut is portion of a line or curve that is introduced in order to define a branch 'F' of a multi-valued function 'f'. Points on the branch cut for F are singular points of F, and any point common to all branch cut is called branch point.

e.g. (i) $f(z) = \dfrac{z^2}{(z-1)^2(z+3)}$

To calculate poles $\rightarrow$

$\lim\limits_{z \to z_0} (z-z_0)^n f(z) =$ non-zero constant

At $z = 1, 3 \rightarrow$ poles

(ii) $f(z) = \tan z$; $C: |z| = 1$

$f(z) = \tan z = \dfrac{\sin z}{\cos z}$

At $z = \pi/2 \rightarrow$ pole

But within $C: |z| = 1 \rightarrow$ we have no pole.

[But we have Zero at $z=0$]

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(iii) $f(z) = \frac{\sin z}{\cos z}$ $C_1 : |z| = 3$

$C_2 : \frac{x^2}{16} + \frac{y^2}{9} = 1$

Within $C_1 \rightarrow$ there is a pole at $z = \frac{\pi}{2}$

$C_2 \rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$

Pole at $z = (2k+1)\frac{\pi}{2}$

For within C_2,

$$\frac{\left[(2k+1)\frac{\pi}{2}\right]^2}{16} + 0 \leq 1$$

$$\Rightarrow (2k+1)^2 \leq \frac{64}{\pi^2}$$

$k \rightarrow 0 \quad \Rightarrow z = \frac{\pi}{2}$ [Pole]

Q Evaluate $\oint_C \frac{5z+7}{(z^2+2z-3)} dz$

where $C : |z-1| = 2$ traversed anti-clockwise

Ans $\oint_C \frac{5z+7}{(z-1)(z+3)} dz = \oint_C \frac{3}{(z-1)} dz + \oint_C \frac{2}{(z+3)} dz$

It has two isolated singularities at $z=1$ & $z = -3$, but $z = 1$ is in circle $C : |z-1| = 2$.

By Cauchy's Theorem,

$\oint_C \frac{2\,dz}{z+3} = 0$ [As Analytic within $C : |z-1| = 2$

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$\therefore \quad I = \oint_c \frac{3}{(z-1)} dz$

Take $|z-1| = 2 \Rightarrow z-1 = 2e^{i\theta}$

$dz = 2ie^{i\theta} d\theta$

$\therefore \quad I = \oint \frac{3}{2e^{i\theta}} \times 2ie^{i\theta} d\theta = \int_0^{2\pi} 3i\, d\theta$

$= 6\pi i$

Q. $\oint \frac{5z+7}{z^2+2z-3} dz \quad ; C : |z| = 2$ Anti-clock wise

Ans. $I = \oint_c \frac{3}{(z-1)} dz + \oint \frac{2\, dz}{(z+3)}$

Again, $\oint_c \frac{2\, dz}{z+3} = 0$ (By Cauchy's theorem)

$\therefore \quad I = \oint \frac{3\, dz}{(z-1)}$

$|z| = 2$ (given)

$\Rightarrow z = 2e^{i\theta}$

$\Rightarrow dz = 2ie^{i\theta} d\theta$

& $z-1 = 2e^{i\theta} - 1$

$\therefore \quad I = \oint \frac{3}{(2e^{i\theta}-1)} \times 2ie^{i\theta} d\theta$

Let $2e^{i\theta} - 1 = t$

$\Rightarrow 2ie^{i\theta} d\theta = dt$

$= \oint \frac{3\, dt}{t} = 3 \ln t$

$= 3\left[\ln(2e^{i\theta}-1)\right]_0^{2\pi}$

$= 3 \ln 1$

$= 3\,[0 + 2\pi i] = 6\pi i$

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Note:- $\oint \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$

where $z_0 \rightarrow$ ~~Isolated~~ point of non-analyticity within the given curve.

Again, Solving above problem,

$$\oint \frac{5z+7}{(z+3)(z-1)} dz = \oint \frac{\frac{5z+7}{(z+3)}}{(z-1)} dz$$

Here $f(z) = \frac{5z+7}{z+3}$ & $z_0 = 1$

$$\therefore \quad I = 2\pi i \left[\frac{5z+7}{z+3}\right]_{at\ z=1}$$

$$= 2\pi i \left[\frac{12}{4}\right] = 6\pi i$$