each point z of which the value $F(z)$ is one of value of $f(z)$.

e.g. (i) $f(z) = \cancel{Lo}\log z$

$\text{Log } z = \ln|z| + i\theta \quad (r > 0, -\pi < \theta < \pi)$

is called principal branch.

(ii) $f(z) = z^{1/2}$

$$= \begin{cases} \sqrt{r}\, e^{i\theta/2} & -\pi < \theta \le \pi \\ \sqrt{r}\, e^{i\theta/2} & \pi < \theta \le 2\pi \end{cases} \Bigg] \text{Branch}$$

~~Principal~~ branch →

$$\alpha < \theta \le \alpha + 2\pi$$
$$\alpha + 2\pi < \theta \le \alpha + 4\pi$$

(iii) $f(z) = z^{1/3}$

$$= \sqrt[3]{r}\, e^{i\theta/3}$$

Argugent → $\theta/3$

$$\left.\begin{array}{l} \alpha < \theta \le \alpha + 2\pi \\ \alpha + 2\pi < \theta \le \alpha + 4\pi \\ \alpha + 4\pi < \theta \le \alpha + 6\pi \end{array}\right\} \text{Branches}$$

To calculate ~~p~~branch point →

e.g. (i) $f(z) = (z^2 + 2z - 3)^{1/2}$

Put $f(z) = 0$

$\Rightarrow z^2 + 2z - 3 = 0 \Rightarrow z = 1, -3$

Branch point

DATE: / /
PAGE NO.:

(ii) $\log(z^2-1) = f(z)$

Put $z^2-1 = 0$

$\Rightarrow z = \pm 1$

(iii) $\log(z-2) = f(z)$

$z-2 = 0 \Rightarrow z = 2 \rightarrow$ Branch point

Branch Cut :-

A branch cut is portion of a line or curve that is introduced in order to define a branch 'F' of a multi-valued function 'f'. Points on the branch cut for F are singular points of F, and any point common to all branch cut is called branch point.

e.g. (i) $f(z) = \dfrac{z^2}{(z-1)^2(z+3)}$

To calculate poles $\rightarrow$

$\lim_{z \to z_0} (z-z_0)^n f(z) = $ non-zero constant

At $z = 1, 3 \rightarrow$ poles

(ii) $f(z) = \tan z$; $C: |z| = 1$

$f(z) = \tan z = \dfrac{\sin z}{\cos z}$

At $z = \pi/2 \rightarrow$ pole

But within $C: |z| = 1 \rightarrow$ we have not pole.

[But we have zero at $z = 0$]

Teacher's Signature

(iii) $f(z) = \frac{\sin z}{\cos z}$ $\quad C_1 : |z| = 3$

$C_2 : \frac{x^2}{16} + \frac{y^2}{9} = 1$

Within $C_1 \rightarrow$ there is a pole at $z = \frac{\pi}{2}$

$C_2 \rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$

Pole at $z = (2k+1)\frac{\pi}{2}$

For within C_2,

$$\frac{\left[(2k+1)\frac{\pi}{2}\right]^2}{16} + 0 \leq 1$$

$$\Rightarrow (2k+1)^2 \leq \frac{64}{\pi^2}$$

$k \rightarrow 0 \Rightarrow z = \frac{\pi}{2}$ [Pole]

Q Evaluate $\oint_C \frac{5z+7}{(z^2+2z-3)} dz$

where $C : |z-1| = 2$ traversed anti-clockwise

Ans $\oint_C \frac{5z+7}{(z-1)(z+3)} dz = \oint_C \frac{3}{(z-1)} dz + \oint_C \frac{2}{(z+3)} dz$

It has two isolated singularities at $z=1$ & $z = -3$, but $z = 1$ is in circle $C : |z-1| = 2$.

By Cauchy's Theorem,

$\oint_C \frac{2\,dz}{z+3} = 0$ [As Analytic within $C : |z-1| = 2$

DATE: / /
PAGE NO.:

$\therefore \quad I = \oint_c \frac{3}{(z-1)} dz$

Take $|z-1| = 2 \Rightarrow \cancel{z} \; z-1 = 2e^{i\theta}$

$dz = 2ie^{i\theta} d\theta$

$\therefore \quad I = \oint \frac{3}{2e^{i\theta}} \times 2ie^{i\theta} d\theta = \int_0^{2\pi} 3i \, d\theta$

$= 6\pi i$

Q $\oint \frac{5z+7}{z^2+2z-3} dz \quad ; C : |z| = 2$ Anti-clockwise

Ans $I = \oint_c \frac{3}{(z-1)} dz + \oint \frac{2\,dz}{(z+3)}$

Again, $\oint_C \frac{2\,dz}{z+3} = 0$ (By Cauchy's theorem)

$\therefore I = \oint \frac{3\,dz}{(z-1)}$

$|z| = 2$ (given)

$\Rightarrow z = 2e^{i\theta}$

$\Rightarrow dz = 2ie^{i\theta} d\theta$

& $z-1 = 2e^{i\theta} - 1$

$\therefore \quad I = \oint \frac{3}{(2e^{i\theta}-1)} \times 2ie^{i\theta} d\theta$

Let $2e^{i\theta} - 1 = t$

$\Rightarrow 2ie^{i\theta} d\theta = dt$

$= \oint \frac{3\,dt}{t} = 3\ln t$

$= 3[\ln(2e^{i\theta}-1)]_0^{2\pi}$

$= 3\ln 1$

$= 3[0 + 2\pi i] = 6\pi i$

Teacher's Signature

Note:- $\oint \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$

where $z_0 \rightarrow$ ~~Isolated~~ point of non-analyticity within the given Curve.

Again, Solving above proe, problem,

$$\oint \frac{5z+7}{(z+3)(z-1)} dz = \oint \frac{\frac{5z+7}{(z+3)}}{(z-1)} dz$$

Here $f(z) = \frac{5z+7}{z+3}$ & $z_0 = 1$

$$\therefore \quad I = 2\pi i \left[\frac{5z+7}{z+3}\right]_{at\ z=1}$$

$$= 2\pi i \left[\frac{12}{4}\right] = 6\pi i$$

★ Cauchy's integral Formula statement:-

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and simply closed curve C in D that encloses z_0.

Analytic part

$$\oint_c \frac{(f(z))}{(z-z_0)} dz = 2\pi i f(z_0)$$

The integration is taken anti-clockwise direction.

e.g. (i) $\oint \frac{\cos z}{(z-1)\, e^z}$; $C: |z| = 2$ in clockwise direction

$$I = \oint_c \frac{\left(\frac{\cos z}{e^z}\right)}{(z-1)} dz$$

DATE : / /
PAGE NO. :

Here $f(z) = \frac{\cos z}{e^z}$; $z_0 = 1$

$$= -2\pi i\, f(1) = -2\pi i \frac{\cos 1}{e}$$

(-ve sign because it is traversed clockwise direction)

(ii) $I = \oint \frac{z}{z^2 - \pi^2}\, dz$

$C : |z-1| = 1$ anti-clockwise

Ans

$$\oint \frac{z}{(z-\pi)(z+\pi)}\, dz$$

We have problem with $z = \pm\pi$ in terms of analyticity.

But $z = -\pi$ does not lie within C.

$$\therefore f(z) = \frac{z}{(z+\pi)}$$

So, $I = 2\pi i \times f(\pi) = 2\pi i \times \frac{\pi}{2\pi}$

$$= \pi i$$

(iii) $I = \oint_C \frac{e^z}{(z^2-1)}\, dz$; $C : |z| = 2$ anti-clockwise

Ans $I = \oint \frac{e^z}{(z-1)(z+1)}\, dz$

$$= \oint \frac{e^z}{2(z-1)}\, dz - \oint \frac{e^z}{2(z+1)}\, dz$$

$$= 2\pi i \frac{e^1}{2} - 2\pi i \frac{e^{-1}}{2}$$

$$= \pi i\,(e - e^{-1})$$

Teacher's Signature

(iv) $I = \oint_c \frac{2z + \sin z}{(z-1)^3}$ $\quad C: |z| = \pi$

Proof of Cauchy's integral formula :-

Let 'C' be closed curve with center z_0 lying in D.

Now,

$$f(z) = f(z) - f(z_0) + f(z_0)$$

$$\oint \frac{f(z)}{(z-z_0)} dz = \oint_C \frac{f(z) - f(z_0) + f(z_0)}{(z-z_0)} dz$$

$$= \oint_C \frac{f(z) - f(z_0)}{(z-z_0)} dz + \oint_C \frac{f(z_0)}{(z-z_0)} dz$$

$$= \oint_C \frac{f(z) - f(z_0)}{(z-z_0)} dz + f(z_0)\, 2\pi i \quad -①$$

$$\left[\because \oint_C \frac{1}{(z-z_0)} dz = \oint (z-z_0)^{-1} dz = 2\pi i\right]$$

Properties

Note :- (i) $|z| \leq 0$ has only one solution $z = 0$

(ii) $|\Sigma(z_1 + z_2 + z_3 + \dots)| \leq \Sigma(|z_1| + |z_2| + |z_3| + \dots)$

(iii) $\left|\int_c f(z)\, dz\right| \leq \int_c |f(z)|\, dz$

DATE : / /
PAGE NO. :

Using (iii) property,

$$\left| \oint \frac{f(z)-f(z_0)}{(z-z_0)} dz \right| \leq \oint \frac{|f(z)-f(z_0)|}{|z-z_0|} dz \quad — ②$$

$f(z)$ is analytic $\Rightarrow f(z)$ is continuous at z_0.

We can get $\varepsilon > 0$ for every $\delta > 0$

Definition of continuity → $|f(z)-f(z_0)| < \varepsilon$, whenever $|z-z_0| < \delta$

Let $|z-z_0| = r < \delta$

$$\therefore \left| \oint \frac{f(z)-f(z_0)}{(z-z_0)} dz \right| \leq \oint \frac{\varepsilon}{r} dz \quad [\text{From } ②]$$

$$\leq \frac{\varepsilon}{r} \oint_C dz$$

$$\leq \frac{\varepsilon}{r} \times 2\pi r$$

$$\leq 2\pi\varepsilon$$

$$\Rightarrow \left| \oint \frac{f(z)-f(z_0)}{(z-z_0)} dz \right| \leq 2\pi\varepsilon$$

When $\varepsilon \to 0$ we get,

$$\oint \frac{f(z)-f(z_0)}{(z-z_0)} = 0 \quad [\text{By (i) property}]$$

So, in ①

$$\oint \frac{f(z)}{(z-z_0)} dz = f(z_0) * 2\pi i \rightarrow \text{H.P.}$$

Teacher's Signature

Generalized form of Cauchy's integral formula

$$\oint \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} \left[\frac{d^n f(z)}{dz^n}\right]_{at\ z=z_0}$$

e.g. $I = \oint \frac{2z + \sin z}{(z-\pi)^3}$ $C: |z| = 2\pi$ anti-clockwise

Ans $f(z) = 2z + \sin z$, $z_0 = \pi$

$$I = \frac{2\pi i}{2!} \left(\frac{d^2[2z + \sin z]}{dz^2}\right)_{z = z_0}$$

$$= \frac{2\pi i}{2!} (-\sin \pi)$$

$$= 0$$

Q 1. $\oint \tan z \, dz$ $C: |z| = 3$ → not solvable by C.I.F.

Q 2. $\oint \frac{e^z}{e^z - 1} dz$ $C: |z| = 1$

★ Cauchy Inequality :- Let $f(z)$ is analytic in D and z_0 be a point in D, and 'C' be a closed curve (disk) of radius 'r' lying in D.

If $|f(z)| \leq M$ for all z inside and on C. Then-

$f^n \rightarrow$ denotes n^{th} derivative of f.

DATE : / /

PAGE NO. :

for any positive integer 'n'

$$|f^n(z_0)| \leq \frac{Mn!}{r^n}$$

Proof :- As $f(z)$ is analytic at z_0 lying in C_0 We can get -

$$|f^n(z_0)| = \left| \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

[By previous Cauchy's generalized integral formula]

$$|f^n(z_0)| \leq \frac{n!}{2\pi i} \oint \frac{|f(z)|}{|z-z_0|^{n+1}} dz$$

$$\leq \frac{n!}{2\pi i} \oint \frac{M}{r^{n+1}} dz = \frac{n!}{2\pi i} \frac{M}{r^{n+1}} (2\pi r)$$

$$= \frac{n!\, M}{r^n} \quad \text{— H.P.}$$

$$\left[\begin{array}{l} |f(z) \leq M \rightarrow \text{given} \\ |z-z_0| = r \end{array} \right]$$

☆ Liouville Theorem :- Every entire & bound function in C is constant.

Proof :- We have given that $f(z)$ is entire in C and $f(z) \leq M$

$$|f'(z_0)| \leq \frac{1!\, M}{r} = \frac{M}{r} \rightarrow 0$$

[As $r \rightarrow \infty$]

↳ As function is entire

Teacher's Signature

Scanned by CamScanner

DATE: / /
PAGE NO.:

$\Rightarrow \quad f'(z_0) = 0$

$f(z_0'') = 0$