15th sept'25

Date

Partial Differential Equations
(PDEs)

DE

ODE   PDE → study of several variable functions.

Linear   Non-linear

$$y'' + a(x)y' + b(x)y = c(x)$$

$y = y(x)$

y ↑ dependent; x ↑ independent variable.

$\{y_1, y_2\}$ = Linearly Independent

sol^n space:

$$\{\alpha_1 y_1 + \alpha_2 y_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$$

PDE : (i) Two or more independent variables

(ii) Sol^n space in no longer finite dimension.

Date

Applications :

(i) Navier - Stoke's Eq^n

$$u_t + (u\cdot\nabla)u = -\nabla p + \nu\Delta u \quad \text{in } \Omega\times[0,T) \quad (\subset \mathbb{R}^3)$$

$$\nabla u = 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

$$\nu > 0$$

(ii) Black - Schole's Equation , Value of green stock

$$f_t + rSf_S + \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad , \quad f = f(S)$$

r = risk free interest

σ = volatility const.

(iii) Image Processing / restoration

There are 3 methods

1. Strategic modelling
2. Wavelet
3. PDE —— (i) Closely related to Physical World
   (ii) Theory of PDE is well established

$$u_t = \operatorname{div}(c(|\nabla u|^2)\nabla u) \quad \Omega\times[0,T)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega\times(0,T)$$

$$u(x,0) = u_0(x) \text{ in } \Omega$$

Date

Stampede Problem

→ to study movement of crowd.

→ Continuity eq^n & Momentum eq^n.

M. Galiardi, 2012

$$\frac{\partial \rho}{\partial t} + \nabla(\rho u) = 0$$ → continuity eq^n. (ρ = density)

$$\rho u_t + \rho u u_x = F_A$$ → Momentum eq^n (F_A = free activity on the object)

PDE: An eq^n involving two independent variables, one ~~and~~ dependent variable and its derivatives.

$$F(x, y, z, z_x, z_y) = 0$$

$$F(x, y, t, \ldots\ldots, z, z_x, z_y, z_t, \ldots) = 0$$

$z = z(x, y, t, \ldots) \in C^n$ on some domain D.

Order of PDE

If highest ordered derivative appears in the PDE, that is its order.

Classification of PDE

(i) Quasi-linear PDE

→ A PDE is said to be Quasi-linear if the highest order derivatives which occur in the eq^n are linear.

What makes you happy?

#HappyCollegeDays #

(We're allowing non-linearity for lower order derivative)

Date

Eg. $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

p = partial derivative of z wrt x = z_x

q = ——"—— y —"— = z_y

$$xy^2z^3p + yzq = e^x yz^5$$

(ii) Semi-linear PDE

→ A quasi linear PDE is said to be semi linear if the coefficients of the highest order derivatives do not contain either the dependent variable or it's derivatives.

Ex. $P(x,y)p + Q(x,y)q = R(x,y,z)$

$$xp + yq = e^x z^2$$

we're allowing Non-linearity.

* All semi-linear PDEs are Quasi-linear PDEs but not the other way round.

(iii) Linear PDE

→ A semi-linear PDE is sd to be linear if it is linear in dependent variable and its derivatives.

$$P(x,y,\cancel{z})p + Q(x,y)q = R(x,y)z + S(x,y)$$

↳ $xp + y^2q = x^2z + xy^3$

#

#

Date

(IV) Non-Linear PDE

→ Eqⁿ which do not come under the previously mentioned 3 PDE categories

$\boxed{f(x, y, z, p, q) = 0}$

Examples :

(1) $\underline{uu_{xx}} + u_t = xt$ → 2nd order

highest order derivative is in linear form ⇒ Quasi-Linear

coeff of u_{xx} = u → dependent vari

& hence it can't be semi-linear

but not linear in derivative of dependent ⇒ not linear

(2) $\underline{xu_{xx}} + uu_t = xt$ → 2nd order

linear ⇒ Quasi-linear

coeff of highest ord derivative is independent ⇒ Semi-Linear

But its not linear in derivative of dep. var so its not linear.

(3) $\underline{xu_{xx}} + u_t = xt$ → 2nd order

Quasi-linear

~~coeff doesn't contain dependent var & its derivative so not semi-linear~~

⇒ linear.

Date

(4) $(u_{xx})^2 + u_t = xt$ → 2nd order

Not linear ⇒ Non-linear ✓

(5) $u_{xx} + (u_t)^2 = xt$ → 2nd order

Quasilinear ⇒ Semi-linear ✓

(Classification – I.N. Sneddon, T. Amarnath)

Genesis of PDE

$$x^2 + y^2 + (z-c)^2 = a^2 \quad \text{--- (1)}$$

a, c → arbitrary const.

(Eqⁿ of spheres whose centres are on z-axis with ~~centre c~~ radius 'a'.)

Differentiating (1) wrt x,

$2x + 2(z-c)z_x = 0$

⇒ $x + (z-c)z_x = 0$ ⇒ $x + (z-c)p = 0$ --- (2)

Differentiating (1) wrt y,

$2y + 2(z-c)z_y = 0$

⇒ $y + (z-c)q = 0$ --- (3)

From (2) and (3), we get,

$$\boxed{xq - py = 0}$$

$F(x, y, z, a, b) = 0$ --- (4)

a, b → constants

Date

Differentiating ④ wrt x.

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\cancel{\frac{\partial y}{\partial x}}^{0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot z_x = 0 \quad —⑤$$

||ly Differentiating ④ wrt y,

$$\Rightarrow \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\cdot z_y = 0 \quad —⑥$$

$f(x,y,z,p,q) = 0$

↳ It can be non-linear.

So the system of surfaces F satisfies the above eqn.

16th sept'15

$F(x,y,z,a,b)=0$ a,b constants

↓

~~give rise to PDE~~ ~~⇒ f( ) is soln of F.~~

$\boxed{f(x,y,z,p,q)=0}$ → It need not be linear

where $p = z_x$; $q = z_y$

Eg.

$z(x,y) = xy + f(x^2+y^2)$

Diff wrt x,

$z_x = p = y + f'(2x)$ —①

Diff wrt y,

$z_y = q = x + f'(2y)$ —②

$y\times① - x\times② \Rightarrow \underline{yp - xq = y^2 - x^2}$

↓

$py - qx = y^2 - x^2 + \cancel{2xy f'(x^2+y^2)} - \cancel{2xy f'(x^2+y^2)}$ Linear first order PDE.

Eg. $z = f(x/y)$

$z_x = p = f'(1/y)$ | $z_y = q = f'\cdot\left(-\frac{x}{y^2}\right)$

$\frac{x}{y} p + q = 0 \Rightarrow \boxed{xp + yq = 0}$

Eg. $z = (x+a)(y+b)$

$z_x = p = (y+b) \Rightarrow b = p-y$

$z_y = q = (x+a) \Rightarrow a = q-x$

$\Rightarrow z = (x+q-x)(y+p-y) \Rightarrow \boxed{z = q\cdot p}$ → Non linear first order PDE

Surface of Revolution

→ All surface of revolutions with z-axis as the axis of revolution can be written as:

$z = F(r)$; $r = \sqrt{x^2+y^2}$

$z_x = p = F'(r)\frac{\partial r}{\partial x} = F'(r)\cdot\frac{x}{r}$

$\frac{\partial r}{\partial l} = \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot 2x = \frac{x}{r}$

$z_y = q = F'(r)\frac{\partial r}{\partial y} = F'(r)\cdot\frac{y}{r}$

Elliminating F', we get $\boxed{yp - xq = 0}$ → First order linear PDE.

Consider the surfaces of the form,

$F(u,v) = 0$ —(1)

where, $u = u(x,y,z)$ and $v = v(x,y,z)$ are known as function of x, y, z.

Now, Differentiating ① wrt x,

Date

$$\frac{\partial F}{\partial u}\cdot\frac{du}{dx} + \frac{\partial F}{\partial v}\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) +$$

$$\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial u}\left(u_x + p u_z\right) + \frac{\partial F}{\partial v}\left(v_x + p v_z\right) = 0 \quad \text{— ②}$$

Q(4)(b

||ly, Differentiating ② wrt y,

$$\frac{\partial F}{\partial u}\left(u_y + q u_z\right) + \frac{\partial F}{\partial v}\left(v_y + q v_z\right) = 0 \quad \text{— ③}$$

Elliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from ① and ②, we get

Jacobian

$$\boxed{\frac{\partial(u,v)}{\partial(y,z)}\,p + \frac{\partial(u,v)}{\partial(x,z)}\,q = \frac{\partial(u,v)}{\partial(x,y)}} \quad \text{— ④}$$

↳ linear first order.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

Q(5

Eg. $(x-a)^2 + (y-b)^2 + z^2 = 1$

Diff wrt $x \Rightarrow$ $2(x-a) + 2z\,z_x = 0 \Rightarrow x - a + z.p = 0$

Diff wrt $y \Rightarrow$ $2(y-b) + 2z\,z_y = 0 \Rightarrow y - b + z.q = 0$

$\Rightarrow a = x + zp$

$b = y + z.q$

$\therefore (x - x - zp)^2 + (y - y - z.q)^2 + z^2 = 1$

$\Rightarrow \boxed{z^2(p^2 + q^2 + 1) = 1}$ ⟶ Non linear P.D.E.

#

A:6 Q2)c)

A:6 Eg. Q2)c) $f(x-z, y-z) = 0$

$u = x - z \quad , \quad v = y - z \quad s.t.$

$f(u,v) = 0$

$$\frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = 1$$

$$\frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} u_z & u_x \\ v_x & v_z \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} = -1$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

PDE is $\boxed{p - q = 1}$ → linear PDE.

Q(4)(b) $(x^2+y^2)\cos^2\theta - (z-c)^2\sin^2\theta = 0$

$x^2+y^2+(z-c)^2 = a^2$

$\boxed{qx - py = 0}$

~~Diff wrt x,~~

$2x\cos^2\theta - 2(z-c)p\sin^2\theta = 0$ —①

Diff wrt y,

$2y\cos^2\theta - 2(z-c)q\sin^2\theta = 0$ —②

We need to elliminate constant 'c' from the two equations

① × q and ② × q

$xq\cos^2\theta - 2(z-c)pq\sin^2\theta = 0$

$yp\cos^2\theta - 2(z-c)pq\sin^2\theta = 0$

$\Rightarrow \boxed{xq - yp = 0}$ → 1^{st} order PDE

Q(5)(c) $u(x,y) = f(x-ay) + g(x+ay)$

Diff wrt x, $u_x = f' + g'$ | $u_{xx} = f'' + g''$

Diff wrt y, $u_y = f'(-a) + g'(-a)$ | $u_{yy} = f''(-a)(-a) + g''(a)(-a)$

$= a(g' - f')$ | $= a^2(f'' + g'')$

$\Rightarrow u_{yy} = a^2 u_{xx}$

$\boxed{u_{yy} - a^2 u_{xx} = 0}$ → Well known wave Eq^n in 1D.

How to find solⁿ of first order PDE

- Very difficult for non linear ; only linear and quasilinear will be covered here.

$f(x, y, z, p, q) = 0$ ——— (1) 1^{st} order PDE

z has to be smooth.

$z = z(x, y) \Rightarrow$ Smooth means continuously diff for $(x, y) \in D$

Defⁿ: A ~~function~~ solution $z = z(x, y)$ in 3D space can be interpreted as surface and hence called integral surface of the PDE.

- **Complete Integral or Complete Solution**

The relⁿ of the type $f(x, y, z, a, b) = 0$ led to PDE of the first order. Any such relⁿ which contains 2 arbitrary constants $\boxed{a \,\&\, b}$ and is a $\boxed{\text{sol}^n \text{ of } 1^{st} \text{ order}}$ is said to be complete solⁿ or complete integral of that eqⁿ.

- **General Integral / General Solution**

$F(u, v) = 0$

→ Involves an arbitrary function F connecting two known functions $u = u(x, y, z)$ & $v = v(x, y, z)$.

→ Provides a solⁿ of 1^{st} order PDE is called general solⁿ or integral.

Date

Singular Integral

We get a sol^n by finding envelope of the parameter family. This is obtained by elliminating a & b from the eq^n.

$\boxed{z = F(x,y,a,b) \quad , \quad F_a = 0 \quad , \quad F_b = 0}$ is called singular sol^n.

envelope of two parameter family

If we elliminate a & b from above three, then we get a new sol^n called singular sol^n.

Eg. $F(x,y,z,p,q) = z - px - qy - p^2 - q^2 = 0$

$\rightarrow \quad z = F(x,y,a,b)$

$F(x,y,a,b) = z = ax + by + a^2 + b^2$

is a complex integral

$F_a = x + 2a = 0 \quad \Rightarrow \quad a = -x/2$

$F_b = y + 2b = 0 \quad \Rightarrow \quad b = -y/2$

$\Rightarrow \quad z = \frac{-x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4} \quad \Rightarrow \quad \boxed{4z = -(x^2+y^2)}$

$\Rightarrow 4z = -(x^2+y^2)$

18th sept '15 — **Cauchy Problem (Initial Value Problem)**

$f(x,y,z,p,q) = 0$

Find an integral of (1) by combining an initial curve, $F: x = x_0(s) \;,\; y = y_0(s) \;,\; z = z_0(s) \;,\; s \in I$

The Cauchy's Problem is to find $z = z(x,y)$ of (1) s.t. $z_0(s) = z(x_0(s), y_0(s)) \quad \forall\, s \in I$

Solution for 1st order Quasilinear PDE

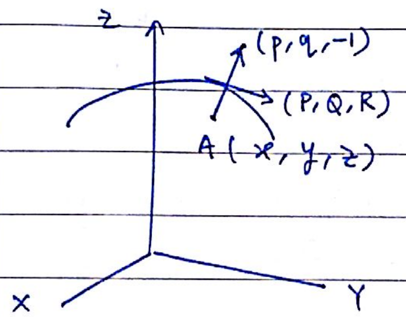
$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$ —— ②

P, Q, R are smooth on some domain D.

$P, Q, R : D \to \mathbb{R}$

$P, Q, R \in C'$ which doesn't vanish simultaneously.

$S = [z(x,y) : (x,y) \in D \subset \mathbb{R} \times \mathbb{R}]$ is the integral surface of PDE ②.



$(p,q,-1)$ is the dir^n of normal derivative to the integral surface S at pt $A \in S$.

Eq^n ② is equivalent to saying that (P,Q,R) and $(p,q,-1)$ are orthogonal at each point on S.

$P\hat{i} + Q\hat{j} + R\hat{k}$ lies on the tangent plane at A.

The curve $S: x = x(t), \; y = y(t), \; z = z(t)$, we have $P\hat{i} + Q\hat{j} + R\hat{k} \parallel \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$

Date

$$\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{z}}{R} \quad (= k' \text{ (say)})$$

$$\Rightarrow \boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \text{ — ③} \qquad \left(= dt \cdot k' = k \right)$$

we'll get P, Q, R from this eqⁿ ②.

Characteristic Equation or Characteristic Curve.

[Quasi-linear PDE → document uploaded on M-3 course website]

Lagrange Method

Theorem P, Q, R are constant function.
$F(u,v)=0$ is the general solⁿ of ② where F is an arbitrary smooth funcⁿ of u & v and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are two independent solⁿ of characteristic eqⁿ ③.

$\boxed{u = c_1 \;\&\; v = c_2}$ — characteristic curve.

Proof $u = c_1$ and $v = c_2$

$\Rightarrow du = 0$ and $dv = 0$

$u_x\,dx + u_y\,dy + u_z\,dz = 0$

Now, $\left.\begin{aligned} dx &= kP, \\ dy &= kQ \\ dz &= kR \end{aligned}\right\}$ from eqⁿ ③

$$\Rightarrow \left.\begin{aligned} &u_x P + u_y Q + u_z R = 0 \\ \text{And } \text{ll}^y \; &v_x P + v_y Q + v_z R = 0 \end{aligned}\right\}$$

After solving these two equations, for P, Q and R

we get,

#

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(x,z)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \quad \text{--- (4)}$$

$F(u,v) = 0$ leads to 1^{st} order linear PDE:

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)} \quad \text{--- (5)}$$

comparing (4) and (5), we get

$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

$f(x, y, z, a, b) = 0$

Ex 6) $x^2p + y^2q - (x+y)z = 0$

$P(x,y,z) = x^2$, $Q(x,y,z) = y^2$

$R(x,y,z) = (x+y)z$

Characteristic curve

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \underbrace{\frac{dz}{(x+y)z} = \frac{dx-dy}{x^2-y^2}}$$

Solving this

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = c_1 = u(x,y,z)$$

$$\frac{d(x-y)}{x-y} = \frac{dz}{z}$$

$$\ln(x-y) = \ln z + \ln c_2$$

$$\Rightarrow x - y = zc_2$$

$$\Rightarrow \frac{x-y}{z} = c_2 = v(x,y,z)$$

$F(u,v)=0$

$\Rightarrow v - G_n(u) = 0$

$$\boxed{\frac{x-y}{z} = G_n\left(\frac{1}{x}-\frac{1}{y}\right)}$$

The general solⁿ $F(c_1,c_2)=0 \Rightarrow \boxed{F\left(\frac{1}{x}-\frac{1}{y}, \frac{x-y}{z}\right)=0}$

Eg 6)(a): $xp + yq = z$

which contains $\Gamma : x_0 = s^2,\ y_0 = s+1,\ z_0 = s$

Solⁿ: Characteristic eqⁿ: $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \frac{y}{z} = c_1 = u(x,y,z)$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{y}{x} = c_2 = v(x,y,z)$$

$F(u,v)=0 \quad = \quad F(c_1,c_2)$

$$\frac{s+1}{s} = c_1 \quad , \quad \frac{s+1}{s^2} = c_2$$

y/x

$\Rightarrow (c_1 - 1)c_1 = c_2$

$$\left(\frac{y}{z}-1\right)\frac{y}{z} = \frac{y}{x}$$

$$\Rightarrow \frac{y-z}{z^2} = \frac{1}{x}$$

$$\Rightarrow xy - xz = z^2$$

$$\Rightarrow \boxed{(y-z)x = z^2}$$

Ass-6

Date 21.09.15

6) c) $yzp + xzq = xy$

LAGRANGE'S Method to solve QUASI-LINEAR :

Characteristic Eq^n : $\dfrac{dx}{yz} = \dfrac{dy}{xz} = \dfrac{dz}{xy}$

$\dfrac{dx}{y\cancel{z}} = \dfrac{dy}{x\cancel{z}} \Rightarrow x\,dx - y\,dy = 0$

$\Rightarrow x^2 - y^2 = c_1 = u(x, y, z)$

$\dfrac{dy}{\cancel{x}z} = \dfrac{dz}{\cancel{x}y} \Rightarrow y\,dy - z\,dz = 0$

$\Rightarrow y^2 - z^2 = c_2 = v(x, y, z)$

General sol^n

$F(u, v) = 0$

i.e. $F(x^2 - y^2, y^2 - z^2) = 0$

$y^2 - z^2 = G(x^2 - y^2)$

Q 6) d) $(z^2 - 2yz - y^2)p + x(y+z)q = x(y-z)$

$$\frac{dx}{z^2 - 2yz - y^2} = \underbrace{\frac{dy}{(y+z)x} = \frac{dz}{x(y-z)}}$$

$(y-z)\,dy = (y+z)\,dz$

$y\,dy - z\,dy - y\,dz - z\,dz = 0$

$y\,dy - z\,dz - \underbrace{(z\,dy + y\,dz)}_{= d(yz)} = 0$

~~2 PDE~~

exact differential

$\Rightarrow \dfrac{y^2}{2} - \dfrac{z^2}{2} - yz = c_1'$

Q (7) (Ass-6)

What makes you happy?

#HappyCollegeDays

Scanned by CamScanner

$$\Rightarrow \quad y^2 - z^2 - 2yz = C_1 = u(x,y,z)$$

Now, using $\left(\dfrac{dx}{z^2-2yz-y^2}\right)\times\dfrac{x}{x}$, $\left(\dfrac{dy}{x(y+z)}\right)\times\dfrac{y}{y}$ and $\left(\dfrac{dz}{x(y-z)}\right)\times\dfrac{z}{z}$

and then adding

$$\frac{x\,dx + y\,dy + z\,dz}{z^2\cancel{x} - 2\cancel{xyz} - \cancel{xy^2} + \cancel{xy^2} + \cancel{xyz} + \cancel{xyz} - \cancel{xz^2}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow \quad \underbrace{x\,dx + y\,dy + z\,dz}_{\text{exact differential}} = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = C_2 = v(x,y,z)$$

General Solⁿ:

$$F(y^2 - z^2 - 2yz,\ x^2 + y^2 + z^2) = 0$$

or $\quad x^2 + y^2 + z^2 = G(y^2 - z - 2yz)$

or $\quad y^2 - z^2 - 2yz = H(x^2 + y^2 + z^2)$

Q(7) (Asd-6) $\quad 2p + 3q + 8z = 0$

(a) $\Gamma : z = 1 - 3x$ and the line $y = 0$

(b) $\Gamma : z = x^2$ on the line $2y = 1 + 3x$

(c) $\Gamma : z = e^{-4x}$ on the line $2y = 3x$

$$\underbrace{\frac{dx}{2} = \frac{dy}{3}} = \frac{dz}{-8z}$$

$$-4x = \ln z + \ln c_2'$$

$$2\,dy - 3\,dx = 0$$

$$\Rightarrow 2y - 3x = c_1 = u(x,y,z)$$

$$e^{-4x} = z c_2'$$

$$\Rightarrow z e^{4x} = c_2 = v(x,y,z)$$

General solⁿ : $F(2y-3x,\ z e^{4x}) = 0$

$$\boxed{z e^{4x} = G(2y-3x)}$$

$s = -3x$
$x = -s/3$

??

(a) $1 - 3x = e^{-4x}\,G(-3x)$

$1 + s = e^{\frac{4s}{3}}\,G(s)$

$\Rightarrow G(s) = e^{-4s/3}(1+s)$

$$z = e^{-4x}\, e^{-\frac{4(2y-3x)}{3}}\,(1+2y-3x)$$

UNIQUE SOLUTION

(b) $z = x^2$

$x^2 = e^{-4x}\,G(1)$

~~Exponential type~~

$\Rightarrow$ It can be bounded.

But x^2 can't be bounded and hence, x^2 can't be written in terms of exponential. hence, NO SOLUTION

(c) $\cancel{e^{-4x}} = \cancel{e^{-4x}}\,G(0)$

$\Rightarrow G(0) = 1$

we can have many G, satisfying this condition such as, $G(t) = e^t$, $\cos t$, $1+t$, $1+t^2$, ...

we can have ∞ solutions for $G(t)$ and hence here its uniqueness fails.

$\Rightarrow$ INFINITE SOLUTIONS

8) i) ?

$(2xy-1)p + (z-2x^2)q = 2(x-yz)$

$\Gamma: \; x_0(s) = 1, \; y_0(s) = 0, \; z_0(s) = s$

Date

sol^n:

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}$$

$$\frac{z\,dx + dy + x\,dz}{2xyz - \cancel{z} + \cancel{z} - 2x^2 + 2x^2 - 2xyz} = \frac{2x\,dx + 2y\,dy + dz}{0}$$

independent solutions

$\Rightarrow \begin{cases} z\,dx + dy + x\,dz & \Rightarrow xz + y \; \cancel{+xz} = c_1 = u(x,y,z) \\ x^2 + y^2 + z = c_2 = v(x,y,z) \end{cases}$

General sol^n: $x^2+y^2+z = G(xz+y)$

$1 + s = G(s)$

$$\boxed{x^2+y^2+z = 1 + xz + y}$$

Non-linear 1st order PDE : Charpit's Method

Second Order PDE

$\rightarrow f(x, y, u, u_{xx}, u_{xy}, u_{x}, u_y) = 0$

where,

$u = u(x, y)$

$\rightarrow f(x, y, t, \ldots, u, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, \ldots, u_x, u_y, u_t \ldots) = 0$

More than 2 independent variables $(x, y, t, \ldots)$

$u = u(x, y, t, \ldots)$

eg

$u = f(x+t)$

$u_x = f'$, $u_{xx} = f''$

$u_t = f'$, $u_{tt} = f'$

$u_{tt} = u_{xx}$

$\Rightarrow \boxed{u_{tt} - u_{xx} = 0}$

eg. vibration of strings.

What makes you happy?
#HappyCollegeDays

Scanned by CamScanner

2^{nd} order derivatives :

Date

Semi-linear 2^{nd} order PDE $\rightarrow Lu + g(\ ) = 0$

$$\underbrace{R(x,y)\, u_{xx} + S(x,y)\, u_{xy} + T(x,y)\, u_{yy}}_{\text{Principal part}} + g(x, y, u, u_x, u_y) = 0 \quad \text{—(1)}$$

$\underbrace{R^2 + S^2 + T^2 \neq 0}_{\text{R, S \& T do not vanish simultaneously}}$, R, S, T are cont. on some domain D.

Regular solution

sol^n: A function $u = u(x,y)$ is said to be regular sol^n of (1) in a domain $D \subset \mathbb{R} \times \mathbb{R}$ if $u \in C^2(D)$ and 'u' satisfies eq^n (1).

CLASSIFICATION :

$S^2 - 4RT < 0$ Ellipse type	$ax^2 + bxy + cy^2 + dx + ey + R = 0$
eg: Laplace Eq^n	$b^2 - 4ac < 0 \rightarrow$ Ellipse
$= 0$ Parabolic type	$= 0 \rightarrow$ Parabola
eg: Heat Eq^n	$> 0 \rightarrow$ Hyperbola
> 0 Hyperbolic type	$\frac{x^2}{a} + \frac{y^2}{b} = 1$
eg: Wave Eq^n	$0 - 4 \cdot \frac{1}{a} \cdot \frac{1}{b} < 0$

Eg. $u_{xx} + x^2 u_{yy} = 0$

$R = 1, \quad S = 0, \quad T = +x^2$

$S^2 - 4RT = 0 - 4(+x^2) = -4x^2 < 0 \ \forall x$ $\rightarrow$ Elliptical type.

Eg. $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$

$R = y^2, \quad S = -2xy, \quad T = x^2$

$S^2 - 4RT = 4x^2y^2 - 4y^2x^2 = 0 \quad \forall x \,\&\, y$ $\rightarrow$ Parabolic type.

Date 22·09·15

Canonical form

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0 \qquad \boxed{\xi, \eta}$$

$$(x,y) \rightsquigarrow (\xi,\eta)$$

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$

$$u(x,y) \longrightarrow u(\xi,\eta)$$

$$\rightarrow \quad u_x = \underline{u_\xi \xi_x} + u_\eta \eta_x$$

$$u_{xx} = (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + u_\xi \xi_{xx} + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\eta \eta_{xx}$$

$$\rightarrow \quad u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{yy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + u_\xi \xi_{yy} + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\eta \eta_{yy}$$

$$\rightarrow \quad u_{xy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + u_\xi \xi_{xy}$$

$$= (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_x + u_\eta \eta_{xy}$$

Now, $R u_{xx} + S u_{xy} + T u_{yy}$

$$= u_{\xi\xi}(R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2) + u_{\xi\eta}\left[2R\xi_x\eta_x + S(\xi_x\eta_y + \eta_x\xi_y) + 2T\xi_y\eta_y\right]$$

$$+ u_{\eta\eta}(R\eta_x^2 + S\eta_y\eta_x + T\eta_y^2) + h(\xi,\eta,u,u_\xi,u_\eta)$$

#

#

eq^n ① becomes

$$A(\xi_x, \xi_y) u_{\xi\xi} + 2B(\xi_x, \xi_y, \eta_x, \eta_y) u_{\xi\eta} + A(\eta_x, \eta_\xi) u_\eta = G(\xi, \eta, u, u_\xi, u_\eta) \quad \text{—②}$$

$$A(u, v) = Ru^2 + Suy + Tv^2$$

$$B(u_1, v_1; u_2, v_2) = R u_1 u_2 + \frac{1}{2} S(u_1 v_2 + u_2 v_1) + T v_1 v_2$$

Eg. $$B^2 - A(\xi_x, \xi_y) A(\eta_x, \eta_y) = \frac{(S^2 - 4RT)(\xi_x \eta_y - \xi_\eta \eta_x)^2}{4} \quad \text{—③}$$

Verify this!

Case I $\quad S^2 - 4RT > 0$

$R\lambda^2 + S\lambda + T = 0 \rightarrow$ using this, get λ_1 & λ_2.

Eq^n has two real & distinct roots, $\lambda(x,y)$ & $\lambda_2(x,y)$

How do we choose $\xi(x,y)$ & $\eta(x,y)$?

Choose ξ & η s.t.

$\xi_x = \lambda_1 \xi_y$ & $\eta_x = \lambda_2 \eta_y$

$\xi = c_1$

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$$

$$= \boxed{\frac{dy}{dx} + \lambda_1 = 0}$$

$f_1(x,y) = c_1$

$\boxed{\xi = \lambda_1(x,y)}$

η is sol^n of the ODE

$$\boxed{\frac{dy}{dx} + \lambda_2 = 0}$$

$\eta = c_1$

$\rightarrow \boxed{\eta = \lambda_2(x,y)}$

$f_2(x,y) = c_2$.

Date

Q. Find ξ & η s.t. the coefficient of $u_{\xi\xi}$ and $u_{\eta\eta}$ vanish
$\xi = f_1(x,y)$, $\eta = f_2(x,y)$ when f_1 and f_2 are solⁿs of ODE.

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \text{respectively}$$

$$A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$$

$$= R\lambda_1^2\xi_y^2 + S\xi_y\lambda_1\xi_y + T\xi_y^2$$

$$\cancel{= R\lambda_1^2} = \xi_y^2\,\underbrace{(R\lambda_1^2 + S\lambda_1 + T)}_{=0}$$

$$A(\eta_x, \eta_y) = 0$$

The Canonical Form will be :

$$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$$

$$u_{\xi\eta} = c$$

$$u_\xi = c\eta + f(\xi)$$

$$u = c\eta\xi + \cancel{\xi}\int f(\xi)\,d\xi + g(\eta)$$

Ass 7 (2)(a)

$u_{xx} - x^2y\,u_{yy} = 0$, $y > 0$

~~first we're to classify this PDE.~~

$R = 1$, $S = 0$, $T = -x^2y$

$S^2 - 4RT = 0 - 4\cdot(-x^2y) = 4x^2y > 0 \quad \forall x$

$\Rightarrow$ Hyperbolic type

What makes you happy?
#HappyCollegeDays
#

Hyperbolic type

$\therefore$ we'll have 2 distinct, real roots

$R\lambda^2 + S\lambda + T = 0$

Using values of R, S, T $\Rightarrow$ $\lambda^2 - x^2 y = 0 \Rightarrow \lambda = \pm x\sqrt{y}$

λ^2

ξ is the sol^n of $\frac{dy}{dx} + x\sqrt{y} = 0$ &

η is the sol^n of this ODE $\frac{dy}{dx} - x\sqrt{y} = 0$

$\frac{x^2}{2} + 2\sqrt{y} = C_1$

$\frac{x^2}{2} - 2\sqrt{y} = C_2$

$\frac{d(\xi,\eta)}{d(x,y)} \neq 0$ hence above two are independent solutions.

$\Rightarrow \xi(x,y) = \frac{x^2}{2} + 2\sqrt{y}$

$\eta(x,y) = \frac{x^2}{2} - 2\sqrt{y}$

$\xi_x = x$	$\eta_x = x$
$\xi_{xx} = 1$	$\eta_{xx} = 1$
$\xi_y = \frac{1}{\sqrt{y}}$	$\eta_{xy} = 0$
	$\eta_y = -\frac{1}{\sqrt{y}}$
	$\eta_{yy} = \frac{1}{2y\sqrt{y}}$

$u_x = u_\xi \xi_x + u_\eta \eta_x = (u_\xi + u_\eta)x$

$u_{xx} = (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)x + (u_\xi + u_\eta)$

$u_{yy} = \frac{1}{y}(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - \frac{1}{2y\sqrt{y}}(u_\xi - u_\eta)$

Date

$\xi + \eta = x^2$

$\boxed{x = \sqrt{\xi + \eta}}$

$\xi - \eta = 4\sqrt{y}$

$\boxed{y = \frac{(\xi - \eta)^2}{16}}$

$$\boxed{u_{\xi\eta} = \frac{(\xi + 3\eta)}{4(\xi^2 - \eta^2)} u_\eta - \frac{(3\xi + \eta)}{4(\xi^2 - \eta^2)}}$$

$u_{\xi\eta} = a u_\xi + b u_\eta$

$u_\xi = a u_{\xi\eta} + bu + f(\xi)$

$u = a\eta \int u_\xi \, d\xi + b u \xi + \int f(\xi) \, d\xi + g(\eta)$

23rd sept'15

2nd order Semi linear PDEs.

$$R(x,y) u_{xx} + S(x,y) u_{xy} + T(x,y) u_{yy} + g(x, y, u, u_x, u_y) = c \quad — (1)$$

$(x, y) \longrightarrow (\xi, \eta)$

$$A(\xi_x, \xi_y) u_{\xi\xi} + 2B(\xi_x, \xi_y ; \eta_x, \eta_y) u_{\xi\eta} + A(\eta_x, \eta_y)$$
$$= G(\xi, \eta, u, u_\xi, u_\eta) \quad — (2)$$

$$B^2 - A(\xi_x, \xi_y) \cdot A(\eta_x, \eta_y) = \frac{(S^2 - 4RT)(\xi_x \eta_y - \xi_y \eta_x)^2}{4}$$

Case I $\quad S^2 - 4RT > 0 \quad$ Hyperbolic Form

Canonical form : $\boxed{u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)}$

Case II $\quad S^2 - 4RT = 0 \quad$ Parabolic Form

$$R\lambda^2 + S\lambda + T = 0 \quad — (*)$$

What makes you happy?
#HappyCollegeDays

#

#

We've only one distinct real root $\lambda(x,y)$

ξ s.t. $\xi_x = \lambda \xi_y \Rightarrow A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$

ξ is sol^n of ODE $\frac{dy}{dx} + \lambda = 0$ —— (4)

Assume sol^n of (4) is $f(x,y) = c$

$\boxed{\xi = f(x,y)}$ How to find η?

Choose $\eta(x,y)$ which is independent to $\xi(x,y)$. $\frac{d(\xi, \eta)}{d(x,y)} \neq 0$

Eq^n (3) implies $B = 0$.

$A(\eta_x, \eta_y) \neq 0$

Eq^n (2) implies the canonical form of Parabolic problem is

Canonical Form

$$\boxed{u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)}$$

Case III $S^2 - 4RT < 0$ Elliptic type

(*) has two distinct roots, Complex

sol.s are λ and $\bar{\lambda}$.

ξ and η s.t. $\xi_x = \lambda \xi_y$ & $\eta_x = \bar{\lambda} \eta_y$

ξ and η are solutions of ODE

$\frac{dy}{dx} + \lambda = 0$ & $\frac{dy}{dx} + \bar{\lambda} = 0$

$\xi = \alpha + i\beta$ $\eta = \alpha - i\beta$

$\alpha = \frac{1}{2}(\xi + \eta)$ $\beta = \frac{i}{2}(\eta - \xi)$

α, β are the real characteristic curves.

Date

$$u_\xi = u_\alpha \alpha_\xi + u_\beta \cdot \beta_\xi$$
$$= u_\alpha \cdot \frac{1}{2} + u_\beta \left(\frac{-i}{2}\right)$$
$$= \frac{1}{2}(u_\alpha - i u_\beta)$$

$\alpha_\xi = \frac{1}{2}$ $\quad \beta_\xi = -\frac{i}{2}$

$\alpha_\eta = \frac{1}{2}$ $\quad \beta_\eta = \frac{i}{2}$

$$u_{\xi\eta} = \frac{1}{2}\left[u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta - i(u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta)\right]$$
$$= \frac{1}{2}\left[u_{\alpha\alpha}\cdot\frac{1}{2} + \cancel{u_{\alpha\beta}}\cdot\frac{i}{2} - i\,\cancel{u_{\beta\alpha}}\frac{1}{2} - i\,u_{\beta\beta}\cdot\frac{i}{2}\right]$$
$$= \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta})$$

$$u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta}) = \phi(\alpha, \beta, u, u_\alpha, u_\beta)$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \phi(\alpha, \beta, \dots)}$$ Canonical Form

Ass.

(2)(b) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$

$R = e^{2x}$, $S = 2e^{x+y}$, $T = e^{2y}$

$S^2 - 4RT = 4e^{2(x+y)} - 4 \cdot e^{2x}e^{2y} = 0$ $\quad \forall\ x \,\&\, y$. Parabolic form

For Hyperbolic type: Canonical form

$$\boxed{u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)}$$

If we do not obtain this form then there must be some problem.

#

#

Parabolic type

$$e^{2x}\lambda^2 + 2e^{x+y}\lambda + e^{2y} = 0$$

$$\lambda^2 + \frac{2e^y}{e^x}\lambda + \frac{e^{2y}}{e^{2x}} = 0$$

$$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$$

$$\lambda = -\frac{e^y}{e^x}$$

$\xi(x,y)$ is the sol^n of ODE.

$$\frac{dy}{dx} - \frac{e^y}{e^x} = 0 \quad \Big| \quad \cancel{\frac{dy}{dx} + \frac{e^y}{e^x} = 0}$$

$$\Rightarrow \cancel{\frac{dy}{dx}} \quad \frac{dy}{e^y} - \frac{dx}{e^x} = 0 \quad \Rightarrow e^{-y}dy - e^{-x}dx = 0$$

$$\Rightarrow \quad e^{-y} - e^{-x} = c$$

$$\boxed{\xi(x,y) = e^{-y} - e^{-x}}$$

$$\boxed{\eta(x,y) = x}$$

$\xi_x = e^{-x}$	$\eta_x = 1$
$\xi_y = -e^{-y}$	$\eta_y = 0$
$\xi_{xx} = -e^{-x}$	$\eta_{xx} = 0$
$\cancel{\xi_{yy}} = e^{-y}$	$\eta_{xy} = 0$
$\xi_{xy} = 0$	$\eta_{yy} = 0$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$= u_\xi e^{-x} + u_\eta$$

$$u_{xx} = (u_\xi)_x e^{-x} - u_\xi e^{-x} + (u_\eta)_x$$
$$= (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)e^{-x} - u_\xi e^{-x}$$
$$+ (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)$$

$$= (u_{\xi\xi} e^{-x} + u_{\xi\eta}) e^{-x} - u_\xi e^{-x} + u_{\eta\xi} e^{-x} + u_{\eta\eta}$$
$$= u_{\xi\xi} e^{-2x} + u_{\xi\eta} e^{-x} - u_\xi e^{-x} + u_{\eta\eta}$$

$$u_{xy} = (u_\xi)_y e^{-x} + (u_\eta)_y$$
$$= (u_{\xi\xi}\xi_y + \cancel{u_{\xi\eta}\eta_y}) e^{-x} + u_{\eta\xi}\xi_y + \cancel{u_{\eta\eta}\eta_y}$$
$$= u_{\xi\xi}(-e^{-y}) e^{-x} + u_{\eta\xi}(-e^{-y})$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$= u_\xi(-e^{-y})$$

$$u_{yy} = (u_{\xi\xi}\xi_y + \cancel{u_{\xi\eta}}\eta_y)(-e^{-y}) + u_\xi e^{-y}$$
$$= u_{\xi\xi}(-e^{-y})(-e^{-y}) + u_\xi e^{-y}$$

$$\boxed{u_{yy} = e^{-2y} u_{\xi\xi} + u_\xi e^{-y}}$$

$$e^{-x} e^{-y} u_{\eta\eta} = (e^{-y} - e^{-x}) u_\xi$$
$$u_\eta \; e^{\eta}(\xi + e^{-\eta}) = (\xi + e^{-\eta} - e^{-\eta}) u_\xi$$

$$\boxed{u_{\eta\eta} = \left(\frac{\xi}{1+\xi e^{\eta}}\right) u_\xi}$$ ✓

2) c) $x^2 u_{xx} + y^2 u_{yy} = 0$, $x, y > 0$

$s^2 - 4RT = -4x^2y^2 < 0 \quad \forall x, y \Rightarrow$ Elliptic

$\lambda = \pm \frac{y}{x} i$

ξ, η are solⁿ of $\frac{dy}{dx} + i\frac{y}{x} = 0$

$\log x \pm i \log y = c$

$\boxed{\begin{array}{l}\xi = \log x \\ \eta = \log y\end{array}}$ $\boxed{\begin{array}{l}x = e^{\xi} \\ y = e^{\eta}\end{array}}$

#
#

$$u_i = \frac{u_\xi}{x}$$

$$u_{xx} = \frac{1}{x^2}(u_{\xi\xi} - u_\xi)$$

$$u_y = \frac{u_\eta}{y}$$

$$u_{yy} = \frac{1}{y^2}(u_{\eta\eta} - u_\eta)$$

$$\boxed{u_{\xi\xi} + u_{\eta\eta} = u_\xi + u_\eta}$$

Canonical Form

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_\xi \frac{1}{x} + u_\eta = 0$$

Ans 3) b) $x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} + x u_x + y u_y = 0$

$$R = x^2, \quad S = -2xy, \quad T = y^2$$

$$S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0 \quad \forall\, x, y$$

Parabolic type

($u_{\eta\eta} = \phi(\xi, \eta, u, \ldots) \rightarrow$ if this comes then ✓ otherwise ✗)

$$R\lambda^2 + S\lambda + T = 0$$

$$x^2\lambda^2 - 2xy\,\lambda + y^2 = 0$$

$$\Rightarrow \lambda^2 - \frac{2y}{x}\lambda + \frac{y^2}{x^2} = 0$$

~~S(x, y) is~~ $\boxed{\lambda = y/x}$

Date

$\xi(x,y)$ is the solⁿ of ODE

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

$$xy = c_1$$

$$\xi(x,y) = xy$$

$$\eta(x,y) = x \longrightarrow \text{Choose it.}$$

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0, \infty \qquad x \neq 0$$

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0, \infty$$

If $x = 0$,

$$y^2 u_{yy} + y u_y = 0$$

$$\Rightarrow u_{yy} = -\frac{1}{y} u_y$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x = y u_\xi + u_\eta$$

$$u_y = x u_\xi,$$

$$u_{xx} = y^2 u_{\xi\xi} + 2y u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = x^2 u_{\xi\xi}$$

$$u_{xy} = u_\xi + xy\, u_{\xi\xi} + x u_{\xi\eta}$$

$$x^2 u_{\eta\eta} + x u_\eta = 0$$

$u = u(\xi,\eta)$

$u_\eta = u_\eta(\xi,\eta)$

$$\boxed{\eta u_{\eta\eta} + u_\eta = 0} \longrightarrow \text{Canonical form}$$

$$\Rightarrow \frac{d}{d\eta}(\eta u_\eta) = 0$$

Integrating wrt η

$\eta u_\eta = f(\xi)$

$\rightarrow u_\eta = \frac{f(\xi)}{\eta}$

$u = f(\xi) \ln|\eta| + g(\xi)$

$\boxed{u(x,y) = f(xy) \ln|x| + g(xy)}$

where, f and g are arbitrary and smooth functions

3) c) $u_{xx} - 2\sin x \, u_{xy} - \cos^2 x \, u_{yy} - \cos x \, u_y = 0$

$R = 1$, $S = -2\sin x$, $T = -\cos^2 x$

$S^2 - 4RT = 4\sin^2 x - 4(-\cos^2 x) = 4 > 0$

Hyperbolic type

$A(\xi_x, \xi_y) = A(\eta_x, \eta_y) = 0$

$\lambda^2 - 2\sin x \, \lambda - \cos^2 x = 0$

$\lambda = \frac{2\sin x \pm \sqrt{4\sin^2 x + 4\cos^2 x}}{2}$

$\boxed{\lambda = \sin x \pm 1}$

ξ & η are sol"s of ODE

$\frac{dy}{dx} + \sin x + 1 = 0$ & $\frac{dy}{dx} + \sin x - 1 = 0$

$y - \cos x + x = c_1$

$y - \cos x - x = c_2$

$\xi(x,y) = y - \cos x + c$

$\eta = y - \cos x - x$

$$u_x = (1+\sin x)u_\xi + (-1+\sin x)u_\eta$$

Date

$$u_y = u_\xi + u_\eta$$

$$u_{xx} = \cos x\,(u_\xi + u_\eta) + (1+\sin x)^2 u_{\xi\xi} - 2\cos^2 x\, u_{\xi\eta} + (1-\sin x)^2 u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

The canonical form is $\boxed{u_{\xi\eta} = 0}$

$$u_\xi = f(\xi)$$

$$u = \int f(\xi)\,d\xi + G(\eta)$$

$$u = F(\xi) + G(\eta)$$

$$\boxed{u(x,y) = F(y - \cos x + x) + G(y - \cos x - x)}$$

One dimensional Wave Equation

Case-I Vibrations of an infinite string

$R=1,\ S=0,\ T=-c^2$

$$u_{tt} = c^2 u_{xx} \quad — (1) \qquad -\infty < x < \infty$$

$$\left\{\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right\} — (2) \qquad -\infty < x < \infty$$

when $t=0$, $u = f(x)$ is the initial position

$u_t = g(x)$ is the initial velocity at x.

$$S^2 - 4RT = 0 + 4c^2 = 4c^2 > 0 \qquad \forall c \neq 0$$

Hyperbolic PDE

$$\lambda^2 - c^2 = 0 \Rightarrow \lambda = \pm c$$

ξ & η are solⁿ of ODEs.

$$\frac{dx}{dt} - c = 0 \quad , \quad \frac{dx}{dt} + c = 0$$

Date

$$x - ct = c_1 \quad , \quad x + ct = c_2$$

$$\boxed{\xi(x,y) = x - ct \quad , \quad \eta(x,y) = x + ct}$$

The canonical form

$$u_{\xi\eta} = 0$$

$$u(\xi,\eta) = F(\xi) + G(\eta)$$

$$\boxed{u(\eta, t) = F(x - ct) + G(x + ct)}$$

$F, G \in c^2$ so that $u(x,t)$ will be a regular sol^n of eq^n 1

$$u(x,0) = f(x) \Rightarrow F(x) + G(x) = f(x) \quad \text{--- } 3$$

$$u_t(x,t) = -c\, F'(x - ct) + c\, G'(x + ct)$$

$$u_t(x,0) = g(x) \Rightarrow cF'(x) + c\,G'(x) = g(x) \quad \text{--- } 4$$

From 3 and 4, we get

$$F(x) = \frac{1}{2c}\left[c\, f(x) - \int_{x_0}^{x} g(s)\, ds\right.$$

$$G(x) = \frac{1}{2c}\left[c\, f(x) + \int_{x_0}^{x} g(s)\, ds\right.$$

$$u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right]$$

$$+ \frac{1}{2}\left[-\int_{x_0}^{x-ct} g(s)\, ds + \int_{x_0}^{x+ct} g(s)\, ds\right]$$

$$= \frac{1}{2}\left(\qquad \right) + \frac{1}{2c}\left[+ \int_{x-ct}^{x_0} g(s)\,ds + \int_{x_0}^{x+ct} \boxed{g(s)\,ds} \right]$$

$$= \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

d - Alembert's

→ soln which describes the vibration of an infinite string.

Case IV. Vibration of semi-infinite string

$$u_{tt} = c^2 u_{xx} \quad —① \qquad 0 < x < \infty$$

$$\text{I.C.} \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} —②$$

—③

—④

B.C.

$u(0,t) = 0$, $t \geq 0$ (i.e. the end pt $x=0$ is fixed)

Note : $u_t(0,t) = 0$

$f(x-ct)$ → has no meaning for $t > \frac{x}{c}$.

Modify the I.C. as below

$$u(x,0) = F(x) \quad , \quad u_t(x,0) = G(x) \qquad -\infty < x < \infty$$

where,

$$F(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x \leq 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases}$$

Odd extensions of $f(x)$ and $g(x)$.

#

#

29th sept : ~~sop~~ answer sheets.

Date

30th sept'15

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty$$

i.e. $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ $\}$ $-\infty < x < \infty$

d-Alembert sol^n

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds}$$

$u \in c^2$

$f \in c^2$, $g \in c$

Case-II Vibration of semi-infinite string

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty$$

I.C. $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $x > 0$

B.C. $u(0,t) = 0$ $\forall$ $t \geq 0$

$u_t(0,t) = 0$

Modify the problem

$$F(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x \leq 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases}$$

F and G are odd extensions of f & g.

Date

$$\left.\begin{aligned} u(x,0) &= F(x) \\ u_t(x,0) &= G(x) \end{aligned}\right\} \quad -\infty < x < \infty$$

$$\boxed{u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds}$$

$F \in c^2$, $G \in c^1$

$\Rightarrow \quad u \in c^2 \qquad \frac{d}{dt}\int_{u(t)}^{v(t)} f(s)\,ds = v'(t) f(v(t)) + u'(t) f(v(t))$

$\underline{x = 0}$

$$u(0,t) = \frac{1}{2}\left[F(-ct) + F(ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} G(s)\,ds = 0$$

$$u_t(x,t) = \frac{1}{2}\left[-cF'(x-ct) + cF'(x+ct)\right]$$

$$+ \frac{1}{2c}\left[cG(x+ct) + cG(x-ct)\right]$$

$$u_t(0,t) = \frac{1}{2}\left[-cF'(-ct) + cF'(ct)\right] + \frac{1}{2}\left[\underbrace{G(ct) + G(-ct)}_{=0}\right] = 0$$

I.C. $\quad t = 0 \quad , \; x > 0$

$$u(x,0) = \frac{1}{2}\left[F(x) + F(x)\right] \ne 0 = F(x) = f(x)$$

$$u_t(x,0) = \frac{1}{2}\left[-cf'(x) + cf'(x)\right] + \frac{1}{2}\left[G(x) + G(x)\right]$$

$$= G(x) = g(x)$$

I.C. $u(x,0) = \cos x$, $u_t(x,0) = 0$ $x > 0$

B.C. $u(0,t) = 0$

$f(x) = \cos x$, $g(x) = 0$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\cos(x-t) + \cos(x+t)\right] & x \geq t \\ \frac{1}{2}\left[\cos(x+t) - \cos(t-x)\right] & x \leq t \end{cases}$$

Eg $u_{tt} = c^2 u_{xx}$ $x > 0$, $t > 0$

$u(x,0) = \sin x$, $u_t(x,0) = \cos x$, $x > 0$

$u(0,t) = 0$ $\forall\, t \geq 0$

$f = \sin x$, $g = \cos x$

when $x \geq ct$

$$u(x,t) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right] + \frac{1}{2}\left[\sin s \Big|_{x-ct}^{x+ct}\right]$$

$$+ \frac{1}{2}\left[\quad\right] + \frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right]$$

when $x \leq ct$

$$u(x,t) = \frac{1}{2}\left[\sin(x-ct) - \sin(ct-x)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

$$\int_{x-ct}^{x} -\cos t \, dt \qquad \int_{x}^{x+ct} \cos s \, ds$$

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] & x \geq ct \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] & x \leq ct \end{cases}}$$

Case III **Vibration of a finite string**

length of string

$u_{tt} = c^2 u_{xx}$ $\quad 0 < x < \cancel{t}\, l \quad , \quad t > 0$

Initial cond^n I.C. $u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad 0 \leq x \leq \cancel{t}\, l$

Boundary cond^n B.C. $\begin{cases} u(0,t) = u(l,t) = 0 & \forall \; t \geq 0 \\ u_t(0,t) = u_c(l,t) = 0 \end{cases}$

$-l \quad 0 \quad l$

Modify the problem

$$F(x) = \begin{cases} f(x) & 0 \leq x \leq l \\ -f(-x) & -l \leq x \leq 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & 0 \leq x \leq l \\ -g(-x) & -l \leq x \leq 0 \end{cases}$$

$$\left.\begin{aligned} F(x+2kl) &= F(x) \\ G(x+2kl) &= G(x) \end{aligned}\right\} \text{ for } k = \pm 1, \pm 2, \ldots$$

$F(x)$ & $G(x)$ are odd extensions of f & g.

Date

F & G are odd functions and periodic functions of period $2l$.

$u(x,0) = F(x)$,

$u_t(x,0) = G(x)$, $-\infty < x < \infty$

$$F(x) = \sum_{n=1}^{x} a_n \sin\left(\frac{n\pi x}{L}\right) \qquad a_n = \frac{x}{L}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$G(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad b_n = \frac{x}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\boxed{u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds}$$

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} a_n \left\{\sin\left(\frac{n\pi(x-ct)}{L}\right) + \sin\left(\frac{n\pi(x+ct)}{l}\right)\right\}\right.$$

$$\left. + \frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi s}{l}\right) ds\right]$$

$$\boxed{u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{l}\right)\right) \quad \frac{l}{\pi c}\sum_{n=1}^{\infty}\frac{b_n}{n}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad \Bigg| \quad b_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Eg $u_{tt} = u_{xx}$ $\quad 0 < x < 1$, $t > 0$

$u(x,0) = u_t(x,0) = 0$

$u(0,t) = \sin\pi$

Date

(i) series must ~~be~~ converge.
(ii) series obtained by turn by turn differentiate piece wise x & t must converge.

A sufficient condition
(i) $f \in C^2$ and f'' is piece-wise continuous
(ii) $g \in C$ and f'' is piece-wise cont.

Duhamel's Principle

Non homogeneous Problem

① — $u_{tt} - c^2 u_{xx} = F(x,t) \qquad x \in \mathbb{R}, \quad t > 0$

homogeneous I.C. $\quad u(x,0) = 0, \quad u_t(x,0) = 0 \quad , x \in \mathbb{R}, \quad t \geq 0$

how to solve Non Homogeneous Problem with Homogeneous data?

Let $v(x,t,\tau)$ is satisfying the following for $t > \tau$

$$v_{tt} - c^2 v_{xx} = 0 \quad , \quad x \in \mathbb{R}, \quad t > \tau > 0$$

I.C. at $\quad t \geq \tau > 0$

$$v(x,t,\tau) = 0 \quad , \quad v_t(x,\tau;\tau) = F(x,\tau)$$

This is Homogeneous Problem with Non Homogeneous Data.

$$\left[u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds \right]$$

$$\boxed{v(x,t;\tau) = \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\, ds}$$

#

#

Define $u(x,t) = \int_0^t v(x,t;\tau)\,d\tau$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} f_t(x,t)\,dt + b'(t) f(b(t),t) - a'(t) f(a(t),t)$$

$$u_t = \int_0^t v_t(x,t;\tau)\,d\tau + v(x,\cancel{t};t)$$

$$u_{tt} = \int_0^t v_{tt}(x,t;\tau)\,d\tau + v_t(x,t;t)$$

$$u_{tt} = \int_0^t v_{tt}(x,t;\tau)\,d\tau + F(x,t)$$

$$u_{tt} - c^2 u_{xx} = \int_0^t v_{tt}(x,t,\tau)\,d\tau + F(x,t) - c^2 \int_0^t v_{tt}(x,t,\tau)\,d\tau$$

$$= \int_0^t (v_{tt} - c^2 v_{xx})\,d\tau + F(x,t) = F(x,t)$$

$\boxed{\text{(2) is sol}^n \text{ of problem (1)}}$

$$v_{tt} - c^2 v_{xx} = 0$$

$$v(x,t,\tau) = 0, \quad v_t(x,\tau;\tau) = F(x,\tau)$$

$$\frac{1}{2}\, f(x - c(t-\tau)) + f(x + c(t-\tau))]$$

Date

Vibration of string.

#. (A) — $\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) & x \in \mathbb{R},\ t \geq 0 \end{cases}$

(B) — $\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & \cancel{x \in \mathbb{R},\ t > 0} \\ u(x,0) = 0, \quad u_t(x,0) = 0 & x \in \mathbb{R},\ t \geq 0 \end{cases}$

principle

Assume, u_1 is solⁿ of (A) and

u_2 — " — (B)

$\rightarrow$ $u = u_1 + u_2$ will be the solⁿ of $\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) & x \in \mathbb{R},\ t \geq 0 \end{cases}$

Verify.

Now sol^n of u_1 is

$$u_1 = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

$$u_2 = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\, ds\, d\tau$$

so, $u = u_1 + u_2$ is the sol^n of problem ①

Non homogeneous problem with Non homogeneous data.

Ass 7

$$\begin{cases} u_{tt} - u_{xx} = x^2 - t = F(x,t) \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} F(s,\tau)\, ds\, d\tau$$

$$= \frac{1}{2c}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} (s-\tau)\, ds\, d\tau$$

$$\boxed{u(x,t) = \frac{10}{12}\left(6x^2t^2 + t^4 - 2t^3\right)}$$

Ass 6

8) $x^3 p + y(3x^2+y)q = z(2x^2+y)$

$$\frac{dx}{x^2} = \frac{dy}{y(3x^2+y)} = \frac{dz}{z(2x^2+y)} = \frac{-x^{-1}dx + y^{-1}dy + z^{-1}dz}{\underbrace{-x^2 + 3x^2 + y - (2x^2+y)}_{=0}}$$

$\Rightarrow -x^{-1}dx + y^{-1}dy - z^{-1}dz = 0$

$\Rightarrow -\ln x + \ln y - \ln z = \ln c_1 \quad \Rightarrow \boxed{\frac{y}{xz} = c_1}$

Date

$$\frac{dx}{x^2} = \frac{dy}{y(3x^2+y)} = \frac{(3x^2+y)\,dx}{x^2} = \frac{dy}{y}$$

$$= \frac{(3x^2+y)dx + dy}{x^2+y}$$

$$\Rightarrow \frac{(3x^2+y)dx + dy + x\,dy}{x^3+y+xy} = \frac{dy}{y}$$

$$\Rightarrow \frac{d(x^2+y+xy)}{x^3+y+xy} = \frac{dy}{y}$$

$$\Rightarrow \ln(x^3+y+xy) = \ln y + \ln c_2$$

$$\frac{x^3+y+xy}{y} = c_2 = v(x,y,z)$$

$F(u,v)=0$

$$F\left(\frac{y}{xz}, \frac{x^3+y+xy}{y}\right) = 0$$

Γ $x_0(s)=1$, $y_0(s)=s$, $z_0(s)=s(1+s)$

$$\frac{s}{s(1+s)} = c_1 \quad \& \quad \frac{1+s+s}{s} \Rightarrow \frac{1+2s}{s} = c_2$$

eliminate ~~c_1 and c_2~~ s

$$\boxed{c_1 c_2 - c_1 - c_2 + 2 = 0}$$

#

#

Date 6 | 10 | 15

(x) Duhamel Principle for solving Non-homogeneous Problems

(i) Non-homogeneous problem with homogeneous I.C.s

(ii) Non-homogeneous problems with non-homogeneous I.C.s

<u>Vibration of finite string</u>

<u>Method of seperation of variables</u>

① — $u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < l, \; t > 0$

② — $u(x,0) = f(x), \quad u_t(x,0) = g(x) \qquad 0 \le x \le l$

B.C. ③ $\left\{ \begin{array}{l} u(0,t) = u(l,t) = 0 \\ u_t(0,t) = u_t(l,t) = 0 \end{array} \right\} \qquad t > 0$

$u(x,t)$ = deflection of string at any point x and at any time $t > 0$.

Assume a solution of the function

$u(x,t) = X(x)\,T(t)$

$u_t = X(x)\,T'(t)$

$u_{tt} = X(x)\,T''(t)$

$u_x = X'(x)\,T(t)$

$u_{xx} = X''(x)\,T(t)$

$X(x)\,T''(t) - c^2 X''(x)\,T(t) = 0$

$\Rightarrow \underbrace{\dfrac{X''}{X}}_{\text{function of } x} = \underbrace{\dfrac{T''(t)}{c^2 T(t)}}_{\text{function of } t} = \lambda \text{ (const)}$

$X'' - \lambda X = 0 \quad \bigg| \quad T'' - \lambda c^2 T = 0$

$u(0,t)=0 \Rightarrow X(0)\,T(t)=0$

$u(l,t)=0 \Rightarrow X(l)\,T(t)=0$

$T(t)\neq 0 \Rightarrow X(0) = X(l)=0$

$$\cancel{\boxed{X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right), \quad n=1,2,\ldots}}$$

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

Case I $\quad \lambda>0$

$m^2-\lambda=0 \qquad m=\pm\sqrt{\lambda}$

$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$

$X(0) = 0 \Rightarrow A+B=0$

$X(l) = 0 \Rightarrow Ae^{\sqrt{\lambda}l} + Be^{-\sqrt{\lambda}l} = 0$

$A=B=0$

$\Rightarrow X=0$, No eigen value

Case II $\quad \lambda=0$

$X''=0$

$X = Ax+B$

$X(0)=0 \Rightarrow B=0$

$X(l) - \Rightarrow A=0$

$X=0$

No eigen value

Case III $\quad \lambda<0$

$m^2-\lambda=0$

$\Rightarrow m=\sqrt{\lambda} = \pm i\sqrt{-\lambda}$

Date

$$X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$$

$X(0) = 0 \Rightarrow A = 0$

$X(l) = 0 \Rightarrow B\sin\sqrt{-\lambda}\,l = 0$

$\Rightarrow B = 0 \qquad \hat{X} = 0$

Assume $B \neq 0$

$\sin\sqrt{-\lambda}\,l = 0 = \sin n\pi$

$\lambda_n = -\left(\frac{n\pi}{l}\right)^2 \qquad n = 1, 2, \ldots$

Eigen values

Eigen function $\boxed{\sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots}$

$$\boxed{X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \qquad n = 1, 2, \ldots}$$

$$T'' + \left(\frac{n\pi c}{l}\right)^2 T = 0$$

$$m^2 + \left(\frac{n\pi c}{l}\right)^2 = 0$$

$$m = \pm\, \frac{2n\pi c}{l}$$

$$T_n(t) = C_n\cos\left(\frac{n\pi ct}{l}\right) + D_n\sin\left(\frac{n\pi ct}{l}\right)$$

$$\boxed{u_n(x,t) = \left[a_n\cos\left(\frac{n\pi ct}{l}\right) + b_n\sin\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)}$$

Date

Principle of Superposition

If u_1 and u_2 are two solutions of a linear, homogeneous ODE then $u_1 + u_2$ is also a sol^n of the linear homogeneous ODE satisfying the same B.C.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$u(x,t)$ is the sol^n of problem ① & ② produced the infinite series converge.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right)}$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

Choose a_n such that $u(x,0)$ becomes fourier series of $f(x)$

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$u_n(x,t) = X_n(x)\, T_n(t)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[a_n\left(-\sin\left(\frac{n\pi ct}{l}\right)\right) \cdot \frac{n\pi c}{l} + b_n \cos\left(\frac{n\pi ct}{l}\right) \cdot \frac{n\pi c}{l}\right] \sin\left(\frac{n\pi x}{l}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \cdot \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right) = g(x)$$

Choose b_n such that $u(x,0)$ becomes fourier series of $g(x)$.

$$\boxed{b_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) . dx}$$

Date

① — $u_{tt} - 2u_{xx} = 0$ $\quad 0 < x < 2\pi \quad , t > 0$

② — $u(x,0) = \cos x - 1 \quad , u_t(x,0) = 0 \quad 0 \le x \le 2\pi$

B.C.

③ $\left\{ \begin{array}{l} u(0,t) = u(2\pi,t) = 0 \\ u_t(0,t) = u_t(2\pi,t) = 0 \end{array} \right\} \; t > 0$

$f(x) = \cos x - 1 \quad , \quad g(x) = 0$

$b_n = 0 \quad \forall n$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi c t}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\cancel{\pi} c t}{2}\right) \sin\left(\frac{n\pi}{2}\right)$$

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} (\cos x - 1) \sin \frac{nx}{2} \, dx$$

$$u = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)(2n+3)} \cos\left[\frac{(2n+1)ct}{2}\right] \sin\left(\frac{(2n+1)x}{2}\right)$$

$$u_n(x,t) = (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin\left(\frac{n\pi x}{l}\right)$$

eigenfunction $\quad$ n^{th} normal mode of vibration

n^{th} harmonic

$$\omega_n = \frac{n\pi c}{l}$$

ω_n = frequency of the n node

ν_1 = fundamental node

ω = frequency of the fundamental node.

Date

$\nu_n = \frac{\omega_n}{2\pi}$ = Angular frequency of node.

7th oct '15

Uniqueness for Wave Equation

① — $u_{tt} - c^2 u_{xx} = F(x,t)$ $\quad 0 < x < t$

② — I.C. $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ $\quad 0 < x \le l$

③ — B.C. $\begin{cases} u(0,t) = u(l,t) = 0 \\ u_t(0,t) = u_t(l,t) = 0 \end{cases}$ $t \ge 0$

sol^n Let u_1 and u_2 are the sol^{n}s of ① - ③

$v = u_1 - u_2$

v satisfies the following problem

④ — $v_{tt} - c^2 v_{xx} = 0$ $\quad 0 < x < l$, $t > 0$

⑤ — $v(x,0) = 0$, $v_t(x,0) = 0$ $\quad 0 \le x \le l$

⑥ — $\begin{cases} v(0,t) = 0 & v(l,t) = 0 \\ v_t(0,t) = 0, & v_t(l,t) = 0 \end{cases}$ $t \ge 0$

$\sin\left(\frac{(2n+1)x}{2}\right)$

claim $v \equiv 0$ i.e. $u_1 \equiv u_2$

$$E(t) = \frac{1}{2}\int_0^l (c^2 v_x^2 + v_t^2)\,dx \rightarrow \text{Energy equation}$$

Total energy of vibrating string at time t

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^l (2c^2 v_x v_{xt} + 2 v_t v_{tt})\,dx$$

$$= \int_{x=0}^{l} v_t v_{tt}\,dx + \underbrace{[c^2 v_x v_t]_0^l}_{0 \text{ (due to BC)}} - \int_0^l c^2 v_{xx} v_t\,dx$$

$$= \int_0^l v_t (v_{tt} - c^2 v_{xx})\,dx = 0$$

#

#

$E(t) = \text{constant} \ \forall t$

$$E(0) = \frac{1}{2}\int_0^l \left(c^2 v_x^2(x,0) + v_t^2(x,0)\right) dx = 0$$

$E(t) = 0 \ \forall t$

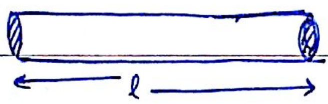
$E(t) = 0 \Rightarrow v_x = 0, \ v_t = 0$

$v = \text{constant}$

$v(x,0) = 0 \Rightarrow v = 0 \ \forall t$

$u_1 = u_2$

Heat Conduction Problem



$u_t = k u_{xx}, \quad 0 < x < l, \quad t > 0$

I.C. $\quad u(x,0) = f(x) \quad 0 \le x \le l$

B.C. $\quad u(0,t) = u(t,t) = 0 \quad t \geq 0$

$u_t(0,t) = u_t(t,t) = 0$

k = Thermal diffusivity

$k > 0$

$$k = \frac{\alpha}{c_p \rho}$$

α = Thermal conductivity

c_p = heat capacity

ρ = mass density

$R = k, \ S = 0, \ T = 0$

$S^2 - 4RT = 0 \Rightarrow 0 - 4k(0) = 0$

$u_{xx} = \frac{1}{k} u_t$

Date

Seperation of Variable Method

Assume a solⁿ of the form

$$u(x,t) = X(x)\,T(t) \neq 0$$

$$u_t = X(x)\,T'(t)$$

$$u_x = X'(x)\,T(t)$$

$$u_{xx} = X''(x)\,T(t)$$

(1) $\Rightarrow X(x)\,T'(t) = k\,X''(x)\,T(t)$

$$\Rightarrow \underbrace{\frac{X''}{X}}_{\text{fun of } x} = \underbrace{\frac{T'}{kT}}_{\text{fun of } t} = \lambda \text{ (const)}$$

$$\begin{cases} X' - \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases} \qquad T' - \lambda k T = 0$$

Case I : $\lambda > 0$

$m^2 - \lambda = 0 \quad \Rightarrow m = \pm\sqrt{\lambda}$

$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$

$X(0) = A + B = 0$

$X(l) = Ae^{\sqrt{\lambda}l} + Be^{-\sqrt{\lambda}l} = 0$

$\Rightarrow A = B = 0 \qquad X \equiv 0$

No eigen values

Case II : $\lambda = 0$

$X'' = 0$

$X(x) = Ax + B$

$X(0) = X(l) = 0$

$\Rightarrow A = B = 0$

$X \equiv 0$

No eigen values

Case III $\lambda < 0$

$m^2 - \lambda = 0$

$m = \pm\sqrt{\lambda} = \pm 2\sqrt{-\lambda}$

$X(x) = A\cos\sqrt{-\lambda}\,x + B\sin\sqrt{-\lambda}\,x$

$X(0) = A = 0$

$X(l) = B\sin(\sqrt{-\lambda}\,l) = 0$

If $B = 0 \Rightarrow X \equiv 0$

Assume $B \neq 0$

$\sin\sqrt{-\lambda}\,l = 0 = \sin n\pi$

$\Rightarrow \lambda_n = -\left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \ldots$

eigen values

Eigen functions : $\sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots$

$$\boxed{X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots}$$

$T' + \left(\frac{n^2\pi^2 k}{l^2}\right)T = 0$

$T_n(l) = C_n \exp\left(\frac{-n^2\pi^2 k}{l^2}\,t\right)$

$u_n(x,t) = a_n \exp\left(\frac{-n^2\pi^2 kt}{l}\right)\sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots$

Principle of superposition implies

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\sin\left(\frac{n\pi x}{l}\right)}$$

be soln of problem of 1 and 3 provided the infinite series converge.

Date

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

Choose a_n such that $u(x,0)$ becomes fourier series of $f(x)$ i.e.

$$\boxed{a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx} \quad n = 1, 2, \ldots$$

$\hookrightarrow$ sol^n of problem ① – ③

eg. $\begin{cases} u_t = u_{xx} \quad , \quad 0 < x < l \quad , t > 0 \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = u(l-x) \quad\quad 0 \leq x \leq l \end{cases}$

find out a_n & then substitute.

12th Oct

$u_t = k u_{xx} \quad\quad 0 < x < l \;, \; t > 0 \;, \; k > 0$

~~I.C. $u(x,0) = f(x)$ $\quad\quad 0 \leq x \leq l$~~

B.C. $\quad u(0,t) = u(l,t) = 0 \quad\quad t \geq 0$

$u_t(0,t) = u_t(l,t) = 0$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 kt}{l}\right) \sin\left(\frac{n\pi x}{l}\right)}$$

$$a_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$u_t - k u_{xx} = F(x,t) \quad , \quad 0 < x < l \quad , \quad t > 0$

I.C. $\quad u(x,0) = f(x) \quad\quad 0 \leq x \leq l \quad$ — ① $\quad t \geq 0$

B.C. $\begin{cases} u(0,t) = u(l,t) = 0 \quad\quad — ② \\ u_t(0,t) = u_t(l,t) = 0 \quad\quad — ③ \end{cases}$

Date

Applying Duhamel's Principle

$$\frac{1}{2\sqrt{k}} \int_0^t \int_{x-\sqrt{k}(t-\tau)}^{x+\sqrt{k}(t-\tau)} F(s,\tau)\, ds\, d\tau$$

Ex. $u_t = u_{xx}$ $\quad 0 < x < 1 \quad , t > 0$

$u(x,0) = x \quad 0 \le x \le 1$

B.C. $u(0,t) = u(1,t) = 0 \quad t \ge 0$

$$a_n = \frac{2}{1} \int_0^1 x \sin\left(\frac{n\pi x}{1}\right) dx = 2\left[\left. \frac{x(-\cos n\pi x)}{n\pi} \right|_0^1 + \frac{1}{n\pi} \int_0^1 \cos n\pi x \, dx \right.$$

$$= 2\left[\frac{-\cos n\pi}{n\pi} + \frac{1}{n\pi} \left. \underbrace{\frac{\sin n\pi x}{n\pi}}_{0} \right|_0^1 \right]$$

$$= -\frac{2}{n\pi}(-1)^n$$

$$= \frac{2}{n\pi}(-1)^{n+1}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \exp(-n^2\pi^2 t) \sin(n\pi x)}$$

Proof: Assume u_1 & u_2 are two sol^n of part ① – ③

$V = u_1 - u_2$ satisfies the following eq^n.

$V_t - kV_{xx} = 0$ — ④ $\quad 0 < x < l \quad , t > 0$

$V(x,0) = 0 \quad 0 \le x \le l$ — ⑤

⑥ — $\begin{cases} V(0,t) = V(l,t) = 0 & t \ge 0 \\ V_t(0,t) = V_t(l,t) = 0 \end{cases}$

$V = 0 \quad \forall \quad x, t > 0 \quad , u_1 \le u_2$

#

#

$$F(t) = \frac{1}{2k}\int_0^t v^2(x,t)\,dx \quad , \quad E(t) \geq 0 \ \forall \ t \geq 0$$

$$\frac{dF}{dt} = \frac{1}{2k}\int_0^t 2 v v_t \, dx = \frac{1}{k}\int_0^l v \cdot k v_{xx} \, dx$$

$$\frac{dE}{dt} = \int_0^l v v_{xx} \, dx = v v_x \Big|_0^l - \int_0^l v_x v_x \, dx$$

$$\frac{dE}{dt} = -\int_0^l v_x^2 \, dx \leq 0$$

$E'(t) \leq 0$

$\Rightarrow E(t)$ is decreasing for $t \geq 0$

$E(0) = 0$

(i) $E(t)$ is decreasing

(ii) $E(t)$ is a +ve function

(iii) $E(t) = 0$ at $t = 0$

$\Rightarrow E(t) \equiv 0 \ \forall \ t \geq 0$

$\Rightarrow v(x,t) \equiv 0 \ \forall \ t \geq 0$

$\Rightarrow u_1 \equiv u_2 \ \forall \ t \geq 0$,

$u_{tt} = c^2 (u_{xx} + u_{yy})$ → 2D wave Eq^n

time variable   space variable

(*) —— $u_{xx} + u_{yy} = 0$ → 2D Laplace Eq^n

Any function satisfying (*) with 2^{nd} order partial derivatives w.r.t x & y is called ~~Analytic function~~ ⇔ harmonic function.
Analytic

All analytic functions are harmonic.

13th Oct '15

Laplace Eqn

$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$

$u \to$ Potential function

eqn ① is called Potential eqn

The theory of soln to eqn ① is called Potential energy

$-\nabla^2 u = u_{xx} + u_{yy} = 0 \qquad (x,y) \in D \subset R^2$

D is bounded domain

D is smooth.

$R = 1, \quad S = 0, \quad T = 1$

$S^2 - 4RT = 0 - 4 < 0$

Canonical form $\Rightarrow u_{\alpha\alpha} + u_{\beta\beta} = \phi(u_\alpha, u_\beta, \alpha, \beta)$

Boundary condn

(i) Drichlet B.C.

$u_{xx} + u_{yy} = 0$ in $(x,y) \in D$

$u(x,y) = f \to$ given funn on $(x,y) \in B$

B is boundary of D

f is continuous on B.

(ii) Neumann B.C.

$u_{xx} + u_{yy} = 0 \qquad (x,y) \in D$

$\frac{\partial u}{\partial n} = g(s)$ on $(x,y) \in D$

$\frac{\partial}{\partial n}$ represent directional derivative along outward normal to the boundary B.

Necessary condn

$\int_B g(s)\, ds = 0$

Date

(iii) Robin B.C.

$$u_{xx} + u_{yy} = 0 \quad \text{in} \quad (x,y) \in D$$

$$\frac{\partial u}{\partial n} + h(s)\, u(s) = 0 \qquad (x,y) \in B$$

$$h(s) \geq 0 \qquad u(s) \neq 0$$

(iv) Mixed B.C.

$$u_{xx} + u_{yy} = 0 \quad \text{in} \quad (x,y) \in D$$

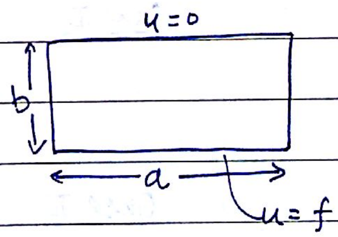
$$u(x,y) = f \quad \text{on } (x,y) \in B_1$$

$$B = B_1 \cup B_2$$

$$\frac{\partial u}{\partial n} = f_2(s) \quad \text{on } (x,y) \in B_2$$

Laplace Equation with Drichlet B.C. rectangular domain

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \qquad 0 < x < a, \; 0 < y < b$$



$$u(x,0) = f(x) \qquad 0 \leq x \leq a$$

$$u(x,b) = 0 \qquad 0 \leq x \leq a$$

$$u(0,y) = 0 \qquad 0 \leq y \leq b$$

$$\Rightarrow u(0,y) = X(0)\, Y(y) = 0$$

$$u(a,y) = 0 \qquad 0 \leq y \leq b \qquad \Rightarrow u(a,y) = X(a)\, Y(y) = 0$$

$$\Rightarrow X(0) = X(a) = 0$$

#

#

Assume a solⁿ of the form

$$u(x,y) = X(x)\, Y(y) \neq 0$$

$u_x = X' Y$ | $u_y = X Y'$

$u_{xx} = X'' Y$ | $u_{yy} = X Y''$

$$X'' Y + X Y'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ (const.)}$$

funⁿ of x alone funⁿ of y alone

$X'' - \lambda X = 0$ $Y'' + \lambda Y = 0$

$X(0) = X(a) = 0$

Case I $\lambda > 0$

$X \equiv 0$ No eigen value

Case II $\lambda = 0$

$X \equiv 0$ No eigen value

Case III $\lambda < 0$

$m^2 - \lambda = 0$

$m = \pm i\sqrt{-\lambda}$

$\mathfrak{X}(x) = A\cos\sqrt{-\lambda}\,x + B\sin\sqrt{-\lambda}\,x$ Date

$\mathfrak{X}(0) = 0 \quad \Rightarrow \quad A = 0$

$\mathfrak{X}(a) = 0 \quad \Rightarrow \quad B\sin\sqrt{-\lambda}\,a = 0$

Assume, $B \neq 0$, then $\sin\sqrt{-\lambda}\,a = 0$

$\sin(\sqrt{-\lambda})a = 0 = \sin n\pi$

eigen value $\lambda_n = -\dfrac{n^2\pi^2}{a^2}$, $n = 1, 2, \ldots$

$\sin\dfrac{n\pi}{a}$ — eigen value

$$\boxed{\mathfrak{X}(x) = B_n \sin\frac{n\pi x}{a}, \quad n = 1, 2}$$

$\mathfrak{X}'' - \lambda\mathfrak{X} = 0 \qquad Y'' + \lambda Y = 0$

$\mathfrak{X}(0) = \mathfrak{X}(a) = 0$

$Y'' - \dfrac{n^2\pi^2 y}{a^2} = 0$

$m^2 = \dfrac{n^2\pi^2}{a^2} \quad \Rightarrow \quad m = \pm\dfrac{n\pi}{a}$

$\left\{ e^{\frac{n\pi y}{a}}, e^{\frac{-n\pi y}{a}} \right\}$ 2 fundamental solutions

$Y_n = t_n \sinh\left(\dfrac{n\pi y}{a}\right)$ $\qquad \boxed{\sinh = \dfrac{e^x - e^{-x}}{2}}$

$Y_n(b) = 0$

$Y_n(y) = t_n \sinh\left(\dfrac{n\pi(y-b)}{a}\right)$

$U_n(x,y) = a_n \sin\left(\dfrac{n\pi x}{a}\right) \sin\left(\dfrac{n\pi(y-b)}{a}\right) \quad n = 1, 2, \ldots$

Date

Superposition principle

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(y-b)}{a}\right)}$$

At $y=0$, $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{-n\pi b}{a}\right)$$

choose a_n s.t. $u(x,0)$ becomes fourier series of $f(x)$.

$$a_n = \frac{-2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$u(x,y) = \sum_{n=1}^{\infty} a_n^* \left(\sinh\left(\frac{n\pi b}{a}\right)\right)^{-1} \sin\left(\frac{n\pi x}{a}\right)$$

$$a_n^* = \frac{-2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Laplace eq^n with Drichlet B.C.

Circular domain (in polar coordinates)

$(x,y) \longrightarrow (r,\theta)$ $\quad r = \sqrt{x^2+y^2}$

$u(x,y) \longrightarrow u(r,\theta)$ $\quad \theta = \tan^{-1}(y/x)$

$u_x = u_r r_x + u_\theta \cdot \theta_x$

$u_{xx} = (u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_r r_{xx}$

$(u_{\theta\theta} \theta_x + u_{\theta r} r_x)\theta_x + u_\theta \theta_{xx}$

Date

$$u_{xx} = \frac{x^2}{r^2} u_{rr} - \frac{2xy}{r^3} u_{r\theta} + \frac{y^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + \frac{2xy}{r^4} u_\theta$$

$$u_{yy} = \frac{y^2}{r^2} u_{rr} + \frac{2xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{x^2}{r^3} u_r - \frac{2xy}{r^4} u_\theta .$$

Laplace Eq^n : $\boxed{\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}}$