$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} = \frac{dz}{2zx^2+zy}$$

(1)

$$\frac{-x^{-1}\,dx + y^{-1}\,dy - z^{-1}\,dz}{-x^2+3x^2+y-2x^2-y} = k$$

$$\Rightarrow -\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0 \longrightarrow ①$$

(2)

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} \Rightarrow \left(\frac{3x^2+y}{x^3}\right) dx = \frac{dy}{y}$$

$$\Rightarrow \frac{(3x^2+y)\,dx + dy}{x^3+y} = \frac{(3x^2+y)\,dx + dy + x\,dy}{x^3+y+xy} = \frac{dy}{y}$$

Exact

$$\Rightarrow \frac{d(x^3+y+xy)}{x^3+y+xy} = \frac{dy}{y}$$

$$\Rightarrow \frac{x^3+y+xy}{y} = c_2$$

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Second Order PDE

The std. second ~~ordrd~~ order PDE in 2 independent variable

$$f(x, y, z, z_{xx}, z_{yy}, z_{xy}, z_x, z_y) = 0$$

(Semi-linear PDE)

↓

$u \to$ dependent variable

$u = u(x,y)$

can be represented as:

(1) —— $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + FU + G = 0$

A, B, C, D, E, F, G : func^n of independent variables x & y

(D, E, F, G may also be func^n of u)

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same eq^n may be classified in diff. form depending on domain

$A^2+B^2+C^2 \neq 0$, A, B, C are continuous as possess continuous partial derivative of as high order as necessary

Def: A func^n $u(x,y)$ is said to be regular sol^n of

② — $Au_{xx} + Bu_{xy} + Cu_{yy} + g(x,y,u,u_x,u_y) = 0$

in $D \subset R \times R$ if $u \in C^2(D)$ (upto 2nd order derivative are continuous) and the func^n u & it's derivatives satisfies ② for all $x,y \in D$

Genesis of 2nd Order PDE

$f \in C^2(D)$ & $u = f(x+at)$

$u_x = f'(x+at)$ $u_t = af'(x+at)$

$u_{xx} = f''(x+at)$ $u_{tt} = a^2 f''(x+at)$

$\Rightarrow \boxed{u_{tt} = a^2 u_{xx}}$: leads to 2nd order PDE

Classification of 2nd Order PDE (Parabola, Ellipse, Hyperbola)

① — $\underline{ax^2+bxy+cy^2} + dx + ey + f = 0$

Principal part (~~do~~ classifications depends on these variables only)

Normal form

→ $b^2-4ac > 0$: Hyperbola $\left(x^2/a^2 - y^2/b^2 = 1\right)$

→ $b^2-4ac = 0$: Parabola $(x^2 = y)$

→ $b^2-4ac < 0$: Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

For PDE :

Principal part of ② :

$Lu = A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy}$

i) $B^2(x,y) - 4A(x,y)C(x,y) > 0 \Rightarrow$ ~~at~~ (x,y) — hyperbolic PDE

ii) $= 0 \Rightarrow$ — parabolic PDE

iii) — elliptic PDE

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Eg. $u_{xx} - x^2 u_{yy} = 0$

$A(x,y) = 1$ $\quad C(x,y) = -x^2$

$B(x,y) = 0$

$B^2 - 4AC = 0 - 4(1)(-x^2) = 4x^2 \geq 0$

$u_{xx} - x^2 u_{yy} = 0 \rightarrow$ hyperbolic $x \neq 0$; Parabolic $x = 0$

Eg. $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \underbrace{\frac{y^2}{x} u_x + \frac{x^2}{y} u_y}$

$A = y^2 \quad B = -2xy \quad C = x^2$

not included in classification but should be defined $(x \neq$

$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0 \Rightarrow$ Parabolic PDE

(But we need to take care of u_x & u_y, & they should be defined, so $x \neq 0$)

Eg. $u_{xx} + x^2 u_{yy} = 0$

$B^2 - 4AC = 0 - 4(1)(x^2) = -4x^2$

$\Rightarrow$ ellipse for $x \neq 0$

parabolic for $x = 0$

Eg. $u_{xx} + x u_{yy} = 0$

$\Rightarrow -4x$

- Parabolic : $x = 0$
- Elliptic : $x > 0$
- Hyperbolic : $x < 0$

Here, we will study 2nd order Semi-Linear PDE with 2 independent variable

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Cannonical (Normal) form of 2nd Order PDE :

good for hyperbolic & parabolic (may not help in case of elliptic)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

$A^2 + B^2 + C^2 \neq 0$ (all A, B, C can't be 0 at same time)
(Otherwise, it won't be 2nd order)

Our aim : $(x, y) \rightsquigarrow (\xi, \eta)$, $\xi = \xi(x, y)$

$u(x, y) \rightsquigarrow u(\xi, \eta)$ $\quad \eta = \eta(x, y)$

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \quad \text{(Assume)}$$

Thus, transformation is invertible

$u_\xi = u_\xi(\xi, \eta)$

$u_x = u_\xi \xi_x + u_\eta \eta_x$ (Chain Rule)

~~$u_{xx} \to u_{\xi\xi}(\xi_x)^2 + u_\xi \xi_{xx} + u_\eta (\eta_x)^2$~~

$$u_{xx} = (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) \xi_x + u_\xi \xi_{xx}$$
$$+ (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x) \eta_x + u_\eta \eta_{xx}$$

$$u_{xx} = u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \eta_x \xi_x + u_\xi \xi_{xx} + u_{\eta\eta} \eta_x^2 + u_\eta \eta_{xx}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_{xy} = (u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y) \xi_x + u_\xi \xi_{xy}$$
$$+ (u_{\eta\xi} \xi_y + u_{\eta\eta} \eta_y) \eta_x + u_\eta \eta_{xy}$$
$$= u_{\xi\xi} \xi_y \xi_x + u_\xi \xi_{xy} + u_{\xi\eta} \eta_y \xi_x + u_{\eta\xi} \xi_y \eta_x$$
$$+ u_{\eta\eta} \eta_y \eta_x + u_\eta \eta_{xy}$$

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$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{yy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + u_\xi \xi_{yy}$$

$$+ (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\eta \eta_{yy}$$

$$= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\xi}\xi_{yy} + u_{\eta\eta}\eta_y^2 + u_\eta\eta_{yy}$$

Principal part :

$$Au_{xx} + Bu_{xy} + Cu_{yy} = u_{\xi\xi}(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2)$$
$$+ u_{\xi\eta}(2A\xi_x\xi_y + B(\xi_x\eta_y + \eta_x\xi_y) + 2\xi_x\eta_y)$$
$$+ u_{\eta\eta}(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2) + h(\xi, \eta, u, u_\xi, u_\eta)$$

We choose ξ and η such that Jacobian $\neq 0$.

$\Rightarrow$ Eqn (1) becomes :

(2) — $\underbrace{\bar{A}(\xi_x;\xi_y)}_{\text{coeff of } u_{\xi\xi} \text{ (func}^n\text{ of something)}} u_{\xi\xi} + 2\bar{B}(\xi_x,\xi_y;\eta_x,\eta_y)u_{\xi\eta} + \bar{A}(\eta_x;\eta_y)u_{\eta\eta}$
$= G(\xi, \eta, u, u_\xi, u_\eta)$ (3)

$$\bar{A}(u;v) = Au^2 + Buv + Cv^2$$
$$\bar{B}(u_1,v_1;u_2,v_2) = Au_1u_2 + \frac{1}{2}B(u_1v_2 + u_2v_1) + Cv_1v_2$$

Eg. $\bar{B}^2(\xi_x,\xi_y;\eta_x,\eta_y) - 4\bar{A}(\xi_x;\xi_y)\bar{A}(\eta_x;\eta_y) = 0$

$= (B^2 - 4AC)(\xi_x\eta_y - \xi_y\eta_x)^2$

> 0 (hyperbolic) ⊛

$\hookrightarrow$ Jacobian $\Downarrow$ > 0 $(\neq 0)$

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Hyperbolic PDE : $B^2 - 4AC > 0$ (wave eqn)

Consider the quadratic eqn

$$A\alpha^2 + B\alpha + C = 0$$

We ~~we~~ 2 real and distinct roots $\lambda_1(x,y)$ $\lambda_2(x,y)$ (say)

We choose $\xi(x,y)$ & $\eta(x,y)$ s.t.

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y} \qquad \text{--- (3)}$$

$$\Rightarrow \xi_x = \lambda_1 \xi_y \quad \text{and} \quad \eta_x = \lambda_2 \eta_y$$

$$\begin{aligned}\bar{A}(\xi_x, \xi_y) &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ &= A\lambda_1^2\xi_y^2 + B\lambda_1\xi_y^2 + C\xi_y^2 \\ &= (A\lambda_1^2 + B\lambda_1 + C)\xi_y^2 \qquad (\lambda_1 : \text{root of eqn}) \\ &= 0\end{aligned}$$

Similarly, $\bar{A}(\eta_x, \eta_y) = 0$

~~$2\bar{B}(\xi_1, \xi_y; u$~~

$$2\bar{B}(\xi_x, \xi_y; \eta_x, \eta_y)\, u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$$

since $\bar{B} > 0$

$$\boxed{u_{\xi\eta} = Q(\xi, \eta, u, u_\xi, \eta)}$$ canonical form for Hyperbolic case.

Eg. $u_{\xi\eta} = k$

$\Rightarrow u_\xi = f(\xi)$

$$\begin{aligned}u &= \int f(\xi)\, d\xi + g(\eta) \\ &= F(\xi) + g(\eta)\end{aligned}$$

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ξ & η are sol^{n}s of eq^{n}s (3)

$\Rightarrow$ $\xi_x - \lambda_1 \xi_y = 0$

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0}$$

$d\xi = 0$

$\Rightarrow \xi = C_2$

$\frac{dy}{dx} + \lambda_1 = 0$

Assume $f_1(x,y) = c_1$ satisfies above eq^n

General sol^n $F(c_1, c_2) = 0 \Rightarrow$

$F(f_1(x,y), \xi) = 0$

$\Rightarrow \xi = G_1(f_1(x,y))$

In particular, the simplest one is

$$\boxed{\xi = f_1(x,y)}$$

$\eta_x - \lambda_2 \eta_y = 0$

The sol^n of this PDE is :

$$\boxed{\eta = f_2(x,y)}$$

where $f_2(x,y)$ is sol^n of $\frac{dy}{dx} + \lambda_2 = 0$

Now, η & ξ are known, so we can find the canonical form.

Ex. $u_{xx} = x^2 u_{yy}$ (Find Canonical form)

$A = 1$ $B = 0$ $C = -x^2$

$B^2 - 4AC = 0 + 4x^2 > 0 \quad \forall \; x \neq 0$

$\Rightarrow$ Hyperbolic Type

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consider eq^n

$1\alpha^2 + 0 + (-x^2) = 0$

$\alpha^2 - x^2 = 0$

$\Rightarrow \cancel{x = \pm 1}$ $\qquad \lambda_1(x,y) = +x$

$\Rightarrow \cancel{\lambda_1, \lambda_2 = \pm 1}$ $\qquad \lambda_2(x,y) = -x$

choose ξ & η s.t

$\xi_x = \lambda_1 \xi_y$ & $\eta_x = \lambda_2 \eta_y$

$\Rightarrow \xi_x - x\,\xi_y = 0$

$\Rightarrow \frac{dx}{1} = \frac{dy}{-x} = \frac{d\xi}{0}$

$+\lambda_2$

$\Rightarrow \frac{dy}{dx} + x = 0$ & $\frac{dy}{dx} - x = 0$

$\Rightarrow y + \frac{x^2}{2} = c_1$ & $y - \frac{x^2}{2} = c_2$

$\quad \parallel \qquad\qquad \parallel$

$\quad f_1 \qquad\qquad f_2$

$\xi(x,y) = y + \frac{x^2}{2}$ $\qquad \eta(x,y) = y - \frac{x^2}{2}$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$= u_\xi (x) + u_\eta (-x) = x(u_\xi - u_\eta)$

$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\eta_x\xi_x + u_\xi \xi_{xx} + u_{\eta\eta}\eta_x^2 + u_\eta \eta_{xx}$

$= u_{\xi\xi}(x^2) + 2u_{\xi\eta}(-x^2) + u_\xi(1) + u_{\eta\eta}(x^2) + u_\eta(-1)$

$u_{xx} = x^2\left[u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}\right] + \left[u_\xi - u_\eta\right]$

Similarly,

$\boxed{u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}}$

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Put in eq^n ①

$A \cdot u_{xx} = x^2 u_{yy}$

$\Rightarrow x^2 \left[\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} \right] + \left[u_\xi - u_\eta \right] = x^2 \left[\cancel{u_{\xi\xi}} + 2u_{\eta\xi} + \cancel{u_{\eta\eta}} \right]$

Checkpoint : coeff. of $u_{\xi\xi}$ & $u_{\eta\eta}$ should be 0 in case of hyperbolic PDE.

$$\Rightarrow 4u_{\xi\eta} x^2 = u_\xi - u_\eta$$

$$\Rightarrow u_{\xi\eta} = \frac{u_\xi - u_\eta}{4x^2} = \boxed{\frac{u_\xi - u_\eta}{4(\xi - \eta)}}$$

canonical form.

$$= Q(\xi, \eta, u, u_\xi, u_\eta)$$

Q) Parabolic PDE : $B^2 - 4AC = 0$ (Heat Eq^n)

$A\alpha^2 + B\alpha + C = 0$: Repeated Real roots $= \lambda(x, y)$

Choose $\xi(x, y)$ s.t.

$$\frac{\partial \xi}{\partial x} = \lambda \frac{\partial \xi}{\partial y} \quad [\text{makes } \bar{A} = 0]$$

This choice of ξ makes the coeff. of $u_{\xi\xi}$ as 0.

$$\bar{A}(\xi_x ; \xi_y) = \xi_y^2 \underbrace{(A\lambda^2 + B\lambda + C)}_{=0}$$

Choose $\eta(x, y)$ s.t. [ξ & η should be independent func^{n}s]

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0 \quad \text{OR} \quad \nabla\xi \times \nabla\eta \neq 0$$

here $\bar{A}(\eta_x, \eta_y)$ may not be equal to 0

From eq^n ⊛,

$(B^2 - 4AC) = 0$

$B^2(\quad) - 4\bar{A}(\quad)\overset{=0}{\bar{C}} \ldots = 0 \Rightarrow B^2 = 0$

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$\Rightarrow \bar{B} = 0$

Using (2), the canonical form is reduced to
$\rightarrow$ non-0

$$\bar{A}(\eta_x, \eta_y) u_{\eta\eta} = G(\xi, \eta, \ldots)$$

$$\Rightarrow \boxed{u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)}$$

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Ques. Find canonical form of (Solve it iff possible)

$$u_{xx} + 2u_{xy} + u_{yy} = 0 \quad A=1 \quad B=2 \quad C=1$$

$B^2 - 4AC = 4 - 4 = 0 \Rightarrow$ parabolic for all x and y

$$\alpha^2 + 2\alpha + 1 = 0$$

$$\Rightarrow (\alpha+1)^2 = 0$$

$$\Rightarrow \alpha = -1$$

Choose $\xi(x,y)$ s.t.

$$\Rightarrow \frac{\partial \xi}{\partial x} = (-1)\frac{\partial \xi}{\partial y}$$

$\Rightarrow \xi_x + \xi_y = 0$: Linear 1st order PDE

$$\frac{dy}{dx} + \lambda = 0$$

$$\Rightarrow \frac{dy}{dx} - 1 = 0$$

$$\Rightarrow \boxed{y - x = c_1}$$

$$\Rightarrow \xi(x,y) = y - x$$

$$\frac{dx}{1} = \frac{dy}{1}$$

$$x - y = c.$$

$\eta(x, y) = ??$

$$\eta_x = \lambda \eta_y - X$$

Choose $\eta(x,y)$ s.t.

$$\frac{\partial(\xi, \eta)}{\partial(x,y)} \neq 0$$

let $\eta(x,y) = x+y$ (can choose any value)

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$\xi = y - x$

$\xi_x = -1 \qquad \xi_{xx} = 0 \qquad \xi_{xy} = 0$

$\xi_y = 1 \qquad \xi_{yy} = 0$

$\eta(x,y) = x + y$

$\eta_x = 1 \qquad$ all other $= 0$

$\eta_y = 1$

$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\eta_x\xi_x + u_{\eta\eta}\eta_x^2 + \cancel{u_\eta \eta_{xx}} + \cancel{u_\xi \xi_{xx}}$

$= u_{\xi\xi}(1) + 2u_{\xi\eta}(-1) + u_{\eta\eta}$

$u_{xy} = u_{\xi\xi}(-1) + 0 + u_{\xi\eta}(-1) + u_{\eta\xi}(1) + u_{\eta\eta}$

$= -u_{\xi\xi} - \cancel{u_{\xi\eta}} + \cancel{u_{\eta\xi}} + u_{\eta\eta}$

$= u_{\eta\eta} - u_{\xi\xi}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

substitute in given problem:

$\Rightarrow u_{\xi\xi} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} + 2\left[\cancel{u_{\eta\eta}} - u_{\xi\xi}\right] + u_{\xi\xi} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}$

$\Rightarrow 4u_{\eta\eta} + 0\,(u_{\xi\xi}) + 0\,(u_{\eta\xi}) = 0$

In case of parabola, these should be 0. (checkpoint)

$\eta = x + y$

$\xi = y - x$

or $\boxed{u_{\eta\eta} = 0} \rightarrow$ Canonical form

$\Rightarrow u_\eta = f(\xi)$

$\Rightarrow u(\xi,\eta) = \int f(\xi)\,d\eta + g(\xi)$

$u(\xi,\eta) = f(\xi)\,\eta + g(\xi)$

$\boxed{u(x,y) = f(y-x)(x+y) + g(y-x)}$

Here, we are able to get explicit form

If we check from here,

$u_x =$

$u_{yy} =$

$\vdots$

this will satisfy parabolic eq^n.

For same eg, take $\eta(x,y) = x$

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0 \Rightarrow \text{Find corresponding sol}^n.$$

3) **Elliptic PDE** : $B^2 - 4AC < 0$

Consider : $A\alpha^2 + B\alpha + C = 0$

Suppose $\lambda_1(x,y)$ and $\lambda_2(x,y)$: ~~two~~ distinct imaginary roots

$\Rightarrow \lambda_1 = \overline{\lambda_2}$

Choose $\xi(x,y)$ and $\eta(x,y)$ s.t.

$\xi_x = \lambda_1 \xi_y$ and $\eta_x = \lambda_2 \eta_y$

$\Rightarrow \bar{A}(\xi_x, \xi_y) = 0 = \bar{A}(\eta_x, \eta_y)$

$\Rightarrow$ Canonical form is:

$$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{—— (1)}$$

complex canonical form

Since $\xi(x,y)$ and $\eta(x,y)$ are complex characteristic curves and we want a real ~~form~~ ch. curve.

(We can add and subtract to get real curve using the superposition principle)

$\xi(x,y) = \alpha + i\beta$ and $\eta(x,y) = \alpha - i\beta$

[ξ & η are conjugate b/c λ_1 & λ_2 are conjugate]

$$\alpha(\xi,\eta) = \alpha = \frac{1}{2}(\xi + \eta) \qquad \beta = \frac{1}{2i}(\xi - \eta) = \beta(\xi, \eta)$$

$\Downarrow$

α & β are two real characteristic curves

We've to get real canonical form in (α, β) form

$$u(x,y) \to \underbrace{u(\xi,\eta) \to u(\alpha,\beta)}_{\text{Using this part:}}$$

$$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi \qquad\qquad u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta$$

$$= \frac{1}{2} u_\alpha + \frac{1}{2i} u_\beta \qquad\qquad = \frac{1}{2} u_\alpha - \frac{1}{2i} u_\beta$$

$$u_{\xi\eta} = \frac{1}{2}(u_{\alpha\alpha}\alpha_\eta + u_{\alpha\beta}\beta_\eta) + \frac{1}{2i}(u_{\beta\alpha}\alpha_\eta + u_{\beta\beta}\beta_\eta)$$

$$= \frac{1}{2}\left(u_{\alpha\alpha}\cdot\frac{1}{2} + u_{\alpha\beta}\left(\frac{-1}{2i}\right)\right) + \frac{1}{2i}\left(u_{\beta\alpha}\cdot\frac{1}{2} + u_{\beta\beta}\left(\frac{-1}{2i}\right)\right)$$

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$= \frac{1}{4} u_{\alpha\alpha} - \frac{1}{4i} \cancel{u_{\alpha\beta}} + \frac{1}{4i} \cancel{u_{\beta\alpha}} + \frac{1}{4} u_{\beta\beta}$

$\Rightarrow \boxed{u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta})}$

The real canonical form is :

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)}$$

$u_{xx} + x^2 u_{yy} = 0$

$A = 1 \quad B = 0 \quad C = x^2$

$B^2 - 4AC = -4x^2 < 0 \quad \forall \quad x \neq 0 \Rightarrow$ Elliptic for all $x \neq 0$

$\lambda^2 + x^2 = 0$

$\lambda = \pm ix \quad \lambda_1 = ix \quad \lambda_2 = -ix$

Choose $\xi(x,y)$ and $\eta(x,y)$ s.t.

~~$\xi_x = ix\,\xi_y$~~

The 2 associated ODE are :

$\frac{dy}{dx} + ix = 0 \quad \& \quad \frac{dy}{dx} - ix = 0$

$\Rightarrow y + i\frac{x^2}{2} = C_1 \quad \& \quad y - i\frac{x^2}{2} = C_2$

$\xi(x,y) = y + i\frac{x^2}{2} \quad \& \quad \eta(x,y) = y - i\frac{x^2}{2}$

$\alpha(\xi,\eta) = \frac{1}{2}(2y) = y$ (real part) $\quad \beta(\xi,\eta) = \frac{x^2}{2}$ (Imaginary part)

directly get $u(x,y) \rightarrow u(\alpha,\beta)$

$u_x = u_\alpha \cancel{\alpha_x}^{0} + u_\beta \beta_x = u_\beta x$

$u_{xx} = 1 \cdot u_\beta + x(\cancel{u_{\beta\alpha}\alpha_x} + u_{\beta\beta}\beta_x)$

$= u_\beta + x^2 u_{\beta\beta}$

$u_y = u_\alpha \alpha_y + \cancel{u_\beta \beta_y}^{0} = u_\alpha$

$u_{yy} = u_{\alpha\alpha}\alpha_y + \cancel{u_{\alpha\beta}\beta_y} = u_{\alpha\alpha}$

substitute :

$\Rightarrow \quad u_\beta + x^2 u_{\beta\beta} + x^2 u_{\alpha\alpha} = 0$

$\Rightarrow \quad \boxed{x^2 (u_{\beta\beta} + u_{\alpha\alpha}) + u_\beta = 0}$ ~~Canonical~~ form.

$x^2 = 2\beta$ (write all x and y in terms of α & β)

$\Rightarrow \quad 2\beta (u_{\alpha\alpha} + u_{\beta\beta}) + u_\beta = 0$

$\Rightarrow \quad \boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\beta}{2\beta}}$ Canonical

* If $u_{xx} + 4u_{yy} = 0$ is given, 'u' is already in canonical form so, no need to proceed further.

Same has to be applied in case of hyperbolic & parabolic

* In elliptic, we can't get solⁿ (can't integrate $u_{\alpha\alpha}$ & $u_{\beta\beta}$) from Canonical form. So, they are of not help.

2nd eg.

$u_{xx} - 2(\sin x) u_{xy} - (\cos^2 x) u_{yy} - u_y \cos x = 0$

$B = -2\sin x \qquad A = 1 \qquad C = -\cos^2 x$

$B^2 - 4AC = 4\sin^2 x + 4\cos^2 x = 4 > 0 \Rightarrow$ Hyperbolic

$\alpha^2 - (2\sin x)\alpha + (-\cos^2 x) = 0 \qquad \alpha = \frac{2\sin x \pm \sqrt{4\sin^2 x + 4\cos^2 x}}{2}$

$\alpha = \sin x \pm 1$

$\lambda_1 = \sin x + 1 \qquad \lambda_2 = \sin x - 1$

$\frac{dy}{dx} + (\sin x + 1) = 0 \quad \& \quad \frac{dy}{dx} + (\sin x - 1) = 0$

$\Rightarrow \quad y - \cos x + x = c_1 \quad \& \quad y - \cos x - x = c_2$

$\xi = y - \cos x + x \qquad \eta = y - \cos x - x$

$u(x,y) \to u(\xi, \eta)$

Find $u_{xx}, u_{yy}, u_{xy}, u_y$, will get $\boxed{u_{\eta\xi} = 0}$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$= u_\xi (1) + u_\eta (1) = u_\xi + u_\eta$

$\Rightarrow u_\eta = f(\eta)$

$u_\xi = \eta f(\xi) + g(\eta)$

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$u_y = u_\xi + u_\eta$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y$

$= \underline{u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}}$

$= u_\xi \sin x + u_\eta \sin x$

$u_{xx} = u_\xi \xi_x + u_\eta \eta_x = u_\xi(\sin x + 1) + u_\eta(\sin x - 1)$

$u_{xx} = \cos x\, u_\xi + u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + \cos x\, u_\eta + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x$

$= \cos x(u_\xi + u_\eta) + u_{\xi\xi}(\sin x + 1) + u_{\eta\eta}(\sin x - 1) + u_{\xi\eta}[$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$= u_\xi(\sin x + 1) + u_\eta(\sin x - 1)$

$= (u_\xi + u_\eta)\sin x + (u_\xi - u_\eta)$

$u_{xx} = \cos x(u_\xi + u_\eta) + \sin x(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x) + (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x - u_{\eta\xi}\xi_x - u_{\eta\eta}\eta_x)$

$= \cos x(u_\xi + u_\eta) + \sin x[u_{\xi\xi}(\sin x + 1) + u_{\xi\eta}(\sin x - 1) + u_{\xi\eta}(\sin x + 1) + u_{\eta\eta}(\sin x - 1)] + [u_{\xi\xi}(\sin x + 1) + u_{\xi\eta}(\sin x - 1) - u_{\xi\eta}(\sin x + 1) - u_{\eta\eta}(\sin x - 1)]$

$= \cos x(u_\xi + u_\eta) + \sin x[u_{\xi\xi}(\sin x + 1) + u_{\eta\eta}(\sin x - 1) + 2u_{\xi\eta}\sin x] + u_{\xi\xi}(\sin x + 1) - u_{\eta\eta}(\sin x - 1) - 2u_{\eta\xi}]$

$u_{xy} = \cancel{(\sin x + 1)}[u$

$(\sin x + 1)[u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y \cancel{+ u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y}] + (\sin x - 1)[u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y]$

Eq^n becoms :

$u_{xx} - 2\sin x\,(u_{xy}) - (\cos^2 x)u_{yy} - u_y \cos x = 0$

13/11/17 One-dimensional wave eq^n

Vibration on a infinite string :

eq^n : $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$, $t > 0$ $(c > 0)$

IC : (1) — $u(x,0) = f(x)$
(2) — $u_t(x,0) = g(x)$ $\}$ $-\infty < x < \infty$

$f(x)$ is the initial position of the string
$g(x)$ is the initial velocity of the string at x.

$A = 1$, $B = 0$, $C = -c^2$

$B^2 - 4AC = +4c^2 > 0 \ \forall \ c \neq 0 \Rightarrow$ Hyperbolic PDE

$A\alpha^2 + B\alpha + C = 0$

$\Rightarrow \alpha^2 - c^2 = 0 \Rightarrow \alpha = \pm c$

$\lambda_1 = c$ $\lambda_2 = -c$ $\boxed{\lambda(t,x)}$

choose ξ and η s.t.

$\frac{dx}{dt}$ $\frac{dy}{dx} + c = 0$ & $\frac{dy}{dx} - c = 0$

$\Rightarrow \boxed{x - ct = \xi}$ $\boxed{x + ct = \eta}$ (let us choose in this way)

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$u(x,y) \longrightarrow u(\xi,\eta)$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x$

$u_t = u_\xi \xi_t + u_\eta \eta_t = c u_\eta - c u_\xi$

$u_{tt} = c\left[u_{\eta\xi}\xi_t + u_{\eta\eta}\eta_t - u_{\xi\xi}\xi_t - u_{\xi\eta}\eta_t\right]$

The canonical form is :

$$\boxed{u_{\eta\xi} = 0}$$

$-2c^2 u_{\eta\xi} + c^2 u_{\eta\eta} + c^2 u_{\xi\xi} = c^2 u_{\xi\xi} + c^2 u_{\eta\eta} + 2u_{\eta\xi} c^2$

$u_{\xi\eta} = 0$

$\Rightarrow u_\xi = f(\xi)$

$\Rightarrow u = F(\xi) + G(\eta)$

$u(x,t) = F(x-ct) + G(x+ct)$

F & G are arbitrary, smooth func^n

Using initial cond^n s :

$u(x,0) = f(x) \Rightarrow$

$u(x,0) = F(x) + G(x) = f(x)$ —— (3)

$u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$

$u_t(x,0) = -cF'(x) + cG'(x) = g(x)$ —— (4)

Integrating (4) wrt x for x_0 to x, we get

$-cF(x) + c\,G(x) = \int_{x_0}^{x} g(s)\,ds$ —— (5)

Solving for $F(x)$ and $G(x)$, we get

$$F(x) = \frac{1}{2c}\left[c f(x) - \int_{x_0}^{x} g(s)\,ds\right]$$

$$G(x) = \frac{1}{2c}\left[c f(x) + \int_{x_0}^{x} g(s)\,ds\right]$$

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Therefore,

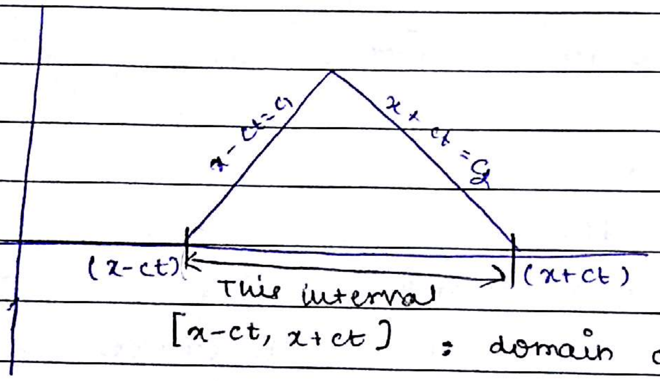
$$u(x,t) = \frac{1}{2c}\Big[\underbrace{c\,f(x-ct) - \int_{x_0}^{x-ct} g(s)\,ds}_{F(x-ct)} + \underbrace{c\,f(x+ct) + \int_{x_0}^{x+ct} g(s)\,ds}_{G(x+ct)}\Big]$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

De-Alembert solⁿ for 1-D wave Eqⁿ

* ~~we~~ $f \in C^2$ (twice differentiable), $g \in C^1$ s.t. $u \in C^2$ will be the regular solⁿ of wave eqⁿ.

$\Longrightarrow$ $x - ct = c_1$ and $x + ct = c_2$



$[x-ct,\ x+ct]$: domain of dependence for (x,t)

PROPERTIES :-

1.) If $f(x)$ and $g(x)$ are odd funcⁿ, then solⁿ $u(x,t)$ is also an odd funcⁿ (Prove $u(-x,t) = -u(x,t)$)

2.) If $f(x)$ and $g(x)$ are even funcⁿ, then $u(x,t)$ is also even ie, $u(-x,t) = u(x,t)$

3.) Periodic initial data yield periodic solⁿ

$f(x+2L) = f(x) \quad \forall x$

$g(x+2L) = g(x) \quad \forall x$

then,

$u(x+2L, t) = u(x,t)$

Infact, $u\left(x, t + \frac{2L}{c}\right) = u(x,t)$ [In this case]

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SPECIAL CASE

(i) Initial position of string is at rest

$f(x) = 0$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

(ii) Initial velocity at x is 0

$g(x) = 0$

then $u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$

Eg. $u_{tt} = c^2 u_{xx}$, $x \in \mathbb{R}$, $t > 0$

(i) IC : $u(x,0) = \sin x$ $x \in \mathbb{R}$

$u_t(x,0) = 0$

$$u(x,t) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right]$$

$$= \frac{1}{2}\left[2\sin x \cos ct\right]$$

(ii) Eg. $u(x,0) = 0$ $u_t(x,0) = \sin x$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \cancel{g(x)} \sin x \, dx$$

$$= \frac{1}{2c}\left[\cos(x-ct) + \cos(x+ct)\right]$$

Vibration on semi infinite string

$u_{tt} = c^2 u_{xx}$ $0 < x < \infty$, $t > 0$, $c > 0$

IC : $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

Zero Condⁿ (boundary condⁿ) $u(0,t) = 0$ $t > 0$

* If we use De-Alembert sol^n for $x - ct < 0 \Rightarrow t > c/x$.
then $f(x-ct)$ will become $-ve$, but we have $f(x) \;\forall\; x$
* It is meaningless for $t > x/c$

Re-write IC's as below

$$\left.\begin{aligned} u(x,0) &= F(x) \\ u_t(x,0) &= G(x) \end{aligned}\right\} \quad -\infty < x < \infty$$

where

$$F(x) = \begin{cases} f(x) & 0 < x < \infty \\ -f(-x) & -\infty < x < 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & 0 < x < \infty \\ -g(-x) & -\infty < x < 0 \end{cases}$$

$F(x)$ and $G(x)$ are odd ~~exten~~ extensions of $f(x)$ & $g(x)$

Now, De-Alembert sol^n becomes

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

We've to verify the above sol^n satisfies ICs. (original)

i) At $t = 0$

~~let t.~~ $u(x,0) = \frac{1}{2}\left[F(x) + F(x)\right] + \frac{1}{2c}\cancel{\int_{x}^{x} G(s)\,ds}^{\,0}$

$= F(x) = f(x) \quad [x > 0]$

ii) diff. wrt t, we get

$$u_t(x,t) = \frac{1}{2}\left[-c\,F'(x-ct) + c\,F'(x+ct)\right]$$

$$+ \frac{1}{2c}\left[0\;G(x+ct) + c\,G(x-ct)\right]$$

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Rule: Leibniz $\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)\,dx = b'(t)\,f(b(t)) - a'(t)\,f(a(t))$

$$u_t(x,0) = \frac{1}{2}\left[-cF'(x) + cF'(x)\right] + G(x)$$
$$= G(x) = g(x) \quad (x>0)$$

iii) $$u(0,t) = \frac{1}{2}\left[F(-ct) + F(ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} G(s)\,ds$$
$$= \frac{1}{2}\left[-F(ct) + F(ct)\right] + 0 = 0 \quad \left[F \,\&\, G : \text{odd func}^{n}\text{s}\right]$$

SPECIAL CASE.

i) initial velocity $g(x) = 0$
$\Rightarrow G(x) = 0$

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] \quad \forall\, x \in \mathbb{R}$$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] & x > ct \\ \frac{1}{2}\left[-f(-x-ct) - f(x+ct)\right] & x < ct \end{cases}$$
$$\Downarrow$$
$$\frac{1}{2}\left[f(x+ct) - f(ct-x)\right]$$
↳ defined for $x < ct$

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Vibration on finite string

(1) → $u_{tt} = c^2 u_{xx}$ $\quad 0 < x < l, \quad t > 0$

$u(x,0) = f(x)$; $u_t(x,0) = g(x)$, $\quad 0 < x < l$

$$\left.\begin{aligned} u(0,t) &= u_t(0,t) = 0 \\ u(l,t) &= u_t(l,t) = 0 \end{aligned}\right\} \; t>0$$

Consider the ICs :
$u(x,0) = F(x)$, $\quad u_t(x,0) = G(x)$, $\quad -l < x < l$

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$$F(x) = \begin{cases} f(x) & 0 < x < l \\ -f(-x) & -l < x < 0 \end{cases} \qquad G(x) = \begin{cases} g(x) & 0 < x < l \\ +g(-x) & -l < x < 0 \end{cases}$$

F & G are odd extension of $f(x)$ & $g(x)$

$F(x+2l) = F(x) \quad \forall x$
$G(x+2l) = G(x) \quad \forall x$ } // Make F and G periodic

F & G are periodic func^n ⇒ can be represented as ~~to~~ a Fourier series.

Therefore, F(x) and G(x) Fourier sine expansions :
(odd func^n)

$\cancel{F(x) = \frac{2}{l}\int_0^l F(x)}$

$$F(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{l}x\right) \; ; \; G(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$

$$a_n = \frac{2}{l}\int_0^l \cancel{F(x)}_{f(x)} \sin\left(\frac{n\pi}{l}x\right)dx \; ; \; b_n = \frac{2}{l}\int_0^l \cancel{G(x)}_{g(x)} \sin\left(\frac{n\pi}{l}x\right)dx$$

Since $F(x) = f(x) \quad 0 < x < l$
$G(x) = g(x) \quad 0 < x < l$

The De-Alembert sol^n, now, will be :

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(\sin\left(\frac{n\pi}{l}(x-ct)\right) + \sin\left(\frac{n\pi(x+ct)}{l}\right)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi s}{l}\right)ds$$

F(x) & G(x) should be periodic and $F, G \in C^2$ for :

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + \frac{1}{2c}\sum_{n=1}^{\infty} b_n \left[\left(-\cos\frac{n\pi s}{l}\right)\frac{l}{n\pi}\right]_{x-ct}^{x+ct}$$

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$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + \frac{l}{\pi c}\sum_{n=1}^{\infty}\frac{b_n}{n}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)}$$

Cond^{n}s required for fourier ~~seu~~ series existence:

$\begin{cases} f \in C^2, & f''' \text{ is piecewise continuous} \\ g(x) \in C^1 & \& \ g'' \ \text{"} \quad \text{"} \end{cases}$

then $u \in C^2$ is regular sol^n of wave eq^n

Separation of variables method (Alternative method)

For above problem:

$u(x,t) = X(x)\,T(t)$: sol^n of PDE (1)

$\Rightarrow$ It satisfies PDE (1)

$u_t(x,t) = X(x)\cdot \dot{T}$ $\qquad u_{tt} = X\ddot{T}$

$u_x = X'T$ $\qquad u_{xx} = X''T$

Satisfying in PDE:

$X\ddot{T} = c^2 X''T$

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = \lambda$$

func^n of X $\qquad$ func^n of T

$\Rightarrow X'' - \lambda X = 0, \quad \ddot{T} - \lambda c^2 T = 0$ (non-trivial sol^{n}s are required)

To find above values, we should have 2 boundary cond^{n}s

$u(0,t) = 0$

$\Rightarrow X(0)\,T(t) = 0$ But $T \neq 0$ (If $T=0$, it'll lead to trivial sol^n)

$\Rightarrow X(0) = 0$ —— (1)

$u(l,t) = 0$

$\Rightarrow X(l)\,T(t) = 0$

$\Rightarrow X(l) = 0$ —— (2)

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Case-I : $\lambda > 0$, $\lambda = +n^2$

$$m^2 - n^2 = 0 \Rightarrow m = \pm n$$

$$X(x) = c_1 e^{nx} + c_2 e^{-nx} \quad \text{— general sol}^n$$

$$\left.\begin{aligned} X(0) &\Rightarrow c_1 + c_2 = 0 \\ X(l) &\Rightarrow c_1 e^{nl} + c_2 e^{-nl} = 0 \end{aligned}\right\} c_1 = c_2 = 0$$

This gives trivial sol^n, we don't need this

Case-II. $\lambda = 0$

$$X'' = 0$$

$$X = A + Bx$$

$$\left.\begin{aligned} X(0) &= c_2 = 0 \\ X(l) &\Rightarrow c_1 \cancel{l} = 0 \end{aligned}\right\} c_1 = c_2 = 0$$

Again we get a trivial sol^n

Case-III : $\lambda < 0$, $\lambda = -n^2$

$$m^2 + n^2 = 0$$

$$m = \pm in$$

$$X = A\cos nx + B\sin nx$$

$$X(0) = A = 0$$

$$X(l) = B\sin nl = 0 \qquad B \neq 0$$

(if $B = 0 \Rightarrow$ trivial sol^n)

$$\Rightarrow \sin nl = 0$$

$$nl = K\pi, \quad K \in \mathbb{Z}$$

$$\boxed{n = \frac{K\pi}{l}}, \quad K \in \mathbb{Z}$$

$$\lambda = -n^2 = -\frac{K^2\pi^2}{l^2} = \cancel{-\frac{n^2\pi^2}{l^2}}, \quad \cancel{n = 1, 2, 3, \ldots}$$

Eigen value : $\lambda_K = -\dfrac{K^2\pi^2}{l^2}$, $K = 1, 2, 3 \ldots$

(For eigen value, we take only +ve K)

Eigen func^n : $X_K(x) = B_K \sin\left(\dfrac{K\pi}{l}x\right)$

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putting in $\ddot{T} - \lambda c^2 T = 0$

$$\ddot{T}_n + \frac{n^2\pi^2 c^2}{l^2} T_n = 0$$

$$m^2 + \left(\frac{n^2\pi^2 c^2}{l^2}\right) = 0$$

$$m = \pm i \left(\frac{n\pi c}{l}\right)$$

$$T_n(t) = C_n \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right) \quad \text{- general sol}^n$$

$$u_n(x,t) = X_n(x) \cdot T_n(x) \quad \& \; n = 1, 2, \ldots$$

$$= \left[a_n \cos\left(\frac{n\pi c t}{l}\right) + b_n \sin\left(\frac{n\pi c t}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right)$$

superposition principle implies :

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c t}{l}\right) + b_n \sin\left(\frac{n\pi}{l} ct\right)\right] \sin\left(\frac{n\pi x}{l}\right)$$

IC: (to eliminate const) $u(x,0) = \underbrace{\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)}_{\Downarrow} = f(x)$

Looks like ~~fourier sine series of~~ $f(x)$
so, we should choose a_n s.t. $f(x)$ becomes fourier series

Choose $u(x,0)$ s.t. a_n become Fourier sine ~~coeff. of~~ $f(x)$

$$\boxed{a_n = \frac{2}{l} \int f(x) \sin\left(\frac{n\pi x}{l}\right) dx}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-a_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi c t}{l}\right) + b_n \left(\frac{n\pi c}{l}\right) \cos\left(\frac{n\pi c t}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{b_n \left(\frac{n\pi c}{l}\right)}_{\text{This should be coeff. of fourier sine series}} \sin\left(\frac{n\pi x}{l}\right) = g(x)$$

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$$\Rightarrow b_n\left(\frac{n\pi c}{l}\right) = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

($\Rightarrow$ Choose $u_t(x,0)$ s.t b_n becomes Fourier ~~sin~~ series coeff. of $g(x)$)

$$\Rightarrow b_n = \left(\frac{2}{l}\right)\left(\frac{l}{n\pi c}\right)\int_0^l g(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$\Rightarrow \boxed{b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx}$$

Ex. $u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < 2\pi, \quad t > 0$

$$\left.\begin{aligned} u(x,0) &= \cos x - 1 = f(x) \\ u_t(x,0) &= 0 = g(x) \end{aligned}\right\} \quad 0 \le x \le 2\pi$$

$$\left.\begin{aligned} u(2\pi, t) &= 0 \\ u(0,t) &= 0 \end{aligned}\right\} \quad t \geq 0$$

Find sol^n using De-Alembert's sol^n & method of separation variable

$$g(x) = 0$$

$$\Rightarrow b_n = 0$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,t) = \sum a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi c t}{l}\right) + \frac{l}{\pi c}\sum_{n=1}^{\infty}\frac{b_n}{n}\sin\frac{n\pi x}{l}\sin\left(\frac{n\pi c t}{l}\right)$$

Non-homogeneous PDE :

with homogeneous ICs :

① PDE : $u_{tt} - c^2 u_{xx} = F(x,t)$ $x \in \mathbb{R}$ $t > 0$

② ICs : $u(x,0) = 0$ $u_t(x,0) = 0$, $x \in \mathbb{R}$

We know the solⁿ for :

$$\begin{cases} \text{PDE} : u_{tt} + c^2 u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ \text{ICs} : u(x,0) = f(x), \; u_t(x,0) = g(x) \end{cases}$$

Consider

$v(x,t;s)$ satisfies the following problem ← boundary condⁿ

$v_{tt} - c^2 v_{xx} = 0$ —③ $x \in \mathbb{R}$ $\underline{s > t > 0}$

$\left.\begin{aligned} v(x, s; s) &= 0 \\ v_t(x,s;s) &= F(x,s) \end{aligned}\right\}$ —④

Initial data is given at $t = s$ (instead of $t = 0$)

Using De-Alembert's solⁿ for PDE ③ - ④

$$v(x,t;s) = \frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} F(r,s) \, dr$$

(1st term will be cancelled)

We define $u(x,t) = \int_0^t v(x,t;s) \, ds$

Claim : $u(x,t)$ is solⁿ of PDE ① - ②

Verification :

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) \, dx = \int_{a(t)}^{b(t)} f_t(x,t) \, dx + b'(t) f(b(t),t) - a'(t) f(a(t),t)$$

at $t = s$, $v(\;) = 0$ (IC)

$$u_t(x,t) = 1 \cdot \cancel{v(x,t;t)} + \int_0^t v_t(x,t;s) \, ds$$

$a(t) = 0$

$b(t) = t$

$$u_{tt}(x,t) = \underbrace{v_{tt}(x,t;t)}_{F(x,t)} + \int_0^t v_{tt}(x,t;s) \, ds$$

$$u_{tt}(x,t) = F(x,t) + \int_0^t c^2 v_{xx}(x,t;s) \, ds$$

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$$u_{xx} = \int_0^t v_{xx}(x, t ; s)\, ds$$

$$\Rightarrow \quad u_{tt}(x,t) = F(x,t) + c^2 u_{xx}$$

$\Rightarrow$ $u(x,t)$ is sol^n of PDE ① − ②

Hence Proved.

→ The sol^n of original problem ① − ② is.

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr\, ds$$

$$\boxed{u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr\, ds}$$

2.) with non-homogenous ICs :

⑤ — IC : $u(x, 0) = f(x)$

$u_t(x,0) = g(x)$

Now, we'll use superposition principle.

Find $u_1(x,t)$ s.t.

$$u_{tt} - c^2 u_{xx} = 0 \quad , \quad x \in R \quad t > 0$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x)$$

Find $u_2(x,t)$ s.t.

$$u_{tt} - c^2 u_{xx} = F(x,t) \quad , \quad x \in \mathbb{R} \quad , t > 0$$

$$u(x,0) = 0 \quad , \quad u_t(x,0) = 0$$

$u(x,t) = u_1(x,t) + u_2(x,t)$ satisfies eq^n ① - ⑤

$$u_1(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

$$u_2(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr\, ds$$

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PDE: $u_{tt} - c^2 u_{xx} = F(x,t)$ $\quad 0 < x < L, \; t > 0$

IC: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ $\quad 0 \le x \le L$

BC: $u(0,t) = 0 \quad u(L,t) = 0$ (Boundary cond^n) $\Rightarrow u_t(0,t) = u_t(L,t) = 0$

Prove that PDE ⑦ - ⑧ has unique sol^n if it exist

Assume u_1 and u_2 are 2 sol^{n}s of ⑦ - ⑧:

$v(x,t) = u_1 - u_2$

$v(x,t)$ will satisfy

$v_{tt} - c^2 v_{xx} = 0$ $\quad$ from $v_t(x,0) = 0$

$v(x,0) = 0 \quad v_t(x,0) = 0$

BC: $v(0,t) = v(L,t) = 0$

$\Rightarrow v_t(0,t) = v_t(L,t) = 0$

→ $f(x) = g(x) = 0 \Rightarrow$ D'Alembert's sol^n will give $v = 0$ but we don't know if it is unique or not

→ Consider $E(t) = \frac{1}{2} \int_0^L \left(c^2 v_x^2(x,t) + v_t^2(x,t) \right) dx$

[Energy term]

$E(t) \ge 0 \quad \forall \; t > 0$

$$\frac{dE}{dt} = \frac{1}{2} \int_0^L \left(2c^2 v_x v_{xt} + 2 v_t v_{tt} \right) dx$$

integrate 1st by parts

$$= \int_0^L v_t v_{tt} \, dt + c^2 v_t v_x \Big|_0^L - c^2 \int_0^L v_{xx} v_t \, dx$$

$$= \int_0^L v_t \left(v_{tt} - c^2 v_{xx} \right) dx = 0$$

$\Rightarrow E(t) = \text{const} \quad \forall \; t \ge 0$ ——— ⑨

$$E(0) = \frac{1}{2} \int_0^L \left(c^2 v_x^2(x,0) + v_t^2(x,0) \right) dx$$

$\Rightarrow E(0) = 0$ ——— ⑩

$\Rightarrow E(t) = 0 \quad \forall \; t$

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$V(x,t) = g(t)$
$V_t'(x,t) = g'(t) = 0$
$\Rightarrow g(t) = c$

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$E(t) = 0 \Rightarrow V_x(x,t) = 0$ & $V_t(x,t) = 0$ [both terms are positive should be individually 0]

$\Rightarrow$ Only possible if

$V_{xt}(x,t) = \text{const.} \quad \forall \; x \in [0,l)$
$t \geq 0$

we know $V(x,0) = 0$

$\Rightarrow \boxed{V(x,t) = 0}$

$\Rightarrow u_1 = u_2$

Eg $u_{tt} = u_{xx} + x^2 - t, \quad x \in \mathbb{R}, t > 0$

$u(x,0) = u_t(x,0) = 0 \quad x \in \mathbb{R}.$

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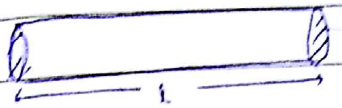
Heat Conduction Problem



$u_t = K u_{xx}$, $0 < x < l$, $t > 0$ $K > 0$

IC $u(x,0) = f(x)$, $0 \le x \le l$ Thermal conductivity

BC $u(0,t) = u(l,t) = 0$, $t \geq 0$ $K = \dfrac{\kappa}{c\rho}$ → mass density

↑ heat capacity

We will solve this PDE by method of separation of variables

~~A = 0 B =~~ $A = K$ $B = 0$ $C = 0$

⇒ $B^2 - 4AC = 0$ ⇒ It is parabolic type

The canonical form is: Can't use D'Alembert's sol^n as RHS ≠ 0

$$u_{\eta\eta} = Q(\xi, \eta, u, u_\xi, u_\eta)$$

We can reduce this PDE into:

$$u(x,t) = X(x)\,T(t)$$

$$u_t = X\dot{T}$$

$$u_{xx} = X''T$$

$$X\dot{T} = K X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{\dot{T}}{KT} = \lambda \text{ (const)}$$

func^n of x alone   ↳ func^n of t alone

$X'' + \lambda X = 0$, $\dot{T} - \lambda K T = 0$

applying ~~BC~~ BC:

$$X(0) = X(l) = 0$$

Case-I : $\lambda > 0$, $\lambda = \alpha^2$

$m^2 - \alpha^2 = 0 \Rightarrow m = \pm\alpha$

$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

~~$X(0)$~~ B.C. ⇒ $X(x) = 0$

Case - II : $\lambda = 0$

$\cancel{Z}\ Z'' = 0$

$\Rightarrow Z = Ax + B$

$\Rightarrow$ BCs $\Rightarrow$ $Z(x) = 0$

Case - III $\lambda < 0$, $\lambda = -\alpha^2$

$m^2 + \alpha^2 = 0 \Rightarrow m = \pm i\alpha$

$Z(x) = A\cos\alpha x + B\sin\alpha x$

$Z(0) = A = 0$

$Z(l) = B\sin\alpha l = 0$

$\Rightarrow \sin\alpha l = 0$ (B can't be 0)

$\Rightarrow \alpha l = n\pi \quad \forall\ n \in \mathbb{Z}$

$\Rightarrow \alpha_n = \frac{n\pi}{L}$

Eigen values: $\lambda_n = -\alpha^2 = -\frac{n^2\pi^2}{l^2}$, $n = 1, 2, 3 \ldots$ (Because n=0 gives $\lambda = 0$) trivial sol^n

$$\boxed{Z_n(x) = B_n \sin\left(\frac{n\pi}{l}\right)x} \quad n = 1, 2, 3, \ldots$$

$\cancel{T}\ \dot{T} - \lambda kT = 0 \Rightarrow \dot{T} + \left(\frac{n^2\pi^2}{l^2}\right)kT_n = 0$

$$\Rightarrow \boxed{T_n(t) = C_n \exp\left(-\frac{n^2\pi^2}{l^2}kt\right)}$$

$U_n(x,t) = Z_n(x)\,T_n(t)$

Superposition principle implies

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum Z_n T_n$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right)}$$

To eliminate a_n, use ICs

$u(x,0) = f(x)$

$\Rightarrow \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$

Choose a_n s.t. $u(x,0)$ become Fourier sine series of $f(x)$

$$\boxed{a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi}{l} x\right) dx}$$

Condn's needs for this soln to exist :

$f(x) \in C^1$

f''' is piecewise continuous

Eg. $u_t = u_{xx}$ $\quad 0 < x < l \quad t > 0$

$u(0,t) = u(l,t) = 0 \quad 0 \leq x \leq l$

$u(x,0) = \underbrace{x(l-x)}_{f(x)} \quad 0 \leq x \leq l$

Find heat eqn

$$a_n = \frac{2}{l}\int_0^l x(l-x) \sin\left(\frac{n\pi x}{l}\right) dx$$

<u>Non-homogenous problem</u> :

1 — $u_t - k u_{xx} = F(x,t) \quad , \quad 0 \leq x \leq l \; , \; t > 0$

2 — $u(x,0) = f(x)$

3 — $u(0,t) = u(l,t) = 0 \quad t \geq 0$

We can use Duhamel principle to find soln of Non-homogenous problem.

→ This problem has unique soln if it exists.

Proof : let $u_1(x,t)$ and $u_2(x,t)$ be 2 soln's of PDE 1 — 3

$\Rightarrow$ $v(x,t) = u_1(x,t) - u_2(x,t)$ will satisfy

$v_t = k v_{xx}$

$v(x,0) = 0$

$v(0,t) = v(l,t) = 0$

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Claim: To show that v is identically zero.

$$E(t) = \frac{1}{2k}\int_0^l v^2(x,t)\,dx$$

$$E(t) \geq 0 \quad \forall \; t \geq 0$$

differentiating wrt t

$$\frac{d}{dt}(E(t)) = \frac{1}{2k}\int_0^l 2v(x,t)\,\underbrace{v_t(x,t)}_{kv_{xx}}\,dx$$

$$= \cancel{2}\int_0^l v(x,t)\,v_{xx}(x,t)\,dx$$

① ②

Integrating by parts

$$= \cancel{v v_x}\Big|_0^l - \int_0^l v_x v_x\,dx$$

$$= -\int_0^l v_x^2\,dx$$

$$E'(t) < 0 \quad \forall \; t > 0$$

Funcⁿ is ↓ng

$$E(0) = \frac{1}{2k}\int_0^l v^2\cancel{(x,0)}\,dx = 0$$

Funcⁿ is positive & ↓ng and $E(0) = 0$

$\Rightarrow \quad E(t) = 0 \quad \forall \; t \geq 0$

$\Rightarrow \quad v^2(x,t) = 0$

$\Rightarrow \quad v(x,t) = 0 \quad \forall \; x \in [0,l] \quad t \geq 0$

$\Rightarrow \quad u_1(x,t) = u_2(x,t)$

Hence Proved

→ With non-homogenous B.C.:

$u_t = k u_{xx}, \quad 0 < x < l, \quad t > 0$ —— ④

IC: $u(x,0) = f(x) \quad 0 \leq x \leq l$ —— ⑤

BC: $u(0,t) = 0, \quad u(l,t) = b \quad t \geq 0$ —— ⑥

*

i) Find a particular solⁿ of PDE & B.C.

ii.) Find solⁿ of corresponding homogenous B.C.

Then add the solⁿ

(if 3 points = polynomial)

(Interpolation to make it line (from 2 points))

i.) Assume $u_p(x,t) = cx + d$ is particular sol^n satisfying BC ⑥

$u_p(0,t) = d = a$

$u_p(l,t) = cl + a = b \Rightarrow c = \frac{b-a}{l}$

Particular sol^n is : $\boxed{u_p(x,t) = \left(\frac{b-a}{l}\right)x + a}$

We can check that it ~~depends~~ satisfies PDE also

ii.) Consider

$V_t - k V_{xx} = 0 \qquad 0 < x < l \qquad t \geq 0$

BC $\quad V(0,t) = 0 \quad , \quad V(l,t) = 0 \qquad t \geq 0$

IC $\quad V(x,0) = f(x) - u_p(x,0) = F(x) \quad , \quad 0 \leq x \leq l$

$$\Rightarrow V(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2 \pi^2 k t}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right)$$

where $a_n = \frac{2}{l} \int_0^l F(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$\boxed{u(x,t) = V(x,t) + u_p(x,t)}$$

Eg

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Ass. 5

(4) Show that $\sum_{n=0}^{\infty} (n+1)^2 z^n = \frac{1+z}{(1-z)^3}$, $|z|<1$

(1) — $\sum_{n=0}^{\infty} (n+1) z^n = \frac{d}{dz}\left(\sum_{z=0}^{\infty} z^{n+1}\right) = \frac{d}{dz}\left(z + z^2 + \dots\right)$

$$= \frac{d}{dz}\left(\frac{1}{1-z} - 1\right) = \frac{1}{(1-z)^2}$$

(2) — $\sum_{n=0}^{\infty} n(n+1) z^n = z\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty} z^{n+1}\right) = \frac{2z}{(1-z)^3}$, $|z|<1$

Add (1) & (2)

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)^2 z^n = \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3}$$

(5) (b) $f(z) = \frac{6z+8}{(2z+3)(4z+5)}$

Not diff.

$f(z) = f(z_0) + (z-z_0) f'(z_0) + \dots$

compare with $f(z)$

(8) claim show that $\forall\ \lambda \in C$ s.t.

i.) $f(z) = \lambda g(z) \in C$

ii.) Determine all entire func's f s.t. $|f'(z)| < |f(z)|$

Liouville's theorem :-

i.) $|f(z)| < |g(z)| \Rightarrow g(z) \neq 0$

Take $h(z) = \frac{f(z)}{g(z)}$ → entire, → non-0 & $h(z)$: entire func^n

$\left|\frac{f(z)}{g(z)}\right| < 1 \Rightarrow h(z)$: bounded by 1

Applying theorem, $h(z) = C$ $\forall$ $z \in C$

$\Rightarrow \frac{f(z)}{g(z)} = \lambda \Rightarrow f(z) = \lambda g(z)$.

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$|f'(x)| < |f(x)|$

$\Rightarrow f'(x) = \lambda f(x)$ (with same argument as above)

$\Rightarrow f'(x) - \lambda f(x) = 0$

$\Rightarrow \dfrac{f'(x)}{f(x)} = \lambda$

$\Rightarrow \cancel{f'} f(x) = e^{-\lambda x}$

Ass - 6

(6) $u(x,t) = \dfrac{1}{2c}\displaystyle\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(r,s)\, dr\, ds$

$c = 1$

$F(x,t) = x^2 - t$

$= \dfrac{1}{2}\displaystyle\int_0^t \int_{x-(t-s)}^{x+(t-s)} (r^2 - s)\, dr\, ds$

$= \dfrac{1}{2}\displaystyle\int_0^t \int_{x-\cancel{(t-s)}} \left(r^2 s - \dfrac{s^2}{2} \Big|_{x-(t-s)}^{x+(t-s)} \right) dr$

$= \dfrac{1}{2}\displaystyle\int_0^t$

Ass - 7 $\longrightarrow$ ~~can't apply~~ Duhamel ~~form~~ (can apply when $F(x,t)$)

(1)(b) $u_{tt} = c^2 u_{xx} - u \qquad 0 < x < L, \quad t > 0$

$u(x,0) = f(x) \qquad u_t(x,0) = 0 \qquad 0 \le x \le L$

$u(0,t) = u(L,t) = 0 \qquad t \geq 0$

→ have to apply separable method

$u(x,t) = X(x)\,T(t)$

$\Rightarrow \ddot{T} X - c^2 X'' T = -XT$

$\Rightarrow \dfrac{\ddot{T}}{T} - c^2 \dfrac{\ddot{X}}{X} = -1$

$\Rightarrow \dfrac{\ddot{T}}{c^2 T} = \dfrac{\ddot{X}}{X} - \dfrac{1}{c^2} = \lambda$

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$$\Rightarrow X'' - \left(\lambda + \frac{1}{c^2}\right) X = 0$$

$X(0) = 0$, $X(l) = 0$

sub. λ in $\ddot{T} - \lambda c^2 T = 0$

② (a) $u_t - u_{xx} = u$ $0 < x < 10$, $t > 0$

$u(x,0) = 3\sin 2\pi x - 7\sin 4\pi x$; $0 \le x \le 10$ $\hookrightarrow f(x)$

$u(0,t) = u(10,t) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left[\left(1 - \left(\frac{n\pi}{10}\right)^2 t\right)\right] \sin\left(\frac{n\pi x}{10}\right)$$ (verify)

1.) $a_n = \frac{2}{l} \int_0^l (3\sin 2\pi x - 7\sin 4\pi x) \sin\left(\frac{n\pi x}{10}\right) dx$

OR

2.) at $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{10}\right) = 3\sin 2\pi x - 7\sin 4\pi x$$

By comparing,

$\frac{n}{10} = 2 \Rightarrow n = 20$ non-zero for $n = 20$

$n/10 = 4 \Rightarrow n = 40$ non-zero for $n = 40$

$a_n \neq 0$ only for $n = 20, 40$

$a_{20} = 3$, $a_{40} = -7$

$$\Rightarrow u(x,t) = 3 \exp\left[\left(1 - \left(\frac{2\pi}{\cancel{10}}\right)^2\right)\right] \sin(2\pi x) - 7 \exp\left[\left(1 - \left(\frac{4\pi}{\cancel{10}}\right)^2\right) \sin(4\pi x)\right]$$

→ if in cos term, try to write in sine terms

(3) $u_t = k u_{xx} = 0 \qquad 0 < x < L, \quad 0 < t < T$

$u(x,0) = f(x) \qquad 0 \le x \le L$

$u(0,t) = u(L,t) = 0, \qquad 0 \le t \le T$

Claim i) $E'(t) \le 0$

ii) If $f(x) = x(2-x)$ & $L = 2$

then, $\int_0^2 u^2\, dx \le \frac{16}{15} \quad \forall\ t \ge 0$

i) $E(t) = \int_0^L u^2\, dx$

$E'(t) = \int_0^L 2 u u_t\, dx$

$= \int_0^L 2uk\, u_{xx}\, dx = 2k \int_0^L u\, u_{xx}\, dx$ ① ②

Integrating by parts

$= 2k \left[\cancel{u u_x} \Big|_0^L - \int_0^L u_x u_x\, dx \right]$

$= -2k \int_0^L u_x^2\, dx$

$\downarrow$ +ve

always (Theorem ....)

$\Rightarrow E'(t) \le 0$

ii) $E(t)$: +ve func^n

$E'(t) \le 0 \Rightarrow E(t)$ is $\downarrow$'g func^n

$\Rightarrow E(t) \le E(0)$

$\le \int_0^L u^2(x,0)\, dx$

$\le \int_0^L f^2(x)\, dx = \int_0^2 x^2 (2-x)^2\, dx$

$\int_0^2 x^2 (4 + x^2 - 4x)\, dx$

$\left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right]_0^2$

$\left[\frac{32}{5} - 16 + \frac{32}{3} \right]$

$= 32 \left[\frac{8}{15} \right] - 16$

$\Rightarrow E(t) \le \frac{16}{15}$

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→ For Heat Eq^n, we can't have De Alembert's sol^n

④ $u_t = u_{xx}$, $0 < x < L$, $t > 0$

IC $u(x,0) = f(x)$

BC $\left\{ \begin{array}{l} u(0,t) = T_1 \\ u(L,t) = T_2 \end{array} \right\} \Rightarrow$ non-hom B.C.

Assume particular sol^n $u_p(x,t) = cx + d$ [Interpolating 2 points]

$u_p(0,t) = t_1 = \cancel{cx}\ d$

$u_p(L,t) = t_2 = cL + d$

$\Rightarrow c = \dfrac{t_2 - t_1}{L}$

$$\Rightarrow \boxed{u_p(x,t) = \left(\frac{t_2 - t_1}{L}\right)x + t_1}$$: satisfying PDE & B.C.

Concider corresponding (making homogenous B.C.)

$V_t = V_{xx}$ $0 < x < L$, $t > 0$

$V(0,t) = 0$, $V(L,t) = 0$

$V(x,0) = f(x) - u_p(x,t)$

$= f(x) - \left(\dfrac{T_2 - T_1}{L}\right)x - T_1 = F(x)$

$$V(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$u(x,t) = u_p(x,t) + V(x,t)$

$$= \left(\frac{T_2 - T_1}{L}\right)x + T_1 + \sum_{n=1}^{\infty} a_n \exp\left(\frac{n^2\pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right)$$

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