M - III

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PDE

Cauchy value Problem (or Initial value Problem)

Objective : To find an integral surface of the given PDE :

$$f(x, y, z, p, q) = 0 \quad \text{—— (1)}$$

which contain an initial curve

$$C : x = x_0(s), \quad y = y_0(s), \quad z = z_0(s), \quad s \in I$$

The cauchy value problem is to find a sol^n $z(x,y)$ of the PDE (1) s.t.

$$z_0(s) = z(x_0(s), y_0(s)) \quad \forall \ s \in I$$

* If we can solve quasilinear $\Rightarrow$ we can solve for linear & semi-linear also.

Langrage Method (for solving Quasilinear Problem)

$$P(x,y,z)\, p + Q(x,y,z)\, q = R(x,y,z) \quad \text{—— (2)}$$

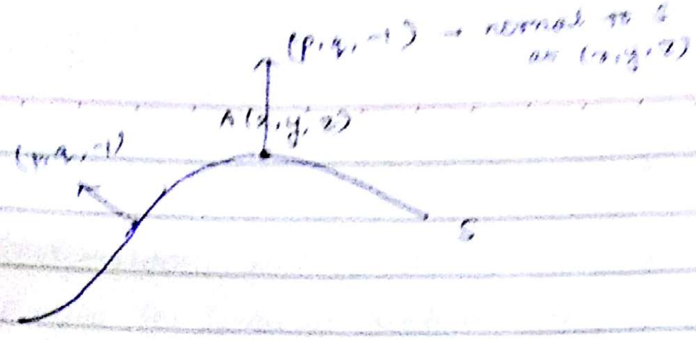
P, Q, R are smooth func^n in $D \subset R^3$.

$P, Q, R \in C^1(D) \rightarrow$ (derivatives of P, Q, R are continuously differentiable)

$P, Q, R : D \rightarrow R$ don't vanish simultaneously.

$Z = z(x,y)$ is an integral surface in xyz- space.

$$S : \{ z = z(x,y) : (x,y) \in D' \subset R \times R^2 \}$$



Eqⁿ ① becomes:

$$Pp + Qq - R = 0$$

We can say that : (P, Q, R) are orthogonal to $(p, q, -1)$ vector. Eqⁿ ② is equivalent to say that vector $(p, q, -1)$ and (P, Q, R) are orthogonal at each point $A \in S$

$\therefore$ $P\hat{i} + Q\hat{j} + R\hat{k}$ lies on the tangent plane at A.
(then only, they'll be orthogonal)

For a curve $C : x = x(t),\ y = y(t),\ z = z(t)$ on S
we have $(P\hat{i} + Q\hat{j} + R\hat{k}) \parallel (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$
diff. wrt t

equivalently,

$$\boxed{\frac{\dot{x}}{P} = \frac{\dot{y}}{Q} = \frac{\dot{z}}{R}}$$ — characteristic eqⁿ of PDE — ② — ③

or $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Thm: The general solⁿ of the PDE ② is
Implicit form ← $F(u, v) = 0$ (or $u = G(v)$ or $v = H(u)$) explicit form
where F is an arbitrary smooth funcⁿ of u and v.
$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solⁿ of characteristic eqⁿ ③ → called characteristic curves (here)

Proof: $u(x, y, z) = c_1$ & $v(x, y, z) = c_2$ are two solⁿs of characteristic eqⁿ ③

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$\Rightarrow$ $du = 0$ and $dv = 0$

$\Rightarrow$ $u_x dx + u_y dy + u_z dz = 0$ and $v_x dx + v_y dy + v_z dz = 0$

$$\text{But } \frac{\overset{dx}{\dot{x}}}{P} = \frac{\overset{dy}{\dot{y}}}{Q} = \frac{\overset{dz}{\dot{z}}}{R} = k \ (\text{let})$$

$$\Rightarrow P u_x + Q u_y + R u_z = 0$$

In the same way,

$$P v_x + Q v_y + R v_z = 0$$

Solving for P, Q and R, we get

$$\cancel{\frac{P}{u_y v_z}}$$

$$k = \frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \quad \text{---} (4)$$

$(u_y v_z - v_y u_z)$

Since $F(u,v) = 0$ leads to PDE of form

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)} \quad \text{---} (5)$$

[seen in revolution of surface Part]

Comparing eq^n (4) & (5), we get

$$\boxed{P(x,y,z)\, p + Q(x,y,z)\, q = R(x,y,z)}$$

Eg. $x^2 p + y^2 q - (x+y) z = 0$

Sol^n: The characteristic is given by :

take on RHS first

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

x^2 ($\hookrightarrow$ coeff. of p)

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow -\frac{1}{x} = -\frac{1}{y} + C$$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = c_1 = u(x,y) \quad \text{--- (1)}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{dx-dy}{x^2-y^2}$$

$$\frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z}$$

$$\Rightarrow \frac{dx-dy}{x-y} = \frac{dz}{z}$$

$$\Rightarrow \frac{d(x-y)}{x-y} = \frac{dz}{z}$$

$$\Rightarrow \log|x-y| = \log|z| + \log k$$

$$\Rightarrow (x-y) = kz = c_2 z$$

$$\Rightarrow \frac{x-y}{z} = c_2 = v(x,y) \quad \text{--- (2)}$$

Now, we have 2 sol^{n}s $u(x,y)$ & $v(x,y)$

$$F(u(x,y), v(x,y)) = F(c_1, c_2)$$

$$= \boxed{F\left(\frac{1}{x} - \frac{1}{y}, \frac{x-y}{z}\right) = 0}$$

This will give general sol^n for given PDE

Only need is it should be ~~soth~~ diff.

To get exact sol^n, we must eliminate F using Cauchy value Problem.

$$F(\quad) \equiv \cancel{\phi}\left(\frac{1}{x}, \frac{-1}{y}\right) = G\left(\frac{x-y}{z}\right)$$

$$\text{or} \quad \frac{x-y}{z} = H\left(\frac{1}{x}, \frac{-1}{y}\right)$$

Ex $xp + yq = z$ containing the curve

$C : x_0 = s^2, \; y_0 = s+1, \; z_0 = 1$

Sol^n Characteristic eq^n :

$$\frac{dx}{x} + \frac{dy}{y} = \frac{dz}{z}$$

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$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log|y| = \log|z| + \log c_1$

$\Rightarrow \frac{y}{z} = c_1 = u(x,y,z)$

$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{y}{x} \cancel{\frac{g}{y}} = c_2 = v(x,y,z)$

Two independent sol^n for this characteristic eq^n.

General sol^n $F(u,v) = 0$

$\Rightarrow F\left(\frac{y}{z}, \frac{y}{x}\right) = 0$

Option 1

$\frac{y}{z} = c_1 \Rightarrow \frac{s+1}{s} = c_1$

$\frac{y}{x} = c_2 \Rightarrow \frac{s+1}{s^2} = c_2$

~~Relation b/w~~ c_1 & c_2 (find)

$\Rightarrow$ we get : $(c_1 - 1)c_1 = c_2$

$\Rightarrow \left(\frac{y}{z} - 1\right)\frac{y}{z} = \frac{y}{x} \Rightarrow \frac{(y-z)y}{z^2} = \frac{y}{x}$

$\Rightarrow \boxed{(y-z)x = z^2}$

Option 2.

$\frac{y}{x} = G\left(\frac{y}{z}\right) \rightarrow$ we have to determine G

$\Rightarrow \frac{s+1}{s^2} = G\left(\frac{s+1}{s}\right)$ $G(t)$: in terms of t

$\frac{s+1}{s} = t \Rightarrow 1 + \frac{1}{s} = t \Rightarrow \frac{1}{s} = t-1$

$\Rightarrow s = \frac{1}{t-1}$

$\Rightarrow \frac{\frac{1}{t-1} + 1}{\frac{1}{(t-1)^2}} = G(t)$

$\Rightarrow t(t-1) = G(t)$ put $t = y/z$

$\Rightarrow G(y/z) = (y/z)(y/z - 1) = \frac{y}{x}$

$\Rightarrow \boxed{(y-z)x = z^2}$

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→ $u(x, y, z) = c_1$ } → characteristic curve
$v(x, y, z) = c_2$

Independent solⁿ :

$\nabla u \times \nabla v \neq 0$

The general solⁿ : $F(u, v) = 0$

$(p, q, -1)$ is normal ?? :.

$F(x,y,z)$: $z = z(x, y)$ — Integral surface (Explicit form)

$F(x, y, z) = z(x, y) - z = 0$ — Implicit solⁿ

$\nabla F = (z_x, z_y, -1) = (p, q, -1)$: normal derivative to S at (x, y, z)

⇒ $P(x,y,z)\, p + Q(x,y,z)\, q = R(x, y, z)$

is equivalent to

$(P, Q, R) \cdot \nabla F = 0$

$(P, Q, R) \parallel (\dot{x}, \dot{y}, \dot{z})$

Eg. $yzp + xzq = xy$

Sol: Characteristic eqⁿ :

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

$\frac{dx}{yz} = \frac{dy}{xz}$ ⇒ $x\,dx - y\,dy = 0$ ⇒ $x^2 - y^2 = c_1 = u(x, y, z)$

$\frac{dy}{xz} = \frac{dz}{xy}$ ⇒ $y\,dy - z\,dz = 0$

⇒ $y^2 - z^2 = c_2 = v(x, y, z)$

$F(x^2 - y^2, y^2 - z^2) = 0$ → Implicit form

⇒ $x^2 - y^2 = G(y^2 - z^2)$ or $y^2 - z^2 = H(x^2 - y^2)$ → Explicit form

→ Not every implicit form can be converted into explicit form

Eg. $(z^2 - 2yz - y^2)p + x(y+z)q = x(y-z)$

Sol^n: Characteristic eq^n:

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} \quad \text{---} (1)$$

$$\frac{dy}{y+z} = \frac{dz}{y-z} \Rightarrow y\,dy - z\,dy = y\,dz + z\,dz$$

$$\Rightarrow \cancel{\frac{y^2}{2} - yz = yz + \frac{z^2}{2} + C}$$

$$\Rightarrow y\,dy - z\,dz - \underbrace{z\,dy - y\,dz}_{d(yz)}$$

$$\Rightarrow \frac{y^2}{2} - \frac{z^2}{2} - yz = C_1$$

or $y^2 - z^2 - 2yz = C_1 = u(x,y,z)$

Modifying (1): (Componendo and Dividendo)

$$= \frac{x\,dx + y\,dy + z\,dz}{xz^2 - 2xyz - xy^2 + xy^2 + xyz + xzy - xz^2} = \frac{}{0} = k$$

0 also makes sense here. {we are not dividing here, it's like some notation.}

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow x^2 + y^2 + z^2 = C_2 = v(x,y,z)$

$$F(y^2 - z^2 - 2yz,\ x^2+y^2+z^2) = 0$$

Eg. $2p + 3q + 8z = 0$

Find the integral curve containing the following curves:

i) σ: $z = 1 - 3x$ on line $y = 0$

ii) σ: $z = x^2$ on line $2y = 1 + 3x$

iii) σ: $z = e^{-4x}$ on line $2y = 3x$

1st, find general sol^n, for this, find characteristic eq^n:

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{-8z}$$

$2dy - 3dx = 0$

$\Rightarrow 2y - 3x = C_1 = u(x,y,z)$

$\frac{dx}{2} = \frac{dz}{-8z} \Rightarrow -4x = \ln z + \ln c_2$

$\Rightarrow e^{-4x} = z c_2$

$\Rightarrow \frac{e^{-4x}}{z} = c_2 = v(x,y,z)$

or $z e^{4x} = c_2 = v(x,y,z)$

general sol^n :

$F(2y-3x, z e^{4x}) = 0$

or $z e^{4x} = G(2y-3x)$ — ①

i) σ : $z = 1-3x$ on $y=0$

$(1-3x) e^{4x} = G(-3x)$

put $-3x = s \Rightarrow x = -s/3$

$\Rightarrow$ ~~$(1+s) e$~~

$G(s) = (1+s) e^{-4s/3}$

From ①

~~$(1+3x) e^{4x} \Rightarrow z e^{4x} = G(1-3x)$~~

From ①,

$z e^{4x} = (1+2y-3x) e^{-4(2y-3x)/3}$

$+4x$

$\Rightarrow z = (1+2y-3x) e^{-8/3 y}$

ii) σ : $z = x^2$ ~~on~~ line $2y = 1+3x$

$(x^2) e^{4x} = G(1+3x-3x) = G(1)$

or $x^2 = G(1) e^{-4x}$. $\Rightarrow$ no possible value of x

polynomial, const, exponential

$\Rightarrow$ NO SOL^n

iii) $z = e^{-4x}$ on $2y = 3x$

$e^{-4x} \cdot e^{4x} = G(0)$

$\Rightarrow G(0) = 1 \Rightarrow$ func^n takes value 1 at $x=0$

$G(t)$:

$\Rightarrow$ can be: e^t, $\cos t$, $1+t$, $1+t^5$ ...

$\Downarrow$

$\boxed{\text{Infinite many sol}^{n}\text{s}}$

Cauchy problem may have:
- unique solⁿ
- no solⁿ
- infinitely many solⁿ

$(2xy-1)p + (z-2x^2)q = 2(x-yz)$

Q. Find Integral surface containing the curve

γ : $x_0(s) = 1$, $y_0(s) = 0$, $z_0(s) = s$

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}$$

try to make denominator = 0 & exact diff eqⁿ in numerator

① $\Rightarrow \dfrac{z\,dx + dy + x\,dz}{\cancel{2xyz} - \cancel{z} + \cancel{z} - \cancel{2x^2} + \cancel{2x^2} - \cancel{2xyz}} = k$

$\Rightarrow z\,dx + x\,dz + dy = 0$

$\Rightarrow xz + y = c_1 = u(x,y,z)$

② $\dfrac{2x\,dx + 2y\,dy + dz}{\cancel{4x^2y} - \cancel{2x} + \cancel{2yz} - \cancel{4x^2y} + \cancel{2x} - \cancel{2yz}}$

$\Rightarrow x^2 + y^2 + z = c_2 = v(x,y,z)$

~~For~~ verify c_1 & c_2 are independent solⁿs.

$F(u,v) = 0$

$\Rightarrow xz + y = G(x^2+y^2+z)$

or $x^2+y^2+z = H(xz+y)$

} both may not help to get exact ⇒ need to take care while choosing.

$xz + y = G(x^2+y^2+z)$

$s = G(1+s)$

$1+s = t$

$G(t) = t-1$

$\Rightarrow xz + y = (x^2+y^2+z) - 1$

$1 + s = H(s)$

$\Rightarrow x^2+y^2+z = 1 + (xz+y)$

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Ex. $x^3 p + y(3x^2+y)q = z(2x^2+y)$

$$\frac{dx}{x^3} + \frac{dy}{y(3x^2+y)} = \frac{dz}{2zx^2+zy}$$

① $$\frac{-x^{-1}dx + y^{-1}dy - z^{-1}dz}{-x^2+3x^2+y-2x^2-y} = k$$

$$\Rightarrow -\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0 \longrightarrow ①$$

② $$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} \Rightarrow \left(\frac{3x^2+y}{x^3}\right)dx = \frac{dy}{y}$$

$$\Rightarrow \frac{(3x^2+y)\,dx + dy}{x^3+y} = \frac{(3x^2+y)\,dx + dy + x\,dy}{x^3+y+xy} = \frac{dy}{y}$$

Exact

$$\Rightarrow \frac{d(x^3+y+xy)}{x^3+y+xy} = \frac{dy}{y}$$

$$\Rightarrow \frac{x^3+y+xy}{y} = c_2$$

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Second Order PDE

The std. second ~~order~~ order PDE in 2 independent variable

$f(x, y, z, z_{xx}, z_{yy}, z_{xy}, z_x, z_y) = 0$
(Semi-linear PDE)
⇓

$u \rightarrow$ dependent variable
$u = u(x,y)$

Can be represented as :

① —— $$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

A, B, C, D, E, F, G : func^n of independent variables x & y
(D, E, F, G may also be func^n of u)

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same eq^n may be classified in diff. form depending on domain

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$A^2+B^2+C^2 \neq 0$, A, B, C are continuous as possess continuous partial derivative of as high order as necessary

Def: A $func^n$ $u(x,y)$ is said to be regular sol^n of

② —— $Au_{xx} + Bu_{xy} + Cu_{yy} + g(x,y,u,u_x,u_y) = 0$

in $D \subset R \times R$ & $u \in C^2(D)$ (upto 2^{nd} order derivative are continuous)

and the $func^n$ u & it's derivatives satisfies ② for all $x,y \in D$

Genesis of 2^{nd} Order PDE

$f \in C^2(D)$ & $u = f(x+at)$

$u_x = f'(x+at)$ $\quad u_t = af'(x+at)$

$u_{xx} = f''(x+at)$ $\quad u_{tt} = a^2 f''(x+at)$

$\Rightarrow$ $\boxed{u_{tt} = a^2 u_{xx}}$: leads to 2^{nd} Order PDE

Classification of 2^{nd} Order PDE (Parabola, Ellipse, Hyperbola)

③ —— $\underbrace{ax^2 + bxy + cy^2}_{\text{principal part}} + dx + ey + f = 0$

Principal part (~~d~~ classifications depends on these variables only)

Normal form

→ $b^2 - 4ac > 0$: Hyperbola $\left(x^2/a^2 - y^2/b^2 = 1\right)$

→ $b^2 - 4ac = 0$: Parabola $(x^2 = y)$

→ $b^2 - 4ac < 0$: Ellipse $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

For PDE :

Principal part of ② :

$Lu = A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy}$

i) $B^2(x,y) - 4A(x,y)C(x,y) > 0 \Rightarrow$ ~~at~~ (x,y) — hyperbolic PDE

ii) $= 0 \Rightarrow$ — parabolic PDE

iii) — elliptic PDE

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eg. $u_{xx} - x^2 u_{yy} = 0$

$A(x,y) = 1$ $\quad$ $C(x,y) = -x^2$

$B(x,y) = 0$

$B^2 - 4AC = 0 - 4(1)(-x^2) = 4x^2 \geq 0$

$\Rightarrow u_{xx} - x^2 u_{yy} = 0$ → hyperbolic $x \neq 0$; → Parabolic $x = 0$

eg. $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \underline{\frac{y^2}{x} u_x + \frac{x^2}{y} u_y}$

$A = y^2$ $\quad$ $B = -2xy$ $\quad$ $C = x^2$

not included in classification but should be defined (x≠

$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0 \Rightarrow$ Parabolic PDE

(But we need to take care of u_x & u_y, & they should be defined, so $x \neq 0$)

eg. $u_{xx} + x^2 u_{yy} = 0$

$B^2 - 4AC = 0 - 4(1)(x^2) = -4x^2$

$\Rightarrow$ ellipse for $x \neq 0$

parabolic for $x = 0$

eg. $u_{xx} + x u_{yy} = 0$

$\Rightarrow -4x$

$$\begin{cases} \text{Parabolic} : x = 0 \\ \text{Elliptic} : x > 0 \\ \text{hyperbolic} : x < 0 \end{cases}$$

Here, we will study 2nd order Semi-Linear PDE with 2 independent variables

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Cannonical (Normal) form of 2nd Order PDE :

good for hyperbolic & parabolic (may not help in case of elliptic)

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0$$

$A^2 + B^2 + C^2 \neq 0$ (all A, B, C can't be 0 at same time) (Otherwise, it won't be 2nd order)

Our aim : $(x, y) \rightsquigarrow (\xi, \eta)$, $\xi = \xi(x,y)$

$u(x, y) \rightsquigarrow u(\xi, \eta)$ $\eta = \eta(x,y)$

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \quad \text{(Assume)}$$

Thus, transformation is invertible

$u_\xi = u_\xi(\xi, \eta)$

$u_x = u_\xi \xi_x + u_\eta \eta_x$ (Chain Rule)

$\cancel{u_{xx} = u_\xi (\xi_x)^2 + u_\xi \xi_{xx} + u_\eta (\eta_x)^2}$

$$u_{xx} = (u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x) \xi_x + u_\xi \xi_{xx}$$

$$+ (u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x) \eta_x + u_\eta \eta_{xx}$$

$$u_{xx} = u_{\xi\xi} \xi_x^2 + 2 u_{\xi\eta} \eta_x \xi_x + u_\xi \xi_{xx} + u_{\eta\eta} \eta_x^2 + u_\eta \eta_{xx}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_{xy} = (u_{\xi\xi} \xi_y + u_{\xi\eta} \eta_y) \xi_x + u_\xi \xi_{xy}$$

$$+ (u_{\eta\xi} \xi_y + u_{\eta\eta} \eta_y) \eta_x + u_\eta \eta_{xy}$$

$$= u_{\xi\xi} \xi_y \xi_x + u_\xi \xi_{xy} + u_{\xi\eta} \eta_y \xi_x + u_{\eta\xi} \xi_y \eta_x$$

$$+ u_{\eta\eta} \eta_y \eta_x + u_\eta \eta_{xy}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{yy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_y + u_\xi \xi_{yy} + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_y + u_\eta \eta_{yy}$$

$$= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\xi}\xi_{yy} + u_{\eta\eta}\eta_y^2 + u_\eta \eta_{yy}$$

Principal part:

$$A u_{xx} + B u_{xy} + C u_{yy} = u_{\xi\xi}(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2) + u_{\xi\eta}\left(2A\xi_x\xi_y + B(\xi_x\eta_y + \eta_x\xi_y) + 2\xi_x\eta_y\right) + u_{\eta\eta}(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2) + h(\xi, \eta, u, u_\xi, \dots)$$

We choose ξ and η such that Jacobian $\neq 0$.

$\Rightarrow$ Eqn (1) becomes:

(2) — $\bar{A}(\xi_x; \xi_y)u_{\xi\xi} + 2\bar{B}(\xi_x, \xi_y; \eta_x, \eta_y)u_{\xi\eta} + \bar{A}(\eta_x; \eta_y)u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$ ... (2)

coeff of $u_{\xi\xi}$ (func^n of something)

$$\bar{A}(u; v) = Au^2 + Buv + Cv^2$$

$$\bar{B}(u_1, v_1; u_2, v_2) = Au_1u_2 + \frac{1}{2}B(u_1v_2 + u_2v_1) + Cv_1v_2$$

Eg. $\bar{B}^2(\xi_x, \xi_y; \eta_x, \eta_y) - 4\bar{A}(\xi_x; \xi_y)\bar{A}(\eta_x; \eta_y) = 0$

$$= \underbrace{(B^2 - 4AC)}_{>0 \text{ (hyperbolic)}}\underbrace{(\xi_x\eta_y - \xi_y\eta_x)^2}_{\text{Jacobian}}$$

Jacobian $\Downarrow$ >0 $(\neq 0)$

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Hyperbolic PDE : $B^2 - 4AC > 0$

consider the quadratic eqn

$$A\alpha^2 + B\alpha + C = 0$$

We've 2 real and distinct roots : $\lambda_1(x,y)$, $\lambda_2(x,y)$ (say)

We choose $\xi(x,y)$ & $\eta(x,y)$ s.t.

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y} \quad \text{— (3)}$$

$\Rightarrow$ $\xi_x = \lambda_1 \xi_y$ and $\eta_x = \lambda_2 \eta_y$

$$\bar{A}(\xi_x ; \xi_y) = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$
$$= A\lambda_1^2 \xi_y^2 + B\lambda_1 \xi_y^2 + C\xi_y^2$$
$$= (A\lambda_1^2 + B\lambda_1 + C)\xi_y^2 \quad (\lambda_1 : \text{root of eq}^n)$$
$$= 0$$

Similarly, $\bar{A}(\eta_x, \eta_y) = 0$

~~$2\bar{B}(\xi_1, \ldots, u$~~

$$2\bar{B}(\xi_x, \xi_y ; \eta_x, \eta_y)\, u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$$

since $\bar{B} > 0$

$$\boxed{u_{\xi\eta} = Q(\xi, \eta, u, u_\xi, \eta)}$$, Canonical form for Hyperbolic case.

Eg. $u_{\xi\eta} = k$

$\Rightarrow$ $u_\xi = f(\xi)$

$u = \int f(\xi)\, d\xi + g(\eta)$

$= F(\xi) + g(\eta)$

ξ & η are sol^{n}s of eq^{n}s (3)

→ $\xi_x - \lambda_1 \xi_y = 0$

$\partial \xi = 0$

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{d\xi}{0} \quad \rightarrow \quad \xi = C_2$$

$$\frac{dy}{dx} + \lambda_1 = 0$$

Assume $f_1(x,y) = c_1$ satisfies above eq^n

General sol^n

$$F(c_1, c_2) = 0 \Rightarrow$$

$$F(f_1(x,y), \xi) = 0$$

$$\Rightarrow \quad \xi = G_1(f_1(x,y))$$

In particular, the simplest one is

$$\boxed{\xi = f_1(x,y)}$$

$$\eta_x - \lambda_2 \eta_y = 0$$

The sol^n of this PDE is :

$$\boxed{\eta = f_2(x,y)}$$

where $f_2(x,y)$ is sol^n of $\frac{dy}{dx} + \lambda_2 = 0$

Now, η & ξ are known, so we can find the canonical form.

Ex. $u_{xx} = x^2 u_{yy}$ (Find canonical form)

$A = 1 \quad B = 0 \quad C = -x^2$

$B^2 - 4AC = 0 + 4x^2 > 0 \quad \forall \; x \neq 0$

$\Rightarrow$ Hyperbolic Type

Consider eq^n :

$$1\alpha^2 + 0 + (-x^2) = 0$$

$$\alpha^2 - x^2 = 0$$

$\Rightarrow$ $\alpha = \pm 1$ $\qquad \lambda_1(x,y) = +x$

$\Rightarrow$ $\lambda_1, \lambda_2 = \pm 1$ $\qquad \lambda_2(x,y) = -x$

Choose ξ & η s.t.

$\xi_x = \lambda_1 \xi_y$ & $\eta_x = \lambda_2 \eta_y$

$\Rightarrow \xi_x - x\,\xi_y = 0$

$\Rightarrow \dfrac{dx}{1} = \dfrac{dy}{-x} = \dfrac{d\xi}{0}$ $\qquad$ λ_2

$\Rightarrow \dfrac{dy}{dx} + x = 0$ & $\dfrac{dy}{dx} - x = 0$

$\Rightarrow \underbrace{y + \dfrac{x^2}{2}}_{f_1} = c_1$ & $\underbrace{y - \dfrac{x^2}{2}}_{f_2} = c_2$

$\xi(x,y) = y + \dfrac{x^2}{2}$ $\qquad \eta(x,y) = y - \dfrac{x^2}{2}$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$= u_\xi (x) + u_\eta (-x) = x(u_\xi - u_\eta)$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\eta_x\xi_x + u_\xi \xi_{xx} + u_{\eta\eta}\eta_x^2 + u_\eta \eta_{xx}$$

$$= u_{\xi\xi}(x^2) + 2u_{\xi\eta}(-x^2) + u_\xi(1) + u_{\eta\eta}(x^2) + u_\eta(-1)$$

$$u_{xx} = x^2\left[u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}\right] + \left[u_\xi - u_\eta\right]$$

Similarly,

$$\boxed{u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}}$$

Put in eq^n ①

A $u_{xx} = x^2 u_{yy}$

$\Rightarrow x^2 [u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}] + [u_\xi - u_\eta] = x^2 [u_{\xi\xi} + 2u_{\xi\eta} + \ldots]$

Checkpoint : coeff. of $u_{\xi\xi}$ & $u_{\eta\eta}$ should be 0 in case of hyperbolic PDE.

$\Rightarrow 4u_{\xi\eta} x^2 = u_\xi - u_\eta$

$\Rightarrow u_{\xi\eta} = \dfrac{u_\xi - u_\eta}{4x^2} = \boxed{\dfrac{u_\xi - u_\eta}{4(\xi - \eta)}}$ canonical form.

$= Q(\xi, \eta, u, u_\xi, u_\eta)$

Parabolic PDE : $B^2 - 4AC = 0$

$A\alpha^2 + B\alpha + C = 0$: Repeated Real roots $= \lambda(x,y)$

Choose $\xi(x,y)$ s.t.

$\dfrac{\partial \xi}{\partial x} = \lambda \dfrac{\partial \xi}{\partial y}$ [makes $\bar{A} = 0$]

This choice of ξ makes the coeff. of $u_{\xi\xi}$ as 0.

$\bar{A}(\xi_x ; \xi_y) = \xi_y^2 (A\lambda^2 + B\lambda + C) = 0$

Choose $\eta(x,y)$ s.t. (ξ & η should be independent func^n's)

$\dfrac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$ OR $\nabla\xi \times \nabla\eta \neq 0$

here $\bar{A}(\eta_x, \eta_y)$ may not be equal to 0

From eq^n ⊛,

$(B^2 - 4AC) = 0$

$B^2(\quad) - 4\bar{A}(\quad) \overset{0}{\bar{C}} \ldots = 0 \Rightarrow B^2 = 0$

$\Rightarrow \bar{B} = 0$

using ②, the canonical form is reduced to

$\underset{\to \text{non-0}}{\bar{A}(\eta_x, \eta_y)} u_{\eta\eta} = G(\xi, \eta, \ldots)$

$\Rightarrow \boxed{u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)}$

Ques. Find canonical form of

$u_{xx} + 2u_{xy} + u_{yy} = 0$

$B^2 - 4AC = 4 - 4 = 0 \Rightarrow$ parabolic

$\alpha^2 + 2\alpha + 1 = 0$

$\Rightarrow (\alpha + 1)^2 = 0$

$\Rightarrow \alpha = -1$

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