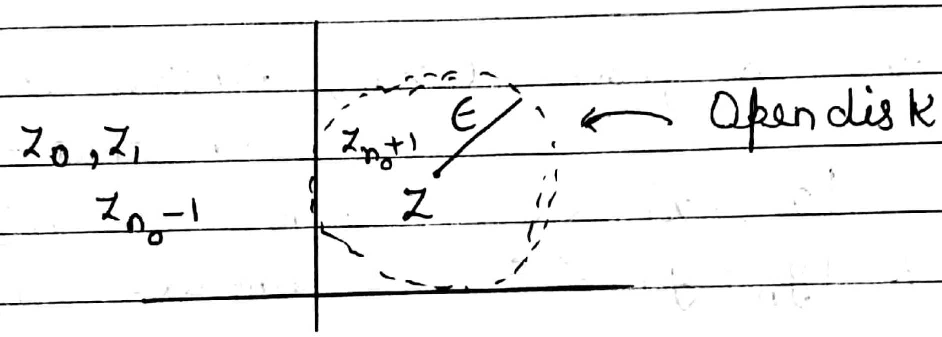
Sequence & Series in Complex Numbers

Convergence of a sequence:- A sequence $\{z_n\}$ of complex number is said to be convergent to z, iff for every $\epsilon > 0$, $\exists\ n \in N$ such that-

$$|z_n - z| < \epsilon \qquad \text{whenever } n \geq n_0$$

Or, $\lim\limits_{n \to \infty} z_n = z$



e.g. (i) $\left\{\dfrac{i^n}{n}\right\}$; $n \in N$

$$= \left\{i, -\frac{1}{2}, \frac{-i}{3}, \frac{1}{4}, \ldots \right\}$$

As $n \to \infty$, $\left\{\dfrac{i^n}{n}\right\} \to 0$ (Converging)

Alternatively,

$$\left|\frac{i^n}{n} - 0\right| < \epsilon \quad , n \geq 0$$

(ii) $\left\{\dfrac{1}{n^2} + i\right\}$

$\left\{\dfrac{1}{n^2} + i\right\} \to i$ for all $n \geq \dfrac{1}{\sqrt{\epsilon}}$

$$\left|\frac{1}{n^2} + i - i\right| < \epsilon \Rightarrow n^2 > \frac{1}{\epsilon} \Rightarrow n \geq \frac{1}{\sqrt{\epsilon}}$$

(iii) $\{i^n\}$ $\rightarrow$ divergent

★ **Theorem :-** Suppose that $z_n = x_n + i y_n$ be a sequence & $z = x + iy$ then-

$$\lim_{n\to\infty} z_n = z \quad \text{iff} \quad \text{— ①}$$

(i) $\lim_{n\to\infty} x_n = x$ & (ii) $\lim_{n\to\infty} y_n = y$ — ②

Proof :- Suppose eq^n ② holds.

For a given $\epsilon > 0$, there exist $n_1, n_2 \in N$ such that -

$|x_n - x| < \epsilon$ whenever $n \geq n_1$

& $|y_n - y| < \epsilon$ whenever $n \geq n_2$

Let $n_0 \geq \max\{n_1, n_2\}$

then-

$|x_n - x| < \epsilon$ whenever $n \geq n_0$

$|y_n - y| < \epsilon$ whenever $n \geq n_0$

Now,

$$|z_n - z| = |(x_n + iy_n) - (x + iy)| = |x_n - x + i(y_n - y)|$$

$$\leq |x_n - x| + |y_n - y| \quad \text{[Triangular inequality]}$$

$$\leq \epsilon + \epsilon$$

$$\leq 2\epsilon \quad \text{whenever } n \geq n_0$$

Hence, $z_n \rightarrow z$.

Conversely,
Assume eqⁿ ② holds.

i.e. for a given $\epsilon > 0$, $\exists$ a $n_0 \in N$ such that -

$$|z_n - z| < \epsilon \quad \text{whenever } n \geq n_0$$

Now,

$$|x_n - x| \leq |x_n - x + i(y_n - y)|$$
$$\leq |(x_n + iy_n) - (x + iy)|$$
$$\leq |z_n - z| < \epsilon \quad \text{whenever } n \geq n_0$$

$\therefore \quad x_n \to x$

Similarly,
$$y_n \to y$$

e.g. $z_n = -2 + i\dfrac{(-1)^n}{n^2}$

Here $x_n = -2$ & $y_n = \dfrac{(-1)^n}{n^2}$

$\downarrow$ bounded sequence

$x_n \to -2$

$y_n \to 0$

$\therefore z_n \to -2 + 0$
$\to -2$

* Convergence of an infinite series :-

$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \dots\dots$ ~~cover~~ converges

iff sequence of partial sum i.e.

$S_n = \sum_{n=1}^{n} z_n = z_1 + z_2 + \dots z_n$ converges.

$$S_1 = Z_1$$
$$S_2 = Z_1 + Z_2$$
$$\vdots$$
$$S_n = Z_1 + Z_2 + \cdots Z_n$$

$\{S_n\} \rightarrow$ Sequence of partial sum

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{n=1}^{n} Z_n = \sum_{n=1}^{\infty} Z_n$$

Remainder term :-

$$\mathcal{S}_n = Z_{n+1} + Z_{n+2} + \cdots$$

$$|S_n - S| = |Z_{n+1} + Z_{n+2} + \cdots|$$
$$= |\mathcal{S}_n|$$

Note :- An infinite series $\sum_{n=1}^{\infty} Z_n$ converges iff sequence of remainder i.e. $\{\mathcal{S}_n\} \rightarrow 0$ as $n \rightarrow \infty$.

e.g. $\sum_{n=1}^{\infty} Z^n = \frac{1}{1-Z}$; $|Z| < 1$

$$1 + Z + \cdots Z^n = \frac{1 - Z^{n+1}}{1-Z} \qquad Z \neq 1$$

Partial sum $\rightarrow$ $S_n = 1 + Z + \cdots Z^{n-1}$

$$= \frac{1 - Z^n}{1-Z} \qquad \& \ S = \frac{1}{1-Z}$$

Remainder $\mathcal{S}_n = S_n - S$
$$= \frac{1-Z^n}{1-Z} - \frac{1}{1-Z} = \frac{Z^n}{1-Z} \rightarrow 0 \text{ when } |Z| < 1$$

For $|z| > 1$, $\sum_{n=1}^{\infty} z^n$ diverges.

Theorem:- Let $z_n = x_n + i y_n$

& $S = x + iy$, then -

$\sum_{n=1}^{\infty} z_n = S$ iff

(i) $\sum_{n=1}^{\infty} x_n = x$ & (ii) $\sum_{n=1}^{\infty} y_n = y$

Theorem:- Let $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$

Proof:- Let $S_n = \sum_{j=1}^{n} z_j$ & $S_{n-1} = \sum_{j=1}^{n-1} z_j$

And, we know,

$$z_n = S_n - S_{n-1}$$

[Consider $\lim_{n\to\infty} \sum_{j=1}^{n} z_j = Z$]

$$\Rightarrow \lim_{n\to\infty} z_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} z_j - \lim_{n\to\infty} \sum_{j=1}^{n-1} z_j$$

$$= Z - Z = 0$$ — H.P.