Function

Single-Valued

Multi-Valued
(Collection of single-valued functions)

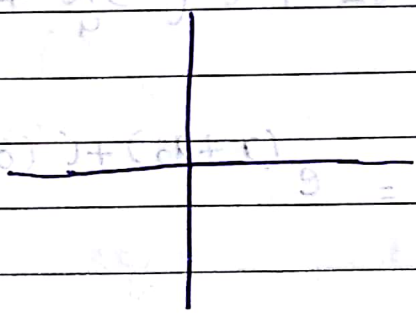
each single-valued function is called branch.

e.g. $f(z) = \sqrt{r}\, e^{\frac{i\theta + 2k\pi i}{2}}$

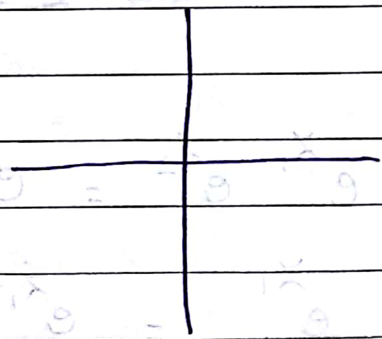
$$= \begin{cases} \sqrt{r}\, e^{i\theta/2} & ; k = 0 \\ \sqrt{r}\, e^{i\theta/2 + \pi i} & ; k = 1 \end{cases} \Bigg] \rightarrow \text{Branches}$$

Note:- To plot $f(z) = z+1$, input space & output space both contain 2 points each, so that has to be plot in 4-Dimensional space.

To plot this we use "Mapping Method."



z
Input space



$f(z)$
Output space

(a) Logarathomic function:-

We know,

$$-1 = e^{\pi i + 2k\pi i}$$

$$\therefore \log_e(-1) = \log_e(\cancel{\pi i} + e^{\pi i + 2k\pi i})$$

$$= \pi i + 2k\pi i$$

(b) Exponential function:-

$$f(z) = e^z = r e^{i\theta}$$

$$e^z = e^{\ln r} e^{i\theta}$$

$$e^z = e^{\ln r + i\theta}$$

$$\therefore z = \ln r + i\theta$$

e.g. $e^z = -1-i$. Find z

Ans

$$e^z = \sqrt{2}\, e^{-\frac{3\pi}{4} i}$$

$$e^x e^{iy} = e^{\ln\sqrt{2}}\, e^{-\frac{3\pi}{4} i}$$

$$\Rightarrow z = \ln\sqrt{2} + i\left(-\frac{3\pi}{4} + 2k\pi\right)$$

(i) $e^{z_1} e^{z_2} = e^{z_1+z_2} = e^{(x_1+x_2)+i(y_1+y_2)}$

(ii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

$f(z) = \ln z$

$$e^{\omega} = z \Leftrightarrow \omega = \log z$$

To calculate z, we need to know 'ω', which we know -

$$\omega = \ln r + i(\theta + 2k\pi)$$

$\log z \to$ ~~Principal~~ General

$\mathrm{Log}\, z \to$ Principal

K = 0

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Q Calculate $\log(-1-\sqrt{3}i)$

Ans Let $e^{\omega} = -1-\sqrt{3}i = 2e^{i\left(-\frac{2\pi}{3}+2K\pi\right)}$

$\Rightarrow \omega = \log_e\left(2e^{i\left(-\frac{2\pi}{3}+2K\pi\right)}\right) = \log(-1-\sqrt{3}i)$

$= \ln 2 + i\left(-\frac{2\pi}{3}+2K\pi\right)$

Principal value of $[\log(-1-\sqrt{3}i)]$

$\mathrm{Log}(-1-\sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3}\right)$ $= \ln 2 + i\left(\frac{4\pi}{3}\right)$ X

~~↳ Arg z is less than π which is not possible.~~

(c) Complex Exponent:-

When $z \neq 0$ & the exponent 'c' is any complex number the function z^c is defined by means of the eqn:-

$$\boxed{z^c = e^{c \log_e z}}$$

e.g. (i) $(i)^{-2i}$

$(i)^{-2i} = e^{\log_e (i)^{-2i}} = e^{-2i \log_e i}$ — (1)

Now, $\log i = \log|i| + i \arg(i)$

$= \log(1) + i\left(\frac{\pi}{2}+2K\pi\right)$

$= \left(\frac{\pi}{2}+2K\pi\right)i$

Into (1),

$(i)^{-2i} = e^{-2i\left(\frac{\pi}{2}+2K\pi\right)i}$

$= e^{(\pi+4K\pi)}$ $[K = 0, \pm 1 \ldots]$

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(ii) $(2)^{\frac{1}{\pi}}$

$$(2)^{\frac{1}{\pi}} = e^{\log_e 2^{\frac{1}{\pi}}} = e^{\frac{1}{\pi}\ln 2}$$

$$= e^{\frac{1}{\pi}\{\ln 2 + i\,2k\pi\}}$$

$$= e^{\frac{1}{\pi}\ln 2}\, e^{2ki}$$

Q What is principal value of $(-1)^{\frac{1}{\pi}}$ & $(-1-i)^{\frac{1}{\pi}}$

Ans

$$(-1)^{\frac{1}{\pi}} = e^{\frac{1}{\pi}\log_e(-1)}$$

$$= e^{\frac{1}{\pi}[\pi i]}$$

$$= e^{i} = \cos 1 + i \sin 1$$

Note:- As we know,

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$ but $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \neq \operatorname{Arg}(z_1 z_2)$

Similarly,

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$ but $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$

(ii) $(z_1 z_2)^c = z_1^c z_2^c$ but $P.V.(z_1 z_2)^c \neq z_1^c z_2^c$

P.V. → Principal Value

But the above properties hold true, when real part of z_1 & $z_2 > 0$.

Q $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = -1-i$

Check (i) $(z_1 z_2)^i = z_1^i z_2^i$

(ii) $(z_2 z_3)^i = z_2^i z_3^i$

Ans

(i) $z_1 z_2 = (1+i)(1-i) = 2$

$(z_1 \cdot z_2)^i = 2^i$

Q $e^z = 0$

$= e^{\log 0} = e^{(\ln|0| + i \operatorname{Arg} 0)}$

[Since, Arg '0' is not defined

$\therefore$ $e^z = 0$ does not have any solution.

$z_1^{\,i} = (1+i)^i = e^{i \log_e (1+i)}$ $= \cancel{\frac{1}{2}\ln 2}$

$= e^{i\left[\frac{1}{2}\ln 2 + i\frac{\pi}{4}\right]}$

$= (\sqrt{2})^i e^{-\pi/4}$

$z_2^{\,i} = (1-i)^i = e^{i \log_e (1-i)}$

$= e^{i\left[\frac{1}{2}\ln 2 + i(-\pi/4)\right]}$

$= (\sqrt{2})^i e^{\pi/4}$

$\therefore (z_1 z_2)^i = z_1^{\,i} z_2^{\,i}$

(ii) $z_2 z_3 = (1-i)(-1-i)$

$= -2$

P.V. $(z_2 z_3)^i = (-2)^i = e^{i \log(-2)} = e^{i[\ln 2 + \pi i]}$

$= e^{i \ln 2} e^{-\pi}$

$(z_2)^i = (\sqrt{2})^i e^{\pi/4} = e^{i/2 \ln 2}$

$(z_3)^i = (-1-i)^i = e^{i \log(-1-i)}$

$= e^{i\left(\frac{1}{2}\ln 2 - \frac{3\pi}{4} i\right)}$

$= e^{i/2 \ln 2} e^{\frac{3\pi}{4}}$

To check $(z_1 z_3)^i = z_2^{\,i} z_3^{\,i}$

$e^{i \ln 2} e^{-\pi} = \left(e^{\frac{i}{2}\ln 2} e^{\frac{\pi}{4}}\right)\left(e^{\frac{i}{2}\ln 2} e^{\frac{3\pi}{4}}\right)$

$e^{i \ln 2} e^{-\pi} \neq e^{i \ln 2} e^{\pi}$

(d) Trignometric function :-

By Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\therefore \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

So, $\cos(i\theta) = \dfrac{e^{i(i\theta)} + e^{-i(i\theta)}}{2} = \dfrac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta$

$\cosh\theta$ → Cosine hyperbolic function

$$\sin(i\theta) = \frac{e^{-\theta} - e^{\theta}}{2i} = i\left[\frac{e^{\theta} - e^{-\theta}}{2}\right]$$
$$= \sinh\theta$$

Note:- ~~Hyperb~~ Hyperbolic Sine & Cosine are both unbounded functions.

(i) $\sin z = \sin(x + iy)$

$\sin z$ → Unbounded

$$= \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + \cos x \sinh y \Rightarrow$$

(ii) $\cos z = \cos(x + iy)$

$\cos z$ → Unbounded

$$= \cos x \cosh y - i \sin x \sinh y$$

e.g. $\log(i)^2 = 2\log i = 2\left(\frac{\pi}{2} i\right) = \pi i$

Or, $\log(i)^2 = \log i^2 = \log(-1) = 0 + \pi i$

Functions — Single-Valued → limit → Continuity → differentiability → Analyticity

Functions — Multi-valued → only $K=0$ (principal) branch

Now,

$\log(i)^3 = 3\log i = 3\left(\frac{\pi}{2}i\right) = \frac{3\pi}{2}i$

Or, $\log(i^3) = \log(-i) = -\frac{\pi}{2}i$

} not same?

Q Find all the roots of the eq: $\log z = i\frac{\pi}{2}$

* Limit:-

Let a function 'f' be defined at all points z in some deleted neighbourhood of z_0. The limit of $f(z)$, as z approaches to z_0, is a number (ω_0) given as-

$$\lim_{z\to z_0} f(z) = \omega_0 \quad — (1)$$

In other words,

for each ~~point~~ positive number ϵ there is a positive number δ such that

$|f(z) - \omega_0| < \epsilon$, whenever $0 < |z - z_0| < \delta$

This definition says that for each ϵ-neighbourhood, there is δ-nbd $0<|z-z_0|<\delta$ of z_0 such that every point z in it has an image ω_0 lying in the ϵ-nbd.

[$f(z)$ plane has 'ω' as point in $|f(z)-\omega_0|<\epsilon$

i.e. soln. of $|f(z)-\omega_0|<\epsilon$ in general is ω

Q 1. $\lim_{z\to 0} \frac{z}{|z_0|}$

Ans $= \lim_{x+iy\to 0} \frac{x+iy}{\sqrt{x^2+y^2}}$

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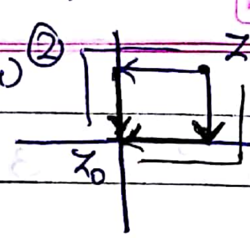
Iterative (repeated) limit :-



$$\lim_{x+iy \to 0} \frac{x+iy}{\sqrt{x^2+y^2}} =$$

First $x \to 0$, Later $y \to 0$ $\Rightarrow$ $\displaystyle\lim_{x \to 0,\, y \to 0} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{y \to 0} \frac{iy}{y} = i$

First $y \to 0$, Later $x \to 0$ $\Rightarrow$ $\displaystyle\lim_{y \to 0;\, x \to 0} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x \to 0} \frac{x}{x} = 1$

different paths, different limits $\therefore$ limit does not exist.

Q2. $\displaystyle\lim_{z \to 0} \frac{(\operatorname{Re} z - \operatorname{Im} z)^2}{|z|^2}$

Ans Let $y = mx$ be the path taken-

$$\lim_{x+iy} \frac{(x-y)^2}{x^2+y^2} = \lim_{x \to 0} \frac{(x-mx)^2}{x^2(1+m^2)} = \frac{(1-m^2)}{1+m^2}$$

For $m = 0$ $\lim \to 1$
$m = 1$ $\lim \to 0$

diff. paths, limit diff. So limit does not exist.

Note:- To prove non-existence of limit of a function we apply "Method of Contradiction":

(i) Iterative method
(ii) Path method

~~which to choose?~~ depends on nature of problem

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To prove existence of a limit :-

Q (i) $\lim_{z \to 1} \frac{i\bar{z}}{2} = \frac{i}{2}$ (Prove)

$f(z) = \frac{i\bar{z}}{2}$, $z_0 = 1$, $\omega_0 = \frac{i}{2}$

Let $0 < |z-1| < \delta$

$|f(z) - \omega_0| = \left| i\frac{\bar{z}}{2} - \frac{i}{2} \right| = \frac{1}{2}|\bar{z}-1| = \frac{\delta}{2}$ (let)

Q (ii) $\lim_{z \to i} z^2 = -1$

Ans $f(z) = z^2$, $\omega_0 = f(z_0) = -1$, $z_0 = i$

$0 < |z-i| < \delta$

$|f(z) - f(z_0)| = |z^2 - (-1)| = |(z-i)(z+i)|$
$= |(z-i)(z-i+2i)|$
$\leq |z-i| \{|z-i| + |2i|\} < \delta(\delta+2)$
$= \varepsilon$

(Triangular inequality)

Here, $\delta(\delta+2) = \varepsilon$

★ Continuity :- A function 'f' is continuous at a point z_0 if all three of the following conditions hold :

(i) $\lim_{z \to z_0} f(z)$ exist

(ii) $f(z_0)$ exist

(iii) $\lim_{z \to z_0} f(z) = \cancel{\lim} f(z_0)$

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In other words,

For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

Q $f(z) = \cancel{\log z}\ \frac{i\bar{z}}{2}$ at $z = i$ (Check for continuity)

Ans $\lim_{z \to i} \frac{i\bar{z}}{2} = \frac{1}{2}$

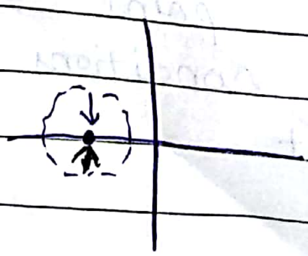
$$f(z) = \frac{1}{2}$$

Q (i) Find points of discontinuity for $f(z) = \log z$
(ii) Check whether $f(z)$ is continuous over negative real axis.

Ans $f(-1) = i\pi$

$$\lim_{z \to -1} f(z) = \lim_{z \to -1} [\ln|z| + i \operatorname{Arg} z]$$

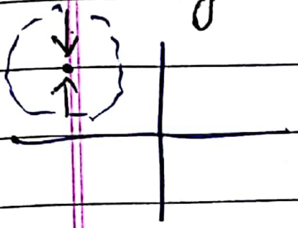
$$\lim_{z \to -1} f(z) = \lim_{\substack{x \to -1 \\ y \to 0}} = \begin{cases} +\pi ; & y > 0, x < 0 \\ -\pi ; & y < 0, x < 0 \end{cases}$$



different paths different limits

$\therefore$ limit does not exist

If ~~at~~ $z = x + iy$ where $x < 0$, $y \neq 0$



limit exists, because there would be at least a circle ~~of~~ of 'δ' radius possible for limit to exist.

★ **Differentiability:-** Let $f(z)$ be a single-valued function defined in a domain D, the function $f(z)$ is said to be differentiable at z_0 if-

$$\lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \text{ exist}$$

Let this limit be denoted by - $f'(z_0)$

~~At $z = z_0$~~ At $z - z_0 = \Delta z_0$

$$\Rightarrow \Delta z + z_0 = z$$

We get

$$f'(z_0) = \lim_{z \to 0} \left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right)$$

In other words,

For every $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that -

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad \text{whenever}$$

$$0 < |z - z_0| < \delta$$

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e.g. (i) $f(z) = z^2$

Sol:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z^2 + 2z\Delta z + (\Delta z)^2) - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

(ii) $f(z) = \bar{z}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \Delta\bar{z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta\bar{z}}{\Delta z}$$ → does this limit exist?

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z \to 0} \frac{x - iy}{x + iy}$$

For $y = mx$

$$= \lim_{x \to 0} \frac{x - xim}{x + imx}$$ → limit does not exist.

$$= \frac{1 - im}{1 + im}$$

Similarly,

$$\lim_{\Delta z \to 0} \frac{\Delta\bar{z}}{\Delta z} = \lim_{\Delta x + i\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Put $\Delta y = m\Delta x$

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x} = \frac{1-im}{1+im}$$

limit does not exist.

☆ Cauchy - Riemann Equation :-

Let $f(z) = U(x,y) + iV(x,y)$ is defined and is continuous at a point $z = x+iy$ & differentiable at z, then at point z, first order partial derivative exists and satisfies the equation

C-R eqⁿ $\longrightarrow$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \& \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

e.g. $f(z) = \bar{z} = x - iy$

$f(z) = \overline{U+iV} = U - iV$

$u = x \quad , \quad V = -y$

$$\frac{\partial U}{\partial x} = 1 \qquad \frac{\partial V}{\partial y} = -1$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial V}{\partial x} = 0$$

Since, C-R eqⁿ does not satify,
So, $f(z)$ is not differentiable.

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Q. $f(z) = |z|^2 = x^2+y^2$
Find all points of differentiability.

Ans $U = x^2+y^2$, $V=0$

$\frac{\partial U}{\partial x} = 2x$, $\frac{\partial V}{\partial y} = 0$ — ①

$\frac{\partial U}{\partial y} = 2y$, $\frac{\partial V}{\partial x} = 0$ — ②

① & ② → Only at $z=0$, $f(z)$ is differentiable

Proof of Cauchy-Riemann Equation:-

$z = x+iy$, $\Delta z = \Delta x + i\Delta y$
$z+\Delta z = (x+\Delta x) + i(\Delta y + y)$

$f(z) = U(x,y) + i\,V(x,y)$

$f(z+\Delta z) = U(x+\Delta x, y+\Delta y) + i\,V(x+\Delta x, y+\Delta y)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{U(x+\Delta x, y+\Delta y) - U(x,y)}{\Delta z} + i\left(\frac{V(x+\Delta x, y+\Delta y) - V(x,y)}{\Delta z}\right)$$

By Iterative limit Concept,

Take $\Delta y \to 0$ (first)

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$\therefore f'(z) = \lim\limits_{\Delta x \to 0} \dfrac{U(x+\Delta x, y) - U(x,y)}{\Delta x}$

$+ i \lim\limits_{\Delta x \to 0} \dfrac{V(x+\Delta x, y) - V(x,y)}{\Delta x}$

$= \dfrac{\partial U}{\partial x} + i \dfrac{\partial V}{\partial x}$ — ①

Now, take $\Delta x \to 0$

$f'(z) = \lim\limits_{\Delta y \to 0} \dfrac{u(x, y+\Delta y) - u(x,y)}{i \Delta y}$

$+ i \lim\limits_{\Delta y \to 0} \dfrac{(V(x, y+\Delta y) - V(x,y))}{i \Delta y}$

$= \dfrac{1}{i} \dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial y} = -i \dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial y}$ — ②

As 'f' is differentiable --

① = ②

$\dfrac{\partial U}{\partial x} + i \dfrac{\partial V}{\partial x} = -i \dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial y}$

So, $\dfrac{\partial U}{\partial x} = \dfrac{\partial V}{\partial y}$

& $\dfrac{\partial U}{\partial y} = -\dfrac{\partial V}{\partial x}$ $\Rightarrow$ H.P.

Q

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} ; & z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Check C-R eq^n at $z = 0$

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Q. $f(z) = e^z$

$e^z = e^x(\cos y + i\sin y)$

$= u + iv$

$\therefore U = e^x \cos y \qquad V = e^x \sin y$

$\frac{\partial U}{\partial x} = U_x = e^x \cos y \quad ; \quad \frac{\partial U}{\partial y} = U_y = e^x(-\sin y)$

$\frac{\partial V}{\partial x} = V_x = e^x \sin y \quad ; \quad \frac{\partial V}{\partial y} = V_y = e^x \cos y$

$\because U_x = V_y \ \& \ U_y = -V_x$

$f(z) = U(x,y) + iV(x,y)$

As per C-R eq^n:- $U_x = V_y \ \& \ U_y = -V_x$

Similarly,

In polar form,

$f(z) = U(r,\theta) + iV(r,\theta)$

then -

$U_r = \frac{1}{r} V_\theta \ \& \ V_r = -\frac{1}{r} U_\theta$

* Analytic at z_0:-

If $f(z)$ is differentiable at each point of some nbd of z_0 then we say $f(z)$ is analytic at z_0.

e.g. $f_1(z) = |z|^2$, $f_2(z) = \bar{z}$

$f_3(z) = z^2$

Check their differentiability at $z_0 = 0$

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$f_1(z) = \|z\|^2$	$f_2(z) = \bar{z}$	$f_3(z) = z^2$
$= x^2+y^2$	$= x+i(-y)$	$= x^2-y^2+i(2xy)$
$U = x^2+y^2,\ V=0$	$U = x,\ V=-y$	$U = x^2-y^2,\ V=2xy$
$U_x = 2x,\ V_x=0$	$U_x = 1,\ V_x=0$	$U_x = 2x,\ V_x=2y$
$U_y = 2y\ \ V_y=0$	$V_y = 0,\ V_y=-1$	$U_y = -2y,\ V_y=2x$
C-R eqⁿ holds only if $z=0$	does not satisfy C-R eqⁿ	satisfies C-R eqⁿ

Ans

Note:- For differentiability, C-R eqⁿ must satisfy → Necessary condition

Q Check analyticity at $z=0$ of above functions.

Ans $f_1(z) \rightarrow$ At $z=0$ differentiable, but in its surrounding not differentiable → Not analytical

$f_2(z) \rightarrow$ Not analytical

$f_3(z) \rightarrow$ Analytic

Q Check if $f(z)$ is differentiable at $z=0$

$$f(z) = \begin{cases} \dfrac{\operatorname{Im} z - \operatorname{Re} z}{|z|}, & z \neq 0 \\ 0, & z=0 \end{cases}$$

Ans Let's check for continuity,

$$\lim_{z\to 0} f(z) = \lim_{z \to 0} \frac{y-x}{\sqrt{x^2+y^2}}$$

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Take $y = mx$

$$\lim_{x \to 0} \frac{mx - x}{\sqrt{x^2 + m^2x^2}} = \frac{m-1}{\sqrt{1+m^2}}$$

$\downarrow$

Not continuous $\rightarrow \therefore$ not differentiable at $z=0$

Q $f(z) = \cancel{\log z}$ $\text{Log } z$

(i) Is it continuous on negative real axis.

(ii) Is it differentiable on negative real axis.

(iii) Is it differentiable ~~on~~ other than negative axis.

Ans (i) We already know, $f(z)$ is discontinuous on negative real axis.

(ii) Discontinuity $\rightarrow$ Not differentiable on $-$ve real axis.

(iii) $\cancel{f(z) = \log z = e^{\ln|z|}}$

$$f(z) = \text{Log } z = \ln|z| + i \,\text{Arg } z$$

$$= \frac{1}{2}\ln(x^2+y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

$$U = \frac{1}{2}\ln(x^2+y^2), \quad V = \tan^{-1}\left(\frac{y}{x}\right)$$

$$U_x = \frac{x}{x^2+y^2}, \quad V_y = \frac{x}{x^2+y^2}$$

$$U_y = \frac{y}{x^2+y^2}, \quad V_x = \frac{-y}{x^2+y^2}$$

It satisfies C-R eq^n

It is evident that C-R eq^n is only necessary condition because in above problem ~~contin~~ without checking continuity of eq^n, we may comment that $f(z) = \log z$ is differentiable everywhere, which isn't true.

Q

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Ans

$$U = \frac{x^3 - y^3}{x^2+y^2}, \quad V = \frac{x^3+y^3}{x^2+y^2}$$

Note that $U_x(0,0)$ calculation by going by partial derivative ∵ approach would be wrong as denominator would become '0' $(x^2+y^2=0)$

$$U_x(a,b) = \lim_{h \to 0} \frac{U(a+h, b) - U(a,b)}{h}$$

$$U_y(a,b) = \lim_{K \to 0} \frac{U(a, b+K) - U(a,b)}{K}$$

That is why limit approach is used to calculate U_x.

At $z = 0$

$$U_x(0,0) = \lim_{h \to 0} \frac{U(0+h, 0) - U(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

$$U_y(0,0) = \lim_{K \to 0} \frac{U(0, 0+K) - U(0,0)}{K}$$

$$= \frac{-K^3/K^2}{K} = -1$$

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$$V_x(0,0) = \lim_{h\to 0} \frac{V(0+h,0) - V(0,0)}{h} = 1$$

$$V_y(0,0) = \lim_{K\to 0} \frac{V(0,0+K) - V(0,0)}{K} = 1$$

↳ satisfies C-R eqⁿ

Now, let's take another approach to solve same problem →

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{x+iy\to 0} \frac{\frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} - 0}{x+iy}$$

$$= \lim_{x+iy\to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$$

Take $y = mx$

↳ not differentiable.

Again, In above problem C-R eqⁿ satisfies still $f(z)$ is not differentiable.

★ Sufficient condition of differentiability :-

(i) Partial derivatives exist U_x, U_y, V_x, V_y
(ii) C-R eqⁿ holds.
(iii) U_x, U_y, V_x, V_y are continuous.

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Q. Is $f(z) = e^{\bar{z}}$ differentiable at any point?

Entire function:- If function is analytic everywhere in Z.

There is no co-relation b/w analyticity & entireness.

Harmonic function:-

A "real-valued function" 'H' of two variable x & y is said to be harmonic in a given domain of xy-plane if throughout the domain it has continuous partial derivative of the first & second order and satisfies the equation-

$$H_{xx}(x,y) + H_{yy}(x,y) = 0 \longrightarrow \text{Laplace eq}^n \quad [\nabla^2 H = 0]$$

Theorem:- If a function $f(z) = u + iv$ is analytic in a domain 'D', then its component function u & v are harmonic in D.

e.g. $f(z) = |z|^2$

$= x^2 + y^2 + i\,0$

$u = x^2 + y^2$, $v = 0$

$u_x = 2x$ $\qquad v_x = 0$

$u_{xx} = 2$ $\qquad v_{xx} = 0$

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$U_y = 2y$ $\qquad\qquad V_{yy} = 0$

$U_{yy} = 2$ $\qquad\qquad V_{xx} + V_{yy} = 0$

$U_{xx} + U_{yy} = 4 \neq 0$

Here, function itself is not analytic at $z=0$
$u \rightarrow$ not even harmonic.

(ii) $f(z) = \bar{z}$
$= x - iy$

$U = x$ $\qquad\qquad V = -y$

$U_{xx} = 0$ $\qquad\qquad V_{xx} = 0$

$U_{yy} = 0$ $\qquad\qquad V_{yy} = 0$

$U_{xx} + U_{yy} = 0$ $\qquad\qquad V_{xx} + V_{yy} = 0$

Here, u & v are harmonic
But function $f(z)$ is not analytic

This Clearly, does not contradict our Theorem as theorem suggests $f(z)$ to be analytic which here isn't the case.

Q For what value of a & b given function is harmonic.

$$f(z) = ax^2 + by^2$$

Ans

$U = ax^2 + by^2$

$U_{xx} = 2a$, $U_{yy} = 2b$

$U_{xx} + U_{yy} = 0$

$2a + 2b = 0 \Rightarrow a = -b$

$\therefore U = x^2 - y^2$

Proof of Theorem:-

$f(z)$ is analytic $\Rightarrow$ u & v are harmonic

Given: $f(z)$ is anlytic $[f(z) = u(x,y) + i\,v(x,y)]$

$\therefore\ U_x = V_y$ — ① $\qquad U_y = -V_x$ — ②

p.d. ① w.r.t. x $\qquad$ P.d. ② w.r.t. y

$U_{xx} = V_{yx}$ $\qquad U_{yy} = -V_{xy}$

$U_{xx} + U_{yy} = V_{yx} - V_{xy} = 0$ [Since V_x & V_y are continuous]

Note:- If f_x & f_y are continuous, then-
$f_{xy} = f_{yx}$

Note:- Converse of Theorem is not true in general.

Q (i) Check whether given function is harmonic -
$U(x,y) = e^x \cos y$.
(ii) If yes, find conjugate harmonic of U.
(iii) Find $f(z) = U + iV$
(iv) Convert $f(z)$ in terms of z.

Ans (i) $U_{xx} = e^x \cos y$ & $U_{yy} = -e^x \cos y$

$\therefore\ U_{xx} + U_{yy} = e^x \cos y - e^x \cos y = 0$
$\downarrow$
Harmonic

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(ii) $U_x = e^x \cos y$

Using C-R eqn $(U_x = V_y)$

$\therefore V_y = e^x \cos y$

$V = e^x \sin y + \phi(x)$ — (1)

$V = 2xy + x^2 + 5$

$\frac{\partial V}{\partial y} = 2x$

$\Downarrow$

$V = 2x + C$ X

So, C has to be fn of x.

$V_x = e^x \sin y + \phi'(x)$ — (2)

Again, ~~E~~ $U_y = -V_x$

$-V_x = -e^x \sin y$

$\Rightarrow V_x = e^x \sin y$ — (3)

From (2) & (3)

$e^x \sin y = e^x \sin y + \phi'(x)$

$\Rightarrow \phi'(x) = 0$

$\phi(x) = C$

So, in (1)

$V = e^x \sin y + C$

(iii) $f(z) = u + iv$

$= e^x \cos y + i e^x \sin y + iC$

$= e^x (\cos y + i e^x \sin y) + iC$

$= e^x e^{iy} + iC$

$= e^{x+iy} + iC$

Note:- Idea of such questions is to construct analytic function if one component function is provided to be harmonic.

[Hence use of C-R eqn]

(iv) $e^{x+iy} + iC = e^z + iC \quad (z = x+iy)$

'v' is conjugate harmonic of u.

Q Check harmonticity for $u = x^2+y^2$

Ans

$u_x = 2x \qquad u_y = 2y$

$v_y = 2x \qquad -v_x = 2y$

$v = 2xy + \phi(x)$ — (A) $\qquad v_x = -2y$ — (2)

$v_x = 2y + \phi'(x)$ — (1)

From (1) & (2)

$2y + \phi'(x) = -2y$

$\phi'(x) = -4y$

$\phi(x) = -4xy + C$ — (B)

From (A) & (B)

$v = -2xy + C$

$\therefore f(z) = (x^2+y^2) - i\,2xy + iC$

$u = x^2+y^2 \quad , v = -2xy$

$u_x = 2x \quad , v_y = -2x$

$u_y = 2y \quad , v_x = -2y$] → does not follow C-R eqn

↓

Hence not analytic

Integration (Complex No.)

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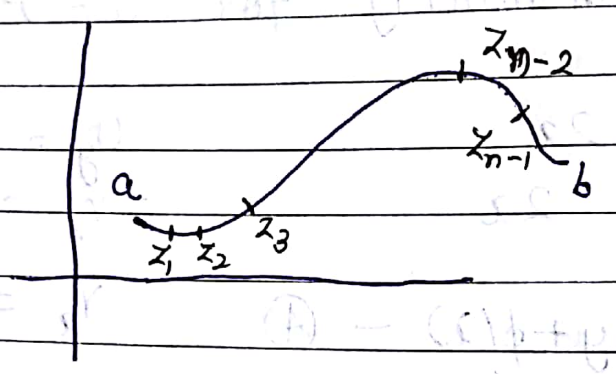
* Complex line integral :-

Let $f(z)$ be continuous at all points of a curve C, which we shall assume has a finite length



Subdivide 'C' into n points by means of points $z_1, z_2, \ldots z_{n-1}$ chosen arbitrarily and call $a = z_0$, $b = z_n$

On each Arc joining z_{k-1} to z_k, choose a point d_k. ~~From~~ Form the sum-

$$S_n = f(d_1)(z_1 - a) + f(d_2)(z_2 - z_1) + \ldots + f(d_{n-1})(z_{n-1} - z_{n-2}) + f(d_n)(b - z_{n-1})$$

Let the number of subdivisions 'n' increases in such a way that the longest of the chord approaches to zero. Then, since $f(z)$ is continuous, sum approaches to $\int_a^b f(z)\,dz$ or $\int_C f(z)\,dz$

$\int f(z)dz$
$= \int (u+iv)\,(dx+idy)$
$= \int (u\,dx - \cancel{v dy}) + \cancel{i(udy +} vdx)$

Let $\Delta Z_K = Z_K - Z_{K-1}$

$$S_n = \sum_{K=1}^{n} f(d_K)\,\Delta Z_K$$

This is called Complex integral or simply line integral of $f(z)$.

e.g. (i) $\int_0^{1+i} z^2 dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{(1+i)^3}{3}$

(ii) $\int_{-\pi i}^{\pi i} \cos z\, dz = \sin z \Big|_{-\pi i}^{\pi i}$

$= \sin \pi i - \sin(-\pi i)$
$= \sin \pi i + \sin \pi i$
$= 2 \sin(\pi i)$
$= 2i \sin h\pi$ $\quad [\because \sin ix = i \sinh x]$

★ Real line integral :-

Let $P(x,y)$ & $Q(x,y)$ be real functions of x and y continuous at all points of Curve 'C'.
Then the real line integral of $Pdx + Qdy$ along curve C can be defined as-

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

or

$$\int_C P dx + Q dy$$

If $x = \phi(t)$ & $y = \psi(t)$, $t_1 \le t \le t_2$

$$= \int_{t_1}^{t_2} P(\phi(t), \psi(t))\,\phi'(t)\,dt + Q(\phi(t), \psi(t))\,\psi'(t)\,dt$$

(To convert it into one variable 't')

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★ Arc :- A set of points $z = (x, y)$ is the complex plane is said to be an arc if -

$$x = x(t) \quad , \quad y = y(t)$$
$$(a \leq t \leq b)$$

where $x(t)$ & $y(t)$ are continuous functions of real parameter 't'.

The arc 'c' is a simple Arc (or Jordan Arc) if it does not ~~intersect~~ cross itself
In other words,

'C' is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$
when C is simple except for the fact that $z(a) = z(b)$. We say that 'C' is simple closed curve. [a, b → end points]

Q. Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along:

(i) The parabola $x = 2t$, $y = t^2 + 3$
(ii) Straight lines from (0, 3) to (2, 3) & then from (2, 3) to (2, 4)
(iii) A straight line from (0, 3) to (2, 4)

Ans (i) $x = 2t \Rightarrow dx = 2\,dt$
$y = t^2 + 3 \Rightarrow dy = 2t\,dt$
$0 \leq t \leq 1$

$$I = \int_0^1 [2(t^2+3) + 4t^2]\,2\,dt + (6t - t^2 - 3)\,2t\,dt$$

$$I = \int_0^1 (4t^2 + 12 + 8t^2)\,dt + (12t^2 - 2t^3 + 6t)\,dt$$

$$= \int_0^1 (-2t^3 + 24t^2 + 6t + 12)\,dt$$

$$= \left[-\frac{t^4}{2} + 8t^3 - 3t^2 + 12t \right]_0^1$$

$$= -\frac{1}{2} + 8 - 3 + 12 = 17 - \frac{1}{2} = \frac{33}{2}$$

(ii) $(0,2)$ to $(2,3)$

$y = 3, \quad dy = 0$

$$\therefore I_1 = \int_0^2 (6 + x^2)\,dx + (3x - 3)\,0$$

$$= \left[6x + \frac{x^3}{3} \right]_0^2 = \frac{44}{3}$$

$(2,3)$ to $(2,4)$

$x = 2, \quad dx = 0$

$$I_2 = \int_3^4 (2y + x^2)\,0 + (6 - y)\,dy$$

$$= \left[6y - \frac{y^2}{2} \right]_0^2 = \frac{5}{2}$$

$$I = I_1 + I_2 = \frac{103}{6}$$

(iii) For eqn of line →

$$\frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - x_0}$$

$$\Rightarrow \frac{y - 3}{1} = \frac{x - 0}{2}$$

$$\Rightarrow 2y - 6 = x \quad \Rightarrow dx = 2\,dy$$

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then-

$$I = \int_3^4 (2y + (2y-6)^2] 2\,dy + [(3(2y-6) - y]\,dy$$

$$= \frac{97}{6}$$

We observe, with three different paths line integral comes out to be different.

Q (a) $\int_C \bar{z}\,dz$ and (b) $\int_C z\,dz$

where C is curve from $z = 0$ to $z = 4+2i$ given by -

(i) $z = t^2 + it$

(ii) The line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4+2i$

Ans

(a) $\int_C \bar{z}\,dz = \int_C (x - iy)\,d(x+iy)$

$$= \int_C (x\,dx + y\,dy) + i(x\,dy - y\,dx)$$

Along (i) path - $z = t^2 + it$

Here $x = t^2$, $y = t$

& $t \to 0$ to 2

$$I = \int_0^2 (t^2 - it)\,d(t^2 + it)$$

$$= \int_0^2 (t^2 - it)(2t + i)\,dt = 10 - \frac{8}{3}i$$

Along (ii) path -

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$$I_1 = \int_0^2 (0\cdot 0 + y\,dy) + i\,(0\cdot dy - y\cdot 0)$$

$$= \left[\frac{y^2}{2}\right]_0^2 = 2$$

$$I_2 = \int_0^4 (x\,dx + \cancel{x}\cdot 0) + i\,(x\cdot 0 - 2\,dx)$$

$$= \int_0^4 x\,dx - 2i\,dx$$

$$= \left[\frac{x^2}{2} - 2xi\right]_0^4 = 8 - 8i$$

$$I = I_1 + I_2 = 10 - 8i$$

(b) $$I = \int_C z\,dz = \int_C (x+iy)(dx+i\,dy)$$

$$= \int_C (x\,dx - y\,dy) + i\,(y\,dx + x\,dy)$$

Along (i) path -

$$z = t^2 + it \quad z = 0 \text{ to } z = 4+2i$$

$$t \to 0 \text{ to } 2$$

$$I = \int_0^2 (t^2 + it)\, d(t^2 + it)$$

$$= \int_0^2 (t^2 + it)(2t + i)\,dt$$

$$= \int_0^2 2t^3 - t + i\,(2t^2 + t^2)$$

$$= \left[\frac{t^4}{2} - \frac{t^2}{2} + i\left(\frac{2t^3}{3} + \frac{t^3}{3}\right)\right]_0^2$$

$$= 8 - 2 + 8i = 6 + 8i$$

Along (ii) path -

$$I_1 = \int_0^2 -y\,dy = -2$$

$$I_2 = \int_0^4 (x\,dx - 2\cdot 0) + i(2\,dx + x\cdot 0)$$

$$= 8 + 8i$$

$$I = I_1 + I_2 = 6 + 8i$$

	$f(z) = \bar{z}$	$f(z) = z$
Continuous	Yes	Yes
Differentiable	No	~~No~~ Yes
Analytic	No	Yes

Note:- In case of Analytic function, along any path chosen the line integral would be same. But In case of non-Analytic function, line integral along different paths would be different.

In simpler words, line integral depend on end points rather than the curve (path)

★ Smooth Curve :-

A curve $z(t)$ $[z: [a,b] \to C]$ ~~$a \leq b$~~ $a \leq t \leq b$ is smooth if the derivative $z'(t)$ is continuous on the closed interval $a \leq t \leq b$ and non-zero on the open $a < t < b$.

Contour or piecewise Smooth Curve :- is an arc consisting of a finite number of smooth arc joined end to end.

Length of a Curve :-

$$z(t) = x(t) + i\,y(t)$$
$$z'(t) = x'(t) + i\,y'(t)$$

length of Curve

$$L = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

★ ML - inequality :- Let 'C' be a piecewise smooth curve $C : z(t) = x(t) + i\,y(t)$

$a \leq t \leq b$

then

$$\left| \int_C f(z)\,dz \right| \leq ML$$

where, L = Length of the Curve

& $|f(z)| \leq M$ everywhere on C.

↳ Here $f(z)$ is bound on the Curve. not necessarily bounded in entire Complex plane

We know,

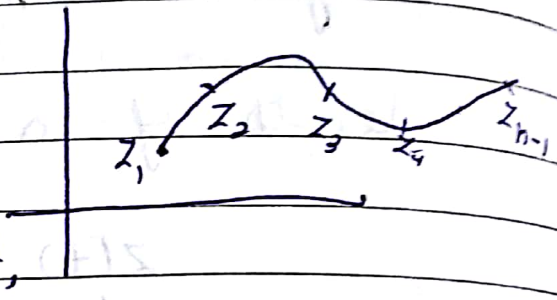
$$S_n = \sum_{k=1}^{n} f(\alpha_k)\,\Delta z_k$$

$$\Rightarrow |S_n| = \left| \sum_{k=1}^{n} f(\alpha_k)\,\Delta z_k \right|$$

$$= \sum^{n} |f(\alpha_k)| \, |\Delta z_k|$$

$$\leq ML$$

Again,



From the definition of Contour integral. We get,

$$S_n = \sum_{k=1}^{n} f(\alpha_k) \, \Delta z_k$$

$$|S_n| = \left| \sum_{k=1}^{n} f(\alpha_k) \, \Delta z_k \right|$$

$$\leq M \left| \sum_{k=1}^{n} \Delta z_k \right|$$

It is clear that $\sum_{k=1}^{n} |z_k - z_{k-1}|$ represent the ~~length~~ length of the chords whole end parts are $z_0, z_1 \ldots z_n$

Since a straight line path is the shortest distance b/w any two paths $|z_k - z_{k-1}|$ does not exceed the length of the arc joined to point z_{k-1} & z_k.

Thus $n \to \infty$ such that -

max. $|\Delta z_k| \to 0$, we have-

$$L^* \to L$$

We get,

$$\lim_{n\to\infty} |S_n| \le M \lim_{n\to\infty} \sum_{K=1}^{n} |x_K - x_{K-1}|$$

$$\left| \int_C f(z)\, dz \right| \le ML$$

~~Let C be the arc of the circle $|z| = 2$~~

Q Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant, then-

$$\left| \int_C \frac{z+4}{z^3-1}\, dz \right| \le \frac{6\pi}{7} \rightarrow ML$$

Ans

$$|f(z)| = \left| \frac{z+4}{z^3-1} \right|$$

$$|f_1(z)| = |z+4| \le |z| + 4 = 2 + 4 = 6$$

$$|f_2(z)| = \frac{1}{z^3-1} \le \frac{1}{|z|^3 - 1} = \frac{1}{8-1} = 7$$

$$\therefore |f(z)| = \left| \frac{f_1(z)}{f_2(z)} \right| \le \frac{6}{7}$$

Q Show that if 'C' is the boundary of the triangle with vertices at the points 0, $3i$ & -4, oriented in the counter-clockwise direction, then-

$$\left| \int_C (e^z - \bar{z})\, dz \right| \le 60$$

Ans

$$f(z) = e^z - \bar{z}$$

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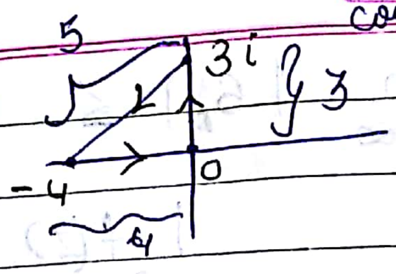
direction $\Rightarrow$ counter-clockwise



Length of Curve (L)

$= 3+4+5 = 12$

$|f(z)| = |e^z - \bar{z}| \geq |e^z| - |\bar{z}|$ → This will give minimum value of function

$\therefore \quad |e^z - \bar{z}| \leq |e^z| + |\bar{z}|$

For maximum value of $|f(z)|$

$\Rightarrow |f(z)| \leq |e^z| + |\bar{z}| = |e^z| + |z|$

Maximum Value of $|z| = 4$

$|e^z| = |e^{x+iy}|$

$= |e^x|\,|e^{iy}|$

$= |e^x|\,|\cos y + i\sin y|$

$= |e^x| \times 1 = |e^x|$

& Maximum Value for $x = 0$

$\therefore \quad |e^z|_{max} = |e^0| = 1$

$\therefore \quad |f(z)| \leq 1+4 = 5$

$|f(z)| \leq 5$

Therefore, by ML - eq^n:

$\left| \int_c (e^z - \bar{z})\, dz \right| \leq M*L = 5\times 12 = 60$

$\cancel{\leq 5\times 12}$

$\cancel{\leq 60}$

Q. Find the length of Curve $C: z(t) = t^3 + it$, $0 \le t \le 1$

Ans. $z'(t) = 3t^2 + i$

Length of Curve $(L) \rightarrow \int_a^b |z'(t)|\, dt$

$$L = \int_0^1 |3t^2 + i|\, dt$$

$$= \int_0^1 \sqrt{(3t^2)^2 + 1}\, dt = \int_0^1 \sqrt{9t^4 + 1}\, dt$$

Q. Find the upperbound for the absolute value of integral:

$$I = \int_L e^{(\bar{z})^2}\, dz \quad ; \quad C: |z| = 1$$

where C is traversed in the anti-clockwise direction.

Ans.

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Q Evaluate the integral $\int_C \frac{2z+3}{z} dz$ where C i[s]
(a) Upper half of the circle $|z| = 2$ is traversed in the clockwise direction.

Ans $|z| = 2 \Rightarrow z = 2e^{i\theta}$

$z'(\theta) = 2ie^{i\theta}$

We know when doing parameters exchange

$$I = \int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

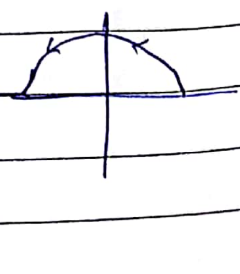
$$I = \int_C \frac{2z+3}{z} dz = \int_C \frac{2(2e^{i\theta})+3}{2e^{i\theta}} (2ie^{i\theta})\,d\theta$$

$$= i\int_{\pi}^{0} (4e^{i\theta}+3)\,d\theta$$

limit from π to 0
as it's upper half of circle
& in clockwise direction

(b) The upperhalf of the circle $|z| = 2$ traversed in the anti-clockwise direction.

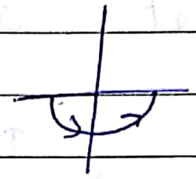
$$I = i\int_0^{\pi} (4e^{i\theta}+3)\,d\theta$$



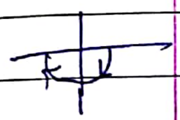
Anti-clockwise → +ve
Clockwise → −ve

(c) Lowerhalf of the circle $|z| = 2$ in the counter-clockwise direction

$$I = i\int_{-\pi}^{0} (4e^{i\theta}+3)\,d\theta$$

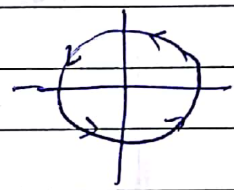


(d) Lowerhalf of the Circle $|z| = 2$ in the clockwise direction



$$I = i\int_{0}^{-\pi} (4e^{i\theta}+3)\,d\theta = i\left[\frac{4e^{i\theta}}{i} + 3\theta\right]_0^{-\pi}$$

(e) Circle $|z| = 2$ in the anti-clockwise direction.



$$I = i\int_{0}^{2\pi} (4e^{i\theta}+3)\,d\theta =$$

$$= i\left[\frac{4e^{i\theta}}{i} + 3\theta\right]_0^{2\pi}$$

$$= 4e^{i2\pi} + 6\pi i = 6\pi i$$

Q Evaluate the integral $\int_C z^2\,dz$ when C is circle $|z| = 2$ is traversed in anti-clockwise direction.

Ans $|z| = 2 \Rightarrow z = 2e^{it}$ $\qquad f(z) = z^2$

$$z'(t) = 2ie^{it}$$

$$I = \int_0^{2\pi} f(z)\,dz = i\int_0^{2\pi} [2e^{it}]^2\,[2ie^{it}]\,dt$$

$$= i2^3 \int_0^{2\pi} e^{(2+1)it}\,dt = \frac{8i}{3}\left[e^{3it}\right]_0^{2\pi}$$

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$$= \frac{8i}{3}[1-1] = 0$$

Let's repeat same problem for $f(z)=z$

$$I = \int_0^{2\pi} [2e^{it}][2ie^{it}]\,dt$$

$$= i\,2^2 \int_0^{2\pi} e^{(1+1)it} = 4i[1-1] = 0$$

Similarly,

For $f(z) = z^n$

$$I = \int_C f(z)\,dz = 0 \quad \text{for } n \neq -1$$

For $f(z) = \frac{1}{z}$

$$I = \int_0^{2\pi} \frac{1}{[2e^{it}]}\, i[2e^{it}]\,dt = i[t]_0^{2\pi}$$

$$= 2\pi i$$

So,

Note:- $$I = \int z^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n = 0, \pm 1, \pm 3, \ldots \end{cases}$$

~~Note~~:- Q Evaluate $I = \int_C z^n\,dz \quad n = 0, \pm 1, \pm 2 \ldots$

where $C: |z| = r$ is traversed in the counter-clockwise direction

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Ans

$$z(t) = r e^{it} \quad ; \quad z'(t) = r i e^{it}$$

$$I = \int_0^{2\pi} [r e^{it}]^n [i r e^{it}] \, dt$$

$$= i r^{n+1} \int_0^{2\pi} e^{(n+1)it} dt$$

$$= \frac{r^{n+1}}{n+1} \underbrace{\left[e^{(n+1)it} \right]_0^{2\pi}}_{} \overset{=0}{(\text{if } n \neq -1)}$$