$\Rightarrow \frac{1}{P(z)}$ is constant [By Liouville's Theorem

$\Rightarrow P(z)$ is constant

It is contradiction of our assumption

$\therefore$ $P(z)$ has at least one root.

Q $f(z) = \dfrac{z^4(z-1)^2}{(z-2)^2(z-3)}$

Ans $z=0$ is a zero of order 4 } Total zeroes $= 6$

$z=1$ is a zero of order 2

$z=2$ is a pole of order 3 } Total poles $= 4$

$z=3$ is a pole of order 1

Pole order $\rightarrow$ $\lim_{z \to z_0} (z-z_0)^n f(z) = \text{Sink} \neq 0$

$n \rightarrow$ order

Q $I_1 = \oint \dfrac{e^z}{e^z - 1}\, dz$; $C: |z| = 10$

Ans $f(z) = \dfrac{e^z}{e^z - 1}$

At $z=0$ what kind of singularity is there?

$\lim_{z \to 0} z^n \dfrac{e^z}{e^z - 1} = \lim_{z \to 0} z \dfrac{e^z}{e^z - 1}$ [For $n=1$]

$= \lim_{z \to 0} \dfrac{e^z + ze^z}{e^z} = 1 \neq 0$

$\therefore$ $z=0 \rightarrow$ pole of 1 order.

Pole exists $\Rightarrow$ Isolated singularity.

We can not find I_1 from the methods we learnt till now.

★ **Argument Theorem :-**

Let 'C' denote a positively oriented simple closed curve and suppose that -

(a) Function $f(z)$ containing 'n' poles inside C counting multiplicity

(b) $f(z)$ contains 'm' zeroes inside 'C' counting multiplicity then -

$$\oint \frac{f'(z)}{f(z)}\,dz = 2\pi i\,(m-n)$$

Q $\quad I = \oint \frac{e^z}{e^z - 1}\,dz \quad ; \quad C : |z| = 10$

Ans $\quad$ Let $f(z) = e^z - 1 \quad , \quad f'(z) = e^z$

No poles But zeroes at $z = 0, \pm 2\pi i$ $\therefore m = 3$

$$\therefore I = \int \frac{f'(z)}{f(z)}\,dz = \int \frac{e^z}{e^z - 1}\,dz = 2\pi i\,(3-0)$$

$$= 6\pi i$$

Q $\quad I = \oint -\tan z\,dz \quad ; \quad C : |z| = 10$

Ans $\quad I = \oint -\frac{\sin z}{\cos z}\,dz$

$f(z) = \cos z \quad , \quad f'(z) = -\sin z$

$f(z) \rightarrow$ no poles

$\rightarrow$ Zeroes at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$

$m = 6$

$\therefore\ I = 2\pi i\,(6-0) = 12\pi i$

Q $\oint \frac{(7z^6 - 12z^2 + 1)}{z^7 - 4z^3 + z - 1}\,dz$; $C : |z| = 1$

Ans $f(z) = z^7 - 4z^3 + z - 1$

But counting multiplicity of zeroes $\rightarrow$ difficult

Rouchi's Theorem :-

Let 'C' denote a simple closed curve and suppose that -

(a) Two function $f(z)$ & $g(z)$ are analytic $\boxed{\text{inside}}$ C.

(b) $|f(z)| > |g(z)|$ $\boxed{\text{on}}$ C (on C $\rightarrow$ on the curve)

then $f(z)$ & $f(z) + g(z)$ has same number of zeroes, counting multiplicity, inside C.

Q $z^5 + z - 16i = 0$ for $|z| = 1$

Ans Choose $f(z)$ & $g(z)$ such that to satisfy (b) condition of Rouchi's Theorem i.e.

$|f(z)| \geq |g(z)|$

Let $f(z) = -16i$

$g(z) = z^5 + z$

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Clearly,

$$|f(z)| > |g(z)| \quad \text{on } C.$$

As $f(z)$ has no roots inside C

$\therefore$ $f(z) + g(z)$ has no roots inside C.

Q $z^5 + z - 16i = 0$, Number of zeroes inside $|z| = 2$?

Ans Here,

$$f(z) = z^5$$

$$g(z) = z - 16i$$

$$|f(z)| > |g(z)| \quad \text{on } |z| = 2$$

Now,

$f(z) = z^5$, so roots of $f(z)$ is $z=0$ with multiplicity 5.

So, $f(z) + g(z)$ has total no. of zeroes $= 5$ inside C.

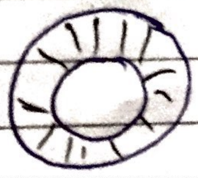
Q $z^5 + z - 16i = 0$. Find number of zeroes inside the region bounded by $|z| = 1$ & $|z| = 2$ i.e. $1 \le |z| \le 2$.

Ans As, inside $|z| = 1$ $\rightarrow$ No zeroes

inside $|z| = 2$ $\rightarrow$ '5' zeroes

$\therefore$ ~~Inside~~ In region $1 \le |z| \le 2$ $\rightarrow$ $5 - 0$

$= 5$ zeroes



Q $\oint \frac{(7z^6 - 12z^2+1)}{z^7 - 4z^3+z-1} dz$; $C : |z| = 1$ (anti-clockwise)

Ans Here, $f(z) = z^7 - 4z^3 + z - 1$

$f'(z) = 7z^6 - 12z^2 + 1$

$\rightarrow$ No poles

For $f(z) = z^7 - 4z^3 + z - 1$

~~Here~~ To find zeroes $\rightarrow$

$X(z) = -4z^3$ (Maximum value $= 4$)

$Y(z) = z^7 + z - 1$ (Maximum Value $= 1+1-1=1$)

$|X(z)| > |Y(z)|$ on C.

$X(z)$ has zeroes at $z=0$ with multiplicity '3'

$\therefore$ $f(z) = X(z) + Y(z)$ has '3' no. of zeroes. inside C.

$m = 3$

So, $I = 2\pi i(3-0)$

$= 6\pi i$

Q If $f(z) = \frac{z+1}{(z^2-2z+1)^3}$, then evaluate -

$\oint \frac{f'(z)}{f(z)} dz$ on $|z| = 3$ (anti-clockwise)

Ans $f(z) \rightarrow$ Poles at $z=1$ $\rightarrow$ Multiplicity $2 \times 3 = 6$

$\rightarrow$ zeroes at $z=-1$ $\rightarrow$ Order $= 1$

$m = 1$, $n = 6$

$\therefore$ $I = 2\pi i(1-6) = -10\pi i$

Repeat above problem for $C_1 : |z+1| = 1$
$C_2 : |z-1| = 1$

$C_1 : |z+1| = 1 \rightarrow f(z) \rightarrow$ poles $\times$ (inside C)
zeroes $\rightarrow$ 1

$\therefore I = 2\pi i(1-0) = 2\pi i$

$C_2 : |z-1| = 1 \rightarrow f(z) \rightarrow$ '6' poles (inside C)
zeroes $\times$

$\therefore I = 2\pi i(0-6) = -12\pi i$