Partial Differential Equations

1. An elementary Course in P.D.E.
~ T. Amarnath

2. (Not sure of book's name) ~ I.N. Sneddon

In real world, P.D.E. seeks many applications. Some of them being:

(i) Black Scholes Equation :-
(1973)

$$f_t + rS f_S + \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

$r \rightarrow$ risk free rate of return

$\sigma \rightarrow$ Constant Valuability

(ii) Navier-Stockes Equation :- For fluid mechanics

$$u_t + (u \nabla) u = -\nabla p + \nu \Delta u$$

$\nabla \rightarrow$ gradient

$\Delta \rightarrow$ Laplace

$$\nabla u = 0 \text{ in } \Omega$$
$$u = 0 \text{ in } \partial\Omega$$

Other than that, there are other applications like - image processing, Computer Vision, Electrodynamics & fluid flow problem.

Definition :- A P.D.E. is an equation of form-

$$F(x, y, \ldots z, z_x, z_y, \ldots z_{xx}, z_{xy}, z_{yy} \ldots) = 0$$

where $x, y \rightarrow$ Independent Variable

$z \rightarrow$ Dependent Variable

↓ depends on independent variable

$$z = z(x, y, \ldots)$$

Order of P.D.E. :- defined by the order of highest order derivatives appearing in the P.D.E.

☆ Classification of P.D.E. :-

(i) Quasi-linear P.D.E. :→ A P.D.E. is called Quasi-linear if the highest order derivative appears linearly in the P.D.E.

(ii) Semi-linear P.D.E. :→ A Quasi-linear P.D.E. is called semi-linear if the coefficient of the highest order derivative is independent of dependent variables & its derivatives.

(iii) Linear P.D.E. :→ A semi-linear P.D.E. is called linear if it is linear in the dependent variables & its

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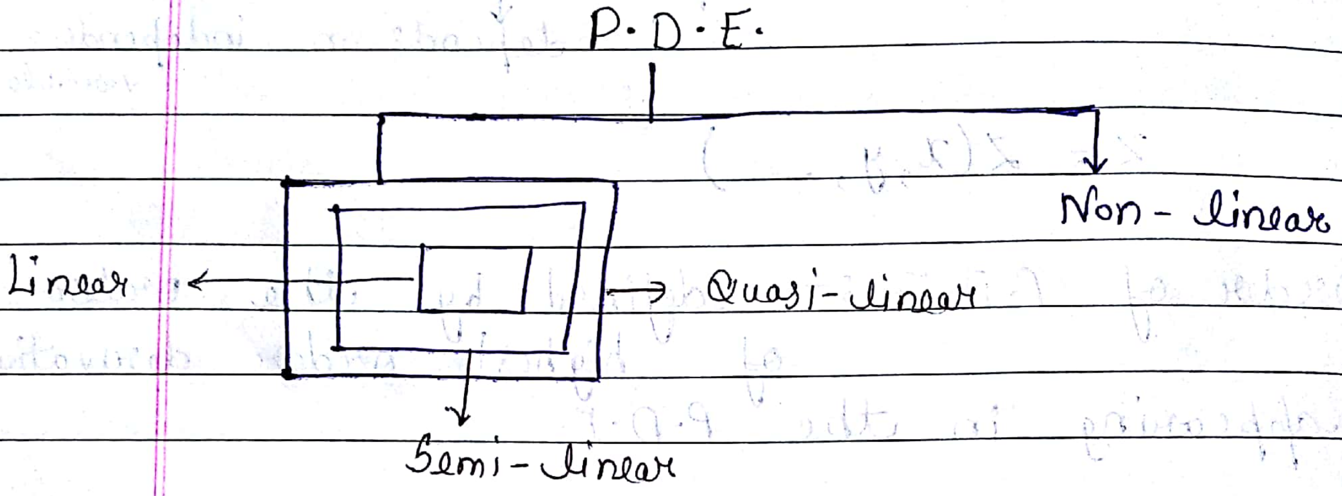
derivatives.

(iv) Non-linear P.D.E. :- A P.D.E. which is not Quasi-linear is called Non-linear P.D.E.



e.g. (i) $u u_{xx} + u_t = xt$ → order = 2

↳ Quasi-linear

[$u u_{xx}$ → linear
but u → dependent variable]

(ii) $u u_{xx} + u^2 u_t = xt$

$\Rightarrow u_{xx} + u u_t = \frac{xt}{u}$ → Semi-linear

(iii) $x u_{xx} + u u_t = xt$ → Semi-linear

(iv) $x u_{xx} + u_t = xt$ → Linear

(v) $(u_{xx})^2 + u_t = xt$ → Non-linear

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(vi) $u_{xx} + (u_t)^2 = xt \rightarrow$ Semi-linear

* **First order P.D.E. :-**

$F(x, y \ldots z \ldots) = 0$

Surface of revolution $\rightarrow$ about z-axis.

$z = F(r)$; $r = \sqrt{x^2+y^2}$

$F \rightarrow$ Arbitary function

Differentiate it w.r.t. x,

$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} = F'(r)\frac{2x}{2(x^2+y^2)^{1/2}}$$

$$= F'(r)\frac{x}{r}$$

& $\frac{\partial z}{\partial y} = F'(r)\frac{y}{r}$

[Let $\frac{\partial z}{\partial x} = p$

$\frac{\partial z}{\partial y} = q$]

$F'(r) = \frac{pr}{x}$

& $q = \frac{pr}{x} \times \frac{y}{r} = \frac{py}{x}$

$\Rightarrow \boxed{xq - py = 0}$

e.g. $x^2 + y^2 + (z-c)^2 = a^2$

↳ Sphere of radius 'a' centred at $(0, 0, c)$

$2x + 2(z-c)p = 0$ & $2y + 2(z-c)q = 0$

$$z = F(r) \quad ; \quad r = \sqrt{x^2+y^2}$$

$$z - F(r) = 0$$

$$\Rightarrow u - v = 0$$

$$\Rightarrow H(u, v) = 0$$

Suppose a surface is denoted as $F(u, v) = 0$ where $u = u(x, y, z)$ & $v = v(x, y, z)$.

F is continuously differentiable function having first order partial derivative w.r.t. x.

$$F(u, v) = 0 \quad \text{—— (1)}$$

To calculate corresponding partial differential eqn:-

Differentiate (1) w.r.t. x & y respectively.

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{dz}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \quad \text{— (2)}$$

Here $z = z(x \,\&\, y)$

& $p = \dfrac{\partial z}{\partial x}$

$$\& \quad \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{(3)}$$

On elimination of $\frac{\partial F}{\partial u}$ & $\frac{\partial F}{\partial u}$, we get -

$$\boxed{\frac{d(u,v)}{d(y,z)}p + \frac{d(u,v)}{d(z,x)}q = 0}$$

First order P.D.E.

where $\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y$

↓

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The above eqn is Quasi-linear.

e.g. $(x-a)^2 + (y-b)^2 + z^2 = 1$ — ①

↳ Family of surface with parameters a & b.

Differentiate w.r.t. x & y.

$2(x-a) + 2zp = 0 \Rightarrow (x-a) = -zp$ — ②

$2(y-b) + 2zq = 0 \Rightarrow (y-b) = -zq$ — ③

From ①, ② & ③

$p^2 z^2 + q^2 z^2 + z^2 = 1$

$z^2 (p^2 + q^2 + 1) = 1 \longrightarrow$ Non-linear P.D.E.

$F(x,y,z,a,b) = 0$ — ①

↳ Surface with parameters a & b.

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p = 0$ — ②

$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q = 0$ — ③

↳ $f(x,y,z,p,q) = 0$

★ Classification of 1st order P.D.E. :-

$$f(x, y, z, p, q) = 0 \quad \text{— (1)}$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

(i) Quasi-linear P.D.E. - Eq. (1) is Quasi-linear if it is in form :

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

e.g. $xz^2 p + x^2 y z q = x^2 y^2 z^2$

(ii) Semi-linear P.D.E. - Eq (1) in the form -

$$P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$$

(iii) Linear P.D.E. - Eq (1) has form

$$P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$$

(iv) Non-linear P.D.E. - If (1) does not have any of the above forms.

e.g. (a) $xy\, p + x^2 y^2 q = xz^2 \rightarrow$ Semi-linear

(b) $xy\, p + x^2 y^2 q = xz \rightarrow$ Linear

* Solution of 1st order P.D.E. :-

$$f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

A function $z = z(x,y)$; $(x,y) \in D \subseteq R^2$ is called solution of eq. (1), if z and its first-order partial derivatives w.r.t. x & y (i.e. p & q) satisfy eq. (1) on D.

Note:- A solution $z = z(x,y)$ can be interpreted as a surface in 3-D space. Thus, solution $z = z(x,y)$ also referred as Integral surface for eq. (1)

Complete integral (Complete solution) :- A two-parameter function of surface -
$z = F(x, y, a, b)$ is called complete integral of (1) if the rank of matrix 'M' :

$$M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} \text{ is } 2.$$

General integral or general solution :-

$$F(u, v) = 0$$

where $u = u(x, y, z)$ & $v = v(x, y, z)$

F is continuously differentiable function which satisfy given P.D.E. implicitly in some domain D.

Singular Integral (or Singular Solution):-

$$z = F(x, y, a, b)$$

$$\frac{\partial F}{\partial a} = 0 \;;\; \frac{\partial F}{\partial b} = 0 \quad - ②$$

$\llcorner$ ①

Singular solution or singular integral is obtained by finding envelope of complete integral by eliminating a & b from

$$z = F(x, y, a, b) \quad - ③$$

Sol. obtained by eq. ①, ② & ③ is called singular solution.

e.g. $z - px - qy - p^2 - q^2 = 0$

Ans $f(x, y, z, q, p) = z - px - qy - p^2 - q^2 = 0$

Since, it is non-linear P.D.E.

[So, we can't be asked to calculate its Complete integral since we are limited to 1st order linear P.D.E.]

Here, Complete integral is -

$$z = F(x, y, a, b) = ax + by + a^2 + b^2 \quad - ①$$

$$F_a = x + 2a = 0 \quad - ②$$

$$F_b = y + 2b = 0 \quad - ③$$

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From ② & ③, we have -

$$a = -\frac{x}{2}, \quad b = -\frac{y}{2}$$

Substituting in ①, we have -

$$\boxed{4z = -(x^2+y^2)} \rightarrow \text{Singular solution}$$

↳ Paraboloid of revolution

★ Cauchy Problem :-

$$f(x, y, z, p, q) = 0 \quad \text{— ①}$$

To find a solution of ①, which contains an intial curve (initial condition):

$$x = x_0(\lambda), \quad y = y_0(\lambda), \quad z = z_0(\lambda)$$
$$\lambda \in I$$

★ Lagrange's method :- [To solve Quasi-linear first-order P.D.E.]

Theorem- Let P, Q, R at continuously differentiable functions of x, y & z. The general solution of P.D.E. -

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

is -

$$F(u, v) = 0$$

where F is any arbitary smooth function and $u(x,y,z) = C_1$ and $v(x,y,z) = C_2$ are two linearly independent solutions of the auxialiary eqⁿ

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

<u>Proof</u>:- Let $u = C_1$ & $v = C_2$ are two linearly independent solutions of Auxiliary eqn-

$$du = 0$$

$$u_x\,dx + u_y\,dy + u_z\,dz = 0$$

Now, let $\frac{dx}{P} = \frac{dy}{q} = \frac{dz}{R} = c$ (let)

$$\Rightarrow u_x(cP) + u_y(cq) + u_z(cR) = 0$$

$$\Rightarrow Pu_x + qu_y + Ru_z = 0 \quad — (1)$$

Similarly,

$$Pv_x + qv_y + Rv_z = 0 \quad — (2)$$

By Cramer's Rule -

$$\cancel{\frac{P}{u_y v_z - u_z v_y} =}$$

$$\frac{P}{u_y v_z - u_z v_y} = \frac{q}{u_z v_x - u_x v_z} = \frac{R}{u_x v_y - u_y v_x}$$

$$\Rightarrow \frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \quad — (1)$$

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Now, recall that $F(u,v) = 0$ is a solution of P.D.E.

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$$\Rightarrow \frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)} R \quad — ②$$

Hence, $Pp + Qq = R$ is solution of P.D.E.

Q $x^2 p + y^2 q = (x+y) z$

Ans Auxiliary eqⁿ :- $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow -\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1 \quad — ①$$

$$u(x,y,z) = c_1$$

Also, $\frac{dz}{(x+y)z} = \frac{dx - dy}{x^2 - y^2}$ [By fraction concept]

$$\Rightarrow \frac{dx - dy}{(x-y)} = \frac{dz}{z}$$

$$\Rightarrow \frac{d(x-y)}{(x-y)} = \frac{dz}{z} \Rightarrow \ln(x-y) = \ln z + \ln c_2$$

$$\Rightarrow x - y = c_2 z$$

$$\Rightarrow \frac{x-y}{z} = c_2 \quad — ②$$

$$v(x,y,z) = c_2$$

So, solution of P.D.E.

$$F\left(\frac{1}{y} - \frac{1}{x}, \frac{x-y}{z}\right) = 0$$

Q. Find the integral surface of -

$$xp + yq = z$$

which contains curve :

$\Gamma : x_0 = s^2 \; ; \; y_0 = s+1 \; ; \; z_0 = s$

Ans Auxiliary eqⁿ :-

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \; u = \frac{y}{z} = C_1 \quad \& \quad v = \frac{y}{x} = C_2$$

~~So,~~ $\cancel{\frac{s+1}{s} = s C_1}$

Now,

if Γ is the curve on integral surface must satisfy both of them.

$$\frac{s+1}{s} = C_1 \quad \& \quad \frac{s+1}{s} = C_2$$

$$\Rightarrow (C_1 - 1) C_1 = C_2$$

$$\Rightarrow (u-1) u = v$$

$$\Rightarrow (y-z) x = z^2 \rightarrow \text{Solution of Cauchy Problem}$$

Q. $yzp + xzq = xy$

Ans Auxiliary eqⁿ : $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$

(i) $\frac{dx}{yz} = \frac{dy}{xz} \Rightarrow x^2 - y^2 = C_1$

(ii) $y^2 - z^2 = C_2$

Note: $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{aa_1 + cc_1 + ee_1}{ba_1 + dc_1 + fe_1}$ → a_1, c_1 & e_1 are multiplier

Sol. → $F(x^2 - y^2, y^2 - z^2) = 0$

$$\frac{dx}{l} = \frac{dy}{m} = \frac{dx - dy}{l - m}$$

1 & -1 are multipliers

Q $(z^2 - 2yz - y^2)p + x(y+z)q = x(y-z)$

Ans Auxiliary eqn : $\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$

(i) $\frac{dy}{y+z} = \frac{dz}{y-z}$

$\Rightarrow y\,dy - z\,dz - (z\,dy + y\,dz) = 0$

$\Rightarrow d\left(\frac{y^2}{2}\right) - d\left(\frac{z^2}{2}\right) - d(yz) = 0$

On integration, we get -

$y^2 - z^2 - 2yz = c_1$

Consider x, y, z as multipliers, we get -

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow x^2 + y^2 + z^2 = c_2$

↳ General Sol. $F(y^2 - z^2 - 2yz, x^2 + y^2 + z^2) = 0$

* Second order P.D.E. :-

Let $D \subseteq R^2$ be a smooth domain. And $R, S, T : D \to R$ be smooth functions. Then a second order P.D.E. is called semi-linear P.D.E. if it has following form:

$$R(x,y)\, u_{xx} + S(x,y)\, u_{xy} + T(x,y)\, u_{yy} + g(x, y, u, u_x, u_y) = 0 \quad \text{①}$$

Also, $R^2 + S^2 + T^2 \neq 0$

$u(x,y)$ defined on D is called solution eq. ① if it satisfies ① on D.

We can classify ① in three categories:

(i) Hyperbolic :
$$S^2 - 4RT > 0 \quad \text{in } D$$

(ii) Parabolic :
$$S^2 - 4RT = 0 \quad \text{in } D$$

(iii) Elliptic : $S^2 - 4RT < 0$ in D

e.g. Consider eq^n for vibration in string as-
$$u_{tt} = c^2 u_{xx} \;;\; c > 0 \qquad u(t,x)$$

Sol. $R = 1, \; S = 0, \; T = -c^2$

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$\therefore\ 0 - 4(-c^2)(1) = 4c^2 > 0$ → Hyperbolic wave eqⁿ

Q Consider 1D- heat conduction eqⁿ is given as

$u_t = \sigma u_{xx}$; $\sigma > 0$

$\sigma \rightarrow$ Diffusivity coefficient

This eqⁿ is hyperbolic, parabolic or elliptic?

Ans $R = 0,\quad S = 0,\quad T = -\sigma$

$\therefore\ S^2 - 4RT = 0 - 0(-\sigma) = 0$ → Parabolic

e.g. $u_{xx} + u_{yy} = 0 \rightarrow$ Laplace eqⁿ in 2D
(Potential eqⁿ)

$R = 1,\quad T = 1,\quad S = 0$

$\therefore\ 0 - 4(1)(1) < 0$ → Elliptic

Q $x u_{xx} + u_{yy} = 0$. Evaluate range of x for which eqⁿ is hyperbolic, parabolic & elliptic.

Ans $R = x,\quad S = 0,\quad T = 1$

$\Rightarrow S^2 - 4RT = 0 - 4x = -4x$

for $x < 0 \rightarrow$ Hyperbolic
$x = 0 \rightarrow$ Parabolic
$x > 0 \rightarrow$ Elliptic

$x<0$	$x>0$
Hyperbolic	Elliptic

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* Canonical Form :-

$$R(x,y)\,u_{xx} + S(x,y)\,u_{xy} + T(x,y)\,u_{yy} + g(x, y, u, u_x, u_y) = 0 \quad \text{--- (1)}$$

Changing the coordinates

Psi $\rightarrow \xi = \xi(x,y)$; $\eta = \eta(x,y)$

η ↓ Eta

$u \rightarrow$ function of (ξ, η)

Above transformation is

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x \neq 0$$

And, $u_x = u_\xi \xi_x + u_\eta \eta_x$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xy} = u_{\xi\xi}\xi_x \xi_y + u_{\xi\eta}\eta_y \xi_x + u_\xi \xi_{xy} + u_{\eta\xi}\xi_y \eta_x + u_{\eta\eta}\eta_y \eta_x + u_\eta \eta_{xy}$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + u_{\xi\eta}\eta_x \xi_x + u_\xi \xi_{xx} + u_{\eta\xi}\xi_x \eta_x + u_{\eta\eta}\eta_x^2 + u_\eta \eta_{xx}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + u_{\xi\eta}\eta_y \xi_y + u_\xi \xi_{yy} + u_{\eta\xi}\xi_y \eta_y + u_{\eta\eta}\eta_y^2 + u_\eta \eta_{yy}$$

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