

After Mid-Sem

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Sequence and Convergence Series

$$\{z_1, z_2, \dots, z_n, \dots, z\}, z_i \in \mathbb{C}$$

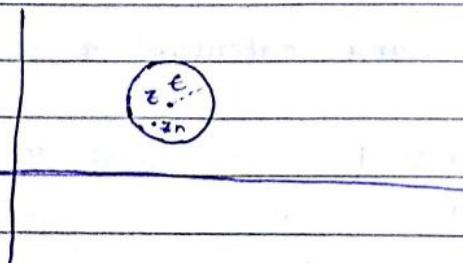
(One-one correspondence with natural no.)

Represented as : $\{z_n\}_{n=1}^{\infty}$

Def'n: An infinite sequence $\{z_n\}_{n=1}^{\infty}$ of complex no. has a limit z if for every $\epsilon > 0$ there is a positive no. n_0 s.t. $|z_n - z| < \epsilon$ whenever $n > n_0$.

Geometrical Interpretation

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If the limit exists and is unique, then we say z_n converges to limit z , or

$$\lim_{n \rightarrow \infty} z_n = z$$

Eg.: $z_n = i^n, \{z_n\}_{n=1}^{\infty}$

$z_n = \{i, -1, -i, 1, i, \dots, z\}$: It is oscillating (non-unique)

It \Rightarrow diverges

Eg.: $z_n = 1 + i^n$

$\{z_n\}$: diverges

$$\text{eg. } z_n = \frac{i^n}{n}$$

$$z_n = \left\{ i, -\frac{1}{2}, -\frac{i}{3}, \frac{1}{4}, \dots \right\}$$

$$\lim_{n \rightarrow \infty} z_n = 0 \Rightarrow \text{Converges}$$

Theorem: Suppose $z_n = x_n + iy_n$, $z = x + iy$
Then,

$$\lim_{n \rightarrow \infty} z_n = z \quad \text{--- (1)}$$

$$\text{iff } \lim_{n \rightarrow \infty} x_n = x \text{ & } \lim_{n \rightarrow \infty} y_n = y \quad \text{--- (2)}$$

$$\begin{aligned} * \lim_{n \rightarrow \infty} z_n &= \lim_{n \rightarrow \infty} (x_n + iy_n) = \lim_{n \rightarrow \infty} x_n + i \lim_{n \rightarrow \infty} y_n \\ &\text{(provided } x_n \text{ & } y_n \text{ are convergent)} \end{aligned}$$

PROOF:

let us assume that eqⁿ (2) holds true.

Claim : eqⁿ (1) holds true

$$\Rightarrow |z_n - z| < \epsilon \text{ whenever } n > n_0$$

(no need to be determined)

$$\text{given } \lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} y_n = y$$

For every $\epsilon > 0$, # positive no. n_1 & n_2 s.t.

$$|x_n - x| < \epsilon/2 \text{ whenever } n > n_1$$

$$|y_n - y| < \epsilon/2 \text{ whenever } n > n_2$$

$$|z_n - z| = |x_n + iy_n - x - iy| = |(x_n - x) + i(y_n - y)|$$

$$\leq |x_n - x| + |y_n - y| = \epsilon \text{ whenever}$$

$$n > \max(n_1, n_2)$$

Hence Proved.

Sum of two convergent sequences is also convergent

Combⁿ

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Converse :- Given $\lim_{n \rightarrow \infty} z_n = z$

For every $\epsilon > 0$, $\exists n_0 > 0$ s.t.

$|z_n - z| < \epsilon$ whenever $n > n_0$.

$$\begin{aligned}|x_n - x| &\leq |(x_n - x) + i(y_n - y)| = |x_n + iy_n - (x + iy)| \\&= |z_n - z| < \epsilon \text{ whenever } n > n_0.\end{aligned}$$

Similarly, $|y_n - y| \leq |(y_n - y)i + (x_n - x)| < \epsilon$ whenever $n > n_0$.

Hence

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y$$

Hence Proved.

Ex. 15

$$z_n = x_n + iy_n = \left(1 - \frac{1}{n^2}\right) + i\left(2 + \frac{4}{n}\right)$$

(Assuming z_n is convergent)

$$\lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} x_n + i \lim_{n \rightarrow \infty} y_n$$

$$= 1 + 2i$$

Ex. 16

How to find n_0 [depends on ϵ]

$$z_n = \frac{1}{n^3} + i, \quad n = 1 \text{ to } \infty.$$

$$\begin{aligned}\lim_{n \rightarrow \infty} z_n &= \lim_{n \rightarrow \infty} \frac{1}{n^3} + i \lim_{n \rightarrow \infty} (1) \\&= 0 + i = i\end{aligned}$$

To get n_0 :-

$$\text{From } |z_n - z| = \left|\frac{1}{n^3}\right| < \epsilon \text{ whenever } n > \frac{1}{(\epsilon)^{1/3}}$$

$$\text{Let } \epsilon = 0.001 \Rightarrow n_0 = 1/(0.001)^{1/3}$$

✓ Series :

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \dots + z_n + \dots$$

$$S_N = \sum_{n=1}^N z_n \rightarrow N^{\text{th}} \text{ partial sol'}$$

Defn: A complex series $\sum_{n=1}^{\infty} z_n$ converges to the sum S if the sequence of partial sum $S_N, N=1, 2, \dots$ converges to S .

$$\sum_{n=1}^{\infty} z_n = S, \quad \lim_{N \rightarrow \infty} S_N = S$$

$$|S_N - S| < \epsilon \text{ whenever } N > N_0.$$

Theorem: Suppose $z_n = x_n + iy_n, n=1, 2, 3, \dots$

$$S = X + iY$$

$$\text{Then } \sum_{n=1}^{\infty} z_n = S$$

if and only if

$$\underbrace{\sum_{n=1}^{\infty} x_n}_{\text{real}} = X \quad \text{and} \quad \underbrace{\sum_{n=1}^{\infty} y_n}_{\text{real}} = Y$$

$$\cancel{\lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} z_n} = \cancel{\lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} x_n} + i \cancel{\lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} y_n}$$

$$\sum_{n=1}^{\infty} (x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

{ Proof is same as }
that in sequences

Theorem: If a series $\sum_{n=1}^{\infty} z_n$ of complex no. converges, then the n^{th} term converges to zero as $n \rightarrow \infty$

If n^{th} term of series doesn't goes to zero, then the given series is divergent.

However, this is necessary cond'n for convergence.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}$, \rightarrow divergent, even though, $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$.

PROPERTIES :

(1) The terms of a convergent series are bounded
 $\exists M > 0 \text{ s.t. } |z_n| \leq M \forall n$

(2) A series is called absolutely convergent if the series

$$\sum_{n=1}^{\infty} |z_n| \text{ converges}$$

Real series

$$\left(\sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \right)$$

Corollary

The absolute convergence of a series of complex no. implies the convergence of that series.

$$\text{Ex: } \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots \quad (\text{converges})$$

(Cauchy's Principle)

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \frac{1}{n} : \text{not convergent}$$

This type of series is called conditionally convergent.

$$\rightarrow \sum_{n=1}^{\infty} z_n = S$$

$$S_N = \sum_{n=1}^N z_n = z_1 + z_2 + \dots + z_N$$

$$\text{define } R_N = \sum_{n=N+1}^{\infty} z_n = z_{N+1} + z_{N+2} + \dots$$

remainder term

$$\text{from here, } R_N = S - S_N, \quad S_N = S - R_N$$

$$S = S_N + R_N$$

$|R_N| \rightarrow$ Error term

We can observe that $S_N \rightarrow S$, when $R_N \rightarrow 0$ (Take ϵ very small)
 $(N$ sufficiently large)

Test for Convergence (Cauchy)

Theorem : For every $\epsilon > 0$, $\exists N \geq 0$ s.t.

$$|z_{n+1} + z_{n+2} + \dots + z_{n+p}| < \epsilon \quad \forall n > N$$

and $p = 1, 2, \dots$

i) Comparison Test

$$\sum_{n=1}^{\infty} z_n$$

If we can find another convergent series $\sum b_n$ with non-negative real terms s.t.

$$|z_n| \leq b_n \quad \forall n = 1, 2, \dots$$

then $\sum z_n$ converges absolutely.

ii) Ratio Test

$$\{z_n\}_{n=0}^{\infty} \text{ with } z_n \neq 0 \quad \text{if} \lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = L$$

z_n converges if $L < 1$

diverges if $L > 1$

Test fails if $L = 1$

iii) Root test

$$\lim_{n \rightarrow \infty} |z_n|^{1/n} = L$$

same results as above.

POWER SERIES

A power series in $(z - z_0)$ is of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \quad \text{--- (1)}$$

a_n 's \rightarrow complex (or real) const.

$z \rightarrow$ complex variable

$z_0 \rightarrow$ complex (or real) const.

centre of series (1).

$$\text{If } z_0 = 0 \quad \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$$

→ Every power series converges at its centre

Convergence of Power Series

Ex. 1: Convergence in a series disk.

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \rightarrow \text{disk}$$

It converges to 0 ($z_0 = 0$) for $|z| < 1$

It diverges for $|z| > 1$

Ex. 2 $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

It converges to e^z for all z

Proof (by Ratio Test):

$$\lim_{n \rightarrow \infty} |z| \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \lim_{n \rightarrow \infty} \frac{|z|}{n+1} = 0$$

$L < 1 \Rightarrow$ it converges for all z .

Ex. 3 $\sum_{n=0}^{\infty} n! z^n$

Ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)! z^{n+1}}{n! z^n} \right| = \lim_{n \rightarrow \infty} (n+1) |z| : \text{diverges}$$

converges at $z = 0$ ($z_0 = 0$)

diverges at all other points.

Radius of Convergence:

smallest circle with centre at z_0 that includes all the points at which power series converges

$$|z - z_0| = R \rightarrow \text{circle of convergence}$$

R: radius of convergence

$R \rightarrow \infty \Rightarrow$ series converges for all z .

$R = 0 \Rightarrow$ series converges only at $z = z_0$.

Earlier :-

Ex 1 : $R = 1$

Ex 2 : $R = \infty$

Ex 3 : $R = 0$

How to find this radius of convergence ?

Cauchy - Hadamard :-

→ we only need to include coeff, not z (proof in Kreyzig)

(i) $\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = L = R$

(Ratio test)

10) $L = 0 \rightarrow R = \infty$: Series converges everywhere

11) $L = \infty \rightarrow R = 0$: series converges only at one point

(ii) $\lim_{n \rightarrow \infty} \left| \frac{1}{\sqrt[n]{a_n}} \right| = \frac{1}{L} = R$

Ex. 12) $\sum_{n=0}^{\infty} \frac{2n!}{n!^2} (z-3i)^n$. Find R ?

$$R = \lim_{n \rightarrow \infty} \left| \frac{2n! \cdot (n+1)!^2}{n!^2 \cdot (2n+2)!} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}$$

13) $= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} = \frac{1}{4}$

14) The series converges for $|z-3i| < \frac{1}{4}$.

diverges outside this disk $|z-3i| > 1/4$

⇒ 15) But, nothing can be concluded at point $|z-3i| = 1/4$
We have to calculate it separately.

How ??

Remark: $\sum_{n=0}^{\infty} a_n z^n$ — non-zero radius of convergence $R > 0$

16) then the sum func' is a func' of z

$$\sum z^n = 1 + z + z^2 + \dots = \frac{1}{1-z}, |z| < 1$$

Here, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is continuous in $|z| < R$, (if convergent in disk, it is cont. also)

Properties

- 1) we can differentiate and integrate the power series termwise, and the resulting series has same radius of convergence as your original series
(Apply ratio test to prove)
- 2) A power series with $R > 0$ represents an analytic funcⁿ at every point interior to its circle of convergence, ie, $|z - z_0| < R$.

Taylor series :

Theorem: Suppose $f(z)$ is analytic in $|z - z_0| < R_0$.

Then, $f(z)$ has a power series representation

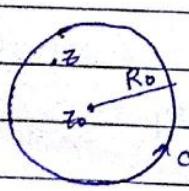
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0) \quad \text{--- ②}$$

where, $a_n = \frac{f^{(n)}(z_0)}{n!}, n=0, 1, 2, \dots$

i.e., the series ② converges to $f(z)$ when z lies in $|z - z_0| < R_0$.

By Cauchy Integral Formula,

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$$



$$\text{Here, } f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \dots$$

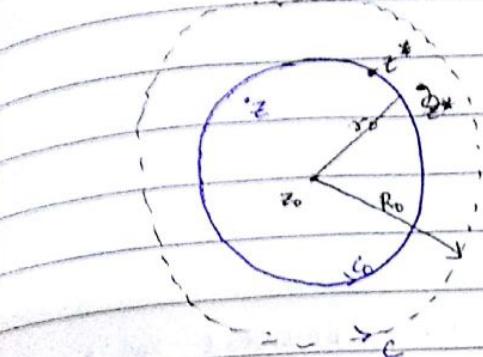
$$+ \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z)$$

↓
Remainder

remainder term is considered as error term & $\rightarrow 0$

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$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \frac{dz^*}{z^* - z}$$



We have 2 discs here:

$$|z - z_0| < R_0 : C$$

$$|z - z_0| < r_0 : C_0$$

$f(z)$ is analytic inside $|z - z_0| < R_0$

We can see that f is analytic at z

$$f(z) = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z^*)}{(z^* - z)} dz^* \quad \text{--- (4)}$$

Write $\frac{1}{z^* - z}$ in power of $(z - z_0)$

$$\frac{1}{z^* - z} = \frac{1}{(z^* - z_0) - (z - z_0)} = \frac{1}{(z^* - z_0)} \left[\frac{1}{1 - \left(\frac{z - z_0}{z^* - z_0} \right)} \right] \xrightarrow{\text{always less than 1 (from curve)}}$$

$$1 + q + q^2 + q^3 + \dots = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q} \xrightarrow{\text{that's why sum formula is used.}}$$

$$\frac{1}{1 - q} = 1 + q + q^2 + \dots + q^n + \frac{q^{n+1}}{1 - q}$$

$$\frac{1}{z^* - z} = \frac{1}{(z^* - z_0)} \left[1 + \left(\frac{z - z_0}{z^* - z_0} \right) + \left(\frac{z - z_0}{z^* - z_0} \right)^2 + \dots + \left(\frac{z - z_0}{z^* - z_0} \right)^n + \frac{\left(\frac{z - z_0}{z^* - z_0} \right)^{n+1}}{1 - \left(\frac{z - z_0}{z^* - z_0} \right)} \right]$$

$$= \frac{1}{(z^* - z_0)} \left[1 + \left(\frac{z - z_0}{z^* - z_0} \right)^n + \frac{(z - z_0)^{n+1}}{(z^* - z_0)} \right] + \frac{1}{z^* - z} \left(\frac{z - z_0}{z^* - z_0} \right)^{n+1}$$

From (4):

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)} dz^* + \frac{(z - z_0)}{2\pi i} \oint_{C_0} \frac{f(z^*)}{(z^* - z_0)^2} dz^* +$$

$$+ \frac{(z - z_0)^n}{2\pi i} \oint_{C_0} \frac{f(z^*)}{(z^* - z_0)^n} dz^* + \frac{(z - z_0)^{n+1}}{2\pi i} \oint_{C_0} \frac{f(z^*)}{(z^* - z_0)^{n+1}} \frac{dz}{z^* - z}$$

$R_n(z)$

→ Taylor's Series

$$\rightarrow f(z) = f(z_0) + (z-z_0)f'(z_0) + \dots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + P_n(z)$$

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If we can prove that $P_n \rightarrow 0$, $\Rightarrow f(z)$ converges to $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ (series) ($a_n = \frac{f^{(n)}(z_0)}{n!}$)

Proof: (Consider previous figure.)

$f(z)$ is analytic in $|z-z_0| < R$, bounded in C

we can observe from fig. $|z^* - z| > 0$

* $f(z^*)$ is analytic & bounded in C

$z^* - z$

$$\text{Assume } \left| \frac{f(z^*)}{z^* - z} \right| \leq \tilde{M}$$

C with radius r_0 is $|z^* - z| = r_0$, i.e. length = $2\pi r_0$,
length of circle.

$$|R_n| = \frac{|z^* - z_0|^{n+1}}{2\pi i} \left| \oint \frac{f(z^*)}{(z^* - z_0)^{n+1}} \frac{dz^*}{z^* - z} \right| \leq \frac{|z - z_0|^{n+1}}{2\pi} \frac{\tilde{M}}{r_0^{n+1}}$$

$\downarrow r_0 \rightarrow 0$

$$\therefore |R_n| \leq \frac{\tilde{M}}{r_0^n} |z - z_0|^{n+1} = \tilde{M} r_0 \left| \frac{z - z_0}{r_0} \right|^{n+1} \xrightarrow{(z^* - z_0)} \text{always less than 1}$$

$$\left| \frac{z - z_0}{r_0} \right| < 1 \quad (\text{from figure})$$

$$\Rightarrow |R_n| \rightarrow 0 \text{ as } n \rightarrow \infty$$

* Taylor's series is unique (proof in Kreyszig)

* By Cauchy's inequality,

$$|f^{(n)}(z_0)| = \left| \frac{1}{2\pi i} \oint \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* \right|$$

$$|a_n| \leq \frac{n! M}{r^n}, \forall n = 0, 1, 2, \dots$$

Corollary:

A power series with non-zero radius of convergence is the Taylor series of the sum $\hookrightarrow f(z)$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^n$$

(We've to prove $a_n = \frac{f^{(n)}(z_0)}{n!}$)

$$f(z_0) = a_0$$

diff. wrt z

$$f'(z) = a_1 + 2a_2(z-z_0) \dots + n a_n(z-z_0)^{n-1}$$

$$f'(z_0) = a_1$$

$$f''(z_0) = 2a_2$$

$$f^{(n)}(z_0) = n! a_n$$

$$\Rightarrow \left\{ a_n = \frac{f^{(n)}(z_0)}{n!} \right\}$$

Comparison with Real Series

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f(x)$ is differentiable for all orders

$$\underline{f^{(n)}(0) = 0 \forall n}$$

$f(x)$ doesn't have MacLaurin Series

\downarrow
(where $z_0 = 0$)

$$10x + x^2 \quad z - \frac{x^3}{3!}$$

Ex. $f(z) = e^z$

$$f^n(z) = e^z \quad \forall n=0, 1, 2, \dots$$

$$f^n(0) = 1 \quad \forall n$$

7. $e^z = \sum_{n=0}^{\infty} z^n \quad (|z| < \infty)$

8. $z = x + i\cdot 0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

Ex. $f(z) = z^2 e^{3z}$

We can use above eg. here.

$$\begin{aligned} z^2 e^{3z} &= \sum_{n=0}^{\infty} \frac{(3z)^n}{n!} \cdot z^2 = \sum_{n=0}^{\infty} \frac{3^n z^{(n+2)}}{n!} \\ &= \sum_{m=2}^{\infty} \frac{3^{m-2}}{(m-2)!} z^m \end{aligned}$$

Ex. $e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} i^n$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}$$

even odd

$$e^{iy} = \cos y + i \sin y$$

Euler's formula

Ex. $f(z) = \frac{1}{1-z}$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad n=0, 1, 2$$

30. $f(z)$ is non-analytic at $z=1$.

We can find $f^{(n)}(0)$ (analytic at $z=0$)

$$f^{(n)}(0) = n!$$

$$\frac{1}{1-(1-z)}$$

MacLaurin Series of $f(z) \equiv$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 \dots \quad |z| < 1$$

Binomial
Expansion

b) Replacing z by $-z$, we get

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 \dots \quad |z| < 1$$

c) Replacing z by $1-z$, we get

$$f(z) = \frac{1}{z}$$

$$\frac{1}{z} = 1 + (1-z) + (1-z)^2 + \dots$$

$$= \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad |z-1| < 1$$

Ex. $f(z) = \frac{1+2z^2}{z^3+z^5}$: Expand into a series involving power of z .

$$f(z) = \frac{1}{z^3} \left(\frac{1+2z^2}{1+z^2} \right) : \text{non-analytic at } z=0$$

$$= \frac{1}{z^3} \left(2 - \frac{1}{1+z^2} \right) : \text{whole is non-analytic, but we can find exp^n of } \frac{1}{1+z^2} \text{ at } z=0.$$

Putting in previous ex.

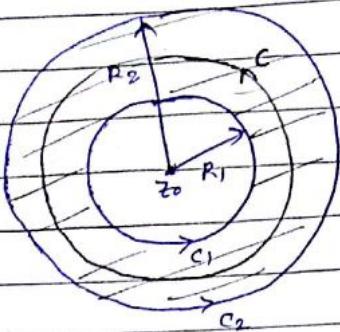
$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 \dots \quad |z| < 1$$

$$f(z) = \frac{1}{z^3} \left(2 - 1 + z^2 - z^4 + z^6 \dots \right) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 \dots$$

$f(z)$ is analytic in $0 < |z| < 1$

→ How to find out power series repⁿ for some region when $f(z)$ is non-analytic? : Laurent series.

Laurent Series



Analytic in

$$R_1 < |z - z_0| < R_2$$

D

known as Annulus Region

(we don't know anything about other regions)

$$D: p_1 < |z - z_0| < R_2$$

 $f(z)$ is analytic throughout the region D.let C be any positively oriented simple closed contour around z_0 and lying inside C .

Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ ($p_1 < |z - z_0| < R_2$)

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = 0, 1, 2, \dots$$

principal part (negative power of $z - z_0$)

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = 0, 1, 2, \dots$$

Replacing n by $-n$ in principal part, we get

$$\sum_{n=-\infty}^{-1} b_{-n} (z - z_0)^n$$

$$b_{-n} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = -1, -2, \dots$$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{-1} b_{-n} (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \quad \left. \right\} R_1 < |z - z_0| < R_2$$

We can find power series everywhere inside the annulus. This is Laurent series

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$$c_n = \begin{cases} b_n & n \leq -1 \\ a_n & n \geq 0 \end{cases}$$

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{n+1}} \quad (n = 0, \pm 1, \dots)$$

Special cases :

(i) $f(z)$ is analytic throughout $|z - z_0| < R_2$
 (2nd term in Laurent series is also valid)

$f(z) (z - z_0)^{-n-1}$ is also analytic
 analytic analytic (also at $z = z_0$)

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = 0 \quad | a_n = \frac{1}{2\pi i} \oint_C f(z) (z - z_0)^{n+1} dz$$

$$\Rightarrow f(z) = \sum a_n (z - z_0)^n$$

(ii) $f(z)$ is not analytic at $z = z_0$. Otherwise, analytic everywhere
 (R_1 you can choose so small so that it only includes z_0)

$$0 < |z - z_0| < R_2$$

$f(z)$ has Laurent series in $0 < |z - z_0| < R_2$

(iii) $f(z)$ is analytic at each point in the finite plane exterior
 to the circle $|z - z_0| = R_1$

$f(z)$ has Laurent series in $R_1 < |z - z_0| < \infty$

(iv) $0 < |z - z_0| < \infty$ (Extending (ii.))

Ex. $f(z) = z^2 e^{1/z}$: problem at $z=0$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, |z| < \infty$$

$$z^2 e^{1/z} = \sum_{n=0}^{\infty} z^2 \left(\frac{1}{n! z^n} \right), \left| \frac{1}{z} \right| < \infty \Rightarrow |z| > 0$$

$$z^2 e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^{n-2}}, |z| > 0$$

Laurent series $= z^2 + \frac{1}{2!} + \frac{1}{3! z} + \frac{1}{4! z^2} + \dots, |z| > 0$

Principal part

Ex. $f(z) = \frac{1}{1-z}$

i) Develop a power series with non-negative power of z

ii) — negative

i) $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 \dots, |z| < 1$

ii) $f(z) = \frac{-1}{z(1-\frac{1}{z})} = -\frac{1}{z} \left(\frac{1}{1-\frac{1}{z}} \right)$ using above expansion, find its expⁿ

$$= -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{(z)^n} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}, |z| < 1$$

$$= -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \dots \right), |z| > 1$$

* Series is unique

Uniqueness:

The Laurent series of a given analytic funcⁿ $f(z)$ in its specified annulus of convergence is unique.

$f(z)$ may have different Laurent series in 2 different annuli with same centre (Eg: Prev. example)

$$\text{Ex: } f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

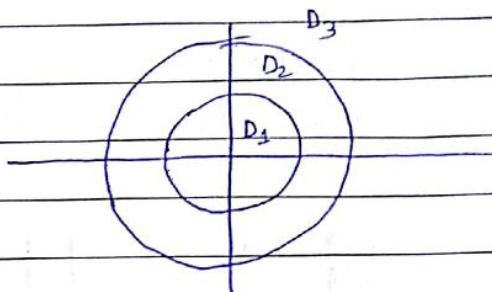
Singular at $z=1, z=2$

$f(z)$ is analytic in following domains:

1) $|z| < 1$

2) $1 < |z| < 2$

3) $|z| > 2$



1) $|z| < 1$: Find Maclaurin Series

$$|z| < 1 \Rightarrow \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{-1}{1-z} + \frac{1}{1-z/2}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n, \quad |z| < 1$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad (|z| < 1)$$

* It doesn't contain negative power ($b_n=0$) as funcⁿ is analytic everywhere in given domain D_1 .

(ii) $D_2 : 1 < |z| < 2$

$\left|\frac{1}{z}\right| < 1 \quad \& \quad \left|\frac{z}{2}\right| < 1$

take care of this while
writing $f(z)$ in next step.

$$f(z) = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}$$

(iii) $D_3 : |z| > 2$

~~$\left|\frac{z}{2}\right| > 1$~~ ~~$\sqrt{z} \neq 1$~~

$$\left|\frac{2}{z}\right| < 1 \Rightarrow \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1-\frac{2}{z})}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \rightarrow \text{we are only getting negative part.}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

Eg. $f(z) = e^{\frac{1}{z}}$, $\left|\frac{1}{z}\right| < \infty \therefore |z| > 0$ Laurent series exist for this region

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \dots \quad |z| > 0$$

diff. from ②

① Coeff. of $\frac{1}{z-z_0}$: Residue (b_1)

from $b_1 = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz \stackrel{c \rightarrow 0}{=} \frac{1}{2\pi i} \oint_C e^z dz = 1 \quad (\text{from } \exp)$$

∴ $\oint_C e^{1/z} dz = 2\pi i$: can find integral of complex func's using this method

Singularity of $f(z)$ at $z=z_0$

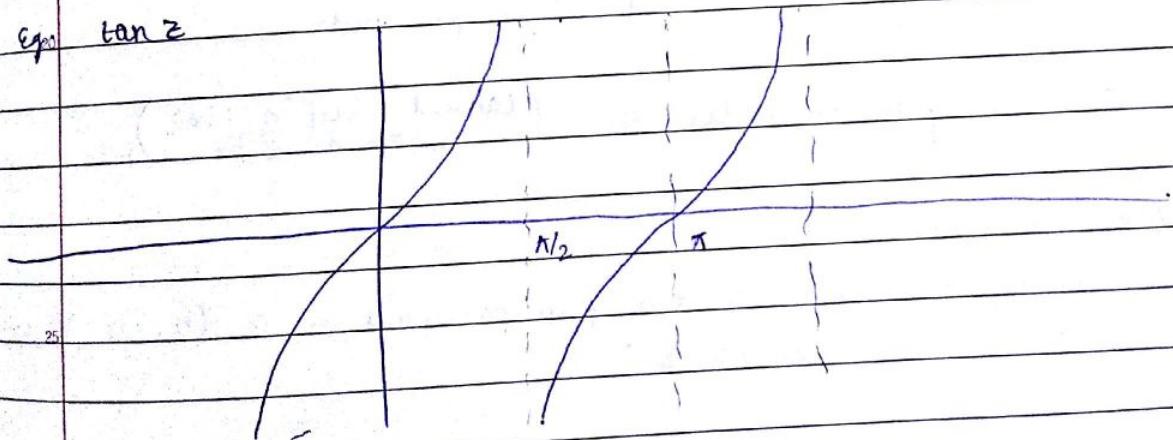
(when $f(z)$ is non-analytic at $z=z_0$)

— ①

② Isolated Singularity :

We call $z=z_0$ as an isolated singularity if $z=z_0$ has a nbd without further singularity of $f(z)$.

e.g. $\tan z$



Isolated singularity cut $z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

* See, why we are getting negative powers. (proof)

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- (i.) If the principal part contains only finitely many terms of the form

$$= \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}$$

then, the singularity of $f(z)$ at $z = z_0$ is called pole of order m .

- (ii.) If principal part contains infinite terms, then $z = z_0$ is an isolated essential singularity of $f(z)$.

Eg. $f(z) = e^{1/z}$: isolated essential singularity ($z = 0$)

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Since $f(z)$ is analytic in

$$0 < |z - z_0| < R$$

$f(z)$ has Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$\frac{1}{(z-2)^8} \quad \frac{1}{(z-2)^7} \quad \frac{1}{(z-2)^6} \quad \frac{1}{(z-2)^5} \quad \frac{1}{(z-2)^4} \quad \frac{1}{(z-2)^3} \quad \frac{1}{(z-2)^2} \quad \frac{1}{(z-2)}$$

Eg. $f(z) = \frac{1}{z(z-2)^5}$ \rightarrow Laurent series ??? $m=1 \Rightarrow$ simple pole

$z = 2$ is pole of order 5. (Laurent series of $f(z)$ around $z = 2$)

(iii) Removable singularity

If every coefficient of the principal part is 0 ($b_n = 0 \forall n$)

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$z = z_0$ is a removal singularity.

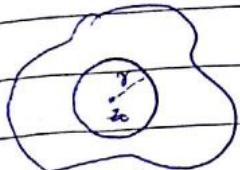
define: $f(z_0) = a_0$ to remove singularity

then $f(z)$ is analytic in entire disk $|z - z_0| < R$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
 $f(z)$ has power series only if $f(z)$ is analytic.

$f(z) = \frac{\sin z}{z}$ find Laurent series about $z=0$, & you will get $a_{-1}=1$

$z=0$ is a removable singularity.



$f(z)$ is analytic everywhere except at z_0 ($|z - z_0| < r$)

Residue :-

$f(z)$ has singularity at $z = z_0$ inside C (counter-clockwise)
 C is inside the region $|z - z_0| < R$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz$$

$$\oint_C f(z) dz = 2\pi i b_1$$

b_1 - Residue of $f(z)$ at $z = z_0$

$$\text{Res } f(z) = b_1$$

Ex. $f(z) = \frac{\sin z}{z^4}$, $C: |z|=1$ with positive orientation,

has singularity at $z=0$

$$\oint_C f(z) dz = ?$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots, |z| < \infty$$

$$\frac{1}{z^4} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-3}}{(2n+1)!} = \frac{1}{z^3} - \frac{1}{3! z} + \frac{z}{5!} - \dots, 0 < |z| < \infty$$

... nbd.

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From expansion, $b_3 = \frac{-1}{3!}$

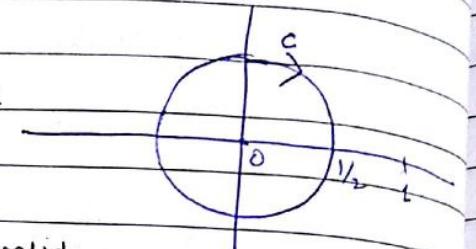
$$\oint_C f(z) dz = 2\pi i \left(\frac{-1}{3!} \right) = -\frac{\pi i}{3}$$

Ex. $f(z) = \frac{1}{z^3 - z^4}$
 C is with clockwise dirⁿ
 C: $|z| = 1/2$
 (different centre of curve)
 $= \frac{1}{z^3(1-z)}$

Singularities are: $z=0$ and $z=1$

Only $z=0$ is included in curve C

so, Laurent series about $z=0$:



$$= \frac{1}{z^3} \cdot \frac{1}{1-z}$$

→ This will be valid throughout the region.

$$= \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-3} \quad 0 < |z| < 1 \quad \begin{cases} |z| < 1 \\ \text{C: } |z|=1 \end{cases}$$

z deleted nbd

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots$$

$$b_1 = 1$$

$$\oint_C f(z) dz = -2\pi i$$

Other possible way :-

Ex. $f(z) = \frac{1}{z^3 - z^4}$

$$= \frac{-1}{z^4(1-\frac{1}{z})}$$

$$= \frac{-1}{z^4} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \quad |z| > 1$$

$$= - \sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = - \left(\frac{1}{z^4} + \frac{1}{z^5} + \frac{1}{z^6} + \dots \right)$$

Residue at $z=0 = 0$

$$\text{res}_{z=0} f(z) = 0$$

$$\oint_C f(z) dz = 0$$

this is wrong, because we have chosen wrong region

so, it is important to choose the right region.

→ $f(z)$ has a simple pole at $z = z_0$.

→ $m = 1$ = Only 1 negative term

$$\Rightarrow f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

$$(z - z_0)f(z) = b_1 + a_0(z - z_0) + a_1(z - z_0)^2 + \dots$$

$$b_1 = \lim_{z \rightarrow z_0} (z - z_0)f(z)$$

We can find residue this way without expanding series if given it has simple pole.

ex. $f(z) = \frac{9z+i}{z(z^2+1)}$

$z = i$: simple pole

$$b_1 = \lim_{z \rightarrow i} \frac{(z-i)(9z+i)}{z(z+i)(z-i)}$$

$$= \lim_{z \rightarrow i} \frac{18z}{z'(zi)} = -5i$$

~~partial~~

zero :-

$f(z)$ has a zero of order n at $z = z_0$ if

$$f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(n-1)}(z_0) = 0$$

and $f^n(z_0) \neq 0$

→ $f(z)$ is analytic & z_0 is a zero of $f(z)$

then $\frac{1}{f(z)}$ has a pole at $z = z_0$

(valid for rational funcⁿ $\frac{p(n)}{q(n)}$ also)

$|z_1| > |z_2|$ (not same as c)

Suppose $f(z) = \frac{p(z)}{q(z)}$,

$p(z)$ is analytic; $p(z_0) \neq 0$

let $q(z)$ has a simple 0 at $z = z_0$

Then, $q(z)$ has Taylor series representation at $z = z_0$

$$\text{Res}_{z=z_0} f(z) = \text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p'(z_0)}{q'(z_0)}$$

$$\frac{p(z_0)}{q'(z_0)}$$

In earlier problem,

$$\text{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \frac{9+i}{(z-i)(3z^2+1)} = \frac{(9+i)i}{-2} = \frac{-9}{2} - 5i$$

$\rightarrow f(z)$ has a pole of order $m > 1$ at $z = z_0$

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

$$\textcircled{1} \quad -(z-z_0)^m f(z) = b_m + (b_{m-1})(z-z_0) + \dots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots$$

Let $\textcircled{1} = g(z)$ ↓
Taylor series of $g(z)$
at $z = z_0$

Our aim is to get b_1 .

$$g(z) = \sum_{m=0}^{\infty} c_m (z-z_0)^m = \sum_{m=0}^{\infty} \frac{g^{(m)}(z_0)}{m!} (z-z_0)^m \quad \textcircled{2}$$

$$b_1 : \text{coeff. of } (z-z_0)^{m-1} \quad \left\{ \begin{array}{l} \text{Comparing } \textcircled{1} \text{ & } \textcircled{2} \\ \Rightarrow b_1 = \frac{g^{(m-1)}(z_0)}{(m-1)!} \end{array} \right.$$

$$b_1 = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-z_0)^m f(z) \right\}$$

Residue of $f(z)$: (One way is using Laurent series)

(i) At simple pole:

$$\text{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

(ii) At a pole of order m

$$\text{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m f(z)\}$$

Ex: $f(z) = \frac{50z}{(z+4)(z-1)^2}$

Order 2

$$\text{Res}_{z=1} f(z) = \frac{1}{1!} \lim_{z \rightarrow z_0} \frac{d}{dz} \left\{ (z-1)^2 \frac{50z}{(z+4)(z-1)^2} \right\}$$

$$= \lim_{z \rightarrow 1} \frac{(z+4)50 - 50z}{(z+4)^2} = \frac{200}{25} = 8$$

Ex: $f(z) = e^{1/z^2} \rightarrow$ Singularity at $z=0$

$\oint_C f(z) dz$ $C: |z|=1$ in counter-clockwise

Laurent Series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!}, \quad |z| < \infty$$

$$e^{1/z^2} = \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n \frac{1}{n!} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \dots \quad \left|\frac{1}{z^2}\right| < \infty$$

$z=0$: isolated singularity (Infinite terms in Laurent series) $\Rightarrow 0 < |z| < \infty$

$\text{Res}_{z=0} f(z) = 0 \quad \{ \text{coeff. of } 1/z = 0 \}$

$$\oint_C f(z) dz = 0$$

* Even though $f(z)$ is non-analytic, we are getting $I=0$

* Analyticity of $f(z)$ within and on a simple closed contour C is a sufficient condition for the value of integral around C to be zero but it is not a necessary cond'.

Ex. $\oint \frac{dz}{z(z-2)^4} dz$

$C: |z-2| = 1 \Rightarrow \frac{|z-2|}{2} < 1$

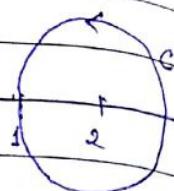
Singularity at $\underset{\downarrow}{z=0}$ & $z=2$

Not included

Laurent series in the region

$$0 < |z-2| < 2$$

↓ larger region than C



15. $\frac{1}{z(z-2)^4} = \frac{1}{(z-2)^4} \left[\frac{1}{z} + \frac{1}{(z-2)} \right]$

$$= \frac{1}{2(z-2)^4} \left[\frac{1}{1 + \frac{(z-2)}{2}} \right]$$

20. $= \frac{1}{2(z-2)^4} \left[1 - \left(\frac{z-2}{2} \right) + \left(\frac{z-2}{2} \right)^2 - \dots \right]$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^{n-4}}{2^{n+1}}$$

25. Res $f(z) = \text{coeff. of } \frac{1}{z-2} = \frac{-1}{16} \quad (n=3)$

$$\int \frac{dz}{z(z-2)^4} dz = 2\pi i \left(\frac{-1}{16} \right) = -\frac{\pi i}{8}$$

30. Alternative

$$\begin{aligned} \text{Res } f(z) &= \frac{1}{3!} \lim_{z \rightarrow 2} \frac{d^3}{dz^3} \left\{ \frac{(z-2)^4}{z(z-2)^4} \frac{dz}{z(z-2)^4} \right\} \\ &= \frac{1}{3!} \lim_{z \rightarrow 2} \frac{d^3}{dz^3} \left\{ \frac{1}{z} \right\}. \end{aligned}$$

$$\frac{1}{z} \rightarrow \frac{-1}{z^2} \rightarrow \frac{-2}{z^3} \rightarrow -$$

$$= \frac{1}{3!} \lim_{z \rightarrow 2} \left[\frac{-3!}{z^4} \right] = \frac{-1}{16}$$

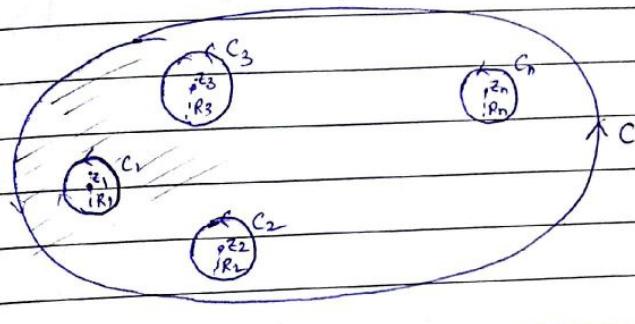
But, what if C includes finite no. of singularities?

Cauchy's Residue Theorem:

$f(z)$ has more than one singularity within a simple closed curve

Let C be simple closed contour, described in +ve sense. If a func' f is analytic inside and on C except for a finite no. of singularities z_k , $k=1, 2, \dots, n$ inside C , then (Assuming that singularities are isolated)

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z)$$



(singularities are isolated, so we can take $R > 0$. all circles are separated)

PROOF:

$f(z)$ is analytic in the multiply connected domain D bounded by C_1, C_2, \dots, C_n

Using Cauchy's Integral Formula,

$$\oint_C f(z) dz + \oint_{C_1} f(z) dz + \dots + \oint_{C_n} f(z) dz$$

$$\oint_C f(z) dz - \sum_{k=1}^n \oint_{C_k} f(z) dz = 0$$

$\left. \begin{array}{l} f=0 \text{ in region where } \\ f(z) \text{ is analytic} \end{array} \right\}$
in closed domain D

→ We are making closed domain D .
(by making clockwise C_1, C_2, \dots, C_n)

$$\oint_C f(z) dz = \sum_{k=1}^n \oint_{C_k} f(z) dz$$

$$\oint_C f(z) dz = \sum_{k=1}^n 2\pi i \operatorname{Res}_{z=z_k} f(z)$$

Ex $\oint_C \frac{5z-2}{z(z-1)} dz$ C: $|z| = 2$ described in counter clockwise dir.

singularities are: 0, 1 (Both included in C)

$$\operatorname{Res}_{z=0} f(z) = ?$$

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \left(-1 - z - z^2 - z^3 \dots \right)$$

$$= \left(5 - \frac{2}{z} \right) \left(-1 - z - z^2 - z^3 \dots \right)$$

$$\operatorname{Res}_{z=0} f(z) = 2$$

$$\operatorname{Res}_{z=1} f(z) = ?$$

need in power of $z-1$

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)}$$

$$= \left[5 + \frac{3}{(z-1)} \right] \left[1 - (z-1) + (z-1)^2 - (z-1)^3 \dots \right]$$

$$\operatorname{Res}_{z=1} f(z) = +3$$

$$\Rightarrow \oint_C f(z) dz = 2\pi i (2+3) = \pm 10\pi i$$

OR

already series in z
Laurent series

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1} \rightarrow \text{already Laurent series in } (z-1)$$

Res of f

$$\oint_C f(z) dz = 2+3 = 5$$

OR

$z=0, 1 \rightarrow$ simple pole

$$\oint_{\text{Res}} f(z) = \lim_{z \rightarrow 0} (z) f(z) = 2$$

$$\text{Res}_{z=1} \oint_{\text{Res}} f(z-1) f(z) = \lim_{z \rightarrow 1} \frac{z-2}{z} = 3$$

$$1+2+3 = 5$$

→ If $|z|=1$, we only need to calculate for $z=0$

$$\oint_C f(z) dz = 2\pi i (2)$$

23/10/17 Applications of Cauchy-P residue Thm

$$\textcircled{1} - \int_0^R f(x) dx = \lim_{R \rightarrow \infty} \int_0^R f(x) dx$$

If the R.H.S. limit exist, then the improper integral converges

$$f \in C(R)$$

$$\textcircled{2} - \int_{-\infty}^{\infty} f(x) dx = \lim_{R_1 \rightarrow \infty} \int_{-R_1}^0 f(x) dx + \lim_{R_2 \rightarrow \infty} \int_0^{R_2} f(x) dx$$

If the R.H.S. limit exist, then the L.H.S. improper integral converges to the sum

Defn: Cauchy Principal value of $\textcircled{2}$ is the no.

$$\text{PV} - \int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

provided RHS limit exists.

- * When the integral ② converges, its Cauchy principal value ③ exists and that value is the no. to which the integral ② converges.
- * CONVERGE IS NOT TRUE.

Eg.
(example
for
converge)

$$\text{PV} \int_{-\infty}^{\infty} x dx = \lim_{R \rightarrow \infty} \int_{-R}^R x dx = 0$$

Acc. to defn ② \rightarrow

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{R_1 \rightarrow \infty} \int_{-R_1}^0 x dx + \lim_{R_2 \rightarrow \infty} \int_0^{R_2} x dx$$

$$= \lim_{R_1 \rightarrow \infty} -\frac{R_1^2}{2} + \lim_{R_2 \rightarrow \infty} \frac{R_2^2}{2}$$

\Rightarrow limit D.N.E.

∴ This improper integral doesn't converge
(although principal value exists)

→ If $f(x)$ is an even func, i.e., $f(x) = f(-x) \forall x$

Suppose Cauchy principal value ③ exists, then

$$\int_{-\infty}^{\infty} f(x) dx = \text{PV} \int_{-\infty}^{\infty} f(x) dx$$

$$\int_0^{\infty} f(x) dx = \frac{1}{2} \left[\text{PV} \int_{-\infty}^{\infty} f(x) dx \right] \quad (\text{try it yourself})$$

25

$$\rightarrow f(x) = \frac{p(x)}{q(x)} \quad (\text{rational func}) \quad \begin{cases} \text{zeros of } q(x) \\ \text{poles of } f(x) \end{cases}$$

(i) $p(x)$ and $q(x)$ are polynomials of x with real coefficients
& no factors in common

(ii) $q(x)$ has no real zeros and atleast one zero above the real axis

(iii) degree of $q(x)$ is atleast 2 units more than degree of $p(x)$.

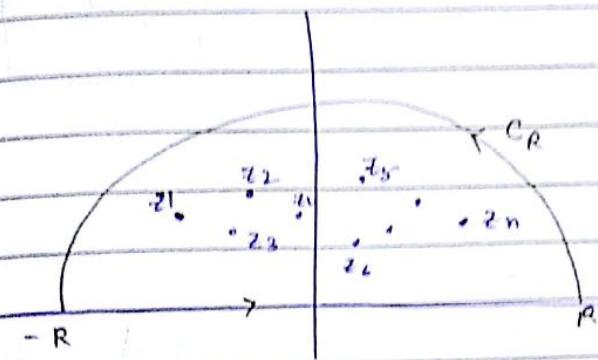
Our aim is to calculate $PV \int_{-\infty}^{\infty} f(z) dz$

Step 1

$$\text{consider } f(z) = \frac{f(z)}{q(z)}$$

(i) find all zeros of $q(z)$ above the real axis (finite no. of them)
 (ii) we assume $q(z)$ has n no. of distinct zeroes above the real axis

$$z_1, z_2, z_3, \dots, z_n \rightarrow \text{zeros of } q(z)$$



Enclose all zeroes in a simple closed contour with a C_R

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z) \quad [\text{Cauchy's Residue Thm}]$$

Here, $C \equiv \text{if } |z|=R \text{ and } z = x, -R \leq x \leq R$

$$\int_{-R}^R f(x) dx + \oint_{C_R} f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z)$$

on real axis

taking limit ~~both sides~~,

$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z) - \lim_{R \rightarrow \infty} \oint_{C_R} f(z) dz$$

$$\text{Let } z = Re^{i\theta}$$

C_R represented by $R : \text{const}$

θ ranges from 0 to π

$$|f(z)| < \frac{k}{|z|^2} \quad \left\{ \text{using point (iii.)} \right.$$

using M-L Inequality :

$$\left| \oint_{C_R} f(z) dz \right| \leq \frac{k}{R^2} (\pi R)^M = \frac{k\pi}{R}$$

$$\lim_{R \rightarrow \infty} \left| \oint_{C_R} f(z) dz \right| < 0$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_R^R f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z)$$

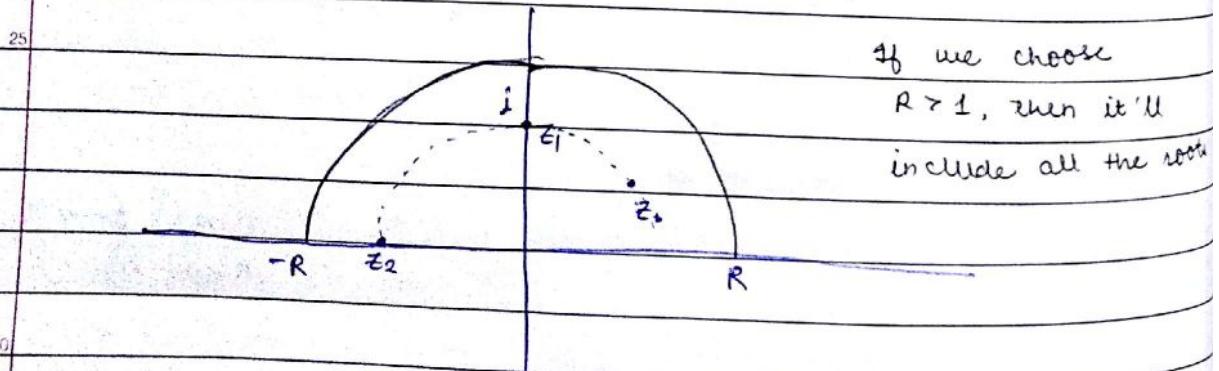
$$\text{Ex. } \int_0^\infty \frac{x^2}{x^6 + 1} dx = \frac{p(z)}{q(z)}$$

$$\text{Consider } f(z) = \frac{z^2}{z^6 + 1}$$

$$\text{zeroes of } z^6 + 1 = 0 \Rightarrow z = (-1)^{1/6}$$

$$z_k = \exp \left[i \left(\frac{\pi}{6} + \frac{2k\pi}{6} \right) \right], \quad k = 0, 1, 2, 3, 4, 5$$

(above real axis) zeroes : $z_0 = e^{i\pi/6} \checkmark (30^\circ)$ other 3 will be below the real axis.
 $z_1 = e^{i\pi/2} \checkmark (90^\circ)$
 $z_2 = e^{i5\pi/6} \checkmark (150^\circ)$



$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = 2\pi i \sum_{k=0}^2 \operatorname{Res}_{z=z_k} f(z) - \lim_{R \rightarrow \infty} \oint_{C_R} f(z) dz$$

Here, degree of $q(z) \geq 2 + p(z)$
 $\Rightarrow f$

z_k are simple poles of $f(z)$

$$\underset{\substack{\text{Res} \\ z=z_k}}{f(z)} = \lim_{z \rightarrow z_k} (z - z_k) f(z) = \frac{p(z_k)}{q'(z_k)}$$

$$\underset{\substack{\text{Res} \\ z=z_0}}{f(z_0)} = \underset{\substack{\text{Res} \\ z=z_0}}{\frac{p(z_0)}{6z^5}} = \underset{\substack{\text{Res} \\ z=z_0}}{\frac{1}{6z^3}}$$

$$= \frac{1}{6} e^{-i\pi/2} = \boxed{-\frac{i}{6}}$$

$$\underset{\substack{\text{Res} \\ z=z_1}}{f(z)} = \frac{1}{6z_1^3} = \frac{1}{6} e^{-3i\pi/2} = \boxed{\frac{i}{6}}$$

$$\underset{\substack{\text{Res} \\ z=z_2}}{f(z)} = \frac{1}{6z_2^3} = \frac{1}{6} e^{-i15\pi/6} = \frac{1}{6} e^{-i5\pi/2} = \boxed{-\frac{i}{6}}$$

$$\int_{-R}^R f(z) dz = 2\pi i \left(\frac{-i}{6} \right) = \boxed{\frac{\pi}{3}} \text{ Ans.}$$

for $\oint_{C_R} f(z) dz$:

$$\text{with } |z| = R, \quad |z|^2 = |z^2| = R^2$$

$$|z^6 + 1| \geq |z^6 - 1| = R^6 - 1$$

$$|\int f(z) dz| = \frac{|z^2|}{|z^6 + 1|} \leq M_R$$

$$\left| \int_{C_R} f(z) dz \right| \leq M_R \cdot R = \frac{R^3 \pi}{R^6 - 1}$$

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} f(z) dz \right| = 0$$

$$\int_{-R}^R f(z) dz = 2\pi i \left(\frac{-i}{6} \right) = \boxed{\frac{\pi}{3}} \text{ Ans.}$$

since $f(z)$
 is an
 even
 func"

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{3} = PV \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx$$

$$\text{also, } \int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6} \quad \{ \text{Even func"} \}$$

Improper analytic integral from Fourier analysis

$$\int_{-\infty}^{\infty} f(x) \sin ax dx, \quad \int_{-\infty}^{\infty} f(x) \cos ax dx$$

$$|\sin ax|^2 = \sin^2 ax + \sinh^2 ay$$

$$|\cos ax|^2 = \cos^2 ax + \sinh^2 ay$$

$$\sinhy = \frac{e^{ay} - e^{-ay}}{2i} : \text{gives problem when } y \rightarrow \infty$$

→ can't apply what we used in previous section for
 $\int_{-\infty}^{\infty} \frac{\operatorname{p.v.}}{q(n)} dx$.

$$\Rightarrow \int_{-R}^R f(x) \cos ax dx + i \int_{-R}^R f(x) \sin ax dx \\ = \int_{-R}^R f(x) e^{iay} dx$$

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{-yt} \cdot e^{iay}| = e^{-ay}$$

$|e^{iaz}|$ is bounded in the upper half plane $y > 0$

e.g. $\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx$

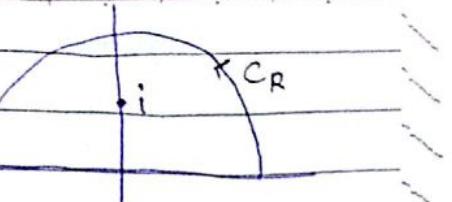
Consider $f(z) = \frac{1}{(z^2+1)^2}$: singularities: $z = \pm i$

e^{izz} $f(z)$ is analytic everywhere on and above real axis except at $z = i$

Applying Cauchy-Riemann theorem,

choose $R > 1$

Cauchy principle value! we don't want it is in general but since it is an even function (the one)



$$\lim_{R \rightarrow \infty} \int_{-R}^R f(z) \cdot e^{iz} dz = 2\pi i \left(\operatorname{Res}_{z=i} (f(z) \cdot e^{iz}) \right) - \oint_{C_F} e^{iz} f(z) dz$$

already real (on real axis)

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{iz}}{(z^2+1)^2} dz = -2\pi i \left[\frac{1}{(z-i)^2} \right]_m$$

$\rightarrow m=2$

$$\operatorname{Res}_{z=i} (f(z) \cdot e^{iz}) = \lim_{z \rightarrow i} \frac{1}{1!} \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{(z-i)^2 \cdot e^{iz}}{(z-i)^2 \cdot (z+i)^2} \right\}$$

$$= \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{e^{iz}}{(z+i)^2} \right\}$$

$$= \lim_{z \rightarrow i} \left\{ \frac{e^{iz} \cdot 3i e^{iz} (z+i)^2 - e^{iz} \cdot 2(z+i)}{(z+i)^4} \right\}$$

$$= \frac{3i e^{-3} (2i)^2 - e^{-3} \cdot 2 \cdot 2i}{16}$$

$$= \frac{3i e^{-3} (-4) - e^{-3} \cdot 4i}{16} = \frac{1}{i e^3}$$

$$\int_{-R}^R f(z) \cdot e^{iz} dz = \frac{2\pi}{e^3} - \operatorname{Res} \left[\oint_{C_F} e^{iz} f(z) dz \right]$$

z is a point on C_F

$$|z| = R$$

$$|f(z)| = \frac{1}{(z^2+1)^2} \leq \frac{1}{(R^2-1)^2}$$

$$r \cdot |e^{iz}| = e^{-3}$$

Using ML inequality,

$$|\operatorname{Res} \left[\oint_{C_F} e^{iz} f(z) dz \right]| \leq \left| \oint_{C_F} e^{iz} f(z) dz \right| \xrightarrow{\text{max-value} = 1}$$

$$\leq \frac{\pi R}{(R^2-1)^2} \rightarrow 0 \text{ as } R \rightarrow \infty$$

given funcⁿ is even funcⁿ

$$\int_{-\infty}^{\infty} \dots = \text{PV} \int_{-\infty}^{\infty} \dots$$

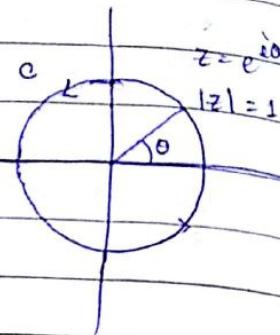
$$= \boxed{\frac{2\pi}{e^3}}$$

→ Some Definite Integrals :-

$$\int_0^{2\pi} F(\sin\theta, \cos\theta) d\theta$$

$$z = e^{i\theta}, r = 1$$

$$0 \leq \theta \leq 2\pi, R = 1$$



$$dz = ie^{i\theta} = iz$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

$$d\theta = \frac{dz}{iz}$$

⇒ the integral becomes :

$$\oint_C F\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right) \frac{dz}{iz} : \text{ If it is rational func, we can apply Cauchy Residue Theorem here.}$$

$$\text{Ex. } I = \int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} \quad z = e^{i\theta}$$

$$\oint_C \frac{dz}{iz\left(\sqrt{2} - \frac{1}{2}(z + z^{-1})\right)} = \oint_C \frac{2z dz}{i\cdot(2\sqrt{2}z - z^2 - 1)}$$

$$= \oint_C \frac{z}{i} \frac{dz}{(z - \sqrt{2} - 1)(z + \sqrt{2} + 1)} = \frac{-2}{i} \int \frac{dz}{z^2 - 2\sqrt{2}z + 1}$$

$$= -\frac{2}{i} \int \frac{dz}{(z - \sqrt{2} - 1)(z + \sqrt{2} + 1)}$$

$$z_0 = \sqrt{2} + 1 \quad z_1 = \sqrt{2} - 1 \quad \checkmark$$

ζ lies outside
the unit circle

ζ lies inside the unit circle

z_1 is simple pole

$$\text{Res } f(z) = \lim_{z \rightarrow z_1} (z - z_1) \cdot \frac{1}{(z - z_1)(z + \sqrt{2} + 1)}$$

$$= \frac{1}{\sqrt{2} - 1 - \sqrt{2} - 1} = -\frac{1}{2}$$

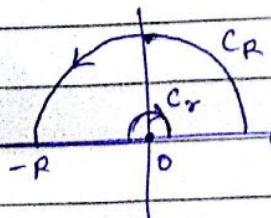
$$\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z + \sqrt{2} + 1)} = 2\pi i \left(-\frac{1}{2}\right)$$

$$\frac{\pi - 2}{i} \oint_C \frac{dz}{(z - \sqrt{2} - 1)(z + \sqrt{2} + 1)} = -\pi i \times \frac{-2}{i} = \boxed{2\pi} \text{ Ans.}$$

Eg: If we have $\cos 2\theta$

$$\cos 2\theta = \frac{e^{i2\theta} - e^{-i2\theta}}{2} = \frac{z^2 - \bar{z}^2}{2}$$

Intended path: if singularity at real axis ($z=0$)



We can handle this by excluding 0 point.

Cauchy Principal Value:

$$\text{PV} \left(\int_{-\infty}^{\infty} f(x) dx \right) = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{R_1 \rightarrow 0} \int_{-R_1}^0 f(x) dx + \lim_{R_2 \rightarrow \infty} \int_0^{R_2} f(x) dx$$

Both of these are equal when $f(x)$ is even func.

Proof: $f(x) = f(-x)$

$$\lim_{R \rightarrow \infty} \int_{-R_1}^0 f(x) dx, \quad \text{let } x = -y \\ dx = -dy$$

$$= \lim_{R_1 \rightarrow \infty} - \int_{R_1}^0 f(-y) dy = \lim_{R_1 \rightarrow \infty} \int_{R_1}^0 f(-y) dy$$

$$= \lim_{R_1 \rightarrow \infty} \int_0^{R_1} f(-x) dx = \lim_{R_1 \rightarrow \infty} \int_0^{R_1} f(x) dx \\ \downarrow \text{even func}$$

$$= \lim_{R_2 \rightarrow \infty} \frac{1}{2} \int_{-R_2}^{R_2} f(x) dx \quad \text{--- (1)}$$

similarly, $\lim_{R_2 \rightarrow \infty} \int_0^{R_2} f(x) dx = \lim_{R_2 \rightarrow \infty} \frac{1}{2} \int_{-R_2}^{R_2} f(x) dx \quad \text{--- (2)}$

$$\text{--- (1)} + \text{--- (2)} = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = \text{PV} \int_{-\infty}^{\infty} f(x) dx$$

Hence Proved

PDE → 1 independent variable
PDE → $\partial z / \partial x = f(x, y)$ "

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PARTIAL DIFFERENTIAL EQUATIONS

T. Amarnath
Kreysig
Shneden

- APP's of PDE :

1. Black - Scholes Eqⁿ : used in mathematical finance to predict value of given stock.

$$f_t + r S f_s + \sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} = rf$$

f : f(s)

r : risk free rate of return

σ : Volatility const.

2. Navier - Stoke's Eqⁿ : used in fluid mechanics

3D

$$u_t + (u \cdot \nabla) u = -\nabla p + \nu \nabla^2 u$$

$$\nabla \cdot u = 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

2/10/12

PDE :-

1st order PDE is of the form:

① — $f(x, y, z, z_x, z_y) = 0$

\downarrow
independent variable

$z = z(x, y) \leftarrow$ dependent variable

$$z_x = \frac{\partial z}{\partial x}, \quad z_y = \frac{\partial z}{\partial y}$$

2nd Order PDE :

$$f(x, y, z(x, y), z_{xx}, z_{xy}, z_{yy}, z_x, z_y) = 0$$

- PDE with more than 2 independent variables :

$$f(x, y, t, \dots, z, z_{xx}, z_{yy}, z_{tt}, \dots) = 0$$

$$z = z(x, y, t, \dots)$$

Order of PDE : Highest order partial derivative which appears in the PDE.

Eg. $z_{xx} + 2x z_x^3 + z_y = 0$ Order : 2
 ↳ Quasilinear & semi-linear

Classification of PDEs1) Quasilinear PDE

PDE is said to be Quasilinear if the highest order derivatives are linear.

2) Semi-linear PDE

A quasilinear PDE is semi-linear if the coefficients of highest order derivatives do not contain dependent variable or its derivative.

* All ~~semi~~ semi-linear are ~~not~~ quasilinear

3) Linear Eqⁿ

A semilinear PDE is said to be linear, if it is linear in the linear dependent variable & its derivatives.

4) Non-linear

If a PDE is not quasilinear, then it is non-linear.

All linear are semilinear
All semilinear are quasilinear

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e.g. highest order

$$u_{xxx} + u_t = xt \Rightarrow \text{2nd Order quasilinear PDE}$$

$$u = u(x, t) : \text{Quasi linear } \checkmark$$

Not semilinear

Not linear

$$x u_{xx} + u u_t = xt$$

not linear ~~independent~~ (product of dependent variables)

2nd Order semi linear \Leftrightarrow PDE

$$u_{xxx} + u_t = xt$$

and Order Linear PDE.

$$(u_{xx})^2 + u_t = xt : \text{Not quasilinear} \Rightarrow \text{Non-linear}$$

2nd order PDE

$$u_{xx} + (u_t)^2 = xt : \text{2nd order semi linear PDE.}$$

$\overline{\text{Not linear}}$

Classification of 1st order PDE:

1) Quasilinear PDE $\rightarrow p$ & q have to be linear

$$f(x, y, z, p, q)$$

$$p = z_x \quad q = z_y \quad z = z(x, y)$$

$$z^2 x p + e^z y q = x^2 \sin y$$

~~non zero~~

$$P(x, y, z) p + Q(x, y, z) q = R(x, y, z) \Rightarrow$$

2) semi linear PDE: \nearrow not funcⁿ of z

$$P(x, y) p + Q(x, y) q = R(x, y, z) \Rightarrow$$

$$xy^2 p + e^x \sin y q = z^3 \sin y$$

3) linear PDE: diff. b/w 2) & 3) \Rightarrow $R(x, y, z)$ is allowed
 \downarrow to be non-linear

$$P(x, y) p + Q(x, y) q = R(x, y) + z S(x, y)$$

note: contains
2, 3

4) Non-linear PDE

$$f(x, y, z, p, q) = 0$$

$$\text{Eg. } pq = 0.$$

Originating 1st Order PDE

Consider an eqn:

 x, y : independent

$$x^2 + y^2 + (z - c)^2 = a^2 \quad \therefore \text{eqn of sphere}$$

diff. wrt x :

$$2x + 2y \frac{\partial}{\partial x} + 2(z - c) \frac{\partial}{\partial z} = 0 \quad \text{--- (1)}$$

Similarly,

$$2x \frac{\partial}{\partial y} + 2y + 2(z - c) \frac{\partial}{\partial z} = 0 \quad \text{--- (2)}$$

$$2x + 2(z - c)p = 0$$

$$2y + 2(z - c)q = 0$$

$$\Rightarrow ap_x + (z - c)pq = 0$$

$$pq + (z - c)pq = 0$$

$$\boxed{q_x - p_y = 0} \quad \text{1st order linear PDE}$$

Characterizes the eqn of spheres
with center at θ z-axis.

Ex. → Consider an eqn of the form:

$$(1) \quad F(x, y, z, a, b) = 0 \quad a, b \rightarrow \text{const.}$$

diff. wrt. x : $x, y \rightarrow$ independent variable

$$\frac{\partial F}{\partial x} \frac{\partial}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial}{\partial z} = 0$$

1 2 ↴
 p

$$\Rightarrow \frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} = 0 \quad \text{--- (2)}$$

diff. wrt. y :

$$\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} = 0 \quad \text{--- (3)}$$

$F, a \& b$ need to be eliminated from these 3 eqns.
 we get a PDE :

$$f(x, y, z, p, q) = 0 \quad (\text{non-linear PDE})$$

(use above eqn here)

Surface of Revolution

All the surfaces with z -axis as the axis of revolution
 are of the form:

$$\textcircled{1} \quad z = F(r) \quad r = \sqrt{x^2 + y^2}$$

diff. wrt x ,

$$p = z_x = \frac{\partial F}{\partial r} \frac{\partial r}{\partial x} = F'(r) \cdot \frac{1}{2} (x^2 + y^2)^{-1/2} \cdot 2x$$

$$p = \left(\frac{x}{r}\right) F'(r) \quad \text{--- (5)}$$

diff. wrt y ,

$$q = z_y = \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial y} = F'(r) \cdot \frac{1}{2} (x^2 + y^2)^{-1/2} \cdot 2y$$

$$q = \left(\frac{y}{r}\right) F'(r) \quad \text{--- (6)}$$

$$\Rightarrow \boxed{py - xq = 0} \quad \text{again, linear 1st order PDE}$$

10/10/17 In general, consider the surface of the form

$$F(u, v) = 0 \quad \text{--- (1)}$$

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are two known
 func' of $x, y \& z$.

$x, y \rightarrow$ independent variable

Diff. ~~eqn~~ (1) wrt x

$$\frac{\partial F}{\partial u} \left[\frac{\partial u}{\partial x} \cdot \frac{\partial}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial}{\partial z} \right] +$$

$$\frac{\partial F}{\partial v} \left[\frac{\partial v}{\partial x} \cdot \frac{\partial}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial}{\partial y} + \frac{\partial v}{\partial z} \cdot \frac{\partial}{\partial z} \right] = 0$$

→ 2-D PDE
 1st order \Rightarrow 1-D solⁿ
 2nd order \Rightarrow 2-D solⁿ
 → But in PDE, it is not so

$$\Rightarrow \frac{\partial F}{\partial u} \left[\left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) \right] = 0 \quad \text{--- (2)}$$

Similarly, diff. wrt y , we get:

$$\frac{\partial F}{\partial v} \left[\left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) + \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right) \right] = 0 \quad \text{--- (3)}$$

Eliminate $\frac{\partial F}{\partial u}$ & $\frac{\partial F}{\partial v}$ from eqⁿ (2) & (3).

10.	$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}$	Not Non Linear P.D.E. (could be Quadratic semi-linear / linear)
-----	---	--

where $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

(Jacobian of u, v w.r.t. x & y)

15.

Ex. If consider : $(x-a)^2 + (y-b)^2 + z^2 = 1$ [1st type]
 leads to non-linear PDE.

sphere

Verify it.

2.) $z = x^n f(\theta/x)$ [2nd type] \hookrightarrow rotation
 20. leads to a 1st order PDE.

Verify it.

Find the PDE which characterizes the above surface?

25.

30.

System of Surfaces

$$f(x, y, z, c) = 0 \quad \xrightarrow{\text{parameter}} \quad (4)$$

$$f(x, y, z, a, b) = 0 : 2 \text{ parameter} \quad (5)$$

Q4) Envelope of one-parameter system :

The surface determined by eliminating the parameter 'c' between the eqⁿ :

$$f(x, y, z, c) = 0 \quad \& \quad \frac{\partial}{\partial c} f(x, y, z, c) = 0$$

is called the envelope of one-parameter system (4).

e.g. $x^2 + y^2 + (z-c)^2 = 1$

diff. w.r.t. c

$$F(x, y, z, c) = x^2 + y^2 + (z-c)^2 - 1 = 0$$

$$\frac{\partial}{\partial c} F(x, y, z, c) = -2(z-c) = 0 \Rightarrow [z=c]$$

Substituting $z=c$, envelope of above eqⁿ is :

$$x^2 + y^2 = 1 \quad (\text{unit circle on } x-y \text{ plane})$$

\hookrightarrow Cross-section of system of surfaces
(cone, sphere)

Defⁿ) Envelope of two-parameter system :

Consider system of surfaces (5)

The surface obtained by eliminating a, and b from eqⁿ's

$$f(x, y, z, a, b) = 0, \quad \frac{\partial f}{\partial a} = 0 \quad \text{and} \quad \frac{\partial f}{\partial b} = 0$$

is called the envelope of two-parameter system (5).

$$\text{Eq: } (x-c)^2 + (y-d)^2 + z^2 = 1$$

$$f(x,y,z,c,d) = (x-c)^2 + (y-d)^2 + z^2 - 1 = 0$$

$$\frac{\partial f}{\partial c} = 2(x-c) = 0 \Rightarrow [x=c]$$

$$\frac{\partial f}{\partial d} = 2(y-d) = 0 \Rightarrow [y=d]$$

→ The envelope is : $z^2 = 1$ or $z = \pm 1$ [^{2 parallel} planes]

Solution of 1st Order PDE

Consider

$$f(x, y, z, p, q) = 0 \quad \text{where } p = z_x, q = z_y \quad (6)$$

- (i) A func' $z = z(x, y)$ should satisfy eqn (6)
- (ii) Since the func' z is continuously differentiable on $(x, y) \in D \subset \mathbb{R} \times \mathbb{R}$ (because p & q must exist)

A solⁿ $z = z(x, y)$ exists in 3D space (ie, $(x, y, z(x, y))$) which can be interpreted as surface & hence, is called integral surface of PDE (6).

Classification based on solutions :-

1. Complete Integral or Complete solⁿ:

$f(x, y, z, a, b) = 0$ lead to PDE of 1st order. Any such relation which contain 2 arbitrary const. a & b , and is a solⁿ of a 1st order PDE is said to be complete integral.

$$\text{Ex: } x^2 + y^2 + (z-c)^2 = a^2$$

solⁿ : $qx - py = 0$: 1st order

2. General Integral or General solⁿ:

$f(u, v) = 0$ provides a solⁿ of 1st order PDE. known as General Integral

$$\text{Ex. } \frac{\partial f}{\partial u} = 0, \frac{\partial f}{\partial v} = 0 \Rightarrow \frac{\partial f}{\partial u} + p \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

will give rise to $f(u, v) = 0$

singular Integral :

The solⁿ obtained from the envelope of the two-parameter family is known as singular Point. This is obtained by eliminating a & b from

$$z = F(x, y, a, b), \quad \frac{\partial F}{\partial a} = 0, \quad \frac{\partial F}{\partial b} = 0$$

$$z - px - qy - p^2 - q^2 = 0$$

$$z = p^2 + q^2 + px + qy$$

$ax + by + a^2 + b^2$ is a complete integral.

$$z = F(x, y, a, b)$$

Find singular solⁿs:

$$\frac{\partial F}{\partial a} = x + 2a = 0, \quad \frac{\partial F}{\partial b} = y + 2b = 0$$

$$\Rightarrow x = -2a$$

$$y = -2b$$

$$\therefore 4z = -(x^2 + y^2) \quad \text{--- singular solⁿ.}$$