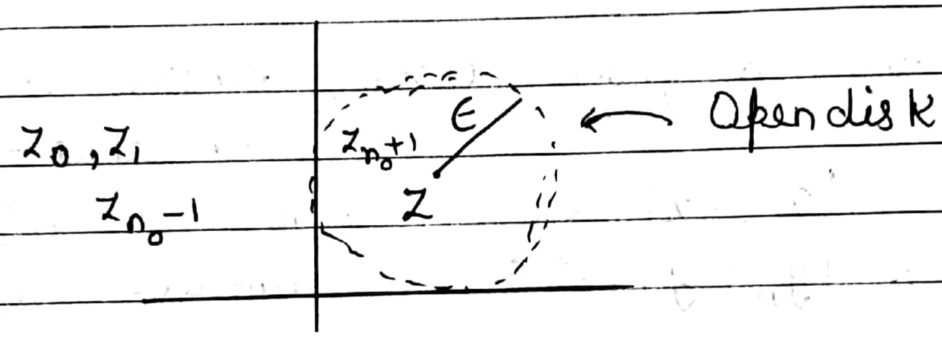
Sequence & Series in Complex Numbers

*** Convergence of a sequence:-** A sequence $\{z_n\}$ of complex number is said to be convergent to z, iff for every $\epsilon > 0$, $\exists$ $n \in N$ such that-

$$|z_n - z| < \epsilon \quad \text{whenever } n \geq n_0$$

Or, $\lim_{n \to \infty} z_n = z$



e.g. (i) $\left\{ \frac{i^n}{n} \right\}$; $n \in N$

$$= \left\{ i, -\frac{1}{2}, \frac{-i}{3}, \frac{1}{4}, \ldots \right\}$$

As $n \to \infty$, $\left\{ \frac{i^n}{n} \right\} \to 0$ (Converging)

Alternatively,

$$\left| \frac{i^n}{n} - 0 \right| < \epsilon \quad , n \geq 0$$

(ii) $\left\{ \frac{1}{n^2} + i \right\}$

$$\left\{ \frac{1}{n^2} + i \right\} \to i \quad \text{for all} \quad n \geq \frac{1}{\sqrt{\epsilon}}$$

$$\left| \frac{1}{n^2} + i - i \right| < \epsilon \Rightarrow n^2 > \frac{1}{\epsilon} \Rightarrow n \geq \frac{1}{\sqrt{\epsilon}}$$

(iii) $\{i^n\}$ → divergent

★ **Theorem:-** Suppose that $z_n = x_n + i y_n$ be a sequence & $z = x + iy$ then-

$$\lim_{n\to\infty} z_n = z \quad \text{iff} \quad \text{—— ①}$$

$$\text{(i)} \lim_{n\to\infty} x_n = x \quad \& \quad \text{(ii)} \lim_{n\to\infty} y_n = y \quad \text{—— ②}$$

Proof:- Suppose eqⁿ ② holds.

For a given $\epsilon > 0$, there exist $n_1, n_2 \in N$ such that -

$|x_n - x| < \epsilon$ whenever $n \geq n_1$

& $|y_n - y| < \epsilon$ whenever $n \geq n_2$

Let $n_0 \geq \max\{n_1, n_2\}$

then-

$|x_n - x| < \epsilon$ whenever $n \geq n_0$

$|y_n - y| < \epsilon$ whenever $n \geq n_0$

Now,

$$|z_n - z| = |(x_n + iy_n) - (x + iy)| = |x_n - x + i(y_n - y)|$$

$$\leq |x_n - x| + |y_n - y| \quad \text{[Triangular inequality]}$$

$$\leq \epsilon + \epsilon$$

$$\leq 2\epsilon \quad \text{whenever } n \geq n_0$$

Hence, $z_n \to z$.

Conversely,
Assume eqⁿ ② holds.
i.e. for a given $\epsilon > 0$, $\exists$ a $n_0 \in N$ such that -

$$|z_n - z| < \epsilon \quad \text{whenever } n \geq n_0$$

Now,

$$|x_n - x| \leq |x_n - x + i(y_n - y)|$$
$$\leq |(x_n + iy_n) - (x + iy)|$$
$$\leq |z_n - z| < \epsilon \quad \text{whenever } n \geq n_0$$

$\therefore \; x_n \to x$

Similarly,

$y_n \to y$

e.g. $z_n = -2 + i\dfrac{(-1)^n}{n^2}$

Here $x_n = -2$ & $y_n = \dfrac{(-1)^n}{n^2}$

$\downarrow$ bounded sequence

$x_n \to -2$

$y_n \to 0$

$\therefore z_n \to -2 + 0$
$\to -2$

* Convergence of an infinite series :-

$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots\cdots$ ~~cover~~ converges

iff sequence of partial sum i.e.

$S_n = \sum_{n=1}^{n} z_n = z_1 + z_2 + \cdots z_n$ converges.

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$S_1 = Z_1$

$S_2 = Z_1 + Z_2$

$\vdots$

$S_n = Z_1 + Z_2 + \cdots Z_n$

$\{S_n\} \rightarrow$ Sequence of partial sum

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{n=1}^{n} Z_n = \sum_{n=1}^{\infty} Z_n$$

Remainder term :-

$$\mathcal{J}_n = Z_{n+1} + Z_{n+2} + \cdots$$

$$|S_n - S| = |Z_{n+1} + Z_{n+2} + \cdots\cdots|$$

$$= |\mathcal{J}_n|$$

Note :- An infinite series $\sum_{n=1}^{\infty} z_n$ converges iff sequen[ce] of remainder i.e. $\{\mathcal{J}_n\} \rightarrow 0$ as $n \rightarrow \infty$.

e.g. $\sum_{n=1}^{\infty} z^n = \frac{1}{1-z}$; $|z| < 1$

$$1 + z + \cdots\cdots z^n = \frac{1 - z^{n+1}}{1-z} \qquad z \neq 1$$

Partial sum $\rightarrow S_n = 1 + z + \cdots\cdots z^{n-1}$

$$= \frac{1 - z^n}{1-z} \qquad \& \; S = \frac{1}{1-z}$$

Remainder $\mathcal{J}_n = S_n - S$

$$= \frac{1-z^n}{1-z} - \frac{1}{1-z} = \frac{z^n}{1-z} \rightarrow 0 \text{ when } |z|<1$$

For $|z| > 1$, $\sum_{n=1}^{\infty} z^n$ diverges.

Theorem:- Let $z_n = x_n + i y_n$

& $S = x + iy$, then -

$$\sum_{n=1}^{\infty} z_n = S \quad \text{iff}$$

(i) $\sum_{n=1}^{\infty} x_n = x$ & (ii) $\sum_{n=1}^{\infty} y_n = y$

Theorem:- Let $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$

Proof:- Let $S_n = \sum_{j=1}^{n} z_j$ & $S_{n-1} = \sum_{j=1}^{n-1} z_j$

And, we know,

$$z_n = S_n - S_{n-1}$$

$$\boxed{\text{Consider } \lim_{n\to\infty} \sum_{j=1}^{n} z_j = z}$$

$$\Rightarrow \lim_{n\to\infty} z_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} z_j - \lim_{n\to\infty} \sum_{j=1}^{n-1} z_j$$

$$= z - z = 0 \quad \text{— H.P.}$$

★ Absolute Convergence:- " " " " An infinite series $\sum_{n=1}^{\infty} z_n$ is called absolute convergent if $\sum_{n=1}^{\infty} |z_n|$ converges.

If $\sum z_n$ converges but $\sum |z_n|$ does not converge, then $\sum z_n$ is said to converge conditionally.

<u>Note:-</u> Absolute convergence $\Rightarrow$ Convergence
$\not\Leftarrow$

e.g. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n} \rightarrow$ converges conditionally.

☆ <u>Various test for convergence:-</u>

(i) <u>Comparison Test:-</u>

Let we have $\sum_{n=1}^{\infty} z_n$ and a convergent series of positive terms say $\sum b_n$ such that -

$$|z_n| \leq b_n \quad \forall\, n \in N$$

then - $\sum z_n$ converges absolutely.

(ii) <u>Ratio Test:-</u>

Let $\sum z_n$ be a series such that -

$$\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = l \text{ then}$$

(a) Series $\sum z_n$ converges absolutely if $l < 1$

(b) Series diverges if $l > 1$

(c) Test fails if $l = 1$.

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e.g. $\sum_{n=1}^{\infty} \frac{(100+75i)^n}{n!}$

$= \lim_{n\to\infty} \left| \frac{(100+75i)^{n+1}}{(n+1)!} \times \frac{n!}{(100+75i)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{100+75i}{n+1} \right| = 0 < 1$

↳ Series converges absolutely.

(iii) <u>Root test</u> :-

Let $\sum z_n$ be a series such that -

$\lim_{n\to\infty} |z_n|^{1/n} = l$ then -

(a) Series $|z_n|$ converges absolutely if $l<1$.
(b) Series diverges if $l>1$
(c) Test fails if $l=1$

e.g. $\sum_{n=1}^{\infty} \frac{(3i)^n}{(n)^n}$

$\lim_{n\to\infty} \left| \left[\frac{(3i)^n}{n^n} \right]^{1/n} \right| = \lim_{n\to\infty} \left| \frac{3i}{n} \right| \to 0$

↳ convergent

★ <u>Power Series</u> :-

An infinite series of the form -

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

is called Power Series.

Where,

$z_0, a_n \rightarrow$ Complex constants

$[z_0 \rightarrow$ Centre$]$

& $z \rightarrow$ Any point in stated region

e.g. (i) $\sum_{n=0}^{\infty} z^n$ ~~Here $a_n = 1$, $z_0 = 0$~~

Here $a_n = 1$, $z_0 = 0$

converges $\rightarrow |z| < 1$

diverges for $|z| > 1$

↳ Can be done by Ratio test too.

(ii) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

$a_n = \frac{1}{n!}$, $z_0 = 0$

By ratio test

$$\lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \times \frac{n!}{z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = 0$$

↓

converges everywhere in the finite complex plane.

(iii) $\sum_{n=0}^{\infty} z^n \lfloor n$

$a_n = n!$, $z_0 = 0$

By ratio test

$$\lim_{n\to\infty} \left| \frac{z^{n+1}}{z^n} \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} |z| \, |n+1|$$

$= 0 < 1$ only for $z = 0$.

* **Radius of Convergence:-** The smallest circle with centre z_0 that includes all points at which a given power series converges is called Circle of Convergence and its radius is called radius of convergence.

$$|z - z_0| < R$$

[Inside Circle → Series convergent
Outside Circle → Divergent
On boundary → May converge or diverge]

Given:-
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \rightarrow \text{Power Series}$$

By Ratio test →
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} (z - z_0) \right| = l$$

Power Series converges if $l < 1$

$$\therefore \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |z - z_0| < 1$$

$$\Rightarrow |z - z_0| < \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$\therefore |z - z_0| < R$$

$$\boxed{R = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}}$$ → Radius of Convergence

By using root test ⇁

$$\boxed{R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}}$$

e.g. $\sum_{n=0}^{\infty} \frac{n!}{n^n} (z+\pi)^n$

Here $z - z_0 = z+\pi$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \qquad a_n = \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \left| \frac{n!}{n^n} (z+\pi)^n \times \frac{(n+1)^{n+1}}{(n+1)!\,(z+\pi)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right| \times \frac{1}{\cancel{|z+\pi|}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$|z - z_0| < R$$

$$|z+\pi| < e$$

☆ Taylor's series :-

Let function $f(z)$ be analytic throughout a disk C contained at z_0. Then $f(z)$ can be expressed as a power series expansion.

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad n = 0, 1, 2 \ldots$$

$\omega \rightarrow$ point of C_1

$z \rightarrow$ point inside C_1

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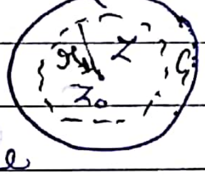
$$\therefore f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

Proof :- 'C' $\rightarrow$ Circle with z_0 at centre oriented anti-clockwise.



Now, construct another circle C_1 centred at z_0 enclosing z.

By using Cauchy's Integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\omega)}{\omega - z}\, d\omega \quad \text{--- (1)}$$

$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{(\omega - z_0)}\left[\frac{1}{1 - \left(\frac{z - z_0}{\omega - z_0}\right)}\right]$$

Note :- $\sum_{n=0}^{N} z^n = \frac{1 - z^n}{1 - z}$

$$\Rightarrow \frac{1}{1 - z} = \sum_{n=0}^{N} z^n + \frac{z^n}{1 - z} \rightarrow \text{Remainder}$$

So,

$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0)}\left[1 + \left(\frac{z - z_0}{\omega - z_0}\right) + \left(\frac{z - z_0}{\omega - z_0}\right)^2 + \dots + \left(\frac{z - z_0}{\omega - z_0}\right)^{n-1} + \left(\frac{z - z_0}{\omega - z_0}\right)^{n} \times \frac{1}{1 - \left(\frac{z - z_0}{\omega - z_0}\right)}\right]$$

$$= \frac{1}{(\omega - z_0)} + \frac{z - z_0}{(\omega - z_0)^2} + \frac{(z - z_0)^{n-1}}{(\omega - z_0)^n} + \dots + (z - z_0)^n$$

Remainder term

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So,

$$\frac{1}{\omega - z} = \frac{1}{\omega - z_0}\left[1 + \left(\frac{z-z_0}{\omega-z_0}\right) + \left(\frac{z-z_0}{\omega-z_0}\right)^2 + \dots \left(\frac{z-z_0}{\omega-z_0}\right)^n + \underbrace{\left(\frac{z-z_0}{\omega-z_0}\right)^n \frac{1}{1-\left(\frac{z-z_0}{\omega-z_0}\right)}}_{\text{Remainder Part}}\right]$$

$$= \frac{1}{\omega - z_0} + \frac{z-z_0}{(\omega-z_0)^2} + \dots \frac{(z-z_0)^{n-1}}{(\omega-z_0)^n} + \frac{(z-z_0)^n}{(\omega-z_0)^n} \times \frac{1}{(\omega - z)}$$

Thus from ①,

$$f(z) = 2\pi i \int_{C_1} \frac{f(\omega)}{\omega - z_0}\, d\omega + \frac{(z-z_0)}{2\pi i}\int_{C_1}\frac{f(\omega)}{(\omega-z_0)}\, d\omega + \dots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_{C_1}\frac{f(\omega)}{(\omega-z_0)^n}\, d\omega + \underbrace{\frac{1}{2\pi i}\int\left(\frac{z-z_0}{\omega-z_0}\right)^n \frac{f(\omega)}{(\omega-z)}\, d\omega}_{X_n}$$

And we know,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

So,

$$f(z) = f(z_0) + \frac{(z-z_0)}{\underline{|1}} f'(z_0) + \frac{(z_0 - z_0)^2}{\underline{|2}} f''(z_0) + \dots \frac{(z-z_0)^{n-1}}{(n-1)} f^{(n-1)}(z_0) + X_n$$

Now, we need to prove, as $n \to \infty$, $X_n = 0$.

Clearly, $\left|\frac{z-z_0}{\omega - z_0}\right| = \gamma < 1$

Let $|f(\omega)| \le M$ on C_1

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$$w - z = (w - z_0) - (z - z_0) = r_1 - (z - z_0)$$

Radius of inner circle

$$|X_n| = \left| \frac{1}{2\pi i} \int_{C_1} \left(\frac{z-z_0}{w-z_0}\right)^n \frac{f(w)}{(w-z_0)} dz \right|$$

$$\leq \frac{1}{2\pi} \int_{C_1} \frac{\gamma^n M}{r_1 - (z-z_0)} \times 2\pi r_1$$

$$\cancel{\leq \frac{1}{2\pi} \int_{C_1} \gamma^n M}$$

$$\leq \int_{C_1} \frac{\gamma^n M}{r_1 - (z-z_0)} \times r_1$$

Since $\gamma < 1$,

$\therefore$ As $n \to \infty$

$|X_n| \to 0$

$$\lim_{n\to\infty} |X_n| = 0$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Note:- If $z_0 = 0$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ & $a_n = \frac{f^{(n)}(0)}{n!}$

Or, $f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \dots$

$\downarrow$

Mclauren expansion

e.g. $f(z) = e^z$; express it in Taylor's series form.

$$f^{(n)}(z) = e^z \quad \forall \; n \in N$$

$\Rightarrow f^{(n)}(0) = 1$ & $a_n = \frac{f^{(n)}(0)}{n!}$

$= \frac{1}{n!}$

And, $f(z) = \sum_{n=0}^{\infty} a_n z^n$

So, $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$; $|z| < \infty$

e.g. $\sinh z = -i \sin iz$

Entire function

$= -i \frac{e^{-z} - e^{z}}{2i} = \frac{e^{z} - e^{-z}}{2}$

$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!}\right]$ when $(|z| < \infty)$

(using previous result)

$= \frac{1}{2}\sum_{n=0}^{\infty}\left[1 - (-1)^n\right]\frac{z^n}{n!}$

Or, $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ when $|z| < \infty$

Q Find out Taylor series expansion for $f(z) = \frac{1}{1-z}$

Ans We know, $f(z) \rightarrow$ Analytic for $|z| < 1$

$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$, $n = 0, 1, 2, \ldots$

$f^{(n)}(0) = n!$

$a_n = \frac{f^{(n)}(0)}{n!} = 1$

$$\therefore f(z) = \sum a_n z^n = \sum_{n=0}^{\infty} z^n \qquad |z|<1$$

Q $$f(z) = \frac{1}{1+z}$$

Ans $$f(z) = \frac{1}{1-(-z)}$$

$$= \sum_{n=0}^{\infty} \cancel{a} a_n z^n = \sum_{n=0}^{\infty} (-1)^n z^n \qquad |z|<1$$

Q $$f(z) = \frac{1}{z}$$

Ans Here, $f(z) \rightarrow$ Non-Analytic at $z=0$

$$\therefore f(z) = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

$$|z-1|<1$$

Q $$f(z) = \frac{1+2z^2}{z^3+z^5}$$

Ans $$= \frac{1+2z^2}{z^3(1+z^2)} = \frac{1}{z^3}\left[\frac{2(z^2+1)-1}{1+z^2}\right]$$

$$= \frac{1}{z^3}\left[2 - \frac{1}{1+z^2}\right]$$

We know,

$$\cancel{2 \frac{1}{1+z^2} = \Sigma}$$

$$\frac{1}{1+z^2} = \left[\cancel{\Sigma} \sum_{n=0}^{\infty} (-1)^n z^{-2n}\right]$$

$$\therefore f(z) = \frac{1}{z^3}\left[2 - \sum_{n=0}^{\infty} (-1)^n z^{2n}\right]$$

$$= \frac{1}{z^3}\left[2-1+z^2-z^4+z^6-\cdots\right]$$

$$= \frac{1}{z^3}\left[1+z^2-z^4+z^6\cdots\right]$$

$$= \underbrace{\frac{1}{z^3}+\frac{1}{z}}_{\text{Principal part}} - z + z^3 + \cdots$$

Laurent series

$0 < |z| < 1$

Or, $$\int_C \frac{1+2z^2}{z^3(1+z^2)}\,dz = \frac{g''(0)}{2!} \times 2\pi i$$

where $g(z) = \frac{1+2z^2}{1+z^2}$

$$= 2\pi i$$

$$\int f(z)\,dz = 2\pi i\, b_1$$

Laurent Series :-

Let $f(z)$ be analytic throughout the annulus region $R_1 < |z-z_0| < R_2$ centred at z_0, then-

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

where, $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \ ; n = 0, 1, 2, \ldots$$

& $$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \ ; \ n = 1, 2, 3 \ldots$$

and C is any +vely oriented simple closed ~~contour~~ contour lying in the region $R_1 < |z-z_0| < R_2$

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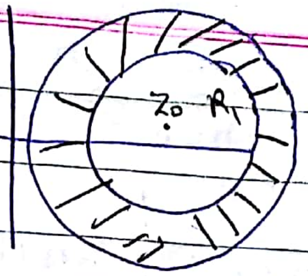


e.g. (i) $f(z) = e^{z}$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad ; \quad |z| < \infty$$

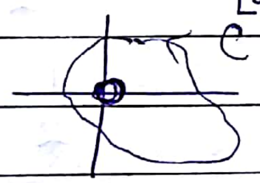
Consider,

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0} \left(\frac{1}{z}\right)^n \times \frac{1}{n!}$$

$$0 < |z| < \infty$$

$$\left[b_n = \frac{1}{n!}\right]$$

$$\int_C e^{\frac{1}{z}} dz = 2\pi i b_1$$



$$[b_1 = 1]$$

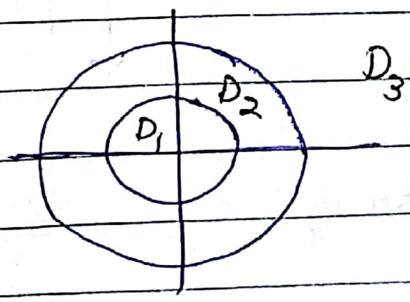
(ii) $f(z) = -\dfrac{1}{(z-1)(z-2)}$

$f(z)$ is analytic everywhere in the region:

$D_1 : |z| < 1$

$D_2 : 1 < |z| < 2$

$D_3 : 2 < |z| < \infty$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$|z| < 1$, then $\frac{|z|}{2} < 1 \quad \forall z \in D_1$

$$f(z) = -\frac{1}{1-z} + \frac{1}{2} \times \frac{1}{1-\frac{z}{2}}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^{n+1}} - 1\right] z^n$$

For D_2 :→ $\quad 1 < |z| < 2$

$$\cancel{\frac{1}{|z|}} \;\cancel{<}\; \cancel{\&}\; \cancel{\frac{1}{|z|}}\; \cancel{z}$$

$$\frac{1}{|z|} < 1 \quad \& \quad \frac{|z|}{2} < 1$$

$$f(z) = \frac{1}{z} \times \frac{1}{1 - z/2} + \frac{1}{2} \times \frac{1}{1 - z/2}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}$$

$$\int_C f(z)\,dz = 2\pi i \qquad [b_1 = 1]$$

For D_3 :→ $\quad 2 < |z| < \infty$

$$\frac{2}{|z|} < 1 \;,\; \frac{1}{|z|} < 1$$

$$f(z) = \frac{1}{z} \times \frac{1}{1 - \frac{1}{z}} - \frac{1}{z} \times \frac{1}{1 - \frac{2}{z}}$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1 - 2^{n-1}}{z^n}\right]$$

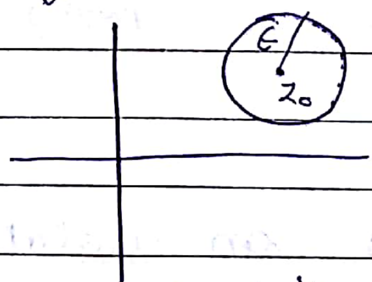
Q $$f(z) = \frac{1}{z(z^2 - 3z + 2)}$$

$$= \frac{1}{z(z-1)(z-2)}$$

★ **Singular point :-**

A point $z = z_0$ is called singular point of $f(z)$ if 'f' fails to be analytic at $z = z_0$, but ~~in same~~ is analytic at some point in every neighbourhood of z_0.

A singular point $z = z_0$ is said to be isolated if there exists a deleted neighbourhood (nbd) of z_0 where $f(z)$ is analytic throughout that nbd.



For $\epsilon > 0$, $\quad 0 < |z - z_0| < \epsilon$

e.g. (i) $$f(z) = \frac{z^2+1}{z^3(z^2+1)}$$

$z = 0, \pm i \rightarrow$ Singular points

$\epsilon \in (0,1)$

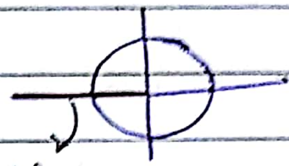
e.g. (ii) $\operatorname{Log} z$

$$= \ln r + i\theta$$

$$[r > 0, \ -\pi < \theta < \pi]$$

$z = 0$ → Non-isolated

(Every nbd of $z = 0$ contains at least one point where function is not-analytic)



Non-analytic region

(iii) $$f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$$

Singular points

↳ $z = 0, \ \frac{1}{n}, \ n = \pm 1, \pm 2, \pm 3 \ldots$

$z = 0$ → Non-isolated → For any given $\epsilon > 0$, $\frac{1}{n} < \epsilon$

$z = \frac{1}{n}$ → Isolated singular points

If $f(z)$ has an isolated singularity at $z = z_0$ then $f(z)$ can be expressed as-

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$ → Laurent Series

$0 < |z - z_0| < R_2$ → Annulus region

★ **Residue:-** Coefficient of $\left(\frac{1}{z-z_0}\right)$ in Laurent series expansion is known as Residue of $f(z)$ at $z = z_0$

$$b_1 = \frac{1}{2\pi i}\int_c f(z)\,dz$$

where C is ~~[illegible]~~ positively oriented simple closed contour in $0 < |z-z_0| < R_2$

$$\Rightarrow \boxed{\int_C f(z)\,dz = 2\pi i\, b_1}$$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

★ **Classification of isolated Singular point:-**

That part which consist -ve power of $(z-z_0)$ is called Principal Part of $f(z)$

i.e. $\sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \rightarrow$ Principal part of $f(z)$.

(i) **Pole:-** If principal part of $f(z)$ contains only finite number of terms with $b_m \neq 0$.

But, $b_{m+1} = b_{m+2} = \dots = 0$

then- $z = z_0$ is called pole of order 'm'.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} b_n (z-z_0)^{-n}$$

where $b_m \neq 0$

in $0 < |z-z_0| < R_2$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots \frac{b_m}{(z-z_0)^m}$$

with $b_m \neq 0$

Alternative definition of pole →

~~A point $z=z_0$ If there exists a +ve~~
If there exists a +ve ~~definitio~~ integer m such that -

$$\lim_{(z \to z_0)} (z-z_0)^m f(z) = l$$

Here $[l = b_m]$
finite & non-zero

then $z = z_0$ is called pole of order m.

e.g. (i) $f(z) = \dfrac{z^2 - 2z + 2}{z-2}$

$$= \frac{z(z-2)+3}{z-2} = z + \frac{3}{z-2}$$

$$= 2 + (z-2) + \frac{3}{(z-2)}$$

Here $a_1 = 2$, $a_2 = 2$, $b_1 = 3$

~~At~~ $z = 2$ is simple pole exits.
(of order 1)

Simple pole → Pole of order '1' is called simple pole.

$$\lim_{z\to 2} (z-2) f(z) = 3 \rightarrow \text{finite}$$

$$C \rightarrow 0 < |z - z_0| < \cancel{R_2}$$

$$\int_C f(z)\,dz = 2\pi i(3) = 6\pi i$$

(ii) $f(z) = \dfrac{\sinh z}{z^4}$

Ans

$$= \frac{1}{z^4}\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right]$$

$$= \underbrace{\frac{1}{z^3} + \frac{1}{z\,3!}}_{\text{principal part}} + \frac{z}{5!} + \frac{z^3}{7!} + \cdots$$

$z = 0 \rightarrow$ ~~one~~ pole of order 3.

$$\lim_{z\to 0} \frac{\sinh z}{z^4} \times z^3 = 1 \quad (\text{finite non-zero})$$

Removable singularity → If principal part of $f(z)$ does not contain any term, then $z = z_0$ is called removable singularity.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$0 < |z - z_0| < R_2$$

That singularity can be removed by defining the function $f(z)$ at $z = z_0$

$$f(z) = \begin{cases} a_0 & ; z = z_0 \\ \sum_{n=0}^{\infty} a_n (z-z_0)^n & ; z \neq z_0 \end{cases}$$

then-

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad ; \quad |z-z_0| < R_2$$

Alternative definition:-

If $\lim_{z \to z_0} f(z)$ exists & is finite

then $z = z_0$ is called removable singularity

e.g. $f(z) = \dfrac{1-\cos z}{z^2}$

$$= \frac{1}{z^2}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)\right]$$

$$= \frac{1}{z^2}\left[\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\cdots\right]$$

$$= \frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots$$

$|z| < \infty$

$$f(z) = \begin{cases} \frac{1}{2} & ; z = 0 \\ \frac{1-\cos z}{z^2} & ; z \neq 0 \end{cases}$$

$$\lim_{z \to 0} \frac{1-\cos z}{z^2} = \lim_{z \to 0} \frac{\sin z}{2z} = \lim_{z \to 0} \frac{\cos z}{2} = \frac{1}{2}$$

(ii) Essential Singularity :-

If principal part of $f(z)$ contains infinite number of terms then point $z = z_0$ is called essential singularity.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$0 < |z-z_0| < R_2$$

Alternatively,

If there $\boxed{\text{does not}}$ exist any +ve integer such that -

$$\lim_{z \to z_0} (z-z_0)^m f(z) = l \text{ (finite \& non-zero)}$$

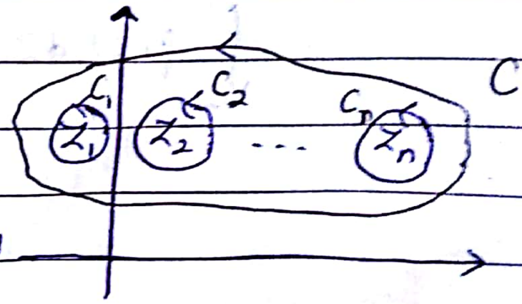
e.g. $e^{\frac{1}{z}} \rightarrow z = 0$ is essential singularity.

* Cauchy's Residue Theorem :-

If $f(z)$ is analytic throughout inside & on a positively oriented simple closed contour 'C' except for a finite number of points (say $z_1, z_2 \ldots z_n$) then -

$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z)$$

Proof :- Let $z_1, z_2 \ldots z_n$ be non-analytic (singular) points inside C each encircled by +vely oriented simple closed contours $C_1, C_2 \ldots C_n$ respectively.



~~If f(z) ~~

It is clear that, $f(z)$ is analytic throughout multi-connected region consituted by $C, C_1, C_2 \dots C_n$.

By Cauchy's theorem for multi-connected domain →

$$\int_C f(z)\,dz - \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 0 \qquad — (1)$$

-ve sign because Both C & C_i $(i=1,2..n)$ are oriented positively.

Inside C_1 :-

$$\int_{C_1} f(z)\,dz = 2\pi i\, B_1$$

where $B_1 = \underset{z=z_1}{\text{Res}}\, f(z)$

⋮

Similary,

$$\int_{C_n} f(z)\,dz = 2\pi i\, B_n$$

where $B_n = \underset{z=z_n}{\text{Res}}\, f(z)$

From (1),

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz = 2\pi i\,(B_1 + B_2 + \dots B_n)$$

H.P.

Q Evaluate : $\int_C \frac{5z-2}{z(z-1)} dz$; $C : |z| = 2$

Ans Singularity $\rightarrow$ $z=0, z=1$ $\rightarrow$ Finite number of points

Both lie inside 'C'

By Cauchy's Residue Theorem -

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i \left[\underset{z=0}{\text{Res}}\, f(z) + \underset{z=1}{\text{Res}}\, f(z) \right]$$

Now,

Let $f(z) = \frac{5z-2}{z(z-1)}$

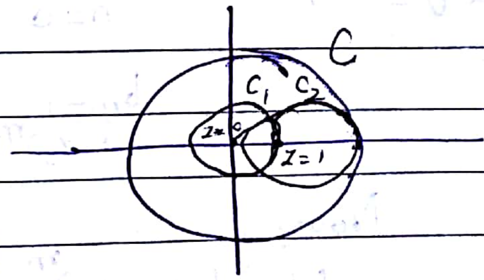
(i) For region $\rightarrow$ $0 < |z| < 1$ $\rightarrow$ $f(z)$ analytic $\downarrow$ Hence can be expanded through power series

$$f(z) = \left(5 - \frac{2}{z}\right)\left(-\frac{1}{1-z}\right)$$

$$= \left(5 - \frac{2}{z}\right)\left(-1 - z - z^2 - z^3 - \dots\right)$$

$$\underset{z=0}{\text{Re}}\, f(z) = 2 = B_1$$

(ii) For region $0 < |z-1| < 1$



$$f(z) = \frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z(z-1)}$$

$$= \left(5 + \frac{3}{z-1}\right)\left(\frac{1}{1+(z-1)}\right)$$

$$= \left(5 + \frac{3}{z-1}\right)\sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

$$= \left(5 + \frac{3}{z-1}\right)\left[1 - (z-1) + (z-1)^2 - (z-1)^3 - \dots\right]$$

$$\operatorname*{Res}_{z=1} f(z) = 3$$

$$\therefore \int_C \frac{5z-2}{z(z-1)}\,dz = 2\pi i\,[2+3) = 10\pi i$$

Alternatively,

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1}$$

By Lauren't series concept,

$B_1 = 2$ & $B_2 = 3$

$$\therefore \int_C f(z)\,dz = 2\pi i\,(3+2) = 10\pi i$$

★ Residue at poles :-

If $f(z)$ has a pole of order m at $z = z_0$, then-

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-z_0)^m f(z)\right\}\right]$$

Proof :- $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \ldots$$

$$\frac{b_{m-1}}{(z-z_0)^{m-1}} + \frac{b_m}{(z-z_0)^m} \quad \cancel{b_m \neq 0} \quad \text{in } 0 < |z-z_0| < R$$

$b_m \neq 0$

Now,

$$f(z)(z-z_0)^m = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + \ldots$$

$$+ b_2 (z-z_0)^{m-2} + \ldots b_{m-1}(z-z_0) + b_m \quad \text{— (1)}$$

with $b_m \neq 0$ & $|z-z_0| < R$

$\Rightarrow (z-z_0)^m f(z)$ is analytic in $|z-z_0| < R_2$

[Since right hand side is nothing but its power series expansion]

Differentiating ① both side $(m-1)$ times i.e. $(m-1)^{th}$ derivative

$$\frac{d^{m-1}}{dz^{m-1}}\left\{f(z)(z-z_0)^m\right\} = (n+m)(n+m-1)\ldots(n+2)\sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} + b_1 (m-1)! + 0 + 0 \ldots 0$$

As $z \to z_0$ $\sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} = 0$

$$\therefore \operatorname*{Re}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}}\left\{f(z)(z-z_0)^m\right\}\right]$$

Q Evaluate: $\int_C \frac{5z-2}{z(z-1)} dz$; $C: |z| = 2$

Ans $z = 0$, $z = 1 \rightarrow$ Singular points

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z \to 0} z\left\{\frac{5z-2}{z(z-1)}\right\} = 2 = B_1$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} (z-1)\left\{\frac{5z-2}{(z-1)z}\right\} = 3 = B_2$$

Note:- $f(z)$ has a zero of order 'm' at z_0 means -

$$f(z_0) = f'(z_0) = f''(z_0) = \dots f^{(m-1)}(z_0) = 0$$

But $f^{m}(z_0) \neq 0$

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* Zeroes of Analytic function:-

Theorem:- Let a function $f(z)$ be analytic at a point z_0. Then $f(z)$ has a zero of order 'm' at z_0, iff there exists an analytic function and non-zero function $g(z)$ at z_0 such that

$$f(z) = (z-z_0)^m g(z)$$

Proof:- $f(z)$ is analytic at z_0.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0| < R_2$$

$$= \sum_{n=m}^{\infty} a_n (z-z_0)^n \qquad |z-z_0| < R_2$$

$$= (z-z_0)^m \left[a_m + a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \dots\right]$$

$$\{a_m \neq 0 \ ; \ |z-z_0| < R_2\}$$

$$= (z-z_0)^m g(z) \rightarrow \text{Power Series function}$$

Conversely,

Let $g(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \dots$

$\hookrightarrow$ Power series expansion since, $g(z)$ is analytic

Also, $g(z)$ is non-zero at z_0

$$g(z_0) \neq 0 \Rightarrow b_0 \neq 0$$

$$f(z) = (z-z_0)^m g(z)$$

$$= b_0(z-z_0)^m + b_1(z-z_0)^{m+1} + b_2(z-z_0)^{m+2} + \cdots \quad \text{--- (1)}$$

From eq^n (1), clearly -

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots f^{(m-1)}(z_0) = 0$$

But, $f^{(m)}(z_0) = (m!)\, g(z_0) \neq 0$

$f(z) \rightarrow$ Analytic at z_0

& $f(z)$ has zero of order 'm'.

<u>Theorem</u> :- Suppose that -

(a) p & q are two analytic functions at z_0

(b) $p(z_0) \neq 0$ & q has zero of order 'm' at z_0.

then - $\frac{p(z)}{q(z)}$ is a pole of order 'm' at z_0.

Proof :- $\frac{p(z)}{q(z)}$ not analytic at z_0, since $q(z)$ has zero at z_0. (order $\rightarrow m$)

z_0 is only singularity in entire domain.

$z_0 \rightarrow$ isolated singular point

By previous theorem,

$$q(z) = (z-z_0)^m g(z)$$

$g(z) \rightarrow$ Analytic

$\therefore \dfrac{P(z)}{q(z)} = \dfrac{P(z)}{(z-z_0)^m g(z)}$ $\quad g(z_0) \neq 0$

Since, $\dfrac{P(z)}{g(z)}$ is analytic at z_0 & can be expressed as power series expansion.

$$\frac{P(z)}{q(z)} = \frac{1}{(z-z_0)^m}\left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right]$$

$a_0 \neq 0$; $|z-z_0| < R_2$

$$= \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots$$

$0 < |z-z_0| < R_2$

$m \rightarrow$ Order of Pole

& $\dfrac{P(z)}{q(z)}$ has pole of order at z_0.

Note:- If $q(z)$ has a zero of order 'm' at $z = z_0$ then $\dfrac{1}{q(z)}$ has a pole of order 'm' at $z = z_0$.

e.g. Show that $f(z) = \dfrac{1}{z(e^z-1)}$ has pole of order 2 at $z = 0$.

Ans

$$z(e^z-1) = z\left[1 + z + \frac{z^2}{2!} + \cdots - 1\right]$$

$$= z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} + \cdots$$

$z(e^z-1)$ has zero of order of 2.

$\therefore \frac{1}{z(e^z-1)}$ has pole at $z_0 = 0$ of order 2.

★ Evaluation of Improper Integrals :-

Improper Integral → (a) If $f(x)$ is continuous over +ve real axis $(0 < x < \infty)$ then improper integral $\int_0^\infty f(x)\,dx$ is defined as-

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx$$

If limit on R.H.S. exists then $\int_0^\infty f(x)\,dx$ is said to be convergent to that limit.

(b) If $f(x)$ is continuous over $-\infty < x < \infty$ then -

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\,dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\,dx$$

If both limit on R.H.S. exist, then $\int_{-\infty}^\infty f(x)\,dx$ is said to converge to sum of these limits.

Cauchy Principal Value (P.V.) is defined as-

$$\boxed{\text{P.V.} \int_{-\infty}^\infty f(x)\,dx = \lim_{R\to\infty} \int_{-R}^R f(x)\,dx}$$

Note:- Convergence of $\int_{-\infty}^{\infty} f(x)\,dx \Rightarrow$ existence of P.V. $\int_{-\infty}^{\infty} f(x)\,dx$

$\not\Leftarrow$

e.g. P.V. $\int_{-\infty}^{\infty} x\,dx$

$$= \cancel{\lim_{R\to\infty} \int_{-\infty}^{\infty}} = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left[\frac{x^2}{2}\right]$$

$$= 0$$

And,

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty} \int_{0}^{R_2} x\,dx$$

$$= \lim_{R_1\to\infty} -\frac{R_1^2}{2} + \lim_{R_2\to\infty} \frac{R_2^2}{2}$$

Both limits don't exist.

$\therefore \int_{-\infty}^{\infty} x\,dx$ is divergent.

That means, P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ exists, still $\int_{-\infty}^{\infty} f(x)\,dx$ is divergent.

Note:- If $f(x)$ is an even function, then-

Existence of P.V. $\int_{-\infty}^{\infty} f(x)\,dx \Rightarrow$ Convergence of $\int_{-\infty}^{\infty} f(x)\,dx$

Proof:-

$\int_{-\infty}^{\infty} f(x)\,dx$ is said to converge iff

$$\lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)\,dx \text{ exists.}$$

Now,

$$\int_{-R_1}^{0} f(x)\,dx + \int_{0}^{R_2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-R_1}^{R_1} f(x)\,dx + \frac{1}{2}\int_{-R_2}^{R_2} f(x)\,dx$$ [In case of even function]

Take $R_1, R_2 \to \infty$

$$= \frac{1}{2}\lim_{R_1\to\infty}\int_{-R_1}^{R_1} f(x)\,dx + \frac{1}{2}\lim_{R_2\to\infty}\int_{-R_2}^{R_2} f(x)\,dx$$

$$= P.V.\int_{-\infty}^{\infty} f(x)\,dx$$

☆ Improper Integral of Rational function :-

$$\int_{-\infty}^{\infty} f(x)\,dx$$

where, $f(x) = \frac{p(x)}{q(x)}$; p & q both are polynomial functions, not having any common zero. Further, let $q(x)$ does not have any zeroes in real axis. and have at least one zero above the real axis.

To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, consider a function

$$f(z) = \frac{p(z)}{q(z)}$$

Let $z_1, z_2, \ldots z_n$ are the zeroes of $q(z)$ which lie above x-axis.

Points $z_1, z_2, z_3 \ldots z_n$ are isolated singularities of $f(z)$.

By Cauchy's Residue Theorem,

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\,(B_1 + B_2 + \ldots B_n)$$

where

$$B_k = \underset{z=z_k}{\text{Res}}\, f(z) \qquad [k = 1, 2 \ldots n]$$

If $\left| \int_{C_R} f(z)\,dz \right| \to 0$ as $R \to \infty$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i\,(B_1 + B_2 + \ldots B_n)$$

$$\Rightarrow \quad P.V. \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,[B_1 + B_2 + \ldots B_n]$$

If $f(x)$ is even function

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,[B_1 + B_2 + \ldots B_n]$$

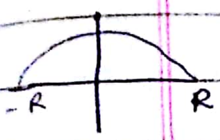
Q. Evaluate: $\int_0^{\infty} \frac{x^2}{1+x^6}\,dx$

Ans Consider

$$f(z) = \frac{z^2}{1+z^6}$$

$$1 + z^6 = 0$$

Roots of $1 + z^6 = 0$ are -



$$C_K = \exp\left[i\left(\frac{\pi}{6} + \frac{2K\pi}{6}\right)\right] \quad ; K = 0, 1, 2 \ldots 5$$

$R > 1$ $\cancel{(\text{because } |1+z^6| > 1)}$

C_0, C_1, C_2 are simple poles of $f(z)$

Now,

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\,(B_0 + B_1 + B_2) \quad \text{--- (1)}$$

$$B_K = \lim_{z \to C_K} (z - C_K)\, f(z)$$

$$= \lim_{z \to C_K} \frac{(z - C_K)\, z^2}{1 + z^6}$$

By L'Hospital rule,

$$= \lim_{z \to C_K} \frac{2z(z - C_K) + z^2}{6z^5}$$

$$= \frac{C_K^2}{6\, C_K^5} = \frac{1}{6\, C_K^3}$$

From eq^n (1)

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right)$$

$$= \frac{\pi}{3}$$

Take $R \to \infty$ both sides

$$P.V. \int_{-R}^{R} f(x)\,dx + \lim_{R \to \infty} \int_{C_R} f(z)\,dz = \frac{\pi}{3}$$

Again,

$$f(z) = \frac{z^2}{1+z^6}$$

$\because \ |1+z^6| \geq ||z^6|-1| = |R^6-1|$

$$\Rightarrow \frac{|z^2|}{|1+z^6|} \leq \frac{R^2}{R^6-1}$$

$$\Rightarrow \left| \int_{C_R} \frac{z^2}{1+z^6}\, dz \right| \leq \int_{C_R} \left| \frac{z^6}{1+z^6} \right| |dz|$$

$$\leq \frac{R^2}{R^6-1} \times \pi R = \frac{\pi R^3}{R^6-1} \to 0$$

as $R \to \infty$.

Also, $f(x) \to$ even function

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\, dx = \frac{\pi}{3}$$

$$\Rightarrow \int_0^{\infty} \frac{x^2}{1+x^6}\, dx = \frac{1}{2} \times \frac{\pi}{3} = \frac{\pi}{6}$$

✱ Improper integral in Fourier analysis:-

$\int_0^\infty f(x) \sin ax\, dx$, $\int_0^\infty f(x) \cos ax\, dx$ — such type of integrals will be evaluated in this section

~~$f(z)e^{iaz}$~~

Here, $f(x) = \frac{P(x)}{q(x)}$; P & q are polynomials having no common factor.

$q(x)$ does not have real zero & has at least one zero which lie above real axis.

Consider

$f(z)\, e^{iaz}$

(in this case)

why $f(z)\, e^{iaz}$?

Since $|\sin az|^2 = \sin^2 ax + \sinh^2 ay$
&
$|\cos az|^2 = \cos^2 ax + \sinh^2 ay$

And,
$$\sinh ay = \frac{e^{ay} - e^{-ay}}{2}$$

$\Rightarrow$ $|\sin az|$ & $|\cos az|$ increase like e^{ay} as $y \to \infty$.

$$|e^{iaz}| = |e^{ia(x+iy)}|$$
$$= |e^{iax}\, e^{-ay}$$
$$e^{-ay} \to 0 \text{ as } y \to \infty.$$

Thus, $\sin az$ & ~~sin~~ $\cos az$ are not bounded.

$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax}\, e^{-ay}| \to 0$ as $y \to \infty$

~~$= |e^{iax - iay}|$~~

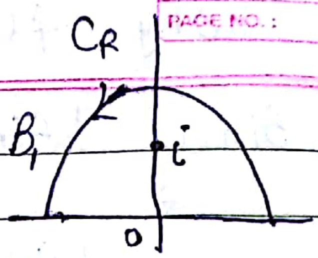
Thus, e^{iaz} is bounded function.

Q Prove that: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\, dx = \frac{2\pi}{e^3}$

Ans Consider $\dfrac{1}{(z^2+1)^2}\, e^{i3z}$ is ~~any~~ analytic everywhere in z-plane except $z = \pm i$.
In fact, both points $z = \pm i$ are ~~points~~ poles of order 2.

By Cauchy's residue theorem

$$\hookrightarrow \int_{-R}^{R} \frac{e^{iax}}{(x^2+1)^2} + \int_{C_R} \frac{e^{iaz}}{(z^2+1)^2}\, dz = 2\pi i B_1$$



$$B_1 = \frac{1}{(1)!} \lim_{z \to i} \left[\frac{d}{dz} \left\{ \frac{e^{i3z}}{(z^2+1)^2} \right\} \right]$$

[By residue at pole formula]

$$= \frac{1}{i e^3}$$

$$\text{So, } \int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\, dx + \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\, dz = \frac{2\pi}{e^3}$$

Equating real parts both sides, we have-

$$Re \int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\, dx + Re \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\, dz = \frac{2\pi}{e^3} \quad \text{--- (A)}$$

Take $R \to \infty$

$$P.V. \int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\, dx + \lim_{R \to 0} Re \int \frac{e^{i3z}}{(z^2+1)^2}\, dz = \frac{2\pi}{e^3}$$

Now,

we know-

$$|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$$

$$\Rightarrow |z^2+1| \geq \big||z|^2 - 1\big|$$

$$\geq |R^2 - 1|$$

$$\Rightarrow \frac{1}{|(z^2+1)^2|} \leq \frac{1}{(R^2-1)^2}$$

$$\Rightarrow \frac{|e^{i3z}|}{(z^2+1)^2} \leq \frac{e^{-3y}}{\cancel{(z^2+1)^2}(R^2-1)^2} \quad \left[\because |e^{iaz}| < e^{-ay}\right]$$

So,

$$\left| Re \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2} dz \right| \leq \left| \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2} dz \right|$$

$$\leq \int_{C_R} \left| \frac{e^{i3z}}{(z^2+1)^2} \right| dz$$

$$\leq \frac{e^{-3y}}{\cancel{(z^2+1)}(R^2-1)^2} \times \pi R$$

$$\to 0 \quad \text{as } R \to \infty$$

$\frac{\cos 3x}{(x^2+1)^2} \longrightarrow$ even function

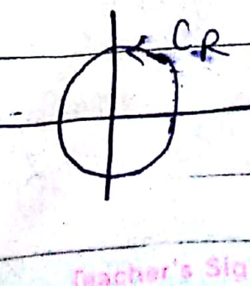
$$\therefore \int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{2\pi}{e^3}$$

Also, $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 3x}{(x^2+1)^2} dx = 0$ (Since, $\sin 3x$ is odd function. Also, (A) shows that only real part can be equated.]

* Indefinite integral :-

Suppose an indefinite integral is in form-

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$ to be evaluated along C_R



$z = e^{i\theta} \qquad 0 \leq \theta \leq 2\pi$

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$$dz = i e^{i\theta} d\theta = iz\, d\theta$$

$$\Rightarrow \quad d\theta = \frac{dz}{iz}$$

Also, $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$

So, the integral becomes

$$\int_{C_R} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz}$$

Q Show that: $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ $(-1<a<1)$

Ans $\sin\theta = \frac{z-z^{-1}}{2i}$ & $d\theta = \frac{dz}{iz}$

$$\therefore \int_{C_R} \frac{dz}{1+a\left(\frac{z-z^{-1}}{2i}\right)} \times \frac{1}{iz} = \int_{C_R} \frac{2/a}{z^2+\left(\frac{2i}{a}\right)z-1}\, dz$$

$C_R : |z| = 1$

[Do not further simplify quadratic eq^n]

Roots of $z^2 + \left(\frac{2i}{a}\right)z - 1$ are –

$$z = \frac{-\left(\frac{2i}{a}\right) \pm \sqrt{\frac{-4}{a^2}+4}}{2} = \frac{-i}{a} \pm \sqrt{1-\frac{1}{a^2}}$$

$$= \left(\frac{-1}{a} \pm \frac{\sqrt{1-a^2}}{a}\right) i$$

$$z_1 = \left(\frac{-1+\sqrt{1-a^2}}{a}\right) i \quad ; \quad z_2 = \left(\frac{-1-\sqrt{1-a^2}}{a}\right) i$$

Clearly, $|z_2| > 1$ $\therefore z_2 \to$ lies outside circle C_R

And, $z_1 z_2 = 1$ (By Quadratic eqn concept)

$\Rightarrow |z_1 z_2| = 1 \Rightarrow |z_1| < 1$

So, $\displaystyle\int_{C_R} \frac{2/a}{z^2 + \left(\frac{2i}{a}\right) z - 1} \, dz = 2\pi i \, B_1$

$\Rightarrow \displaystyle\int_{C_R} \frac{2/a}{(z-z_1)(z-z_2)} \, dz = 2\pi i \, B_1$

$B_1 = \underset{z = z_1}{\operatorname{Res}} f(z) = \displaystyle\lim_{z \to z_1} \frac{2/a}{(z - z_2)}$

$= \dfrac{2/a}{z_1 - z_2} = \dfrac{1}{(\sqrt{1-a^2})\, i}$

$\Rightarrow \displaystyle\int_{C_R} \frac{2/a}{(z-z_1)(z-z_2)} \, dz = 2\pi i \times \frac{1}{(\sqrt{1-a^2})\, i}$

$= \dfrac{2\pi}{\sqrt{1-a^2}}$