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Introduction to complex Nos.

• $i^{422} + 2i^{203} + 2i + 1$

$\cancel{i^2} + 2i^3 + 2i + \cancel{1}$

$\Rightarrow \quad -2i + 2i = \underline{\underline{0}}$

★ Complex nos. are written in the form $\underbrace{a}_{\text{Real}} + \overset{\text{Imaginary}}{bi}$

All Real nos. are complex and all Imaginary nos. are complex.

Product of Complex no. :-

$(x_1, y_1) \times (x_2, y_2) = (\overbrace{x_1x_2 - y_1y_2}^{\text{Real}} + \underbrace{y_1x_2 + x_1y_2}_{\text{Imaginary}})$

Complex no. is a natural extension of real number.

* $(1,0) \rightarrow 1 \qquad * (0,1) \rightarrow i$

$(0,1) \times (0,1) = i \times i = i^2 = -1$

Ans. is $(-1,0)$

Inverse of a complex number :-

For any non zero complex no. $z = (x,y)$ there is a number z^{-1} such that $\boxed{zz^{-1} = 1}$

Following group properties :-

$(Z, +)$.

(i) $a+b \in Z \quad \forall \quad a,b \in Z$ — closure

(ii) $(a+b)+c = a+(b+c)$ — associative

(iii) $a+e = a = e+a \quad \forall \quad a \in Z$

(iv) $a + a^{-1} = e = a^{-1} + a \Rightarrow$ Identity $(e=0)$

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For complex identity also belong to complex no.

$e \in C$ $\quad e = (1,0)$

also $zz^{-1} = e = (1,0) = z^{-1}z$

$z = (0,0)$ have no Inverse

eg $\quad$ Must satisfy :-

$(x,y)(u,v) = (1,0)$

Then,

$xu - yv = 1 \qquad xv + uy = 0$

$\boxed{u = \frac{x}{x^2+y^2}} \qquad \boxed{v = \frac{-y}{x^2+y^2}}$

<u>Conjugate of complex no.</u>

$z = x + yi \qquad \bar{z} = x - yi$

$|z| = \sqrt{x^2+y^2} = |\bar{z}|$

$zz^{-1} = 1 = z^{-1} = \frac{1}{z} \qquad z \neq (0,0)$

$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{x-yi}{x^2+y^2}$

* $\boxed{z\bar{z} = |z^2| = x^2+y^2}$

<u>For subtraction</u> like $z_1 - z_2$

we take negative of $z_2 = z_1 + (-z_2)$

$(x_1, y_1) + (-x_2, -y_2)$

<u>Complex Plane</u> :-

x axis = Real axis

y axis = Imaginary axis

also called Argand Plane

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Reflection about x axis

complex conjugate = $\theta = -\theta$

Negative of complex no = Reflection about Origin

→ How to get Polar Form.

eg $(1+i) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Modulus of complex no. ← $= \sqrt{2}\, e^{i\pi/4}$ (euler's form)

argument of complex no.

• $(1+i)^2 = 2e^{i\pi/2}$ (By squaring any complex no. argument doubles)

$= 2i$

Properties for complex conjugate :

$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

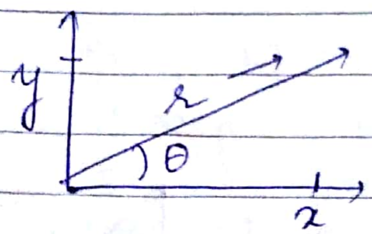
$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ $(z_2 \neq 0)$

$z = f + ig \longrightarrow \overline{z} = \overline{f} + \overline{ig} = \overline{f} - i\overline{g}$

Polar form of complex no. :-

$x = r\cos\theta$ $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$ $\tan\theta = y/x$



$z = x + iy$

$z = r\cos\theta + ir\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$\boxed{z = re^{i\theta}}$

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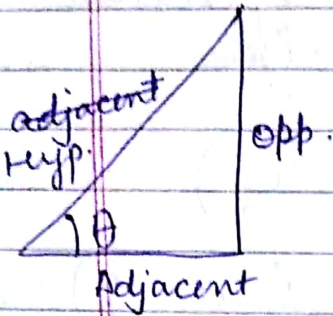
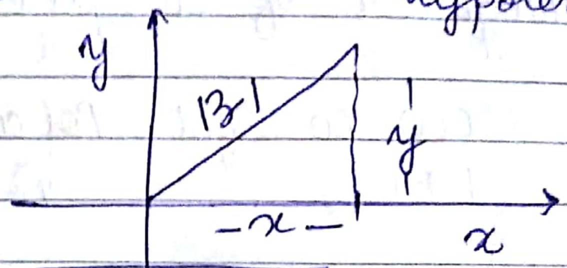
Dheerendra Sir class :-

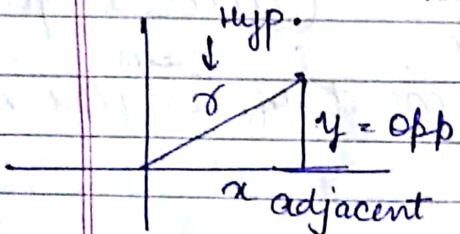
$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$





$\boxed{Z = x + yi}$

$|z| = \sqrt{x^2 + y^2}$



$r = |z| = \sqrt{x^2 + y^2}$

$\sin\theta = y/r = \boxed{y = r\sin\theta}$

$\cos\theta = \frac{x}{r}$ $\boxed{x = r\cos\theta}$

$Z = x + yi$

$\boxed{Z = r\cos\theta + r\sin\theta}$

$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$

$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots$

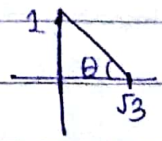
$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ euler's form

$\boxed{Z = re^{i\theta}}$

ex $Z = \sqrt{3} + i$

Calculate z^{61}



$Z = re^{i\theta}$

$Z = 2e^{i(\pi/6)}$

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Method :

$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = (r_1 r_2)\, e^{i\theta_1} \cdot e^{i\theta_2}$

$$\boxed{z_1 z_2 = r_1 r_2\, e^{i(\theta_1+\theta_2)}}$$

During Multiplication arguments get added.

$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$

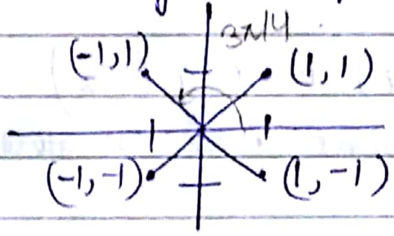
$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}}$$

1) $z = 1+i$ 2) $z = -1-i$ → This negative Sometimes Make trouble

3) $z = 1-i$ 4) $z = -1+i$

all four have same Modulus

If we try to plot all four of these.



By euler's formula :- 1) $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

2) $\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \theta = \pi/4$ → Here will be use the concept of Argument.

3) $\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$ 4) $\theta = -\frac{\pi}{4}$ So $-\frac{\pi}{4}+\pi$ $3\pi/4$

when x is positive no trouble when x is negative some problem occur in plotting.

So concept of "Argument".

Argument — argument (arg z)

— principle argument (Arg z) ↓ Capital

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$$\boxed{\arg z = \operatorname{Arg} z + 2n\pi}$$
$n = 0, \pm 1; \pm 2;$

$$-\pi \leq \operatorname{Arg} z \leq \pi$$

$$\operatorname{Arg} z = \begin{cases} \tan^{-1}(y/x) & x \geq 0 \\ \tan^{-1}(y/x) + \pi & x < 0, y \geq 0 \\ \tan^{-1}(y/x) - \pi & x < 0, y < 0 \end{cases}$$

Q

	$\sqrt{3}+i$ $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$	$\sqrt{3}-i$ $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$	$-\sqrt{3}-i$ $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$	$-\sqrt{3}+i$ $-\frac{\pi}{6}+\pi$
Arg →	$\pi/6$	$-\pi/6$	$\pi/6 - \pi$	$5\pi/6$

Properties of Arguments :-

① $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

True in case of argument 'not' in the case of principal argument.

eg $z_1 = -i$ $\quad z_2 = -1$

$z_1 z_2 = +i$

$\operatorname{Arg}(z_1 z_2) = +\pi/2$

Now $\operatorname{Arg}(z_1) = \operatorname{Arg}(i) = \pi/2$

$\operatorname{Arg}(z_2) = \operatorname{Arg}(-1) = 0 + \pi = \pi$

$\operatorname{Arg} z_1 + \operatorname{Arg} z_2$

$\Rightarrow \frac{\pi}{2} + \pi = \frac{3\pi}{2}$

Because in gen argument we have addition of $2n\pi$ That'why This property hold in gen argument But not in principle argument.

$Z = \sqrt{2}e^{3\pi/4 \, i} = Z = \sqrt{2}e^{(3\pi/4 + 2\pi)i}$

So we get same Results

$$e^{i\theta} = (\cos\theta + i\sin\theta)$$
$$(e^{i\theta})^n = e^{in\theta}$$
$$= \cos n\theta + i\sin n\theta$$
$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

De Moivre's Theorem

Prove De Moivre's Theorem :-

By Induction Method

- $n = 1 \quad (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$
$$= \cos 1\theta + i\sin 1\theta$$

- $n = \cancel{k}\,2 \quad (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

$\Rightarrow (\cos 2\theta + i\sin 2\theta)$

Let's assume it is true for $n = k$.

$n = k$

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

Now we have to prove for $k+1$

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$$
$$\Rightarrow (\cos k\theta + i\sin k\theta) \cdot (\cos\theta + i\sin\theta)$$

$\Rightarrow \cos k\theta \cdot \cos\theta + \cos k\theta \cdot i\sin\theta + i\sin k\theta\cos\theta - \sin\theta\sin k\theta$

$\Rightarrow \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$\Rightarrow \cos(k+1)\theta + i\sin(k+1)\theta$

Hence Proved.

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For $n=3$

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

Q $(\sqrt{3}+i)^{61} = \left(2e^{\pi/6\, i}\right)^{61}$

$(2^{61})\left[e^{\frac{\pi}{6}\times 61 i}\right] = (2^{61})\left(e^{10\pi i} \cdot e^{\pi/6\, i}\right)$ → it's value is 1

$e^{10\pi i} \downarrow 1$

$\Rightarrow 2^{61}\times 1\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$2^{61}\times\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = 2^{60}(\sqrt{3}+i)$

\# $\left(\frac{Z_1}{Z_2}\right)$ $\left(\frac{i}{-1+i}\right)$

$\frac{i}{-1+i}\times\frac{-1-i}{-1-i} = \frac{i(-1-i)}{1+1} \Rightarrow \frac{1-i}{2}$

$(a+b)(a-b)$
(a^2)
a^2+b^2

Roots :- n^{th}

$\Rightarrow$ w is a n^{th} root of z if

$\star\ \boxed{w^n = z}\ \star$

$w = R(\cos\phi + i\sin\phi)$

$w = Re^{i\phi}$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$

$\Rightarrow w^n = z$

$w = z^{1/n}$

$Re^{i\phi} = \left(re^{i\theta}\right)^{1/n}$

$Re^{i\phi} = r^{1/n}\cdot\left(e^{i[\theta+2k\pi]}\right)^{1/n}$

$Re^{i\phi} = r^{1/n}\, e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}$ k = integer

$\boxed{R = r^{1/n}}$

$\boxed{\phi = \frac{\theta}{n} + \frac{2k\pi}{n}}$

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$Z = 1 = 1.e^{0n}$

we take $2k\pi$ with θ because then we will get only one root but with $2k\pi$ we also get other roots.

only when $2k\pi$ is ignored

$R = 1 \qquad \phi = \frac{0}{2} + \frac{k\pi}{2}$

$\omega = Re^{i\phi} = 1\,e^{0} = 1 \quad = 1\,e^{\pi i}$

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Given : $(-1)^{1/3}$

So $z = -1$

$= e^{\pi i}$

$\omega = z^{1/3}$

$\omega = e^{\pi/3\, i}$

$\omega = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\omega = e^{(\pi i + 2k\pi i)1/3}$

$\omega = e^{\pi/3\, i + 2k\pi/3\, i}$

$\omega = \quad k = 0, \pm 1, \pm 2 \text{ -----}$

at $k=0$;

$\omega = e^{\pi/3\, i}$

at $k = 1$

$\omega = e^{\pi/3\, i + 2\pi/3\, i} = e^{\pi i}$

at $k = 2$

$\omega = e^{\pi/3\, i + 4\pi/3\, i} = e^{5\pi/3\, i}$

at $k = 3$

$\omega = e^{\pi/3\, i} \cdot e^{6\pi/3\, i} = e^{7\pi/3\, i}$ $\quad e^{6\pi/3\, i} \to 1$

$\omega = e^{\pi/3\, i}$ it it equal to the first Root

when we are putting higher values of k ; we are getting Repeatation.

at $k = -1$;

$\omega = e^{\pi/3 i - 2\pi/3 i}$

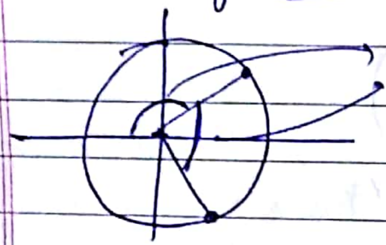
$\omega = e^{-\pi/3 i}$

When we calcultate cubic Roots we get Repeatation after 3 values in 4 Roots we get Repeatation after 4 values.

$e^{-\pi/3 i}$ is same as $e^{5\pi/3}$ (ω_2)

All the roots appear on the circle of radius $R = (r)^{1/n}$

In this question; we get all Roots on the circle of radius 1.



This difference of angle is always same.

Q n^{th} root of unity!

$$(1)^{1/n} = (e^{0i})^{1/n} = (e^{0i + 2k\pi})^{1/n}$$

$R = 1 \qquad \phi = 2k\pi/n$

Q Roots $(-8i)^{(1/3)} \rightarrow$ 1/3 can be easily calculated if $\frac{1}{130}$ is given Than;

Q can be find all Roots or find its geometric expression.

So, First we need to simplify the expression

$\omega = (r)^{1/n} e^{i\theta/n} e^{\frac{2k}{n}}$

let $\alpha = e^{\frac{2\pi}{n} i}$

$\alpha^k = e^{2\pi k i/n}$

$$\omega_0 = \sqrt[n]{r}\, e^{i\theta/n}$$

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$$w_1 = \sqrt[n]{r}\, e^{i\theta/n}\, e^{2\pi i/n}$$

$$w_1 = w_0 \cdot d$$

$$w_2 = \sqrt[n]{r}\, e^{i\theta/n}\, e^{(2\pi i/n)^2}$$

$$\boxed{w_2 = w_0 d^2}$$

$$w_3 = \sqrt[n]{r}\, e^{i\theta/n}\, e^{(2\pi i/n)^3}$$

$$\boxed{w_3 = w_0 . d^3}$$

So we are able to Represent all our Roots as the multiple of w_0.

Sum of Roots :-

$$S = w_0 + w_1 + w_2 + \cdots w_{n-1}$$

$$S = w_0 + w_0 d + w_0 d^2 + w_0 d^3 - - w_0 d^{n-1}$$

$$S = w_0 (1 + d + d^2 + d^3 - - d^{n-1})$$

$$\boxed{S = w_0 \left(\frac{1-d^n}{1-d}\right)}$$

$$d^n = e^{2\pi i} = 1 \qquad \boxed{d^n = 1}$$

$$S = w_0 (0) \qquad * \boxed{S = 0} *$$

Product of Roots :-

$$P = w_0 . w_1 . w_2 . - - w_{n-1}$$

$$P = w_0 . w_0 d . w_0 d^2 * - - w_0 d^{n-1}$$

$$P = w_0^n (d . d^2 . d^3 . - - d^{n-1})$$

$$P = w_0^n (d^{1+2+3 - - n-1})$$

$$P = w_0^n\, d^{(n-1)(n)/2}$$

$$\cancel{P = w_0^n . e^{\pi i (n-1) \times 2}}$$

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$$P = \left(\sqrt[n]{r}\, e^{i\theta/n}\right)^n \left(e^{2\pi i/n}\right)^{\frac{n(n-1)}{2}}$$

$$P = \underbrace{\left(r e^{i\theta}\right)}_{z} e^{(n-1)\pi i}$$

$$P = z(-1)^{n-1} \qquad \boxed{P = (-1)^{n-1} z}$$

Q. $z^3 = -8i$ find all its Roots and then Represent them graphically.

$$z^3 = -8i \qquad z = (-8i)^{1/3}$$

$$z = 2(-i)^{1/3} = 2\left[e^{-\pi/2\, i + 2k\pi i}\right]$$

$$z = 2\left[e^{-\pi/6\, i + 2k\pi/3\, i}\right]$$

$k = 0, 1, 2$

for $k = 0$; ~~$W_0 = 2$~~

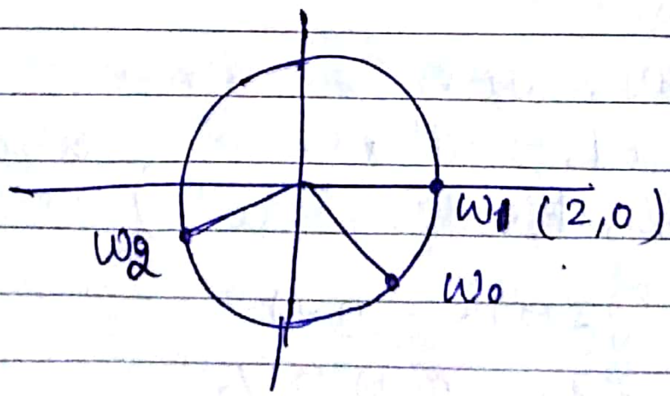
$$W_0 = 2e^{-\pi/6\, i} = 2\left(\cos\left[\frac{-\pi}{6}\right] + i\sin\left(\frac{-\pi}{6}\right.\right.$$

$k = 1$

$$w_1 = 2e^{\left[-\pi/6\, i + \frac{2\pi}{3} i\right]} = 2e^{3\pi/6\, i}$$

$k = 2$

$$w_2 = 2e^{\left[-\pi/6\, i + \frac{4\pi}{6} i\right]} = 2e^{7\pi/6\, i}$$



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Q (a) $(-1+i)^{1/3}$ (b) $(-1-i)^{1/4}$

find all the roots and locate them graphically

(a) $z^5 = 32$

(b) $z^4 = (-2\sqrt{3} - 2i)$

(c) $(-2\sqrt{3} - 2i)^{1/4}$

(a) Argument in this case $\pi/4$

$$\text{Arg } z = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

Polar Representation $\rightarrow (\sqrt{2}\, e^{3\pi/4\, i})$

1. $(-1+i)^{1/3} \Rightarrow (\sqrt{2})^{1/3} e^{i\pi/4}$

$\boxed{(-1+i) \quad -\frac{\pi}{4} + \pi = \frac{3\pi}{4}}$ $\Rightarrow 2^{1/6} e^{i\pi/4}$

$$z = 2^{1/6}\left[e^{i\pi/4 + 2k\frac{\pi}{3} i}\right]$$

$k=0 \qquad z = 2^{1/6} e^{i\pi/4}$

$k=1 \qquad z = 2^{1/6} e^{(\pi/4 + 2\pi/3)i}$

$k=2$

$\Rightarrow z^5 = 32 \qquad z = (32)^{1/5}$

$z = (32)^{1/5} (1)^{1/5}$

$z = 2(e^{0i})^{1/5}$

$z = 2(e^{0i + 2k\pi i})^{1/5}$

$z = 2(e^{0/5 i + 2k\pi/5 i})$

$k = 0, 1, 2, 3, 4, 5$

$\Rightarrow z^4 = (-2\sqrt{3} - 2i)$

$z^4 = 2(-\sqrt{3} - i)$

$z^4 = 2(2)(e^{-2\pi/3 i + 2k\pi i})$

$z^4 = (4)^{1/4} (e^{-2\pi/3 i + 2k\pi i})^{1/4}$

$\frac{\pi}{4}$

$\frac{\pi}{3} - \pi$

$-\frac{2\pi}{3}$

$k =$

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Functions :-

Neighbourhood (Nbd) : A deleted Nbd of a pt. z_0 with set of all points z such that $|z - z_0| < \delta$, where δ is any positive real number. $N_\delta(z_0) = \{ z : |z - z_0| < \delta \}$

Interior point : A point z_0 is called interior point of a set S if we can find a δ-nbd of z_0 all of whose points belongs to S.

Boundary : If every nbd of z_0 consists same point of S and some other then S.

Exterior point : If there exist a nbd of z_0 which do not contain any pt. of S

Open set : which consist only interior pts.

Connected Set : If any two points of the set can be joined by a path containing of straight line segment all pt. of which are in S.

Domain : An open & connected set.

Bounded Set : If there exist a constant m such that $|z| < m \ \forall \ z \in S$.

Some sets : $S_1 = \{ z : |z - z_0| = r \}$

$S_2 = \{ z : |z - z_0| < r \}$

$S_3 = \{ z : |z - z_0| \leq r \}$

All Three of them represent different - different circles

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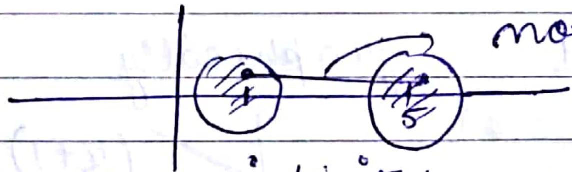
A collection which does not have centre point

Deleted nbd = $\{ z : 0 < |z - z_0| < \delta \}$

comprises all points other than the set z_0.

eg $S = \{ |z-1| < 1$ or $|z-5| < 1 \}$



not connected because they does not contain all pts of S.

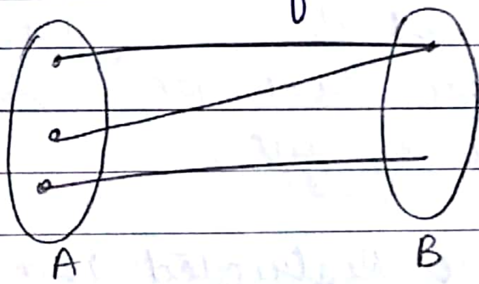
They have infinitely many points

If we can draw a path which does not contain all the points of S Then they are not connected.

Function Mapping :-

$f : A \rightarrow B$

Let A & B are two non-empty sets than a rule ~~that~~ which assigns each element of A to some ~~each~~ element of B (uniquely)



[2 Elements of A can combine to same element]

no ~~two~~ one element of A can combine to two other elements of B

2 types of functions — Single valued / Multi valued

$f(z) = z^{1/2}$ $f(z) = z^{1/3}$ $\longrightarrow$ Multivalued

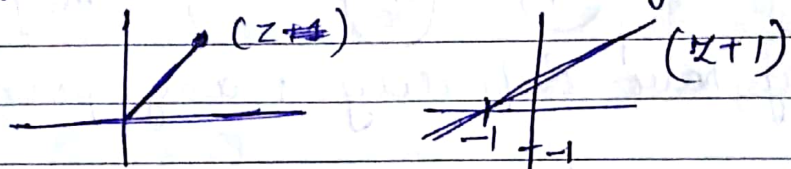
$f(z) = z+1$ (single valued)

$$f(z) = \sqrt{r}\, e^{i\theta + 2k\pi i / n=2} \qquad k = 0, 1, 2 \ldots$$

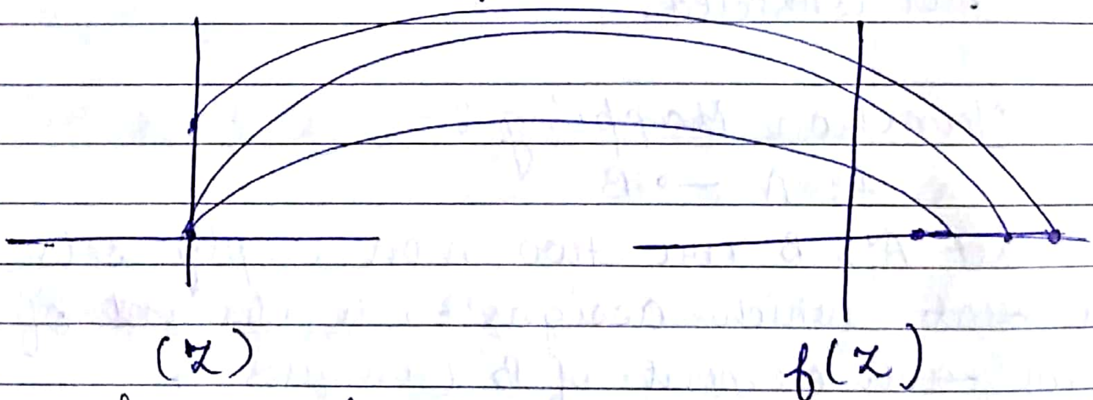
we get diff values

For Individual θ; we get diff. value of f^n
But $f(z)$ have more than one value so
<u>Multivalued f^n</u> is collection of single valued f^n

$f(z) = z+1$ graphically



We need 4D to Represent This; so we use Mapping concept.



Input = 2 Dimensional Output = 2 Dimensional
so we take one plane for Input and one plane for output.

earlier we were Restricted and we were not able to find some values
eg $\log(-1)$
But now $(-1) = e^{\pi i + 2k\pi i}$

$$\log\left(e^{\pi i + 2k\pi i}\right) = \pi i + 2k\pi i$$

smallest value $k = 0$

$$\ln(-1) = \pi i$$

(imaginary values)

Behaviour for algebric & polynomial f^n remains same

But change for $\mathfrak{I}$

Exponential functions :-

$$f(z) = e^{z} = re^{i\theta}$$

O/P will always be a complex

$$e^{z} = e^{\ln r} e^{i\theta}$$

$$e^{z} = e^{\ln r + i\theta}$$

$$\boxed{z = \ln r + i\theta}$$

$e^{z} = -1 - i$

$$e^{z} = \sqrt{2}\left(e^{-3\pi/4\, i}\right)$$

$$e^{z} = e^{\ln\sqrt{2}} e^{-3\pi/4\, i}$$

$$\boxed{z = \ln\sqrt{2} - \frac{3\pi}{4} i}$$

~~$\ln\sqrt{2} = y$~~

$$e^{x} \cdot e^{y} = \sqrt{2}\, e^{-3\pi/4\, i}$$

$$e^{x} = \sqrt{2}$$

$$x = \ln\sqrt{2} \qquad x = \frac{1}{2}\ln 2$$

also $z = \dfrac{\ln 2}{2} + i\left(-\dfrac{3\pi}{4} + 2k\pi\right)$

$$f(z) = e^{z} = re^{i\theta}$$

$$e^{z_1} e^{z_2} = e^{z_1 + z_2} = e^{(x_1 + x_2) + i(y_1 + y_2)}$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2} = e^{(x_1 - x_2) + i(y_1 - y_2)}$$

$$f(z) = \ln z \qquad \ln z = w \qquad z = e^{w}$$

$$\boxed{e^{w} = z} \iff w = \log z \mid \ln z$$

$$e^{w} = re^{i\theta}$$

$$\boxed{w = \ln r + i(\theta + 2k\pi)}$$

continuing this part after 4 pages

Somnath Sir

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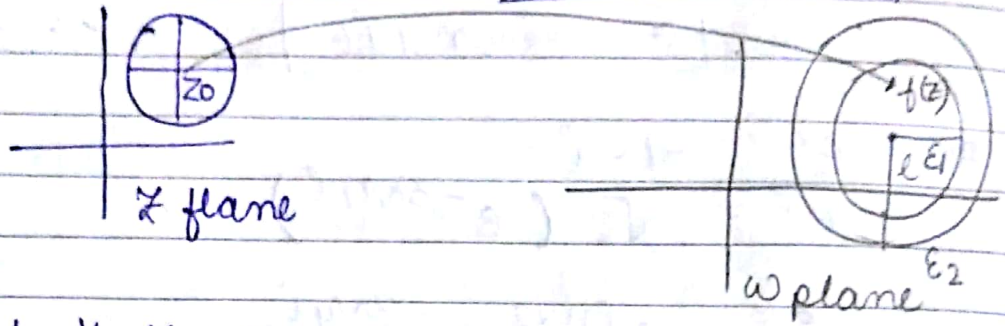
limit, continuity ;

$w = f(z)$ has a limit l when z approaches to z_0 (denoted) by $\boxed{\lim_{z \to z_0} f(z) = l}$



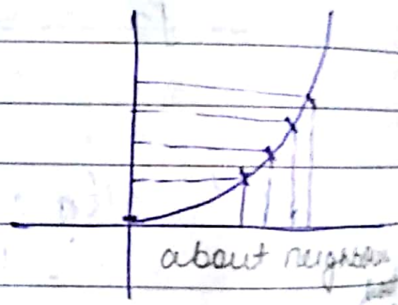
if $\forall z \neq z_0$ such that

$$|z - z_0| < \delta , \quad |f(z) - l| < \varepsilon$$

Ques

$y = x^2$

$\lim_{x \to 1} y$



Now; $f(z) = iz/2$

$\lim_{z \to 1} f(z)$, $l = \frac{i}{2}$

By ε, δ Method;

$$|f(z) - l| = |iz/2 - i/2| = \frac{|i|}{2} [|z-1|]$$

$$\Rightarrow \frac{1}{2} |z-1| < \varepsilon$$

$$\Rightarrow |z-1| < 2\varepsilon$$

choose, $\delta \leq 2\varepsilon$

$\lim_{z \to 1} f(z) = \frac{i}{2}$ $\quad \varepsilon > 0 ; \exists \delta$

$|z-1| < \delta \Rightarrow |f(z) - l| < \varepsilon$

Suppose; you choose

$$f(n) = \frac{1}{n} \; ; \; n \in N$$

$\rightarrow n = 1 \qquad f(n) = 1$

$\rightarrow n = 2 \qquad f(n) = 1/2$

$\rightarrow n = 10 \qquad f(n) = 1/10$ so; The value of $f(n) \rightarrow 0$ on ↑ing n;

We have to take δ either 2ϵ or very less.

CONTINUITY :-

If f is defined at $z = z_0$; $\lim_{z \to z_0} f(z) = l$ and if

$l = f(z_0)$ i.e. $\lim_{z \to z_0} f(z) = f(z_0)$

Then we call f is continous at $z = z_0$. If a f^n is continous at every pt. of a set A; then f is called (continous) if function is continous at every pt of open set in that set (continous in set A)

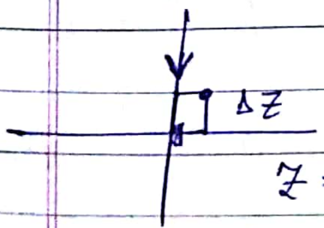
Ques $f(z) = \frac{z}{\bar{z}}$

$\lim_{z \to 0} f(z)$

Path

$z = x$

$\lim_{x \to 0} \frac{x}{x} = 1$



$z = x + yi$

$z = iy$

$\lim_{y \to 0} \frac{-iy}{iy} = -1$

limit does not exist.

Derivative of $f(z)$:-

$f(z)$ has a derivative at $z = z_0$ written as $f'(z_0)$ and defined by

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided the limit ~~the~~ exists

$$\Delta z = z - z_0$$

$$z = \Delta z + z_0$$

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

Exp $\quad f(z) = z^2 \qquad z = z_0$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z z_0 \Delta z + \Delta z^2}{\Delta z} \qquad (\Delta z \neq 0)$$

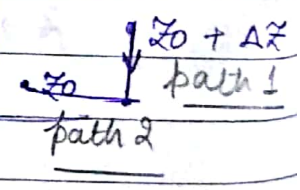
$$= \lim_{\Delta z \to 0} (z z_0 + \Delta z)$$

$$\to \underline{\underline{z z_0}}$$

Ques $\quad f(z) = \bar{z}$

$$\lim_{\Delta z \to 0} \frac{\overline{z_0 + \Delta z} - \overline{z_0}}{\Delta z}$$

$$\lim_{\Delta z \to 0} = \frac{\overline{\Delta z}}{\Delta z} \qquad \Delta z = \Delta x$$



for first path ; $\Delta y \to 0$ then $\Delta x \to 0$

2nd path ; $\Delta x \to 0$ then $\Delta y \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

II $\quad \Delta z = i\Delta y$

$$\lim_{\Delta y \to 0} \cdot \frac{-i\Delta y}{\Delta y} = -1$$

$f'(z_0)$ does not exist

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But $f(z) = \bar{z}$ is continous everywhere (show it)
Hence continuity does not imply derivative of a function. Moreover, if f has derivative at $z = z_0$, then it must be continous at $z = z_0$.

Let $f'(z_0)$ exist

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\lim_{z \to z_0} \left(f(z) - f(z_0) \right) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \times (z - z_0) \quad (z \neq z_0)$$

$$\Rightarrow \lim_{z \to z_0} (z - z_0) = f'(z_0) \times 0$$
$$= 0$$

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$$ f is continous at z_0

Derivative f^n must be continous.

<u>Show that</u>; $f(z) = |z|^2$ has derivative at $z = 0$

Dheerendra sir

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$\Rightarrow$ $\log z = \ln r + i(\theta + 2k\pi)$

$\log(-1) = \ln(-1) + i(\pi + 2k\pi)$

$0 + i(\pi + 2k\pi)$; $k = 0, \pm 1, \pm 2 \ldots$

Principal logarithm $\log z$

$\log(-1) = i(\pi + 2k\pi)$

$\log(-1) = i\pi$

$Arg(z_1 z_2) = Arg\, z_1 + Arg\, z_2$

$\log(z_1 z_2) = \log z_1 + \log z_2$

$z_1 = z_2 = -1$

1. exponential : $e^z = f(z) = e^z$

$e^z = e^{x+iy} = e^x . e^{iy}$

$\Rightarrow e^x(\cos y + i \sin y)$

$= e^x \cos y + i e^x \sin y$

2. Logarithmic : $f(z) = \log y$

$e^w = z \Rightarrow w = \log z$

$* \rightarrow \log z = \ln|z| + i \arg z$

Capital letter $\leftarrow$ $\boxed{\mathrm{Log}\, z = \mathrm{Ln}|z| + i \arg z}$ — Principal Logarithm

• Complex exponents $\rightarrow$ when $z \neq 0$ & the exponent c is any complex no., then f^n z^c is defined by means of the equation

$z^c = e^{c \log z}$

$z^c = e^{\log z^c}$

Calculate $\rightarrow$ i^{-2i}

Soln $= e^{\log i^{-2i}} = e^{-2i \log i}$

first $\log(i) = \log|i| + i \arg(i)$

$= \frac{\pi}{2} + 2k\pi i$

Now; $i^{-2i} = e^{-2i(\pi/2 + 2k\pi) i}$

$2(\pi/2 + 2k\pi)$

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(2) i^{1+i}

$= e^{\log i^{1+i}}$

$= e^{(1+i)\log i}$

$= e^{(1+i)(\pi/2+2k\pi)i}$

$= e^{(i-1)(\pi/2+2k\pi)}$

$= e^{\pi/2 i+2k\pi i-\pi/2-2k\pi}$

$= ie^{-\pi/2}$

(3) 2^{1+i}

$= e^{\log 2^{1+i}}$

$= e^{(1+i)\log 2}$

$= e^{\log 2+i\log 2}$

$= e^{\log 2} \cdot e^{i\log 2}$

$= e^{\log 2}(\cos\log 2 + i\sin\log 2)$

$= 2\cos\log 2 + i\sin\log 2$

(4) $(-1)^{1/\pi}$

$e^{1/\pi \log(-1)}$

$e^{1/\pi(\pi i+2k\pi i)}$

$e^{i(1+2k)}$

$\cos(1+2k)+i\sin(1+2k)$

(5) $(1+i)^{i}$

Ans $e^{i(\log\sqrt{2}+i(\pi/4+2k\pi i))}$

Note • $\log(z_1z_2) = \log z_1+\log z_2$, but $\mathrm{Log}(z_1z_2) \neq \mathrm{Log}\, z_1+\mathrm{Log}\, z_2$

• $\arg(z_1z_2) = \arg z_1+\arg z_2$, but $\mathrm{Arg}(z_1z_2) \neq \mathrm{Arg}\, z_1+\mathrm{Arg}\, z_2$

?• $(z_1z_2)^c = z_1^c z_2^c$ for principal value of $(z_1z_2)^c \neq z_1^c z_2^c$

ex $z_1 = (1+i)$ $z_2 = (1-i)$ $z_3 = (-1-i)$

$PV = (z_1z_2)^i = (2)^i = \cos(\log 2)+i\sin(\log 2)$

$(z_2z_3)^i = 2^i = \cos(\log 2)+i\sin(\log 2)$

then $z_1^i z_2^i$ & $z_2^i z_3^i$ & verify the property.

• in exponential f^n problem arg will differ by 2π in power of e.

Trigonometric function of complex No.

$e^{i\theta} = \cos\theta+i\sin\theta$ —(1) $e^{-i\theta} = \cos\theta-i\sin\theta$ —(2)

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$\log z = \ln|z| + i \arg z$

capitals ← $\text{Log}\, z = \ln|z| + i\, \text{Arg}\, z$

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1+2. $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

1−2. $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\quad$ $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + i \cos x \sinh y$

$\cos iy = \dfrac{e^{-y} + e^{y}}{2} = \dfrac{e^{y} + e^{-y}}{2} = \cosh y$

Hyperbolic function: always unbounded f^n

$\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$ $\quad$ $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2i}$

$\sin(iy) = \dfrac{e^{-y} - e^{y}}{2i} = i^2\left[\dfrac{e^{y} - e^{-y}}{2i}\right] = \sin iy = i \sinh y$

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{e^{-y+ix} - e^{y-ix}}{2i}$

$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$
$= \cos x \cosh y - i \sin x \sinh y$

14/8/17 Dheerendra Sir Class

Q $\log(i)^3 - 3\log i = 3\left(\dfrac{\pi}{2}\right)i$

{ using $\log z = \ln|z| + i \arg z$ }

$\log(i^3) = \log(-i) = -\dfrac{\pi}{2} i$

Q $\log(i)^2 = 2\log i = 2\left(\dfrac{\pi}{2}\right)i = \pi i$

$\log(i^2) = \log(-1) = \pi i$

Q Find all the ~~Roots~~ solutions of the equation

$\log z = \dfrac{i\pi}{2}$ $\quad$ proceed as $e^{z} = 1 + i$

Both were not coming same because

$\log(i^2 i) = \log(-1 \cdot i)$

$\log(-1) + \log(i)$ $\quad$ Because Real part negative

→ Trying to calculate principal values only.

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$z = (-1)^{1/3} = \boxed{z^3 = 1}$.

[log = Multi valued function]
because $\log z = \ln|z| + i \arg z$
(infinite values)

Functions

Single valued → limit → Continuity → Differentiability → Analyticity

Multivalued (we will solve for principal value only)

\# We will go sequentially.

Limit : Let a function 'f' be defined at all points z in some deleted neighbourhood of z_0. The limit of $f(z)$ as z approaches to z_0 is a no. w_0,

$$\boxed{\lim_{z \to z_0} f(z) = w_0} \quad — ①$$

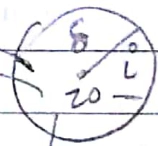
In other words → For each positive no. ε there is a positive no. δ such that $|f(z) - w_0| < \varepsilon$, whenever $0 < |z - z_0| < \delta$.

This definition says that for each ε-neighbourhood there is deleted δ nbd $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε-nbd.

(eg) $|f(z) - w_0| < 1/2$ whenever $0 < |z - z_0| < 1/4$

(Input/Domain circle)

→ In our domain



any one pt points outside this circle, then limit does not exist.

all points of this circle are mapping inside that circle then limit exist

Q $\lim_{z \to 0} \dfrac{z}{|z|} =$ (we can't put 0 directly on both of the Num and den)

$\{z \to 0;\ x \to 0;\ y \to 0\}$

we will see some alternative way

→ If limit exist on every path then we say limit exists

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→ $\lim_{z \to 0} \frac{z}{|z|} = \lim_{x+yi \to 0} \frac{x+yi}{|x+yi|}$

→ Repeated (iterative) limit :-

$\lim_{x \to 0 ; y \to 0} \frac{x+yi}{\sqrt{x^2+y^2}} = \lim_{y \to 0} \frac{yi}{y} = i$

$\boxed{\text{limit}_{z \to 0} \frac{z}{|z|} = i}$ (when we approach through $x \to 0$ then $y \to 0$)

Now

$\lim_{y \to 0 ; x \to 0} \frac{x+yi}{\sqrt{x^2+y^2}} = \lim_{x \to 0} \frac{x}{x} = 1$

$\lim_{z \to 0} \frac{z}{|z|} = 1$ (when we approach through $y \to 0$ then $x \to 0$)

limit does not exist, because we are getting diff values for diff. paths.

Q-2 $\lim_{z \to 0} \frac{(\text{Re} z - \text{Im } z)^2}{|z|^2}$

Let $y = mx$ (Taking this path)

$\lim_{x+yi \to 0} \frac{(x-y)^2}{x^2+y^2}$

$\lim_{x \to 0} \frac{(x-mx)^2}{x^2+(mx)^2} = \frac{(1-m)^2}{1+m^2}$

∴ Now we have to check that can we apply this Repeated limit approach in this ques. Is this sufficient?

so along diff-diff paths, for diff. values of m we get diff. values of limit; so limit does not exist.

\# If from Repeated method limits are not equal then limit does not exist but if limits come equal then it is not sufficient method to tell so we need to check the limit by path ($y = mx$) method.

$\lim_{x \to 0\, y \to 0} \frac{(x-y)^2}{x^2+y^2} = \lim_{y \to 0} \frac{y^2}{y^2} = 1$

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eg like $\lim_{z\to 0} \frac{(\text{Im } z)^2}{|z|^2}$

$\lim_{z\to 0} \frac{\text{Re } z}{|z|}$

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Q-3 $\lim_{z\to 0} \left[\frac{1}{1-e^{1/x}} + iy^2 \right]$
?

Ques 1 $\lim_{z\to 1} f(z) = i/2$
$f \downarrow i\bar{z}/z$

$f(z) = \frac{i\bar{z}}{z}$; $z_0 = 1$
; $w_0 = i/2$

Let;
$0 < |z-1| < \delta$ — ①

$|f(z) - w_0| = \left|\frac{i\bar{z}}{z} - \frac{i}{2}\right|$

Don't solve them;
just try to find Relationship in between them

$= \frac{1}{2}|\bar{z}-1| = \frac{1}{2}|z-1| = \frac{\delta}{2}$ {from eq^n ①}

we can assume that Let $\frac{\delta}{2} = \varepsilon$ $\boxed{\delta = 2\varepsilon}$

② $\lim_{z\to i} z^2 = -1$

$z_0 = i$; $w_0 = -1$; $f(z) = z^2$

$0 < |z-i| < \delta$ — ①

$|f(z) - w_0| = |z^2 - (-1)|$

$\to |(z - i)(z+i)|$

$\to |z-i|\,|z - i + 2i|$

$\to \leq$

Q-2 Applying triangular Inequality

$\leq |z-i| \{ |z-i| + |2i| \} = \varepsilon$

$\leq \delta(\delta + 2) = \varepsilon$

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Continuity :- A function f is continous at a point z_0 if all three of the following condition holds:

① $\lim_{z\to z_0} f(z)$ exist

② $f(z_0)$ exist

③ $\lim_{z\to z_0} f(z) = f(z_0)$

∴ Continuity is something that value at that particular pt. exist.

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we deleted nbd $\leftarrow$ $0 < |z - z_0| < \delta$ $\Rightarrow$ It does not include any point
of z_0

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The difference is only nbd and deleted nbd of z_0 in b/w limit and continuity.

Eg ② $f(z) = \log z$
① $f(z) = \frac{i\bar{z}}{2}$ at $z = i$

$$\lim_{z \to i} = \frac{i\bar{z}}{2} = \frac{1}{2}$$
Also $f(i) = 1/2$

It is continous

$f(z) = \text{Log}\, z$; find set of all points of discontinuity.

limit of function does not exist.

© wether $f(z)$ is continous over negative Real axis.

- $$\lim_{z \to -1} f(z) = \lim_{z \to -1} \left[\ln|z| + i \,\text{Arg}\, z \right]$$

$f(-1) \Rightarrow \underline{\underline{i\pi}}$ (value at the point exist)

- $$\lim_{z \to -1} = \lim_{\substack{x \to -1 \\ y \to 1}} \begin{cases} +\pi & y > 0,\ x < 0 \\ -\pi & y < 0,\ x > 0 \end{cases}$$

for any z; we have fix argument But the path to Reach that point can be different.

Log Z is not continous on negative Real Axis including zero.

Differentiability :- Let $f(z)$ be a single valued function defined in a domain D. The function $f(z)$ is said to be differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

$z = z_0 + \Delta z$

and let denotes this limit by $f'(z_0)$ and $z - z_0 = \Delta z$ so $z = z_0 + \Delta z$, we get;

$$\boxed{f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z) - f(z_0)}{\Delta z}}$$

→ Ratio of diff. of o/p by diff of i/p.

For every $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t.

$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$; whenever

$0 < |z - z_0| < \delta$

Q ex $f(z) = z^2$ is this function diff?

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - (z)^2}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\cancel{z^2} + \Delta z^2 + 2z\Delta z - \cancel{z^2}}{\Delta z} \right)$$

$$\rightarrow \lim_{\Delta z \to 0} (2z + \Delta z) = 2z \text{ Ans}$$

Q $f(z) = \bar{z}$ is this f^n continous?

yes it is continous.

→ is this f^n diff at any point?

$f(z) = \bar{z}$ $f(z + \Delta z) = \overline{z + \Delta z}$

$$\left| \lim_{\Delta z \to 0} \frac{(\overline{z + \Delta z}) - (\bar{z})}{\Delta z} \right| = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \frac{x - yi}{x + yi}$$ } next page

Taking $y = mx$

$$\lim_{x \to 0} \frac{1 - mx}{1 + mx}$$

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Now for going along different paths ; we get different results so function is discontinuous

Now; $\lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z} = \lim_{\Delta x + i\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$

Take path $\Delta y = m\Delta x$

then : $\lim_{\Delta x \to 0} \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x} = \frac{1 - im}{1 + im}$

For diff m; diff values.

Cauchy- Reimann equation :-

(Necessary But not Suff)

Let $f(z) = u(x,y) + iv(x,y)$ is defined and continous at a point $z = x + yi$ and differentiable at z. Then a pt z, first order partial derivative exist and satisfy the C-R equation.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

If f^n does not satisfy this eq^n Then it is non-differentiable.

$f(z) = \bar{z}$

$\bar{z} = x - yi$ $\quad f(z) = u + iv$

$u = x$ $\quad v = -y$

diff w.r.t x consider y as a constant $\boxed{\frac{\partial u}{\partial x} = 1}$ $\quad \boxed{\frac{\partial v}{\partial y} = -1}$

$\frac{\partial u}{\partial y} = 0$

$\partial v / \partial x = 0$

Both the conditions should satisfy simultaneously.

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Q. for $f(z) = |z|^2$ find out the point of differentiability?

$|z|^2 = x^2+y^2$ $\qquad f(z) = u + iv$

$\underline{u = x^2 + y^2}$. $\qquad \underline{v = 0}$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial x} = 0$

$\Bigg\}$ Not equal, f^n not differentiable

$\Rightarrow$ It does not satisfy CR equation

for $z=0$; all conditions are true so we can say $z=0$ May be a pt. of differentiable But at all other points non differentiable

$\rightarrow$ $z = x + iy \qquad \Delta z = \Delta x + i\Delta y$

$z + \Delta z = (x+\Delta x) + (y+\Delta y)i$

$f(z) = u(x,y) + i v(x,y)$

$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iy(x+\Delta x, y+\Delta y)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$*\ \boxed{f'(z) = \lim_{\Delta z \to 0} \left(\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta z} + i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta z} \right)}$$

we got this expression

If limit exist along the diff paths; then limit exists.

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Take $\Delta y \to 0$, first

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \quad — ①$$ partial differentiation

Now take $\Delta x \to 0$ first

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y i} + i\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y i}$$

$(-i)$

$$\boxed{f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}} \quad — ②$$

Now we are getting 2 diff values; along 2 diff paths. But

As f^n is diff ① = ②

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Now, these are CR equations (Derive CR equations)

Q $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & ; z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

f^n diff or not.

Q $$\frac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2} \to \underbrace{\frac{x^3-y^3}{x^2+y^2}}_{u(x,y)} + i\underbrace{\left(\frac{x^3+y^3}{x^2+y^2}\right)}_{v(x,y)}$$

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$\rightarrow \dfrac{(x-y)(x^2+y^2+xy)}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2(3x^2)-(x^3-y^3)(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{3x^4+3x^2y^2-2x^4+2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{x^2+y^2(3x^2)-(x^3+y^3)(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2}$$

$\partial u/\partial x \neq \partial v/\partial y$ (Not differentiable)

$$u_x = \frac{3x^2(x^2+y^2)-(x^3-y^3)(2x)}{x^2+y^2}$$

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x^3/x^2}{x} = 1$$

Somnath Sir Class 18/08/17

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$\lim_{y\to 0} \frac{-y^3/y^2}{y} = -1$$

$u_x(0,0) = 1 = v_y(0,0)$

$u_y(0,0) = -1 = -v_x(0,0)$

$$f'(0) = \lim_{\Delta z\to 0} \frac{f(\Delta z)-f(0)}{\Delta z}$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\left(\dfrac{\Delta x^3-\Delta y^3}{\Delta x^2+\Delta y^2} + i\,\dfrac{\Delta x^3+\Delta y^3}{\Delta x^2+\Delta y^2}\right) - 0}{\Delta x + i\Delta y}$$

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Path 1 →

$\Delta z = \Delta x$

$$\lim_{\Delta x \to 0} \frac{\Delta x^3}{\Delta x^2} + \frac{i\Delta x^3}{\Delta x^2} \Big/ \Delta x$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x^3}(1+i)}{\cancel{\Delta x^3}}$$

$$\rightarrow (1+i) \quad \{\Delta x \neq 0\}$$

Path II :-

$\Delta z = \Delta x + i\Delta y$ $\qquad \Delta y = \Delta x$

$$f'(0) = \lim_{\Delta x \to 0} \frac{\dfrac{\cancel{x} i \Delta x^3}{\cancel{x}\Delta x^2}}{\Delta x(1+i)}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{i\Delta x^3}{\Delta x^3(1+i)} \Rightarrow \frac{i}{1+i} \qquad (\Delta x \neq 0)$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{i(1-i)}{(1+i)(1-i)} \Rightarrow \frac{-i^2+i}{2} \Rightarrow \frac{1+i}{2}$$

Since Both the limits are different ⇒ limit does not exist. $f'(0)$ also does not exist

Now $\quad u_x = \dfrac{x^4 + 3x^2y^2 + 2xy^3}{(x^2+y^2)^2} \qquad u_x = \dfrac{du}{dx}$

If we go along the path $y = mx$

$$\lim_{\substack{x \to 0 \\ y \to 0}} u_x = \lim_{x \to 0} \frac{x^4 + 3m^2x^4 + 2m^3x^4}{(1+m^2)^2 x^4}$$

It is not continous at $(0,0)$ $\quad = \dfrac{1 + 3m^2 + 2m^3}{(1+m^2)^2}$

function has no derivative at $(0,0)$.

Q $\quad f(z) = \dfrac{z^2+1}{(z^2-3)(z^2+4)}$ $\Big\} \rightarrow$ Polynomial function

No limit $\Rightarrow$ Not differentiable

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$$(f/g)' = \frac{gf' - fg'}{g^2} \quad \text{if } g(z) \neq 0$$

so, at pt.

$z^2 - 3 = 0 \quad z = \pm\sqrt{3}$ f^n is not differentiable

$z^2 + 4 = 0 \quad z = \pm 2i$ f^n not differentiable

$f(z)$ has derivative except at the points $z = \pm\sqrt{3}, \pm 2i$

at the points function will also lose its analyticity.

(2) $f(z) = e^z = e^{xe^{iy}}$

$z = xe^{iy}$

$= e^{x(\cos y + i\sin y)}$

$= e^x (\cos y + i \sin y)$

$= u + iv$

$u = e^x \cos y \quad ; \quad v = e^x \sin y$

$u_x = e^x \cos y \qquad u_y = -e^x \sin y$

$v_x = e^x \sin y \qquad v_y = e^x \cos y$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \begin{cases} u_x = u_x(x_0, y_0) \\ v_x = v_x(x_0, y_0) \\ u_y = u_y(x_0, y_0) \\ v_y = v_y(x_0, y_0) \end{cases}$$

{ At all points function is continous }

$\boxed{v_x = -u_y} \qquad \boxed{u_x = v_y} \quad \forall (x,y)$

$f'(z) = u_x + i v_x = v_y - i u_y$

$f'(z) = e^x \cos y + i e^x \sin y$

$= e^x (e^{iy}) = e^z$

Now $f(z) = e^{cz}$ $\quad c = a+bi \in \mathbb{C}$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$\Delta z = \Delta x + i\Delta y$$

$$f = u(x,y) + iv(x,y)$$

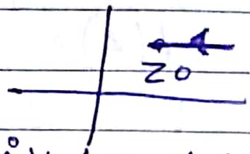
★ $f(z) = z^2 = (x^2-y^2) + 2xyi$

$$= u(x,y) + iv(x,y)$$

$$f(z_0+\Delta z) = u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x + y_0+\Delta y)$$

If $f'(z_0)$ exists

Path I $\to$ $\Delta z = \Delta x$



$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y_0) - u(x_0,y_0) - iv(x_0,y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} +$$

$$i \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

So, this $f'(z) = \frac{df}{dz} = v_x + iv_x$ at (x_0, y_0)

$$\Delta z = i\Delta y$$

$$f'(z_0) = v_y - i\underline{u_y}$$

Q e^{cz} $\quad c = a+bi \in \mathbb{C}$

$$= e^{(a+bi)(x+yi)}$$

$$= e^{(ax-by) + i(ay+bx)}$$

$$= u + vi$$

$$f(z) = u_x + iv_x$$

$$= \underline{ce^{cz}}$$

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continuing CR equations:-

- satisfaction of C-R equations for a function $f(z)$ at $z_0 \Rightarrow$ $f(z)$ May or May not be differentiable at z_0.

- Dissatification of C-R eq^n is
  $\Rightarrow$ $f(z)$ never differentiable at z_0

Theorem $\rightarrow$ Let the function $f(z) = u + iv$ be defined in some neighbourhood of z_0, if partial derivative u_x, u_y, v_x, v_y exist throughout that neighbourhood & satisfy CR equation as well as are continous at (x_0, y_0) then $f'(z)$ exists at z_0 & $f'(z_0) =$
$u_x(x_0, y_0) + i v_x(x_0, y_0)$

Proof If $f(x,y)$ has continous partial derivative f_x, f_y at a pt. (x_0, y_0)

then $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) =$
$$f_x(x_0, y_0)\Delta x + f_y(x_0, y_0) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$
where $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $\Delta x \cdot \Delta y \rightarrow 0$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
$$\Delta z = \Delta x + i \Delta y$$

$$f(z_0 + \Delta z) - f(z_0) = u(x_0 + \Delta x, y_0 + \Delta y) + i v(x_0 + \Delta x, y_0 + \Delta y) - [u(x_0, y_0) + i v(x_0, y_0)]$$

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$$= \left[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)\right] + i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right] \quad -(1)$$

From above Result →

$$u_x(x_0,y_0)\,\Delta x + u_y(x_0,y_0)\,\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y$$ where

each of $\varepsilon_1, \eta_1 \to 0$ as both $\Delta x, \Delta y \to 0$

In the same manner

$$v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0) = v_x(x_0,y_0)\,\Delta x + v_y(x_0,y_0)\,\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y$$

where $\varepsilon_2, \eta_2 \to 0$ as Both $\Delta x, \Delta y \to 0$

from (1)

$$f(z_0+\Delta z) - f(z_0) = \left[u_x(x_0,y_0)\,\Delta x + u_y(x_0,y_0)\,\Delta y\right] + i\left[v_x(x_0,y_0)\,\Delta x + v_y(x_0,y_0)\,\Delta y\right] + \varepsilon \Delta x + \eta \Delta y$$

where

$$\varepsilon = \varepsilon_1 + i\varepsilon_2 \qquad \eta = \eta_1 + i\eta_2$$

ε & $\eta \to 0$ as $\Delta x, \Delta y \to 0$

$$\Rightarrow \left[u_x(x_0,y_0)\,\Delta x + u_y(x_0,y_0)\,\Delta y\right] + i\left[-u_y(x_0,y_0)\,\Delta x + \right. \quad \varepsilon \Delta x + \eta$$

$$\Rightarrow \left[u_x(x_0,y_0) + i v_x(x_0,y_0)\right]\left[(\Delta x + i\Delta y) + \varepsilon \Delta x + \eta \Delta y\right]$$

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$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u_x(x_0, y_0) + i v_x(x_0, y_0) + \frac{\epsilon \Delta x + \eta \Delta y}{\Delta x + i \Delta y}$$

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

$$z = re^{i\theta}$$

$$z \neq 0$$

$$f(z) = f(re^{i\theta}) \; ; \; u(r,\theta) + i v(r,\theta)$$

$$f'(re^{i\theta})\, e^{i\theta} = \underline{u_r + i v_r}$$

$$f'(re^{i\theta})\, i r e^{i\theta} = u_\theta + i v_\theta$$

$$ir(u_r + i v_r) = u_\theta + i v_\theta$$

$$\boxed{r u_r = v_\theta} \qquad \boxed{-v_r r = u_\theta}$$

<u>Analyticity at pt z_0</u> :-

A f^n $f(z)$ is said to be analytic at pt z_0 if $\exists$ a neighbourhood $|z - z_0| < \delta$ such that $f(z)$ is differentiable at every pt. of that neighbourhood.

eg ① $f(z) = |z|^2$ no where analytic
differentiable at $z = 0$ proved earlier

② $f(z) = \frac{1}{z}$ is analytic everywhere in z-plane except at origin.

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Entire function :-

A function is said to be entire function if it is analytic everywhere in the finite complex plane eg $\sin z$, $\cos z$, polynomial etc.

Q If $f(z)$ satisfies Cauchy-Reimann eq at z_0 so does $(f(z))^n$ for every positive integer n.

Proof $f(z)$ $u(x,y)+iv(x,y)$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) f(z) = 0$$

at z_0

$$\left(\partial/\partial x + i\,\partial/\partial y\right)\left(f(z)\right)^n$$

→ If $f(z)$ is continous so $|f(z)|$

Ans $f(z) = u(x,y) + i\, v(x,y)$

Since $f(z)$ is continous at $z_0 = x_0 + y_0 i$ =

$u(x,y)$ & $v(x,y)$ are continous at x_0, y_0

→ $(u(x,y))^2 + V(x,y)^2 =$

$|f(z)| = \sqrt{u(x,y)^2 + V(x,y)^2}$

CR eq^n $\left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right) f(z) = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

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$f(z) = u(x,y) + iv(x,y)$
Then,

$u_x = v_y$ & $u_y = -v_x$

similarly in polar, $f(z) = u(r,\theta) + i\,v(r,\theta)$
Then,

$u_r = \frac{1}{r} v_\theta$ & $v_r = -\frac{1}{r} u_\theta$

Analytic at z_0,
If $f(z)$ is differentiable at each point of some nbd of, then we say that $f(z)$ is analytic at z_0.

eg)

$f_1(z) = |z|^2$
$f_1(z) = x^2+y^2+i0$
$u = x^2+y^2$
$v = 0$
$u_x = 2x$, $v_x = 0$
$u_y = 2y$, $v_y = 0$

$\boxed{u_x \neq v_y}$

only possible at $x=0$,
so $x=0, y=0$ are the only conditions
CR conditions only hold at $z=0$
$z=0$ only pt. of differentiability
at $z=0$, → anglytic

$f_2(z) = \bar{z}$
$f_2(z) = x - iy$
$u = x$
$v = -y$
$u_x = 1$, $v_x = 0$
$u_y = 0$, $v_y = -1$

It also does not satisfy C-R equations
so it is not analytic function

$f_3(z) = z^2$
$= x^2 - y^2 + 2xyi$
$u = x^2 - y^2$
$v = 2xy$
$u_x = 2x$
$v_x = 2y$
$u_y = -2y$
$v_y = 2x$

It satisfy C-R equation at each and every pt.
so function is differentiable also it is analytic

even in small circles it is differentiable so analytic

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Analyticity means we have a small circle and check f^n is differentiable in it or not.

For differentiability ; we only check at a pt. but for analyticity we check neighbourhood.

Q Check wether $f(z)$ is analytic at $z=0$

$$f(z) = \begin{cases} \dfrac{\operatorname{Im} z - \operatorname{Re} z}{|z|} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

$$\lim_{z\to 0} f(z) = \lim_{z \to 0} \frac{y-x}{\sqrt{x^2+y^2}}$$

Take, $y = mx$

$$\lim_{x\to 0} \frac{mx - x}{\sqrt{x^2+m^2x^2}} = \frac{m-1}{\sqrt{1+m}}$$

So, for different paths limit is different our function is not continous $\Rightarrow$ f^n is not differentiable , we can't check analyticity

Q $f(z) = \operatorname{Log} z$

1) It is continous on negative real axis ?

2) It is differentiable " " " " ?

3) Is it differentiable , other than negative axis ?

Sol $\Rightarrow$ On negative axis f^n is not continous so we can't say anything about differentiability so not analytic.

$z = x + yi$

$\operatorname{Log}(x+iy) = u + vi$

capital 'L' $\rightarrow$ Represent principal argument

$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$

$\ln|z| + i \operatorname{Arg} z = u + vi$

$\frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) = u + vi$

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$$u = \frac{\ln(x^2+y^2)}{2} \qquad V = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, we can check through C-R equations

$$u_x = \frac{x}{x^2+y^2} \qquad V_y = \frac{x}{x^2+y^2}$$

$$u_y = \frac{y}{x^2+y^2} \qquad V_x = \frac{-y}{x^2+y^2}$$

$$\underline{u_x = V_y} \qquad \underline{u_y = -V_x}$$

It satisfies CR equation ; But we must know the terms & conditions as By these eqn f^n look to be diff. in whole plane But it is not diff. in Real negative axis.

Q ⇒ $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

$$u = \frac{x^3 - y^3}{x^2+y^2} \quad , \quad v = \frac{x^3+y^3}{x^2+y^2}$$

$$\underbrace{u_x(a,b)}_{} = \lim_{h\to 0} \frac{u(a+h,b) - u(a,b)}{h}$$

derivative at a single pt.

$$u_y(a,b) = \lim_{k\to 0} \frac{u(a, b+k) - u(a,b)}{k}$$

$$\boxed{\frac{du}{dx} = \frac{x^2+y^2(3x^2) - (x^3-y^3)(2x)}{(x^2+y^2)^2}}$$ not Required

Ux →
$$\lim_{h\to 0} \frac{\dfrac{(a+h)^3 - b^3}{(a+h)^2+b^2} - \left(\dfrac{a^3-b^3}{a^2+b^2}\right)}{h}$$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h}$$

$$\lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = \underline{\underline{1}}$$

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$$U_y(0,0) = \lim_{k\to 0} \frac{u(0,0+k) - u(0,0)}{k}$$

$$\Rightarrow \frac{-k^3}{k^2} \Big/ k = -1$$

$$V_x(0,0) = \lim_{h\to 0} \frac{V(0+h,0) - V(0,0)}{h} = 1$$

$$V_y(0,0) = \lim_{k\to 0} \frac{V(0,0+k) - V(0,0)}{k} = 1$$

$$\underline{U(x) = V_y} \quad \& \quad \underline{U_y = -V_x}$$

So CR equations are satisfied here.

But it is differentiable only if it satisfies the definition of differentiability.

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$\Rightarrow \lim_{x\to 0\, y\to 0} \frac{\frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} - 0}{x+yi}$$

$$\lim_{(x+yi)\to 0} \frac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+yi)}$$

Take path $y = mx$

$$\lim_{x\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{(x^2+m^2x^2)(x+mxi)}$$

$$= \frac{(1+i) - m^3(1-i)}{(1+m^2)(1+mi)}$$

Take $m = 0 = 1+i$

$m = 1$, (some other output)

So f^n is not differentiable

By this eg, we know that CR equations are not sufficient conditions.

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~~Necessary~~ & sufficient condition

[So CR equation satisfied → Partial derivati exist → They are also continious]

Then f^n is always differentiable

- Sufficient conditions :-

1) Partial derivative exist — must
2) CR eq^n holds — Must 3) u_x, u_y, v_x, v_y are also continious

Q $f(z) = e^{\bar{z}}$ 12.4 Section Try

$f(z) = e^{x-yi}$

$u + vi = \frac{e^x}{e^{yi}} = re^{i\theta} =$

$e^z = re^{i\theta}$

$e^{\bar{z}} = re^{-i\theta}$

Necessary condition → CR equation

$f(z) = u(x,y) + iv(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

sufficient eq^n are written above :-

entire functions :- e^z, $\cos z$, $\sin z$ functions which are analytic everywhere in Z.

Harmonic functions :- A real valued f^n H of two variables x and y is said to be Harmonic in a given domain of x-y plane if throughout the domain, it has continous partial derivative of the 1^{st} and 2^{nd} Order and satisfies the equation

$$\boxed{H_{xx}(x,y) + H_{yy}(x,y) = 0}$$

Laplace equation

x and y are Real parameters

z → complex parameter.

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if $f(z) = u(x,y) + iv(x,y)$

u_x and u_{xx} → continous
1st order 2nd order

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Theorem :-

If a function $f(z) = u + iv$ is analytic in a domain D then its component function u and v are Harmonic in D.

eg * $f(z) = |z|^2 = x^2 + y^2 \rightarrow$ not analytic

$u = x^2 + y^2$ $v = 0$

$u_x = 2x$ | $v_x = 0$

$u_{xx} = 2$ | $v_{xx} = 0$

$u_y = 2y$ | $v_{yy} = 0$

$u_{yy} = 2$ | $v_{xx} + v_{yy} = 0$

$u_{xx} + u_{yy} = 4 \neq 0$

u is not Harmonic V is Harmonic

→ If a f^n satisfies CR eq^n need not that it is differentiable. But if diff ⇒ Must satisfy CR equation

Q $f(z) = \bar{z} = x - yi$ → no where differentiable

$u = x$ $v = -y$

$u_{xx} = 0$ $v_{xx} = 0$

$u_{yy} = 0$ $v_{yy} = 0$

$\boxed{u_{xx} + u_{yy} = 0}$ $\boxed{v_{xx} + v_{yy} = 0}$

eg that a f^n is differentiable no where but still its component f^n are Harmonic in nature.

Result →** If u and V are Harmonic Then f^n is not always analytic but if f^n is analytic Than u and V Must be Harmonic

$f(z)$ analytic $\Rightarrow$ u and v Harmonic but converse is not true.

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Q For what value of a and b given f^n is Harmonic?

$$u = ax^2 + by^2$$

$u_{xx} = 2a$ $\qquad$ $u_{yy} = 2b$

For Harmonic $u_{xx} + u_{yy} = 0$

$$2a + 2b = 0$$
$$a + b = 0$$
$$\boxed{a = -b}$$

$$\boxed{u = x^2 - y^2} \qquad \boxed{u = 2x^2 - 2y^2}$$

or $\quad u = ax^2 - ay^2$

Given $\to$ $f(z)$ is analytic $\Rightarrow$ satisfies CR eq^n

$$u_x = v_y \quad -① \qquad u_y = -v_x \quad -②$$

partially differentiating functions

$$\boxed{u_{xx} = v_{yx}} \quad -③ \qquad \boxed{u_{yy} = -v_{xy}} \quad -④$$

Adding 3 + 4

$$u_{xx} + u_{yy} = v_{yx} - v_{xy}$$

\# If f_x and f_y are continous Then $f_{xy} = f_{yx}$ (Result)

So we have v_x and v_y are continous

$$u_{xx} + u_{yy} = v_{xy} - v_{xy}$$

$$\boxed{u_{xx} + u_{yy} = 0}$$

Similarly $\to$

$v_x = -u_y$ $\qquad$ $v_y = u_x$

$v_{xx} = -u_{yx}$ $\qquad$ $v_{yy} = u_{xy}$

$v_{xx} + v_{yy} = u_{xy} - u_{yx}$ $\to$ Both are equal

$$\boxed{v_{xx} + v_{yy} = 0}$$

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Q Check wether given f^n is Harmonic ?
$$u(x,y) = e^x \cos y$$
② if yes! find conjugate Harmonic of u.
③ find $f(z) = u + iv$
④ and $f(z)$ in terms of z.

Sol

$u(x,y) = e^x \cos y$

$u_x = e^x \cos y$ $\qquad$ $u_y = -e^x \sin y$

$u_{xx} = e^x \cos y$ $\qquad$ $u_{yy} = -e^x \cos y$

$u_{xx} + u_{yy} = 0$

$e^x \cos y - e^x \cos y = 0$ ✓

Satisfying laplace and also continous so;
u is Harmonic

② Conjugate of u ! Using CR equations

$u_x = e^x \cos y$

Using CR $\qquad u_x = v_y$

$\Rightarrow v_y = e^x \cos y$

Integrating this we will get wrt y

$$\int v_y = \int e^x \cos y \, dy.$$

$$\boxed{v = e^x \sin y + c}$$

As we considered x constant differentiation will give zero so writing $\phi(x)$ instead of c

$$\boxed{v = e^x \sin y + \phi(x)}$$

$u_y = -e^x \sin y$ $\qquad$ using CR eq^n

$u_y = -v_x$

$v_x = e^x \sin y$

$u_x = v_y$

$v_y = e^x \cos y$

$v = e^x \sin y + \phi(x)$ $\qquad v_x = e^x \sin y + \phi'(x)$

$$v = \int e^x \sin y \, dx$$

$\cancel{v = -e^x \cos y + \phi}$ $\qquad \boxed{v = e^x \sin y + \phi(x)}$

$v_x = -u_y$

$v_x = e^x \sin y + \phi'(x)$

$e^x \sin y + \phi'(x) = e^x \sin y$

$\phi'(x) = 0 \qquad \boxed{\phi(x) = c}$

Now we have $\boxed{v = e^x \sin y + c}$ Harmonic conjugate

$f(z) = u + iv = e^x \cos y + i(e^x \sin y) + ic$

$= e^x(\cos y + i \sin y) + ic$

$= e^x \cdot e^{iy} + ic = e^{x+iy} + ic$

$\boxed{f(z) = e^z + ic}$

$\Rightarrow$ v is conjugate Harmonic of u.

Q. $u = x^2 + y^2$ find all parts as before

$u_x = 2x \qquad u_y = 2y$

$u_{xx} = 2 \qquad u_{yy} = 2$

$u_{xx} + u_{yy} \neq 0$

But function can be analytic or cannot.

$u_x = v_y$

$v_y = 2x$ $\qquad v = 2xy + \phi(x)$

w.r.t y $\qquad v_x = 2y + \phi'(x)$

$u_y = -v_x$

$2y = -v_x$

$\cancel{2y = -2x - \phi'(x)}$

$\cancel{\phi'(x) = -2y - 2x}$

$\cancel{\phi'(x) = -2(x+y)}$

$\cancel{\phi(x) = -\{x^2 + 2xy\}}$

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$Vx = 2y + \phi'(x)$

$Vx = -4y$

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$V = \cancel{x^2 - x^2 - 2xy}$

$\cancel{V = -2xy}$

$\phi'(x) + 2y = -2y$

$\phi'(x) = -4y$

$\phi(x) = -4xy + c$

$V = 2xy - 4xy + c$

$\boxed{V \Rightarrow -2xy + c}$

$f(z) = u + iv$

$= x^2 + y^2 + (-2xy + c)i$

$u = x^2 + y^2$ $\qquad V = -2xy + c$

$u_x = 2x$

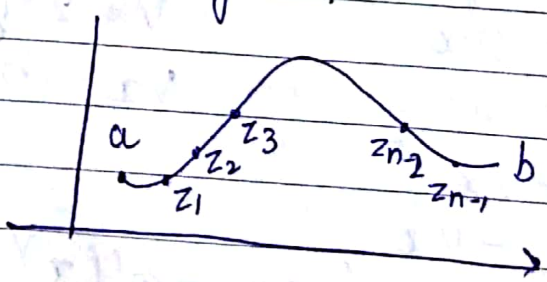
$u \cdot$

It does not satisfies CR equation

so we cannot proceed it further; f^n is not Harmonic

25/8/17 **Complex Line Integral :-**

Let $f(z)$ be continous at all points of a curve C, which we shall assume has a finite length.



Subdivide C into n points by means of points $z_1, z_2 \ldots z_{n-1}$ chosen arbitrarily and call $z_0 = a$ & $z_n = b$

on each arc joining z_{k-1} to z_k

- choose a pt α_k. from the sum

$$S_n = f(\alpha_1)(z_1 - a) + f(\alpha_2)(z_2 - z_1) + \cdots f(\alpha_n)(b - z_{n-1})$$

on writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(\alpha_k)\,\Delta z_k$$

let the no. of subinterval n increases in such a way that the largest of the chord length $|\Delta z_k|$ approaches zero then

$$\int_a^b f(z)\,dz \quad \text{or} \quad \oint_C f(z)\,dz$$ C – closed path

<u>Arc</u> : A set of point $z = (x, y)$ in the complex plane is said to be arc if

$$x = x(t) \; ; \quad y = y(t)$$
$$(a \le t \le b)$$

where $x(t)$ and $y(t)$ are continous f^n of the Real parameter t.

<u>Simple Arc</u> (<u>Jordan Arc</u>) :

If it does not cross itself

If $z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$

- complex Integration has Many advantages using it we can solve Real integral problem also.

<u>Integration of function $\omega(t)$</u> :-

$$\omega(t) = u(t) + v(t) \; ; \; a \le t \le b$$

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$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + \int_a^b v(t)\,dt$$

eg $\int_0^1 (1+it)^2\,dt - \int_0^1 (1-t^2+2it)\,dt$

$$\Rightarrow \int_0^1 (1-t^2)\,dt + \int_0^1 2it\,dt$$

$$\Rightarrow \left[t-\frac{t^3}{3}\right]_0^1 + i\left[t^2\right]_0^1$$

$$\Rightarrow 1-\frac{1}{3} + i = \frac{2}{3}+i$$

eg $e^z = 0$ Find all possible Roots

$e^z = 0$

$e^z = e^{\log 0} = e^{\ln|z| + i\arg z}$

No possible Roots as log is not defined at 0.

<u>Line Integral</u> :-

Ques Evaluate $\int_{0,3}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along the

(a) parabola $x=2t$, $y=t^2+3$

(b) straight line from $(0,3)$ to $(2,3)$ and then from $(2,3)$ to $(2,4)$

(c) A straight line from $(0,3)$ to $(2,4)$

Sol (a) $x = 2t$ $\qquad y = t^2+3$

$dx = 2dt$ $\qquad dy = 2t\,dt$

$$I = \int_0^1 \left(2(t^2+3) + 4t^2\right)dt \times 2 + \left(3(2t) - t^2 - 3\right)2t\,dt$$

This limit is 0 to 1 now because

$t = x/2 \Rightarrow 2/2 = 1$

also $y-3 = t^2 = \quad t^2 = 4-3 = \quad t = 1$

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$$I = \int_0^1 \left(-2t^3 + 24t^2 - 6t + 12\right)dt$$

$$I = 3/2$$

(ii) from $(0,3)$ to $(2,3)$

x from 0 to 2

& $y = 3 \quad dy = 0$

$$I_1 = \int_0^2 (6+x^2)dx + (\cancel{3x-3})^0 \times 0$$

$$I_1 = \left[6x + \frac{x^3}{3}\right]_0^2 = 12 + 8/3 \Rightarrow 44/3$$

from $(2,3)$ to $(2,4)$

$x = 2 \quad dx = 0$

$$I_2 = \int 0 + \int_3^4 (6-y)dy$$

$$I_2 = \left[6y - y^2/2\right]_3^4$$

$$I_2 = 24 - 8 - 18 - 9/2$$

$$I_2 = 5/2$$

$$I_0 = I_1 + I_2 = 44/3 + 5/2 = 103/6$$

(c) from $\underset{}{\overset{x_1 y_1}{(0,3)}}$ to $\overset{x_2\ y_2}{(2,4)}$

$$\rightarrow \frac{x - x_0}{x_0 - x_1} = \frac{y - y_0}{y_0 - y_1}$$

$$\rightarrow \cancel{x-0} \quad \frac{x-0}{0-2} = \frac{y-3}{3-4}$$

$$\rightarrow \frac{x}{2} = \frac{y-3}{1} \Rightarrow \underline{2y - x = 6}$$

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$$x = 2y - 6$$
$$dx = 2dy$$

$$I = \frac{4}{3} \int \left(2y + (2y-6)^2\right) 2dy + \left[3(2y-6) - y\right] dy$$
$$= \frac{97}{6}$$

Q evaluate $\int_C \underbrace{f(z)}_{\to \bar{z}} dz$

(a) from $z = 0$ to $z = 4 + 2i$ along the curve $z = t^2 + it$

(b) The line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$

Let $f(z) = \bar{z}$

Through line

$$\int \bar{z}\, dz = \int (x - yi)(dx + i dy) = \int x dx + y dy + i \int x dy - y dx$$

if $f(z) = z$

$$\int z\, dz = \int (x + yi)(dx + i dy) = \int x dx - y dy + i \int x dy + y dx$$

→ $$\int_0^{4+2i} x dx + y dy + i \int x dy - y dx$$

$$I_a = \int_0^2 t^2 (2t dt) + (t dt) + i \int_0^2 t^2 dt - t(2t dt)$$

\# When we change parameters our curve should be smooth curve

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$\int \bar{z}\,dz$ from $z=0$ to $z=4+2i$

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Ans

Through curve

t is from 0 to 2

$$I = \int_0^2 \cancel{4-2i}\ (t^2-it)\ \cancel{\xi}\, d(t^2+it)$$

$$I = \int_0^2 (t^2-it)(2t+i)\,dt$$

$$I = \int_0^2 (2t^3 + it^2 - 2it^2 + t^2)\,dt$$

$$I = 10 - \frac{8i}{3}$$

(b) $(x-yi)(dx+idy) = (xdx+ydy) + i(xdy - ydx)$

Path from 0 to $2i$ $\qquad x \to 0 \quad dx \to 0$

$$I_1 = \int_0^2 y\,dy$$

$$I_1 = \frac{y^2}{2} \Rightarrow \frac{4}{2} = 2$$

(c) Path from $2i$ to $4+2i$

y is fixed $\quad dy = 0$

$$\int_0^4 (xdx) + i(-2dx)$$

$$\int_0^4 xdx - 2dx\,i$$

$$\left[\frac{x^2}{2} - 2xi\right]_0^4$$

$$\Rightarrow \frac{16}{2} - 8i = \underline{8-8i}$$

$$I = I_1 + I_2$$

$$\Rightarrow \underline{10 - 8i}$$

This f^n is not analytic we can't say that Both Integrals are be equal or not because ~~curve~~ Integral does not depend only on end pts.

In a & b part paths are same but we get diff value of Integrals

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continuously differentiable curve

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Q for $\int f(z)\,dz$

Ans $I = \int_0^2 (t^2+it)\,d(t^2+it) = \int_0^2 (t^2+it)(2t+i)\,dt$

$I = \int_0^2 (2t^3-t) + i(t^2+2t^2)\,dt$

$I = 6+8i$

(b) from 0 to $2i$

$x=0$ $dx=0$ only y changing

$\int_C x\,dx - y\,dy + i\int x\,dy + y\,dx$

$\int_0^2 -y\,dy \rightarrow -\left[\frac{y^2}{2}\right] = -2 = I_1$

from $2i$ to $4+2i$

y fixed $dy \to 0$

$\int_0^4 x\,dx + i\int_0^4 y\,dx$

$\left[\frac{x^2}{2}\right]_0^4 + i\left[2x\right]_0^4$

$I_2 = \underline{8+8i}$

$I \Rightarrow I_1 + I_2 = \underline{\underline{6+8i}}$

This f^n is analytic & Integral depends only on end points not on path so both integrals come same for same path.

→ **Smooth curve** :- A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be smooth curve if $\gamma(t)$, $a \le t \le b$ is continuously differentiable.

differentiation of curve is also continous o/p is also continuous

(eg) $\frac{d}{dt}\left(t^2+i2t\right) = 2t + 2i$

continuously defferentiable curve

\# when a function is continously differentiable in some intervals but have problem at some pt. it is called peice wise smooth curve. In that case we break our curve in 2 parts. We do not integrate it directly.

→ Peice wise smooth curve → If a f^n can be defined on the finite no. of subintervals such that on each sub interval curve is smooth.

Theorem : ML- Inequality →

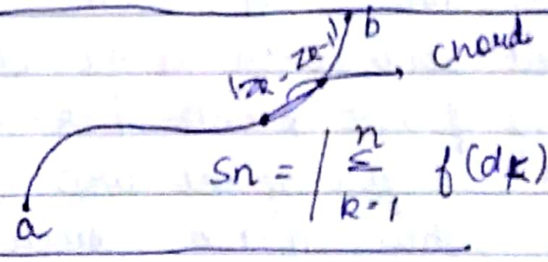
Let C be a peicewise smooth curve

$$C : z = z(t) = x(t) + iy(t) \quad a \le t \le b$$

then, $$\left| \int_C f(z)\,dz \right| \le ML$$

where L is the length of the curve and $|f(z)| \le M$ everywhere on C.

discussion



$$S_n = \left| \sum_{k=1}^{n} f(\alpha_k)\,\Delta z_k \right| \le \sum_{k=1}^{n} |f(\alpha_k)|\,|\Delta z_k|$$

A smooth curve is always peicewise smooth but converse is not true.

We also have that our curve is bounded.

$$S_n \le \sum_{k=1}^{n} M\,|\Delta z_k| = M \sum_{k=1}^{n} |\Delta z_k|$$

$$= M \sum_{k=1}^{n} |z_k - z_{k-1}|$$

chord length of a curve is always Min/less than the length of curve.

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so we take very-very small sub Intervals / distance also $n \to \infty$ so for each subinterval length of chord = length of curve for that interval

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} M \sum_{k=1}^{n} |z_k - z_{k-1}|$$

$$\boxed{\left|\int f(z)\,dz\right| = ML}$$

Proof of Theorem → From the definition of contour interval, we get;

$$S_n = \sum_{k=1}^{n} f(\alpha_k)(\Delta z_k)$$

Given $$|S_n| = \left|\sum_{k=1}^{n} f(\alpha_k)(\Delta z_k)\right|$$

$$\leq \sum_{k=1}^{n} |f(\alpha_k)(\Delta z_k)|$$

$$\leq M \sum_{k=1}^{n} |\Delta z_k|$$

It is clear that; $L^* = \sum_{k=1}^{n} |z_k - z_{k-1}|$ Represents the sum of the length of the chords whose end points are $z_0, z_1 \dots z_n$.

since a straight line path is the shortest distance b/w any ~~f~~ two points $|z_k - z_{k-1}|$ does not exceed the length of the arc joining the pts z_{k-1} & z_k, Thus $n \to \infty$ such that the Max $|\Delta z_k| \to 0$ ~~we~~ we have $L^* = L$ (L is the length of the curve) Hence we get,

$$\lim_{n\to\infty} |S_n| = M \lim_{n\to\infty} \sum_{k=1}^{n} |\Delta z_k|$$

$$\boxed{\left|\int_C f(z)\,dz\right| \leq ML}$$

overall f^n Max

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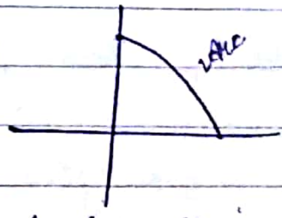
Ex. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant then show that

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}$$

Sol. Radius is 2; length of the curve is

$$\frac{\pi(r^2)}{4} = \frac{\pi(4)}{4} = \pi$$



* Max value of an exp f^n is ∞. But for this curve Max value can be.

We Break our function →

$f_1(z) = |z+4|$ → Max of upper side

$f_2(z) = \dfrac{1}{|z^3-1|}$ → Min of lowest side

* $f_1(z) = |z+4| \leq |z| + 4 = 2 + 4 = 6$

* $f_2(z) = \dfrac{1}{|z^3-1|} \leq \dfrac{1}{|z^3|-1} \leq \dfrac{1}{7}$

we try to make $|z^3-1| \geq |z^3| - 1$

$8 - 1 = 7$

to Minimize the lower side; we need to Maximize the actual value

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq ML$$

$$\leq \frac{6}{7} \times \pi = \frac{6\pi}{7}$$

\# Calculate length of curve & the values of functions

30/8/17 $\left| \int f(z) dz \right| \leq ML$

If $f(z)$ is bounded on C i.e. $f(z) \leq M$ and L is the length of the peicewise smooth curve C.

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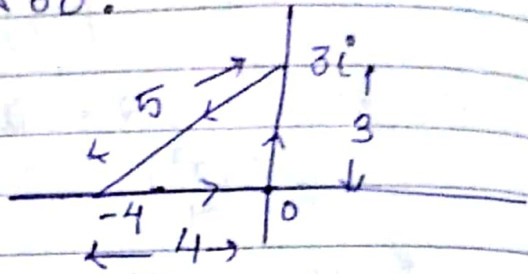
eg. Show that if c is the boundary of the triangle, vertices at the points 0, $3i$ & -4 oriented in the counterclockwise direction, then $\left| \int (e^z - \bar{z})\, dz \right| \leq 60$.



Sol $I = \int_c (e^z - \bar{z})\,dz$

length of the curve in this case is $5+3+4 = 12$ units

Now, we need to find bound of the f^n.

$$|f(z)| = |e^z - \bar{z}| \leq |e^z| + |\bar{z}|$$
$$= |e^z| + |z|$$

$|z|$ is checked from origin, so its max value can be 4.

$$|e^z| = |e^{x+iy}| = |e^x|\,|e^{iy}| \quad \to \cos^2\theta + \sin^2\theta = 1$$
$$= |e^x|\,\underbrace{|\cos\theta + i\sin\theta|}$$

Max value of this can be one

on 0 to -4 max value of e^z can be $\underline{e^0} = 1$.

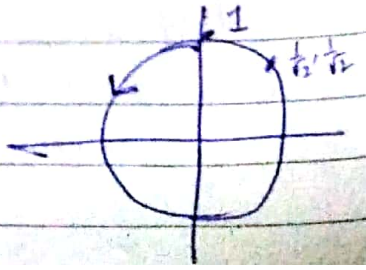
$$|f(z)| \leq 1+4 = 5$$

using ML inequality

$$\left|\int f(z)\,dz\right| \leq ML$$
$$\leq 12 \times 5 = 60$$

Q Find the upper bound for the absolute value of the integral $I = \int e^{(\bar{z})^2}\, dz$

$C: |z| = 1$ traversed in the anticlockwise direction.



→ Anticlockwise direction = +ve direction

$z(t) = t^2 + it$

Length of curve $= \int_a^b |z'(t)|\,dt$ > diff of curve

$= \int_a^b \sqrt{\cancel{\frac{dx(t)}{dt}} \left(\frac{d}{dt}(x(t))\right)^2 + \left(\frac{d}{dt}(y(t))\right)^2}\,dt$

Sol Length of the curve is 2π.

$$|f(z)| = |e^{(\bar{z})^2}|$$

$$\Rightarrow |e^{(x-iy)^2}|$$

$(x-iy)^2$

$x^2 + i^2y^2$

$$\rightarrow |e^{x^2-y^2-2ixy}|$$

$$\rightarrow \frac{|e^{x^2-y^2}|}{|e^{2xiy}|}$$

$i\,2xy$

cos

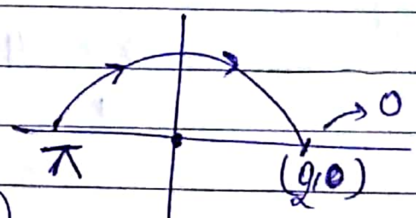
$x = 1$, $y = 0$ $\rightarrow \dfrac{|e^{x^2-y^2}|}{|\cos(2xy) + i\sin(2xy)|} \longrightarrow$ (1) $\boxed{e}$ **Ans**

Q Evaluate the foll. Integral

$$I = \int_C \frac{2z+3}{z}\,dz \quad \text{where } c \text{ is}$$

(a) upper half of the circle $|z| = 2$ in the clockwise direction.

Sol $|z| = 2$ Taking parametric variables



$x = 2\cos(t)$ $y = 2\sin(t)$

$z(t) = 2e^{it}$

$z'(t) = 2ie^{it}\,dt$

$$\boxed{I = \int z\,dz = \int_a^b [z(\phi(t))]\,\phi'(t)\,dt}$$

$$I = \int \frac{2(2e^{it})+3}{2e^{it}}\,d(2e^{it}) = \int \frac{4e^{it}+3}{2e^{it}} \cdot 2ie^{it}$$

$$I = \cancel{2}i \int \frac{4e^{it}+3 \times e^{it}}{\cancel{2}e^{it}} = i\int \frac{4e^{it}+3 \times \cancel{e^{it}}}{\cancel{e^{it}}}$$

$$\boxed{I = \int_c f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt}$$

We put limits in form of angle so

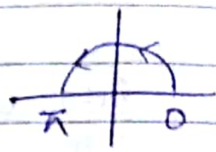
$I = i \int_{\pi}^{0} \left(\frac{4e^{it} + 3}{\cancel{e^{it}}} \right) dt$ $\Rightarrow$ $\underline{5 - 3\pi}$

$\cancel{I = i \int [4dt] + [3e^{-it}] dt}$ $4e^{i\pi} - 3\cancel{\pi}$

$\cancel{i \left([4t]_{\pi}^{0} - [3e^{-it}]_{\pi}^{0} \right)}$ $+8 - 3\pi i$

$\cancel{i[-4\pi] - 3[1 - e^{-\pi i}]}$

(b) upper half of the circle in the anti clockwise direction



Sol $\rightarrow$ $I = \int_{0}^{\pi} (4e^{it} + 3)\, dt$

e^{it} $\rightarrow 4 \left[\frac{e^{it}}{i} \right]_{0}^{\pi} \times i + 3[t]_{0}^{\pi} \times i$

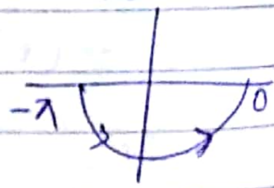
$\cos \pi i$ $\rightarrow -4i \left[e^{\pi i} - 1 \right] + 3\pi$

$\rightarrow (-4i[-1-1] + 3\pi) i$

$\rightarrow \cancel{3\pi\, 8i}$ $\rightarrow -8 + 3\pi i$

(c) lower half of the circle $|z| = 2$ in anticlockwise direction

$I = i \int_{-\pi}^{0} (4e^{it} + 3)\, dt$



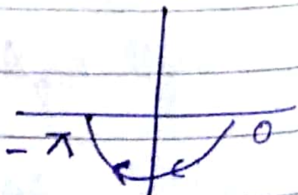
$-4i \left[e^{it} \right]_{-\pi}^{0} + 3(t)_{-\pi}^{0}$

$+1 \cos(0) = 1$ $\rightarrow \cancel{-4i\,[e^{\pi}}$

$-4i[1 - e^{-\pi i}] + 3\pi = \boxed{+4i[2] + 3\pi}$

(d) lower half of the circle $|z| = 2$ in clockwise direction

$I = \int_{0}^{-\pi} (4e^{it} + 3) dt$



$-4i \left[e^{it} \right]_{0}^{-\pi} + 3[t]_{0}^{-\pi}$

$-4i \left[e^{-\pi i} - 1 \right] + 3(-\pi)$

$\Rightarrow -4i[-2] - 3\pi$

$\Rightarrow 8i - 3\pi$

Parametric → According to curve on which we want to find Integral.

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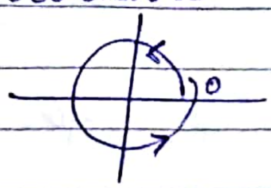
(e) In the circle $|z| = 2$ in the anticlockwise direction →



$I \rightarrow \int_0^{2\pi} (4e^{it} + 3)\, dt = \underline{\underline{6\pi i}}$

Now, if we take $f(z) = z^2$

$I = \int_c z^2 dz$ where c is the circle in $\underline{\underline{|z|=2}}$ the anticlockwise direction.

$f(z) = z^2$ $\qquad z(t) = 2e^{it}$

$\qquad\qquad z'(t) = 2ie^{it}$

→ $I = \int_c f(z) dz = \int_a^b f(z(t))\, z'(t)\, dt$

$= \int_0^{2\pi} (2e^{it})^2 (2ie^{it}) dt$

$= \int_0^{2\pi} (8i)\, e^{3it} dt$

$= \cancel{\int_0^{2\pi}} 8i \left[\frac{e^{3it}}{3i}\right]_0^{2\pi} = \underline{\underline{0}}$

If we take z^3 in place of z^2 Then also Answer is zero.

so $f(z) = z, z^2, z^3, z^4$ -- for positive power of z ; over a closed circle Ans is always zero.

Q $I = \int_0^{2\pi} \frac{1}{2e^{it}} (i\, 2e^{it})\, dt$ when question is $I = \int_c \frac{1}{z} dz$

But circle is same. use this

when $I = \int_c \frac{1}{z^2} dz = \int_0^{2\pi} \frac{1}{(2e^{it})^2} (i2e^{it}) dt$

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$$I = \frac{1}{2} \times i \int_0^{2\pi} e^{-it}\, dt = \frac{i}{2}\left[\frac{e^{-it}}{-i}\right]_0^{2\pi} = \underline{\underline{0}}$$

for $\frac{1}{z^3}$ Ans is also $\underline{\underline{\text{zero}}}$

✱ $$\boxed{I = \int_C z^n\, dz = \begin{cases} 2\pi i & \text{when } n = -1 \\ 0 & \text{otherwise} \end{cases}}$$ ✱

$C : |z| = r$ is traversed in the counterclockwise direction.

→ Prove the above Result!

Parametric $z(t) \to re^{it}$
$z'(t) \to rie^{it}$

$$I = i \int_0^{2\pi} (re^{it})^n (re^{it})\, dt$$

$$I = i \int_0^{2\pi} r^{n+1} e^{(n+1)it}\, dt$$

$$I = i\left(\frac{r^{n+1}}{n+1}\right)\left[e^{(n+1)it}\right]_0^{2\pi}$$

$n = -1$

$$I = i\left(\frac{r^0}{0}\right)\left[e^{0 \times 2\pi} - e^{0 \times 0}\right]$$

$$I = i \times \frac{1}{0}[1 - 1] = \underline{0/0}$$

$i\left[r^{n+1} e^{n+1\, it}\right]$
$i\left[1 \cdot e^{0it}\right]$

Take $z - z_0 = \omega$
$dz = d\omega$

$\int_C \omega^n\, d\omega$, $C : |\omega| = r$

$$|z - z_0| = r$$
$$z - z_0 = re^{it}$$
$$z = z_0 + re^{it}$$
$$dz = rie^{it}\, dt$$

Simple closed path → That does not intersect or touch itself.

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$$I = \int_C (z-z_0)^n dz$$

$$= \int_0^{2\pi} (re^{it})^n rie^{it} dt$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt$$

$$= \frac{ir^{n+1}}{i(n+1)} \left[e^{i(n+1)t} \right]_0^{2\pi} \quad \text{when } n \neq 1$$

$$= 0$$

$$\oint z^2 dz = 0$$

when $n = -1$

$$I = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} dt$$

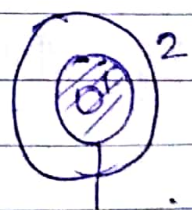
$$= \int_0^{2\pi} i dt = 2\pi i$$

$$\oint_C z^2 dz = 0$$

<u>1A simply connected domain</u> :- ~~such that every simple~~ closed path in D Contains only pts. of D encloses only pt of D

In a complex plane a domain (open & connected set) that every simple closed curve in domain encloses only pt. of domain.

$1 < |z| \leq 2$



simply connected

Not simply connected

<u>Multiply connected</u> :- → contains pt other ~~than~~ D

Domain that is not simply connected.

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Green's Theorem :-

Let C be a positively oriented, peicewise smooth, simple closed curve (does not intersect except at end points) in a plane and let D be a region bounded by C. If (L) and (M) are functions of (x, y) defined on an open region containing D and have continous partial derivatives then,

line Integral around a simple closed curve $\leftarrow$ $$\oint_C L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$ double Integral over the plane D

D is the region bounded by C

$z(t) \quad a<t<b$

$\frac{d}{dt}(z(t)) = z'(t)$ is continous & $z'(t) \neq 0$ smooth

Cauchy's Theorem :-

Let $f(z)$ is analytic in a simply connected domain D and $f'(z)$ is continous in D then for every simple closed curve C in D.

$$\oint_C f(z)\,dz = 0$$

eg 1) $I = \oint_C \sin z\,dz \qquad C: |z+1| = 5$

$= 0$

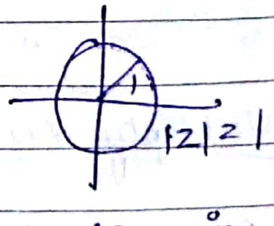
simple closed curve & function is analytic

2) $I = \oint_C \frac{\cos z}{z+3}\,dz \qquad C: |z| = 1$

Orientation is anticlock-wise

$f(z) = \frac{\cos z}{z+3}$

$= 0$



Function is analytic in the domain

In this Analytic sufficient, but not necessary
Simple connectedness essential

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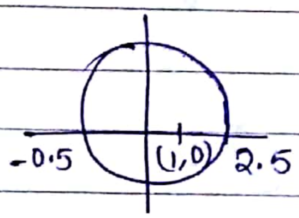
→ $I = \oint \frac{z^2}{z^2+4}$ C: $|z-i| = 1.5$

centre at $(1,0)$

$f(z) = \frac{z^2}{z^2+4} = \frac{z^2}{(z+2i)(z-2i)}$



Not defined at $(2,0)$, $(-2,0)$

- Analyticity is not necessary but sufficient condition to prove Integeration $=0$.

- $f(z) = \frac{1}{z}$ $1<|z|<2$

- Simply connected set is necessary condition.

Proof → $f(z) = u+iv$
$dz = dx + i dy$

$I = \oint_c f(z)dz = \oint_c (u+iv)(dx+idy)$

$= \oint_c (udx - vdy) + i(udy + vdx)$

Apply green's theorem as u and V have continous partial derivatives.

$\int udx + (-v)dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

Using C-R equation
$u_x = v_y$ & $v_x = -u_y$

subtraction

$= \iint \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$

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Similarly,

$$\int u\,dy + v\,dx = \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Using C-R eq^n $\rightarrow$ $\iint \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$

$\underline{I = 0 + 0}$

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$$\sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}\, e^{i\theta/2}$$

Assignment Question $\rightarrow$ $= \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Replace θ by $\theta/2$

$\cos\theta/2 = \sqrt{\frac{1+\cos\theta}{2}}$ $\qquad$ $\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$

$$\sqrt{z} = \sqrt{r}\left(\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right)$$

$$\sqrt{z} = \sqrt{\frac{r+r\cos\theta}{2}} + i\sqrt{\frac{r-r\cos\theta}{2}}$$

$$= \sqrt{\frac{|z|+x}{2}} + i\,(\ )\sqrt{\frac{|z|-x}{2}} \rightarrow \underline{\text{sig } y}$$

$$e^z = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i\sin y)$$

$$e^{\bar{z}} = e^{x-iy} = e^x(\cos y - i\sin y)$$

$\rightarrow$ In this term we ~~co~~ are not using any term Related to y That's why we use signature of y it can either be positive or negative

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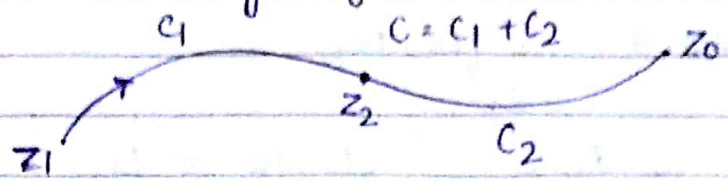
① <u>Partitioning of the Path</u> →



Then $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

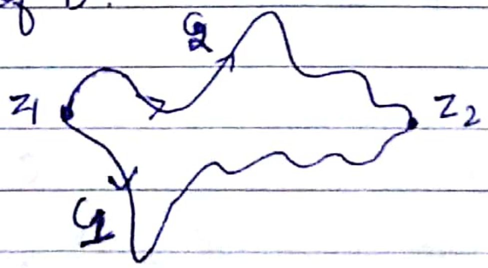
This Theorem says that either you can find the Integral of whole path or you can do that by partitioning of the given path like (circle) example we done before.

② Sence <u>~~Sign~~ Reversal</u> → If we change direction of integral value remains same with opp. signs

$$\int_{z_0}^{z_1} f(z)dz = -\int_{z_1}^{z_2} f(z)dz$$

\# Because of analyticity $\int z\,dz$ → only depends on end points of the path but not for $\int \bar{z}\,dz$

③ <u>Independence of Path</u> :- If $f(z)$ is analytic in a simply connected domain D then the integral of $f(z)$ is independent of path of D.



If we Move on the way to C_2 and then we Move on C_1 we are just Reversing the direction. same curve with opp direction

$$C: C_1 + (-C_2)$$

$$\int_C f(z)dz = 0 = \int_{C_1 - C_2} f(z)dz =$$

4 Linearity also followed

$$\int_C k_1(x_1) + k_2(x_2) = k_1\int_C x_1 + k_2 \int_C x_2$$

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz$$

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = 0$$

Proof : Let z_1 and z_2 be any point in D. Consider two paths C_1 and C_2 in D from z_1 to z_2 without further common points. Denote by path C_1 and C_2 where C_2^* is the path C_2 with orientation Reversed. Integrating from z_1 over C_1 to z_2 and over C_2^* back to z_1.

This $C = C_1 + C_2^*$ become simple closed path in simply connected domain D. ~~There~~

Then, by cauchy's theorem

$$\int_C f(z)dz = 0 \quad -①$$

but $\int_C f(z)dz = \int_{C_1+C_2^*} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2^*} f(z)dz$

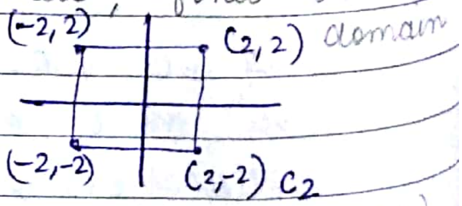
$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz \quad -②$$

From ① and ② we have $\boxed{\int_{C_1} f(z)dz = \int_{C_2} f(z)dz}$

ⓐ $\int_{C_1} e^z dz = 0$ $\quad C : |z| = 2$

ⓑ $\int_{C_2} \frac{z^2}{z+8} dz = 0$ function is differentiable & Analytic

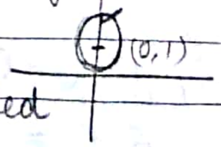
domain is Rectangular Area ; f^n is defined in this domain



ⓒ $\int_{C3} \frac{z}{z^2+4} dz =$ ~~+2i, −2i not defined~~

2i

~~2i included~~



$|z-i| = 2$

0, 1 + 1.14

$\sqrt{2}$ 1.14

ⓓ $\int_{C_4} \tan z \, dz = 0$ $\quad C_4 : |z| = 1$

Singularity → f^n is not Analytic at a pt. but it must be analytic at a surrounding pt.

① Our f^n is analytic and Integration over closed path by Cauchy's Theorem Ans = 0

⑨ $f(z) = \frac{\sin z}{\cos z}$ → Both analytic in our domain
so Ans = 0   Path is closed.

Singularity :-

① * $\frac{z^2}{(z+8)^3}$ → Order of singularity - 1
$z = -8$

② * $\frac{z}{(z^2+4)^2}$ singularity $z = \pm 2i$
Order of singularity = 2

③ * $\tan z$ singularity as $(2n+1)\pi/2$
Order of singularity = ∞

* Singularity point → where our ~~path~~ function is not defined, The pt at which f^n is not Analytic

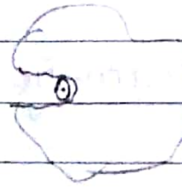
④ e^z → no pt of singularity in finite Range

⑤ $f(z) = \bar{z}$ ⇒ f^n is not Analytic anywhere
f^n does not satisfies CR equation.
It does not Mean f^n is singular in whole plane

⑥ $f(z) = Re\,\bar{z}$ ⇒ f^n does not satisfies CR eq^n

Isolated singularity → Only at one pt. function is sigular we can Reject that pt. But in neighbourhood f^n is analytic.

⑦ $f(z) = |z|^2$ ⑧ $f(z) = \log z$



Holomorphic Means analytic

function $\bar{z}$ is nowhere differentiable.

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(1) $f(z) = e^{\bar{z}}$ → This function is continous every but in them no pt. is singular pt.

(2) $f(z) = \bar{z}$

A pt. where f^n is not defin

(3) $f(z) = \dfrac{e^z}{(z-1)(z-2)}$ 1, 2 are singular pts

(4) $g(z) = \tan z$ $(2n+1)\pi/2$

(5) $g(z) = \dfrac{1}{z}$ $z = 0$

(6) $\log z$ → Analytic not in negative Axis singularity in all the pts in negative Real Axis

Singularity :- A single valued function is said to have a singularity at a pt. if the function is not analytic at the point while every neighbourhood of that pt contains at least one point at which the f^n is analytic.

Example: $f(z) = \dfrac{1}{z}$ has singularity at $z=0$, as every nbd of $z=0$ contains some Regular (analytic) pt.

For Instance :- $f(z) = \bar{z}$ this function is nowhere differentiable, hence no nbd contain any Regular pt. Thus $f(z) = \bar{z}$ has no singularity.

→ There are two types of singularity :-

1. Isolated Singularity
2. Non Isolated singularity

Analytic → f^n should be continous

But in neighbourhood no problem

at 0 f^n is not continous so not Analytic

Ex

$$f_1(z) = \begin{cases} z+1 & \text{for } z \neq 0 \\ 3 & \text{for } z = 0 \end{cases}$$

Isolated singularities

$$f_2(z) = \left\{ \frac{1}{z} \quad \text{for } z \neq 0 \right.$$

$$f_3(z) = \sin(1/z) \quad z \neq 0$$

$$f_4(z) = \frac{z^2}{(z-1)^2(z-3)} \;\Big/\; \frac{5}{z(z-1)^2(z-3)}$$

3 singularities

Non Isolated

∞ singularities

$g(z) = \log z$

$$g(z) = \frac{1}{\sin(1/z)}$$

Real line is dense so it is always possible even if we draw a very small circle

$z=0$ singularity

→ <u>Isolated singularity</u> → If f^n is analytic in the deleted nbd of the singularity.

\# $1/\sin(1/z)$ → All the values where $\sin(1/z)$ is zero are points of singularities

$z = 1/\pi,\ 2/\pi$

<u>Nature of (isolated) singularities can also be different.</u>

1) <u>Pole</u> : If z_0 is an isolated singularity and we can find a positive integer n such that

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0 \quad \text{(finite value)}$$

$z \to z_0$ → pt of singularity

Then $z = z_0$ called pole of order n.

ex $\dfrac{z^2}{(z-1)^2(z-3)}$

$$\lim_{z \to 1} \frac{(z-1)^3 z^2}{(z-1)^2(z-3)} = \underline{0}$$

Addition value

$z=0$ zero

But definition is $\neq 0$

So, $$\lim_{z \to 1} \frac{(z-1)z^2}{(z-1)^2 z-3} \to \infty$$

$$\lim_{z \to 1} \frac{\cancel{(z-1)^2} z^2}{\cancel{(z-1)^2}(z-3)} \to \underline{1/-2}$$

So we say that $z = -1$; is a pole of order 2

$$\lim_{z \to 3} \frac{\cancel{(z-3)}(z^2)}{(z-1)^2\cancel{(z-3)}}$$

$z=3$ is a simple <u>pole</u>

$e^{1/z}$ singularity at ____

tanz have isolated singularities

Is $z = \pi/2$ a pole?

→ $f(z) = \tan z$

$\lim_{z \to \pi/2} (z - \pi/2) \frac{\sin z}{\cos z}$

Zeros $n\pi$

Poles $(2n+1)\pi/2$ Applying L'HOSPITAL RULE

$(z - \pi/2)\left(\frac{\cos z}{-\sin \pi}\right) = \sin z\,(1)$

$\Rightarrow -1$ Ans

$z = \pi/2$ is a pole of order 1

All other points at which this function is ~~not defined~~ is upcoming zeros are called zeros of this f^n.

7. Zeros of function :- $z = z_0$ is said to be a zero of order m where $m > 1$ if

$\lim_{z \to z_0} (z - z_0)^{-m} f(z) \neq$ finite non zero value $\neq 0$

$\lim_{z \to z_0} \cancel{\frac{f(z)}{(z-z_0)^m}} \neq \emptyset$

$\lim_{z \to z_0} \frac{1}{(z-z_0)^m} f(z) \neq$ finite

eg $f(z) = \frac{z^2}{(z-1)(z-2)}$ has a zero of order 2

Q find all zeros & poles of

(a) $f(z) = \tan z$

(b) $f(z) = e^{1/z}$

Removable singularity :- An isolated singular z_0 is called Removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z_0)$ exists (not zero)

Ex $f(z) = \frac{\sin z}{z}$ (not zero)

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eg $f(z) = \dfrac{z^2+1}{(2z-2i)(z+2)}$

singularity at $z = i$

$f(z) = \dfrac{(z+i)\cancel{(z-i)}}{2\cancel{(z-i)}(z+2)} = \dfrac{z+i}{2(z+2)}$

so Removable singularity

eg $\lim_{z\to 0} z \sin\frac{1}{z} = \lim_{z \to 0} z \left(\lim_{z\to 0} \underbrace{\sin \frac{1}{z}} \right)$

Non Removable Type of singularity = does not exist

unbounded f^n so limit DNE

Essential singularity :- Any isolated singularity that is not a pole or not a Removable singularity.

when you express your f^n in series & your series has ∞ values

eg $f(z) = e^{1/z}$ → neither Removal nor non Removable singularity

It is not a pole even

eg $f(z) = \sin 1/z$ a finite no. can't be multiplied to Remove the singularity

- Singularity at Infinity :-

$f(z) = z$

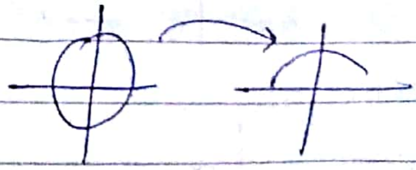
if $z = \infty \Rightarrow f(z) = \infty$

The type of singularity of $f(z)$ at $z = \infty$ is the same as that of $f\left(\frac{1}{\omega}\right)$ and $\omega = 0$.

Branch of a function :- (a type of singularity)

$f(z) = z^{1/2}$

$= \sqrt{r}\, e^{i\theta/2}$



I/P	O/P
$\pi/2$	$\pi/4$
0	0
π	$\pi/2$
2π	π

Branch :-

A branch of a multivalued function f is any single valued function F that is analytic in some domain at each pt z of which the value $F(z)$ is one of the value of $f(z)$

ex This function

$\text{Log} z = \log|z| + i\theta \quad (r>0 \quad -\pi<\theta<\pi)$

principal branch

similarly

$f(z) = z^{1/2}$

$\sqrt{r}\, e^{i\theta/2} \qquad -\pi<\theta<\pi$

Branch Point :-

$f(z) = z^2 = r^2 e^{2i\theta}$

Input	o/p
$\theta = \pi/4$	$\pi/2$
$\pi/2$	π
π	2π
2π	4π

complete circle

o/p is completing the circle twice

$0 \leq \theta < 2\pi$

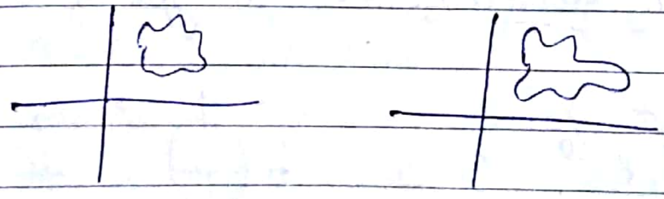
$f(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2}$

i/p	o/p
$\pi/2$	$\pi/4$
π	$\pi/2$
2π	π

complete circle

Non complete circle

while i/p covers $\{0, 2\pi\}$ o/p covers only the half circle. for it to cover the entire circle i/p have to go over the circle twice i.e. $\{0, 4\pi\}$



No Branch pt

z plane

w plane

Branch point

To calculate Branch pt. put $f(z) = 0$

f^n is never analytic at Branch pt.

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→ A Branch cut is a portion of a line or curve that is introduced in order to define a branch F of multivalued f^n of $f(z)$.

$\sqrt{z}$ has branch pt at 0 and also at ∞ because f^n is not analytic.

→ A Branch Points are points on the Branch cut for f are singular points & any pt that is common to all Branch cuts of $f(z)$ is called a Branch pt.

The origin and the ray $\theta = \alpha$ make up the Branch cut for the branch of a multi valued f^ns.

eg $(z-2)^{1/2}$ has a branch pt. at $z = 2$

$\log(z^2 + z - 2)$ has Branch

$z^2 + z - 2 = 0$

$z = 1 \quad z = -2$

→ In multivalued f^n we need to cover more values in order to reach our starting pt. again

eg $z^{1/2}$

• In order to reach end pt. again we need to draw one more curve.

→ $\alpha < \theta \le \alpha + 2\pi$

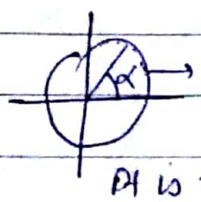
$\alpha + 2\pi < \theta \le \alpha + 4\pi$ → Requires 2 seats to come to the Beginning pt again

$\alpha + 4\pi < \theta \le \alpha + 6\pi$ →

$f(z) = z^{1/3}$: Requires 3 seats

eg $f(z) = \log z$ → It is really a multivalued f^n.

$= \ln|z| + i \operatorname{Arg} z$



We have freedom to choose this angle we can add 2π to complete our curve

At is to come to Begining pt. Again

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$\sqrt{x} \rightarrow$ +ve side is Root Branch

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$f(z) = z^{1/3}$

$\alpha < \theta \leq 2\pi + \alpha$ — 1^{st} Branch

$2\pi < \theta \leq 4\pi + \alpha$ — 2^{nd} Branch

$4\pi < \theta \leq 6\pi + \alpha$ — 3^{rd} Branch

Common pt. of the Branch cuts is Branch pt.
in case of circle '0' is the Branch pt.
(? Barrier is Branch line)

Now, $f(z) = \log(z-3)$

In this case we need to shift so $z-3=0$

Branch pt. $\leftarrow$ $z=3$

$f(z) = \log(z^2+2z-3)$

$z = 1, -3$ are branch points

$f(z) = (z-2)^{1/4}$

$\underline{z=2}$

$f(z) = \cot z$ in the Region of $|z|=3$ How Many times Zero?

__Ans__ Zeros $\pm \pi/2$

Poles 0

Now Region is $|z-3| = 2$

$f(z) = \dfrac{(z-2)(z)}{(z-3)(z-5)}$ $2, 0 \rightarrow$ zeros

$3, 5 \rightarrow$ poles

Region is $|z-6| = 2$

Pole at $z=5$

In Region $|z-8| = 2$ This function is analy[tic]

__Countour Integrals :-__

We need to identify 2 things:

•

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eg Evaluate $\oint_C \frac{5z+7}{z^2+2z-3} dz$

where c is a circle $|z-1|=2$ traversed in anticlockwise direction.

Some Results : $\oint_{C-|z|=r} z^n dz = \begin{cases} 2\pi i & n=-1 \\ 0 & \text{otherwise} \end{cases}$

$$\oint_{|z-z_0|=r} (z-z_0)^n dz = \begin{cases} 2\pi i & n=-1 \\ 0 & n \neq -1 \end{cases}$$

$$\oint_C f(z) dz = 0$$

- $z^2+2z-3=0$ $\quad (z-1)(z+3)=0$

$z=-3, 1$ are isolated singularities

In $|z-1|=2$ we have only one singularity That is $z=1$. (problem)

$\oint_C \frac{z+1}{z^2+2z-3}$ $\quad C: |z-5|=1$

z^2+2z-3 has isolated singularity at $z=1,-3$ but none of them is inside the circle $|z-5|=1$ Hence in $|z-5|=1$ our $f(z)$ is analytic so $\oint_C f(z)=0$ (by cauchy's Theorem)

→ Continuing previous eg → $\frac{5z+7}{z^2+2z-3} = \frac{3}{(z-1)} + \frac{2}{(z+3)}$

$$\oint_C \frac{5z+7}{z^2+2z-3} = \oint_C \frac{3}{z-1} dz + \oint_C \frac{2}{z+3} dz$$

$= \oint_{|z-1|=2} \frac{3}{z-1} dz \longrightarrow 3(2\pi i) = 6\pi i$

by using property $\oint (z-z_0)^n dz$ $\quad n=-1$

This part is analytic so by cauchy's Theorem this Integral is Zero

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$$z-1 = 2e^{i\theta}$$
$$dz = 2ie^{i\theta}$$
$$\int_0^{2\pi} \frac{3}{2\cancel{e^{i\theta}}} \cdot 2i\cancel{e^{i\theta}}\, d\theta = 3\int_0^{2\pi} i\, d\theta$$
$$= 3i\, 2\pi = \underline{6\pi i}$$

Q $\oint_C \frac{5z+7}{z^2+3z-3}$; $C: |z| = 2$

only $z=1$ is our problem

$$3\oint_C \frac{1}{(z-1)}dz + \underbrace{2\oint \frac{dz}{z+3}}_{0 \text{ by cauchy's Theorem}}$$

$C: |z| = 2$

Ans

$= 3(2\pi i) = \boxed{6\pi i}$ $\underline{z = 2e^{i\theta}}$? Here this thing will not help because in previous question N & D were becoming same

→ Method by Cauchy :-

\# Let $f(z)$ be analytic in a simply connected domain D. Then for any pt. ~~on~~ z_0 in D and any simple closed curve C in D that encloses z_0

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)}dz = 2\pi i\, f(z_0)}$$

Integration is taken anti clockwise.

\# Here direction matters because ans is not coming zero ; Magnitude same with opp sign.

eg $f(z) = \oint \frac{z^2}{(z-5)} dz$ $C: |z-4| = 3$ anti clockwise

$f(z) = z^2+1$

$$\oint \frac{z^2+1}{(z-5)} dz$$ $z_0 = 5$ is isolated singulari

$$= 2\pi i\, f(z_0)$$
$$= 2\pi i\,(5^2+1) = \underline{\underline{52\pi i}}$$

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eg $$I = \oint \frac{2z+1}{(z-1)^3} dz$$

For this type of questions we do like

diff wrt z0

$$2\pi i f'(z_0) = \oint_c \frac{f(z)}{(z-z_0)^2} dz$$

Again diff wrt z0

$$2\pi i f''(z_0) = 2! \oint_c \frac{f(z)}{(z-z_0)^{2+1}} dz$$

Again

$$2\pi i f'''(z_0) = 3! \oint_c \frac{f(z)}{(z-z_0)^{3+1}} dz$$

$$\vdots$$

$$2\pi i f^n(z_0) = n! \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\boxed{\oint \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)}$$

question → $\oint \frac{2z+1}{(z-1)^3} = \oint \frac{2z+1}{(z-1)^{2+1}} dz$

$f(z) = z+1$; $n=2$

$f''(z) = 0$

$$I = \frac{2\pi i}{2!} f''(z_0) = \frac{2\pi i}{2!}(0) = 0$$

~~Cauchy's Integral~~

Ex $\oint \frac{\cos z}{(z-1)e^z} dz$, $C: |z|=2$ clockwise direction

$$I = \oint \frac{\left(\frac{\cos z}{e^z}\right)}{(z-1)} dz$$

$f(z) = \frac{\cos z}{e^z}$; $z_0 = 1$

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due to clockwise direction

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$I = -2\pi i f(1) = -2\pi i \frac{\cos 1}{e}$

(2) $I = \oint_C \frac{z}{z^2-\pi^2} dz$ C: $|z-3| = 1$ anticlockwise

→ we have 2 singularities

• $\oint_C \frac{z}{(z-\pi)(z+\pi)} dz$ → partial fraction

→ In $|z-3| = 1$ only $z = \pi$ comes inside the curve but $z = -\pi$ does not come inside this curve.

$f(z) = \frac{z}{z+\pi}$; $z_0 = \pi$

$I = 2\pi i f(\pi)$

$= 2\pi i \times \frac{\pi}{2\pi} = \underline{\pi i}$ Ans

(3) $I = \oint_C \frac{e^z}{z^2-1} dz$; C: $|z| = 2$ Anti clockwise

$I = \oint_C \frac{e^z}{(z-1)(z+1)} dz$

Now we will use Partial fraction here.

→ $\oint_C \left(\frac{e^z}{2(z-1)} - \frac{e^z}{2(z+1)}\right) dz$

→ $\oint_C \frac{e^z}{2(z-1)} - \oint \frac{e^z}{2(z+1)} dz$ → each of the integral has one singularity at a time

$f(z)$

→ $2\pi i f(1) - 2\pi i f(-1)$

→ $\frac{2\pi i e^1}{2} - \frac{2\pi i e^{-1}}{2}$

→ $\boxed{e\pi i - e^{-1}\pi i} = \pi i (e - e^{-1})$

Q $I = \oint \frac{2z + \sin z}{(z-\pi)^3} dz$ C: $|z| = \pi$ anticlockwise

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Using definition of continuity :-

Proof: Let C be a closed curve with center z_0 lying in D. Now, $f(z) = f(z) - f(z_0) + f(z_0)$

Take

- $$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_C \frac{f(z)-f(z_0)+f(z_0)}{(z-z_0)}\,dz$$

- $$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz + \underbrace{\oint_C \frac{f(z_0)}{(z-z_0)}\,dz}$$

- $$\oint_C \frac{f(z)-f(z_0)}{(z-z_0)}\,dz + f(z_0)\underbrace{\oint_C (z-z_0)^{-1}\,dz}_{2\pi i \text{ Ans}}$$

because $n=-1$

- $$\boxed{\oint_C \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{0}\,dz + \underbrace{2\pi i}\,f(z_0)}$$

$\therefore$ $|z| \leq 0$ only possible value of $z=0$;

$$\left|\Sigma (z_1+z_2)\right| \leq \Sigma |z_1| + |z_2|$$

$$\rightarrow \left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,dz$$

Let $$\left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \oint_C \frac{|f(z)-f(z_0)|}{z-z_0}\,dz$$

$f(z)$ is also continous at z_0, we can get $\varepsilon > 0$ for every $\delta > 0$ s.t.

$$|f(z)-f(z_0)| < \varepsilon, \text{ whenever } |z-z_0| < \delta$$

Let $|z-z_0| = r < \delta$

Now
$$\left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\right| dz \leq \oint_C \frac{\varepsilon}{r}\,dz$$

$$\leq \frac{\varepsilon}{r}(2\pi \cancel{r})$$

$$\Rightarrow 2\pi\varepsilon$$

$$\boxed{\left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq 2\pi\varepsilon}$$

when $\varepsilon \rightarrow 0$ we get this integral $\rightarrow \underline{0}$

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$$\boxed{\oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0}$$

→ so only Integral we get $2\pi i\, f(z_0)$

Generalised Result →

$$\boxed{\oint \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!}\left[\frac{d^n f(z)}{dz^n}\right]_{z=z_0}}$$

Q = $I = \oint \frac{2z + \sin z}{(z-\pi)^3}\, dz$

$f(z) = 2z + \sin z$

$$I = \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left[2z + \sin z\right]_{z=z_0}$$

$$I = \pi i \left[-\sin \pi\right]_{z=\pi}$$

$$\boxed{I = 0}$$

Q ① $\oint \tan z\, dz$ C : $|z| = ①\ 3$ → Taking this

On this case no singularity

It has 2 poles $\pi/2$ and $-\pi/2$

→ But by cauchy's Integral we can't find this Integral

② $\oint \frac{e^z}{e^z - 1}\, dz$ C : $|z| = 1$

→ It's solution is not determined by Cauchy Integral

Cauchy Inequality :-

Let $f(z)$ is analytic in D and z_0 be a point in D, and C be a closed curve (disk) of radius 'r' lying in D. If $|f(z)| \leq M$ for all z inside and on C then for any positive integer n

$$\left| f^{(n)}(z_0) \right| \leq \frac{M n!}{r^n}$$

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Proof :- As $f(z)$ is analytic and z_0 is lying in C, we can get

$$\left| f^{(n)}(z_0) \right| \leq \left| \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \oint \frac{|f(z)|}{|z-z_0|^{n+1}} dz$$

$$\leq \frac{n!}{2\pi} \oint \frac{M}{r^{n+1}} dz$$

$$\boxed{\left| f^{n}(z_0) \right| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} (2\pi r) = \frac{n! M}{r^n}}$$

Liouville Theorem :- ~~Let f be a bounded~~ Every ~~Analytic~~ entire and Bounded function in C is constant.

- Given $f(z)$ is entire if we Prove derivative zero Job done

Proof we have given $f(z)$ is ~~analytic~~ entire in C so by Cauchy Inequality →

$$\left| f'(z_0) \right| \leq \frac{1! M}{r} = \frac{M}{r} \left\{ \text{for } n=1 \right\}$$

r → radius of Region where f^n is analytic & our f^n is analytic in whole plane. so we are taking our whole plane → $r \to \infty$

$$\boxed{\left| f'(z_0) \right| = 0}$$

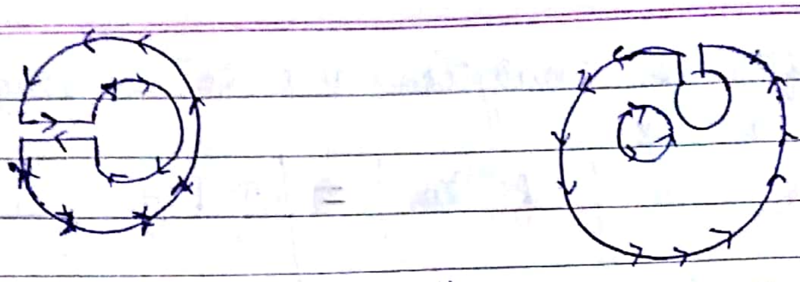
z_0 = arbitrary value

$$\Rightarrow \boxed{f(z_0) = c}$$

15/9/17 If $f(z)$ is analytic,

$$\oint_c f(z) = 0$$

Doubly connected →



Multiconnected to simply connected & then apply cauchy Integral theorem →

- $\oint_c \frac{e^2}{z-5} \, dz$ $\quad c: |z-2| = 0$

  $I = 0$

- $\oint_c g(z) = 0$

  g(z) is defined

  $\not\Rightarrow$ it is analytic

  eg. $f(z) = z^n$

- $\oint_c g(z) = 0$

  g(z) is defined and continous

  $\Rightarrow$ g(z) is analytic

- $\frac{1}{z^2}, \frac{1}{z^3}$ - - - -

- $\oint_{|z|=r} z^n dz$ $\quad n = -2, -3 \ldots$

  $\oint_{|z-z_0|=R} (z-z_0)^n dz = 0$ $\quad$ for $n = -2, -3 \ldots$

Morera's Theorem :-

Let f(z) be continous in a simply connected domain D and suppose that

$$\oint_c f(z) \, dz = 0$$

around every simple closed curve C in D, Remember some standard eg to disprove something.

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then $f(z)$ is analytic.

Ques If $f(z)$ is analytic everywhere in C and $|f(z)| < 13$ Prove that $f(z)$ is constant.

Given $|f(z)| < 13 = M$

Now cauchy inequality for $n=1$ & closed curve C such that $C: |z| = R$.

we get,

$$|f'(z_0)| = 13/R$$

As $R \to \infty$

$$|f'(z_0)| = 0$$

$\Rightarrow$ $f(z_0)$ is constant.

Fundamental Theorem of Algebra :-

Eg Every non constant polynomial has at least 1 root

Proof $\to$ Let $f(z)$ be a non constant polynomial

Let us assume that $P(z)$ does not have any root $P(z) \neq 0$ for any z.

$\Rightarrow$ $\frac{1}{P(z)}$ exist

Also, we know that polynomial in z are analytic

$\Rightarrow$ $P(z)$ is analytic

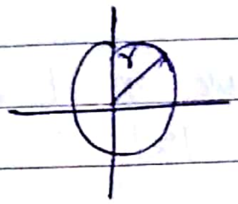
$\Rightarrow$ $\frac{1}{P(z)}$ is also analytic for all z

- Analytic functions are not always bounded.
  For eg $f(z) = |z|$

→ If the function $P(z) \to \infty$ as $z \to \infty$
there exist some M s.t.

$$|P(z)| > M \text{ for all } |z| > r$$

$$\Rightarrow \frac{1}{|P(z)|} < M \text{ for all } |z| > r \quad -(1)$$



If function is bounded inside the circle also, then we can apply Liouville's Theorem.

- $\frac{1}{|P(z)|}$ is differentiable $|z| \leq r$.
- $\frac{1}{P(z)}$ is continous $|z| \leq r$
- $\frac{1}{P(z)}$ is bounded in $|z| \leq r \quad -(2)$

By (1) and (2) $\frac{1}{P(z)}$ is bounded in entire complex plane.

$\Rightarrow \frac{1}{P(z)}$ is entire & bounded in entire complex plane.

$\Rightarrow \frac{1}{P(z)}$ is constant by Liouville's Theorem.

$\Rightarrow P(z)$ is constant

$\Rightarrow$ It is contradiction of our assumption

$\Rightarrow P(z)$ has atleast 1 root.

eg $f(z) = z^7 - 2z + 12$ inside $|z| = 1$

How many zeros inside $|z| = 1$?