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Partial Differential Equations

1. An elementary Course in P.D.E.
~ T. Amarnath

2. (Not sure of book's name) ~ I.N. Sneddon

In real world, P.D.E. seeks many applications. Some of them being:

(i) Black Scholes Equation :- (1973)

$$f_t + rS f_S + \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

$r \rightarrow$ risk free rate of return

$\sigma \rightarrow$ Constant Valuability

(ii) Navier - Stockes Equation :- For fluid mechanics

$$u_t + (u \nabla) u = -\nabla p + \nu \Delta u$$

$\nabla \rightarrow$ gradient

$\Delta \rightarrow$ Laplace

$\nabla u = 0$ in Ω

$u = 0$ in $\partial\Omega$

Other than that, there are other applications like - image processing, Computer Vision, Electrodynamics & fluid flow problem.

Definition :- A P.D.E. is an equation of form-

$$F(x, y, \ldots z, z_x, z_y, \ldots z_{xx}, z_{xy}, z_{yy} \ldots) = 0$$

where $x, y \to$ Independent Variable
$z \to$ Dependent Variable
depends on independent variable

$$z = z(x, y, \ldots)$$

Order of P.D.E. :- defined by the order of highest order derivatives appearing in the P.D.E.

★ Classification of P.D.E. :-

(i) Quasi-linear P.D.E. :→ A P.D.E. is called Quasi-linear if the highest order derivative appears linearly in the P.D.E.

(ii) Semi-linear P.D.E. :→ A Quasi-linear P.D.E. is called semi-linear if the coefficient of the highest order derivative is independent of dependent variables & its derivatives.

(iii) Linear P.D.E. :→ A semi-linear P.D.E. is called linear if it is linear in the dependent variables & its

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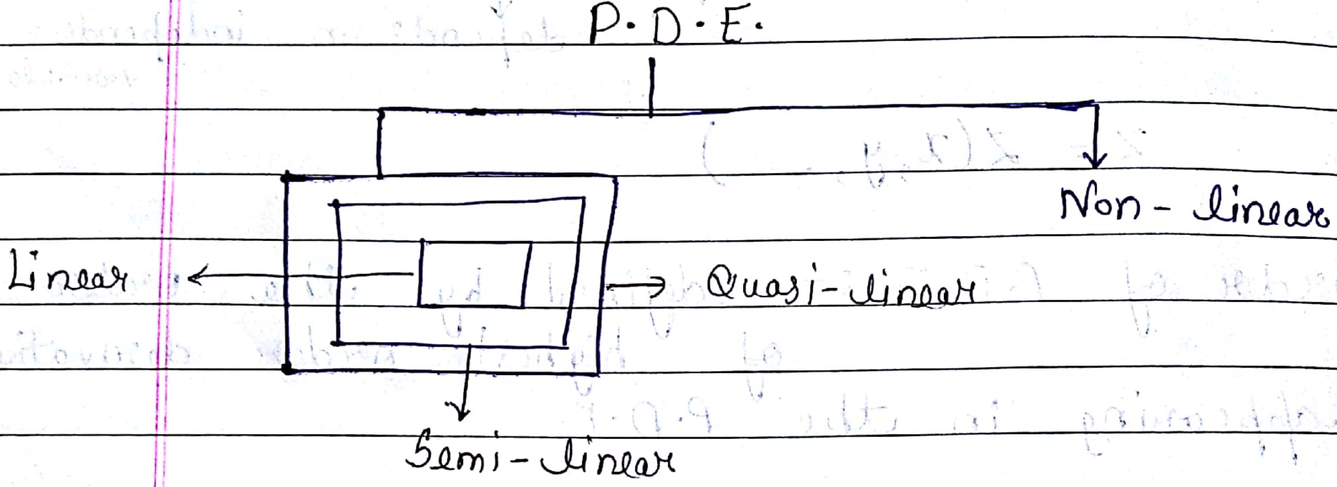
derivatives.

(iv) Non-linear P.D.E. :- A P.D.E. which is not Quasi-linear is called Non-linear P.D.E.



e.g. (i) $u u_{xx} + u_t = xt$ → order $= 2$

↳ Quasi-linear

[$u u_{xx}$ → linear
but u → dependent variable]

(ii) $u u_{xx} + u^2 u_t = xt$

$\Rightarrow u_{xx} + u u_t = \frac{xt}{u}$ → Semi-linear

(iii) $x u_{xx} + u u_t = xt$ → Semi-linear

(iv) $x u_{xx} + u_t = xt$ → Linear

(v) $(u_{xx})^2 + u_t = xt$ → Non-linear

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(vi) $u_{xx} + (u_t)^2 = xt \rightarrow$ Semi-linear

* First order P.D.E. :-

$F(x, y \ldots z \ldots) = 0$

Surface of revolution $\rightarrow$ about z-axis.

$z = F(r) \quad ; \quad r = \sqrt{x^2+y^2}$

$F \rightarrow$ Arbitary function

Differentiate it w.r.t. x,

$\cancel{\frac{\partial z}{\partial x}} \cancel{=}$

$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} = F'(r)\frac{2x}{2(x^2+y^2)^{1/2}}$$

$$= F'(r)\frac{x}{r}$$

& $\frac{\partial z}{\partial y} = F'(r)\frac{y}{r}$

$\left[\text{Let } \frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q\right]$

$F'(r) = \frac{pr}{x}$

& $q = \frac{pr}{x} \times \frac{y}{r} = \frac{py}{x}$

$$\Rightarrow \boxed{xq - py = 0}$$

e.g. $x^2 + y^2 + (z-c)^2 = a^2$

$\hookrightarrow$ Sphere of radius 'a' centred at $(0, 0, c)$

$2x + 2(z-c)p = 0$ & $2y + 2(z-c)q = 0$

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$z = F(r)$; $r = \sqrt{x^2+y^2}$

$z - F(r) = 0$

$\Rightarrow$ $u - v = 0$

$\Rightarrow$ $H(u, v) = 0$

Suppose a surface is denoted as $F(u, v) = 0$ where $u = u(x, y, z)$ & $v = v(x, y, z)$.

F is continuously differentiable function having first order partial derivative w.r.t. x.

$$F(u, v) = 0 \quad \text{—} \quad (1)$$

To calculate corresponding partial differential eqn:-

Differentiate (1) w.r.t. x & y respectively.

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{dz}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \quad \text{—} (2)$$

Here $z = z(x \,\&\, y)$

& $p = \dfrac{\partial z}{\partial x}$

$$\& \ \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0 \quad (3)$$

On elimination of $\frac{\partial F}{\partial u}$ & $\frac{\partial F}{\partial u}$, we get -

$$\boxed{\frac{d(u, v)}{d(y, z)} p + \frac{d(u, v)}{d(z, x)} q = 0}$$

First order P.D.E.

where $\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y$

$\downarrow$

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The above eqⁿ is Quasi-linear.

e.g. $(x-a)^2 + (y-b)^2 + z^2 = 1$ — ①

↳ Family of surface with parameters a & b.

Differentiate w.r.t. x & y.

$2(x-a) + 2zp = 0 \Rightarrow (x-a) = -zp$ — ②

$2(y-b) + 2zq = 0 \Rightarrow (y-b) = -zq$ — ③

From ①, ② & ③

$p^2z^2 + q^2z^2 + z^2 = 1$

$z^2(p^2+q^2+1) = 1 \rightarrow$ Non-linear P.D.E.

$F(x,y,z,a,b) = 0$ — ①

↳ Surface with parameters a & b.

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p = 0$ — ②

$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q = 0$ — ③

↳ $f(x,y,z,p,q) = 0$

★ Classification of 1st order P.D.E. :-

$$f(x, y, z, p, q) = 0 \quad — ①$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

(i) Quasi-linear P.D.E. - Eq. ① is Quasi-linear if it is in form :

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

e.g. $xz^2 p + x^2 y z q = x^2 y^2 z^2$

(ii) Semi-linear P.D.E. - Eq ① in the form -

$$P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$$

(iii) Linear P.D.E. - Eq ① has form

$$P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$$

(iv) Non-linear P.D.E. - If ① does not have any of the above forms.

e.g. (a) $xy\, p + x^2 y^2 q = xz^2 \rightarrow$ Semi-linear

(b) $xy\, p + x^2 y^2 q = xz \rightarrow$ Linear

* Solution of 1st order P.D.E. :-

$$f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

A function $z = z(x, y)$; $(x, y) \in D \subseteq R^2$ is called solution of eq. (1), if z ~~has~~ and its ~~deri~~ first-order partial derivatives w.r.t. x & y (i.e. p & q) satisfy eq. (1) on D

Note:- A solution $z = z(x,y)$ can be interpreted as a surface in 3-D space. Thus, solution $z = z(x,y)$ also referred as Integral surface for eq. (1)

Complete integral (Complete solution) :- A two-parameter function of surface -

$$z = F(x, y, a, b)$$ is called complete integral of (1) if the rank of matrix 'M' :

$$M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$$ is 2.

General integral or general solution :-

$$F(u, v) = 0$$

where $u = u(x, y, z)$ & $v = v(x, y, z)$

F is continuously differentiable function which satisfy given P.D.E. implicitly in some domain D.

Singular Integral (or Singular Solution):-

$$z = F(x, y, a, b)$$

$$\frac{\partial F}{\partial a} = 0 \text{ ; } \frac{\partial F}{\partial b} = 0 \quad - (2)$$
(1)

Singular solution or singular integral is obtained by finding envelope of complete integral by eliminating a & b from
$$z = F(x, y, a, b) \quad - (3)$$

Sol. obtained by eq. (1), (2) & (3) is called singular solution.

e.g. $\quad z - px - qy - p^2 - q^2 = 0$

Ans $\quad f(x, y, z, q, p) = z - px - qy - p^2 - q^2 = 0$

Since, it is non-linear P.D.E.
[So, we can't be asked to calculate its complete integral since we are limited to 1st order linear P.D.E.]

Here, Complete integral is -
$$z = F(x, y, a, b) = ax + by + a^2 + b^2 \quad - (1)$$

$$F_a = x + 2a = 0 \quad - (2)$$
$$F_b = y + 2b = 0 \quad - (3)$$

From ② & ③, we have -

$$a = -\frac{x}{2}, \quad b = -\frac{y}{2}$$

Substituting in ①, we have -

$$\boxed{4z = -(x^2+y^2)} \rightarrow \text{Singular solution}$$

↳ Paraboloid of revolution

e^{-z}

★ Cauchy Problem :-

$$f(x, y, z, p, q) = 0 \quad \text{— ①}$$

To find a solution of ①, which contains an intial curve (initial condition):

$$x = x_0(\lambda), \quad y = y_0(\lambda), \quad z = z_0(\lambda)$$

$$\lambda \in I$$

★ Lagrange's method :- [To solve Quasi-linear first-order P.D.E.]

Theorem- Let P, Q, R at continuously differentiable functions of x, y & z. The general solution of PDE -

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

is -

$$F(u, v) = 0$$

where F is any arbitary smooth function and $u(x,y,z) = C_1$ and $v(x,y,z) = C_2$ are two linearly independent solutions of the auxialiary eqⁿ

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Proof:- Let $u = c_1$ & $v = c_2$ are two linearly independent solutions of Auxiliary eqn-

$$du = 0$$

$$u_x\,dx + u_y\,dy + u_z\,dz = 0$$

Now, let $\frac{dx}{P} = \frac{dy}{q} = \frac{dz}{R} = c$ (let)

$$\Rightarrow u_x(cP) + u_y(cq) + u_z(cR) = 0$$

$$\Rightarrow P u_x + q u_y + R u_z = 0 \quad \text{—} \quad (1)$$

Similarly,

$$P v_x + q v_y + R v_z = 0 \quad \text{—} \quad (2)$$

By Cramer's Rule -

$$\cancel{\frac{P}{u_y v_z - u_z v_y} =}$$

$$\frac{P}{u_y v_z - u_z v_y} = \frac{q}{u_z v_x - u_z v_z} = \frac{R}{u_x v_y - u_y v_x}$$

$$\Rightarrow \frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \quad \text{—} \quad (4)$$

Jacobian

Now, recall that $F(u,v) = 0$ is a solution of P.D.E.

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$$\Rightarrow \frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)} R \quad — ③$$

Hence, $Pp + Qq = R$ is solution of P.D.E.

Q. $x^2 p + y^2 q = (x+y)z$

Ans. Auxiliary eqn :- $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow -\frac{1}{x} = -\frac{1}{y} + C_1$$

$$\frac{1}{y} - \frac{1}{x} = C_1 \quad — ①$$

$$u(x,y,z) = C_1$$

Also, $\frac{dz}{(x+y)z} = \frac{dx - dy}{x^2 - y^2}$ [By fraction concept]

$$\Rightarrow \frac{dx - dy}{(x-y)} = \frac{dz}{z}$$

$$\Rightarrow \frac{d(x-y)}{(x-y)} = \frac{dz}{z} \Rightarrow \ln(x-y) = \ln z + \ln C_2$$

$$\Rightarrow x - y = C_2 z$$

$$\Rightarrow \frac{x-y}{z} = C_2 \quad — ②$$

$$v(x,y,z) = C_2$$

So, solution of P.D.E.

$$F\left(\frac{1}{y} - \frac{1}{x}, \frac{x-y}{z}\right) = 0$$

Q. Find the integral surface of -

$$xp + yq = z$$

which contains curve :

$\Gamma : x_0 = s^2$; $y_0 = s+1$; $z_0 = s$

Ans Auxiliary eqn :- $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

$\Rightarrow$ $u = \frac{y}{z} = C_1$ & $v = \frac{y}{x} = C_2$

~~So,~~ $\cancel{\frac{s+1}{s}} = \cancel{s}C_1$

Now,

if Γ is the curve on integral surface must satisfy both of them.

$\frac{s+1}{s} = C_1$ & $\frac{s+1}{s} = C_2$

$\Rightarrow$ $(C_1 - 1) C_1 = C_2$

$\Rightarrow$ $(u-1) u = v$

$\Rightarrow$ $(y - z) x = z^2$ $\rightarrow$ Solution of Cauchy Problem

Q $yzp + xzq = xy$

Ans Auxiliary eqn : $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$

(i) $\frac{dx}{yz} = \frac{dy}{xz}$ $\Rightarrow$ $x^2 - y^2 = C_1$

(ii) $y^2 - z^2 = C_2$

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Note:- $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{aa_1 + cc_1 + ee_1}{ba_1 + dc_1 + fe_1}$ → a_1, c_1 & e_1 are multipliers

Sol. → $F(x^2 - y^2, y^2 - z^2) = 0$

$\frac{dx}{l} = \frac{dy}{m} = \frac{dx - dy}{l - m}$

1 & -1 are multipliers

Q $(z^2 - 2yz - y^2)p + x(y+z)q = x(y-z)$

Ans Auxiliary eqn : $\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$

(i) $\frac{dy}{y+z} = \frac{dz}{y-z}$

$\Rightarrow y\,dy - z\,dz - (z\,dy + y\,dz) = 0$

$\Rightarrow d\left(\frac{y^2}{2}\right) - d\left(\frac{z^2}{2}\right) - d(yz) = 0$

On integration, we get -

$y^2 - z^2 - 2yz = c_1$

Consider x, y, z as multipliers, we get -

$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} = \frac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow x^2 + y^2 + z^2 = c_2$

↳ General Sol. $F(y^2 - z^2 - 2yz, x^2 + y^2 + z^2) = 0$

* Second order P.D.E. :-

Let $D \subseteq R^2$ be a smooth domain. And $R, S, T : D \to R$ be smooth functions. Then a second order P.D.E. is called semi-linear P.D.E. if it has following form:

$$R(x,y)\, u_{xx} + S(x,y)\, u_{xy} + T(x,y)\, u_{yy} + g(x, y, u, u_x, u_y) = 0 \qquad (1)$$

Also, $R^2 + S^2 + T^2 \neq 0$

$u(x,y)$ defined on D is called solution eq. (1) if it satisfies (1) on D.

We can classify (1) in three categories:

(i) Hyperbolic:

$$S^2 - 4RT > 0 \quad \text{in } D$$

(ii) Parabolic:

$$S^2 - 4RT = 0 \quad \text{in } D$$

(iii) Elliptic : $S^2 - 4RT < 0$ in D

e.g. Consider eq^n for vibration in string as-

$$u_{tt} = c^2 u_{xx} \; ; \; c > 0 \qquad u(t,x)$$

Sol. $R = 1, \quad S = 0, \quad T = -c^2$

$\therefore\ 0 - 4(-c^2)(1) = 4c^2 > 0$ → Hyperbolic (wave eqⁿ)

Q. Consider 1D-heat conduction eqⁿ is given as

$u_t = \sigma u_{xx}\ ; \quad \sigma > 0$

$\sigma \rightarrow$ Diffusivity coefficient

This eqⁿ is hyperbolic, parabolic or elliptic?

Ans. $R = 0,\quad S = 0,\quad T = -\sigma$

$\therefore\ S^2 - 4RT = 0 - 0(-\sigma) = 0$ → Parabolic

e.g. $u_{xx} + u_{yy} = 0$ → Laplace eqⁿ in 2D (Potential eqⁿ)

$R = 1,\quad T = 1,\quad S = 0$

$\therefore\ 0 - 4(1)(1) < 0$ → Elliptic

Q. $x u_{xx} + u_{yy} = 0$. Evaluate range of x for which eqⁿ is hyperbolic, parabolic & elliptic.

Ans. $R = x,\quad S = 0,\quad T = 1$

$\Rightarrow\ S^2 - 4RT = 0 - 4x = -4x$

for $x < 0$ → Hyperbolic

$x = 0$ → Parabolic

$x > 0$ → Elliptic

$x<0$	$x>0$
Hyperbolic	Elliptic

* Canonical Form :-

$$R(x,y)\,u_{xx} + S(x,y)\,u_{xy} + T(x,y)\,u_{yy} + g(x,y,u,u_x,u_y) = 0 \quad \text{--- (1)}$$

Changing the coordinates

Psi $\rightarrow \xi = \xi(x,y)$; $\eta = \eta(x,y)$
Eta

$u \rightarrow$ function of (ξ, η)

Above transformation is

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x \neq 0$$

And, $u_x = u_\xi \xi_x + u_\eta \eta_x$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xy} = u_{\xi\xi}\xi_x \xi_y + u_{\xi\eta}\eta_y \xi_x + u_\xi \xi_{xy} + u_{\eta\xi}\xi_y \eta_x + u_{\eta\eta}\eta_y \eta_x + u_\eta \eta_{xy}$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + u_{\xi\eta}\eta_x \xi_x + u_\xi \xi_{xx} + u_{\eta\xi}\xi_x \eta_x + u_{\eta\eta}\eta_x^2 + u_\eta \eta_{xx}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + u_{\xi\eta}\eta_y \xi_y + u_\xi \xi_{yy} + u_{\eta\xi}\xi_y \eta_y + u_{\eta\eta}\eta_y^2 + u_\eta \eta_{yy}$$

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Again,

$$R(x,y)\,u_{xx} + S(x,y)\,u_{xy} + T(x,y)\,u_{yy} + g(x,y,u,u_x,u_y) = 0 \quad \text{——} (1)$$

Consider $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ such that -

Jacobian → $\dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x \neq 0$

i.e. ξ, η are linearly independent

And, this transformation in terms of ξ, η is reversible (invertible)

Substituting values of u_{xy}, u_{xx}, u_{yy} in (1),

$$\underbrace{(\qquad)}_{A} u_{\xi\xi} + 2\underbrace{(\qquad)}_{B} u_{\xi\eta} + \underbrace{(\quad)}_{C} u_{\eta\eta} = G(\xi,\eta,u,u_\xi,u_\eta)$$

$$A\,u_{\xi\xi} + 2B\,u_{\xi\eta} + C\,u_{\eta\eta} = G(\;) \quad — (2)$$

$\cancel{S^2 - 4RT}$

$$4B^2 - 4AC = (S^2 - 4RT)(\xi_x \eta_y - \xi_y \eta_x)^2$$

$$\Rightarrow B^2 - 4AC = \frac{(S^2-4RT)(\xi_x\eta_y - \xi_y\eta_x)^2}{4} \quad — (3)$$

Case 1 :- $S^2 - 4RT > 0$ → hyperbolic

i.e. $R\lambda^2 + S\lambda + T = 0$ has two distinct real roots, say λ_1 & λ_2

If we choose ξ & η is a manner such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y} \quad \& \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}$$

$$\xi_x - \lambda_1 \xi_y = 0 \qquad \eta_x - \lambda_2 \partial \eta_y = 0$$

$$\Downarrow$$

$$\frac{dx}{1} = \frac{dy}{\lambda} = \frac{d\xi}{0} \qquad \frac{dx}{1} = \frac{dy}{-\eta} = \frac{d\eta}{0}$$

$$\Rightarrow \frac{dy}{dx} + \lambda_1 = 0 \qquad \frac{dy}{dx} + \lambda_2 = 0$$

$$\Downarrow$$

$$f_1(x, y) = c_1 \qquad f_2(x, y) = c_2$$

$$\xi = f_1(x, y) \quad \& \quad \eta = f_2(x, y)$$

$$\boxed{u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)} \rightarrow \text{Coeff. } A \,\&\, C = 0$$

Case 2:- $S^2 - 4RT = 0 \rightarrow$ parabolic

$R\lambda^2 + S\lambda + T = 0$ has real root say λ_1

$$\xi_x = \lambda_1 \xi_y$$

$\xi = f_1(x, y)$ Since A is zero, then-

$B^2 - AC = 0$ iff $B = 0$. We choose such η which is linearly independent of $\xi = f_1(x, y)$

$$\boxed{u_{\eta\eta} = \psi(\xi, \eta, u, u_\xi, u_\eta)} \rightarrow \text{Coeff. } B \,\&\, C = 0$$

Case 3: $S^2 - 4RT < 0 \rightarrow$ Ellipse

$R\lambda^2 + S\lambda + T = 0$

↳ roots are complex which are conjugate to each other.

$$\lambda_1, \lambda_2 = \alpha \pm i\beta$$

$$\frac{dy}{dx} + (\alpha + i\beta) = 0, \quad \frac{dy}{dx} + (\alpha - i\beta) = 0$$

$$f_1(x,y) = y + (\alpha + i\beta)x = c_1$$
$$f_2(x,y) = y + (\alpha - i\beta)x = c_2$$

$\xi_1 = f_1(x,y)$ & $\eta = f_2(x,y)$

then both ξ & η are complex. But we need real transformation. Hence,

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{i}{2}(\eta - \xi)$$

$$\boxed{\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = \phi(\alpha, \beta, u, u_\alpha, u_\beta)}$$

e.g. $u_{xx} - x^2 u_{yy} = 0$ — (1)

Here $R = 1$, $S = 0$, $T = -x^2$

$S^2 - 4RT = 4x^2 > 0 \rightarrow$ Hyperbolic

Consider

$R\lambda^2 + S\lambda + T = 0$ i.e.

$\lambda^2 - x^2 = 0 \Rightarrow \lambda_1, \lambda_2 = \pm x$

So, $\frac{dy}{dx} + \lambda_1 = 0$ & $\frac{dy}{dx} + \lambda_2 = 0$

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$\Rightarrow \quad \frac{dy}{dx} + x = 0 \quad \& \quad \frac{dy}{dx} - x = 0$

$\Downarrow$

$y + \frac{x^2}{2} = c_1 \qquad y - \frac{x^2}{2} = c_2$

Let $\xi = x + \frac{x^2}{2}$ — (i) & $\eta = y - \frac{x^2}{2}$] (ii)

Here ξ & η are linearly independent.

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$= x u_\xi - x u_\eta$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$= u_\xi + u_\eta$$

$$u_{xx} = u_{\xi\xi} x^2 - u_{\xi\eta} x^2 + u_\xi - u_{\eta\xi} x^2$$

$[u_{\xi\eta} = u_{\eta\xi}]$ $+ u_{\eta\eta} x^2 + \cancel{u_{\eta\xi} x^2} - u_\eta$

$$= x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} + u_\xi - u_\eta$$

$$u_{yy} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Thus from eqn 1,

$$x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} + u_\xi - u_\eta$$
$$- x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} - x^2 u_{\eta\eta} = 0$$

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$$\Rightarrow \quad 4x^2 u_{\xi\xi} = u_{\xi} - u_{\eta}$$

Also, $\xi - \eta = x^2$ (from (i) & (ii))

$$\Rightarrow \quad \boxed{u_{\xi\eta} = \frac{u_{\xi} - u_{\eta}}{4(\xi - \eta)}}$$

Here Cleary, $A = 0$ & $C = 0$

Q $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$ — ①

$R = y^2$, ~~z~~ $S = -2xy$, $T = x^2$

$S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0 \rightarrow$ Parabolic

Consider, $y^2\lambda^2 - 2xy\lambda + x^2 = 0 \Rightarrow \lambda = \frac{x}{y}$

$\therefore \frac{dy}{dx} + \frac{x}{y} = 0 \Rightarrow x^2 + y^2 = c_1$

Let $\xi = x^2 + y^2$ Choose $\eta = x^2 - y^2$ so that ξ & η are linearly independent.

$u_x = u_{\xi}\, 2x + u_{\eta}\, 2x = 2x\, u_{\xi} + 2x\, u_{\eta}$

$u_y = u_{\xi}\, 2y + u_{\eta}(-2y) = 2y\, u_{\xi} - 2y\, u_{\eta}$

$u_{xx} = 4x^2 u_{\xi\xi} + 4x^2 u_{\xi\eta} + 2u_{\xi} + 4x^2 u_{\eta\xi}$
$+ 4x^2 u_{\eta\eta} + 2u_{\eta}$

$[u_{\xi\eta} = u_{\eta\xi}] \rightarrow$ Since u_x, u_y are continuous

$u_{yy} =$

$u_{xy} =$

Reduce eq^n ① to $u_{\eta\eta} = 0$

Integrate it -

$u_\eta = f(\xi)$

$u = \eta f(\xi) + g(\xi)$

So, solution of ①

$\cancel{u = (x^2 - y^2) f(\xi)}$

$u = (x^2 - y^2) f(x^2 + y^2) + g(x^2 + y^2)$

Q $u_{xx} + x^2 u_{yy} = 0$ — ①

Ans $S = 0, \quad R = 1, \quad T = x^2$

$S^2 - 4RT = -4x^2 < 0 \longrightarrow$ Elliptic

Consider quadratic eq^n →

$R\lambda^2 + S\lambda + T = 0$

$\Rightarrow \lambda^2 + x^2 = 0 \Rightarrow \lambda_{1,2} = \pm ix$

$\frac{dy}{dx} + \lambda_1 = 0$ & $\frac{dy}{dx} + \lambda_2 = 0$

$\Rightarrow \frac{dy}{dx} + ix = 0$ & $\frac{dy}{dx} - ix = 0$

$\Rightarrow y + i\frac{x^2}{2} = c_1$ & $y - i\frac{x^2}{2} = c_2$

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Let $\xi = y + i\frac{x^2}{2}$, $\eta = y - i\frac{x^2}{2}$

For real transformation →

$$\beta \cancel{\alpha} = \frac{1}{2}(\xi + \eta) = y$$

$$\alpha \cancel{\beta} = -\frac{i}{2}(\xi - \eta) = \frac{x^2}{2} \qquad (u \to \beta, \alpha)$$

$\cancel{u_x = u_\alpha}$

Now, $u_x = u_\alpha \cdot x$

$u_y = u_\beta$

$$u_{xx} = u_{\alpha\alpha} \cdot x^2 + u_\alpha$$

$$u_{yy} = u_{\beta\beta}$$

Substitute it in eqn ①

$$u_{\alpha\alpha} \cdot x^2 + u_\alpha + x^2 u_{\beta\beta} = 0$$

$$\Rightarrow (u_{\alpha\alpha} + u_{\beta\beta})\, x^2 + u_\alpha = 0$$

$$\Rightarrow (u_{\alpha\alpha} + u_{\beta\beta})\, 2\alpha = -u_\alpha \qquad \left[\because \alpha = \frac{x^2}{2}\right]$$

$$\Rightarrow \boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{2\alpha}}$$

Q. $e^{2x} u_{xx} + 2e^{x+y} u_{xy} + e^{2y} u_{yy} = 0$

Ans. $R = e^{2x}$, $S = 2e^{x+y}$, $T = e^{2y}$

$S^2 - 4RT = 4e^{2(x+y)} - 4e^{2(x+y)} = 0$ → Parabolic

Consider $R\lambda^2 + S\lambda + T = 0$

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$\Rightarrow \quad e^{2x}\lambda^2 + 2e^{x+y}\lambda + e^{2y} = 0$

$\Rightarrow \quad (\lambda e^{x} + e^{y})^2 = 0$

$\Rightarrow \quad \lambda = -\dfrac{e^{y}}{e^{x}} = -e^{y-x}$

Let $\dfrac{dy}{dx} + \lambda = 0 \rightarrow \dfrac{dy}{dx} + e^{y-x} = 0$

$\Rightarrow \dfrac{dy}{e^{y}} = -e^{-x}dx \quad \Rightarrow e^{-y} - e^{-x} = C_1$

let $\xi = e^{-y} - e^{-x}$ Choose $\eta = x$

$u_x = e^{-x}u_\xi + u_\eta$

$u_y = -e^{-y}u_\xi$

$u_{xx} = e^{-2x}u_{\xi\xi} + 2e^{-x}u_{\xi\eta} + u_{\eta\eta} - e^{-x}u_\xi$

$u_{yy} = e^{-2y}u_{\xi\xi} + e^{-y}u_\xi$

$u_{xy} = -e^{-y}(e^{-x}u_{\xi\xi} + u_{\xi\eta})$

Substitute it in eq^n 1

$e^{x}e^{-y}u_{\eta\eta} = (e^{-y} - e^{-x})u_\xi \quad — 2$

Also, $\eta = x$

& $e^{-y} = \xi + e^{-x} = \xi + e^{-\eta}$

$\therefore$ eq^n 2 becomes,

$u_{\eta\eta} = \dfrac{\xi}{1+\xi e^{\eta}} \cdot u_\xi$

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* Vibrations of an infinite string :-

$$y_{tt} = c^2 y_{xx} \quad -\infty < x < \infty \; ; t > 0$$

Here $R = 1$, $S = 0$, $T = -c^2$

$$S^2 - 4RT = 4c^2 > 0 \rightarrow \text{hyperbolic}$$

Consider $\rightarrow R\lambda^2 + S\lambda + T = 0 \Rightarrow \lambda^2 - c^2 = 0$

$R U_{xx} + S U_{xy} + T U_{yy} = g(x, y, u, u_x, u_y)$
dependent $(x, y \rightarrow$ independent$)$

$\Rightarrow \lambda_{1,2} = \pm c$

So, $\frac{dx}{dt} - c = 0$ & $\frac{dx}{dt} + c = 0$

$$\Rightarrow x - ct = c_1 \quad \& \quad x + ct = c_2$$

Let $\xi = x - ct$ & $\eta = x + ct$

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = 0 \Rightarrow \frac{\partial y}{\partial \xi} = h(\xi)$$

$$\Rightarrow y(\xi, \eta) = F(\xi) + G(\eta)$$

$$\Rightarrow y(x, t) = F(x - ct) + G(x + ct) \quad \text{--- (1)} \rightarrow \text{general solution}$$

Let's consider initial conditions:

At time $t = 0 \rightarrow$ $y(x, 0) = f(x)$
displacement

$\Rightarrow y_t(x, 0) = g(x)$

These are given conditions

Initial Velocity of string

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So, From eqⁿ ①

$$y(x,0) = F(x) + G(x) = f(x) \quad —②$$

$$y_t(x,0) = -cF'(x) + cG'(x) = g(x) \quad —③$$

Integrate ③ w.r.t. x

$$-cF(x) + cG(x) = \int_{x_0}^{x} g(s)\,ds \quad —④$$

From ② & ④

$$F(x) = \frac{1}{2c}\left[cf(x) - \int_{x_0}^{x} g(s)\,ds\right]$$

$$\& \quad G(x) = \frac{1}{2c}\left[cf(x) + \int_{x_0}^{x} g(s)\,ds\right]$$

$$\boxed{y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

→ ~~Vibration~~

→ d'Alembert's solution of wave which describes the vibration of an infinite string

★ Vibrations of Semi-infinite string :-

Semi-infinite → One end fixed; one free

$$y_{tt} - c^2 y_{xx} = 0 \quad ; \quad 0 < x < \infty, \; t > 0$$

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The initial conditions are:-

$y(x,0) = u(x)$; $y_t(x,0) = v(x)$

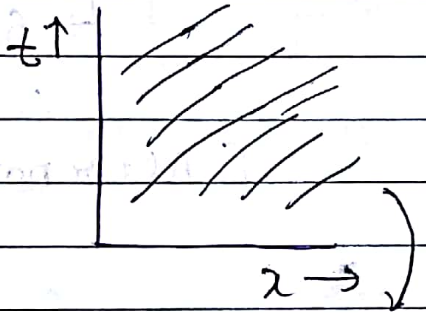
Also, the fixed end will not make displacement with time

So, $y(0,t) = 0$ → Boundary Condition

Above boundary condition leads to:-

$y_t(0,t) = 0$

Remember, one of the function we'll obtain $u(x-ct)$



Region under discussion

If $t > \frac{x}{c}$, then $u(x-ct)$ is meaningless. [$x \to -ve$; which is not the case assumed]

However, in this case, we can modify our problem to problem of infinite string in the following manner:

$$U(x) = \begin{cases} u(x) & ; x \geq 0 \\ -u(-x) & ; x \leq 0 \end{cases}$$

$$V(x) = \begin{cases} v(x) & ; x \geq 0 \\ -v(-x) & ; x \leq 0 \end{cases}$$

Since, problems is changed to infinite one.

Assume, $U(x,t)$ & $V(x,t)$ are odd functions

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$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)\,ds$$

↳ Sol. of infinite problem

$$y(0,t) = \frac{1}{2}\left[U(0-ct) + U(0+ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} V(s)\,ds = 0$$

↳ Since, U & V are odd

$\therefore\ U(-x) = -U(x)$

[Alternatively, ~~to~~ ~~as~~ satisfy $y(0,t)=0$ we assumed U & $V \to$ odd]

$$y_t(x,t) = \frac{c}{2}\left[U'(x-ct) + U'(x+ct)\right] + \frac{1}{2c}\left[c\,V(ct) + V(-ct)\right]$$

$\therefore\ y_t(x,0) = 0$

And, $y(x,0) = \frac{1}{2}\left[U(x) + U(x)\right]$

$= U(x) = u(x)$ for $x \geq 0$

$y_t(x,0) = v(x)$ for $x \geq 0$

Lebnitz rule :-

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = F(x, b(x)) \times \frac{d\,b(x)}{dx} - F(x, a x) \times \frac{d\,a(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} F(x,t)\,dt$$

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* Vibrations of a finite string :-

$$y_{tt} - c^2 y_{xx} = 0 \qquad 0 < x < l \; ; \; t > 0 \quad — (1)$$

Initial condition → $y(x,0) = u(x)$, $y_t(x,0) = v(x)$ } $0 \le x \le l$ — (2)

Boundary condition → $y(0,t) = y(l,t) = 0$, $y_t(0,t) = y_t(l,t) = 0$ } $t \ge 0$ — (3)

[Displacement → 0 at both the ends
Velocity = 0 " "]

We reduced above problem to a problem of vibration of an infinite string by extending the interval data as 'odd periodic' functions as follows:

$$U(x) = \begin{cases} u(x) & 0 \le x \le l \\ -u(-x) & -l \le x \le 0 \end{cases}$$

$$\therefore U(x + 2rl) = U(x) \; ; \; -l \le x \le l$$
$$r = \pm 1, \pm 2, \pm \ldots$$

Same way, we extend $v(x)$ as odd periodic functions.

Note:- A periodic function can be expressed as Fourier expansion.

If function is odd, ~~cosine~~ sine function will be there.

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~~So, u(x)~~

Concept of Fourier →

$f(x)$ is periodic over period $-l \leq x \leq l$,
then –

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Where, $a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

— X — X — X — X — X — X —

Now,

$$u(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$a_n = \frac{2}{l} \int_{0}^{l} u(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

And,

$$V(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l} \int_{0}^{l} v(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

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$$y(x,t) = \frac{1}{2}\left[u(x-ct) + u(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\,ds$$

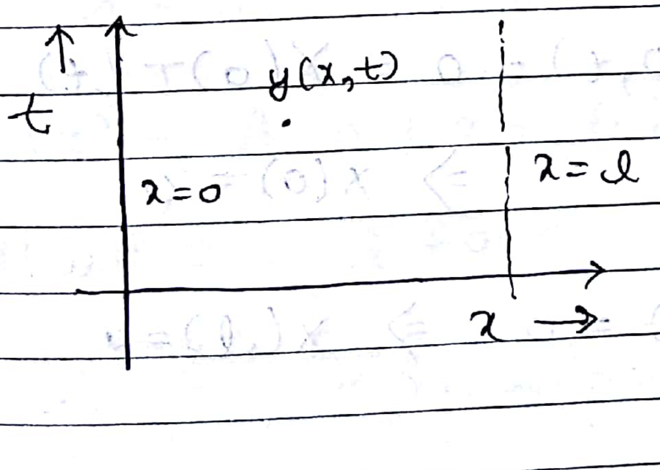
$$\frac{1}{2}\left[u(x-ct) + u(x+ct)\right] = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$$

&

$$\frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\,ds = \frac{l}{\pi c}\sum_{n=1}^{\infty} \frac{b_n}{n} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right)$$

So,

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + \frac{l}{\pi c}\sum_{n=1}^{\infty} \frac{b_n}{n} \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$



* Method of Separation of Variable →

Consider $y(x,t) = X(x)\,T(t)$

Substitute it in eqⁿ ① [of vibration in finite length string]

$$X T'' - c^2 X'' T = 0$$

T'' ↓ w.r.t. t ; X'' ↓ w.r.t. x

$$\Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = \lambda \rightarrow \text{constant}$$

$\frac{T''}{c^2T}$ ↓ function of T ; $\frac{X''}{X}$ ↓ function of X

⇒ can be equal only when they are equal to some constant

$$\Rightarrow X'' - \lambda X = 0 \quad \& \quad T'' - c^2\lambda T = 0$$

By boundary condition → $y(0,t) = 0 = X(0)T(t)$, $T(t) \neq 0$

$$\Rightarrow X(0) = 0$$

$\&$ $y(l,t) = 0 \Rightarrow X(l) = 0$

Boundary Value Problem → $X'' - \lambda X = 0$, $X(0) = 0 = X(l)$ → Sturm-Liouville problem in Eigen function value problem

We'll be considering only those values of eigen function (Eigen Value) that gives non-trivial or non-zero solution.

→ Real roots case

Case 1 :- $\lambda > 0$; $\lambda = \mu^2$

$\therefore X(x) = A e^{\mu x} + B e^{-\mu x}$ → Solution for S-L problem

$X(0) = 0 = A + B$ — (i)

& $X(l) = 0 = A e^{\mu l} + B e^{-\mu l}$ — (ii)

~~Aor~~

$\Rightarrow A = B = 0$ → Trivial sol.

$\therefore \lambda > 0$ is not eigen-value for above problem

Case 2 :- $\lambda = 0$

$\therefore X = Ax + B$

$X(0) = 0 = B$

$X(l) = 0 = Al + B = Al$

$\Rightarrow A = 0$

$\therefore \lambda = 0$ is not eigen-value for above problem.

Case 3 :- $\lambda < 0$; $\lambda = -\mu^2$

$X(x) = A \cos \mu x + B \sin \mu x$ → roots complex in Auxilliary eq^n

$X(0) = 0 = A + 0$

$\Rightarrow A = 0$

$X(l) = 0 = \cancel{B \cos \mu l}\ B \sin \mu l$

~~B can't be zero (if that happens trivial sol.)~~

$\therefore \sin \mu l = 0 = \sin n\pi$

$\Rightarrow \mu l = n\pi$

$\Rightarrow \mu = \frac{n\pi}{l}$

$\therefore \lambda_n = -\left(\frac{n\pi}{l}\right)^2 \rightarrow$ Eigen Value

$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow$ Eigen function corresponding to λ.

Now,

$$T_n'' - c^2 \lambda T_n = 0$$

$$\Rightarrow T_n'' - c^2\left[-\left(\frac{n\pi}{l}\right)^2\right] T_n = 0$$

$$\Rightarrow T_n'' + \left(\frac{cn\pi}{l}\right)^2 T_n = 0$$

$$T_n(t) = C_n \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right)$$

So,

$$y_n(x,t) = X_n(x)\, T_n(t)$$

$$= \left[a_n \cos\left(\frac{n\pi c t}{l}\right) + b_n \sin\left(\frac{n\pi c t}{l}\right)\right] \sin\frac{n\pi x}{l}$$

$; n = 1, 2, 3 \ldots$

$\{a_n = B_n C_n \;\; ; \;\; b_n = D_n B_n\}$

$$\sum_{n=1}^{\infty} y_n(x,t) = y(x,t)$$

$$\Rightarrow y(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi c t}{l}\right) + b_n \sin\left(\frac{n\pi c t}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right)$$

By Initial condition $\rightarrow$ (To calculate a_n & b_n)

$$y(x,0) = u(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \rightarrow \text{Fourier sine series}$$

where $\boxed{a_n = \frac{2}{l}\int_0^l u(x) \sin\left(\frac{n\pi x}{l}\right)}$

$\left[\begin{matrix} u(x) \\ \text{odd} \end{matrix}\right]$

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And,

$$y_t(x,0) = g(x) = \sum_{n=1}^{\infty} v(x)$$

$$y_t(x,0) = v(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \quad ; 0 < x < l$$

$$\boxed{b_n = \frac{2}{n\pi c} \int_0^l v(x) \sin\left(\frac{n\pi x}{l}\right) dx}$$

Q Solve initial-Boundary Value problem:

$$u_{tt} = c^2 u_{xx} \quad ; \quad 0 < x < 2\pi$$

Initial conditions: $u(x,0) = \cos x - 1$ & $u_t(x,0) = 0$

Boundary Conditions: $u(0,t) = u(2\pi, t) = 0$
$[t \geq 0]$

Ans Here $l = 2\pi$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{2\pi}\right) \sin\left(\frac{n\pi x}{2\pi}\right)$$

[$b_n = 0 \rightarrow$ since, Velocity $u_t(x,0) = 0$]

Here,

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} (\cos x - 1) \sin\left(\frac{nx}{2}\right) dx$$

$$= \frac{16 \sin^2\left(\frac{n\pi}{2}\right)}{n\pi(n^2-4)}$$

For even $n \rightarrow a_n = 0$

For odd $n \rightarrow$

$$a_n = \frac{16}{n\pi(n^2-4)}$$

(For all n)

$$\Rightarrow a_n = \frac{16}{\pi} \times \frac{1}{(2n-1)(2n+1)(2n+3)}$$

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$$u(x,t) = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+3)(2n+3)} \cos\left(\frac{(2n+1)ct}{2}\right) \sin\left(\frac{(2n+1)x}{2}\right)$$

★ Duhamel's principle :- used for non-homogenous eq^n / PDE

Duhamel's principle for wave eq^n :- Consider the non-homogenous wave eq^n :-

$$u_{tt} - c^2 u_{xx} = F(x,t) \quad ; t \geq 0, \quad -\infty < x < \infty \quad \text{--- (1)}$$

For simplicity, we consider homogenous

Case 1 :- Initial condition

$$u(x,0) = u_t(x,0) = 0 \quad \text{--- (2)}$$
$$-\infty < x < \infty$$

We, now, consider the function $v(x,t,\tau)$ which satisfies the following eq^n w.r.t. x & t for $t > \tau$

$$v_{tt} - c^2 v_{xx} = 0 \quad ; \quad -\infty < x < \infty, \quad t > 0 \quad \text{--- (3)}$$

and following condition at $t = \tau$

$$v(x,\tau,\tau) = 0 \quad \& \quad v_t(x,\tau,\tau) = F(x,t) \quad \text{--- (4)}$$

Since $t > \tau$

By d' Alembert's solution

$$v(x,t,\tau) = \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\, ds$$

$\boxed{A}$

Consider,

$$\boxed{u(x,t) = \int_0^t v(x,t,\tau)\, d\tau}$$

We claim that it is the solution of eq^n 1

$$\Rightarrow u(x,t) = \frac{1}{2c} \int_0^t \left(\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\, ds \right) d\tau$$

By Lebnitz rule,

$$u_t = v(x,t,t) + \int_0^t v_t(x,t,\tau)\, d\tau$$

We know, for $\tau = t$,

$v(x,t,t) = 0 = v(x,\tau,\tau)$

$$\therefore u_t = \int_0^t v_t(x,t,\tau)\, d\tau$$

Again, By Lebnitz rule,

$$u_{tt} = v_t(x,t,t) + \int_0^t v_{tt}(x,t,\tau)\, d\tau$$

$$= F(x,t) + \int_0^t v_{tt}(x,t,\tau)\, d\tau$$

~~Hence, in eq^n 1~~

And,

$$u_{xx} = \int_0^t v_{xx} \cancel{v(x,t,\tau)}\, d\tau$$

~~$u_{tt} - c^2$~~

Hence, in eqⁿ ①

$$u_{tt} - c^2 u_{xx} = F(x,t) + \int_0^t v_{tt}(x,t,\tau)\,d\tau - c^2 \int_0^t v_{xx}(x,t,\tau)\,d\tau$$

$$= F(x,t) + 0 \quad [\text{From eq}^n \text{ ③}]$$

$$= F(x,t)$$

Hence eq. $\boxed{A}$ is the sol. of eq. ①

Case 2 :- Initial condition non-homogeneous

$$u(x,0) = f(x) \; ; \; u_t(x,0) = g(x)$$

$$-\infty < x < \infty$$

Solution

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds + \int_0^t v(x,t,\tau)\,d\tau}$$

e.g. $u_{tt} = u_{xx} + x^2 - t \; ; \; -\infty < x < \infty \; ; \; t > 0$

And, $u(x,0) = u_t(x,0) = 0 \; ; \; -\infty < x < \infty$

Ans Here $c = 1$, $F(x,t) = x^2 - t$

Hence, sol. ⇒

$$u(x,t) = \frac{1}{2}\int_0^t \left(\int_{x-(t-\tau)}^{x+(t-\tau)} (s^2 - \tau)\,ds \right) d\tau$$

$$= \frac{10}{12}\left(6x^2t^2 + t^4 - 2t^3\right)$$

★ Heat conduction problem (Finite rod case):-

(i) The position of the rod coincides with x-axis and rod is homogenous.

(ii) It is so thin such that heat is uniformly distributed over its cross-section at time 't'.

(iii) The rod is insulated on surface to prevent the loss of heat at the boundary.

I.B.V.P. :

$$u_t - K u_{xx} = 0 \quad ; \quad 0 < x < l \, , \, t > 0 \quad \text{---(1)}$$

$$u(x,0) = f(x) \quad ; \quad 0 \le x \le l$$

$$u(0,t) = u(l,t) = 0 \quad , \quad t \ge 0$$

$K \to$ heat conduction coefficient

By method of separation of variable →

$$u(x,t) = X(x)\,T(t)$$

Substituting it in eqn (1) gives us-

$$XT' - KX''T = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{KT} = \text{constant} = \lambda \text{ (say)}$$

$$\left.\begin{aligned} X'' - \lambda X &= 0 \\ T' - \lambda k T &= 0 \end{aligned}\right\}$$

$\because u(0,t) = 0 \Rightarrow X(0)\, T(t) \Rightarrow X(0) = 0$

And $u(l,t) = 0 \Rightarrow X(l) = 0$

So, $\left.\begin{aligned} X'' - \lambda X &= 0 \\ X(0) = X(l) &= 0 \end{aligned}\right\}$ Sturm-Liouville problem (SLP)

Here $\lambda > 0$ & $\lambda = 0$ are not eigen values because corresponding to these λ, we obtained only trivial solutions for SLP.

Let $\lambda < 0 \Rightarrow \lambda = -\alpha^2 \;\; \forall\, \alpha \in R$

$X'' + \alpha^2 X = 0 \Rightarrow X(x) = A\cos\alpha x + B\sin$

$X(0) = 0 = A$

$X(l) = 0 \Rightarrow B\sin\alpha l = 0$

B can't be zero (to avoid trivial soln)

$\therefore \sin\alpha l = 0 = \sin n\pi$

$\Rightarrow \alpha_n = \frac{n\pi}{l}$; $n = 1, 2, 3 \ldots$

$\lambda_n = \frac{-n^2\pi^2}{l^2} \rightarrow$ eigen-value

$\boxed{X_n(x) = B_n \sin\left(\frac{n\pi}{l}\right)x} \longrightarrow$ Eigen function

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For $T_n' + \frac{n^2\pi^2}{l^2} k T_n = 0$

$$\boxed{T_n = C_n \exp.\left(-\frac{n^2\pi^2}{l^2} kt\right)}$$

Now,

$$u_n(x,t) = X_n(x)\, T_n(t)$$

$$= a_n \exp\left(-\frac{n^2\pi^2}{l^2} kt\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$[a_n = B_n C_n]$$

$$\sum_{n=1}^{\infty} u_n(x,t) = u(x,t)$$

$$\therefore \boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2}{l^2} kt\right) \sin\left(\frac{n\pi x}{l}\right)}$$

Now, we only need to identify a_n.

We know, $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$ → Fourier Sine Series

where, $\boxed{a_n = \frac{2}{l}\int_0^l f(x) \times \sin\left(\frac{n\pi x}{l}\right) dx}$

★ Uniqueness of the solution of Non-homogenous wave eq^n:-

Theorem:- The solution of the following problem (if it exists) is unique:

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PDE
$$u_{tt} - c^2 u_{xx} = F(x,t); \quad 0 < x < l, \; t > 0$$

I.C $\quad u(x,0) = f(x) \; ; \; u_t(x,0) = g(x) \quad 0 \le x \le l$

B.C $\quad u(0,t) = u(l,t) = 0 \quad ; \; t \ge 0$

Proof :- Let u_1 & u_2 be two solutions of the above problem.

Consider $v = u_1 - u_2$, then v will satisfy the following problem:

$$v_{tt} - c^2 v_{xx} = 0, \quad 0 < x < l, \; t > 0$$

$$\begin{cases} v(x,0) = 0, \; v_t(x,0) = 0, & 0 \le x \le l \\ v(0,t) = v(l,t) = 0 \end{cases}$$

Consider,

Energy function (for wave eqn) $\rightarrow$ $E(t) = \frac{1}{2} \int_0^l (c^2 v_x^2 + v_t^2)\, dx$

$$\frac{dE}{dt} = \int_0^l (c^2 v_x v_{xt} + v_t v_{tt})\, dx$$

$$= \int_0^l v_t v_{tt}\, dx + c^2 \int_0^l v_x v_{xt}\, dx$$

$$= \int_0^l v_t v_{tt}\, dx + c^2 \left[v_x v_t \Big|_0^l - \int_0^l v_{xx} v_t\, dt \right]$$

(By parts method)

Since, $v(0,t) = v(l,t) = 0$

$\Rightarrow v_x(0,t) = v_x(l,t) = 0$

$\therefore \left[v_x v_t \right]_0^l = 0$

$$\therefore \frac{dE}{dt} = \int_0^l v_t \left(v_{tt} - c^2 v_{xx} \right) dx = 0$$

$\Rightarrow$ $E \rightarrow$ constant function

Again,

$$v(x,0) = 0 \Rightarrow v_x(x,0) = 0$$
$$\& \quad v_t(x,0) = 0$$

$$\therefore E(0) = \frac{1}{2} \int_0^l \left(c^2 (0) + (0) \right) dx = 0$$

$$\Rightarrow \quad E \equiv 0 \quad \forall \quad 0 < x < l \quad t \geq 0$$

That implies, $v_x \equiv 0$ & $v_t \equiv 0$ $\forall$ $0 < x < l$, $t > 0$

Since derivative of v w.r.t. both x & t is zero $\Rightarrow$ $v \rightarrow$ constant

$v(x) \rightarrow$ constant function.

Now, $v(x,0) = 0 \Rightarrow v \equiv 0 \quad \forall \quad 0 < x < l, t \geq 0$

$$\Rightarrow \quad u_1 - u_2 = 0 \quad \Rightarrow \quad \boxed{u_1 = u_2}$$

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* Uniqueness of the solution of non-homogenous heat conduction eqⁿ :-

Theorem :- The solution of the following problem if it exists, is unique:

$$\begin{cases} u_t - K u_{xx} = F(x,t) \; ; \; 0 < x < l \; ; \; t > 0 \\ u(x,0) = f(x) \, , \; 0 < x < l \\ u(0,t) = u(l,t) = 0 \quad t > 0 \end{cases}$$

Proof :- Let u_1 & u_2 be two solutions of the above problem, then $v = u_1 - u_2$, v will be solution of following problem:

$$\begin{cases} v_t - K v_{xx} = 0 \, , \; 0 < x < l \, , \; t > 0 \\ v(x,0) = 0 \, , \quad 0 \leq x \leq l \\ v(0,t) = v(l,t) = 0 \end{cases}$$

Consider,

$$E(t) = \frac{1}{2K} \int_0^l v^2 \, dx \quad ; \; E(t) \geq 0 \;\; \forall \, t \geq 0$$

$$\frac{dE}{dt} = \frac{1}{K} \int_0^l v v_t \, dx$$

$$= \int_0^l v v_{xx} \, dx$$

$$= v v_x \Big|_0^l - \int v_x^2 \, dx = - \int_0^l v_x^2 \, dx \leq 0$$

$[\because v_x(0,t) = v_x(l,t) = 0]$

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$\Rightarrow$ E is a decreasing function

$E(0) = 0$

Since E is decreasing functions.

So, $E(t) \leq 0 \quad \forall\ t \geq 0$

$\Rightarrow E(t) \equiv 0 \quad \forall\ t \geq 0$

$\Rightarrow v = 0$

$\Rightarrow u_1 = u_2$