Ans $z(t) = re^{it}$; $z'(t) = rie^{it}$

$$I = \int_0^{2\pi} [re^{it}]^n [ire^{it}]\, dt$$

$$= ir^{n+1} \int_0^{2\pi} e^{(n+1)it}\, dt$$

$$= \frac{r^{n+1}}{n+1} \left[e^{(n+1)it} \right]_0^{2\pi} = 0 \quad (\text{if } n \neq -1)$$

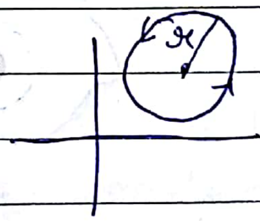
Q $I = \int_C (z - z_0)^n dz$; $C : |z - z_0|\ r$ anti-clockwise

Ans Take $z - z_0 = \omega$
$dz = d\omega$

$\int_C \omega^n d\omega$; $C : |\omega| = r$

It is same problem only which we have calculated.

$|z - z_0| = r$
$\therefore\ z - z_0 = re^{it}$
$z = z_0 + re^{it}$



$dz = rie^{it}\, dt$

$$I = \int_C (z - z_0)^n dz$$

$$= \int_0^{2\pi} [re^{it}]^n\, rie^{it}\, dt$$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\, dt$$

$$= \frac{r^{n+1}}{n+1} \left[e^{i(n+1)t} \right]_0^{2\pi}$$

$$= 0 \quad [\text{when } n \neq -1]$$

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When $n = -1$

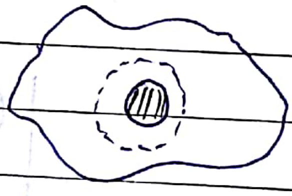
$$I = \int_0^{2\pi} \frac{1}{r e^{it}} \, r i e^{it} \, dt$$

$$= \int_0^{2\pi} i \, dt = 2\pi i$$

Note:- Neither circle radius (r) nor its centre (z_0) location changes the value of integration $\int_C (z - z_0)^n \, dz$; $C : |z - z_0| = r$

A simply Connected domain:-

In a Complex plane a domain (open & connected) that every simple closed curve in Domain encloses only point of domain.



D_1

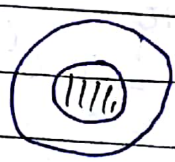
//// → region represents not included in domain

( ) → Contains points other than domain D_1 too.

~~not con~~

Thus, not simply connected Domain (D_1)

e.g. (i) $1 \leq |z| \leq 2$ → not simply connected



(ii) $|z| < 5$ → Simply Connected

Multiply Connected Domain :-

Domain that is not simply connected is called Multiply Connected.

e.g. $1 < |z| < 7$

★ Green's Theorem :-

Let 'C' be a positively oriented, piecewise smooth, simple closed curve in a plane and let 'D' be a region bounded by 'C'.

If L & M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives, then -

$$\boxed{\oint_C L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy}$$

★ Cauchy's Theorem :-

Let $f(z)$ is analytic in a simply connected domain D & $f'(z)$ is continuous in D, then for every simple closed curve ~~E~~ 'C' in D -

$$\oint_C f(z)\,dz = 0$$

Q 1. $I = \oint \sin z\,dz$; $C : |z-1| = 5$

$= 0$ [As $\sin z$ is analytic in its domain]

Teacher's Signature

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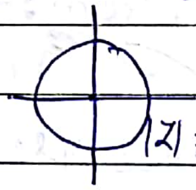
Q. $I = \oint \frac{\cos z}{z+3}\, dz$; $C : |z| = 1$
oriented in anti-clockwise direction

$f(z) = \frac{\cos z}{z+3}$ → not defined for $z = -3$

But $z = -3$ isn't in the domain of Curve



$|z| = 1$

$\therefore\ I = 0$ (Cauchy's Theorem)

Q. $I = \oint_C \frac{z^2}{z^2+4}\, dz$; $C : |z - i| = 1.5$

Ans $f(z) = \frac{z^2}{z^2+4} = \frac{z^2}{(z+2i)(z-2i)}$

↳ not defined for $z = 2i$ & $z = -2i$

In domain of 'C'
$z = 2i$ is confide there.

$\therefore\ I \neq 0$

Note:- In Cauchy's Theorem, simply connected domain is 'Necessary' condition & Analyticity is 'Sufficient' condition for $\int_C f(z)\, dz = 0$ to hold.

e.g. $f(z) = \frac{1}{z}$ $C : \frac{1}{2} < |z| < 1$

↳ In domain of 'C' function is analytic.

Still, $I \neq 0$

Therefore, simply connected domain is required.

Proof of Cauchy's Theorem:-

Let $f(z) = u + iv$

$dz = dx + i\,dy$

Consider $f(z) \rightarrow$ Analytic (as per Cauchy's Theorem initial given conditions)

$\therefore u, v \rightarrow$ ~~Analytic~~ partial derivatives exist & continuous

$$I = \oint_C f(z)\,dz = \oint_C (u+iv)(dx + i\,dy)$$

$$= \oint (u\,dx - v\,dy) + i \int (u\,dy + v\,dx)$$

Apply Green's Theorem as u & v have continuous partial derivatives.

$$\int u\,dx + (-v)\,dy = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

using C-R eq^n [$U_x = V_y$ & $V_x = -U_y$

$$= \iint_D \left(-\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dx\,dy = 0$$

Similarly,

$$\int u\,dy + v\,dx = \iint \left(\frac{\partial u}{\partial x} - \frac{\partial V}{\partial y}\right) dx\,dy = 0$$

(Using C-R eq^n)

$\therefore I = 0 + 0 = 0$ - Hence proved

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$\sqrt{z} = \sqrt{re^{i\theta}}$

$$= \sqrt{r}\, e^{i\theta/2}$$

$$= \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad — (1)$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Replace θ by $\theta/2$

$$\therefore \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

So, in (1)

$$\sqrt{z} = \sqrt{r}\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{r}\sqrt{\frac{1-\cos\theta}{2}}$$

$$= \sqrt{\frac{r + r\cos\theta}{2}} + i\sqrt{\frac{r - r\cos\theta}{2}}$$

$$= \sqrt{\frac{|z|+x}{2}} + i(\text{Sig } y)\sqrt{\frac{|z|-x}{2}}$$

It is important to add signature (sign) of y in the expression as there is no 'y' envolvement in expression

e.g. $\sqrt{z} = \sqrt{1-i}$; $\sqrt{z} = \sqrt{1+i}$

Both will have same solution if sign of 'y' is ignored.

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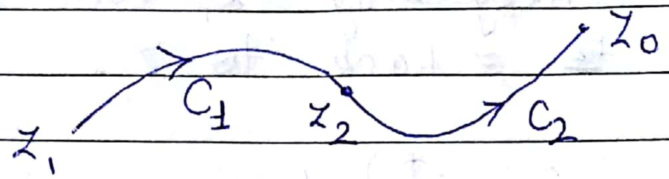
Properties :-

1. Partition of Path :-



$C_1 : z_1 \to z_2$
$C_2 : z_2 \to z_0$

$C = C_1 + C_2$ $\quad$ $C : z_1 \to z_0$

then - $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

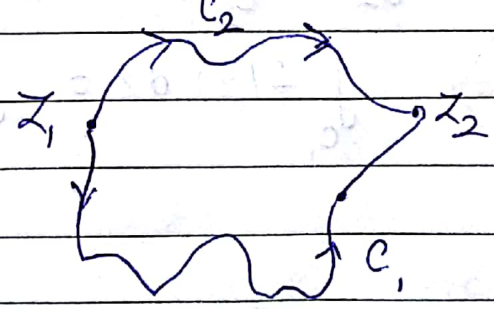
2. Sense reversal :-

$$\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz$$

3. Independent of path :-

If $f(z)$ is analytic in a simply ~~convert~~ connected domain D then the integral of $f(z)$ is independent of path in D.

Proof:



Let z_1 & z_2 be any point in D.
Consider two path C_1 & C_2 in D from z_1 to z_2 ~~&~~ without further common points.

Denote path C_1 & C_2, where C_2^* is the path C_2

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$C_2 \rightarrow z_1$ to z_2

$C_2^* \rightarrow z_2$ to z_1

with orientation reversed.

Integrating z_1 over C_1 to z_2 and over C_2^* ~~to~~ back to z_1.

This ($C = C_1 + C_2^*$) becomes simple closed path in simply connected domain D then -

By Cauchy's Theorem -

$$\int_C f(z)\,dz = 0 \qquad \text{— (1)}$$

but

$$\int_C f(z)\,dz = \int_{C_1 + C_2^*} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz \qquad \text{— (2)}$$

From (1) & (2)

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

In case of Analytic function, this integral depends on end points but not on path.

Q Find the solution of following:

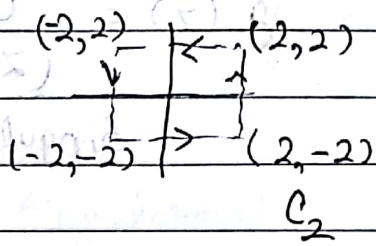
(i) $\int_{C_1} e^z \, dz$; $C_1 : |z| = 2$

Ans $e^z \rightarrow$ Analytic

$\therefore$ By Cauchy's Theorem

$I = 0$

(ii) $\int_{C_2} \frac{z^2}{z+8} \, dz$



Ans $f(z) = \frac{z^2}{z+8}$

$\downarrow$ Analytic in C_2 [Singularity $z = -8$]

So, By Cauchy's Theorem

$I = 0$

(iii) $\int \tan z \, dz$ $C_3 : |z| = 1$

Ans $\int \frac{\sin z}{\cos z} \, dz$

$f(z) = \frac{\sin z}{\cos z}$ $\rightarrow$ not differentiable at $z = \frac{\pi}{2}$ which is not in C_3

[Singularity $(2n+1)\frac{\pi}{2}$]

So, $f(z) \rightarrow$ Analytic in C_3

So, $I = 0$ (By Cauchy's Theorem)

(iv) $f_1(z) = e^{\bar{z}}$

$\hookrightarrow$ not differentiable anywhere (but continuous everywhere)

(v) $f_2(z) = \bar{z}$

↳ ~~not~~ continuous but not differentiable anywhere

(iv), (v) → No point is singular point.

(vi) $g_1(z) = \dfrac{e^z}{(z-1)(z-2)}$

Singular point → $z = 1, 2$

(vii) ~~$g_2(z) = \tan z$~~ $g_2(z) = \dfrac{1}{z}$

Singular point $z = 0$

(viii) $\log z$

↳ differentiable everywhere except -ve real axis.

& ↳ all points on -ve real axis are singular points.

★ <u>Singularity</u> :-

A single valued function is said to have a singularity at a point if the function is not analytic at the point while every neighbourhood of that point contains at least one point at which function is analytic.

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e.g. (i) $f(z) = \frac{1}{z}$ has singularity at $z=0$ as every nbd of $z=0$ contains ~~infinitely~~ some regular (analytic) point.

(ii) $f(z) = \bar{z}$ This function is nowhere differentiable. Hence no nbd contains any regular points. Thus it has no singularity.

There are two types of singularities:
(i) Isolated Singularity
(ii) Non-isolated Singularity

(i) Isolated Singularity: In every nbd of a particular singularity if there is ~~not~~ no other singularity, then such singularity is isolated.

e.g. (a) $$f(z) = \begin{cases} z+1 & \text{for } z \neq 0 \\ 3 & \text{for } z = 0 \end{cases}$$

Singularity at $z=0$

(b) $f(z) = \frac{1}{z}$ for $z \neq 0$

Singularity at $z=0$

(c) $f(z) = \sin\left(\frac{1}{z}\right)$, $z \neq 0$

Singularity at $z=0$

(d) $f(z) = \frac{5}{z(z-1)^2(z-3)}$

↳ Singularity at $z = 0, 1, 3$ (Seperately)

We can get at least one nbd of $z=0,1,3$ where ~~function~~ there is no singularity.

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In other words, if function is analytic in the deleted nbd of the singularity → Isolated singularity.

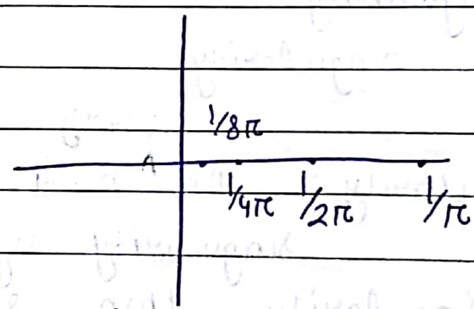
(ii) Non-isolated singularity: If singularity is not isolated, it is non-isolated.

e.g. (i) $g(z) = \operatorname{Log} z$

(ii) $g(z) = \dfrac{1}{\sin\left(\frac{1}{z}\right)}$

Why? Why non-isolated?

$\sin\left(\frac{1}{z}\right) = 0 \Rightarrow z = \frac{1}{n\pi}, n \in \mathbb{Z}$



$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{4\pi}$ ---- → Isolated

$z = 0$ → Non-isolated $\left[\,|z| = r\right.$ $r = \frac{1}{\pi\, 1000000}$

Another singularity in nbd of $z=0$

$\therefore z = 0$ → non-isolated

★ **Pole:-** If z_0 is an isolated singularity and we can find a positive integer 'n' such that - $(n>0)$

$$\lim_{(z \to z_0)} (z - z_0)^n f(z) = A \neq 0 \quad [\text{finite Value}]$$

then $z = z_0$ is called pole & 'n' is its order.

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e.g. (i) $\frac{z^2}{(z-1)^2 (z-3)}$

$\Rightarrow$ For $z_0 = 1$,

$$\lim_{z \to 1} (z-1)^n \frac{z^2}{(z-1)^2 (z-3)}$$

For $n = 2$

$$\lim_{z \to 1} (z-1)^2 \frac{z^2}{(z-1)^2 (z-3)} = -\frac{1}{2} \neq 0$$

Thus $z = 1$ is a pole of order 2.

$\Rightarrow$ For $z_0 = 3$ & $n = 1$

$$\lim_{z \to 3} (z-3) \frac{z^2}{(z-1)^2 (z-3)} = \frac{9}{4} \neq 0$$

Thus, $z = 3$ is a pole of order 1

(ii) $f(z) = \tan z$; Is $z = \frac{\pi}{2}$ a pole?
And, if yes what's order?

Ans $\lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \frac{\sin z}{\cos z}$ [For $n = 1$]

Applying L'Hospital rule -

$$= \lim_{z \to \frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right) \cos z + 1 \times \sin z}{-\sin z} = -1 \neq 0$$

$z = \frac{\pi}{2} \rightarrow$ a pole of order of 1

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★ Zero of function :-

$z = z_0$ is said to be zero of order m $(m>0)$ if

$$\lim_{z \to z_0} (z-z_0)^{-m} f(z) \neq 0 \quad \text{(finite non-zero value)}$$

e.g. (i) $f(z) = \tan z$

Zeroes → $n\pi$, $n \in \mathbb{Z}$

Poles → $(2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$

(ii) $f(z) = \cot z$

$= \frac{\cos z}{\sin z}$

Poles → $n\pi$, $n \in \mathbb{Z}$

Zeroes → $(2n+1)\frac{\pi}{2}$ $n \in \mathbb{Z}$

Removable type discontinuity :-

Function is originally discontinuous at particular point, but by redefining the function we can make it continuous

e.g. (i) $f(x) = \frac{\sin x}{x}$ ↳ discontinuous at $x=0$ [not defined]

But redefining

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$ ✓ continuous

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(ii) $f(x) = \begin{cases} x+1 & x \neq 0 \\ 3 & x = 0 \end{cases}$ → not continuous

By redefining → $f(x) = \begin{cases} x+1, & x \neq 0 \\ 1, & x = 0 \end{cases}$

✓ Continuous

e.g. (a) $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ [$\sin \frac{1}{x}$ → bounded $x \to 0$]

(b) $\lim_{z \to 0} z \sin \frac{1}{z} \neq 0$ [$\sin \frac{1}{z}$ → not bounded but $z \to 0$]

* Removable singularity :-

An isolated singularity or singular point z_0 is called a removable of $f(z)$ if $\lim_{z \to z_0} f(z)$ exist.

e.g. (i) $f(z) = \frac{\sin z}{z}$ has singularity at $z = 0$

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$ [Here singularity $z_0 = 0$ is removable]

(ii) $f(z) = \frac{z+2}{(z^2-4)\, z^2}$

$$= \frac{z+2}{(z-2)(z+2)(z^2)}$$

$z = -2$ is removable singularity

as –

$$\lim_{z \to -2} \frac{z+2}{(z-2)(z+2)z^2} = \lim_{z \to -2} \frac{1}{(z-2)z^2}$$

$$= \frac{-1}{16}$$

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(c) $f(z) = z \sin\frac{1}{z}$ does not have removable singularity at $z=0$

[Reason:- $\sin\frac{1}{z} \to$ unbounded

hence $\lim_{z\to 0} f(z)$ does not exist]

(d) $f(z) = e^{\frac{1}{z}}$

$$= 1 + \underset{\downarrow \infty}{\frac{1}{z}} + \underset{\downarrow \infty}{\frac{1}{2!\,z^2}} + \underset{\downarrow \infty}{\cdots}$$

For Each term in series expansion the lim approaches to ~~...~~ ∞.

↳ not removable singularity at $z=0$

↳ $z=0$ is not even pole

↳ It is "essential singularity"

(e) $f(z) = \sin\frac{1}{z}$

↳ not removable singularity at $z=0$

↳ $z=0$ not even pole

↳ It is categorized under "Essential Singularity"

Essential singularities:- An isolated singularity which is not pole or removable singularity.

Singularity at infinity:- The type of singularity of $f(z)$ at $z=\infty$ is the same as that $f(\frac{1}{z})$ at $z=0$.

e.g. $f(z) = z$ At $z = \infty$, $f(z) \to \infty$.

[Note that, $f(z)$ has to be analytic in z, for singularity to exist at ∞]

$f_1(z) = z^{1/3}$ $\qquad$ $f_2(z) = z^{1/2}$

$\downarrow$ Triple Valued $\qquad$ $\downarrow$ Double Valued

(Their roots repeat as they move in circular fashion)

$f(z) = \log z$

$= \ln|z| + i(\text{Arg } z + 2k\pi)$

$\downarrow$

roots move in linear fashion $\rightarrow$ Hence they don't get repeated

$\downarrow$

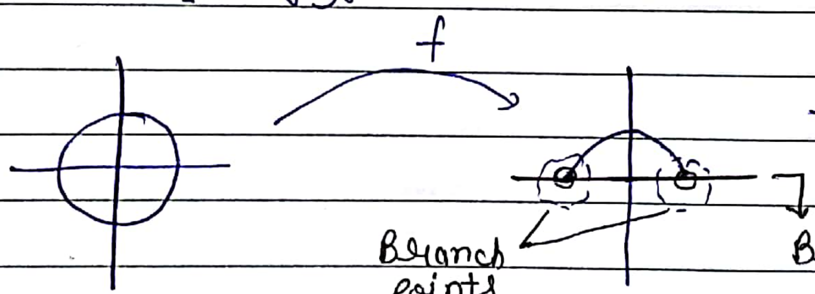
Multi-Valued fn

For $f(z) = z^{1/2}$

$= \sqrt{r}\, e^{i\theta/2}$

Half Circle (Branch)

Full circle Completed $\rightarrow$



$\rightarrow$ Half Circle completed

Branch points

Branch line

z-plane (Input) $\qquad$ ω-plane (Output)

z-plane (Input)	ω-plane (Output)
$\theta = 0°$	$\frac{\theta}{2}$ $\quad 0°$
$= \frac{\pi}{2}$	$\frac{\pi}{4}$
$= \pi$	$\frac{\pi}{2}$
$= 2\pi$	π

Branch points $\downarrow$ intersection with branch line.

Branch :-

A branch of a multi-valued function f is any single-Valued function 'F' that is analytic in some domain at each point z of which the value $F(z)$ is one of value of $f(z)$.

e.g. (i) $f(z) = \cancel{Lo} \text{Log } z$

$\text{Log } z = \ln|z| + i\theta \quad (r > 0, -\pi < \theta < \pi)$

is called principal branch.

(ii) $f(z) = z^{1/2}$

$$= \begin{cases} \sqrt{r}\, e^{i\theta/2} & -\pi < \theta \le \pi \\ \sqrt{r}\, e^{i\theta/2} & \pi < \theta \le 2\pi \end{cases} \Bigg] \text{Branches}$$