Assignment 1 - MAT-MEK4270

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Stationary Waves

1.2.3 Exact Solution

We want to show that

$$u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$$

satisfies the 2D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Time derivatives

$$\frac{\partial u}{\partial t} = -i\omega e^{i(k_x x + k_y y - \omega t)}$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 e^{i(k_x x + k_y y - \omega t)} = -\omega^2 u$$

Spatial derivatives

$$\frac{\partial u}{\partial x} = ik_x e^{i(k_x x + k_y y - \omega t)}, \quad \frac{\partial^2 u}{\partial x^2} = -k_x^2 u$$

$$\frac{\partial u}{\partial y} = ik_y e^{i(k_x x + k_y y - \omega t)}, \quad \frac{\partial^2 u}{\partial y^2} = -k_y^2 u$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -(k_x^2 + k_y^2)u$$

We have

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u, \quad \text{and} \quad c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = -c^2(k_x^2 + k_y^2)u$$

We require

$$\omega^2 = c^2(k_x^2 + k_y^2)$$

So

$$u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$$

with

$$\omega = c\sqrt{k_x^2 + k_y^2}.$$

1.2.4 Dispersion coefficient

Assume $m_x = m_y$ so that $k_x = k_y = k$. The discrete wave equation scheme is

$$\frac{u_{ij}^{n+1} - 2u_{ij}^{n} + u_{ij}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{h^2} + \frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{h^2} \right)$$

with the discrete version of the wave equation

$$u_{ij}^{n} = e^{i(kh(i+j) - \tilde{\omega} n \Delta t)}.$$

Temporal part

We compute the second-order time difference:

$$\begin{aligned}
u_{ij}^{n+1} &= e^{i(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} = e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} e^{-i\tilde{\omega}\Delta t} = u_{ij}^{n}\, e^{-i\tilde{\omega}\Delta t}, \\
u_{ij}^{n-1} &= e^{i(kh(i+j) - \tilde{\omega}(n-1)\Delta t)} = e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} e^{i\tilde{\omega}\Delta t} = u_{ij}^{n}\, e^{i\tilde{\omega}\Delta t}.
\end{aligned}$$

The discrete second derivative in time is

$$\begin{aligned}
\frac{u_{ij}^{n+1} - 2u_{ij}^{n} + u_{ij}^{n-1}}{\Delta t^2} &= \frac{u_{ij}^{n} e^{-i\tilde{\omega}\Delta t} - 2u_{ij}^{n} + u_{ij}^{n} e^{i\tilde{\omega}\Delta t}}{\Delta t^2} \\
&= \frac{u_{ij}^{n} \left(e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} \right)}{\Delta t^2} \\
&= \frac{u_{ij}^{n} \left(2\cos(\tilde{\omega}\Delta t) - 2 \right)}{\Delta t^2} \\
&= \frac{2\left(\cos(\tilde{\omega}\Delta t) - 1 \right)}{\Delta t^2}\, u_{ij}^{n}.
\end{aligned}$$

Spatial part

Compute the discrete second differences in space. For the x-direction:

$$\begin{aligned}
u_{i+1,j}^{n} &= e^{i(kh((i+1)+j) - \tilde{\omega} n \Delta t)} = u_{ij}^{n} e^{ikh}, \\
u_{i-1,j}^{n} &= e^{i(kh((i-1)+j) - \tilde{\omega} n \Delta t)} = u_{ij}^{n} e^{-ikh}.
\end{aligned}$$

Thus, the discrete second derivative in x is

$$\begin{aligned}
\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{h^2} &= u_{ij}^{n}\, \frac{\left(e^{ikh} - 2 + e^{-ikh} \right)}{h^2} \\
&= u_{ij}^{n}\, \frac{\left(2\cos(kh) - 2 \right)}{h^2} \\
&= u_{ij}^{n}\, \frac{2(\cos(kh) - 1)}{h^2}
\end{aligned}$$

Similarly, for the y-direction:

$$\begin{aligned}
u_{i,j+1}^{n} &= u_{ij}^{n} e^{ikh}, \\
u_{i,j-1}^{n} &= u_{ij}^{n} e^{-ikh}, \\
\frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{h^2} &= u_{ij}^{n}\, \frac{2(\cos(kh) - 1)}{h^2}
\end{aligned}$$

The total RHS of the scheme is then

$$c^2 \left(\frac{u^n_{i+1,j} - 2u^n_{ij} + u^n_{i-1,j}}{h^2} + \frac{u^n_{i,j+1} - 2u^n_{ij} + u^n_{i,j-1}}{h^2} \right) = u^n_{ij} \; \frac{4c^2(\cos(kh) - 1)}{h^2}$$

We now get

$$\begin{aligned} \frac{2(\cos(\tilde{\omega}\Delta t) - 1)}{\Delta t^2} &= \frac{4c^2(\cos(kh) - 1)}{h^2} \\ \cos(\tilde{\omega}\Delta t) - 1 &= 2\frac{c^2 \Delta t^2}{h^2}(\cos(kh) - 1) \\ \cos(\tilde{\omega}\Delta t) &= 1 + 2C^2(\cos(kh) - 1) \end{aligned}$$

where $C = \frac{c\Delta t}{h}$ is the CFL number.

With $C = \frac{1}{\sqrt{2}}$, we get

$$\begin{aligned} 2C^2 &= 2 \cdot \frac{1}{2} = 1, \\ \cos(\tilde{\omega}\Delta t) &= 1 + (\cos(kh) - 1) \\ &= \cos(kh) \\ \tilde{\omega}\Delta t &= kh \implies \tilde{\omega} = \frac{kh}{\Delta t} \end{aligned}$$

We already know that

$$\omega = c\sqrt{k_x^2 + k_y^2}.$$

and since we assume $k_x = k_y = k$, we get

$$\omega = c\sqrt{k^2 + k^2} = c\sqrt{2}\ k.$$

Using the definition of the CFL number

$$C = \frac{c\Delta t}{h} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \Delta t = \frac{h}{\sqrt{2}\,c},$$

we can substitute this into $\tilde{\omega}$:

$$\tilde{\omega} = \frac{kh}{\Delta t} = \frac{kh}{h/(\sqrt{2}\,c)} = k\sqrt{2}\,c.$$

Comparing this with the continuous frequency computed above:

$$\omega = c\sqrt{2}\,k,$$

we see that

$$\tilde{\omega} = \omega.$$