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Title: Invariants of Diophantine Approximation

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I. Diophantine exponents ($e>0$)

$$\mathcal{D}_e = \left\{ \alpha \in \mathbb{R} : \exists\, c>0 \text{ s.t. } \left|\alpha - \tfrac{p}{q}\right| > \tfrac{c}{q^e} \ \forall\, \tfrac{p}{q} \in \mathbb{Q} \right\}.$$

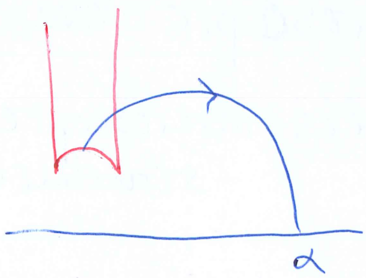
($\neq \emptyset$ iff $e \geq 2$)

$$\boxed{\alpha \in \mathcal{D}_e \iff a_{k+1} = O(q_k^{e-2}) \iff q_{k+1} = O(q_k^{e-1}).}$$

$\alpha \in \mathcal{D}_2 \iff \alpha$ badly approx. $\quad \mathrm{Leb}(\mathcal{D}_2) = 0$, $\ \mathrm{Hdim}(\mathcal{D}_2) = 1$

$\alpha \in \mathcal{D}_{2+\varepsilon}\ (\forall \varepsilon>0) \iff \alpha$ Roth $\quad \mathrm{Leb}(\mathcal{D}_{2+\varepsilon})$ full, $\ \mathrm{Hdim}(\mathcal{D}_{2+\varepsilon}) = \frac{2}{2+\varepsilon}$

$\alpha \in \left(\bigcup \mathcal{D}_e \cup \mathbb{Q}\right)^c \iff \alpha$ Liouville



$$\boxed{\alpha \in \mathcal{D}_e \iff \limsup_{t\to\infty} \frac{d_{hyp}(x_t, x_0)}{t} \leq \frac{e-2}{e}}$$

(variation on) DANI CORRESPONDENCE

$\theta \in \mathbb{R}^d \rightsquigarrow \Lambda_\theta = \underbrace{\begin{pmatrix} 1 & & -\theta_1 \\ & \ddots & \vdots \\ & & -\theta_d \\ & & 1 \end{pmatrix}}_{h_\theta} \mathbb{Z}^{d+1}$ "sheared integer lattice"

$h_\theta \in$ unstable horospherical subgroup (of g_t)

$d(x_0, x_t) \asymp -\log \operatorname{sys}(g_t \Lambda_\theta)$ where $g_t = \begin{pmatrix} e^t & & & \\ & \ddots & & \\ & & e^t & \\ & & & e^{-dt} \end{pmatrix}$

sys: λ_1 = first successive minimum

$$\boxed{\begin{array}{lcl} \text{ordinary exponents} & \longleftrightarrow & \limsup\limits_{t\to\infty} \dfrac{d(x_0,x_t)}{t} \\[2ex] \text{uniform exponents} & \longleftrightarrow & \liminf\limits_{t\to\infty} \dfrac{d(x_0,x_t)}{t} \end{array}}$$

linear speed

Liouville # $\Longleftrightarrow$ ordinary is largest possible

Roth # $\Longleftrightarrow$ ordinary $= 0$.

(Rem: VWA $\Longleftrightarrow$ ordinary > 0
WA $\Longleftrightarrow$ $\neg$ BA)

WHILE DEFINITION OF THESE EXPONENTS DEPEND ON A CHOICE OF NORM ON $\mathbb{R}^d$, THEIR VALUES DO NOT.

Advantage: Meaningful macroscopic invariants

Disadvantage: Too crude, ignores microscopic structure.

crude $\rightarrow$ logarithmic / sublinear speeds

Other exponents

no higher dim'l analog

Khintchine : $\lim_{n\to\infty} \frac{1}{n} \log a_1 \cdots a_n$

Levy : $\lim_{n\to\infty} \frac{1}{n} \log q_n$ ← best approx. denominators.

Lyapunov : $\lim_{n\to\infty} -\frac{1}{n} \log \|q_n \overset{\theta}{\alpha}\|$

$$\boxed{\text{When } d=1, \text{ all these can be expressed in terms of } (a_k) \text{ or equivalently.. in terms of } (q_k).}$$

As such, convergent denominators may be thought of as $\boxed{\text{more fundamental}}$ of these "invariants".

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singular on average ⟺ uniform is largest possible

(?) Dirichlet improvable ⟺ uniform > 0

(Das-Fishman-Simmons-Urbanski)

$$\boxed{\begin{array}{c}\liminf_{t\to\infty} \frac{d(x_0, x_t)}{t} \\ \text{(unif. exponent)}\end{array}} = \text{proportion of time spent near } \infty$$

(= minimum amount of loss of mass)

II. hD Convergents vs Best Approximants

Definition $p/q \in \mathbb{Q}^d$ is a hD convergent of $\theta \in \mathbb{R}^d$

if • p_i = nearest integer to $q\theta_i$ $\forall i = 1, \dots, d$.

$\boxed{\begin{matrix}\forall n \\ \exists i\end{matrix}}$ • $|q\theta_i - p_i| < |n\theta_i - m_i|$ $\forall \frac{m}{n} \in \mathbb{Q}^d$, $1 \le n < q$.

Remarks: $d=1 \iff$ best approx. (of the 2^{nd} kind)

$\iff$ convergent (Lagrange)

Definition does NOT involve any choice of norm.

Good candidate in our search for more fundamental invariants

Definition $p/q \in \mathbb{Q}^d$ is a best approx. to $\theta \in \mathbb{R}^d$ (rel. to $N : \mathbb{R}^d \to \mathbb{R}_{\ge 0}$) if

• $d_N(q\vec{\theta}, \mathbb{Z}^d) = d_N(q\vec{\theta}, \vec{p})$

• $N(q\vec{\theta} - \vec{p}) < N(n\vec{\theta} - \vec{m})$ $\forall \frac{m}{n} \in \mathbb{Q}^d$, $1 \le n < q$.

We can order the sequence of best approx. by increasing denominators (q_k). They are the seq. of nearest returns to the origin of $\mathbb{R}^d / \mathbb{Z}^d$ under $\vec{x} \longmapsto \vec{x} + \vec{\theta}$. (w.r.t. N).

Rem: It is slightly less natural to order the sequence of convergents by increasing height.

$\boxed{\text{(G. Santello, 2018) Assume } N \text{ is } \underline{\text{axis-symmetric}}^{*}. \text{ THEN every BA wrt } N \text{ is a convergent.}}$

* N $\underline{\text{axis-symmetric}}$ $:=$ $\boxed{N \circ \Phi = N}$ where

$\Phi : \mathbb{R}^d \to \mathbb{R}^d$ "$\underline{\text{folding}}$ $\underline{\text{map}}$"

$\vec{x} \rightsquigarrow (|x_i|)_{i=1}^{d}$

$\underline{\text{Geometric}}$ $\underline{\text{interpretation}}$ $\Lambda \subset \mathbb{R}^n$ $(n = d+1)$

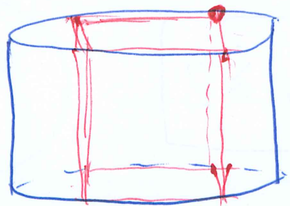
$r \in \mathbb{R}^n_{\geq 0}$ $\boxed{B(r) = \prod_{i=1}^{n} [-r_i, r_i]}$ Λ-$\underline{\text{box}}$

$(r,h) \in \mathbb{R}^2_{\geq 0}$ $\boxed{C(r,h) = \overline{B}_N(0,r) \times [-h,h]}$ Λ-$\underline{\text{cylinder}}$

$\left.\begin{array}{l} \cdot\ \Lambda \cap \operatorname{Int} B = \{0\} \\ \cdot\ \Lambda \cap \partial B \neq \emptyset \end{array}\right\} \rightsquigarrow$ POSET, order by inclusion

$\boxed{\text{convergents/ best approx} \longleftrightarrow \text{minimal elements}}$

N axis-symmetric $\Rightarrow$



$B(u) \subset C(u)$

III Levy constant

P = partition of $\{1, \dots, n\}$

$$A = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} : \prod_1^n a_i = 1,\ a_i > 0 \right\} \quad \text{positive diagonal}$$

$$A_P = \{ a \in A : a_i = a_j \text{ for } i \sim j \ (P) \}.$$

$$n = n_1 + \dots + n_k \qquad (\dim_{\mathbb{R}} A_P = k-1).$$

For each $i = 1, \dots, k$ fix a norm N_i on $\mathbb{R}^{n_i}$

(If $n_i = 1$, we always use standard $|\cdot|$.)

$\boxed{\text{Mostly interested in the extreme cases } k=2 \text{ or } n}$

Folding map
$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^k_{\geq 0}.$$
$$\vec{x} \rightsquigarrow (N_i(\vec{x}))_{i=1}^k$$

(C-Chevallier, 2022) For a.e. $\theta \in \mathbb{R}^d$, BA-denominators satisfy
$$\lim_{k \to \infty} \frac{1}{k} \log q_k^\theta = \text{const}.$$

(Remark: Proof uses transversal defined by maximal Λ-cylinders)

Analog for hD convergents NOT OBVIOUS.

$\boxed{\dim A_P \geq 2}$

$$\begin{tikzcd} \text{continuous flow} \arrow[r, bend left, "\text{Poincaré section}"] & \text{discrete dyn. sys} \arrow[l, bend left, dashed, "\text{suspension}"] \end{tikzcd}$$

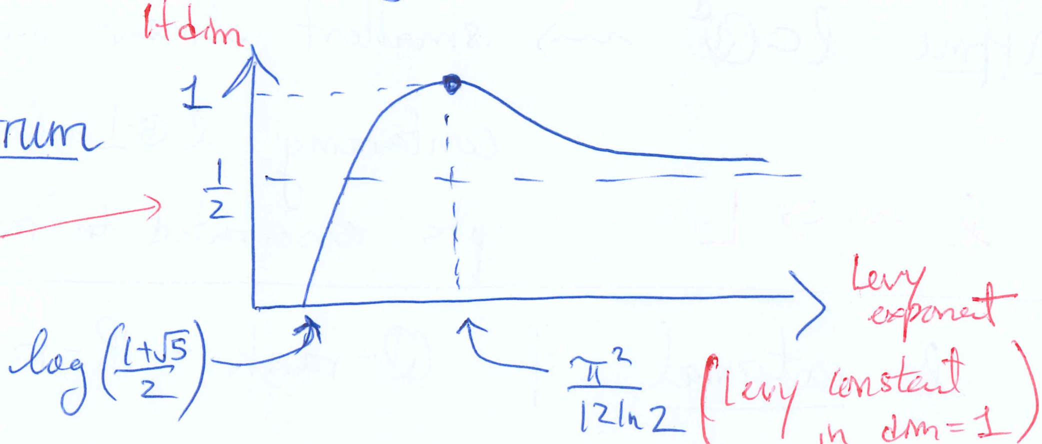
$\boxed{\begin{array}{l} \underline{\text{CONJECTURE}}: \exists\ 0 < C_d < \infty \text{ s.t. for a.e. } \theta \in \mathbb{R}^d \\ \text{Convergent denominators satisfy } \boxed{\lim_{k\to\infty} \frac{(\log q_k^\theta)^d}{k} = C_d} \end{array}}$

DFSU completed the determination of the dimension spectrum (for unif. exponents) in the cases $(d \geq 2, c=1)$ & $(d=1, c\geq 2)$.

<u>Remark</u>: (Fan - Liao - Wang - Wu, 2009)

<u>Levy spectrum</u>

singular on average



There are many interesting questions remaining even when the <u>dynamics</u> is 1D $(k=2)$

BUT we are mostly interested in exploring what can be done when the <u>dynamics</u> is higher "rank". (<u>STAIRCASES</u>)

IV Dual convergents $\theta \in \mathbb{R}^d$ $(d \geq 2)$

$\ell \subset \mathbb{Q}^d$ rational affine subspace of codim 1

$$\boxed{\text{DEF: } \ell \text{ is a } \underline{\text{dual convergent}} \text{ to } \theta \text{ if } \dots (\text{tbd}) \dots}$$

$\frac{\vec{p}}{q} \in \mathbb{Q}^d \rightsquigarrow v = \langle p_1, \dots, p_d, q \rangle \in \mathbb{Z}^{d+1}_{pr}$ (or $\mathbb{Z}v \subset \mathbb{R}v$)

affine points

$\theta \in \mathbb{R}^d \rightsquigarrow \mathbb{R}\langle \theta_1, \dots, \theta_d, 1 \rangle$ ← 1D linear subspaces

$\theta \oplus 1$

affine $\ell \subset \mathbb{Q}^d \rightsquigarrow$ smallest linear subspace of $\mathbb{R}^{d+1}$ containing $\ell \oplus 1$ (or all integer pts associated to rational pts in ℓ).

$\ell \rightsquigarrow L$

ℓ <u>rational</u> if $\mathbb{Q}$-rank $=$ $\mathbb{R}$-rank.

$L \subset \mathbb{Z}^{d+1}$ (primitive) sublattice, (i.e. $L_{\mathbb{R}} \cap \mathbb{Z}^{d+1} = L$) of codimension one.

$$\boxed{L = \ker v^* \quad \text{for some} \quad v^* \in \Lambda^* \coloneqq L(\Lambda, \mathbb{Z}).}$$

v^* is uniquely determined up to sign by requiring $v^* = \pm 1$ on generators of $\mathbb{Z}^{d+1}/L$

Alternatively, may think of L as "oriented."

Option to think of L

EQUIV DEF: ℓ is a *dual convergent* of ϑ if $B(h_\vartheta^{-1} v^*)$ is a minimal Λ_ϑ^*-box.

$\Lambda \subset \mathbb{R}^n$ • $u \in \Lambda \setminus \{0\}$ *pivot* if $B(u)$ is a Λ-box.

• u is a *stable pivot* if $B(u)$ is a minimal Λ-box.

p/q convergent of $\vartheta \iff h_\vartheta v$ stable pivot of Λ_ϑ

ℓ dual convergent of $\vartheta \iff h_\vartheta^{-1} v^*$ — " — Λ_ϑ^*

GOAL: Exploit rich structure of L to gain insightful information about...

Q: What condition in terms of L makes $h_\vartheta^{-1} v^*$ a stable pivot of Λ_ϑ^*?

(P.-H. Lee, 2018) provides a geometric characterization of these *dual pivots*.

V $\boxed{\text{Q: Does every dual convergent necessarily contain a convergent?}}$

This is a question about Λ_θ.

$\boxed{\text{(Yuming WEI) } \exists^* \text{ lattices } \Lambda \subset \mathbb{R}^3 \text{ containing stable dual pivots that do not contain any pivots.}}$

*In fact, a 3-parameter family of codim 3.

Note: Not clear if construction can be adapted to Λ_θ.

Remark: $\boxed{\text{Margulis Conj.: } A\text{-orbit of } \Lambda \in G/\Gamma \text{ is compact if it is relatively cpt}}$

$\boxed{\text{Littlewood Conj: } A_+\text{-orbit of } \Lambda_\theta \text{ rel. cpt.}}$

Isolated vertices of BI-PARTITE GRAPH

$v \in \Pi(\Lambda) =$ ~~...~~ stable pivots of Λ

$\Pi^*(\Lambda) \simeq \Pi(\Lambda^*)$ stable dual pivots

$\begin{array}{cc} \Pi^*(\Lambda) & \Pi(\Lambda^*) \\ \cup & \cup \\ L & v^* \end{array}$

edge relation $\boxed{v \in L \iff v^*(v) = 0}$

fundamental invariant

Geometric group theory

(Y. WEI) : $\exists\ \Lambda \subset \mathbb{R}^3$ s.t.

$$\boxed{\chi(\Lambda) := \sup_{v^* \in \Pi(\Lambda^*)}\ \inf_{v \in \Pi(\Lambda)} |v^*(v)| > 0}$$

integer-valued A-invariant function

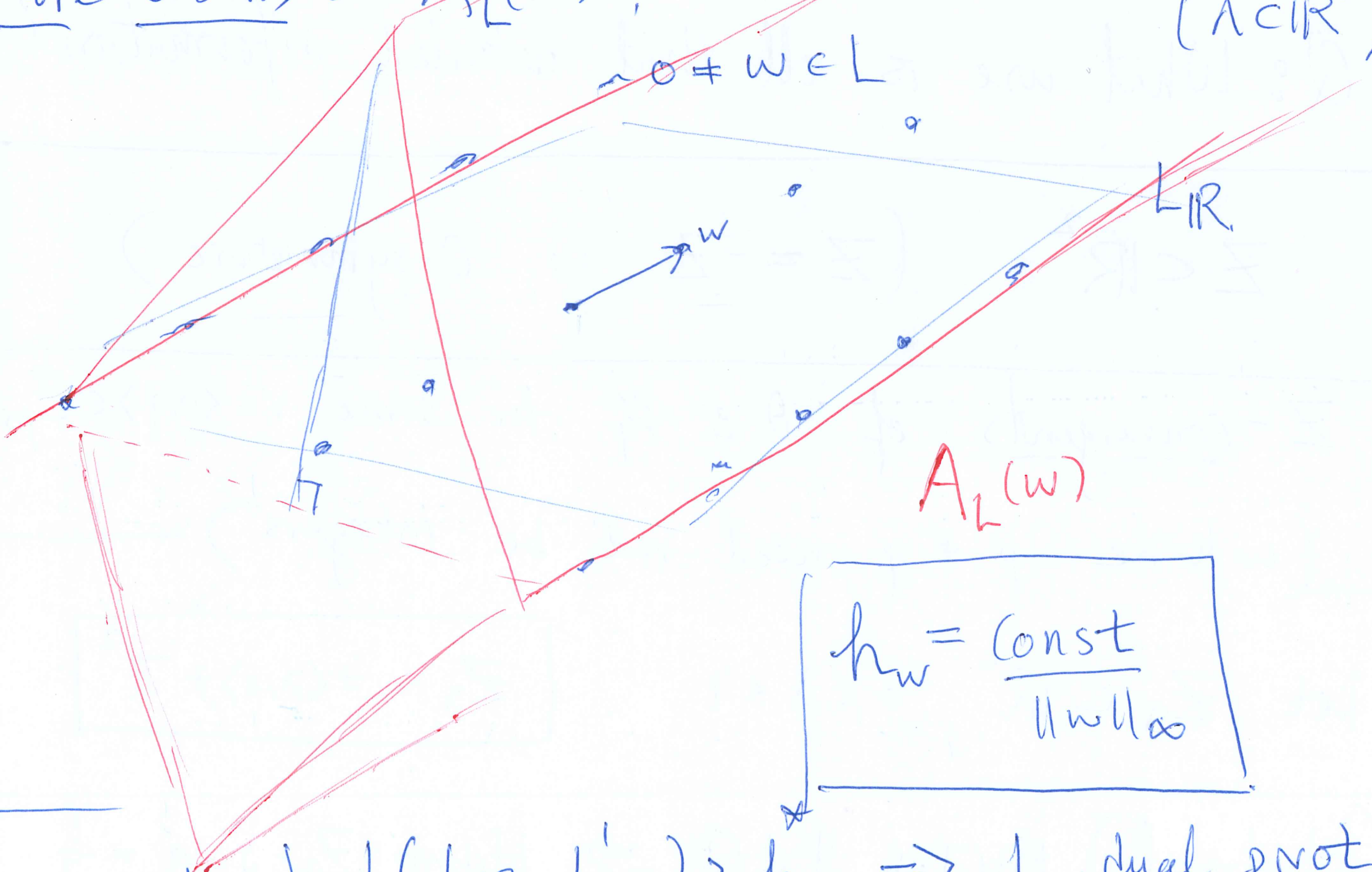
$$\boxed{\chi : A \backslash G / \Gamma \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}}$$

Q: What is the generic value for χ ?

Q: Is it the same for $\{\Lambda_\theta : \theta \in T^d\}$?

<u>More</u> <u>details</u> : $A_L(w)$, $L \subset \Lambda$ 2D sublattice ($\Lambda \subset \mathbb{R}^3$)

$0 \neq w \in L$

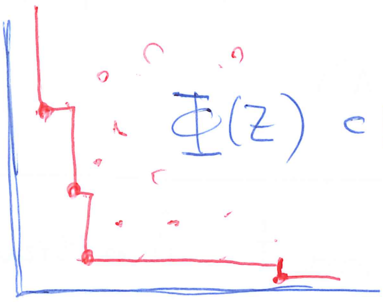


$$\boxed{h_w = \frac{\text{const}}{\|w\|_\infty}}$$

$\forall$ $h_L = \max_{0 \neq w \in L} h_w$ | $\forall$ $d(L_{\mathbb{R}}, L'_{\mathbb{R}}) \geq h_L \Rightarrow L$ dual pivot

VI Staircases $\Phi : \mathbb{R}^n \to \mathbb{R}^k_{\geq 0}$ ← standard poset

$Z \subset \mathbb{R}^n$ closed discrete subset



$\Phi(Z) \subset \mathbb{R}^k_{\geq 0}$

∂ (smallest left (order) ideal containing $\Phi(Z)$) $\approx \mathbb{R}^{k-1}$ (staircase)

Applications: Z = • lattice

• $V(X,\omega)$ = saddle connections on a translation surface

• shifted lattice (Ostrowski expansions)

quasi-crystals

(order of difficulty)

$$\boxed{\text{Q: What use is all that detailed information?}}$$

$Z \subset \mathbb{R}^2$ ($Z = -Z$, i.e. 0-symmetric)

$$\boxed{\begin{array}{l}Z\text{-convergents of } \theta:\ p/q \text{ for some } v = \langle p, q\rangle \in Z_{pr} \\ (\text{where } p \ \& \ q \text{ need not be integers.})\end{array}}$$

Let $Z_\lambda \subset \mathbb{R}^2$, $0 < \lambda < 1$ $\quad \boxed{Z_\lambda = \pm\langle \lambda, 0\rangle + \mathbb{Z}^2}$

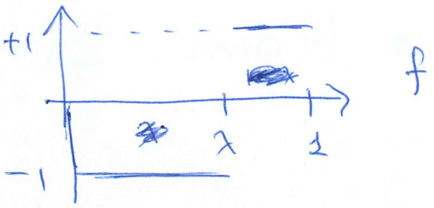
(Nontrivial) FACT: $\boxed{\lambda \notin \mathbb{Q} \implies \text{Hdim } Z_\lambda\text{-Sing} = \frac{1}{2}}$

(C-H-M, 2011)

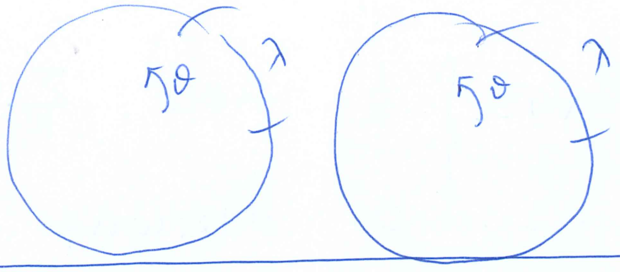
VII Nonergodic Parameters (for $\mathbb{Z}/2$-skew products)

$$S_{\lambda,\theta}(x,\varepsilon) = (x+\theta,\ f(x+\theta)\varepsilon)$$

$$S_{\lambda,\theta} : \mathbb{R}/\mathbb{Z} \times \{\pm 1\} \circlearrowleft$$



Veech '69



$$\boxed{NE_\lambda = \{\theta : S_{\lambda,\theta} \text{ is not (uniquely) ergodic}\}.}$$

(C-Eskin, 2007) $\theta \in NE_\lambda \Rightarrow Z_\lambda$-convergents are eventually even

(w_j) $\exists k_0$ s.t. $\forall j > k_0$ $w_j \in Z_\lambda^{even} = \pm\langle \lambda, 0\rangle + (2\mathbb{Z})^2$

Moreover, $NE_\lambda \subset Z_\lambda^{even}$-singular

$\subset Z_\lambda$-singular (Masur '92)

In fact, $\boxed{\theta \in NE_\lambda \iff \sum |w_j \times w_{j+1}| < \infty}$

$\boxed{NE_\lambda}$ $K_1(\theta) = \left\{ \lambda = \langle m; b_1, b_2, \dots \rangle_\theta : |b_j| \le a_{j+1},\ b_j \text{ eventually even},\ \sum_j \frac{|b_j|}{a_{j+1}} < \infty \right\}$

$\cap$

Ostrowski expansion

$\boxed{Z_\lambda^{even}\ \text{sing}}$ $K_0(\theta) = \left\{ \text{— '' — '' —} \quad \lim_{j\to\infty} \frac{b_j}{a_{j+1}} = 0 \right\}$

(C-Hubert-Masur, 2011)

- Hdim Z_λ-Sing $= \begin{cases} \frac{1}{2} & \lambda \notin \mathbb{Q} \\ 0 & \lambda \in \mathbb{Q} \end{cases}$

(Same for sing on average)

- Hdim $NE_\lambda = \begin{cases} \frac{1}{2} & \lambda \notin \mathbb{Q}, \ \overbrace{\sum_k \frac{\log\log q_{k+1}}{q_k} < \infty}^{\text{Perez Marco}} \\ 0 & \text{otherwise}. \end{cases}$

Note: $\left.\begin{array}{l} \lambda = \frac{p}{q} \in \mathbb{Q} \\ w_j \times w_{j+1} \neq 0 \end{array}\right\} \Rightarrow |w_j \times w_{j+1}| \geq \frac{1}{q}$.

$\therefore \boxed{\lambda \in \mathbb{Q} \implies NE_\lambda \text{ countable}}$

Boshernitzan: $\exists\ \lambda \notin \mathbb{Q}$ s.t. Hdim $NE_\lambda = 0$.

KEY NEW IDEA: $\boxed{\text{Non-ergodic directions} \Rightarrow \text{Liouville.}}$

(C-Yan HUANG) Hdim $NE_{(\lambda,\mu)} = 0$ if

$\sum_k \frac{\log\log q_{k+1}}{q_k} \neq \infty$ where (q_k) = seq. of BA-denominators to the pair (λ,μ).

Note: divergence for BA-denom $\Rightarrow$ divergence for convergents denominators.

Q: Expect dichotomy but what is the sharp condition?

VIII

Q1 : $\mathrm{Hdim}\ Z_\lambda^{(3)}\text{-Sing} = 4/3$?

$$\| $$

$$\pm\langle \lambda, 0, 0\rangle + \mathbb{Z}^3$$

$\theta \in Z_\lambda^{(3)}\text{-Sing}$ if $\forall\ \delta > 0\ \exists\ T_0$

$\theta' = \langle \theta_1, \theta_2, 1\rangle$

s.t. $\forall\ T > T_0$

$$\begin{cases} \angle \theta' v < \dfrac{\delta}{|v|\,T^{1/2}} \\ 1 \le |v| < T \end{cases}$$

has a solution $v \in Z_\lambda^{(3)}$

Q2 : $\exists\ \lambda$ s.t. $\mathrm{Hdim}\ NE_\lambda^{(3)} \in (0, \frac{4}{3})$?

$\theta \in NE_\lambda^{(3)}$ if $\theta \in Z_\lambda^{(3)}\text{-Sing}$ <u>and</u>

$$\sum_{1}^{\infty} |w_j \times w_{j+1}| < \infty$$

where (w_j) = sequence of $Z_\lambda^{(3)}$-convergents

IX Domains of Approximation

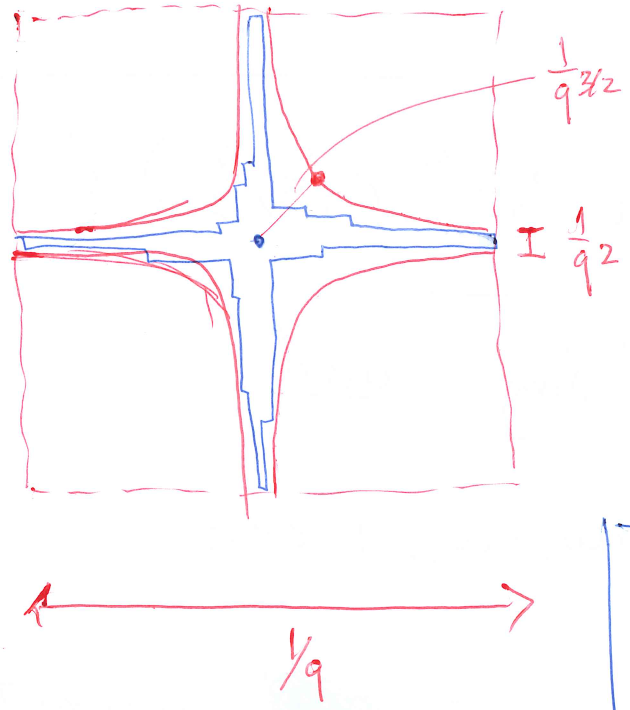
$\frac{p}{q} \in \mathbb{Q}^d \qquad \Delta_N(p/q) := \{ \theta \in \mathbb{R}^d : p/q \text{ is } N\text{-BA of } \theta \}.$

(C 2011) $\quad B_N(\frac{p}{q}, \frac{r}{2}) \subset \Delta_N(p/q) \subset B_N(\frac{p}{q}, 2r) \qquad r = \frac{\lambda_1^N(\mathbb{Z}\frac{p}{q} + \mathbb{Z}^d)}{q}$

(Bita Nosratieh, 2010) $\Delta_N(p/q)$ not always convex.

THEIR STRUCTURE IS TOO COMPLEX TO BE INTERESTING

$\Delta(p/q) = \{ \theta \in \mathbb{R}^d : p/q \text{ is a (d-dim'l) convergent of } \theta \}.$



Only have

$\Delta(p/q) \subset B(\frac{p}{q}, r)$

geometric norm

(Oliver Knitter, 2019)
Complexity of $\Delta(p/q) = O(\log q)$

Q: Can we find a more restrictive notion than convergents that ~~gives~~ allows for an inclusion of the form $B(\frac{p}{q}, r) \subset \Delta(p/q)$?