Diophantine Equations and Homogeneous Dynamics

September 30, 2020

Hyperbolic Actions of Higher Rank Groups

(joint with Kurt Vinhage)
arXiv:1901.06559

— or —

How to homogenize?
RIGIDITY IN DYNAMICS

Primary Goal: is NOT to analyze homogeneous actions

Rather: When is a dynamical object FORCED to be homogeneous

Basic Examples

1. Ratner measure rigidity

measure rigidity thms

Einsiedler – Lindenstrauss

techniques to be discussed are similar to meas rigidity tools

2. Anosov Smale Conjecture

on Anosov diffeos φ

$$TM = E^s \oplus E^u$$

$\Rightarrow$

$c, \lambda > 0$ $\quad v \in E^s \iff$

$n \geq 0 \quad \|d\varphi^n(v)\| < c \cdot e^{-\lambda n}\|v\|$

$n \leq 0 \quad v \in E^u$

Conj.: Anosov diffeos are C^0-conjugate to an automorphism of an N/Γ

Nilpotent

$\boxed{\text{Note:}}$ no such conjecture for flows

3 Katok Entropy Conjecture

SM = unit tangent bundle

M closed Riem. mf., $K < 0$

h_{top}(geo flow) = $h_{Liouville}$(geo flow)

$\overset{?}{\Rightarrow}$ M is isometric to symm. space.

geo flow C^∞-conj. to $M \backslash G / \Gamma$

M - compact group

Katok: $n=2$

4 BCG + BFL

Benoist Foulon Labourie

Besson Courtois Gallot

Main Thms close to 3

Create

* Symmetries of the geometry and flow
* create a geometric structure

well intertwined with the dynamics

Crucial: Gromov Rigid Structures

$\boxed{\text{transitive local symmetries on a dense open set}}$

dynamics $\rightsquigarrow$ dense open homogeneous set.

Symmetry in Dynamics?

$\varphi_t : M \circlearrowleft$,

possibly: diffeos commuting with φ_t.

φ has trivial centralizer

if $\text{Zentralizer}_{\text{Diff}^\infty(M)}(\varphi) = \{\varphi^n\}$

5 Smale's Conjecture on Centralizers:

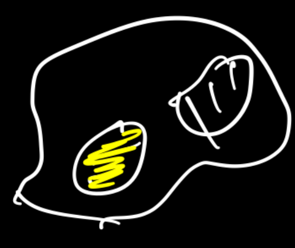
<u>THINK</u>: systems with symmetries

~2010

<u>Thm</u> (Bonatti-Crovisier-Wilkinson)

The generic C^1 diffeo has trivial centralizer.

(generic = residual set)

Stupid Example: 

But maybe can do better with hyperbolicity

Zimmer Program

G semisimple Lie group

Actions are superspecial

Not: Some Symmetries are built in — from G.

Need: $\boxed{\text{Higher rank}}$

e.g. $SL(n,\mathbb{R}) \supset \Delta^{n-1}$
$n \geq 3$

i.e. a split Cartan of dim ≥ 2.

$$\boxed{\begin{array}{l}\underline{\text{Hope:}}\quad G\text{-actions with} \\ \text{"uniformly hyperbolic" dynamics} \\ \text{are homogeneous.}\end{array}}$$

<u>Ex:</u> $SL(n,\mathbb{Z})$ on T^n, $n \geq 3$

Hurder, Katok-Lewis

locally rigid.

Relied on local rigidity of

$n-1 \geq 2$ $\mathbb{Z}^{n-1} \subset SL(n,\mathbb{Z})$ actions on T^n.

$\rightsquigarrow$ Back to Smale's Question.

<u>PROBLEM:</u>

Smale's problem was generic

Can we improve — under tighter

assumptions? Uniform hyperbolicity?

Anosov $\mathbb{R}^k \times \mathbb{Z}^\ell$ - actions

M closed mfl.

$\| \ \|$ Riemannian metric

Call $a \in \mathbb{R}^k \times \mathbb{Z}^\ell$ Anosov

if there exists a splitting and constants $C, \lambda > 0$ s.t.:

$$TM = E^s_a \oplus E^0_a \oplus E^u_a$$

s.t. orbit $E^0_a(x) = T(\mathbb{R}^k . x)$

$v \in E^s_a \iff \|da^n(v)\| < Ce^{-\lambda n}$ for $n \geq 0$

$v \in E^u_a \iff \ldots$

Note: $\mathbb{Z}^\ell$ — just Anosov

Examples $\mathbb{R}^k$ — products of flows *

Δ-action on G/Γ *

e.g. $G = SL(n,\mathbb{R}), n \geq 3$

Γ — uniform lattice.

$\mathbb{Z}^\ell$ — actions on tori

90's: Local rigidity of Algebraic Anosov Actions

(Katok-S)

i.e. C^1-perturbations are C^∞-conjugate.

(up to an automorphism of $\mathbb{R}^k$)

Partial Hyperbolicity:

(Damjanovic - Katok — on tori

" — " — on G/Γ some G

with a KAM tool

Wang — other G/Γ

Vinhage Wang — G/Γ

S – Lei Yang — nil manifolds

~~rep theory~~ ⇝ exp mixing + KAM.
Green–Tao ↑ Gorodnik–S

Katok–S Conjecture: Anosov $\mathbb{R}^k, \mathbb{Z}^\ell$ actions are C^∞-conjugate to algebraic actions, ~~KRRZ~~
unless they have a rank 1 factor.

Thm: True for $\mathbb{Z}^k$-actions on tori & nilmanifolds (Rodriguez-Hertz & Wang) *
Based on work by Fisher–Kalinin–S.

Global Rigidity

Such Global Rigidity is not known for G/Γ, G semisimple.

Work with Vinhage

Generalizes earlier works by Kalinin–S, Related: Kalinin–Sadovskaya, Damjanovic–Xu

~~totally~~ Cartan actions — special Anosov actions

Require that

totally * dense set of Anosov elts.

* maximal nontrivial intersection of stable spaces $\bigcap_{a_1,\dots,a_k} E^s_{a_i}$

$$\dim \bigcap E^s_{a_i} = 1$$

Δ on $SL(n,\mathbb{R})/\Gamma$

Thm. A: (S-Vinhage) $\mathbb{R}^k \times \mathbb{Z}^\ell$ totally Cartan with a dense orbit. Then Katok-S conj. is true.

factor here is a smooth factor

Thm. B: Structure: $\mathbb{Z}^\ell$ tot Cartan is C^∞-conjugate to a product of Anosov diffeos on tori and actions by automorphisms on nil manifolds

Thm. B': Structure: Similar for $\mathbb{R}^k$, and $\mathbb{R}^k \times \mathbb{Z}^\ell$

Cor.: $\exists!$ (up to C^∞-conjugacy) tot. Cartan action on $SL(n,\mathbb{R})/\Gamma$ (up to auto of $\mathbb{R}^k$)

First Global Rigidity for semisimple

$\boxed{\text{PROBLEMS}}$

① Get rid of $\boxed{\text{totally}}$ HARD

② Get rid of $\boxed{\text{Cartan}}$

! BIQUOTIENTS !

③ Classify Anosov G actions, G s.s., higher rk.

(on the way, Butler-Damjanovic-S-Vinhage-Xu)

④ .....

$\Delta \curvearrowright SL(n,\mathbb{R})/\Gamma$

$$\begin{pmatrix} e^{t} & & 0 \\ & e^{t} & \\ 0 & & \ddots \end{pmatrix} \qquad e_1 - e_2$$

IDEAS FROM THE PROOF(S)

MOST IMPORTANT

Co-existence of hyperbolic & isometric behavior

roughly

E^α : $\dim \bigcap_{a_1,\dots,a_\ell} E^s_{a_i} = 1$

coarse Lyapunov spaces

hyperbolicity gives Lyapunov directions and other dynamical structures.

Lyapunov directions scale according to a linear functional $\boxed{\alpha : \mathbb{R}^k \to \mathbb{R}}$

$a \in \ker \alpha$ is isometric

reminiscent of: Margulis' SR

Part I: Dichotomy

$\mathbb{R}^k \curvearrowright M$

either $\exists$ a smooth rank 1 factor

or functionals $\alpha : \mathbb{R}^k \to \mathbb{R}$

and Holder norms $\|\ \|_\alpha$ on $E^\alpha \ni v$

that rescale precisely

uses a $\boxed{\| da(v) \|_\alpha = e^{\alpha(a)} \|v\|_\alpha}$

Cocycle Rigidity Result from Kalinin-S (based ideawise on a Livsic thm).

$\boxed{\text{dense orbit for } \ker\alpha}$ $\rightsquigarrow$ functionals

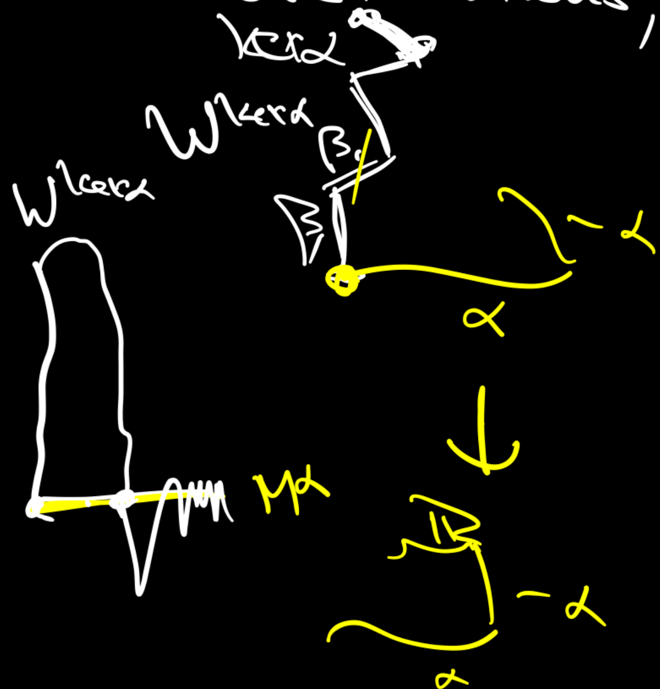
(where $\ker \alpha = \partial$ (half space that contracts E^α).

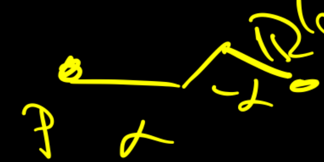
$W^{\ker}$-leaves

$\downarrow$

$\ker\alpha$ orbit not dense $\rightsquigarrow \ldots \rightsquigarrow$ integrability of all E^β

distributions, β not $c\cdot\alpha$, $c\in\mathbb{R}$



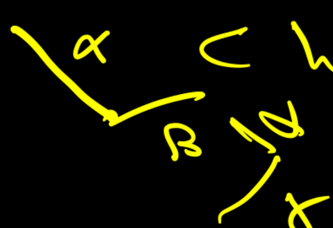


$M^\alpha(p)$

3-dim Anosov flow.

---

basic observation: $\alpha, -c\alpha$ integrate $\Leftarrow \mathbb{R}^k$



$\subset W^\alpha$ foliation tangent to E^α

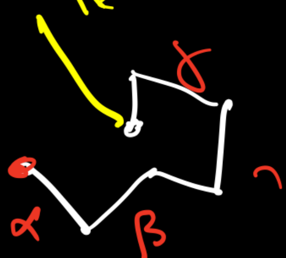
Part II: Free Products of Lie groups

The Hölder metrics $\| \; \|_\alpha$ furnish W^α with a canonical action of $\mathbb{R}$

pathgroup

Thus $\mathbb{R}^k * \cdots * \mathbb{R}$ acts on M

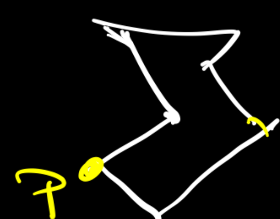
topological gp
inf dim, not loc comp.



Gleason–Palais: loc p-c top gp $\hookrightarrow$ findim top space then Lie

ite ensional

Constant Cycles:



stabilizer (p) =: cycle at p

This idea

If cycles are the same everywhere, get a normal comes from Kurt's wonderful thesis

subgp of $\mathbb{R}^k * \mathbb{R}$

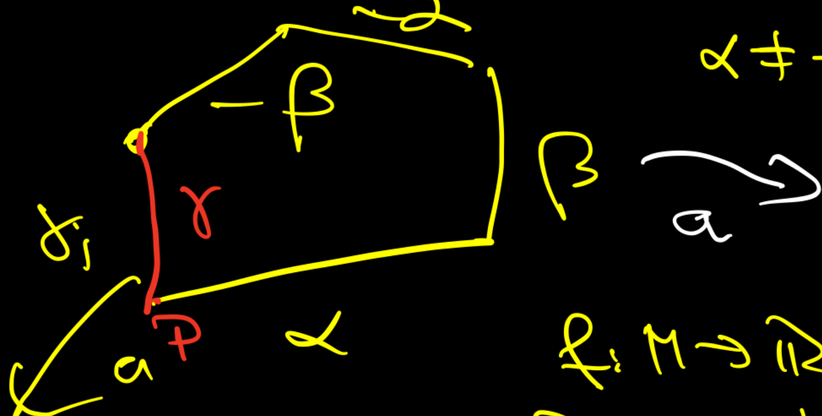
Then apply Gleason-Palais

$(\mathbb{R}^k * \mathbb{R})/\text{cycls}$

W: only Holder v.f.'s

Geometric Commutators

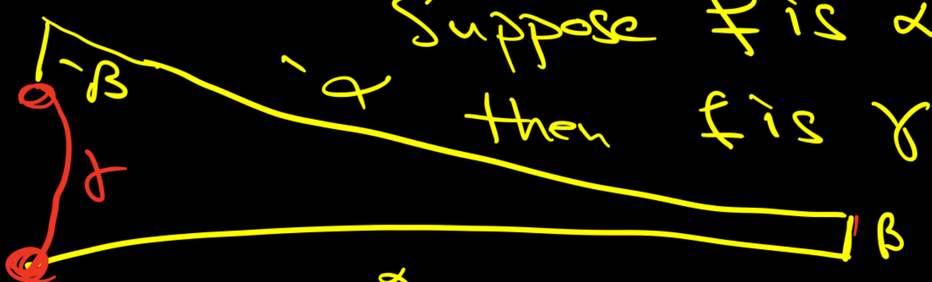
$\alpha \neq -c\beta$



$f: M \to \mathbb{R}$ cts.

Suppose f is α-inv

then f is γ_i inv.



ap inv)

Ideals, Factors, Structure of towers

Part III: Structure Theorem

$\mathbb{R}^k \hookrightarrow$ homogeneous $\times \prod$ Anosov flows

Proof is not trivial, but not as hard as the others.

---

cocycle result of Kalinin-S not clear

<u>general idea</u>

how to combine homogeneous structures on different foliations?

algebraic action

1 $\mathbb{Z}^k$-action $\curvearrowright X$

induce to an $\mathbb{R}^k$-action

$\mathbb{R}^k \curvearrowright \mathbb{R}^k \times_{\mathbb{Z}^k} X$

$\mathbb{R}^k$-alg action is

G/Γ $\quad$ G Lie gp

$\mathbb{R}^k \to G$

$\rightsquigarrow$ $\mathbb{R}^k \curvearrowright G/\Gamma$

Anosov actions

more tricky

$$\begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_n} \end{pmatrix} \curvearrowright M = SL(n,\mathbb{C})/\Gamma$$

$t_i \in \mathbb{R}$ $\sum t_i = 0$

M

α $\quad f$ α-inv.

$$I^\alpha = \{\gamma \mid f \ \gamma\text{-inv}\}$$

$\forall \beta$ $\quad [\alpha, \beta] = \{\gamma$ in gram commutator of α & $\beta\}$

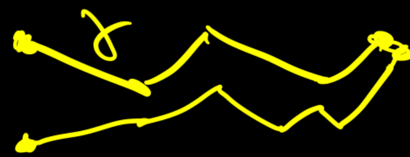
ideal property $\quad \boxed{[\alpha, \beta] \subset I^\alpha}$

max ideal I inside hyp funct.

$M/\sim$

$\sim$ equiv rel generated by

path $\subset I^\alpha \ni \gamma$

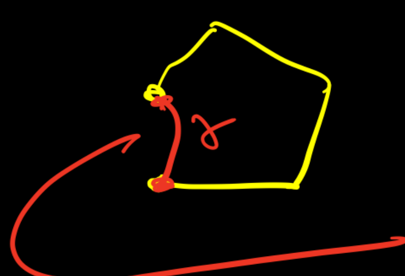


transfinite induction

$\mathbb{R}^k$ $\qquad M$

$\qquad\qquad \downarrow$

$\mathbb{R}^k$ $\qquad X^I$



constant length on X^I

geom comm with $\alpha, \sim c\alpha$ cycles

$\rightsquigarrow$ constancy of cycle relations.

(replaces K-theory from local rigidity arguments)

$$\begin{tikzcd} F \arrow[r] & M \arrow[d] \\ & X^I \end{tikzcd}$$

homog

homog

$$\begin{tikzcd} M \arrow[d, phantom, "\vdots"] \\ {} \arrow[d] \\ X^{I_1} \arrow[d] \\ X^F \end{tikzcd}$$