A Quick Review of Sets and Equivalence Relations

Liver Szabo

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Definition 0.1. ***Set Subtraction.*** *Given two sets A, B, the* difference set *$A - B$ is the set $\{x \in A | x \notin B\}$.*

Lots of high schools only teach unions and intersections.

Example 0.2. *Prove that for any set B, the set A satisfies $A = (A \cap B) \cup (A - B)$.*

Proof: $A \cap B$ is $\{x \in A | x \in B\}$, while $A - B$ is $\{x \in A | x \notin B\}$. So performing the union operation on these two sets will return A. ■

Definition 0.3. ***Cartesian Product.*** *The* Cartesian Product *$A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.*

Definition 0.4. ***Equivalence Relation.*** *A binary relation $\sim$ over a set A, defined to be a subset of $A \times A$, is an* equivalence relation *on A if $\forall a, b, c \in A$:*

1. *$a \sim a$ (Reflexivity)*
2. *$a \sim b \Rightarrow b \sim a$ (Symmetry)*
3. *$a \sim b \wedge b \sim c \Rightarrow a \sim c$ (Transitivity)*

Definition 0.5. ***Equivalence Class.*** *For a set A and an equivalence relation $\sim$ on A, the* equivalence class *of $a \in A$ is $\{x \in A | x \sim a\}$, also written as $[a]$.*

Theorem 0.6. *Given an equivalence relation $\sim$ on A, the distinct equivalence classes of $\sim$ are mutually disjoint subsets of A that, when combined with cup, form A in its entirety. Conversely, given a decomposition of A as a union of mutually disjoint, nonempty subsets, we can define an equivalence relation such that these subsets form different equivalence classes.*

Proof Outline: This is a dense (but important) result. First, let's consider what happens when two equivalence classes are not disjoint. Let's call these equivalence classes $[a], [b]$ for $a, b \in A$. There must be an element in both, denoted z. Since $z \in [a] \wedge z \in [b]$, $a \sim z$ and $z \sim b$. But this means that $a \sim b$ (transitivity). Therefore, $[a] \subset [b]$. Since this argument goes both ways (symmetry), $[a] = [b]$. All distinct equivalence classes must thus be disjoint.

For the converse, let $A = \bigcup A_i$. We can define the equivalence relation $a \sim b$ as "a and b are in the same A_i". This is obviously an equivalence relation that creates distinct subsets (using an argument along the lines of that above.) ■