A New Look at Organ Transplantation Models and Double Matching Queues

Onno J. Boxma*, Israel David†, David Perry‡and Wolfgang Stadje§

April 16, 2010

Abstract

In this paper we propose a prototype model for the problem of managing waiting lists for organ transplantations. Our model captures the double-queue nature of the problem: there is a queue of patients, but also a queue of organs. Both may suffer from "impatience": the health of a patient may deteriorate, and organs cannot be preserved longer than a certain amount of time. Using advanced tools from queueing theory, we derive explicit results for key performance criteria: the rate of unsatisfied demands and of organ outdatings, the steady-state distribution of the number of organs on the shelf, the waiting time of a patient, and the long-run fraction of time during which the shelf is empty of organs.

1 Introduction

The problems of managing a list of patients waiting for a transplantation have attracted the attention of operations researchers from the mid eighties [8, 20, 21, 9, 27, 13, 30, 16, 24]. Zenios et al. [28] contains an excellent introduction to the modeling of live-organ transplantations by means of a waiting list. The problem is a hybrid of queueing and inventory aspects, the inventory being a collection of organs "on the shelf". Considering patients who suffer from organ failure (kidneys, livers, hearts, etc.) and who register on a transplantation list as 'customers' waiting for service, one is tempted to take recourse to queueing models, but several non-standard features need to be taken into account. First, the 'servers' are organs usually donated from the dead; they arrive sequentially and randomly, and the transplantation itself only takes a negligible time. One

*EURANDOM and Department of Mathematics and Computer Science, Eindhoven University of Technology, HG 9.14, P.O. Box 513, 5600 MB Eindhoven, The Netherlands (boxma@win.tue.nl)

†Department of Industrial Engineering and Management, Ben-Gurion University, Beer Sheva, Israel (idavid@bgu.ac.il)

‡Department of Statistics, University of Haifa, Haifa 31909 Israel (dperry@haifa.ac.il)

§Department of Mathematics and Computer Science, University of Osnabrück, 49069 Osnabrück, Germany (wolfgang@mathematik.uni-osnabrueck.de)

thus faces a *double matching queue* whose two lines (one of organs and one of patients) can be both empty but, normally, not be both non-empty simultaneously. Second, both queues are affected by influences which may cause reductions in their lengths without matchings taking place. One major issue is the health deterioration of waiting patients. In the language of queueing theory, one speaks of *impatience*, a random positive time-lag, drawn from a general distribution, which is assigned to any arriving customer (patient). If a customer's 'patience' runs out before being served, he reneges. A second, dual, major issue is that live organs such as kidneys or livers cannot be preserved for more than a certain fixed period of time (often about two days), since freezing them is not possible. This establishes a link with stochastic inventory theory where one speaks of a perishable inventory system (PIS) whose output process is split into satisfied demands and outdatings. Obviously, the two types of untimely departures – impatience and outdating – are not symmetric.

The model that we propose and analyze in this paper captures in full the double-queue nature of the problem and takes health deterioration (customer impatience) as well as organ outdating (server removal) into account. In the basic model presented here we assume a FIFO regime of patients and show how tools available from queueing theory can be used to derive explicit results regarding the processes of unsatisfied demands, outdated organs and of waiting patients, and other important information about the efficacy of the issuing process.

One might criticize the above model in the following two ways.
(1) In most Western countries, the line of organs on the shelf is empty most of the time. However, this situation is bound to change, due to changes in societal convention, recent trends in religious attitudes regarding transplantation, but mostly due to possible new legislation (e.g., if permission of the family to use an organ from a deceased person is no longer required). In such a case the arrival rate of organs will increase dramatically – as is already seen in some countries – and the double-matched queue feature will be a very natural one.
(2) In most organ transplant situations, the issuing policy of organs is not FIFO. For example, the condition of the patient may play a role, and the level of matching. Generally speaking, that gives rise to very complicated stochastic models which are not analytically tractable. However, for a large subset of the population of patients, a FIFO policy still provides a good representation (see also Section 7). Our model may be viewed as a prototype model that provides a useful first approximation of reality, and that allows a methodologically novel approach that could provide the basis for much further research. In Section 7 we discuss various extensions, among other things returning to the FIFO issue.

Methodology
Our starting-point is the age process of the model, i.e., the evolution of the age of the oldest organ on the shelf and, as long as the shelf is empty, minus the waiting time of a (virtual) arriving patient whose patience would be long enough to eventually receive the first available organ. It should be noticed that age here can be negative, hence age should be interpreted in a broad sense. A

key idea is to flip its graphical representation over by 180 degrees (cf. Figure 1 in Section 2), yielding the virtual outdating process (VOP). The VOP provides the time until the next outdating of an organ if the demand process is stopped right now. The VOP leads to the key performance criteria of the model. A second key idea is to note the analogy of the VOP with the workload process in a certain $M/G/1$-type queue with impatience. We use level-crossing, martingales and other techniques to analyze the latter queue. This leads to explicit results for important performance criteria of the organ-transplant system.

Literature overview

In the literature one can find approaches to several aspects of organ allocation. Historically, researchers have sought to formulate and solve the problem of a single candidate first, that is, assuming only one patient with known relevant traits, who has no competitors on the donor stream, in the form of an optimal stopping problem [8, 13]. One current approach to the multi-candidate case is to use the solutions for single candidates as building blocks for an overall heuristic allocation policy. In [11] it is proposed to sort the line of candidates by FIFO, and then to implement the individually optimal policy sequentially for these candidates, such that rejected organs are offered down the line. An alternative approach is to conduct simulation studies, based on large amounts of real data (e.g., the US Registry combined with the US UNOS – United Network of Organ Sharing – official simulation system for kidneys, UKAM [25, 19, 26]).

Returning to analytical contributions, a significant body of research in this area is due to Su and Zenios [30, 22, 23, 24], who use a variety of approaches. Originally, the multi-attribute nature of the problem was treated in [28] by assuming many simultaneous streams of customers and applying fluid approximations to the generated system. The obtained set of linear differential equations leads, after further simplifying assumptions, to a Gittins index policy, the performance of which is then tested by simulation for various weight combinations of tri-criteria optimization, accounting for both "efficiency" and "equity." Similar techniques of fluid and Brownian approximation are used in further papers (e.g. [30]). In a different approach [23] they adopted the framework of analysis known as sequential stochastic assignment, which dates back to [12] and had been used in earlier investigations see [20, 21, 9, 10]. In [23] these models were extended significantly and an important feature was added by allowing for "patient choice" in kidney allocation and analyzing the tension due to this option between the social planner and the individual planner. For performance evaluation and numerical studies [23] again resorted to simulation. In [29] "impatience" is taken into account in the context of a random allocation policy without real interaction between kidney types and customer types.

Summarizing, the existing literature consists, apart from pure simulation studies, either of simplified models which provide important insights but which do not capture certain important facets of the complicated problem, or of more elaborate approaches that focus on competing allocation policies, which are evaluated exclusively by simulation. In this paper we present the problem in the framework of queueing theory, which allows an analytic treatment.

Organization of the paper
In Section 2 we describe the model, expound the key ideas mentioned above, and introduce the main performance criteria of the organ-transplant system that we will derive: the rate of the instants at which demands leave unsatisfied and that of organ outdatings, the steady-state laws of the number of organs on the shelf and of the waiting time of a patient, and the long-run fraction of time during which the shelf is empty of organs. The VOP is analyzed in Sections 3 and 4. The performance criteria are then explicitly computed in Section 5. In the special case of Poisson arrival of organs and exponential patience times we study the outdating process in Section 6. Finally, Section 7 presents possible extensions of the basic model which we intend to pursue in the future.

2 Model description and preliminaries

We consider the following model. Perishable items (donated organs) arrive at the organ bank according to a renewal process with interrenewal time distribution function G, having mean $1/\lambda$ and Laplace-Stieltjes transform (LST) G^*. The arrival times of the demands (patients needing transplantation) are independent of those of the items (in the case of kidneys this may not be quite true) and form a Poisson process, of rate μ, which is independent of the item arrival times. A demand that upon its arrival finds the shelf of items not empty is satisfied immediately by the oldest organ present. Demands that arrive at an empty organ bank join the line of waiting demands (if any); newly arriving organs are assigned on the spot to waiting demands on a first-come-first-served basis.

Each demand possesses its own *random patience time*, which can be interpreted as the time until the patient's physical condition no longer allows carrying out a transplantation. Denoting by P_n the patience time of the nth arriving patient, we assume that $\{P_n : n = 1, 2, ...\}$ is a sequence of independent, identically distributed, positive random variables which are independent of the arrival times of organs and demands. P_n has the distribution function H, the mean $1/\eta$ and the LST H^*. If the waiting time of the nth patient exceeds his patience he *abandons* the waiting line without receiving treatment. On the other hand, the 'shelf lifetime' of the stored organs, i.e., their maximum usage time, is a prespecified constant that we set equal to 1 (without loss of generality). Thus, each organ is stored until it either satisfies some demand or, after one time unit on the shelf, is outdated (and then scrapped).

According to the above description *two* connected queueing systems of the FCFS type are generated (for a related but slightly different model see [18]). The first queue consists of the stored organs on the shelf while the second one is given by the line of patients waiting for these organs. The two queues cannot be simultaneously non-empty, but it is possible that both of them are empty.

In our paper we focus on five *performance criteria* which are key characteristics of the efficiency of the organ bank:

1. the rate of the times of organ outdatings;
2. the rate of the times at which demands leave unsatisfied;
3. the steady-state law of the number of organs on the shelf;
4. the steady-state law of the waiting time of a patient;
5. the length and long-run fraction of time during which the shelf is empty of organs.

We now give a global outline of our approach to study these performance criteria, explaining the basic ideas by referring to Figure 1. We use the following notation:

- A_n is the arrival time of the nth organ;
- D_n is the arrival time of the nth demand;
- P_n is the patience time of the nth patient;
- O_n is the time of the nth outdating.

Let $A(t)$ be defined as follows. As long as the shelf is not empty, $A(t)$ is the shelf age of the oldest organ in the system. When the shelf is empty, $A(t)$ is minus the waiting time of a (virtual) patient arriving at time t whose patience would be long enough to eventually receive the first available organ. In Figure 1(a) an exemplary sample path of this process is depicted. At time A_1 the first organ arrives at an empty shelf, and there are no waiting demands. At time D_1 a demand arrives that takes this organ. The organ which had arrived at time $A_2 < D_1$ becomes the oldest one, and now *its* age is drawn in the picture. At D_2, it is taken. At time O_1 the organ which had arrived at A_3 becomes outdated as its age reaches 1. At D_3 the fourth organ is taken by the third demand and the system again becomes empty of organs and demands; both queues are now empty. $A(t)$ jumps down to the negative of the time until the arrival of the next organ. The patience time P_4 of the fourth demand (arriving at D_4) is not large enough so that this patient leaves unsatisfied. The fifth demand has enough patience to wait for the next (the fifth) organ arrival and the process jumps down to the negative of the residual waiting time for the sixth organ arrival. This organ is taken upon its arrival by the sixth demand. The seventh organ arrives at an empty shelf without waiting demands and the process $A(t)$ starts anew.

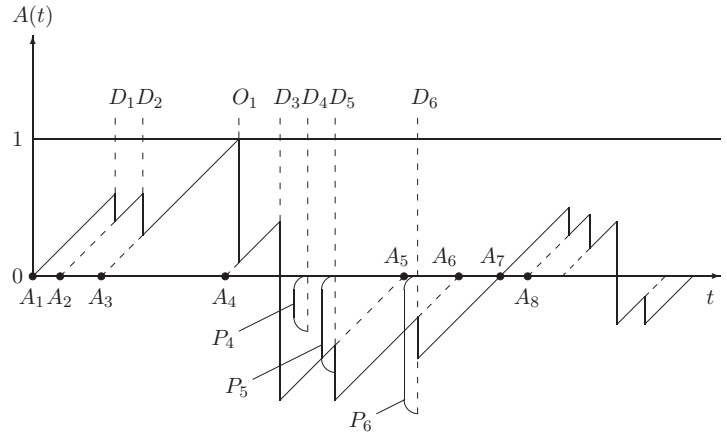

Figure 1a

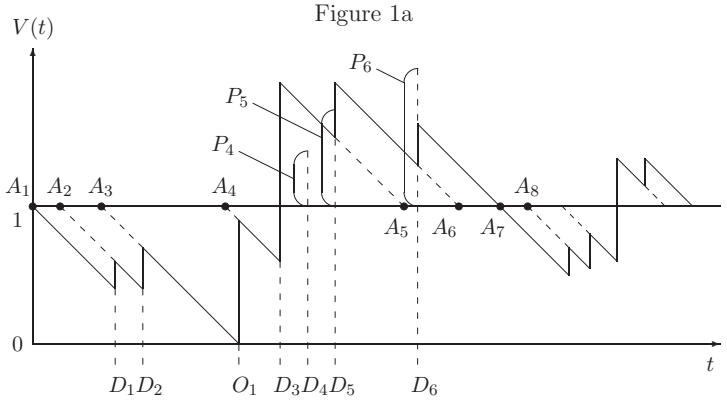

Figure 1b

Figure 1: a typical sample path of $A(t)$ (1a) and the corresponding sample path of the VOP (1b).

In Figure 1(b) the sample path of 1(a) has been reflected along the horizontal line at height 1/2 and shows the resulting *virtual outdating process* (VOP) $\mathbf{V} = \{V(t) : t \geq 0\}$. Clearly, $V(t) = 1 - A(t)$ is the time from t till the next outdating of an organ that would occur if the demand arrival process was completely stopped at t. We shall mainly concentrate on $\mathbf{V}$, deriving results for the above-mentioned performance criteria from it. A crucial observation is that, because of the Poisson nature of the demand arrivals, $\mathbf{V}$ is a Markov process whose sample path can also be interpreted as the *workload process* of a special $M/G/1$-type queueing system with abandonments (or impatience). In this system customers arrive according to a Poisson process with rate μ and have service requirements with distribution G (having mean $1/\lambda$). These service requirements are not observable: upon arrival the customers don't *see* their waiting times. Let W_n be the waiting time of the n*th* arriving customer. The patience of the nth

customer is $1 + P_n$, where the P_n are i.i.d. with distribution H. Therefore, the nth customer is admitted to the system if $W_n \leq 1$; if $W_n > 1$ the nth customer is refused if $P_n < W_n - 1$ and admitted otherwise. We call this system an $M/G/1 + G$ queue. Finally, the idle periods are deleted and the busy periods are glued together. The resulting workload process has the same law as $\mathbf{V}$.

Given $V(t) \leq 1$, the age of the oldest organ on the shelf at time t is given by $1 - V(t)$. Note that (i) an outdating of an organ occurs at time t if and only if $V(t) = 0$ and (ii) if $V(t) > 1$ the shelf is empty of organs at t. By interpreting the VOP of the organ-transplant model as the work process $\mathbf{V}$ of the special queueing model with abandonments as described above, the outdating times of organs form a renewal process whose interrenewal times have the same law as that of a busy period in that queue. It constitutes our performance criterion 1, and is studied in Section 5. In particular, we will obtain the rate $\lambda^* = \lambda^*(\lambda, \mu)$ of the outdating times. The unsatisfied demand process (Criterion 2), which in our interpretation counts the abandoned customers, is in general not a renewal process – not even in the special case of Poisson arrivals of organs and of demands. Its law seems difficult to obtain. However, once the law of the outdating process is known we can derive the value of $\lambda^*(\lambda, \mu)$ and then the rate $\mu^* = \mu^*(\lambda, \mu)$ of unsatisfied demands is obtained from the following *conservation law*:

$$\lambda - \lambda^*(\lambda, \mu) = \mu - \mu^*(\lambda, \mu). \tag{1}$$

To see (1) note that since λ is the arrival rate of organs into the system, it is also their departure rate out of the system in steady state. Thus, the left-hand side of (1) represents the long-run average rate of organs which are not outdated, i.e., are used to satisfy demands, while the right-hand side of (1) is just the rate of satisfied demands. As a result, once we have computed $\lambda^*(\lambda, \mu)$, the rate $\mu^*(\lambda, \mu)$ is also known.

Criteria 3 and 4 are also closely related to the age process, and hence to the VOP, and are studied in Section 5. Criterion 5 is also considered there.

3 The steady-state law of V

The important basic quantity for our analysis is the steady-state law of $\mathbf{V}$. Evidently, by *level crossing theory* (LCT; see e.g. [6]) this steady-state distribution is absolutely continuous and its density f at level x is equal to the long-run average number of *downcrossings* of x, which is of course equal to the long-run average number of *upcrossings* of x (similar statements hold for other steady-state densities which will be considered furtheron). This approach leads to

Theorem 1 *The stationary density f of* $\mathbf{V}$ *satisfies the integral equation*

$$f(x) = \begin{cases} \mu \int_0^x [1 - G(x-w)] f(w)\ dw + f(0)[1 - G(x)], & 0 < x \leq 1, \\ \mu \int_0^1 [1 - G(x-w)] f(w)\ dw + f(0)[1 - G(x)] & \\ \quad + \mu \int_1^x [1 - G(x-w)][1 - H(w-1)] f(w)\ dw, & x > 1. \end{cases} \tag{2}$$

Proof. The long-run average number of upcrossings of x is given by the right-hand side of (2). To see this, first note that the arrival rate of upward jumps is μ, and that by PASTA the density at a jump epoch is also f. When starting a jump from $w \in (0, \min(x, 1)]$, the probability to end up above x is $1 - G(x-w)$. When starting from $w \in (1, x)$, the probability to jump at all upon a demand arrival is $1 - H(w-1)$ and then, given that a jump takes place, the probability to jump above x is $1 - G(x-w)$. Multiplying we get $[1 - H(w-1)][1 - G(x-w)]$. Moreover, level x can also be upcrossed from level 0 at the end of a cycle. The rate of cycle endings is $f(0)$ and $1 - G(x)$ is the probability to upcross level x jumping from 0. The proof is complete. ■

In case that the arrival times of organs form a Poisson process, the VOP is the workload process in an $M/M/1 + G$ queue with deleted idle periods. Eq. (2) becomes

$$f(x) = \begin{cases} \mu \int_0^x e^{-\lambda(x-w)} f(w)\ dw + f(0) e^{-\lambda x}, & 0 < x \leq 1, \\ \mu \int_0^1 e^{-\lambda(x-w)} f(w)\ dw + f(0) e^{-\lambda x} & \\ \quad + \mu \int_1^x e^{-\lambda(x-w)} [1 - H(w-1)] f(w)\ dw, & x > 1. \end{cases} \tag{3}$$

Solving for f in (3) we obtain (see also Section IV of [2])

$$f(x) = \begin{cases} k_0 e^{-(\lambda - \mu)x}, & 0 < x \leq 1, \\ k_1 \exp\left\{ -[\lambda x - \mu \int_1^x (1 - H(z-1)) dz] \right\}, & x > 1, \end{cases} \tag{4}$$

for certain constants k_0 and k_1. To find k_0 and k_1 note that $f(x)$ is continuous at 1. We get $k_0 = k_1 e^{\mu}$ and k_0 can be easily calculated via the normalizing condition for f. We find that

$$k_0 = \left[\int_0^1 e^{-(\lambda - \mu)x} dx + e^{\mu} \int_1^{\infty} \exp\left\{ -[\lambda x - \mu \int_1^x (1 - H(z-1)) dz] \right\} dx \right]^{-1}.$$

In particular, if the patience is $\exp(\eta)$-distributed, (4) yields

$$f(x) = \begin{cases} k_0 e^{-(\lambda-\mu)x}, & 0 < x \leq 1, \\ k_0 \exp\left\{-[\lambda x - \mu - \frac{\mu}{\eta}(1 - e^{-\eta(x-1)})]\right\}, & x > 1. \end{cases}$$

4 Decomposition and explicit computation of f

The workload density f for general G and H can be obtained from Theorem 1 in the following way. (i) solve the integral equation (2) in the interval $[0, 1]$ (in terms of an infinite series of convolutions and the constant $f(0)$); (ii) insert this solution in the equation for $x \in (1, \infty)$, which can then also be solved in terms of an infinite series of convolutions in which the first series occurs as under the integral sign; and (iii) determine $f(0)$ from the normalization condition $\int_0^\infty f(x)\ dx = 1$. We will proceed somewhat differently by providing a more intuitive argument for the decomposition of f.

The VOP $\mathbf{V}$ can be separated into two different processes. The first one is constructed by deleting the time periods in which the shelf is empty of organs, i.e., $\mathbf{V} > 1$, and gluing together the remaining time periods in which the shelf is not empty, i.e., in which $\mathbf{V} \leq 1$. The second one is generated by gluing together the time periods in which $\mathbf{V} > 1$. We denote these two processes by $\mathbf{V_D} = \{V_D(t) : t \geq 0\}$ and $\mathbf{V_P} = \{V_P(t) : t \geq 0\}$, respectively. D indicates a relation to finite Dams, and P a relation to models with (im)Patience. Indeed, one easily sees that $\mathbf{V_D}$ describes the content level in an $M/G/1$ queue with finite capacity 1 (and idle periods deleted), also called finite dam, while $\mathbf{V_P} - 1$ represents the workload of an $M/G/1 + G$ queue with deleted idle periods. However, $\mathbf{V_P} - 1$ has a special feature because the overflow of $\mathbf{V}$ above level 1 is the residual service time of the first customer in the busy period of $\mathbf{V_P} - 1$, so that $\mathbf{V_P} - 1$ is the workload process of an $M/G/1 + G$ queue (with deleted idle periods) in which the first service time of a busy period has a different distribution.

Let f_D and f_P be the conditional densities of $\mathbf{V}$ given that $\mathbf{V} \leq 1$ and that $\mathbf{V} > 1$, respectively. With $I.$ denoting an indicator function, and with ν being the fraction of time that $\mathbf{V} > 1$, we have

$$f(x) = (1 - \nu) f_D(x) I_{x \leq 1} + \nu f_P(x) I_{x>1}.$$

Let us now compute $f_D(x)$, $f_P(x)$ and ν.

(A) Computation of $f_D(x)$
The appropriate balance equation for $f_D(x)$ is given by

$$f_D(x) = \mu \int_0^x [1 - G(x - w)] f_D(w) dw + f_D(0)[1 - G(x)], \quad 0 \leq x \leq 1 \qquad (5)$$

and coincides with the corresponding equation for the workload density in the standard $M/G/1$ queue with finite capacity 1. One possible approach is to

distinguish between $\rho = \mu/\lambda < 1$ and $\rho \geq 1$ and to observe the following. If $\rho < 1$ then the workload density is proportional to the steady-state workload density of the unrestricted $M/G/1$ queue. If $\rho \geq 1$ then the steady-state distribution of the workload still exists (recall that we consider a system with finite capacity) but the solution procedure has to be adapted slightly (see [7], pp. 72-73 or Section 2 of [5]).

However, we follow a more direct approach. Let g_e be the equilibrium density of G, i.e., $g_e(x) = \lambda[1 - G(x)]$. Then (5) can be rewritten as

$$f_D = kg_e + \rho g_e * f_D$$

where k is a certain constant, and $*$ denotes convolution. Solving for f_D we get

$$\begin{aligned} f_D &= kg_e + \rho g_e * [kg_e + \rho g_e * f_D] \\ &= k[g_e + \rho g_e^{*(2)} + \rho^2 g_e^{*(3)} + \ldots] \\ &= kQ, \end{aligned}$$

where $Q = \sum_{n=1}^{\infty} \rho^{n-1} g_e^{*(n)}$, and the constant k can be computed from the normalizing condition $\int_0^1 kQ(x)\ dx = 1$, yielding

$$f_D(x) = \frac{Q(x)}{\int_0^1 Q(y)\ dy}. \tag{6}$$

Remark 1 *In the special case of a Poisson arrival process of organs, $f_D(x)$ is the workload density in the $M/M/1$ queue with capacity 1 and with idle periods deleted, which is a truncated exponential density:*

$$f_D(x) = \begin{cases} \frac{(\lambda-\mu)e^{-(\lambda-\mu)x}}{1-e^{-(\lambda-\mu)}}, & \lambda \neq \mu, \\ 1, & \lambda = \mu. \end{cases} \tag{7}$$

It should be noticed that the density is uniform when $\lambda = \mu$, in this case of an $M/M/1$ queues with barriers 0 and 1 and idle periods deleted.

(B) Computation of $f_P(x)$

We first need to determine the distribution function, say A, of the overshoot of $\mathbf{V}$ above level 1, which is also the distribution function of the first service time in a busy period of $\mathbf{V_P} - 1$ (recall that it is in general different from the other ones). For this purpose we construct a modified work process $\mathbf{V}_{const} = \{V_{const}(t) : t \geq 0\}$ which is a special case of $\mathbf{V}$. We define $\mathbf{V}_{const}$ as the workload process of the $M/G/1 + D$ type queue with deleted idle periods, arrival rate μ, service rate λ and constant patience times equal to 1. In this queue the workload can jump above 1 but customers do not admit to the system if they have to wait in line more than 1. In other words, customers leave unsatisfied if they see an empty shelf at their arrival. It turns out that there is a simple relation between the distribution of the first service in the busy period of $\mathbf{V_P} - 1$ and the steady-state law of $\mathbf{V}_{const}$. Let $f_{const}(x)$ be the steady-state density of $\mathbf{V}_{const}$. In the next lemma we express the law of the first service time of the busy period of $\mathbf{V_P} - 1$ in terms of f_{const}.

Remark 2 *The workload process $\mathbf{V}_{const}$ is the same process as that described in [14] (under a different motivation). By LCT, the balance equation of f_{const} is given by*

$$f_{const}(x) = \mu \int_0^{x\wedge 1} [1 - G(x-w)] f_{const}(w)dw + f_{const}(0)[1 - G(x)]. \qquad (8)$$

The balance equation (8) is given in [14] but without its solution. We will introduce the solution here (in terms of the Neumann series) after the proof of the lemma.

Lemma 1 *Let A be the distribution function of the first service time in the busy period of $\mathbf{V_P} - 1$. Then, for all $x > 0$,*

$$A(x) = 1 - \frac{f_{const}(x+1)}{f_{const}(1)}.$$

Proof. Let

$$F_{const}(x) = \int_0^x f_{const}(w)dw$$

be the steady-state distribution function of $\mathbf{V}_{const}$. On the one hand, the function

$$a_e(x) := \frac{f_{const}(x+1)}{1 - F_{const}(1)} \qquad (9)$$

is the conditional steady-state density of f_{const} given $\mathbf{V}_{const} > 1$. By the strong Markov property, the overflows above level 1 are i.i.d. random variables so that if we delete the time periods in which $\mathbf{V}_{const} \leq 1$ and glue together the time periods in which $\mathbf{V}_{const} > 1$ we get a sample path of a renewal process in which the asymptotic density of the forward recurrence time is given by (9). On the other hand, as $a_e(x)$ is the equilibrium density of the overflow, we have

$$a_e(x) = \zeta[1 - A(x)] \qquad (10)$$

where $\zeta^{-1} = \int_0^\infty [1 - A(x)]dx$. Taking $x = 0$ in (9) and (10) yields

$$\zeta = \frac{f_{const}(1)}{1 - F_{const}(1)}. \qquad (11)$$

Substitute (11) in (10) and the lemma follows by equating the right hand side of (9) with the right hand side of (10). ■

Note that, by Lemma 1, $f_{const}(x+1)$ is a monotone decreasing function.

To solve for f_{const} in (8) we start with the solution for $x \leq 1$. We designate $\bar{\rho} = \mu/\lambda$ and get

$$f_{const} = \bar{\rho} g_e * f_{const} + C g_e \qquad (12)$$

where $g_e := \lambda[1 - G(x)]$ and the constant $C := f_{const}(1)/\lambda$. The unique solution of (12) is known to be

$$f_{const}(x) = C \sum_{n=1}^{\infty} \bar{\rho}^{n-1} g_e^{n*}(x). \qquad (13)$$

For $x > 1$ we get in (8)

$$f_{const} = \bar{\rho} \int_0^1 g_e(x-w) f_{const}(w) dw + C g_e(x). \tag{14}$$

By substituting (13) in (14) we get

$$f_{const}(x) = \bar{\rho} \int_0^1 g_e(x-w) C \sum_{n=1}^{\infty} \bar{\rho}^{n-1} g_e^{n*}(w) dw + C g_e(x)$$

where C is obtained via the normalizing condition

$$\begin{aligned} C &= \Big(\textstyle\sum_{n=1}^{\infty} \bar{\rho}^{n-1} \int_0^1 g_e^{n*}(x) dx \\ &\quad + \textstyle\sum_{n=1}^{\infty} \bar{\rho}^{n} \int_1^{\infty} \int_0^1 g_e(x-w) g_e^{n*}(w) dw dx + \int_1^{\infty} g_e(x) dx \Big)^{-1}. \end{aligned}$$

We are now in a position to compute $h_P(x) = f_P(x+1)$. Its balance equation contains the distribution function A:

$$h_P(x) = \mu \int_0^x [1 - G(x-w)] h_P(w)\ dw + h_P(0)[1 - A(x)]. \tag{15}$$

To solve (15) for h_P we write it as

$$h_P = \bar{k} a_e + \rho g_e * h_P,$$

so that

$$h_P = \bar{k} a_e * \sum_{n=1}^{\infty} \rho^{n-1} g_e^{n*} = \bar{k} a_e * Q,$$

and the constant $\bar{k}$ can again be calculated from the normalizing condition. We obtain

$$f_P(x) = (a_e * Q)(x-1) / \int_0^{\infty} (a_e * Q)(y)\ dy. \tag{16}$$

(C) The value of ν

By regenerative theory ν is equal to the ratio of the expected busy period of $\mathbf{V_D}$ and the sum of the expected busy periods of $\mathbf{V_P}$ and $\mathbf{V_D}$, and this ratio is equal to

$$\nu = \frac{1/f_D(0)}{(1/h_P(0)) + (1/f_D(0))} = \frac{f_P(1)}{f_D(0) + f_P(1)}.$$

Summarizing, we have proved

Theorem 2 *The steady-state density f is given by*

$$f(x) = \begin{cases} \dfrac{f_P(1)}{f_D(0) + f_P(1)} f_D(x), & 0 \le x \le 1 \\ \dfrac{f_D(0)}{f_D(0) + f_P(1)} f_P(x), & x > 1 \end{cases}$$

where $f_D(x)$ and $f_P(x)$ are given by (6) and (16).

Remark 3 *In the special case of a Poisson arrival process of organs, there is no need to calculate $f_{const}(x)$, because the distribution A of the overshoot above 1 is $exp(\mu)$ by the memoryless property. Finally $h_P(x)$ follows from the known results for an $M/M/1+G$ queue; see, e.g., [3].*

5 The performance criteria

From the steady-state law of $\mathbf{V}$ we can now obtain several performance characteristics.

1. **The rate of organ outdatings**, λ^*, is given by

$$\lambda^* = f(0),$$

and hence, using Theorem 2,

$$\lambda^* = \frac{f_D(0) f_P(1)}{f_D(0) + f_P(1)}.$$

In the special case of Poisson organ arrival times of intensity λ we obtain from the example in Section 3

$$\lambda^* = \left[\int_0^1 e^{-(\lambda-\mu)x} dx + e^{\mu} \int_1^{\infty} e^{-[\lambda x - \mu \int_1^x (1-H(z-1))dz]} dx \right]^{-1}.$$

2. **The rate of unsatisfied demands**, μ^*, is, by (1),

$$\mu^* = \mu - \lambda + f(0).$$

Also, by using level crossing theory, $f(1)$ can be interpreted as the rate of organ arrivals at an empty system.

3. **The number of organs on the shelf** is zero as long as $\mathbf{V} > 1$ and equal to one plus the number of arrivals during the age of the oldest organ if $\mathbf{V} \leq 1$. Thus, if N_0 denotes the number of organs on the shelf in steady state, its generating function is given by

$$\begin{aligned} \mathbb{E}z^{N_0} &= \int_1^{\infty} f(x)\ dx + \int_0^1 \sum_{n=1}^{\infty} z^n \mathbb{P}(n-1 \text{ arrivals in a time interval of} \\ &\qquad\qquad \text{length } 1-x) f(x)\ dx \\ &= \int_1^{\infty} f(x)\ dx + \int_0^1 \sum_{n=1}^{\infty} z^n (G^{(n-1)*}(1-x) - G^{n*}(1-x)) f(x)\ dx. \quad (17) \end{aligned}$$

Hence, in the $M/M/1+G$ case (organs arrive according to a Poisson process),

$$\mathbb{E}z_0^N = \int_1^{\infty} f(x)\ dx + z \int_0^1 e^{-\lambda(1-z)(1-x)} f(x)\ dx.$$

4. **The steady-state law of the waiting time** can be determined as follows. A patient arriving at time t does not have to wait if $A(t) \geq 0$ and has to wait for $-A(t)$ time units, provided his patience does not run out before. The steady-state probability that $A(t) \geq 0$ is equal to $1 - \nu = \dfrac{f_D(0)}{f_D(0) + f_P(1)}$, and the steady-state probability of having to wait between x and $x + dx$ time units (assuming sufficient patience) is $\nu f_P(x-1)\ dx$.

For the $M/M/1 + M$-type system, with $\exp(\lambda)$ organ interarrival times and $\exp(\eta)$ patience times, **the steady-state law of the number of waiting patients**, say N_c, (including those who are going to leave unsatisfied) can be easily derived. The conditional law of N_c, given that the system is empty of organs, can be calculated from the linear set of balance equations

$$\begin{aligned} \mu p_0 &= \lambda p_1 \\ \mu p_n &= (\lambda + n\eta) p_{n+1}, \qquad n = 1, 2, \ldots, \end{aligned} \tag{18}$$

where p_n is the conditional steady-state probability of n customers being present. So $p_n = p_0 \mu^n / \prod_{j=0}^{n-1} (\lambda + j\eta)$, and

$$EN_c = \int_1^\infty f(x)\ dx \cdot \sum_{n=1}^\infty n p_n = p_0 \int_1^\infty f(x)\ dx \sum_{n=1}^\infty \frac{n\mu^n}{\prod_{j=0}^{n-1}(\lambda + j\eta)},$$

where

$$p_0 = \left[1 + \sum_{n=1}^\infty \frac{\mu^n}{\prod_{j=0}^{n-1}(\lambda + j\eta)} \right]^{-1}.$$

Finally notice that $\tilde{p}_n$, the probability of n customers waiting for organs, equals p_{n+1}, $n = 0, 1, \ldots$; the number of waiting patients equals the number of customers in the corresponding $M/M/1 + M$ queue minus one.
In the case of general patience one can obtain the generating function of the number of waiting patients, $E[z^{N_c}] = \Theta(z)/z$, where $\Theta(z)$ is the generating function of the number of customers in $M/M/1 + G$. $\Theta(z)$ has been derived in [3]:

$$\Theta(z) = \int_0^\infty \exp\{-\lambda(1-z) \int_0^t [1 - H(s)] \mathrm{d}s\} \mathrm{d}\tilde{F}(t),$$

with $\tilde{F}(t)$ denoting a distribution with LST

$$\int_0^\infty \mathrm{e}^{-\alpha t} \mathrm{d}\tilde{F}(t) = \mathbb{E}[\mathrm{e}^{-\alpha V} H(V)] + G^*(\alpha) \mathbb{E}[\mathrm{e}^{-\alpha V}(1 - H(V))],$$

where V is the steady-state workload – which is the same as $V_P - 1$. If organs don't arrive according to a Poisson process, we have to consider an $M/G/1 + G$ queue in which the first service time in a busy period has a special distribution. This requires further research; in [3] we only considered the number of customers in the ordinary $M/G/1 + G$ queue.

5. For our last performance criterion, **the long-run fraction of time the shelf is empty**, we restrict ourselves to the case of Poisson arrivals of organs.

The consecutive periods of time during which the shelf is empty and non-empty form an alternating renewal process. Whenever $\mathbf{V} \leq 1$ (ON period) the shelf is not empty and whenever $\mathbf{V} > 1$ (OFF period) the shelf is empty. From the description above the OFF period has the same law as the busy period in the $M/M/1+G$ queue with arrival rate μ, service rate λ and patience distribution $H(\cdot)$. It should be noted that the busy period distribution for $M/M/1+G$ is only known in special cases, in particular for discrete patience times (see [4]; see [3] for preliminary results on the busy period for $M/G/1+G$). Let $\hat{f}(\cdot)$ be the steady-state density of the workload of this queue. Then, using a similar argument as in the proof of Theorem 1 we have for all $x > 0$

$$\hat{f}(x) = \mu \int_0^x [1 - H(w)] e^{-\lambda(x-w)} \hat{f}(w)\ dw + \hat{f}(0) e^{-\lambda x}. \tag{19}$$

Solving for $\hat{f}(\cdot)$ in (19) we get

$$\hat{f}(x) = \hat{k} e^{-[\lambda x - \mu \int_0^x (1-H(y))\ dy]} \tag{20}$$

where

$$\hat{k} = \left[\int_0^\infty e^{-[\lambda x - \mu \int_0^x (1-H(y)) dy]} \right]^{-1}. \tag{21}$$

By renewal theory,

$$\mathbb{E}[\ \text{length of an OFF period}\ ] = 1/\hat{f}(0),$$

and

$$\mathbb{E}[\ \text{cycle length}\ ] = 1/f(1).$$

The renewal reward theorem now shows that the long-run fraction of time that the shelf is empty is equal to $\sigma = f(1)/\hat{f}(0)$, where $f(1)$ and $\hat{f}(0) = \hat{k}$ are given by (4) and (21), respectively.

6 The outdating process in the Markovian case

In this section we derive the LST of the inter-outdating times in the special case of Poisson arrivals of organs and of $H(x) = 1 - e^{-\eta x}$. Recall that the outdating process of organs is a renewal process and let τ be the time period between two successive outdatings. To compute its LST $\Gamma(\beta) = \mathbb{E}e^{-\beta\tau}$, we write it as

$$\Gamma(\beta) = \int_0^\infty \mathbb{E}_x e^{-\beta\tau} \lambda e^{-\lambda x} dx, \tag{22}$$

where $\mathbb{E}_x$ denotes conditional expectation given that $V(0) = x$. Let L be the time from a moment the shelf becomes empty of organs (say, for the first time) until the next arrival of an organ (see also Item 4 in Section 5). From our assumptions, the strong Markov property and the lack-of-memory property of

the jump sizes it follows that L has the same law as the busy period of the $M/M/1+M$ queue with arrival rate μ, service rate λ and patience rate η. For its LST $\Psi(\beta)$ no analytic expression is known; however, in Sect. 3 of [4] it is given in the form of a continued fraction that can be derived from the recursion $\Psi(\beta)=\Psi_1(\beta)$ and

$$\Psi_n(\beta)=\frac{\mu+(n-1)\xi}{\mu+(n-1)\xi+\beta+\lambda(1-\Psi_{n+1}(\beta))},\quad n\geq 1.$$

We note that $\mathbb{E}L$ is known in closed form [4]:

$$\mathbb{E}L=\sum_{k=0}^{\infty}\frac{\mu^k}{\prod_{j=0}^{k}(\lambda+j\eta)}.$$

Lemma 2

$$\Gamma(\beta)=\int_0^1 \mathbb{E}_x e^{-\beta\tau}\lambda e^{-\lambda x}dx+e^{-\lambda}\mathbb{E}e^{-\beta L}\mathbb{E}_1 e^{-\beta\tau}.$$

Proof. By (22) it is enough to show that

$$\int_1^{\infty}\mathbb{E}_x e^{-\beta\tau}\lambda e^{-\lambda x}dx=e^{-\lambda}\mathbb{E}e^{-\beta L}\mathbb{E}_1 e^{-\beta\tau}.$$

If the first jump in the cycle is greater than 1, the excess beyond level 1 is $\exp(\lambda)$-distributed and the cycle starts with an interval of length L. After that interval it follows by the strong Markov property that the LST of the time until the next outdating is $\mathbb{E}_1 e^{-\beta\tau}$. Finally, the probability that a cycle will start with a jump above level 1 is $e^{-\lambda}$. ■

By Lemma 2, it is enough to compute $\mathbb{E}_x e^{-\beta\tau}$ for $0<x\leq 1$. Define the stopping time $T=\inf\{t>0: V(t)=0 \text{ or } V(t)>1\}$ and

$$\phi_*(x,\beta)=\mathbb{E}_x e^{-\beta T}1_{\{V(T)=0\}},$$

$$\phi^*(x,\beta)=\mathbb{E}_x e^{-\beta T}1_{\{V(T)\geq 1\}}.$$

Then we have for any $0<x\leq 1$:

$$\mathbb{E}_x e^{-\beta\tau}=\phi_*(x,\beta)+\phi^*(x,\beta)\mathbb{E}e^{-\beta L}\mathbb{E}_1 e^{-\beta\tau}.$$

In particular, for $x=1$,

$$\mathbb{E}_1 e^{-\beta\tau}=\frac{\phi_*(1,\beta)}{1-\phi^*(1,\beta)\mathbb{E}e^{-\beta L}},$$

so that for any $0<x<1$

$$\mathbb{E}_x e^{-\beta\tau}=\phi_*(x,\beta)+\phi^*(x,\beta)\frac{\phi_*(1,\beta)\mathbb{E}e^{-\beta L}}{1-\phi^*(1,\beta)\mathbb{E}e^{-\beta L}}.$$

This leads to

Theorem 3

$$\Gamma(\beta)=\int_0^1\left[\phi_*(x,\beta)+\phi^*(x,\beta)\frac{\phi_*(1,\beta)\mathbb{E}e^{-\beta L}}{1-\phi^*(1,\beta)\mathbb{E}e^{-\beta L}}\right]\lambda e^{-\lambda x}dx+e^{-\lambda}\mathbb{E}e^{-\beta L}\mathbb{E}_1e^{-\beta\tau}. \tag{23}$$

Thus, in order to obtain $\Gamma(\beta)$ explicitly in terms of $\mathbb{E}e^{-\beta L}$ it remains to compute $\phi_*(x,\beta)$, $\phi^*(x,\beta)$ and the integrals $\int_0^1\phi_*(x,\beta)e^{-\lambda x}dx$ and $\int_0^1\phi^*(x,\beta)e^{-\lambda x}dx$. To this end we use the fact that for $\varphi(\alpha)=\alpha-\frac{\mu\alpha}{\lambda+\alpha}$ the process

$$(\varphi(\alpha)-\beta)\int_0^t e^{-\alpha V(s)-\beta s}ds+e^{-\alpha V(0)}-e^{-\alpha V(t)-\beta t},\quad t\geq 0$$

is a *Kella-Whitt* martingale (see [15] or [1], Sect. IX.3). Applying to this martingale the stopping time T we get the identity

$$(\varphi(\alpha)-\beta)\mathbb{E}_x\int_0^T e^{-\alpha V(s)-\beta s}ds=-e^{-\alpha x}+\mathbb{E}_xe^{-\alpha V(T)-\beta T}. \tag{24}$$

The solutions of $\varphi(\alpha)-\beta=0$ are

$$\theta_1,\theta_2=\frac{-(\lambda-\mu-\beta)\pm\sqrt{(\lambda-\mu-\beta)^2+4\lambda\beta}}{2}.$$

Inserting θ_1,θ_2 in (24) we obtain the two equations

$$e^{-\theta_i x}=\phi_*(x,\beta)+\frac{\lambda}{\lambda+\theta_i}\phi^*(x,\beta),\quad i=1,2.$$

This yields

$$\phi^*(x,\beta)=\frac{e^{-\theta_1 x}-e^{-\theta_2 x}}{\frac{\lambda}{\lambda+\theta_1}-\frac{\lambda}{\lambda+\theta_2}},$$

and

$$\phi_*(x,\beta)=\frac{(\lambda+\theta_1)e^{-\theta_1 x}-(\lambda+\theta_2)e^{-\theta_2 x}}{\theta_1-\theta_2}.$$

Finally,

$$\int_0^1\phi_*(x,\beta)e^{-\lambda x}dx=\frac{e^{-(\lambda+\theta_2)}-e^{-(\lambda+\theta_1)}}{\theta_1-\theta_2},$$

and

$$\int_0^1\phi^*(x,\beta)e^{-\lambda x}dx=\frac{(\lambda+\theta_2)(1-e^{-(\lambda+\theta_1)})-(\lambda+\theta_1)(1-e^{-(\lambda+\theta_2)})}{\lambda(\theta_2-\theta_1)}.$$

Now insert all these terms in (23) to get $\Gamma(\beta)$.

7 Possible extensions

Our model calls for various extensions to incorporate the special characteristics of transplant allocation. We finally present a list of some of the features that we intend to investigate in the future.

1. Issuing policies. Any blood transfusion is prohibitively conditioned by an asymmetric ABO-matching between the donor and the recipient, as blood-type O can be donated universally, blood-type A only to blood-types A and AB, etc. In addition, immunology recognizes the importance of HLA (Human Leukocyte Antigens) tissue matching, especially in kidney transplantation. Human tissue cells contain proteins that vary from person to person and are considered to be potential transplant antigens. When transplanted in another individual, they can cause an immunological response resulting in the formation of antibodies that can lead to the rejection of the transplanted organ. The HLA system contains these immunologically relevant antigens and arranges them in sites ("loci"). Every site contains two antigens (as formed by two "allelic" genes, one contributed by the father and one by the mother). It is traditionally held that any antigen present in the donor which is foreign to the recipient can trigger an immune response. The higher the number of such antigens, the lower the chance of a successful transplant. Thus, some finite number of possible match-levels can be identified. The ideal situation would be the case of no mismatches. In this paper we have assumed a FIFO regime, but it is a challenging problem to introduce and analyze matching policies based entirely or partially on the number of foreign antigens in the organs. However, it is to be noted that priority point systems (implemented worldwide for allocation and organ sharing), while seeking to take into account antigen matching, all prioritize patients who wait the longest. Thus, due to the increasing gap between supply and accelerating demand, the waiting time criterion (subject to ABO) is still the dominating one, effectively allocating the organs according to the FIFO rule.

Another feature that implies that the organ issuing policy is not FIFO is that the condition of the patients may play a role. To incorporate that into our model is another topic for further research.

2. Variable arrival rate. The arrival rate of new organs may depend on the current waiting times of patients in order to increase the number of transplantations. For example, the arrival rate of organs could be changed from λ to some $\lambda' > \lambda$ whenever the waiting time of all patients present gets larger than some constant or random b, and back to λ when the waiting times become again sufficiently small. This could represent situations in which temporary promotional efforts are made to increase donation, or even situations in which organs are bought from abroad. The analytical investigation of the latter is important especially since the ethics involved are under debate. In our model this means that the arrival rate of organs is dependent on the state of the VOP, being $\lambda(x)$ at time t if $V(t) = x$.

3. Non-constant shelf life. In our basic model it is supposed that any organ is scrapped after spending a constant period of time unused on the shelf. It is possible to generalize this assumption to the case of random shelf lifetimes.

For example one could assume that every organ is rechecked when reaching age 1 and is deleted with probability p but still found suitable for transplantation with probability $1-p$ in which case it will stay on the shelf for another a time units.

4. Performance measures and objectives. The widespread and high-tone debates on transplantation stem directly from the different and often contradicting relevant performance measures or objective functions. We refer to [28] for the important notions of QALY, discounted-QALY and the (post-transplant) k-year graft-survival criterion as well as for equity criteria related to waiting times and to the likelihoods of transplantation for different sub-populations. Our present study does not yet lead to these performance measures. A more refined matching queue model could serve as the basis for a quantitative analysis of these criteria.

Acknowledgements. The research of O.J. Boxma was done within the framework of the BRICKS project. D. Perry gratefully acknowledges a visitor grant from the Netherlands Organisation for Scientific Research NWO. W. Stadje was supported by the Deutsche Forschungsgemeinschaft.

References

[1] S. Asmussen (2003) *Applied Probability and Queues*. 2nd ed. Springer.

[2] F. Baccelli and G. Hebuterne (1981) On queues with impatient customers. In: F.J. Kylstra (ed.). *Proceedings of Performance '81*. North-Holland, Amsterdam, 159-179.

[3] O.J. Boxma, D. Perry and W. Stadje (2010). The $M/G/1+G$ queue revisited. EURANDOM Report 2010-xy, February 2010.

[4] O.J. Boxma, D. Perry, W. Stadje and S. Zacks (2009) The busy period of an $M/G/1$ queue with customer impatience. EURANDOM Report 2009-012. To appear in: *Journal of Applied Probability*.

[5] O.J. Boxma, D. Perry, W. Stadje and S. Zacks (2009) The $M/G/1$ queue with quasi-restricted accessibility. *Stochastic Models* **25**, 151-196.

[6] P.H. Brill (2008) *Level Crossing Methods in Stochastic Models*. Springer.

[7] J.W. Cohen (1976) *On Regenerative Processes in Queueing Theory*. Springer.

[8] David, I. and U. Yechiali (1985) A time dependent stopping problem with application to live organ transplants. *Operations Research* **33**, 491-504.

[9] I. David and U. Yechiali (1990) Sequential assignment match processes with arrival of candidates and offers. *Probability in the Engineering and Informational Sciences* **4**, 413-430.

[10] I. David (1995) A sequential assignment match processes with general renewal arrival times. *Probability in the Engineering and Informational Sciences* **9**, 475-492.

[11] I. David and M. Moatty-Assa (2009) A stylistic queueing-like model for the allocation of organs on the public list. *Technical Report*, Department of Industrial Engineering and Management, Ben-Gurion University, Israel.

[12] C. Derman, G.J. Lieberman and S.M. Ross (1972) A sequential stochastic assignment problem. *Management Science* **18**, 349-355.

[13] J.C. Hornberger and J. H. Ahn (1997) Deciding eligibility for transplantation when a donor kidney becomes available. *Medical Decision Making* **17**, 160-170.

[14] H. Kaspi and D. Perry (1984) Inventory systems of perishable commodities with renewal input and Poisson output. *Advances in Applied Probability* **16**, 402-421.

[15] O. Kella and W. Whitt (1992) Useful martingales for stochastic storage processes. *Journal of Applied Probability* **29**, 396-403.

[16] C.H. Lee, G.M. Chertow and S.A. Zenios (2008) Optimal initiation and management of dialysis therapy. *Operations Research* **56**, 1428-1449.

[17] D. Perry and S. Asmussen (1995) Rejection rules in the $M/G/1$ type queue. *Queueing Systems* **19**, 105-130.

[18] D. Perry and W. Stadje (1999) Perishable inventory systems with impatient demands. *Mathematical Methods of Operations Research* **50**, 77-90.

[19] J. Ratcliffe, T. Young, M. Buxton, T. Eldabi, R. Paul, A. Burroughs, G. Paratheodoridis and K. Rolles (2001) A simulation modelling approach to evaluating alternative policies for the management of the waiting list for liver transplantation. *Health Care Management Science* **4**, 117-124.

[20] R. Righter (1987) The stochastic sequential assignment problem with random deadlines. *Probability in the Engineering and Informational Sciences* **1**, 189-202.

[21] R. Righter (1989) A resource allocation problem in a random environment. *Operations Research* **37**, 329-338.

[22] X. Su and S.A. Zenios (2004) Patient choice in kidney allocation: the role of the queueing discipline. *Manufacturing and Service Operations Management* **6**, 280-301.

[23] X. Su and S.A. Zenios (2005) Patient choice in kidney allocation: a sequential stochastic assignment model. *Operations Research* **53**, 443-455.

[24] X. Su and S.A. Zenios (2006) Recipient choice can address the efficiency-equity trade-off in kidney transplantation: a mechanism design model. *Management Science 52*, 1647-1660.

[25] S.E. Taranto et al. (2000) Developing a national allocation model for cadaveric kidneys. In: *Proceedings of the 2000 Winter Simulation Conference*, J. A. Joines, R. R. Barton, K. Kang, and P. A. Fishwick (eds.).

[26] D. Thompson, L. Waisanen, R. Wolfe, R.M. Merion, K. Mccullough and A. Rodgers (2004) Simulating the allocation of organs for transplantation. *Health Care Management Science* **7**, 331-338.

[27] S. Ur, M. Trick and D. Sleator (1992) Analysis of online algorithms for organ allocation. In: J. van Leeuwen (ed.), *Algorithms, Software, Architectures. Information Proceedings* 92, Vol. I, Elsevier (North-Holland), 458-464.

[28] S.A. Zenios, G.M. Chertow and L.M. Wein (2000) Dynamic allocation of kidneys to candidates on the transplant waiting list. *Operations Research* **48**, 549-569.

[29] S.A. Zenios (1999) Modeling the transplant waiting list: a queueing model with reneging. *Queueing Systems* **31**, 239-251.

[30] S.A. Zenios (2002) Optimal control of a paired-kidney exchange program. *Management Science* **48**, 328-342.