Title: Something with Unexpected Inflation

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1 Introduction

2 Unexpected Inflation Decomposition

2.1 Connection with the Value of Debt

I start by establishing the connection between inflation and the value of debt. Consider an economy with a homogeneous final good which households value. The economy has a government that manages a debt composed by bonds that pay one unit of currency in the next period. Currency is a commodity that only the government can produce, at zero cost. There is nothing special about currency. Households do not value it, and it provides no convenience for trading. Households cannot burn currency.

At the beginning of period t, the face value of debt issued in the previous period is V_{t-1}. The government redeems it for currency, which moves to the hands of households. After redeeming bonds, the government announces that each household i must pay $T_{i,t}$ goods in taxes, payable in currency. It also announces it will sell V_t new bonds and purchase G_t units of the final good at market prices.[1]

Nothing binds the government's choices of $T_{i,t}$ and V_t. Note the difference from the case with bonds payable in goods, in which the government *must* raise tax revenue or issue new debt to redeem old bonds. Additionally, if households accept currency as payment for final goods, nothing binds the government's choice of G_t either.

Let M_t be private holdings of currency at the end of t. As there is no free disposal of currency, the quantity used by the government to redeem $t-1$ bonds and purchase goods is used to either pay taxes, buy new bonds or increase currency holdings:

$$\begin{aligned} V_{t-1} + G_t &= P_t T_t + Q_t V_t + \Delta M_t \\ \implies V_{t-1} &= P_t s_t + Q_t V_t + \Delta M_t \end{aligned} \tag{1}$$

[1] That the government forces households to pay for taxes and new bonds in currency is not necessary for the argument. All else follows if the government accepts payment in goods but stands ready to exchange these goods for currency at market prices.

where T_t are aggregate taxes, $s_t = T_t - G_t$ is the primary surplus, P_t is the final good's price and Q_t is the price of new bonds (I state prices in currency units). Equation (1) provides a low of motion for public debt. It holds for all prices and all choices of money holdings, new debt sales, and primary surplus.[2]

If $P_t = 0$, real public debt V_{t-1}/P_t equals infinity. For now, that is a possiblity.

Suppose $P_t > 0$. Define $\beta_t \equiv Q_t P_{t+1}/P_t$ as the real discount for public bonds, and $\beta_{t,t+k} \equiv \prod_{\tau=t}^{t+k} \beta_\tau$. Since V satisfies (1), it also satisfies

$$\frac{V_{t-1}}{P_t} = \sum_{i=0}^{k} \beta_{t,t+i-1} \left(s_{t+i} + \Delta M_t\right) + \beta_{t,t+k} \frac{V_{t+k}}{P_{t+k+1}} \qquad \text{for any } k \geq 0 \tag{2}$$

regardless of prices and choices. Expressions (1) and (2) do not represent a constraint on the path of surpluses $\{s_t\}$ and bond sales $\{V_t\}$ the government chooses to follow. They merely express future debt given paths for quantities and prices.

So far, all we did was public finances accounting. I now move to economic behavior. If $P_t = 0$, households demand infinite final goods and there is no equilibrium. Therefore $P_t > 0$.

Given a utility function over consumption paths $U(\{c_t\})$, the optimal consumption-savings choice involves two conditions. First: $\beta_{t,t+k}$ = marginal rate of substitution between time-t and time-$t+k$ consumption. Second, the transversality condition $\lim_{k\to\infty} \beta_{t,t+k} V_{t+k}/P_{t+k+1} \leq 0$. Coupled with a no-Ponzi condition that establishes the reverse inequality, it implies

$$\lim_{k\to\infty} \beta_{t,t+k} \frac{V_{t+k}}{P_{t+k+1}} = 0 \text{ at every period } t. \tag{3}$$

If bonds were redeemable in goods, (3) would represent a debt sustainability condition, as it forces the government to eventually raise the resources to pay for past borrowing.

With nominal debt, however, the government has no constraints on its choice of debt and surplus, as already pointed out. Replacing (3) on (2) and taking expectation yields

$$\frac{V_{t-1}}{P_t} = \sum_{i=0}^{\infty} E_t \left[\beta_{t,t+i-1} \left(s_{t+i} + \Delta M_{t+i}\right)\right]. \tag{4}$$

Given the face value of maturing public debt, the relative price of the consumption good is given by the expected β-discounted stream of real primary surpluses. This latter term - the right-hand side of (4) - I call the *real value of public debt.*In the case of nominal debt, (4) is a *valuation equation.* Hence, expression (3) implies not a no-default condition, but a no-bubble condition: market prices should reflect fundamental value.

[2]Strictly speaking, (1) holds for all *feasible* choices of money holdings, that is, choices that respect households' budget constraint. Otherwise, one could point out that, for $B_{t-1} > 0$, $M_t = M_{t-1}$ and $s_t = B_t = 0$ violates (1). That would nevertheless involve households burning up currency.

Now, define the inflation rate $\Pi_t = P_t/P_{t-1}$, and take innovations on both sides

$$\frac{V_{t-1}}{P_{t-1}}\Delta E_t \Pi_t^{-1} = \sum_{i=0}^{\infty} \Delta E_t \left[\beta_{t,t+i-1}\left(s_{t+i} + \Delta M_{t+i}\right)\right]. \tag{5}$$

Any shock can lead the government to change the conditional distribution of future surpluses (in units of goods) backing the stock of public nominal liabilities. The relative price of bonds in terms of goods $1/P_t$ then adjusts to reflect that change, much in the same way that news of future dividends (in currency units) per stock change the relative price of stocks in terms of currency (Cochrane (2005)). Unexpected inflation $\Delta E_t \Pi_t$ follows. Also like stocks, changes in stochastic discounting β also affect fundamental value, and thus affect prices.

Importantly, (4) and (5) do not depend to equilibrium selection mechanisms. Both hold on all models in which (3) holds, inclusing the standard New-Keynesian model.

2.2 Inflation Decomposition in the Simplest Environment

I linearize the law of motion (2).

$$\beta(v_t + s_t) = v_{t-1} + i_t - \pi_t \tag{6}$$

where v_t is *end-of-period* stock of real debt, $i_t = -\log(Q_t)$ and $\pi_t = \log(\Pi_t)$. I assume $\Delta M_t = 0$ (households do not hold currency). Note that v and s are both in levels - I assume them to be stationary for simplicity. Moreover, I linearize around the point $v = 1$, which I take to be the average real debt level.

The interpretation of (6) is the same as before. Previous period debt accrues by the mean real interest $(1/\beta)$ plus its local variation $i_t - \pi_t$. 1% more real interest raises debt by 1% on average because debt is on average one. The government redeems bonds for currency, and then sells new debt v_t and runs a surplus s_t to soak it up.

2.3 Generalizing Public Financing Instruments

2.3.1 Currency Denomination

The debt process considered so far is too unrealistic to be taken to the data. I generalize the financing instruments available to the government in two dimensions: currency denomination and maturity structure. I do not consider the case of debt with stage-contingent nominal payoffs.[3]

Starting here, I recycle part of the notation already established. Fix the case of a country's government. Let P_t be the price of the consumption basket in terms of domestic currency.

[3]Publicly-ran social security and pension systems can be modelled as cases of public debt with state-contigent payoffs or, in a fashion compatible with my framework, as an additional layer of uncertainty in the primary surplus process.

Symbol	Description	Nominal Debt	Real Debt	Dollar-Linked Debt
j	Index Symbol	N	R	D
	Notation	δ, ω	δ_R, ω_R	δ_D, ω_D
P_j	Price per Good	P	1	P_t^{US}
$\mathcal{E}_j$	Nominal Exchange Rate	1	P	Dollar NER
H_j	Real Exchange Rate	1	1	Dollar RER
π_j	Log Variation in Price	π	0	π_t^{US}
Δh_j	Log Real Depreciation	0	0	Δh_t

Notes: P = price of consumption basket in domestic currency. P^{US} = price of consumption basket in US dollars. NER = Nominal Exchange Rate. RER = Real Exchange Rate.

Table 1: Public Debt Denomination

The payoff of public bonds can be indexed to different currencies, enumerated by j. Let $P_{j,t}$ be the price of the consumer price index in units of currency j. Let $Q_{j,t}^n$ be the discount rate for a zero-coupon public bond paying one unit of currency j after n periods. Without loss of generality, all bonds are zero-coupon. Let $\mathcal{E}_{j,t}$ be the price of currency j in units of domestic currency.

The notation is general enough to accomodate currency-linked bonds *per se*, but also inflation-linked securities such as American TIPS. In this case, one can interpret the "currency" as the final goods basket (so that $P_j = 1$ and $\mathcal{E}_j = P_t$). In the empirical exercises that follow, I consider only nominal bonds ($j = N$), inflation-linked (or real) bonds ($j = R$) and US-dollar-denominated bonds ($j = D$). Table 1 shows the value or interpretation of the variables defined above for these three cases.

I do not assume the price index for government's purchases (denoted P_t^S) and levied aggregates (such as income) to be the same as the price index for households' consumption (P_t). While governments do tend to tax private consumption, they also tax and transfer income, and purchase baskets of goods different than that of households. The fiscal effect of variations in the relative price of government's to households' final goods basket must be accounted for.

With these changes, the law of motion of government debt (1) becomes

$$\sum_j \mathcal{E}_{j,t} B_{j,t-1}^1 = P_t^s S_t + \sum_j \mathcal{E}_{j,t} \sum_{n=2}^{\infty} Q_{j,t}^{n-1} \left(B_{j,t}^{n-1} - B_{j,t-1}^n \right),$$

where S_t is now the real primary surplus and $B_{j,t}^n$ is the face value of bonds issued in currency j, period t, payable n periods in the future. The term on the left represents the cost of debt in period t; the second term on the right represents proceeds from the selling of new bonds.

Let $\mathcal{V}_{j,t} = \sum_n Q_{j,t}^n B_{j,t}^n$ be the end-of-period market value of nominal debt issued in currency j, $i_{j,t}$ the risk-free rate in bonds issued in currency j and $rx_{j,t} = \mathcal{V}_{j,t-1}^{-1} \sum_{n=1}^{\infty} Q_{j,t}^{n-1} B_{j,t-1}^n - i_{j,t-1}$

the realized excess return on portfolios that mimic the composition of j-currency debt. We can re-write the law of motion in terms of the $\mathcal{V}_j$ and its corresponding returns:

$$\sum_j (1 + rx_{j,t} + i_{j,t-1})\mathcal{E}_{j,t}\mathcal{V}_{j,t-1} = P_t^s S_t + \sum_j \mathcal{E}_{j,t}\mathcal{V}_{j,t}$$

For empirical work, it is ideal to write the law of motion in terms of real and stationary variables. So, let Y_t be real GDP and let $g_t = Y_t/Y_{t-1} - 1$ be its growth rate. Define the real "exchange rate" $H_{j,t} = \mathcal{E}_{j,t}P_{j,t}/P_t$ and its growth rate $\Delta h_{j,t} = H_{j,t}/H_{j,t-1} - 1$. Define the detrended real value of j-indexed debt $V_{j,t} = \mathcal{V}_{j,t}/P_{j,t}Y_t$, the real value of total debt $V_t = \sum_j H_{j,t}V_{j,t}$ and the j-indexed share $\delta_{j,t} = H_{j,t}V_{j,t}/V_t$.

By properly dividing the whole above equation by P_tY_t, and multiplying and dividing the j sum on the left by $P_{j,t-1}$, $P_{j,t}$, Y_{t-1} and $H_{j,t-1}$, we arrive at a final version for the law of motion of public debt:

$$V_{t-1}\sum_j \frac{(1 + rx_{j,t} + i_{j,t-1})(1 + \Delta h_{j,t})}{(1 + \pi_{j,t})(1 + g_{Y,t})}\delta_{j,t-1} = \frac{P_t^s}{P_t}s_t + V_t.$$

The law of motion above generalizes (2) for $k = 1$. During period t, the government must "pay" V_{t-1} plus realized returns and eventual changes to the relative value of currency j.[4]

I linearize the debt law of motion. Let $\beta_j = \exp\{-(rx_j + i_j - \pi_j - g)\}$ be the steady-state real and growth-adjusted discounting for public debt issued in currency j. I linearize around a steady-state - assumed to exist - with $\beta_j = \beta$ for all j and $P^s = P$. This leads to

$$\beta\left(v_t + s_t + s(p_t^s - p_t)\right) = v_{t-1} + v\left[\sum_j \delta_j\left(rx_{j,t} + i_{j,t-1} + \Delta h_{j,t} - \pi_{j,t}\right) - g_t\right], \tag{7}$$

which generalizes (6). Parameter v is the steady-state level of public debt.

2.3.2 Geometric Term Structure

I assume the government keeps a geometric maturity structure of its debt. The geometric term structure implies a tractable relationship between short-term interest rate and the excess return on public bonds.

The term structure is constant over time, but can vary across the different currency portfolios $\{j\}$ of public debt. Specifically, for the slice of public debt linked to currency j, suppose the outstanding volume of bonds decays at a rate ω_j, so that $B_{j,t}^n = \omega_j B_{j,t}^{n-1}$. Define $Q_{j,t} = \sum_{n=1}^{\infty} Q_{j,t}^n \omega_j^{n-1}$ as the weighted-average price of currency j public bonds. Then, $\mathcal{V}_{j,t} = Q_{j,t}B_{j,t}^1$. The total return on currency-j bonds then is $1 + rx_{j,t} + i_{j,t-1} = (1 + \omega_j Q_{j,t})/Q_{j,t-1}$, which I linearize as

$$rx_{j,t} + i_{j,t-1} = \omega_j\beta q_{j,t} - q_{j,t-1} \tag{8}$$

[4] "Pay" comes in paranthesis here because, unlike in (1), the government does not actually redeem the entire term on the left at period t. It only pays for bonds maturing at t.

where $q = \log Q$ and I use the log approximation of level returns to effectively re-define $rx_j + i_j$.

Equation (8) above defines the excess return on holdings of the j-currency portfolio of public debt. Given a model for the risk premium $E_t rx_{j,t+1}$, it also defines the price of the debt portfolio as a function of short-term interest:

$$\begin{aligned} q_{j,t} &= \omega_j \beta E_t q_{j,t+1} - E_t rx_{j,t+1} - i_{j,t} \\ &= -\sum_{i=0}^{\infty} (\omega_j \beta)^i E_t \left[rx_{j,t+1+i} + i_{j,t+i} \right]. \end{aligned} \tag{9}$$

The second equation in (9) which clarifies the connection between short-term interest and returns on the market price of debt showing up in (7). Given news of, say, higher interest rates, the discount of public bond increases, and q falls. Equation (8) then prescribes a low excess return on j debt.

3 Empirical Model and Estimation

3.1 Public Finance Model

Governments typically report the par or book value of debt, and this is the data I gather. Because theory is based on the *market* value of debt, some adjustment is necessary. The Dallas Fed provides estimates of the market value of debt for the United States following Cox and Hirschhorn (1983) and Cox (1985).[5] I follow a similar methodology.

Let $\mathcal{V}^b_{j,t}$ be the par value of the j-currency portfolio debt, and let $i^b_{j,t}$ be its average yield-to-maturity by which book values are updated from period to period. The adjustment equation is

$$\mathcal{V}_{j,t} = \mathcal{V}^b_{j,t} \times \frac{\text{market value of bonds}}{\text{book value of bonds}} = \frac{1 + \omega_j Q_{j,t}}{(1 + i^b_{j,t-1}) Q_{j,t-1}} = \frac{1 + rx_{j,t} + i_{j,t-1}}{1 + i^b_{t-1}}.$$

I detrend the $\mathcal{V}$'s, convert to real, sum across portfolios and linearize to arrive at:

$$v_t = v^b_t + \frac{v}{b} \left[\sum_j \delta_j \left(rx_{j,t} + i_{j,t-1} - i^b_{j,t-1} \right) \right]. \tag{10}$$

Estimates of the VAR provide an equation for the law of motion of par-value debt. I use (10) to infer a law-of-motion of market-value debt.

The average interest $i^b_{j,t}$ is not observed, so we cannot estimate an equation for it. Instead, I use a model. With a geometric maturity structure of debt and in steady state, every period the government rolls over the volume of bonds coming to maturity (which had maturity $n = 1$ in the previous period). That accounts for a share $1 - \omega$ of total outstanding bonds. In a

[5]Web link: https://www.dallasfed.org/research/econdata/govdebt.

steady state, the government then issues the same amount of new bonds ($1-\omega$ of total debt) at the prevailing interest rate i_t. The average interest therefore satisfies

$$i^b_{j,t} = (1-\omega)i_{j,t} + \omega_j i^b_{j,t-1} = (1-\omega_j)\sum_{k=0}^{\infty}\omega_j^k i_{j,t-k} \tag{11}$$

for $j \in \{N, R, D\}$.

3.2 The Bayesian-VAR

3.2.1 Empirical Model

I gather data for a set of twenty-eight economies, and estimate a ten-equation VAR in which the debt law of motion (7) holds by construction. From the ten variables in the VAR, five are observed: the nominal interest (i_t), the inflation rate (π_t), par-value public debt (v^b), the real exchange rate to the dollar (Δh) and GDP growth (g). I select these variables based on (7).

I use annual data. Although quarterly data is available, for many developing countries it does not go back many years (especially public debt data). With a focus on long-term debt sustainability, going further into the past is more important than accounting for quarterly dynamics. Additionally, with annual data there is no danger of measurement errors due to seasonality adjustments.

Data period range changes from country to country (table), with the longest panel starting in 1970 (United States), and the shortest beginning in 2000 (Russia). For some emerging economies, I arbitrarily trim data to remove periods of hyperinflation, always in the 1990s.

Inflation is the log variation in the consumer price index. The dollar real exchange rate is the nominal exchange rate to the US dollar multiplied by the ratio of US-to-domestic CPI. The nominal interest rate is the log of 1+ interest data. GDP growth is in log too. Public debt data is provided by ratio of GDP by the source, and requires no transformation.[6]

With the exception of exchange rate depreciation, I demean all series. This is to assume that the stationary steady-state (around which I linearize) is given by the sample mean of the series of each country (with the exception of real exchange rate growth, which I assume to be zero).

The functional format of the VAR is

$$x_t = ax_{t-1} + bu_{t-1} + k\, e_t. \tag{12}$$

Both x and u are vectors with ten entries. Five of them are the observed variables enumerated above.

[6]Real GDP (constant 2015 prices), GDP deflator, public spending and the nominal exchange rate data come from the United Nations's National Accounts Main Aggregates Database. Consumer price index and and primary surplus data come from the IMF's WEO Database. Public debt (as ratio of GDP) comes Ali Abbas et al. (2011) database, which is kept up-to-date. The sources for interest rate vary from country to country; they are usually the central bank, but also from the IMF's International Financial Statistics database. Appendix B provides further details.

Vector u_t groups the same set of variables as x, but for the United States. I often use the "u" notation to refer to the US case. Because the public debt process of each country has a dollar component, and hence depends on dollar interest and inflation, u and ε_u enter the regression of all countries.

There are five shocks hitting the domestic economy: $\varepsilon \sim N(0, \Sigma)$, independent over time. The shocks hitting the US economy are $\varepsilon_u \sim N(0, \Sigma_u)$. I group them in $e_t = [\varepsilon_t' \; \varepsilon_{u,t}']'$. Matrix $k_{10\times10}$ serves to properly reproduce the law of motion of unobserved variables.

The model for US variables is:

$$u_t = a_u u_{t-1} + k_u \, \varepsilon_{u,t} \tag{13}$$

(I use the same notation x to the VAR of all countries and differentiate only in the US case). In (13), k_u is a 10×5 matrix.

3.2.2 Estimation Procedure

Cochrane (2022) imposes the law of motion of public debt in the VAR by using primary surplus "data" calculated from it. Primary deficit equals the change in public debt net of realized return. This procedure bypasses the problem of dealing with accounting conventions in primary surplus reporting that, in most cases, make its data inconsistent with that of public debt. Since the OLS estimator is linear, his VAR satisfies the debt law of motion by construction.

In my VAR, the procedure needs to be adjusted. First, I do not measure excess return, or real interest rates. Second, the debt equation depends on US interest and inflation. Hence, the unrestrictive estimation of (12) spuriously projects these two US variables on domestic ones, which is inconsistent with (13). For these reasons, instead of measuring implied surpluses and then estimating the model, I estimate the model and then fill the equation for primary surpluses that ensures the debt law of motion (7) holds. Before doing that, I also need to include the adjustment equation for market-value debt (10) (the estimated equation is for par-value, not market-value debt!) as well as the three definitions of average interest rates (11) required to do it. These five unobserved variables (surplus s_t, market-value debt v_t, and the average interest $i^b_{j,t}$) complete the ten variables of the VAR.

Note that the estimated equation for par-value public debt represents its law of motion *after* replacing the equation determining tax proceeds, or its equilibrium law of motion.

The estimation has four steps.

Step 1. I estimate the VAR

$$\begin{aligned} \tilde{x}_t &= \tilde{a}\tilde{x}_{t-1} + \tilde{b}\tilde{u}_{t-1} + \varepsilon_t \\ \tilde{u}_t &= \tilde{a}_u \tilde{u}_{t-1} + \varepsilon_{u,t} \end{aligned} \tag{14}$$

where $\tilde{x}$ is a vector with the five observed variables, and $\tilde{u}$ is defined similarly. Matrices $\tilde{a}$, $\tilde{b}$ and $\tilde{a}_u$ are submatrices of a, b and a_u, with the rows and columns of these observed variables.

I also estimate $\text{cov}(\varepsilon) = \Sigma$ and $\text{cov}(\varepsilon_u) = \Sigma_u$.

Step 2. I assume a constant risk-premium: $E_t rx_{j,t+1} = 0$ for all j. I use the already estimated VAR (14) to compute $E_t i_{j,t+i}$ and apply (9) to compute $q_{j,t}$. Equation (8) then yields expressions for excess return of the form

$$rx_{j,t} = \varphi_j' e_t.$$

An equation for real debt is also necessary. I use

$$i_{R,t} = i_t - E_t \pi_{t+1} = \zeta' X_t.$$

where $X_t = [x_t' \ u_t']'$ stacks domestic and US variables. In the United States case, $X_t = u_t$. In appendix C, I present the formulas of the φ's and of ζ.

Step 3. Using the estimated model of step 1, I compute the equations for average interest using (11), and fill the corresponding rows of a, b and k (a_u and k_u in the US case). With the equations for average interest filled, I can do the same for the market-price debt using the par-value adjustment equation (10). With the equation for the market-price debt, I use the law of motion (7) to fill the equation row for the primary surplus.

This completes the estimation of a, b and k in the general case, a_u and k_u in the US case. For each country, we can stack the equations into a single system for X:

$$X_t = A(L) X_{t-1} + K e_t. \tag{15}$$

If we order unobserved variables x^o at the top of the x, we can write (15) more explicitly:

$$\begin{pmatrix} x_t \\ u_t \end{pmatrix} = \begin{bmatrix} a & b \\ 0 & a_u \end{bmatrix} \begin{pmatrix} x_{t-1} \\ u_{t-1} \end{pmatrix} + \begin{bmatrix} & k \\ 0 & k_u \end{bmatrix} \begin{pmatrix} \varepsilon_t \\ \varepsilon_{u,t} \end{pmatrix}$$

$$\text{or yet} \begin{pmatrix} x_t^o \\ \tilde{x}_t \\ u_t^o \\ \tilde{u}_t \end{pmatrix} = \begin{bmatrix} & & * & \\ 0 & \tilde{a} & 0 & \tilde{b} \\ 0 & 0 & & * \\ 0 & 0 & 0 & \tilde{a}_u \end{bmatrix} \begin{pmatrix} x_{t-1}^o \\ \tilde{x}_{t-1} \\ u_{t-1}^o \\ \tilde{u}_{t-1} \end{pmatrix} + \begin{bmatrix} & * \\ I & 0 \\ 0 & * \\ 0 & I \end{bmatrix} \begin{pmatrix} \varepsilon_t \\ \varepsilon_{u,t} \end{pmatrix}.$$

Symbol * indicates the coefficients are filled to ensure that (7), (10) and (11) hold. In appendix C I provide their formulas.

Step 4. I compute sample residuals $\hat{e}$ ($\hat{e}_u$ for the US) from (14), and estimate $\text{cov}(\varepsilon, \varepsilon_u) = \Sigma_{xu} = \sum_i \hat{e}_i \hat{e}_{u,i}/(N-1)$, where N is the sample size. Then:

$$\Omega = \text{cov}(e) = \begin{bmatrix} \Sigma & \Sigma_{xu} \\ \Sigma_{ux} & \Sigma_u \end{bmatrix}.$$

3.3 The Minnesota Prior

I interpret model parameters $\tilde{a}$ and $\tilde{b}$ as being random, and estimate (14) by esblishing a prior distribution, and then using data likelihood to compute the posterior.

Estimating the model using Bayesian methods has two advantages. First, parameter shrinkage reduces the volatility of estimated coefficients and over-fitting, an invaluable feature when samples contain 20 to 50 observations.[7] Second, the fiscal policy literature estimates highly persistent public debt processes (Bohn (1998), Uctum et al. (2006), Yoon (2012)); in the time period I analyze (1970-2019) the sovereign debt of many economies increased significantly. For these reasons, OLS estimates often estimate explosive dynamics, which is inconsistent with the assumption of debt sustainability. By properly choosing the parameters of the prior distribution, we can ensure stability at the same time we search for parameters that provide the best fit.

I use an adapted version of the Litterman (1979) (or Minnesota) prior. The Minnesota prior formalizes the view that the variables of interest follow a random walk, $x_t = c + x_{t-1} +$ shock, or a white noise $(1 - L)x_t = c+$ shock if differenced.[8] I assume variables of the VAR to be $I(0)$, and adopt a white noise version of the prior.

The prior distribution centers matrices $\tilde{a}$ and $\tilde{b}$ around zero, which implies stable dynamics. Since surpluses are inferred from $\tilde{a}$ and $\tilde{b}$, and since other variables do not respond to debt (all coefficients are zero), the economic content of the prior is that primary surpluses adjust to stabilize debt. I regard that as a plausible view for a prior on fiscal sustainability.

The prior is of the Normal-Inverse-Wishart distribution family, with general format

$$\begin{aligned} \Sigma &\sim IW(\Phi; d) \\ \theta|\Sigma &\sim N(\bar{\theta}, \Sigma \otimes \Omega). \end{aligned}$$

where $\theta = [\text{vec}(\tilde{a}')' \ \text{vec}(\tilde{b}')']'$ and vec means stacking the columns.

The mean of the IW distribution is $\Phi/(d - n - 1)$, where $n = 5$ is the dimension of ε and larger values of d represent tighter priors. I choose Φ to be the identity matrix (one percent standard deviation for all shocks, which are uncorrelated) and select $d = n + 2$, the lowest integer possible that leads to a well-defined distribution mean - which therefore equals Φ.

The prior for $\tilde{a}$ and $\tilde{b}$ is centered around zero. Therefore, $\bar{\theta} = 0$. The conditional covariance between the coefficients in $\tilde{a}$ is

$$\text{cov}\left(\tilde{a}_{ij}, \tilde{a}_{kl} \mid \Sigma\right) = \begin{cases} \lambda^2 \frac{\Sigma_{ij}}{\Phi_{jj}} & \text{if } j = l \\ 0 & \text{otherwise.} \end{cases}$$

The format allows the loadings on the variable in different equations to be correlated. Loadings

[7] Giannone et al. (2015) show that priors of the Normal-Inverse-Wishart family, such as the Minnesota prior, lead to posterior distributions that can be decomposed as posterior = model fit term + expectation volatility term.

[8] The literature about the Minnesota prior is vast. Interested readers can see del Negro and Schorfheide (2011) or Karlsson (2013) for a survey-like approach.

on the different variables on the same equation are independent. Hyperparameter λ governs the overall tightness of the prior over linear coefficients, with lower values yielding tighter priors.

The conditional mean of $\tilde{b}$ is zero. Its conditional covariance is

$$\operatorname{cov}\left(\tilde{b}_{ij}, \tilde{b}_{kl} \mid \Sigma\right) = \begin{cases} (\xi\lambda)^2 \frac{\Sigma_{ij}}{\Phi_{u,jj}} & \text{if } j = l \\ 0 & \text{otherwise} \end{cases}$$

where $\Phi_u = \Phi = I$ is the mean of the IW distribution in the US case. Hyperparameter ξ governs the tightness of the prior that US variables do not affect domestic ones more than the mechanical effect of US interest and inflation on domestic public debt. If $\xi = 1$, the prior is just as tight as that of $\tilde{a}$.

Finally, the covariance between $\tilde{a}$ and $\tilde{b}$ is zero.

It is straightforward to set Ω so that the conditional covariance structures above hold.

4 Empirical Results

4.1 Covariance Decomposition

In the baseline specification, I calibrate $\beta = 0.98$ for all countries, and set b tightness parameter $\xi = 1/3$. I calibrate parameters δ and ω based on debt structure data gather from various sources (see appendix B). They are reported in Table 2 along with average debt.

The mean debt-to-GDP ratio in the sample was 0.50, with developed countries slightly more indebted on average. Nominal debt tends to account for the bulk of sovereign debt, Chile being a notable exception. Emerging markets' governments tend to rely relatively more on real and especially foreign debt, and issue securities with higher maturity, on average.

Priors of the Normal-Inverse-Wishart class are conjugate and admit closed-form solutions for both the posterior distribution and the marginal likelihood. I start by setting λ so as to maximize the marginal likelihood.[9] Then, if the mode of the posterior leads to an unstable VAR, I progressively reduce λ in 0.01 steps until it leads to a stable VAR. You can check the resulting λ's in table 2.

[9]Giannone et al. (2015) show that, in the case of Normal-Inverse-Wishart priors, the marginal likelihood can be decomposed in a goodness-of-fit term and a model-complexity term that penalizes conditional forecast variance. By maximizing the marginal likelihood, we ensure we cannot improve one of these terms without reducing the other.

Country	v (%)	δ_N (%)	δ_R (%)	δ_D (%)	Avg. Term (Years)	λ
Aggregates						
Median	46	78	4	11	5.6	
Average	51	74	10	16	6.6	
Advanced	58	86	6	8	5.8	
Developing	45	64	13	23	7.2	
Advanced						
Australia	24	90	10	0	7.2	0.20
Canada	79	92	5	3	6.5	0.18
Denmark	50	84	0	16	5.6	0.24
Germany (Euro)	68	99	1	0	6	0.42
Iceland	83	37	12	51	4	0.06
Japan	139	100	0	0	5.5	0.20
New Zealand	34	82	6	13	4.3	0.22
Norway	38	99	0	1	3.7	0.20
Republic of Korea	24	97	0	3	4	0.23
Sweden	57	69	16	14	4.8	0.13
Switzerland	47	100	0	0	6.9	0.24
United Kingdom	56	76	24	0	12.3	0.17
United States	60	93	7	0	5	0.04
Developing						
Brazil	68	70	25	5	2.6	0.14
Chile	14	10	57	33	12.8	0.20
Colombia	39	45	23	32	5.6	0.12
Czech Republic	30	91	0	9	5.6	0.14
Hungary	68	76	0	23	4.1	0.19
India	72	90	3	7	10.1	0.17
Indonesia	40	44	0	56	9.2	0.30
Israel	80	43	34	23	6.6	0.15
Mexico	46	65	10	26	5.5	0.13
Poland	48	79	1	20	4.2	0.19
Romania	28	50	0	50	4.8	0.11
Russia	18	76	2	22	10.9	0.13
South Africa	41	70	20	10	12.9	0.23
Turkey	40	47	23	30	3.6	0.09
Ukraine	43	100	0	0	9.1	0.06

Notes: v is the average public debt-to-GDP, δ_N is the share of nominal debt, δ_R is the share of real or inflation-linked debt, δ_D is the share of dollar-denominated debt (I also include debt denominated in other currencies, but the amounts are negligible in almost all cases). "Avg. Term" is the average maturity across currency portfolios, calculated as $\sum_j \delta_j(1-\omega_j)^{-1}$. λ is the prior tightness parameter.

Table 2: Debt Structure Parameters and Prior Tightness

Country	$\sigma(\Delta E_t \pi_t)$ (%)	$p_\pi(r_0)$ (%)	$p_\pi(rx)$ (%)	$p_\pi(r)$ (%)	$-p_\pi(s)$ (%)
Aggregates					
Median	1.31	**-0.05**	**-0.88**	1.11	1.02
Average	1.55	**-0.05**	**-0.89**	1.45	0.49
Advanced	0.96	0.33	**-0.77**	1.38	0.06
Developing	2.05	-0.37	**-0.99**	1.51	0.85
Advanced					
Australia	1.31	**-0.06**	**-1.28**	1.02	1.32
Canada	0.80	**-0.14**	**-0.90**	0.94	1.11
Denmark	0.81	**-0.78**	*-0.90*	1.68	1.00
Germany (Euro)	0.43	**-0.03**	-1.46	-0.44	2.93
Iceland	1.83	**5.26**	*0.12*	17.08	-21.5
Japan	0.61	0	**-1.40**	2.13	0.27
New Zealand	0.66	*-0.34*	**-0.28**	-1.75	3.37
Norway	0.99	**-0.01**	**-0.47**	-1.12	2.59
Republic of Korea	0.84	**0.12**	**-0.63**	-1.82	3.33
Sweden	1.17	0.13	**-0.61**	1.09	0.39
Switzerland	0.56	0.00	**-0.86**	1.68	0.17
United Kingdom	0.88	0.12	**-0.45**	-4.64	5.98
United States	1.66	**-0.04**	**-0.85**	2.06	-0.16
Developing					
Brazil	1.43	**-0.30**	**-0.18**	1.50	-0.02
Chile	1.11	*-5.36*	**-0.66**	3.50	3.52
Colombia	1.35	-0.68	**-2.63**	5.66	-1.35
Czech Republic	1.30	*-0.15*	**-0.94**	1.27	0.82
Hungary	1.30	**-0.65**	**-0.93**	0.98	1.60
India	1.59	0.05	**-0.67**	-0.11	1.74
Indonesia	3.59	**1.89**	*-0.22*	-1.22	0.55
Israel	1.44	**-0.88**	**-1.13**	1.13	1.87
Mexico	1.33	-0.08	**-1.07**	0.24	1.91
Poland	1.20	-0.06	**-1.37**	2.90	-0.47
Romania	2.05	-0.55	**-1.05**	-0.81	3.41
Russia	1.92	**0.84**	**-0.52**	1.95	-1.27
South Africa	1.33	-0.08	**-1.71**	3.19	-0.40
Turkey	4.19	**0.40**	**-0.82**	1.56	-0.15
Ukraine	5.68	**0.00**	**-0.95**	0.91	1.04

Notes: v is the average public debt-to-GDP, δ_N is the share of nominal debt, δ_R is the share of real or inflation-linked debt, δ_D is the share of dollar-denominated debt (I also include debt denominated in other currencies, but the amounts are negligible in almost all cases). "Avg. Term" is the average maturity across currency portfolios, calculated as $\sum_j \delta_j (1-\omega_j)^{-1}$. λ is the prior tightness parameter.

Table 3: Unexpected Inflation Volatility Decomposition

Country	$-p_\pi(\pi)$ (%)	$p_\pi(s)$ (%)	$p_\pi(r)$ (%)	$p_\pi(\Delta h)$ (%)	$-p_\pi(\pi^{US})$ (%)
Aggregates					
Median	-0.54	1.02	0.54	0.00	-0.01
Average	-0.48	0.49	0.74	0.34	-0.09
Advanced	-0.43	0.06	1.09	0.25	0.03
Developing	-0.53	0.85	0.44	0.41	-0.18
Advanced					
Australia	-0.75	1.32	0.43	0.00	0.00
Canada	-0.62	1.11	0.66	-0.11	-0.03
Denmark	-0.47	1.00	1.30	-0.70	-0.14
Germany (Euro)	-0.53	2.93	-1.37	-0.01	-0.01
Iceland	-0.25	-21.46	17.96	4.16	0.59
Japan	-0.73	0.27	1.46	0.00	0.00
New Zealand	-0.01	3.37	-1.97	-0.34	-0.05
Norway	-0.54	2.59	-1.05	-0.01	0.00
Republic of Korea	0.13	3.33	-2.54	0.09	0.00
Sweden	-0.31	0.39	0.72	0.23	0.02
Switzerland	-0.35	0.17	1.18	0.00	0.00
United Kingdom	-0.82	5.98	-4.16	0.00	0.00
United States	-0.34	-0.16	1.51	0.00	0.00
Developing					
Brazil	-0.05	-0.02	0.99	0.08	0.00
Chile	-0.09	3.52	-5.72	5.27	-1.98
Colombia	-1.31	-1.35	2.96	0.81	-0.11
Czech Republic	-0.13	0.82	0.33	0.00	-0.01
Hungary	-0.67	1.60	1.11	-0.92	-0.12
India	-0.84	1.74	0.04	0.01	0.05
Indonesia	0.44	0.55	0.07	-0.06	0.00
Israel	-0.63	1.87	0.13	-0.26	-0.12
Mexico	-0.74	1.91	0.21	-0.37	-0.01
Poland	-0.74	-0.47	2.23	0.03	-0.05
Romania	-1.17	3.41	-1.24	0.49	-0.50
Russia	0.00	-1.27	1.41	0.66	0.20
South Africa	-0.83	-0.40	2.08	0.20	-0.05
Turkey	-0.86	-0.15	1.79	0.27	-0.05
Ukraine	-0.30	1.04	0.26	0.00	0.0

Notes: v is the average public debt-to-GDP, δ_N is the share of nominal debt, δ_R is the share of real or inflation-linked debt, δ_D is the share of dollar-denominated debt (I also include debt denominated in other currencies, but the amounts are negligible in almost all cases). "Avg. Term" is the average maturity across currency portfolios, calculated as $\sum_j \delta_j(1-\omega_j)^{-1}$. λ is the prior tightness parameter.

Table 4: Unexpected Inflation Volatility Decomposition

4.2 Response to Reduced-Form Shocks

5 New-Keynesian Model Benchmarks

6 Robustness

7 Conclusion

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A Linearization

B Data Sources and Treatment

C Additional Details of the BVAR Estimation

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E Deriving the SOE-NK Model