Convert Book to Market Value Debt

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Abstract

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1. Introduction

2. A General Budget Constraint

I start by assuming that the government does not issue bonds that pay coupons. This simplification is economically inconsequential. If the government owes one dollar due tomorrow, it makes no difference if it calls this one dollar "principal" or a "coupon". In addition, working with zero-coupon bonds simplifies the expressions significantly, and indeed it has become a common practice in the macroeconomic literature. So I start by looking at a general budget constraint involving only zero-coupon bonds and then I consider the existence of coupons. Suppose each bond pays one unit of currency (or "dollar", for brevity) in a given future date and nothing more. The difference between the current period and this future date equals the bond's maturity. Let $\mathcal{B}_{n,t}$ be the number of outstanding bonds in period t with maturity n. If the government does not issue or redeem these bonds before they expire, it will need to pay $\mathcal{B}_{n,t}$ dollars in period $t+n$. For now, I consider only the existence of nominal public debt.

In period t, the government must come up with $\mathcal{B}_{1,t-1}$ dollars to pay that amount to holders of maturing debt. It can accomplish that by selling new bonds, running a primary surplus or simply issuing new currency. The notation is: M_t is the amount of dollars at the hands of households at the end of period t, $\mathcal{S}_t^*$ is the nominal value of the primary surplus and $Q_{n,t}$ is the market price of a bond with maturity n. Because the distinction between revenue from primary surpluses and seignorage is not relevant for this paper, and because reported public debt does not include outstanding currency, I further simplify and define $\mathcal{S}_t = \mathcal{S}_t^* + \Delta M_t/P_t$ as the seignorage-adjusted primary surplus, which I will just refer to as primary surplus. The budget constraint faced by the government is the following:

$$\sum_{n=1}^{\infty} \left(\mathcal{B}_{n,t} - \mathcal{B}_{n+1,t-1}\right) Q_{n,t} + \mathcal{S}_t = \mathcal{B}_{1,t-1}.$$

(Note that an $n+1$-maturity bond in $t-1$ becomes an n-maturity bond in

t.) We can re-arrange that equation and re-write it as

$$\mathcal{V}_t + \mathcal{S}_t = (1 + r_t^n)\mathcal{V}_{t-1},$$

where

$$\mathcal{V}_t = \sum_{n=1}^{\infty} Q_{n,t}\mathcal{B}_{n,t} \qquad \text{and} \qquad 1 + r_t^n = \frac{\sum_{n=1}^{\infty} Q_{n-1,t}\mathcal{V}_{n,t-1}}{\sum_{n=1}^{\infty} Q_{n,t-1}\mathcal{V}_{n,t-1}}$$

are, respectively, the end-of-period *market value* of public debt and the nominal return on holdings of the basket of public bonds. Next, we convert nominal into real variables, and detrend to make them stationary. To do this, define P_t as the price of the basket of goods in terms of currency (that is, the price level), and Y_t as real GDP (or any variable that plausibly renders public debt stationary). Let $B_{n,t} \equiv \mathcal{B}_{n,t}/P_tY_t$, and define $V_{n,t}$, V_t and S_t similarly. Now, V_t is debt-to-GDP and S_t is the surplus-to-GDP. Our final budget constraint for the government is

$$V_t + S_t = \frac{1 + r_t^n}{(1 + \pi_t)(1 + g_t)} V_{t-1} = \begin{matrix}\text{Beginning-of-period real} \\ \text{market value of public debt,}\end{matrix} \tag{1}$$

where $1 + \pi_t = P_t/P_{t-1}$ is the inflation rate and $1 + g_t = Y_t/Y_{t-1}$ is the GDP growth rate.

The importance of the market value of public debt is that in most models it corresponds to the discounted sum of expected future primary surpluses. It is therefore informative about households' expectation of future fiscal policy (as well as discount rates) much in the same way that firm value is informative of future dividends (Cochrane (2005)). Market value of debt = discounted surpluses is *not* a condition particular to fiscal theory of the price level models; the proposition is far more general. Indeed, let $m_{t,t+j}$ be a stochastic discount factors (assume no arbitrage; a discount factor therefore exists). We can replace the pricing condition $Q_{n,t}/P_t = E_t m_{t,t+1} Q_{n-1,t+1}/P_{t+1}$ inside the definition of V_t in equation (1) and solve it forward to find that the beginning-of-period market value of public debt (the right-hand side of

(1)) equals discounted surpluses

$$\sum_{j=0}^{\infty} E_t \left[m_{t,t+j} S_{t+j} \right].$$

(This result depends on $E_t \left[m_{t,t+n} V_{t+n} \right]$ converging to zero as $n \to \infty$, which is guaranteed by households' transversality condition when $m =$ marginal utility growth. Otherwise, the convergence is a separate assumption. See Bohn (1995).)

The key motivation for this paper is that V_t is *not* the quantity reported in public finance statistics. Instead, governments usually report the *book value* (sometimes called the "par" value) of public debt, which is simply the sum of outstanding bonds' principal payments. Coupons are considered "interest" and do not enter the statistic. Of course, from the economist's perspective, coupons and the interest rate are two highly different concepts. Additionally, the book value does not take into account variation in the price of existing bonds $Q_{n,t}$. For these reasons, the book value of public debt cannot be considered (at least in theory) as precise of a measure of expected future surpluses as its market value counterpart.

3. Coupons and Book Value

We now consider the case of a government that issues bonds that pay coupons plus a principal payment (or face value) upon maturity. Because it is always easier to work with the zero-coupon structure of the previous section, we start with a sequence of zero-coupon payments $\{B_{n,t}\}$ and ask how we can replicate it using principal and coupon installments given a rule for how the government determines coupon rates. To avoid dwelling into functional analysis, I assume there is an N such that $B_{n,t} = 0$ for $n > N$. If $B_{n,t} \to 0$ as $n \to \infty$ uniformly in time (in a model or in reality), we can pick a large N to get an arbitrarily small error.

The notation is: $\mathcal{A}_{n,t}$ is the sum of principal payments promised by bonds of maturity n, $\Delta\mathcal{A}_{n,t}$ is the sum of principals of new n-maturity bonds, and $c_{n,t}$ is the coupon rate of new bonds. Coupons are constant over payment

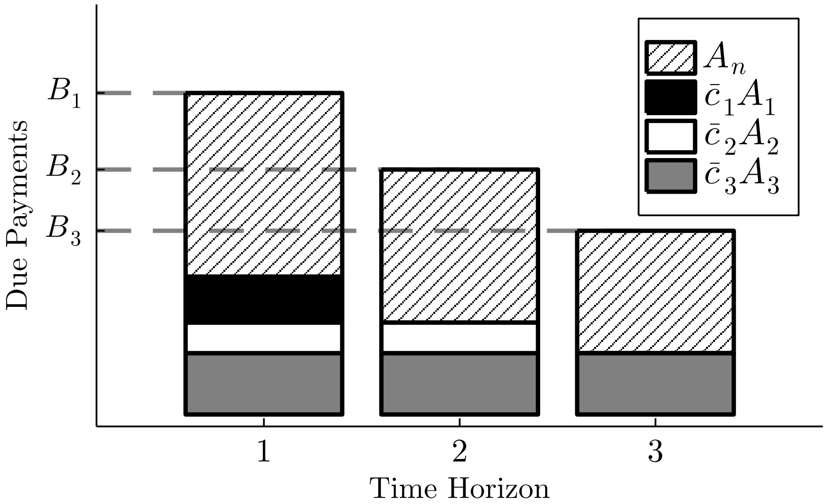


Figure 1: Example of Coupon + Principal Structure

horizons. For example, a one-dollar bond with maturity $n = 2$ promisses $c_{2,t}$ dollars after one period (coupon) and $1 + c_{2,t}$ dollars after two periods (principal + coupon). The point is that the $c_{2,t}$ is the same. We do not need to keep track of coupon rates of bonds issued in the past. More conveniently, I call $\bar{c}_{n,t}$ the *average* coupon rate of bonds with maturity n: the government must pay $\bar{c}_{n,t}\mathcal{A}_{n,t}$ in each period from $t+1$ to $t+n$ corresponding to coupons from all outstanding bonds with maturity n.

Like before, $A_{n,t} \equiv \mathcal{A}_{n,t}/P_tY_t$ and the same for $\Delta A_{n,t}$. Denominators don't matter much, therefore I work directly with normalized variables A and ΔA. If that bothers you, you can just set $\pi_t = g_t = 0$ in what follows and treat A, B, etc as nominal variables in levels instead of GDP ratios. Figure 1 depicts an $N = 3$ example of public debt payment structure. The constraint that bonds pay a constant stream of coupons implies that the sizes of the bars representing their corresponding coupon payments are the same in all horizons. The book value of public debt is the sum of principals:

$$A_t = \sum_{n=1}^{N} A_{n,t},$$

or the sum of hatchet bars in the figure. It is also convenient to define the ratio of market-to-book value k_t:

$$k_t = \frac{V_t}{A_t}.$$

As figure 1 illlustrates, the total payment due after n periods comprises the face value of bonds of maturity n, their coupons, plus the coupons from bonds with maturity superior to n. To replicate the zero-coupon payment structure, we therefore need:

$$B_{n,t} = A_{n,t} + \sum_{j=n}^{N} \bar{c}_{j,t} A_{j,t}. \tag{2}$$

Equation (2) establishes the connection between the volume of outstanding bonds in each formulation (with and without coupons). But we are ultimately interested in the connection between the book value and the market value of public debt, k_t. The definition of debt at market value involves multiplication of each payment by a discount price $Q_{n,t}$ that is usually lower than one. The definition of book value does not. This tends to make the market value of debt *smaller* than the book value. On the other hand, the book value ignores coupon payments, whereas the market value does not. This tends to make the market value *greater* than the book value. There should be a benchmark upon which these two forces cancel out.

Definition 1 (Par Value)**:** The coupon rate schedule $\{c_{n,t}\}$ is at par if, for every n:

$$Q_{n,t} + \sum_{j=1}^{n} Q_{j,t} c_{j,t} = 1. \tag{3}$$

Proposition: If the average coupon schedule $\{\bar{c}_{n,t}\}$ is at par, the market and book values of public debt coincide: $V_t = A_t$ (and $k_t = 1$).

A coupon rate schedule is at par when the market prices of bonds promising that schedule coincide with their face values. Indeed, note that the left side of (3) is the sum of the market prices of a one-dollar principal ($Q_{n,t} \times 1$) and associated coupons ($Q_{n,t} \times c_{n,t} \times 1$). The definition asks this sum to be equal to the principal (1). The proposition is in that sense intuitive: if the market price of the average bond (average in terms of coupon rates) equals its principal, then the market value of debt (which adds up the former) equals the book value (which adds up the latter). To prove it, replace (2) in the definition of the market value of debt.

$$V_t = \sum_{n=1}^{N} Q_{n,t} B_{n,t} = \sum_{n=1}^{N} D_{n,t} A_{n,t}, \tag{4}$$

where

$$D_{n,t} = Q_{n,t} + \sum_{j=1}^{N} Q_{j,t} \bar{c}_{j,t}$$

is the market value of a bond that promises the average coupon rate schedule $\{\bar{c}_{n,t}\}$. If $\{\bar{c}_{n,t}\}$ is at par, $D_{n,t} = 1$ for every n, and therefore $V_t = A_t$.

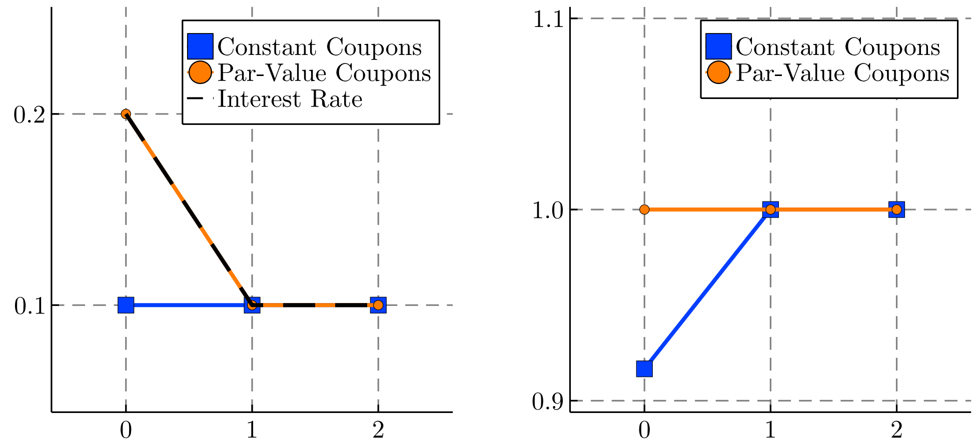
3.1. Example: One-Period Debt

Let $1 + i_t = 1/Q_{1,t}$ be the economy's nominal interest rate. We consider an $N = 1$ example. The government issues only one-period debt. Suppose it wishes to replicate a series of zero-coupon paying $B_{1,t}$ by issuing coupon-paying bonds with coupon rates $c_{1,t}$. In period t, the government repays the principal and coupon of bonds sold in $t-1$, and issues new bonds maturing in $t+1$. It rolls over public debt entirely every period. The face value of the new bonds is $A_{1,t}$ and the volume of coupons is $c_{1,t}A_{1,t}$. Therefore, the book value of public debt is $A_t = A_{1,t}$ and the average coupon rate is $\bar{c}_{1,t} = c_{1,t}$. In particular, the book value satisfies

$$V_t = Q_{1,t} B_{1,t} = \frac{1 + c_{1,t}}{1 + i_t} A_t.$$

Therefore, the market value of public debt is greater than the book value when $i_t < c_{1,t}$. Importantly, the market-to-value ratio $k_t = (1 + c_{1,t})/(1 + i_t)$ does no depend on the size of public debt - this changes when $N > 1$.

As an example, consider a steady-state equilibrium in which the government sells bonds at par, with coupon rate = interest rate = 0.1. Then, a temporary monetary policy shock hits: interest rate jumps to 0.2 in period zero, and returns to 0.1 subsequently. Figure 2 depicts two limit cases regarding coupon rate policy. In the first case (blue squares), the government keeps the coupon rate constant despite the increased interest. In period zero,



(a) Coupon Rates ($\bar{c}_{1,t} = c_{1,t}$) (b) Market-to-Book Ratio (k_t)

Figure 2: Monetary Policy Shock with One-Period Debt

one-period bonds supplied to the market do not sell at par. Instead, their prices decline relative to face value. Since public debt consists only of these one-period bonds, the market value of debt V_t declines relative to book value A_t, $k_t < 1$ (see panel 2b). In the second case (orange circles), the government raises the coupon rate one-to-one with the interest change (see panel 2a). Higher coupons prevent the price of one-period bonds from declining in spite of the increase in the discount rate. Hence, the market-to-book ratio k_t remains unchanged.

3.2. The General Case

The main difficulty that arises when $N > 1$ is that the average coupon rate $\bar{c}_{n,t}$ does not coincide with those offered by new bonds $c_{n,t}$. Instead, the average coupon rate changes over time, as the government redeems old bonds and sells new ones. We look at how these new bonds affect average coupons. In period t, the government inherits a commitment to pay $\bar{c}_{n+1,t-1}\mathcal{A}_{n+1,t-1}$ dollars in $t+n$. We must add to that the payments promised by newly-issued bonds $c_{n,t}\Delta\mathcal{A}_{n,t}$. Adding up and normalizing denominators for GDP growth and inflation yields:

$$\bar{c}_{n,t}A_{n,t} = \bar{c}_{n+1,t-1}\frac{A_{n+1,t-1}}{(1+\pi_t)(1+g_t)} + c_{n,t}\Delta A_{n,t}. \tag{5}$$

The sum of principals of n-maturity bonds adds up the face value of past debt and that of new issues:

$$A_{n,t} = \frac{A_{n+1,t-1}}{(1+\pi_t)(1+g_t)} + \Delta A_{n,t}. \tag{6}$$

In all, the average coupon rate is the weighted combination of the average inherited from $t-1$ with the coupon rate offered by new bonds. The government can also redeem bonds, $\Delta A_{n,t} < 0$. In that case, we take the average coupon rate among the set of retired bonds to be $c_{n,t}$ - this avoids introducing a nonlinear law of motion.

The point is: the key difference between $N = 1$ and $N > 1$ is that when $N > 1$ the evolution of public debt matters to determine the average coupon rate and, by (4), the ratio of market-to-book values k_t. We must therefore pay greater attention to V_t and $B_{n,t}$. I again consider the example of a monetary policy shock. Suppose the interest rate follows an AR(1)

$$i_t - i^* = \rho_i(i_{t-1} - i^*) + \epsilon_t.$$

The government adopts a surplus policy rule that responds to V_t (otherwise public debt spirals away):

$$S_t = S_t^* + \alpha\left(V_t - V^*\right).$$

In this example, I set $N = 2$ and assume the government maintains $B_{2,t} = 0.5 \times B_{1,t}$. I also ignore inflation $\pi_t = 0$ for clarity. Bond prices respect the expectations hypothesis in levels.[1]

[1] That means that the forward rate is determined by $f_{n,t} = E_t i_{t+n-1}$, and bond prices follow $Q_{n,t} = Q_{n-1}/(1+f_{n,t})$ with $Q_{0,t} = 1$.

4. Algorithms

4.1. Computing Coupon Structures

4.2. Estimating the Market Value of Public Debt

5. The US Case

6. Multiple Currencies

7. Concluding Remarks

Market value of debt:

$$V_t = \sum_{n=1}^{N} Q_{n,t} B_{n,t}$$

Flow equation for the market value of public debt:

$$V_{t-1} + S_{t-1} = \frac{(1 + r_t^n)}{(1 + \pi_t)(1 + g_t)} V_t$$

Definition of B_n:

$$B_{n,t} = A_{n,t} + \sum_{j=n}^{N} \bar{c}_{j,t} A_{j,t} = \chi_{n,t} + \sum_{j=n+1}^{N} \bar{c}_{j,t} A_{j,t}$$

In the iteration for maturity n, equation above determines $\chi_{n,t} \equiv (1+\bar{c}_{n,t})A_{n,t}$, which is the amount due in n period in repayment of maturing bonds only.

Definition $\bar{c}$:

$$\chi_{n,t} \equiv (1 + \bar{c}_{n,t}) A_{n,t} = \frac{1 + \bar{c}_{n+1,t-1}}{1 + g_t} A_{n+1,t-1} + (1 + c_{n,t}) \Delta A_{n,t}$$

In the iteration for maturity n, equation above determines $\Delta A_{n,t}$, which then gives $A_{n,t} = A_{n+1,t-1} + \Delta A_{n,t}$. *This step requires the inner iteration to be on time, since we need* $A_{n+1,t-1}$.

Compute $\bar{c}_{n,t}$:

$$\bar{c}_{n,t} = \frac{\chi_{n,t}}{A_{n,t}} - 1.$$

$$B_{n,t} = A_{n,t} + \sum_{j=n}^{N} \bar{c}_{j,t} A_{j,t}$$

Market value of debt:

$$V_t = Q_t B_{1,t} = \sum_{n=1}^{N} Q_{n,t} B_{n,t} = \sum_{n=1}^{N} D_{n,t} A_{n,t} = \underbrace{\left[\sum_{n=1}^{N} D_{n,t} M_{n,t} \right]}_{K_t} A_t$$

where

$$D_{n,t} = Q_{n,t} + \sum_{j=1}^{n} Q_{j,t} \bar{c}_{j,t}$$

and $M_{n,t} = A_{n,t}/A_t$

References

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