Convert Book to Market Value Debt

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Abstract

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1. Introduction

2. Zero-Coupon Bonds and Market Value

I start with a budget constraint for the government that involves only zero-coupon bonds. This simplification is economically inconsequential. If the government owes one dollar, it makes no difference if it calls this one dollar "principal" or "coupon". Moreover, the zero-coupon structure has become common in the macroeconomic literature, which makes it a natural starting point. Each bond pays one unit of currency (or "dollar") in a future expiration date, and nothing more. The difference between the current period and the expiration date is the bond's maturity. Let $\mathcal{B}_{n,t}$ be the number of outstanding bonds at the end of period t with maturity n. If the government does not issue or redeem these bonds before they expire, it will need to pay $\mathcal{B}_{n,t}$ dollars in the expiration date $t+n$. For now, all debt is nominal.

In period t, the government must come up with $\mathcal{B}_{1,t-1}$ dollars to redeem maturing bonds. It can raise revenue by selling new bonds, running a primary surplus or issuing new currency (seignorage). The notation is: M_t is the volume of currency owned by households at the end of period t, $\mathcal{S}_t^*$ is the nominal value of the primary surplus and $Q_{n,t}$ is the market price of the zero-coupon bond with maturity n (I also call it discount rate to later differentiate it from the price of coupon-paying bonds). The distinction between primary surplus revenue and seignorage is not relevant, and reported public debt usually excludes outstanding currency. So define $\mathcal{S}_t = \mathcal{S}_t^* + \Delta M_t$ as the seignorage-adjusted primary surplus, which I simply call primary surplus. The government's budget constraint is

$$\sum_{n=1}^{\infty} \left(\mathcal{B}_{n,t} - \mathcal{B}_{n+1,t-1}\right) Q_{n,t} + \mathcal{S}_t = \mathcal{B}_{1,t-1}.$$

(An $n+1$-maturity bond in $t-1$ becomes an n-maturity bond in t. The term in parethenses represents new bond issues.) We can re-write the budget constraint as

$$\mathcal{V}_t + \mathcal{S}_t = (1 + r_t^n)\mathcal{V}_{t-1},$$

where

$$\mathcal{V}_t = \sum_{n=1}^{\infty} Q_{n,t}\mathcal{B}_{n,t} \qquad \text{and} \qquad 1 + r_t^n = \frac{\sum_{n=1}^{\infty} Q_{n-1,t}\mathcal{V}_{n,t-1}}{\sum_{n=1}^{\infty} Q_{n,t-1}\mathcal{V}_{n,t-1}}$$

are, respectively, the end-of-period *market value* of public debt and the nominal return on holdings of the basket of public bonds. More concretely: at the end of $t-1$, the market value of debt is $\mathcal{V}_{t-1}$; at the beginning of t, bond prices change and the market value of debt becomes $(1+r^n)\mathcal{V}_{t-1}$. Next, we convert nominal into real variables, and detrend to make them stationary. Let P_t be the price of the basket of goods in terms of currency (or the price level), and Y_t real GDP (or any variable that plausibly renders public debt stationary). Let $B_{n,t} \equiv \mathcal{B}_{n,t}/P_tY_t$, and define $V_{n,t}$, V_t and S_t similarly. Now, V_t is the debt-to-GDP ratio and S_t is the surplus-to-GDP. The final version of the zero-coupon budget constraint is

$$V_t + S_t = \frac{1 + r_t^n}{(1+\pi_t)(1+g_t)} V_{t-1} = \substack{\text{Beginning-of-period real} \\ \text{market value of public debt,}} \tag{1}$$

where $1 + \pi_t = P_t/P_{t-1}$ is the inflation rate and $1 + g_t = Y_t/Y_{t-1}$ is the rate of GDP growth.

Measuring the market value of public debt should be of interest to economists because, in most models, it corresponds to the discounted sum of expected future primary surpluses. It is therefore informative about households' expectation of future fiscal policy (as well as discount rates), much in the same way that firm value is informative of expected firm performance (Cochrane (2005)). Market value of debt = discounted surpluses is not a condition particular to fiscal theory of the price level models; the proposition is far more general. Indeed, let $m_{t,t+j}$ be a stochastic discount factors (assume no arbitrage; a discount factor therefore exists). We can replace the pricing condition $Q_{n,t}/P_t = E_t m_{t,t+1} Q_{n-1,t+1}/P_{t+1}$ inside the definition of V_t in equation (1) and solve it forward to find that the beginning-of-period market value of public debt (the right-hand side of (1)) equals discounted surpluses

$$\sum_{j=0}^{\infty} E_t \left[m_{t,t+j} S_{t+j} \right].$$

(This result depends on $E_t\left[m_{t,t+n}V_{t+n}\right]$ converging to zero as $n \to \infty$, which is usually guaranteed by households' transversality condition when $m =$ marginal utility growth. Otherwise, the convergence is a separate assumption. See Bohn (1995).)

The market value of public debt V_t is *not* the quantity traditionally reported in public finance statistics. Instead, governments report its *book value*, or the sum of outstanding bonds' principal payments. Coupons are considered "interest" and do not enter the statistic. Additionally, the book value does not take into account variation in the price of existing bonds $Q_{n,t}$. For these reasons, the book value of public debt cannot be considered a precise measure of expected future surpluses - this observation motivates this paper. To estimate market value using book value data, we first need a model that distinguishes principal and coupon payments.

3. Coupons and Book Value

We now consider the case of a government that issues bonds that pay coupons plus a principal payment (or face value) in the expiration date. Because it is always easier to work with the zero-coupon structure of the previous section, we start with a sequence of zero-coupon payments $\{B_{n,t}\}$ and ask how we can replicate it using principal and coupon installments given a rule for how the government determines coupon rates. I assume there is a maturity N such that $B_{n,t} = 0$ for $n > N$. If $B_{n,t} \to 0$ as $n \to \infty$ uniformly in t (in a model or in reality), we can pick a large N to get an arbitrarily small error.

The notation is: $\mathcal{A}_{n,t}$ is the sum of principal payments promised by bonds of maturity n, $\Delta\mathcal{A}_{n,t}$ is the sum of principals of new n-maturity bonds, and $c_{n,t}$ is the coupon rate of new bonds. Coupons are constant over payment horizons. For example, a one-dollar bond issued in t with maturity $n = 2$ promisses the same $c_{2,t}$ dollars in $t+1$ and $t+2$, plus the one dollar principal in $t+2$. We need not keep track of the entire distribution of coupon rates of bonds issued in the past. We only keep track of the *average* coupon rate $\bar{c}_{n,t}$. The government must pay $\bar{c}_{n,t}\mathcal{A}_{n,t}$ dollars in each period from $t+1$ to $t+n$. If it issues new bonds with expiration date $t+n$, the value of promised coupons changes and we adjust $\bar{c}_{n,t}$ accordingly.

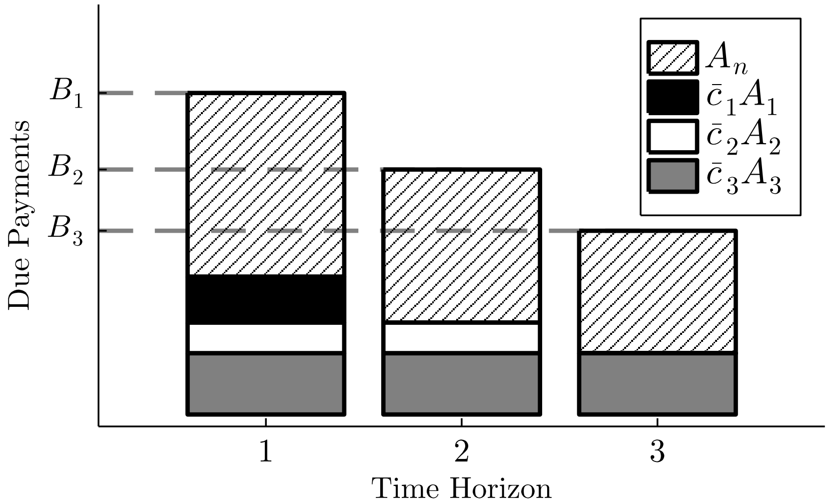


Figure 1: Example of Coupon + Principal Structure

Like before, $A_{n,t} \equiv \mathcal{A}_{n,t}/P_t Y_t$ and the same for $\Delta A_{n,t}$. Denominators don't matter much, so I work directly with normalized variables A and ΔA. If that bothers you, you can just set $\pi_t = g_t = 0$ in what follows and treat A, B, etc as nominal variables in levels instead of GDP ratios. Figure 1 depicts an example of public debt payment structure with $N = 3$. You can think we are in $t = 0$, and the graph plots the values the government is currently committed to pay in periods $t = 1, 2, 3$. The constraint that bonds pay a constant stream of coupons implies that the bars representing total coupon payments have the same size across horizons. The book value of public debt is the sum of principals:

$$A_t = \sum_{n=1}^{N} A_{n,t}, \tag{2}$$

or the sum of hatchet bars in the figure. I also define the ratio of market-to-book value D_t:

$$D_t = \frac{V_t}{A_t}.$$

As figure 1 illustrates, the total payment due after n periods comprises the face value of bonds of maturity n, their coupons, plus the coupons from bonds with maturity superior to n. To replicate the zero-coupon payment structure, we therefore need:

$$B_{n,t} = A_{n,t} + \sum_{j=n}^{N} \bar{c}_{j,t} A_{j,t}. \tag{3}$$

Equation (3) establishes the connection between the volume of outstanding bonds in each formulation (with and without coupons). We are ultimately interested in the connection between the book value and the market value of public debt, captured by D_t. The definition of debt at market value involves multiplying each payment by a discount price $Q_{n,t}$ that is usually lower than one. The definition of book value does not. This difference tends to make the market value of debt *smaller* than the book value. On the other hand, the book value ignores coupon payments, whereas the market value does not. This difference tends to make the market value *greater* than the book value. There should be a benchmark upon which these two forces cancel out.

Definition 1 (Par Coupon)**:** The par coupon rate $c^*_{n,t}$ is a coupon rate that satisfies

$$Q_{n,t} + c^*_{n,t} \sum_{j=1}^{n} Q_{j,t} = 1. \tag{4}$$

Since $Q_{n,t} > 0$, the par coupon rate is unique. A coupon rate is at par if it is the par coupon rate. A coupon rate schedule $\{c_{n,t}\}$ is at par if all its coupon rates $c_{n,t}$ are at par.

Proposition: If the average coupon schedule is at par $\{\bar{c}_{n,t}\} = \{c^*_{n,t}\}$, the market and book values of public debt coincide: $V_t = A_t$ (and $D_t = 1$).

The defining property of the par coupon rate is that the market price of the associated bond coincides with its face value. Indeed, the left side of (4) is the sum of the market price of a one-dollar principal ($Q_{n,t} \times 1$) with that of its coupons ($Q_{j,t} \times c_{n,t} \times 1$). The definition asks this sum to be equal to the principal (1). This observation explains the proposition. If the market price of the average n-maturity bond equals its principal, then the market value of debt (which adds up market values across n) equals its book value (which adds up principals across n). To prove it, I first define

$$D_{n,t} = Q_{n,t} + \bar{c}_{n,t} \sum_{j=1}^{n} Q_{j,t} \tag{5}$$

as the market value of a coupon-paying bond that promises a one-dollar principal and the average coupon rate $\bar{c}_{n,t}$. The bond price is linear in its cash flow, therefore $D_{n,t}$ is also the *average price* of outstanding n-maturity bonds. Replacing (3) in the definition of the market value of debt yields:

$$V_t = \sum_{n=1}^{N} Q_{n,t} B_{n,t} = \sum_{n=1}^{N} D_{n,t} A_{n,t}. \tag{6}$$

If $\{\bar{c}_{n,t}\}$ is the par schedule, $D_{n,t} = 1$ for every n, and therefore $V_t = A_t$. Divide both sides of (6) by A_t to get the formula for D_t:

$$D_t = \sum_{n=1}^{N} M_{n,t} D_{n,t} \tag{7}$$

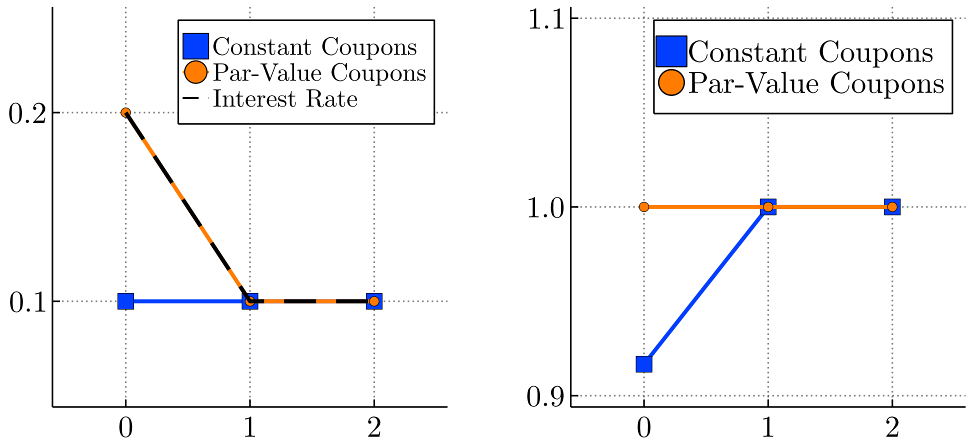
where $M_{n,t} = A_{n,t}/A_t$ is the share of principals due in n periods. These shares sum up to one; the market-to-book value ratio D_t is therefore determined by the weighted average of the average price $D_{n,t}$ of n-maturity coupon-paying bonds (hence the notation D_t and $D_{n,t}$). In simpler terms, the market-to-book ratio is a weighted average of outstanding bond prices.

3.1. Example: One-Period Debt

Let $1 + i_t = 1/Q_{1,t}$ be the economy's nominal interest rate. I consider an example in which the government issues only one-period debt ($N = 1$). It wishes to replicate a series of zero-coupon payments $B_{1,t}$ by issuing coupon-paying bonds with coupon rates $c_{1,t}$. In period t, the government repays the principal and coupon of bonds sold in $t - 1$, and issues new bonds maturing in $t + 1$. It rolls over public debt entirely every period. The face value of new bonds equals $A_{1,t}$, and their coupon value $c_{1,t}A_{1,t}$. Therefore, the book value of public debt is $A_t = A_{1,t}$ and the average coupon rate is $\bar{c}_{1,t} = c_{1,t}$. In particular, the book value satisfies

$$V_t = Q_{1,t} B_{1,t} = \frac{1 + c_{1,t}}{1 + i_t} A_t = D_t A_t.$$

Therefore, the market value of public debt is greater than the book value when $i_t < c_{1,t}$.



(a) Coupon Rates $\bar{c}_{1,t} = c_{1,t}$

(b) Market-to-Book Ratio D_t

Figure 2: Monetary Policy Shock with One-Period Debt

As an example, consider a steady-state equilibrium in which the government sells bonds at par, with coupon rate = interest rate = 0.1. Then, a temporary monetary policy shock hits: interest rate jumps to 0.2 in period zero, and returns to 0.1 in $t = 1$ onward. Figure 2 depicts two limit cases regarding coupon policy. In the first case (blue squares), the government keeps the coupon rate constant despite the increased interest: $c_{1,t} = i^*$ (panel (panel 2a)). In period zero, one-period bonds supplied to the market do not sell at par. Higher discounting means their price declines relative to face value, $D_{1,t} < 1$. Since public debt consists only of these one-period bonds, the market value of debt V_t declines relative to book value A_t. Hence $D_t = D_{1,t} < 1$ (panel 2b). In the second case (orange circles), the government raises the coupon rate one-to-one with the interest rate: $c_{1,t} = i_t$. Higher coupons prevent the price of one-period bonds from declining in spite of the increase in the discount rate. Hence, the market-to-book ratio D_t remains unchanged.

3.2. The General Case

With $N > 1$, the government no longer rolls over the entire stock of public debt each period. The average coupon rate $\bar{c}_{n,t}$ changes slowly over time, as the government redeems old bonds and sells new ones. We no longer have

$\bar{c}_{n,t} = c_{n,t}$. Instead, in period t the government inherits a commitment to pay $\bar{c}_{n+1,t-1}\mathcal{A}_{n+1,t-1}$ dollars from coupons of bonds with expiration date $t+n$. We add to this value the flow of payments promised by newly-issued bonds $c_{n,t}\Delta\mathcal{A}_{n,t}$. Adding up and normalizing denominators gives:

$$\bar{c}_{n,t}A_{n,t} = \bar{c}_{n+1,t-1}\frac{A_{n+1,t-1}}{(1+\pi_t)(1+g_t)} + c_{n,t}\Delta A_{n,t}. \tag{8}$$

(If $n = N$, $\bar{c}_{n,t} = c_{n,t}$.) The combined face values of n-maturity bonds equals the sum of the face value of past debt and that of new bond issues:

$$A_{n,t} = \frac{A_{n+1,t-1}}{(1+\pi_t)(1+g_t)} + \Delta A_{n,t}. \tag{9}$$

(If $n = N$, $A_{n,t} = \Delta A_{n,t}$.) In all, the average coupon rate is the weighted combination of the average inherited from $t-1$ with the coupon rate offered by new bonds. The government can also redeem bonds before the expiration date, $\Delta A_{n,t} < 0$. In this case, we take the average coupon rate among the set of redeemed bonds to be $c_{n,t}$. This clause avoids the introduction of a nonlinear law of motion.

To illustrate the connection between slowly-moving average coupons $\bar{c}_n$ and the market-to-value ratio D_t, I again consider the example of a monetary policy shock. The interest rate follows the AR(1)

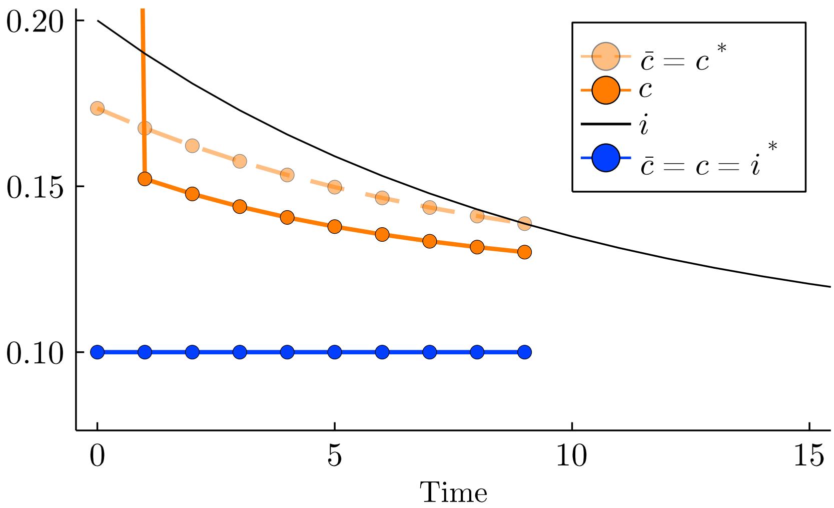
$$i_t - i^* = 0.9\ (i_{t-1} - i^*) + \epsilon_t,$$

with $i^* = 0.10$. Bond prices respect the expectations hypothesis in levels:

$$Q_{n,t} = \frac{Q_{n-1,t}}{1 + E_t i_{t+n-1}},$$

with $Q_{0,t} = 1$. In the steady state, $Q_n^* = (1+i^*)^{-n}$, and the government issues bonds at par, which implies $c_n^* = \bar{c}_n^* = i^*$ and $D_n^* = 1$ for all n.

In period zero, $\epsilon_0 = 0.10$ so the interest jumps to $i_0 = 0.20$. Given the AR(1) path of interest, $Q_{n,t}$ falls for all n and t, but converges to Q_n^* as t grows. The government adopts a primary surplus rule that precludes public debt from spiralling: $S_t = S_t^* + 0.2\ (V_{t-1} - V^*)$. It also maintains a constant



Notes. In blue: constant coupons $\bar{c}_n = c_n = \bar{i}$. Orange: average coupons at par. Dashed curve is the average coupon rate = par rate.

Figure 3: Monetary Shock: Average and New Issue Coupons (Expiration Date $t = 10$)

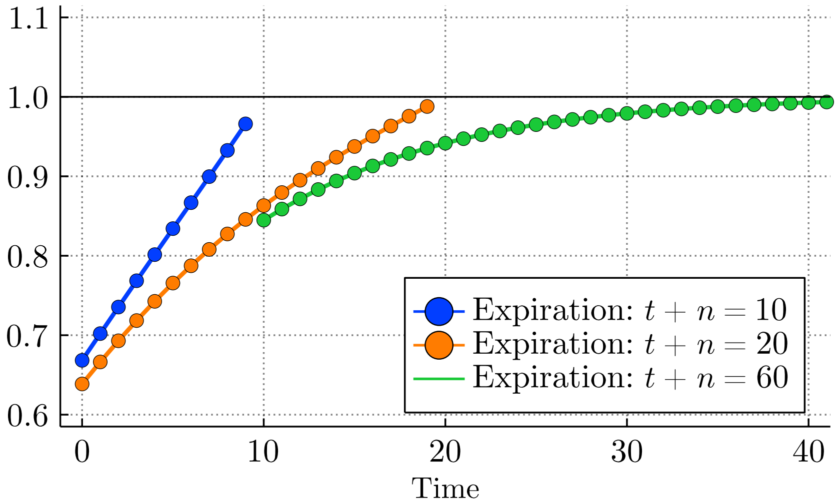
geometric term structure to its debt:

$$B_{n,t} = \omega B_{n-1,t}, \tag{10}$$

with $\omega = 0.8$. I set $N = 50$ (leading to a negligible error) and ignore inflation and GDP growth $\pi_t = g_t = 0$ for clarity.

Figure 3 shows the impulse-response function (IRF) of interest rates and, as an example, coupon rates for the set of bonds expiring in $t = 10$ (that is, $c_{n,t}$ such that $t + n = 10$). The x-axis represents time, not maturity. The blue curve corresponds to the constant coupon rule: $c_{n,t} = \bar{c}_{n,t} = i^*$. In this case, like the $N = 1$ example, higher discounting lowers average bond prices $D_{n,t}$ as the government fails to raise their coupon rates. Figure 4 plots the IRFs of average bond prices $D_{n,t}$. Each curve corresponds to a fixed expiration. In all cases, $D_{n,t} < D^*_n = 1$, but bond prices slowly return to par. Replacing $\bar{c}_{n,t} = i^*$ in (7) shows that the market-to-book ratio is a combination of discount rates $Q_{n,t}$:

$$D_t = \sum_{n=1}^{N} \kappa_n \, Q_{n,t} \qquad \text{with } \kappa_n > 0.$$



Notes. Each curve plots the average price of bonds maturing at a fixed expiration date. The green curve starts at $t = 10$ because that is when bonds with expiration date $t + 60$ are first issued.

Figure 4: Monetary Shock: Average Bond Prices D_n (Constant Coupons $c_n = \bar{i}$)

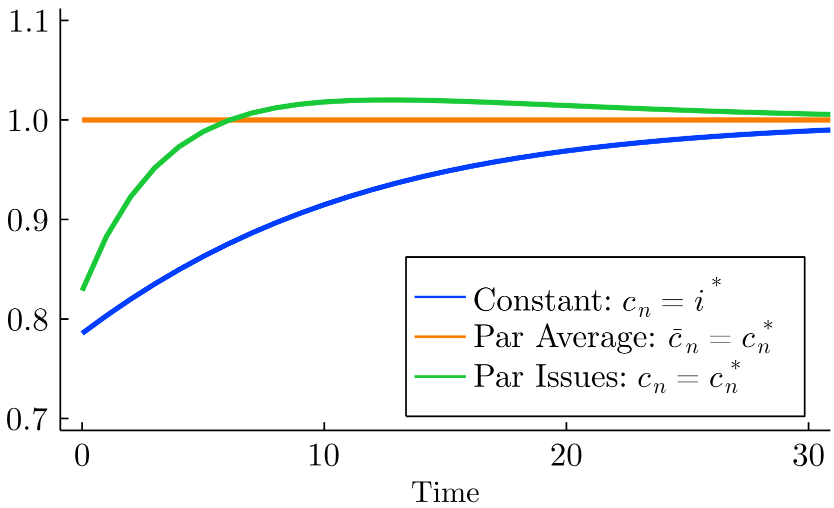
Coefficients κ_n are positive. Hence, given the AR(1) interest dynamics + expectations hypothesis, the market-to-book ratio simply replicates the behavior of current interest i_t: high $i_t \implies$ low $Q_{n,t} \implies$ low D_t.

Back to figure 3, the orange curves correspond to a different coupon policy. In it, the government forces the average coupon rate to be at par: $\bar{c}_{n,t} = c^*_{n,t}$ (plotted by the dashed curve), which implies $D_{n,t} = 1$.[1] The IRFs of average bond prices in figure 4 would be drawn on top of the $D_{n,t} = 1$ horizontal line, and thus $D_t = 1$ as advertised by the proposition. But keeping the *average* coupon rate at par calls the government to promote unrealistically large changes to the coupon rate on new bonds $c_{n,t}$ (orange, solid curve). The reason is that the volume of such new issues might represent a small share of the outstanding total. In this particular case, equation (8) reads:

$$\bar{c}_{10,t=0} = 0.937\ \bar{c}_{11,t=-1} + 0.063\ c_{10,t=0}.$$

Hence, bringing $\bar{c}_{10,t=0}$ from $i^* = 0.1$ to the par coupon rate $c^*_{10,t=0} = 0.174$

[1]For example: in period $t = 9$, bonds expiring at 10 have maturity $n = 1$. By (4) $c^*_{1,t} = i_t$, therefore the dashed curve touches the interest IRF.



Notes. Different curves correspond to different coupon rules (constant, par average and par issues).

Figure 5: Monetary Shock: Market-to-Bond Value Ratio D_t

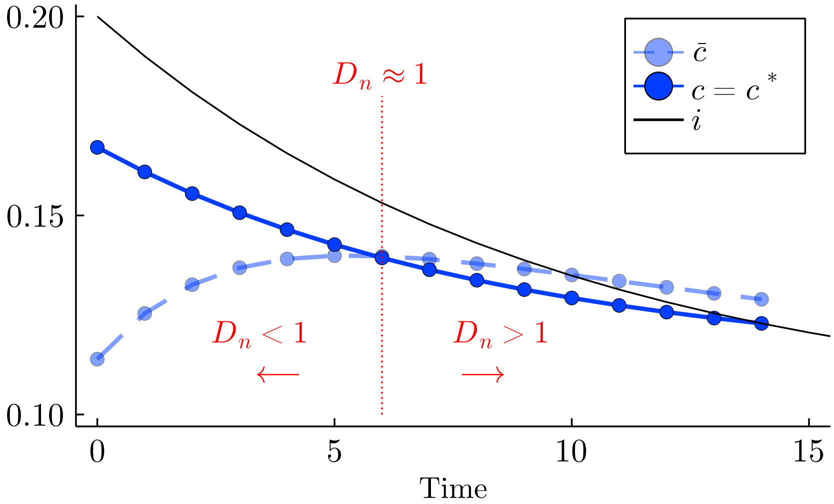
requires the off-the-chart $c_{10,t=0} = 1.27$. That is, in $t = 0$ the government must issue bonds that pay 1.27 times their face value each period!

Figure 5 plots market-to-book value ratios D_t. With constant coupons, D_t is visibly the negative of the interest IRF. With average coupon rates at par, book and market values always coincide, $D_t = 1$. In terms of sensitivity to the interest rate, these two policies are opposing limit cases. A plausible intermediate model beacons.

I therefore consider a coupon policy that prescribes *new bond issues* to be at par: $c_{n,t} = c^*_{n,t}$. Applying this rule, figure 6 plots coupon rates for bonds with expiration date set to $t = 15$. The solid curve shows par rates offered by new bonds $c_{n,t} = c^*_{n,t}$. Higher interest leads the government to offer higher coupons on new bond sales. The average coupon rate (dashed curve) grows for as long as new bonds offer higher rates, $c_{n,t} > \bar{c}_{n,t}$. Since $c_{n,t}$ is at par, we have that $c_{n,t} > \bar{c}_{n,t}$ if and only if $D_{n,t} < 1$:

$$1 = Q_{n,t} + c_{n,t} \sum_{j=1}^{n} Q_{j,t} \; > \; Q_{n,t} + \bar{c}_{n,t} \sum_{j=1}^{n} Q_{j,t} = D_{n,t}.$$

So the dashed and solid curves conveniently determine when $D_n < 1$.

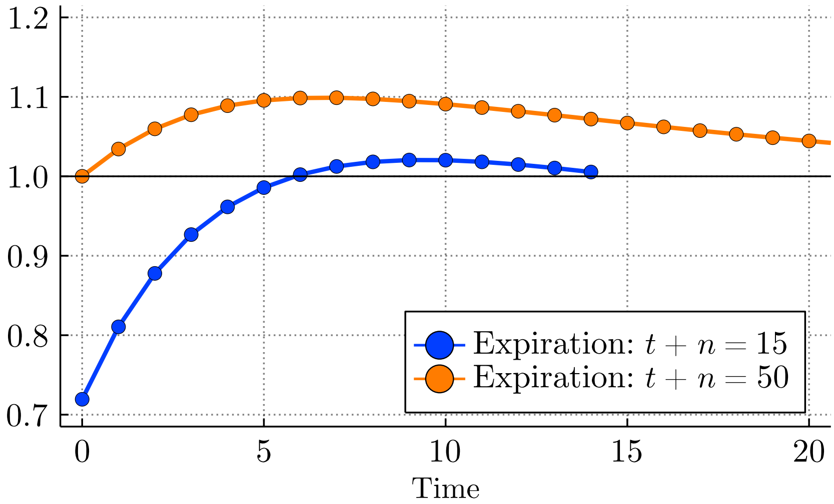


Notes. The solid curve is coupon rate on newly-issued bonds, at par. Dashed curve is the average coupon rate.

Figure 6: Monetary Shock: Coupons (New Issues at Par, Expiration Date $t = 15$)

Figure 7 plots the IRF of average bond prices D_n. The blue IRF refers to bonds expiring in $t = 15$. In accordance with the coupon curves shown in figure 6, $D_n < 1$ prior to $t = 6$, and $D_n > 1$ afterwards. Unlike the constant coupons policy, here the positive interest shock eventually leads some groups of bonds to have an average coupon rate above the par rate ($D_{n,t} > 1$). More evidently, consider bonds expiring in $t = 50$ (orange curve). Since $N = 50$, these bonds are first issued exactly in period zero. Their average coupon is therefore initially at par $\bar{c}_{n,t=0} = c^*_{n,t=0}$. From then on, the par coupon $c^*_{n,t=0}$ declines as the interest rate converges back to the steady state. The average coupon rate declines more slowly, hence $D_{n,t} > 1$ throughout the transition.

To close the analysis, we look at the IRF for the market-to-book ratio, plotted in figure 5. By (7), D_t is an average of $D_{n,t}$. It starts below one, as old bonds weight down average coupon rates, but eventually grows above one as debt issued after the interest shock becomes predominant.



Notes. Each curve plots the average price of bonds maturing at a fixed expiration date.

Figure 7: Monetary Shock: Average Bond Prices D_n (New Issues at Par, $c_n = c_n^*$)

3.3. Mixed Debt Composition

Governments often supply a mix of coupon and zero-coupon bonds. In the US, Treasury Notes pay coupons; Treasury Bills do not. Our framework accomodates a mixed debt profile without further notation. Suppose the government wishes to sell a share $z_{n,t}$ of n-maturity bonds as zero coupon. Let $\hat{c}_{n,t}$ be the coupon rate offered by the remaining $1 - z_{n,t}$ bonds. For instance, $\hat{c}_{n,t} = c^*_{n,t}$. The value of committed coupons is the product of the coupon rate by the number of new coupon-paying bonds: $\hat{c}_{n,t} \times (1 - z_{n,t})\Delta A_{n,t}$. By setting

$$c_{n,t} = \hat{c}_{n,t}(1 - z_{n,t}) \tag{11}$$

we add to the stock of committed coupons the correct value of new installments. Equation (2) still determines the book value of debt, and $D_{n,t}$ continues to represent the average price of n-maturity bonds, but this average is now taken across coupon paying and zero-coupon bonds.

4. Algorithms

This section presents the two central algorithms of the paper. The first one solves the set of equations presented in the previous section. The second algorithm estimates the market value of public debt.

4.1. Computing Coupon Structures

The setup for the following algorithm is similar to that of the previous section. We start with a quantity schedule $\{B_{n,t}\}$ for zero-coupon bonds. The algorithm returns quantities of coupon-paying bonds $\{A_{n,t}\}$ and average coupon rates $\{\bar{c}_{n,t}\}$. We also need times series for inflation and output growth, a rule for new coupon issues $\{c_{n,t}\}$ and initial conditions $\{A_{0,n}\}$ and $\{\bar{c}_{0,n}\}$. I use this algorithm to solve the equations in two cases of the example in the previous section: the case of constant coupons and that of par-coupons on new bond issues. We later see an alternative algorithm that starts with an average coupon target and returns the required schedule for new bonds.

Algorithm 1 Compute Coupon Structure

Input: π_t, g_t, $\{B_{n,t}\}$, $\{c_{n,t}\}$, $\{A_{n,0}\}$, $\{\bar{c}_{n,0}\}$
Output: $\{A_{n,t}\}$, $\{\bar{c}_{n,t}\}$

for $t = 1, \dots, T$, $n = N, N-1, \dots, 1$ **do**
  if $n = N$ **then**
    $A_{N,t} \leftarrow B_{N,t}/(1 + c_{N,t})$
    $\bar{c}_{N,t} \leftarrow c_{N,t}$
  else
    Auxiliary $\chi \leftarrow B_{n,t} - \sum_{j=n+1}^{N} \bar{c}_{j,t} A_{j,t}$
    Auxiliary $\psi \leftarrow A_{n+1,t-1}(1 + \bar{c}_{n+1,t-1})/\left[(1 + \pi_t)(1 + g_t)\right]$
    $\Delta A_{n,t} \leftarrow (\chi - \psi)/(1 + c_{n,t})$
    $A_{n,t} \leftarrow \Delta A_{n,t} + \psi/(1 + \bar{c}_{n+1,t-1})$
    $\bar{c}_{n,t} \leftarrow \chi/A_{n,t} - 1$
  end if
end for

The algorithm iterates time (forward) and bond maturity (backward). The selection of the inner loop is irrelevant. Auxilary variable χ represents $(1 + \bar{c}_{n,t})A_{n,t}$. We compute it from (3). To distinguish $\bar{c}_{n,t}$ from $A_{n,t}$, we use

(9) and (8). These equations involve $A_{n+1,t-1}$ and $\bar{c}_{n+1,t-1}$, which is why we need initial conditions for both.

The following alternative version of algorithm 1 starts with a target for average coupon rates $\{\bar{c}_{n,t}\}$, and returns the coupon schedule offered by new bond issues $\{c_{n,t}\}$. I use it to solve the equations in the case of par-value average coupons in the example of the previous section.

Algorithm 2 Compute Coupon Structure (Average Coupon Target)

Input: π_t, g_t, $\{B_{n,t}\}$, $\{\bar{c}_{n,t}\}$, $\{A_{n,0}\}$, $\{\bar{c}_{n,0}\}$
Output: $\{A_{n,t}\}$, $\{c_{n,t}\}$

for $t = 1, \ldots, T$, $n = N, N-1, \ldots, 1$ **do**
    if $n = N$ **then**
        $A_{N,t} \leftarrow B_{N,t}/(1+\bar{c}_{N,t})$
        $c_{N,t} \leftarrow \bar{c}_{N,t}$
    else
        Auxiliary $\chi \leftarrow B_{n,t} - \sum_{j=n+1}^{N} \bar{c}_{j,t} A_{j,t}$
        Auxiliary $\psi \leftarrow A_{n+1,t-1}(1+\bar{c}_{n+1,t-1}) / \left[(1+\pi_t)(1+g_t)\right]$
        $A_{n,t} \leftarrow \chi/(1+\bar{c}_{n,t})$
        $\Delta A_{n,t} \leftarrow A_{n,t} - \psi/(1+\bar{c}_{n+1,t-1})$
        $c_{n,t} \leftarrow (\chi - \psi)/\Delta A_{n,t} - 1$
    end if
end for

Auxiliary variables χ and ψ are the same as in algorithm 1. But now we can compute $A_{n,t}$ directly from χ, since the average coupon rate is known beforehand.

4.2. Estimating the Market Value of Public Debt

The next algorithm computes an estimate of the market value of public debt. We start with time series for the book value of debt A_t, the inflation rate π_t and GDP growth g_t. The complete term structure of face values $\{A_{n,t}\}$ is unknown. The algorithm estimates V_t as the solution to the fixed point problem

$$V_t = \mathbf{D_t}(\{B_{n,t}(V_t)\})A_t \qquad \text{for all } t. \tag{12}$$

Function $\mathbf{D_t}$ returns the market-to-book ratio D_t using algorithm 1 (or 2, but I focus here on the former). Algorithm 1 alone does not give us D_t. But we can use its outputs $\{A_{n,t}\}$ and $\{\bar{c}_{n,t}\}$, along with a model for discount rates $\{Q_{n,t}\}$, to compute bond prices $D_{n,t}$ using (5), and then the market-to-book ratio D_t using (7). However, the schedule of zero-coupon payments $\{B_{n,t}\}$ that enters algorithm 1 as an input must be consistent with the aggregate value of debt V_t given the schedule of discount rates $\{Q_{n,t}\}$. This is what the fixed-point problem (12) is all about.

Two new ingredients are necessary. First, a model for discount rates $\{Q_{n,t}\}$. Second, a schedule $\{\omega_{n,t}\}$ for the term structure of zero-coupon payments, which then satisfy

$$B_{n,t} = \omega_{n,t} B_{1,t}.$$

In the example of section 3, I use $\omega_{n,t} = \omega^{n-1}$. Schedules $\{Q_{n,t}\}$ and $\{\omega_{n,t}\}$ may depend on available data, but not on variables being estimated. Hence, both are in practice given. By the definition of the market value of public debt, we have

$$V_t = \sum_{n=1}^{N} Q_{n,t} B_{n,t} = Q_t B_{1,t}$$

where $Q_t = \sum_{n=1}^{N} \omega_{n,t} Q_{n,t}$ is a weighted sum of discount rates. The two expressions above imply that

$$B_{n,t} = \frac{\omega_{n,t}}{Q_t} V_t. \tag{13}$$

Equation (13) gives meaning to the term $B_{n,t}(V_t)$ in (12).

Algorithm 3 explains the iteration procedure, which simply repeats the operation

$$TV_t = \mathbf{D_t}(\{B_{n,t}(V_t)\}) A_t$$

until $TV = V$. The iteration updates V_t along with the initial condition for principals $\{A_{n,0}\}$. I find that updating $\{\bar{c}_{n,0}\}$ as well leads to unstable results. So in my applications I fix $\bar{c}_{n,0}$ at the par coupon rate corresponding to the bond price schedule in period one, $\{Q_{n,t=1}\}$. Finally, the updating step of V_t and $\{A_{n,0}\}$ might require some damping.

Algorithm 3 Estimate Market Value of Public Debt

Input: π_t, g_t, A_t, $\{\omega_{n,t}\}$, $\{Q_{n,t}\}$, $\{c_{n,t}\}$, $\{\bar{c}_{n,0}\}$

Output: V_t

```
Compute Q_t = Σ_n Q_{n,t} ω_{n,t}
Guess V_t^(0)
Guess {A_{n,0}^(0)}
for iteration i do
    {B_{n,t}^(i)} ← equation (13) using V_t^(i)
    {A_{n,t}^(i)} and {c̄_{n,t}^(i)} ← algorithm 1 using {B_{n,t}^(i)} and {A_{n,0}^(i)}
    {D_{n,t}^(i)} ← equation (5) using {c̄_{n,t}^(i)}
    D_t^(i) ← equation (7) using {D_{n,t}^(i)} and {A_{n,t}^(i)}
    TV_t^(i) ← D_t^(i) × A_t
    if ||TV_t^(i) − V_t^(i)|| < ϵ then
        return V_t
    end if
    Update V_t^(i+1) ← V_t^(i)
    Update {A_{n,0}^(i+1)} ← {A_{n,t=1}^(i)}
end for
```

5. The US Case

6. Multiple Currencies

7. Concluding Remarks

References

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