Convert Book to Market Value Debt

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Abstract

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1. Introduction

2. A General Budget Constraint

I start by assuming that the government does not issue bonds that pay coupons. From an economic perspective, this simplification is inconsequential. If the government owes one dollar due tomorrow, it makes no difference if it calls this one dollar a "principal" or a "coupon" payment. In addition, working with zero-coupon bonds simplifies the expressions significantly and indeed has become a common practice in the macroeconomic literature. So I start by looking at a general budget constraint involving only zero-coupon bonds and then I consider the existence of coupons. Suppose each bond pays one unit of currency in a given future date and nothing more. The difference between the current period and this future date equals the bond's maturity. Let $\mathcal{B}_{n,t}$ be the number of outstanding bonds in period t with maturity n. If the government does not issue or redeem these bonds before they expire, it will need to pay $\mathcal{B}_{n,t}$ units of currency in period $t+n$. For now, I consider only the existence of nominal public debt.

In period t, the government must come up with $\mathcal{B}_{1,t-1}$ units of currency to pay that amount to holders of maturing debt. It can accomplish that by selling new bonds, running a primary surplus or simply issuing currency. The notation is: M_t is the amount of currency at the hands of households at the end of period t, $\mathcal{S}_t^*$ is the nominal value of the primary surplus and $Q_{n,t}$ is the market price of a bond with maturity n. Because the distinction between revenue from primary surpluses and seignorage is not relevant for this paper, and because reported public debt does not include outstanding currency, I further simplify and define $\mathcal{S}_t = \mathcal{S}_t^* + \Delta M_t / P_t$ as the seignorage-adjusted primary surplus, which I will just refer to as primary surplus. The budget constraint faced by the government is the following:

$$\sum_{n=1}^{\infty} \left(\mathcal{B}_{n,t} - \mathcal{B}_{n+1,t-1}\right) Q_{n,t} + \mathcal{S}_t = \mathcal{B}_{1,t-1}.$$

We can re-arrange that equation and re-write it as

$$\mathcal{V}_t + \mathcal{S}_t = (1 + r_t^n)\mathcal{V}_{t-1}$$

where

$$\mathcal{V}_t = \sum_{n=1}^{\infty} Q_{n,t}\mathcal{B}_{n,t} \qquad \text{and} \qquad 1 + r_t^n = \frac{\sum_{n=1}^{N} Q_{n-1,t}\mathcal{V}_{n,t-1}}{\sum_{n=1}^{N} Q_{n,t-1}\mathcal{V}_{n,t-1}}$$

are, respectively, the end-of-period *market value* of public debt and the nominal return on holdings of the basket of public bonds. Next, we convert nominal into real variables, and de-trend them to make them stationary. For this, define P_t as the price of the basket of goods in terms of currency (that is, the price level), and Y_t as real GDP (or any variable that plausibly renders public debt stationary). Let $B_{n,t} \equiv \mathcal{B}_{n,t}/P_tY_t$, and define $V_{n,t}$, V_t and S_t similarly. Now, V_t is debt-to-GDP and S_t is the surplus-to-GDP. Our final budget constraint is

$$V_t + S_t = \frac{1 + r_t^n}{(1 + \pi_t)(1 + g_t)} V_{t-1} = \begin{array}{c}\text{Beginning-of-period real}\\ \text{market value of public debt.}\end{array} \tag{1}$$

The importance of the market value of public debt is that in most models it corresponds to the discounted sum of expected future primary surpluses. It is therefore informative about households' expectation of future fiscal policy (as well as discount rates) much in the same way that firm value is informative of future dividends (Cochrane (2005)). Market value of debt = discounted surpluses is *not* a condition particular to fiscal theory of the price level models; the proposition is far more general. Indeed, let $m_{t,t+j}$ be a stochastic discount factors (assume no arbitrage; a discount factor therefore exists). We can replace the pricing condition $Q_{n,t}/P_t = E_t m_{t,t+1} Q_{n-1,t+1}/P_{t+1}$ inside the definition of V_t in equation (1) and solve it forward to find that the beginning-of-period market value of public debt (the right-hand side of (1)) equals

$$\sum_{j=0}^{\infty} E_t \left[m_{t,t+j} S_{t+j} \right].$$

(This result depends on $E_t \left[m_{t,t+n} V_{t+n} \right]$ converging to zero as $n \to \infty$, which is guaranteed by households' transversality condition when m = marginal utility growth. Otherwise, the convergence is a separate assumption.)

The key motivation for this paper is that V_t is *not* the quantity reported in public finance statistics. Instead, governments usually report the *book value* (sometimes called the "par" value) of public debt, which is simply the sum of principal payments. Coupons are considered "interest" and do not enter the sum. Of course, from the economist's perspective, coupons and the interest rate are two highly different concepts. Additionally, the book value does not take into account variation in the price of existing bonds $Q_{n,t}$. For these reasons, the book value of public debt cannot be considered (at least in theory) as precise of a measure of expected future surpluses as its market value counterpart.

3. Coupon and the Book Value of Public Debt

We now consider the case of a government that issues bonds that pay coupons plus a principal payment upon maturity. Because it is always easier to work with the zero-coupon structure of the previous section, we start with a sequence of desired zero-coupon payments $\{B_{n,t}\}$ and ask how we can map it into principal and coupon installments given a rule for how the government determines coupon rates. To avoid dwelling into functional analysis, I assume there is an N such that $B_{n,t} = 0$ for $n > N$. If $B_{n,t} \to 0$ uniformly in time as $n \to \infty$ (in a model or in reality), large N will involve an arbitrarily small error.

The notation is: $\mathcal{A}_{n,t}$ is the sum of principal payments promised by bonds of maturity n, $\Delta\mathcal{A}_{n,t}$ is the sum of principals of new n-maturity bonds, and $c_{n,t}$ is the coupon rate of the new bonds. Coupons are constant over payments horizons. For example, a one-dollar bond with maturity $n = 2$ promisses $c_{2,t}$ dollars after one period (coupon) and $1 + c_{2,t}$ dollars after two periods (principal + coupon). I call $\bar{c}_{n,t}$ the *average* coupon rate of bonds with maturity n: the government must pay $\bar{c}_{n,t}\mathcal{A}_{n,t}$ in each period from $t + 1$ to $t + n$ corresponding to coupons from all outstanding bonds with maturity n.

Like before, $A_{n,t} \equiv \mathcal{A}_{n,t}/P_tY_t$ and the same for $\Delta A_{n,t}$. Denominators don't

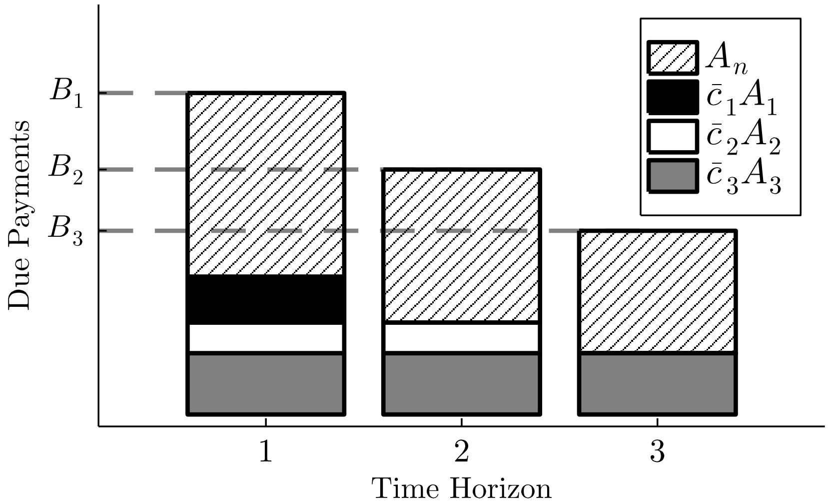

Figure 1: Example of Coupon + Principal Structure

matter much, therefore I work directly with normalized variables A and ΔA - if that bothers you too much, you can just set $\pi_t = g_t = 0$ in what follows and treat A, B, etc as nominal variables in levels instead of GDP ratios. Figure 1 depicts an $N = 3$ example. The constraint that bonds pay a constant stream of coupons implies that the size of the coupon bars is the same across payment horizons.

In this environment, the book value of public debt is the sum of principals:

$$A_t = \sum_{n=1}^{N} A_{n,t}.$$

Our goal is to find a sequence of principals $\{A_{n,t}\}$ and average coupons $\{\bar{c}_{n,t}\}$ that attain a desired level of zero-coupon payments $\{B_{n,t}\}$ given a coupon rate schedule $\{c_{n,t}\}$ for new bonds. To avoid dwelling into functional analysis, suppose there is an N such that $B_{n,t} = 0$ for $n > N$. Governments usually limit the maturity of bonds they issue, and we can always make N large enough. After n periods, the government must pay principal and coupon for n-maturity bonds, plus coupons for bonds with maturity superior to n. Hence:

$$B_{n,t} = A_{n,t} + \sum_{j=n}^{N} \bar{c}_{j,t} A_{j,t}.$$

3.1. Single-Maturity Term Structure

3.2. Double-Maturity Term Structure

3.3. The General Case

Market value of debt:

$$V_t = \sum_{n=1}^{N} Q_{n,t} B_{n,t}$$

Flow equation for the market value of public debt:

$$V_{t-1} + S_{t-1} = \frac{(1 + r_t^n)}{(1 + \pi_t)(1 + g_t)} V_t$$

Definition of B_n:

$$B_{n,t} = A_{n,t} + \sum_{j=n}^{N} \bar{c}_{j,t} A_{j,t} = \chi_{n,t} + \sum_{j=n+1}^{N} \bar{c}_{j,t} A_{j,t}$$

In the iteration for maturity n, equation above determines $\chi_{n,t} \equiv (1+\bar{c}_{n,t})A_{n,t}$, which is the amount due in n period in repayment of maturing bonds only.

Definition $\bar{c}$:

$$\chi_{n,t} \equiv (1 + \bar{c}_{n,t})A_{n,t} = \frac{1 + \bar{c}_{n+1,t-1}}{1 + g_t} A_{n+1,t-1} + (1 + c_{n,t})\Delta A_{n,t}$$

In the iteration for maturity n, equation above determines $\Delta A_{n,t}$, which then gives $A_{n,t} = A_{n+1,t-1} + \Delta A_{n,t}$. *This step requires the inner iteration to be on time, since we need* $A_{n+1,t-1}$.

Compute $\bar{c}_{n,t}$:

$$\bar{c}_{n,t} = \frac{\chi_{n,t}}{A_{n,t}} - 1.$$

$$B_{n,t} = A_{n,t} + \sum_{j=n}^{N} \bar{c}_{j,t} A_{j,t}$$

Market value of debt:

$$V_t = Q_t B_{1,t} = \sum_{n=1}^{N} Q_{n,t} B_{n,t} = \sum_{n=1}^{N} D_{n,t} A_{n,t} = \underbrace{\left[\sum_{n=1}^{N} D_{n,t} M_{n,t}\right]}_{K_t} A_t$$

where

$$D_{n,t} = Q_{n,t} + \sum_{j=1}^{n} Q_{j,t} \bar{c}_{j,t}$$

and $M_{n,t} = A_{n,t}/A_t$

References

Cochrane, J. H. (2005). Money as stock. *Journal of Monetary Economics*, 52(3):501–528.