1. Environment

The economy is populated by households and a government. They live for two periods, $t = 0$ and $t = 1$, and trade identical consumption goods and public bonds. Public bonds promise their holder one unit of the consumption good in the following period. There is no money in this economy. Agents trade public bonds using consumption goods.

A word on notation: each variable in the model takes a value in period zero and a value in period one, as indicated by their subscript (a process that is a function of time is called a *time series*). For example, x_0 and x_1. The vector $x = (x_0, x_1)$ refers to the time series as whole.

1.1. The Government

The government demands $g = (g_0, g_1)$ consumption goods (*i.e.*, g_0 in period zero and g_1 in period one). To finance its purchases, it charges lump-sum taxes $\tau = (\tau_0, \tau_1)$ on households. Households cannot avoid paying taxes. The pair g and τ characterize *fiscal policy* in this model.

The government also raises revenue from selling new public bonds. In period 0, the price of one bond is q_0 units of the consumption good. Usually $q_0 < 1$: you pay less than one good in $t = 0$, to get one good in $t = 1$. As such,

$$1 + r_0 = \frac{1}{q_0}$$

is the interest rate implied by public bonds. In period 1, agents have no incentive to save since the world ends in the following period. Since bonds have no demand we can set its equilibrium price to zero: $q_1 = 0$.

We make two critical assumptions on government behavior. First, it can *credibly* commit to fully repay previously issued debt. "Credibly" means that households believe in the commitment, and act accordingly. Second, the government indeed never defaults.

The government brings to period zero a debt of b_{-1} bonds, and must therefore come up with b_{-1} consumption goods to pay bondholders. To that end, it can either sell new bonds b_0 and raise $q_0 b_0$ goods in revenue, or run a *primary surplus*. The primary surplus is defined as the difference between tax proceeds and non-interest spending. In this model, it corresponds to the quantity $\tau_0 - g_0$. The government avoids a default in period zero if

$$q_0 b_0 + \tau_0 - g_0 = b_{-1}. \tag{1}$$

The revenue from selling new bonds plus the revenue from taxes in excess of public spending must be enough to redeem old bonds. Since the government will not default, condition (1) represents a budget constraint. It restricts the government's choice of how much to tax, how much to spend, and how much to borrow.

Like in period zero, in period one the government again must pay bondholders, which are now due b_0 units of the consumption good. But, in period one, the government cannot sell new bonds, since there is no demand for them (the bond price is zero $q_1 = 0$, so the government would not raise any revenues anyway). Therefore, to pay bondholders, the government must run a primary surplus of b_0 in period one:

$$\tau_1 - g_1 = b_0. \tag{2}$$

Like (1), expression (2) is a budget constraint for the government.

1.2. Households

The consumption good is non-durable (households can only enjoy them for a single period), and perishable (agents cannot store them). Households value the consumption good in the period they make use of them. The utility function

$$u(c_0) + \beta u(c_1)$$

captures households' preferences over the amount consumed in period zero c_0 and period one c_1. Period utility $u(c)$ is an increasing, strictly concave and twice differentiable function. Parameter $\beta \in (0, 1]$ discounts the flow of future consumption, and therefore captures households' impatience.

Each household receives an endowment of $y = (y_0, y_1)$ consumption goods. You can think as households producing these goods at home; we later model firms, production and labor income more realistically.

We normalize the number of households to one, which avoids the introduction of unnecessary notation. If each household consumes c_0 goods, aggregate consumption will be

$$c_0 \times \text{Number of Households} = c_0 \times 1 = c_0.$$

The same symbol c_0 represents both individual and aggregate consumption. Likewise, (y_0, y_1) represent aggregate production in the economy.

In period zero, each household brings a_{-1} public bonds from the previous period, and redeem them for the same quantity of consumption goods. Add to that their after-tax income $y_0 - \tau_0$ and we find the amount of available goods to each household in period zero. They can use these goods to consume or purchase public bonds from the government. Let a_0 be the household's choice of how many public bonds to purchase. There is no other asset in the economy, so a_0 also represents the household's savings and its net wealth. The following equation is the budget constraint faced by each household in period zero:

$$q_0 a_0 + c_0 \leq a_{-1} + y_0 - \tau_0. \tag{3}$$

Equation (3) restricts the households' decision of how much to consume and how much to save in period zero. In period one, households redeem a_0 public bonds, and do not demand new ones, as the world ends thereafter. Hence:

$$c_1 \leq a_0 + y_1 - \tau_1. \tag{4}$$

Households can borrow too, and the government can lend. While we have referred to b_0 as government "borrowing" and a_0 as household "savings", nothing precludes these variables from being negative (in which case, the household borrows and the government lends).

Suppose households exhaust their available resources, that is, that their budget constraints hold with equality. By equation (4), the maximum amount of goods a household can repay from previsouly acquired debt is $y_1 - \tau_1$ (in that case, the household would consume zero goods in period one, $c_1 = 0$). If the household's debt is larger than $y_1 - \tau_1$, the household defaults. Knowing that, potential lenders (other households or the government) refuse to purchase bonds from (*i.e.*, lend to) a household whose debt exceeds this value. Therefore, the largest size of any household debt position is $y_1 - \tau_1$. We incorporate this *borrowing constraint* in the model by establishing a lower bound $\underline{a}$ on period-zero savings a_0:

$$a_0 \geq \underline{a} = -(y_1 - \tau_1). \tag{5}$$

(If you get confused with signs, think of an example; if after-tax income equals 5 goods, then debt cannot be higher than 5, so net wealth cannot be lower than $\underline{a} = -5$.)

Economists often refer to a household's maximum repayable debt as its *natural borrowing limit*. In our model, the natural borrowing limit is $-\underline{a} = y_1 - \tau_1$. Other choices of borrowing limit $-\underline{a}$ are possible, and often more realistic. However, adopting the natural borrowing limit is a convenient starting point to analyze households' allocation decisions, because any choice that involves a positive consumption in period one ($c_1 > 0$) necessarily satisfies it. Consequently, if we prove the period-one consumption is not zero, we can safely ignore the borrowing limit.

Households decide how much to consume $c = (c_0, c_1)$ and how many bonds to purchase (or issue) a_0 taking into account their budget and borrowing constraints (3)-(5). They take as given the price of public bonds q_0.[1]

In period zero, households decide how much to consume c_0 and how much to save (or borrow) in the bond market a_0. They understand that different bond positions at the beginning of period one entail different future consumption levels c_1. The choice of how much to consume solves

[1] We refer to price-taking behavior as *competitive* behavior.

the following utility maximization problem:

$$\max_{c \geq 0, a_0} \quad u(c_0) + \beta u(c_1) \tag{6}$$

$$\text{s.t.} \qquad q_0 a_0 + c_0 \leq a_{-1} + y_0 - \tau_0 \tag{3}$$

$$c_1 \leq a_0 + y_1 - \tau_1 \tag{4}$$

$$a_0 \geq \underline{a}. \tag{5}$$

Optimization problems similar to (6) are often referred to as *consumption-savings* problems.

Since u is an increasing, strictly concave function, optimization (6) has a single solution.[1] In that solution, budget constraints (3) and (4) hold with equality - otherwise households could raise consumption and get more utility. Let $c(a_{-1}; q_0, \tau)$ and $a_0(a_{-1}; q_0, \tau)$ be the pair of consumption levels (c_0, c_1) and public bond purchases that solve (6). The arguments underscore how households' choices depend on their initial net wealth, the price of public bonds and taxes.

2. Present-Value Budget Constraints

2.1. Government and Fiscal Policy Sustainability

Let us return to the government's budget constraints, repeated below for convenience:

$$q_0 b_0 + s_0 = b_{-1} \tag{1}$$

$$s_1 = b_0. \tag{2}$$

($s = \tau - g$ is the primary surplus sequence). Equations (1) and (2) are examples of *sequential* budget constraints ("sequential" because we have one of them in each period).

[1] We assume income y and initial wealth b_{-1} are large enough so that the household can choose non-negative amounts of consumption goods.

Sequential budget constraints focus on the interaction between surpluses and wealth. But they also indirectly capture the possibilities of *intertemporal allocation* available to the government. For example: if it wants to lower period-zero surpluses by one ($\Delta s_0 = -1$, Δ means "a change in"), it must issue the necessary volume of new bonds $\Delta b_0 = 1/q_0 = 1 + r_0$; and then raise period-one surpluses by $\Delta s_1 = \Delta b_0 = 1/q_0$ to pay the additional debt.

It is often useful to represent the restrictions involving current and future surpluses more directly, with a single expression. Replace (2) on (1) to get:

$$b_{-1} = s_0 + q_0 s_1. \tag{7}$$

Equation (7) is the government's *present-value* budget constraint. It immediately shows that $\Delta s_0 = -1$ demands $\Delta s_1 = 1/q_0$.

We say "present-value" because we are converting spending in different points in time to their corresponding value in period zero. Indeed, the value in $t = 0$ of the delivery of X goods in $t = 1$ is $q_0 X$, since any agent can purchase X bonds for that amount, and get the X goods in $t = 1$.[1]

We say "budget constraint" because expression (7) is a sufficient and necessary condition to ensure that the government does not default. Conveniently, it does not depend on the b_0 term, only on fiscal policy objects τ and g through the surplus terms $s = \tau - g$. In that sense, the present-value budget constraint implies and is implied by fiscal policy *sustainability*.

Let us check this important claim. If the government does not default, then s and b_0 must respect the sequential budget constraints (1) and (2). Together, they imply (7). Thus, no default $\implies$ the present-value budget constraint.

In the opposite direction, suppose we have a surplus process $s = (s_0, s_1)$ that satisfies (7). We use the period-zero sequential budget constraint (1) to

[1]This is a *no-arbitrage* argument: If the value was $A > q_0 X$, you could sell the period-one delivery of X goods for A and purchase the required bonds for $q_0 X$ to make a something-for-nothing profit.

find the necessary volume of bonds the government needs to issue:

$$b_0 = \frac{b_{-1} - s_0}{q_0}.$$

The above b_0 ensures that the government does not default in period zero. Does it default in period one? By assumption, the surplus pair satisfies (7). So:

$$b_{-1} = s_0 + q_0 s_1 \implies s_1 = \frac{b_{-1} - s_0}{q_0} = b_0.$$

Since $s_1 = b_0$, period-one sequential budget constraint (2) holds. In conclusion, validity of the present-value budget constraint $\implies$ no government default.

2.2. Re-Stating Households' Consumption-Savings Problem

Consider now the sequential budget constraints faced by households, expressions (3) and (4). The conclusions we find above for the government apply somewhat similarly. The sequential budget constraints imply the present-value budget constraint:

$$a_{-1} \geq [c_0 - (y_0 - \tau_0)] + q_0 [c_1 - (y_1 - \tau_1)]. \tag{8}$$

Each term in brackets represents the household's expenditure in excess of its after-tax income (you can think of it as the household's own "primary deficit"). The present value of its excess consumption must be lower or equal to the initial wealth a_{-1}. Intuitively, if its exceeds a_{-1}, then households default in period one.

Like in the government's case, a consumption process $c = (c_0, c_1)$ that satisfies the present-value budget constraint (8) also satisfies the sequential budget constraints, if we choose the right net wealth a_0. For instance, we can use period-one budget constraint, expressed with equality:

$$a_0 = c_1 - (y_1 - \tau_1). \tag{9}$$

The equivalency between restricting households' consumption choice using sequential or present-value budget constraints opens the door to writing the consumption-savings problem (6) in terms of the c only:

$$\underset{c\geq 0}{\text{Max}} \quad u(c_0) + \beta u(c_1) \tag{10}$$

$$\text{s.t.} \qquad a_{-1} \geq [c_0 - (y_0 - \tau_0)] + q_0\,[c_1 - (y_1 - \tau_1)] \tag{8}$$

$$(a_0 =)\; c_1 - (y_1 - \tau_1) \geq \underline{a}. \tag{5}$$

(We have used (9) to replace a_0 in the borrowing constraint.[1]) The solution $c(a_{-1}; q_0, \tau)$ to problem (6) also solves problem (10). We can then use (9) again to recover the optimal demand for public bonds $a_0(a_{-1}; q_0, \tau)$.

3. Intertemporal Choice and Equilibrium

3.1. Household Optimality

We want to characterize the *competitive equilibrium* of our two-period economy. The competitive equilibrium is defined by market prices and quantities that cover two properties. First, agents choose the quantities optimally, taking prices as given. The "taking prices as given" part makes the equilibrium "competitive". Second: all markets clear, which means that quantities optimally supplied equal quantities optimally demanded.

When computing an equilibrium, we fix fiscal policy (g, τ). We will later study how the government can choose fiscal policy to generate the "best" equilibrium possible. For now, we take g and τ as given, assuming that they respect the present-value budget constraint (7).

Consider households' optimal choices, $c(a_{-1}; q_0, \tau)$ and $a_0(a_{-1}; q_0, \tau)$. Because they solve the consumption-savings problem (6) (or (10)), they must satisfy the first-order optimality condition associated with that problem. In an interior solution (*i.e.*, in a solution with $c_0 > 0$, $c_1 > 0$), that

[1] (9) is the only level of bond purchases consistent with a consumption choice because the sequential budget constraints hold with equality in the solution of (6).

condition is the *Euler equation*

$$q_0 u'(c_0) = \beta u'(c_1). \tag{11}$$

We interpret the Euler equation (11) as a condition of consumption smoothing. Since the utility function u is increasing and concave, *marginal utility* u' is a positive, but *decreasing* function.[1] Intuitively, consuming more always makes the household "happier", but the amount of extra "happiness" an additional unit of consumption provides declines as it consumes more. Equating marginal utility therefore means balancing value over time. If you are lost in the desert, do not empty the waterskin in the first night.

To prove (11) is the first-order condition for optimality, consider the following variational argument. The utility gain of marginally increasing period-one consumption by Δc_1 is $\beta u'(c_1)\Delta c_1$. According to the present-value budget constraint (8), to increase period-one consumption by Δc_1, the household must give up $\Delta c_0 = -q_0 \Delta c_1$ units of period-zero consumption.

$$\begin{aligned}
a_{-1} &= [c_0 - (y_0 - \tau_0)] + q_0 \, [c_1 - (y_1 - \tau_1)] \\
\Delta a_{-1} &= \Delta \, [c_0 - (y_0 - \tau_0)] + q_0 \, \Delta \, [c_1 - (y_1 - \tau_1)] \\
0 &= \Delta c_0 + q_0 \Delta c_1
\end{aligned}$$

The utility loss of reducing period-zero consumption is

$$u'(c_0)\Delta c_0 = -q_0 u'(c_0)\Delta c_1.$$

For a choice of c to be optimal, the marginal gain cannot be lower or higher then the marginal loss. Thus, $q_0 u'(c_0)\Delta c_1 = \beta u'(c_1)\Delta c_1$, as we wanted to show.

The Euler equation (11) establishes a positive relationship between period-zero and period-one consumption.

$$c_0 \uparrow \implies u'(c_0) \downarrow \implies u'(c_1) \downarrow \implies c_1 \uparrow$$

[1]Technicaly, marginal utility could be zero even though utility is increasing. Here, I am assuming $u' > 0$.

To find the actual solution $c(a_{-1}; q_0, \tau)$ to the consumption-savings problem, we impose the fact that the present-value budget constraing must hold with equality. Households exhaust their resources. We then have two equations determining two variables.

The optimal choice of period-zero savings $a_0(a_{-1}; q_0, \tau)$ follows from the sequential budget constraint (3).

3.2. The Competitive Equilibrium

In equilibrium, prices adjust so that the two markets of our economy - consumption goods and publics bonds - clear in both periods.

Now that we understand households' demand for public bonds and consumption goods, we need to find prices that clear all markets.

Although, we have allowed initial public debt b_{-1} and households' savings a_{-1} to differ, it is natural to restrict these parameters to be equal to each other: $a_{-1} = b_{-1}$. If the government sold bonds in $t = -1$, it must have sold them to households.

Having studied the equilibrium of the model, we now examine two fiscal policy issues.

4. Ricardian Equivalence

In general terms, *Ricardian equivalence* is the proposition that households' consumption choices are unaffected by the timing of taxation. In this section, we model Ricardian equivalency in our two-period setup and discuss which conditions are key to make it hold.

We start with a government that fixes a fiscal policy pair g and $\tau = (\tau_0, \tau_1)$. Fiscal policy is sustainable, in that the present-value budget constraint (7) is satisfied. We can write it as:

$$[\tau_0 + q_0 \tau_1] = b_{-1} + [g_0 + q_0 g_1] \,.$$

On the left, the present value of tax proceeds; on the right, the present value of outlays divided between spending and old debt redemption.

Households observe the path of due taxes, and plan how much to consume $c(\tau)$ and how much to save $a_0(\tau)$.[1]

Suppose that, still at the beginning of period zero, the government announces a different, *but still sustainable*, path to lump-sum taxes, $\hat{\tau} = (\hat{\tau}_0, \hat{\tau}_1)$. Spending g remains unaltered.

How do households revise their consumption plans in response to the government announcement? It turns out that, in the conditions of our two-period model, *they don't*: $c(\tau) = c(\hat{\tau})$. We say that Ricardian equivalency holds.

The key to prove the proposition is to show that different but equally sustainable taxation paths do not change the set of consumption levels affordable by households. Formally, any c that satisfies the constraints of the consumption-savings problem (10) under τ will continue to satisfy them under $\hat{\tau}$, and vice-versa.

Let's check that claim. We start with the present-value budget constraint (8), which holds with equality. We can re-write it as:

$$[c_0 + q_0 c_1] + [\tau_0 + q_0 \tau_1] - [y_0 + q_0 y_1] = a_{-1}.$$

The middle term on the left-hand side is the present value of charged taxes. Since both τ and $\hat{\tau}$ are fiscally sustainable, and since g is unchanged, that quantity must stay constant:

$$[\tau_0 + q_0 \tau_1] = [\hat{\tau}_0 + q_0 \hat{\tau}_1] = b_{-1} + [g_0 + q_0 g_1].$$

Therefore, the household's present-value budget constraint is unchanged.

Next, consider the borrowing constraint (5). Since we use the natural borrowing limit, they read:

$$\begin{aligned} c_1 - (y_1 - \tau_1) = \quad & a_0 \geq \underline{a} \quad & = -(y_1 - \tau_1) \\ c_1 - (y_1 - \hat{\tau}_1) = \quad & a_0 \geq \underline{a} \quad & = -(y_1 - \hat{\tau}_1) \end{aligned}$$

[1]In this section only, I ignore the arguments a_{-1} and q_0 of the optimal solutions for brevity.

Both restrictions above are satisfied whenever $c_1 \geq 0$ (this is how we define the natural borrowing limit!). Hence, the borrowing limit is effectively unchanged.

Since the restrictions of the consumption-savings problem (10) remain the same, the optimal level of consumption cannot be different. In conclusion, $c(\tau) = c(\hat{\tau})$.

5. The Fiscal Multiplier