Title

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1 Unexpected Inflation in a Benchmark NK Model

1.1 Connection with the Value of Debt

I start by establishing the connection between inflation and the value of debt.

Accounting. Consider a government that manages a public debt composed of one-period bonds, denominated in a specific currency. There is no constraint on the nature of this currency, and the assumption of one-period bonds buys simplicity but is not critical for the argument.

At the beginning of period t, the face value of debt issued in the previous period is V_{t-1}. The government finances the payment of V_{t-1} by either raising new debt maturing in the following period at a price Q_t or by running a (nominal) primary surplus S_t.

$$V_{t-1} = Q_t V_t + S_t \tag{1}$$

Let $P_t > 0$ be the relative price of an arbitrary basket of goods in terms of our selected currency and define $\beta_t \equiv Q_t P_{t+1}/P_t$ and $\beta_{t,t+k} \equiv \prod_{\tau=t}^{t+k} \beta_\tau$. Variable β_t represents a real discount for public bonds relative to the basket of goods. Since V satisfies (1), it also satisfies

$$\frac{V_{t-1}}{P_t} = \sum_{i=0}^{k} \beta_{t,t+i-1} \left(\frac{S_{t+i}}{P_{t+i}} \right) + \beta_{t,t+k} \frac{V_{t+k}}{P_{t+k+1}} \qquad \text{for any } k \geq 0 \tag{2}$$

regardless of the paths of public debt prices and surpluses. Importantly, expressions (1) and (2) do not represent a constraint on the path of nominal surpluses $\{S_t\}$ the government chooses to follow. They merely express the value of future debt given $\{S_t\}$, prices and current face value V_{t-1}.

Economics. I now make two assumptions about agents' behavior.

Assumption 1: At period t economic agents form expectations over the path of future, unknown variables through an operator $\tilde{E}_t$ which satisfies the linearity condition $\tilde{E}_t(X+Y) = \tilde{E}_t(X) + \tilde{E}_t(Y)$ and $\tilde{E}_t(X) = X$ if X is in the information set at t. Such operator can be heterogeneous across agents as long as these two conditions are satisfied.

Assumption 2: $\lim_{k\to\infty} \tilde{E}_t \left(\beta_{t,t+k} \frac{V_{t+k}}{P_{t+k+1}} \right) = 0$ at every period t. Assumption 2 is tipically referred to a no-bubble condition. In most micro-founded models, it is not even an assumption, but a result of optimal intertemporal consumption choice by households.

Assumptions 1 and 2 added to (2) lead to the valuation equation of public debt:

$$\frac{V_{t-1}}{P_t} = \sum_{i=0}^{\infty} \tilde{E}_t \left(\beta_{t,t+i-1} \frac{S_{t+i}}{P_{t+i}} \right). \tag{3}$$

Given the (pre-determined) face value of maturing public debt, the relative price of the basket of goods in terms of debt currency is determined by the expected β-discounted stream of surpluses in terms of the basket of goods. This latter term - the right-hand side of (3) - I call the *real value of public debt*.

Equation (3) provides the connection between fiscal and inflation shocks I explore in the paper. (TODO: Mention importance of revision of public bonds' prices, absent here)

(Incomplete: Maybe include "What about Japan?" footnote)

1.2 The New-Keynesian Model

I start with the two usual equations of the New-Keynesian model. All variables should be interpreted as deviations from a steady-state equilibrium.

$$c_t = E_t c_{t+1} - \sigma \left[i_t - E_t \pi_{t+1} \right] \tag{4}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{5}$$

along with an equation for market clearing for goods market:

$$y_t = \gamma c_t + g_t, \tag{6}$$

where y, c, i, π and g represent respectivelly log-output, log-consumption, the interest rate, the inflation rate and government spending in levels.[1]

The stock of real public debt v follows the law of motion (reference to earlier equation, reference to the use of rational expectations - use of E, not $\tilde{E}$, reference to why assumption 2 above holds here)

$$\beta v_t = v_{t-1} + i_{t-1} - \pi_t - s_t \tag{7}$$

where $s_t \equiv \tau_t - g_t$ is the public primary surplus (which does not include interest payments on debt). τ_t are total tax proceeds in levels. In the stationary equilibrium of the NK model assumption 2 above holds, and, hence, v coincides with the real value of public debt.

Policy. *Observed* monetary policy is muted, except for a white-noise shock: $i_t = \epsilon_{i,t}$.

Fiscal policy prescribes the following rules for taxation (which I assume to be entirely *lump-sum*) and public expenditures:

$$\tau_t = \rho_\tau \tau_{t-1} + \alpha_\tau v_t + \epsilon_{\tau,t} \tag{8}$$

$$g_t = \rho_g g_{t-1} - \alpha_g v_t + \epsilon_{g,t}. \tag{9}$$

Stability of public debt requires either $\alpha_\tau > 0$ or $\alpha_g > 0$.

The equations above by themselves do not determine unexpected changes to the real value of public debt, and hence they do not pin down unexpected inflation. The last equation characterizing policy solves that issue:

$$\pi_t = E_{t-1} \pi_t + \eta_t, \tag{10}$$

[1] γ represents the steady-state consumption-to-output ratio. The choice of linearizing equilibrium conditions around the level of government spending and not its log makes the connection with the rest of the paper clearer. I also linearize around an equilibrium with output = real debt.

for an exogenous term η_t. The term $\Delta E_t \pi_t = (E_t - E_{t-1})\pi_t$ I call unexpected current inflation. In this case, unexpected current inflation is given by η_t.

There are two existing selection mechanisms that justify equation (10) and provide an interpretation to it: fiscal selection and the spiral threat selection. Both imply (10) while leaving other equations unchanged (observational equivalence, Cochrane (2011), Cochrane (1998)) and, more importantly, both interpret η as part of public policy, as a government *choice*.

Fiscal selection, or the fiscal theory of the price level, arrives at (10) by means of (3), with causality coming from right to left. Any economic shock can change the conditional distribution of discounted future surpluses (in units of goods) backing the stock of public nominal liabilities. It can thus change its real value. The relative price of public debt in terms of goods $1/P_t$ then adjusts to reflect that change, much in the same way that news of future dividends (in money units) per stock change the relative price of stocks in terms of money (Cochrane (2005)). (Maybe include a reading of 2022 US inflation through the lenses of FS and STS)

Spiral threat selection is the approach that most of the monetary economics literature has adopted so far. It arrives at (10) by means of an explosive root introduced by an interest policy equation of the format $i_t = \phi\pi_t, \phi > 1$. The equation was incorrectly associated to the famous Taylor (1993) rule, for its role in the NK model is by no means to stabilize "demand" shocks via fast, pro-cyclical real interest rates. *Au contraire*, the policy rule here introduces the instability required by the NK model to pin down unexpected inflation. Assuming muted monetary policy $i_t = 0$, the system of equations (4)-(5) (with $c = y$ for simplicity) is "too stable": it contains one explosive eigenvalue for two forward-looking variables. Any choice of unexpected inflation forms a stable equilibrium path that converges to the zero steady state.[2] Equation $i_t = \phi\pi_t$ solves that issue when $\phi > 1$.

Importantly, the *selection of equilibrium* is completely unrelated to the *observed* interest rate. This is why $i_t =$ white noise as above is a perfectly fine specification for *observed* interest. More rigorously, consider the basic NK system (4)-(5), with $c = y$. Add to that the following equations:

$$i_t = i_t^* + \phi(\pi_t - \pi_t^*) \qquad \phi > 1 \tag{ST-1}$$

$$i_t^* - E_t\pi_{t+1}^* = i_t - E_t\pi_{t+1} \tag{ST-2}$$

$$i_t^* \text{ given for all } t \tag{ST-3}$$

$$\pi_t^* \text{ given at } t. \tag{ST-4}$$

Format (ST-1) is due to King and William (1996); i^* is the central bank's desired observed interest rate. The term π_t^* is a stochatic inflation target. Equation (ST-2) asks that the government's choices respect private market conditions and expectations formation. It forces the government to elect *unexpected* inflation only.[3]

Mechanically, one can combine (ST-1) and (ST-2) to find $E_t\pi_{t+1} - E_t\pi_{t+1}^* = \phi(\pi_t - \pi_t^*)$; $\phi > 1$

[2] Economists have interpreted this feature as admissibility of "sunspot" shocks. Without a selection mechanism, (4)-(5) will only determine the unexpected component of one variable, *if* it is fed the unexpected component of the other.

[3] The attentive eye may have noticed an apparent modelling sin: system (4)-(5), (ST-1)-(ST-4) presents six equations, for only five variables: y, π, π^*, i, i^*. There is no over-identification, nevertheless. Target inflation enters the system both as a static (= forward-looking) variable π_t^* *and* as a state variable, in expected value $E_{t-1}\pi_t^*$. Another way to write (ST-3) would be $E_{t-1}\pi_t = i_t^* - (i_t - E_{t-1}\pi_t)$. It becomes evident then that (ST-4) only really picks the unexpected component of inflation.

and Blanchard and Kahn (1980)'s razor then champion the unique stationary path $\pi = \pi^*$, $i = i^*$, which form the *observed* equilibrium. Parameter ϕ remains unidentified (Cochrane (2011)).

Researchers have interpreted (ST-1) as a threat of nominal spiral - hence my name choice "spiral threat" selection. Different papers discuss if central banks can indeed rule out nominal spirals, but the key assumptions here do not really relate to what the central bank can do, but what *households believe* it can and would. Indeed, note that there is nothing particularly special about inflation in (ST-1)-(ST-4). One could as well write the whole system using an output target instead:

$$i_t = i_t^* + \phi(y_t - y_t^*) \qquad \phi > 1 \tag{ST-1'}$$

$$i_t^* - E_t y_{t+1}^* = i_t - E_t y_{t+1} \tag{ST-2'}$$

and now the "threat" is not that of a nominal spiral, but of a *real* spiral. Obviously, the central bank cannot trigger a "hyperproduction" (as in hyperinflation) process. Neither could it stop one, say if productivity for some reason started to grow at exponential rates. But, if the central bank vacuously threatens hyperproduction, and it is the case that agents believe its threat; and if then the central bank vacuously promisses to stop the hypothetical hyperproduction it has vacuously threatened to create, and again agents trust its word; then and only then does the Blanchard and Kahn (1980) equilibrium arranged by (ST-1')-(ST-2') arises. The actual powers of the central bank are irrelevant.

While I favor a fiscal selection interpretation of unexpected inflation throughout the article, the takeaway from this discussion is that both equilibrium selection mechanisms interpret as a government *choice* - even if an indirect one - the determination of unexpected inflation.

1.3 Unexpected Current Inflation

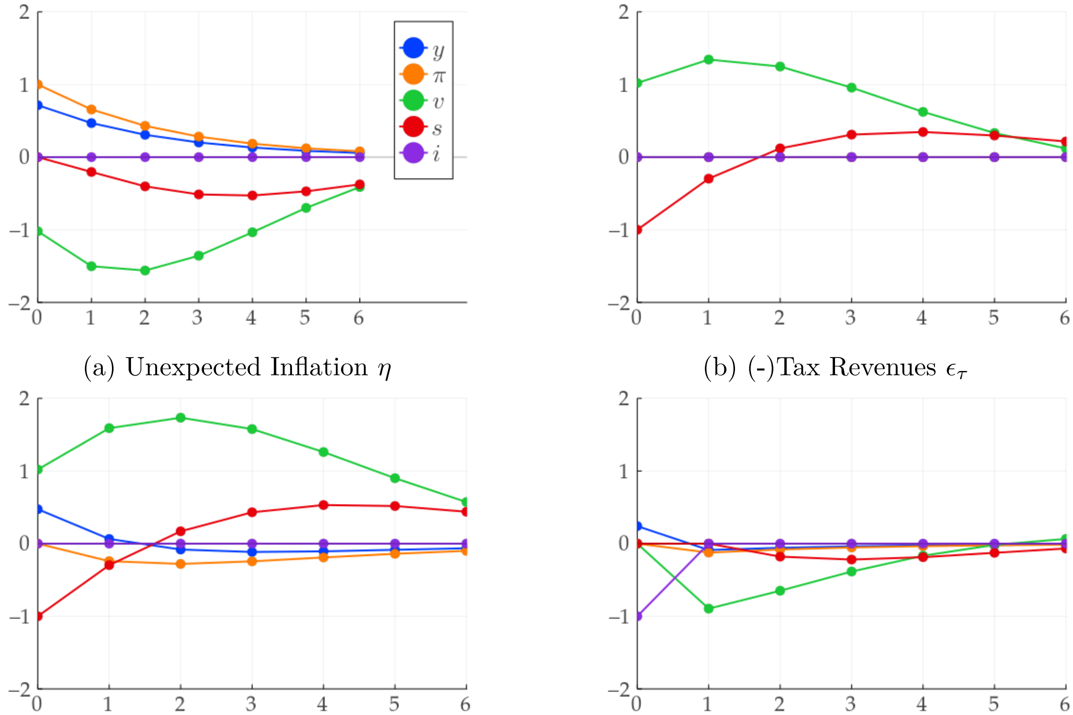
Consider the response of the New-Keynesian model to η, and ϵ shocks, one at a time, plotted in figure 1. Calibration follows literature standards: $\sigma = 0.5$, $\beta = 0.98$, $\gamma = 0.75$ and $\rho_\tau = \rho_g = 0.5$. Momentarily, I set $\kappa = 0.50$ to make figures pretty. In this benchmark case, I consider fiscal adjustment via taxation only: $\alpha_\tau = 0.2$ and $\alpha_g = 0$.

Panel 1a plots the response to the unexpected inflation shock η. Inflation jumps by assumption, the fiscal interpretation being that agents foresee a reduced stream of surpluses. Accordingly, the real value of public debt v jumps down on spot. A lower debt leads taxation τ to decline (not plotted) via the $\alpha_\tau v_{t-1}$ term. The government runs deficits starting in the first period following the shock (I refer to $s < 0$ as a fiscal deficit). These deficits 1. slowly bring v back to zero and 2. validade agents' expectation at period zero of a lower value of public bonds - indeed primary surpluses were lower.

The impact on economic activity resembles the typical Keynesian "demand" shock, combining an increase in inflation and output at the same time. Positive inflation in period zero leads to a negative real interest rate; the IS curve (4) then implies output larger than future output - output is large and declining.[4] Large output implies large marginal costs, and, by (5), inflation greater than future inflation - inflation is thus positive and declining.

Panels 1b and 1c show that, in the absence of unexpected inflation in period zero, expansionary fiscal policy fails to generate inflation at all in the basic NK model. A negative shock to taxation - the model version of COVID checks - simply leads to an increase in public debt, subsequently paid through taxes that turn positive in period two. Households are unconstrained and have zero marginal

[4] The apparently small response of output follows from the choice of κ. Values that are lower than my choice lead to more pronounced responses of equal sign.



(a) Unexpected Inflation η

(b) (-)Tax Revenues ϵ_τ

(c) Government Spending ϵ_g

(d) (-)Nominal Interest ϵ_i

Figure 1: NK Model - Impulse-Response Function to Policy Shocks

propensity to consume out of their checks. Output thus stays put, which implies $\pi_t = E_t\pi_{t+1}$ by the Phillips curve. $\pi_0 = 0$ follows from the absence of unexpected inflation.

A positive shock to public spending g does affect output and inflation, as the government directly purchases goods from firms (equation (6)). We can think of government spending as a transfer to a fictional "public household" with constant marginal propensity to consume equal to one. Output increases in period zero. The Phillips curve then says that current inflation must be greater than future. But since current inflation is zero (no unexpected jump by assumption), that means inflation *declines* from period zero to one. In the absence of unexpected inflation, the NK model predicts below-average inflation, or even deflation, as a consequence of increased public expenditure.

Lastly, panel 1d corresponds to an expansionary monetary policy shock. Without unexpected inflation, the effect of a monetary policy shock is purely Fisherian (references of Fisherian interest shock): lower interest forecasts lower inflation. Stimulative interest does stimulate output, albeit for a single period, as low inflation produces a contractionary effect thereafter.

Figure 1 assumes fiscal adjustment is carried out entirely through tax instead of spending adjustments ($\alpha_\tau > 0$, $\alpha_g = 0$). The symmetric opposite assumption little changes the predictions of the model. The only qualitative change happens in the case of the tax reduction shock ϵ_τ. Since government spending changes with debt, there are small output effects that lead to lower inflation in the transition - again the "wrong" sign. The quantitative effects are nevertheless small, and I leave the IRF figures to the appendix.

1.4 Unexpected Total Inflation

Economic news lead not only to the revision of expectations of current inflation, but also of its entire future path. As Sims (2011) exemplifies with his study of the effects of monetary policy in US inflation in the 1970s, the short-run innovation can be remarkably different than the one long-run one.

(Incomplete)

We can solve the linearized law of motion of public debt (7) forward and apply assumptions 1 and 2 to arrive at

$$v_{t-1} = \sum_{i=0}^{\infty} \beta^i E_t s_{t+i} - \sum_{i=0}^{\infty} \beta^i E_t i_{t-1+i} + \sum_{i=0}^{\infty} \beta^i E_t \pi_{t+i},$$

which generalizes (3). The inflation revaluation of public debt term (the denominator on the left side of (3)) corresponds to the first term of the inflation sum on the right-hand side. Taking innovations:

$$0 = \sum_{i=0}^{\infty} \beta^i \Delta E_t s_{t+i} - \sum_{i=0}^{\infty} \beta^i \Delta E_t i_{t-1+i} + \underbrace{\sum_{i=0}^{\infty} \beta^i \Delta E_t \pi_{t+i}}_{\text{Unexpected Total Inflation}}. \tag{11}$$

Expression (11) is from Cochrane (2022a). The sum $\sum \beta^i \Delta E_t \pi_{t+i}$ I call *unexpected total inflation*, which is really shorthand for revision of expectations over the discounted inflation path. I also refer to unexpected future surpluses and interest in a similar manner.

Decomposition (11) simply states that surprises to the relative price of public liabilities in terms of goods, captured by unexpected current inflation $\Delta E_t \pi_{t+1}$, reflect changes in the real value of public debt. Re-writting it as

$$-\underbrace{\Delta E_t \pi_t}_{\text{Unexpected Current Inflation}} = \underbrace{\sum_{i=0}^{\infty} \beta^i \Delta E_t s_{t+i} - \sum_{i=0}^{\infty} \beta^i \Delta E_t i_{t-1+i} + \overbrace{\sum_{i=1}^{\infty} \beta^i \Delta E_t \pi_{t+i}}^{\text{Unexpected Future Inflation}}}_{\text{Innovation to the Real Value of Public Debt}}. \tag{12}$$

makes that point clear. It also reminds us that equation (11) continues to be a debt valuation equation, not a budget constraint. Yet, the use of language such as "higher total inflation *pays* for lower total surpluses" can often simplify the exposition.

The β discounting is the linearized version of the $\beta_{t,t+k}$ discounting term of equations (2) and (3). It means that we mark-to-(steady-state-)market each revision of expectation, so that we may interpret each sum in terms of current-period market value.

Table 1 shows the decomposition for the policy shocks of NK model. Note the minus in front of the unexpected total interest; each row sums to zero. I re-calibrate $\kappa = 3.8$ to a more realistic value, since the quantitative aspect is more relevant now.[5]

The first panel corresponds to the case of figure (1). Taxes respond to real debt variation, not spending. The 1% unexpected current inflation shock leads to a roughly 1.5% increase in unexpected total inflation. Bondholders pay for the unexpected decline in total surpluses.

[5] I calibrate $\kappa = (1-\theta)(1-\beta\theta)/\theta$ using the price rigidity parameter $\theta = 0.65^4$ estimated by Smets and Wouters (2007), adjusted for annual frequency.

Policy Shock	$\sum \beta^i \Delta E_t \pi_{t+i}$	$\sum \beta^i \Delta E_t s_{t+i}$	$-\sum \beta^t \Delta E_t i_{t-1+i}$
Fiscal Adjustment via Taxes ($\alpha_\tau > 0$, $\alpha_g = 0$)			
Unexpected inflation η_t	1.46	-1.46	0
(-) Tax revenue ϵ_τ	0	0	0
Government spending ϵ_g	-2.06	2.06	0
(-) Monetary policy ϵ_i	-0.66	-0.32	0.98
Fiscal Adjustment via Spending ($\alpha_\tau = 0$, $\alpha_g > 0$)			
Unexpected inflation η_t	1.46	-1.46	0
(-) Tax revenue ϵ_τ	-0.02	0.02	0
Government spending ϵ_g	-2.17	2.17	0
(-) Monetary policy ϵ_i	-0.66	-0.32	0.98
Fiscal Adjustment via Taxes + Taylor Rule ($i_t = \phi\pi_t$)			
Unexpected inflation η_t	2.49	-1.27	-1.22
(-) Tax revenue ϵ_τ	0	0	0
Government spending ϵ_g	-3.9	2	1.92
(-) Monetary policy ϵ_i	-1.24	-0.35	1.59

Table 1: Unexpected Total Inflation

The taxation shock leads to a "zero-zero-zero" decomposition as inflation and interest are unchanged, and future taxes pay for the current negative shock. A 1% increase in government spending leads to a *positive* unexpected total surplus of about 2%, which pay for the unexpected total *deflation*. Finally, a 1% unexpected decline in interest creates fiscal space consumed by lower total inflation and surpluses in a two-to-one ratio. The reported measures quantify how the three expansionary policy shocks, which fail to create unexpected current inflation by assumption, actually create unexpected total *disinflation*, by result.

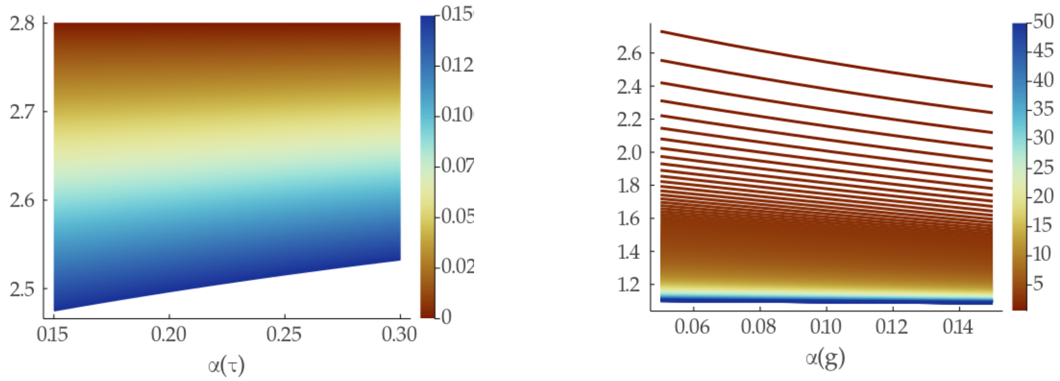
The second panel considers the case of debt stabilization via changing expenditure g. I set $\alpha_g = 0.07$, so that the decomposition of the unexpected inflation shock is about the same. Switching the variable of adjustment does little to change the decomposition of the other shocks.

The third panel returns to tax adjustment with the same $\alpha_\tau = 0.2$, but includes a more realistic Taylor rule to monetary policy $i_t = \phi\pi_t$. I use $\phi = 0.50$. Active monetary policy leads to larger reactions of each term of the decomposition to our policy shocks in comparison with the baseline case. Results also reveal the Fisherian character of the NK model. Unexpected total inflation and unexpected total interest have the same signal in all cases.

1.4.1 A Ricardian Equivalence Result for Inflation Decomposition

As (re-)stated by Barro (1974), the Ricardian Equivalence theorem says that different taxation plans fail to change households' perception of their own wealth, or the real value of debt, which is pinned down by the stock of bonds inherited from the previous period and the assumption of debt sustainability.[6] Thus they do not affect households' consumption path. The value of debt cannot jump unexpectedly at the beginning of any period.

[6] In the environemnt where Ricardian Equivalency is usually stated, debt is real. Thus, debt sustainability means a no-default condition.



(a) UTI as function of α_τ for different α_g

(b) UTI as function of α_g for growing κ

Figure 2: Ricardian Equivalence of Unexpected Total Inflation

In the environment with nominal debt, that is not case: the real value of public debt suddenly changes, and equation (12) shows that such change is given by unexpected current inflation. Therefore, in the NK model, that aspect of Ricardian Equivalence fails to hold, which is why early studies of the fiscal selection mechanism referred to it as being "Non-Ricardian" (although, again, (3) and (12) do not depend on the selection mechanism).

However, under one key condition, the second and often more celebrated aspect of Ricardian Equivalence does hold in the NK model. (For the following statements, we can generalize the interest rate policy to a Taylor Rule $i_t = \phi_1 \pi_t + \phi_2 y_t + \epsilon_{i,t}$.)

Proposition 1. *Given any exogenous innovation vector to the NK model, if $\alpha_g = 0$, the equilibrium paths of consumption, inflation, output and interest rates do not depend on ρ_g, ρ_τ or α_τ. The result only holds for combinations of parameters the lead to unique, stationary equilibria of the NK model.*

Corollary 1 (Ricardian Equivalence of Total Inflation)**.** *Under the assumptions of proposition 1, the innovation terms of the debt value decompositions* (11) *and* (12) *do not depend on ρ_g, ρ_τ or α_τ.*

The proof of proposition 1 is straightforward: if $\alpha_g = 0$, the system of equations (4)-(6), (9) plus the interest rule, by itself, has a unique, stationary solution. The solution of the overall system is unique by assumption, so they must coincide.

In addition, since any equilibrium path of the linearized NK model is given by an initial condition plus the responses of each sequence of innovations (the model is linear), proposition 1 essentially says that equilibrium paths do not depend on ρ_g, ρ_τ or α_τ. So, even if the value of public debt fluctuates in the NK model: 1. unexpected current inflation is a sufficient statistic for all other changes of behavior due to re-valuation of public debt; and 2. the conclusion of the Ricardian Equivalence theorem still applies: parameters of debt-repayment timing ρ_g, ρ_τ, α_τ do not affect households' consumption decision.

The key condition that proposition 1 asks is that $\alpha_g = 0$.

(Incomplete)

For this reason, when considering only stable, unique solutions, as $\kappa \to \infty$, the terms of the inflation decomposition become less sensitive to *all* parameters governing tax and spending (including α_g).

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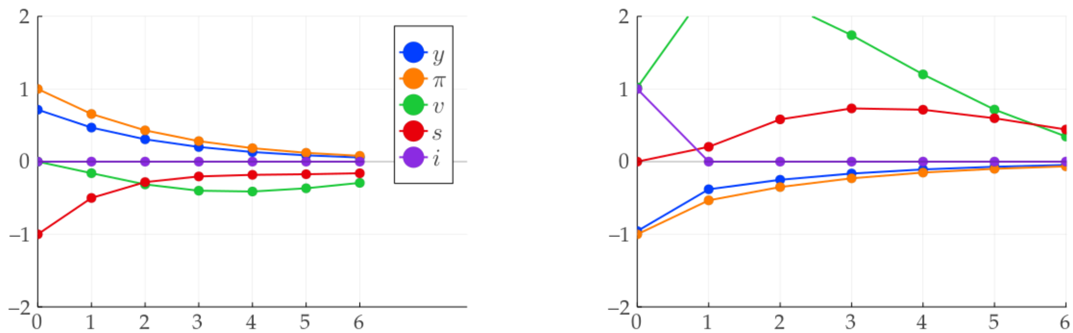

(a) ...COVID checks and inflation (b) ...and the monetary-fiscal effort to fight it

Figure 3: Combining shocks in the NK Model successfully reproduces...

1.5 Unexpected Inflation as a Choice

Maybe this section should be cut

1.5.1 Relationship with Sims (1980) Orthogonalization Method

The view that unexpected inflation is a choice automatically microfounds the orthogonalization process proposed by Sims (1980).

(Incomplete)

1.5.2 Combining Policy Shocks

The understanding of unexpected inflation as a integrand part of public policy opens the question of how it relates to other policy choices. The correlation with other policy shocks (the ϵ's) is particularly important in the NK model.

The IRF and the decomposition of the debt valuation equation show that having policy shocks be accompanied by unexpected current inflation is critical for the NK model to deliver responses consistent with most economists' view of the effects of "stimulative" policy on the price level and empirical evidence. (PAPERS WITH IDENTIFIED FISCAL SHOCKS)

(Incomplete)

In light of the connection between inflation and the value of debt, one might even expect η and the ϵ's to be correlated. Say, unexpectedly large spending leads to lower surpluses, hence a lower value of debt, hence unexpected inflation. However intuitive, the claims requires empirical inquiry. This is where we go next.

2 Empirical Models

I study two Bayesian estimation that differ in the information contained in the choice of prior. (Incomplete)

I estimate the model for a set of 24 economies, listed in table (REFERENCE).

2.1 A Bayesian VAR: Estimates of Unexpected Current Inflation

The first model is a Bayesian VAR. The autoregression is

$$X_t = \mu + \Psi(L)X_{t-1} + \Lambda w_t + \varepsilon_t. \tag{13}$$

Of course, domestic inflation is one of the variables in X, and the corresponding residual is a measure of unexpected current inflation. The term w_t is a forcing process that I use to model the impact of international variables over domestic ones. I assume $\varepsilon \sim N(0, \Sigma)$ and independent over time.

Estimating (13) using Bayesian methods has two advantages for us. First, we can interpret different specifications of the prior distribution over random parameters μ, Ψ, Λ and Σ as different views of the economy by the economic agents. Second, parameter shrinkage (*i.e.*, reducing the dependency of estimated parameters on the data) allows the econometrician to reduce the imprecision of out-of-sample forecasts in detriment of model fit, a good deal to analysts less concerned in explaining GDP forty years in the past and more interested in anticipating interest rates and inflation one year in the future. That latter point, and the fact that it has been most commonly adopted distribution in recent BVAR models - so it is a natural starting point -, justifies my choice for the Litterman (1979) (or Minnesota) prior.[7]

Minnesota Prior. The prior formalizes the view that the variables of interest follow a random walk, or a white-noise process if we difference them.[8] The distribution is part of the Normal-Inverse-Wishart family, in that the prior has the general format

$$\begin{aligned} \Sigma &\sim IW(\Phi; d) \\ \theta|\Sigma &\sim N(b, \Sigma \otimes \Omega). \end{aligned}$$

where $\theta = [\mu' \ \mathrm{vec}(\Psi')' \ \mathrm{vec}(\Lambda')']'$ and vec means stacking the columns.

The mean of the IW distribution is $\Phi/(d - N - 1)$, where N is the dimension of the square matrices and larger values of d represent tighter priors. I choose Φ to be the identity matrix (one percent standard deviation for all shocks, which are uncorrelated) and select $d = N + 2$, the lowest integer possible that leads to a well-defined distribution mean - which therefore equals Φ.

Prior parameters b and Ω reflect the choices that follow. Given Σ, parameter μ has zero mean, 10^6 variance and is uncorrelated with Ψ or Λ. Let $\Psi_{p,ij}$ be the (i, j) element of the p-th matrix in $\Psi(L)$. The conditional expectation is $E(\Psi_{p,ij} \mid \Sigma) = 0$ if $p > 1$ or if $i \neq j$.

In the baseline specification, I assume variables are $I(0)$. Hence, $E(\Psi_{1,ij} \mid \Sigma) = 0$ even when $i = j$. When I later test priors that prescribe some variables to have unit roots, I set $E(\Psi_{1,ii} \mid \Sigma) = 1$ for their corresponding indexes.

The conditional covariance between the coefficients in Ψ is

$$\mathrm{cov}\left(\Psi_{p,ij}, \Psi_{q,kl} \mid \Sigma\right) = \begin{cases} \frac{\lambda^2}{p^2}\frac{\Sigma_{ij}}{\Phi_{jj}} & \text{if } p = q \text{ and } j = l \\ 0 & \text{otherwise.} \end{cases}$$

[7] Giannone et al. (2015) show that priors of the Normal-Inverse-Wishart family, such as the Minnesota prior, lead to posterior distributions that can be decomposed as posterior = model fit term + penalty for model complexity.

[8] The literature about the Minnesota prior is vast. Interested readers can see del Negro and Schorfheide (2011) or Karlsson (2013) for a survey-like approach.

Symbol	Description	Nominal Debt	Inflation-Linked Debt	Dollar-Linked Debt
j	Index Symbol	N	R	D
	Notation	δ, ω	δ_R, ω_R	δ_D, ω_D
P_j	Price per Good	P	1	P_t^{US}
$\mathcal{E}_j$	Nominal Exchange Rate	1	P	Dollar NER
H_j	Real Exchange Rate	1	1	Dollar RER
π_j	Log Variation in Price	π	0	π_t^{US}
Δh_j	Log Real Depreciation	0	0	Δh_t

Notes: P = price of consumption basket in domestic currency. P^{US} = price of consumption basket in US dollars. NER = Nominal Exchange Rate. RER = Real Exchange Rate.

Table 2: Public Debt Denomination

The conditional mean of Λ is zero. Its conditional covariance is

$$\operatorname{cov}\left(\Lambda_{ij}, \Lambda_{kl} \mid \Sigma\right) = \begin{cases} \lambda^2 \Sigma_{ij} & \text{if } j = l \\ 0 & \text{otherwise} \end{cases}$$

and zero with the elements of Ψ.

I base the decision of which variables to include in the VAR on a more general law of motion for public debt than (1) which, it turns out, involves several variables economists usually include in macroeconometric research.

2.1.1 Generalizing Public Debt Instruments

Starting here, I recycle all notation established in section 1.

Fix the case of a country and its government. Let P_t be the price of the final goods basket in terms of the country's domestic currency.

I generalize the class of financing instruments available to the government in two dimensions: currency denomination and maturity structure. I do not consider the case of debt with stage-contingent nominal payoffs.[9]

The value of public bonds can be linked to different currencies, enumerated by j. Let $P_{j,t}$ be the price of the consumer price index in units of currency j. Let $Q_{j,t}^n$ be the discount rate for a zero-coupon public bond paying one unit of currency j after n periods. Without loss of generality, all bonds are zero-coupon. Let $\mathcal{E}_{j,t}$ be the price of currency j in units of domestic currency.

The notation is general enough to accomodate currency-linked bonds *per se*, but also inflation-linked securities such as American TIPS. In this case, one can interpret the "currency" as the final goods basket (so that $P_j = 1$ and $\mathcal{E}_j = P_t$). In the empirical exercises that follow, I consider consider domestic currency bonds ($j = 1$), inflation-linked (or real) bonds ($j = 2$) and US-dollar-denominated bonds ($j = 3$). Table 2 shows the value or interpretation of the variables defined above for these three cases.

[9] Publicly-ran social security and pension systems can be modelled as cases of public debt with state-contigent payoffs or, in a fashion compatible with my framework, as an additional layer of uncertainty in the primary surplus process.

I do not assume the price index for government's purchases (denoted P_t^S) and levied aggregates (such as income) to be the same as the price index for households' consumption (P_t). While governments do tend to tax private consumption, they also tax and transfer income, and purchase baskets of goods different than that of households (insert reference). The fiscal effect of variations in the relative price of government's to households' final goods basket must be accounted for.

With these changes, the law of motion of government debt (1) becomes

$$\sum_j \mathcal{E}_{j,t} B_{j,t-1}^1 = P_t^s S_t + \sum_j \mathcal{E}_{j,t} \sum_{n=2}^{\infty} Q_{j,t}^{n-1} \left(B_{j,t}^{n-1} - B_{j,t-1}^n \right), \tag{14}$$

where S_t is now the real primary surplus and $B_{j,t}^n$ is the face value of bonds issued in currency j, period t payable n periods in the future. The term on the left represents the cost of debt in period t; the second term on the right represents the selling of new bonds of all possible maturities.

Let $\mathcal{V}_{j,t} = \sum_n Q_{j,t}^n B_{j,t}^n$ be the end-of-period market value of nominal debt issued in currency j, $i_{j,t}$ the risk-free rate in bonds issued in currency j and $rx_{j,t} = \mathcal{V}_{j,t-1}^{-1} \sum_{n=1}^{\infty} Q_{j,t}^{n-1} B_{j,t-1}^n - i_{j,t-1}$ the realized excess return on portfolios that mimic the composition of j-currency debt. We can re-write (14) in terms of the $\mathcal{V}_j$ and its corresponding returns:

$$\sum_j (1 + rx_{j,t} + i_{j,t-1}) \mathcal{E}_{j,t} \mathcal{V}_{j,t-1} = P_t^s S_t + \sum_j \mathcal{E}_{j,t} \mathcal{V}_{j,t}$$

For empirical work, it is ideal to write the law of motion in terms of real and stationary variables. So, let Y_t be a trend such that $s_t \equiv S_t/Y_t$ is reasonably stationary, and let $g_{Y,t} = Y_t/Y_{t-1} - 1$ be its growth rate. Define the real "exchange rate" $H_{j,t} = \mathcal{E}_{j,t} P_{j,t}/P_t$ and its growth rate $\Delta h_{j,t} = H_{j,t}/H_{j,t-1} - 1$. Define the de-trended real value of j-indexed debt $V_{j,t} = \mathcal{V}_{j,t}/P_{j,t}Y_t$, the real value of total debt $V_t = \sum_j H_{j,t} V_{j,t}$ and the j-indexed share $\delta_{j,t} = H_{j,t}V_{j,t}/V_t$.

By properly dividing the whole above equation by $P_t Y_t$, and multiplying and dividing the j sum on the left by $P_{j,t-1}$, $P_{j,t}$, Y_{t-1} and $H_{j,t-1}$, we arrive at a final version for the law of motion of public debt:

$$V_{t-1} \sum_j \frac{(1 + rx_{j,t} + i_{j,t-1})(1 + \Delta h_{j,t})}{(1 + \pi_{j,t})(1 + g_{Y,t})} \delta_{j,t-1} = \frac{P_t^s}{P_t} s_t + V_t. \tag{15}$$

Stated now in real quantities, (15) generalizes (1). During period t, the government must "pay" V_{t-1} plus realized returns and eventual changes to the relative value of currency j.[10]

2.1.2 Variable Selection and Data Treatment

Based on the law of motion (15) I select eight variables for the VAR: public debt, inflation, nominal interest, real exchange rate, tax proceeds, government spending, relative price of public good basket and gross domestic product (GDP). From these, only GDP does not show up in (15). Conversely, I do not include data on the excess return of the public debt portfolios rx_j as it is not available to the majority of countries in the sample. Under the assumption of an exponential maturity structure for public debt, unexpected excess return is determined by unexpected interest rate movements. (Show that.) I also de-trend non-stationary variables using a constant linear trend of log-GDP - so I

[10] "Pay" comes in paranthesis here because, unlike in (1), the government does not actually redeem the entire term on the left at period t. It only pays for bonds maturing at t.

assume a constant log-linear trend - which then implies $g_Y = 0$.

Data is annual, with period ranges varying from country to country (Table with sample size). Inflation is the log variation in the consumer price index. The dollar real exchange rate is the nominal exchange rate to the US dollar multiplied by the ratio of US-to-domestic CPI. The nominal interest rate is the log of 1+ interest data. I take the price index for public spending and tax revenue to be the GDP deflator, since income taxation accounts for the bulk of tax proceeds for most countries. I normalize the average log relative price of the public basket to zero. The series for GDP is the log deviation from trend.[11] Public spending and public debt data are both divided by the GDP trend described in the appendix. Finally, I use primary surplus data, divided by the GDP deflator, to build a tax revenue series as government spending plus primary surplus.[12]

2.1.3 Unexpected Current Inflation

Given the annual model, I assume a single lag in the $\Psi(L)$ polynomial in the baseline specification. The forcing process contains lagged interest rate and current inflation, both from the United States $w_t = [i_{t-1}^{US}, \pi_t^{US}]$. When I estimate the US case itself, I omit w_t.

Priors of the Normal-Inverse-Wishart class are conjugate and admit well-known closed-form solutions for both the posterior distribution and the marginal likelihood, which allows us to rapidly perform the estimation for a given choice of prior tightness λ. As I attempt to approximate private agents' expectation formation through the prior distribution, I set λ so as to maximize the marginal likelihood, which, as Giannone et al. (2015) show, weights model fit and forecast variance.

Finally, in the baseline estimation, I assume all eight variables of the VAR to be $I(0)$. The prior distribution for Ψ is thus centered around $0_{8\times 8}$.

Table 3 reports estimated statistics related to the reduced-form residuals ϵ. I am interested in the stochastic properties of unexpected current inflation $\Delta E_t \pi_t = \epsilon_{\pi,t}$. I am interested in the stochastic properties of ϵ_π, the residual of the inflation equation. (Discussion of whether it is structural, here or in section 1.)

(Calculate statistical significance of averages and means)

All columns show the statistic calculated at the mode of the posterior distribution. The asterisk indicates that the sign of the reported value is statistically significant at the 10% confidence level. I perform inference by simulating a 10,000 sample for each country.

The first column reports $\sqrt{\Sigma_\pi}$, the standard deviation of $\epsilon_p i$. The remaining five columns report the statistic $b(\pi, x) \equiv \Sigma_{\pi,x}/\Sigma_x$, which is the projection coefficient of ϵ_π on ϵ_x. The value $b(\pi, x)$ answers the question "what do you predict unexpected current inflation $\epsilon_{\pi,t}$ to be if you observe $\epsilon_{x,t}$ and no other period-t data?"

The table shows that...

[11] I run OLS on $\log(gdp)_t = c_0 + c_1 t + \nu_t$ using all data available for GDP (which usually covers a longer period than that of the balanced panel) and define $\exp(\hat{c}_0 + \hat{c}_1 t)$ as the economy's trend, or Y_t in the notation of the previous subsection. The series for GDP is $\hat{\nu}_t$.

[12] Real GDP (constant 2015 prices), GDP deflator, public spending and the nominal exchange rate data come from the United Nations's National Accounts Main Aggregates Database. Consumer price index and and primary surplus data come from the IMF's WEO Database. Public debt (as ratio of GDP) comes Ali Abbas et al. (2011) database, which is kept up-to-date. The sources for interest rate vary from country to country; they are usually the central bank, but also from the IMF's International Financial Statistics database.

Country	Std(π)	$b(\pi, Tx)$	$b(\pi, G)$	$b(\pi, i)$	$b(\pi, gdp)$	$b(\pi, p^s - p)$
Advanced						
Denmark	0.63*	-0.08	0.08	0.10	0.04	-0.36*
Norway	0.82*	0.01	-0.71*	0.34*	0.02	-0.00
Sweden	0.98*	0.10	0.42	-0.23*	0.09	-0.42*
Switzerland	0.54*	0.31*	-0.35	0.46*	0.26*	-0.36*
United Kingdom	0.71*	0.04	0.29	0.36*	-0.03	-0.11
Iceland	0.73*	-0.11*	-0.05	-0.08	-0.06	-0.17
Canada	0.67*	0.08	-0.07	0.24*	0.05	0.10
United States	0.88*	0.06	-1.32*	0.34*	0.11	-1.05*
Australia	0.68*	0.13	-0.50	0.69*	-0.26*	-0.20*
Japan	0.57*	0.15*	-0.88*	0.48*	0.15*	-0.47*
Republic of Korea	0.82*	-0.57*	-0.24	0.54*	-0.28*	-0.27*
Developing						
Hungary	0.89*	0.24*	-0.26	0.57*	-0.08	-0.17
Poland	0.90*	-0.14	0.24	0.57*	0.00	-0.24
Ukraine	4.96*	1.98*	2.09	0.53*	-0.38*	-0.82*
Romania	10.84*	5.03*	9.19*	2.47*	-1.87*	-2.03*
Turkey	1.57*	0.29	-0.48	0.49*	0.21	0.33
Russia	1.07*	0.10	-0.75*	0.79*	-0.04	-0.02
Brazil	0.88*	-0.29*	-0.18	0.52*	-0.14	-0.49*
Colombia	1.13*	0.31	-1.10*	0.70*	0.44*	-0.78*
Chile	0.70*	-0.09	-0.73	0.55*	0.05	-0.22*
Mexico	0.94*	-0.43*	0.43	0.40*	0.03	-0.12
South Africa	0.71*	0.05	-0.08	1.12*	0.20	-0.55*
India	0.96*	-0.41*	0.29	0.16	0.20*	-0.26*
Indonesia	3.11*	0.23	-1.67	1.36*	-1.70*	0.60*
Average						
All	1.53	0.29	0.15	0.56	-0.12	-0.34
Advanced	0.73	0.01	-0.30	0.29	0.01	-0.30
Developing	2.20	0.53	0.54	0.79	-0.24	-0.37
Median						
All	0.88	0.07	-0.21	0.50	0.02	-0.25
Advanced	0.71	0.06	-0.24	0.34	0.04	-0.27
Developing	0.96	0.10	-0.18	0.57	0.00	-0.24

Table 3: BVAR Estimation: Current Unexpected Inflation

2.1.4 Robustness

2.2 A Tighter Prior: Estimates of Unexpected Total Inflation

I linearize (15). Let $\beta_j = \exp\{-(rx_j + i_j - \pi_j - g_Y)\}$ be the steady-state real and growth-adjusted discounting for public debt issued in currency j. I linearize around a steady-state - assumed to exist - with $\beta_j = \beta$ for all j and $P^s = P$. This leads to

$$\beta\left(v_t + s_t + s(p_t^s - p_t)\right) = v_{t-1} + v\left[\sum_j \delta_j\left(rx_{j,t} + i_{j,t-1} + \Delta h_{j,t} - \pi_{j,t}\right) - g_{Y,t}\right], \tag{16}$$

which generalizes (7). Since $\beta < 1$, equation (16) introduces an unstable eigenvalue to the system of equations it is part of.

In general, public debt data will not respect the laws of motion (15) and (16) for a few reasons. Government agencies in the vast majorities of cases report data on the book value of debt, while the theoretical law of motion of public debt (15) leads to expressions that involve the *market value* of debt. Moreover, measurement errors related to changing accounting conventions, applicability of different rates to the accrual of debt than the short-term interest rate we use, incorrect specification of the term and currency strucuture of debt and so on.

Yet, we can use knowledge of the dynamics of public debt to refine our parameter search by asking that it implies *debt sustainability*. More deeply, nothing guarantees that the estimated VAR (13) leads to a stable path for a variable with an explosive eigenvalue as introduced by (16). Any model that predicts an explosive debt dynamics while theoretically possible, does not satisfy the basic assumption 2 I make in this paper. Automatically, the decomposition of the valuation equation does not hold as in (11): the three terms do not necessarily sum to zero.

Therefore, at this point, I explicitly assume that assumption 2 holds for all economies considered in this empirical exercise. In practice, that means we can change our prior to filter out the estimation combinations of parameters that do not lead to a stable debt dynamics. In the second empirical exercise, I pursue this variation of the estimation process.

Sadly, in breaking the VAR format (13), we can no longer take advantage of the convenience provided by the conjugate Minnesota prior and its closed-form solution.

Debt Value Decomposition.. The generalized law of motion of public debt leads to a different decomposition of the valuation equation (11). Solve (16) for forward, apply assumption 2 and take innovations to arrive at the following expression.

$$\begin{aligned} 0 = {} & \sum_{i=0}^{\infty}\beta^i \Delta E_t s_{t+i}^p \\ & - \frac{v}{\beta}\left[\sum_{i=0}^{\infty}\sum_{j\neq N}\beta^i\delta_j\Delta E_t r_{j,t+i} + \delta\left(\Delta E_t rx_t + \sum_{i=0}^{\infty}\beta^i\Delta E_t i_{t-1+i} - \underbrace{\sum_{i=0}^{\infty}\beta^i\Delta E_t\pi_{t+i}}_{\substack{\text{Unexpected Total}\\ \text{Inflation}}}\right)\right] \end{aligned} \tag{17}$$

In (17), $s_t^p = s_t + v(p_t^s - p_t)$ is the price-adjusted surplus deviation and $r_{j,t} = rx_{j,t} + i_{j,t-1} + \Delta h_{j,t} - \pi_{j,t}$ is the *ex-post* real return on holdings of the j-currency portfolio, in domestic currency.[13] I simplify

[13] I also apply the assumption of this particular application to save space: $g_{Y,t} = 0$ for all t and $\Delta E_t rx_T$ for $T > t$.

Median	Std(π)	$b(\pi, Tx)$	$b(\pi, G)$	$b(\pi, i)$	$b(\pi, gdp)$	$b(\pi, p^s - p)$
(a) Baseline BVAR						
All	0.88	0.07	-0.21	0.50	0.02	-0.25
Advanced	0.71	0.06	-0.24	0.34	0.04	-0.27
Developing	0.96	0.10	-0.18	0.57	0.00	-0.24
(b) Restrictive Prior: Debt Law of Motion Prior						
All	1.34	-0.05	0.13	0.35	0.18	-0.20
Advanced	1.00	-0.05	0.36	0.40	0.22	-0.25
Developing	1.50	-0.05	0.11	0.18	0.01	-0.18
(c) Robustness on Baseline BVAR						
VAR lags = 2						
All	0.76	0.03	-0.04	0.45	0.01	-0.33
Advanced	0.68	0.05	-0.06	0.35	0.05	-0.27
Developing	0.83	-0.14	-0.02	0.58	-0.06	-0.22
VAR lags = 3						
All	0.60	0.09	-0.01	0.48	0.04	-0.30
Advanced	0.56	0.08	-0.10	0.37	0.05	-0.29
Developing	0.68	0.19	0.11	0.60	0.04	-0.32
Increased shrinkage of Λ						
All	0.89	0.10	-0.25	0.49	0.04	-0.24
Advanced	0.74	0.09	-0.25	0.35	0.09	-0.26
Developing	1.08	0.17	-0.29	0.61	0.02	-0.23
Deviation from SS, no intercept						
All	0.95	0.09	-0.10	0.51	0.03	-0.22
Advanced	0.77	0.12	-0.11	0.36	0.05	-0.22
Developing	1.11	0.06	0.08	0.59	-0.01	-0.23
Increased overall shrinkage, $E_t x_{t-1} \approx$ *constant*						
All	1.83	0.14	0.38	0.50	-0.02	-0.03
Advanced	1.31	0.14	0.54	0.39	0.01	0.07
Developing	3.06	0.15	0.04	0.63	-0.10	-0.06

Table 4: BVAR Estimation and UCI: Robustness to Prior (Cross-Country Medians)

notation by setting, $rx_t = rx_{1,t}$ $i_t = i_{1,t}$, $\pi_t = \pi_{1,t}$.

Again, we can highlight the innovation to the real value of public debt at the beginning of period t.

$$
\begin{aligned}
\frac{v}{\beta}\left[\sum_{j\neq N}\delta_j\Delta E_t r_{j,t} + \delta\left(\Delta E_t rx_t - \overbrace{\Delta E_t \pi_t}^{\substack{\text{Unexpected}\\ \text{Current}\\ \text{Inflation}}}\right)\right] &= \sum_{i=0}^{\infty}\beta^i \Delta E_t s^p_{t+i} \\
&+ \frac{v}{\beta}\left[\sum_{i=1}^{\infty}\sum_{j\neq N}\delta_j\beta^i\Delta E_t r_{j,t+i} + \delta\left(\sum_{i=1}^{\infty}\beta^i\Delta E_t i_{t-1+i} - \underbrace{\sum_{i=1}^{\infty}\beta^i \Delta E_t \pi_{t+i}}_{\substack{\text{Unexpected Future}\\ \text{Inflation}}}\right)\right]
\end{aligned} \tag{18}
$$

Currency-linked and long-term maturity debt breaks the equality (12) between unexpected current inflation and the change in the real value of debt (right-hand side of the expression above). The price of the public nominal liabilities is no longer the only variable that can translate debt sustainability. News of lower discounted surpluses can be met with lower nominal ($\Delta E_t rx_t$) or real ($\Delta E_t r_{j,t}$) bond prices. This is a key mechanism explored by Sims (2011) and Cochrane (2022b) to generate a negative response of inflation to interest rates.

2.2.1 Geometric Maturity Structure of Public Debt

I assume the government keeps a geometric maturity structure of its debt. The geometric term structure implies a tractable relationship between short-term interest rate and the excess return on public bonds.

The term structure is constant over time, but can vary across the different currency portfolios $\{j\}$ of public debt. Specifically, for the slice of public debt linked to currency j, suppose the outstanding volume of bonds decays at a rate ω_j, so that $B^n_{j,t} = \omega_j B^{n-1}_{j,t}$. Define $Q_{j,t} = \sum_{n=1}^{\infty} Q^n_{j,t}\omega_j^{n-1}$ as the weighted-average price of currency j public bonds. Then, $\mathcal{V}_{j,t} = Q_{j,t}B^1_{j,t}$. The total return on currency-j bonds then is $1 + rx_{j,t} + i_{j,t-1} = \frac{1+\omega_j Q_{j,t}}{Q_{j,t-1}}$, which linearizes to

$$rx_{j,t} + i_{j,t-1} = \bar{\omega}_j q_{j,t} - q_{j,t-1} \tag{19}$$

where $\bar{\omega}_j = \omega_j \exp\{-(rx_j + i_j)\}$, $q = \log Q$ and I use the log approximation of level returns to effectively re-define $rx_j + i_j$.

Equation (19) above defines the excess return on holdings of the j-currency portfolio of public debt. Given a model for the risk premium $E_t rx_{j,t+1}$, it also defines the price of the debt portfolio as a function of short-term interest:

$$q_{j,t} = \bar{\omega}_j E_t q_{j,t+1} - E_t rx_{j,t+1} - i_{j,t}. \tag{20}$$

Equation (20) implies that $q_t = -\sum_{i=0}^{\infty}\bar{\omega}_j^i E_t\left[rx_{j,t+1+i} + i_{j,t+i}\right]$, which clarifies the connection between short-term interest and returns on the market price of debt showing up in (16). Given news of, say, higher interest rates, the discount of public bond increases, and q falls. Equation (19) then prescribes a low excess return on j debt.

(Debt Decomposition 2)

2.2.2 The Rational-Expectations Model

In the second empirical exercise, I keep the equations in (13) except for the debt equation. In its place, I introduce the law of motion (16) for the market value of debt and the equations

$$\begin{aligned} v_t^b &= v_t + u_t \\ u_t &= \rho u_{t-1} + \zeta v \delta \left(i_t - i_{t-1}\right) + \epsilon_{v,t} \end{aligned} \tag{21}$$

for the observed book value of debt v_t^b. The market value of debt v_t is not observed. The second equation is motivated by the fact that the market and book value of debt tend to differ in periods of changing nominal interest.

I assume a constant risk premium for all debt portfolios: $E_t \pi_{j,t+1} = 0$, and let equation (20) determine its price. Equation (19) defines the excess return.

I assume the government issues real bonds with interest rates consistent with agents' expectations: $i_{2,t} = i_t - E_t \pi_{t+1}$. Finally, we need a law of motion for the dollar interest-inflation pair $x_t^{US} = [i_t^{US}, \pi_t^{US}]$. I use a simple two-equation VAR $x_t^{US} = \psi x_{t-1}^{US} + \epsilon_{US,t}$.

In total, the rational-expectations model contains twenty-two equations and ten shocks.[14] The ten shocks in $\epsilon = [\epsilon_{-v,t}, \epsilon_{v,t}, \epsilon_{US,t}]$ contain: the shocks to the seven reduced-form equations ($\epsilon_{-v,t}$), the shock to the book value of debt ($\epsilon_{v,t}$) and the two shocks to US variables ($\epsilon_{US,t}$).

Given a set of parameters that lead to a unique and stationary equilibrium, I find the solution in state-space representation

$$x_t = \Phi x_{t-1} + \Gamma \epsilon_t$$

using Klein (2000)'s method.

With a solution, I compute the data likelihood using the Kalman filter. The dataset is the same as in the no-intercept BVAR of the previous section.[15]

US case. (Incomplete)

2.2.3 Steady State and Fixed Parameters

I do not use an intercept in the estimation. Instead, I transform all variables into deviations from a steady state.

In the steady state, public debt corresponds to the average public debt in the dataset (country by country), the same is true for government spending, the relative price of public basket, interest, inflation and gross domestic product. The steady-state surplus must be consistent with public debt, so I set $s = (1-\beta)v/\beta$. Steady-state taxation follows: $T = G + s$. Finally, the steady-state variation in real exchange rate is zero.

A subset of the model's parameters are fixed. Steady-state real discouting β I fix at 0.98. The currency and term structure parameters of public debt (the δ's and ω's) for each country I collect

[14] The seven reduced-form equations (for inflation, interest, the relative price of the public basket, GDP, currency depreciation, tax proceeds and public spending), the law of motion for debt, the two equations in (21) for the book value of debt, equations (20), (19) and $E_t rx_{j,t+1} = 0$ for the three portfolios, the short-term real interest $i_{2,t}$ definition, and the two-equation VAR for US interest and inflation. In practice, the solution contains a few more auxiliary variables to lag q and π, which show up with subscripts t and $t-1$ in the model.

[15] The Gaussian distribution for the initial state of the chain is centered around zero, with the covariance matrix equal to the unconditional covariance matrix of the system (hence dependent on estimated parameters).

from a myriad of sources compising OECD panel data, official websites and individual government reports. All sources are listed in the appendix.

The parameters of equation (21), $\rho \approx 0.76$, $\zeta \approx 0.23$, $\text{std}(\epsilon_v) \approx 1.17$, I estimate using US data, a case in which both book and market values of debt available.[16]

The parameter of the US VAR ψ and $\text{cov}(\epsilon_{US})$ I estimate by OLS. They are the same for all countries, with the exception on the US itself.

2.2.4 Adjusting the Prior Distribution

The new format of the model requires a few adaptations to the Minnesota Prior.

First, there is no reason to believe that the ϵ_v which affects the discrepancy between book and market value of debt *after* controlling for interest variation has any correlation with the other shocks of the model. Sadly, the Inverse-Wishart distribution does not offer enough flexibility to control the variance of each element of the square matrix individually. For this reason, in the new prior distribution $\epsilon_{-v,t}$ continues to follow a IW distribution, centered around the identity as before, with $d = N + 2$ and $N = 7$; and the correlation between shocks in $\epsilon_{-v,t}$ and $\epsilon_{v,t}$ is zero. The prior for the correlation between $\epsilon_{-v,t}$ and $\epsilon_{US,t}$ is a uniform with limits plus and minus one.

The prior distribution for the autoregressive parameters of the seven variables in the reduced-form VAR is the same as before, but slightly altered to ensure that its mode leads to a stable model. I set the loading of tax proceeds on public debt to 0.025; and set the loading of public spending to -0.025. The tightness parameter λ is the one that maximizes the marginal likelihood of the BVAR model for steady-state deviation (the same I use in the "no-intercept" case in table 4).

Lastly, I attribute zero density to parameters that lead to solutions with unrealistically large term in decomposition (17). Specifically, the terms of the decomposition are $C(I-\beta\Phi)^{-1}\Gamma\epsilon_t$ for a properly specified C (see the appendix). For any ϵ in the unit circle, I require $\|C(I-\beta\Phi)^{-1}\Gamma\epsilon\|_2 < M$, where $\|.\|_2$ is the Euclidian norm.[17] I set $M = 10$, which binds the estimation in the case of six countries.

2.2.5 Unexpected Total Inflation

For each country, I use a Metropolis-type adaptive sampelr to draw from the posterior distribution. Following Andrieu and Thoms (2008), the algorithm randomly alternates between three different update procedures of the parameter vector. All increments are symmetric and centered around the previous draw. They can update the entire parameter vector, a single entry or a given *direction* determined by principal component decomposition. The covariance matrices are updated after each new draw, so that the average acceptance rate is 20%. To ensure convergence to the asymptotic distribution, adaptation eventually vanishes. I provide details of the algorithm in the appendix.

3 Conclusion

Path forward:

[16]The Dallas Fed provides estimates of the market value of debt at https://www.dallasfed.org/research/econdata/govdebt. They are calculated by Jonah Danziger and Tyler Atkinson, using the methodology in Cox and Hirschhorn (1983) and Cox (1985).

[17]The condition can be summarized by requiring that the matrix norm induced by the vector 2-norm of $C(I-\beta\Phi)^{-1}\Gamma$ is bounded by M.

Median	Inflation	Primary Surplus	Nominal Interest	Excess Return	Real Debt
Unexpected Current Inflation ($\Delta E_t \pi_t = \epsilon_{\pi,t} = 1$)					
All	0.06	-0.38	0.56	-0.21	0.01
Advanced	0.34	-0.66	0.56	-0.22	0.02
Developing	-0.19	-0.21	0.68	-0.21	0
Lower Taxation ($\epsilon_{T,t} = -1$)					
All	0.39	0.18	-0.64	0	-0.03
Advanced	0.48	0.25	-1.10	0.15	0
Developing	0.07	-0.04	-0.11	-0.05	-0.15
Higher Public Spending ($\epsilon_{G,t} = 1$)					
All	0.30	-0.41	-0.98	0.23	0
Advanced	0.67	-0.17	-1.68	0.78	-0.13
Developing	0.10	-0.53	0.26	-0.15	0.18
Looser Monetary Policy ($\epsilon_{i,t} = -1$)					
All	-0.04	0.18	0.15	-0.34	0
Advanced	0.35	0.16	0.09	-0.54	0
Developing	-0.05	0.21	0.20	-0.20	0
Recession ($\epsilon_{gdp,t} = -1$)					
All	0	0.15	-0.16	-0.01	-0.01
Advanced	-0.28	0.24	0.22	-0.09	-0.01
Developing	0.58	0.10	-0.59	0	-0.02

Table 5: Debt Law Prior and Value Decomposition (Cross-Country Medians)

- Consider variation in risk-premia, particularly important for emerging markets

References

Ali Abbas, S. M., Belhocine, N., El-Ganainy, A., and Horton, M. (2011). Historical Patterns and Dynamics of Public Debt—Evidence From a New Database. *IMF Economic Review*, 59(4):717–742.

Andrieu, C. and Thoms, J. (2008). A tutorial on adaptive MCMC. *Statistics and Computing*, 18(4):343–373.

Barro, R. J. (1974). Are Government Bonds Net Wealth? *Journal of Political Economy*, 82(6):1095–1117.

Blanchard, O. J. and Kahn, C. M. (1980). The Solution of Linear Difference Models under Rational Expectations. *Econometrica*, 48(5):1305–1311.

Cochrane, J. H. (1998). A Frictionless View of U.S. Inflation. *NBER Macroeconomics Annual*, 13:323–384.

Cochrane, J. H. (2005). Money as stock. *Journal of Monetary Economics*, 52(3):501–528.

Cochrane, J. H. (2011). Determinacy and Identification with Taylor Rules. *Journal of Political Economy*, 119(3):565–615.

Cochrane, J. H. (2022a). The fiscal roots of inflation. *Review of Economic Dynamics*, 45:22–40.

Cochrane, J. H. (2022b). A fiscal theory of monetary policy with partially-repaid long-term debt. *Review of Economic Dynamics*, 45:1–21.

Cox, W. M. (1985). The behavior of treasury securities monthly, 1942–1984. *Journal of Monetary Economics*, 16(2):227–250.

Cox, W. M. and Hirschhorn, E. (1983). The market value of U.S. government debt; Monthly, 1942–1980. *Journal of Monetary Economics*, 11(2):261–272.

del Negro, M. and Schorfheide, F. (2011). Bayesian Macroeconometrics. In Geweke, J., Koop, G., and Van Dijk, H., editors, *The Oxford Handbook of Bayesian Econometrics*, pages 292–389. Oxford University Press.

Giannone, D., Lenza, M., and Primiceri, G. (2015). Prior Selection for Vector Autoregressions. *The Review of Economics and Statistics*, 97(2):436–451.

Karlsson, S. (2013). Forecasting with Bayesian Vector Autoregression. In *Handbook of Economic Forecasting*, volume 2, pages 791–897. Elsevier.

King, R. G. and William, K. (1996). Limits on Interest Rate Rules in the IS Model. *FRB Richmond Economic Quarterly*, 82(2):47–75.

Klein, P. (2000). Using the generalized Schur form to solve a multivariate linear rational expectations model. *Journal of Economic Dynamics and Control*, 24(10):1405–1423.

Litterman, R. (1979). Techniques of forecasting using Vector Auto Regression. *Federal Reserve Bank of Minneapolis Working Paper*, 115.

Sims, C. A. (1980). Macroeconomics and Reality. *Econometrica*, 48(1):1–48.

Sims, C. A. (2011). Stepping on a rake: The role of fiscal policy in the inflation of the 1970s. *European Economic Review*, 55(1):48–56.

Smets, F. and Wouters, R. (2007). Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach. *The American Economic Review*, 97(3):586–606.

Taylor, J. B. (1993). Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy*, 39:195–214.

Appendices

A Additional Plots of the NK Model

B Data Treatment

Calulation of GDP trend

Table with data sources for each country.

Report parameters of public debt structure. List of sources for public debt structure.

C Additional Details of the Estimation

Closed-form solution of the BVAR posterior and marginal likelihood. Show the decomposition of the marginal distribution by Giannone et al. (2015).

OLS Regression on the wedge between book and market value of debt, to estimate ζ, ρ and σ_v.

How to compute decomposition and explain prior constraint.

FIgure with data and conditional expectation for a few countries.

Details of the Metropolis sample.