

Bond Risk Premia and Discount Rate Variation in Brazil

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February 4, 2022

Abstract

Long-duration public bonds denominated in local currency have been traded with liquidity in Brazil since the late 2000s. I use the twelve years or so of now available data to study bond risk premia. As previously documented for the US, a single tent-shaped combination of forward rates predicts bonds' excess returns. R^2 surpasses 0.40. The same return-forecasting factor also predicts equity stock excess returns ($R^2 \approx 0.18$), and returns on the Real-Dollar carry trade ($R^2 \approx 0.28$), even after controlling for domestic interest, dividend-yield and US risk premium variation. The return factor is slightly correlated with the MSCI index and domestic interest, but disconnected from other macroeconomic and international variables. I input it along with term structure principal components on a exponential-affine model, and characterize model dynamics with a simple pair of co-movements involving the return factor and the level principal component, a variable I show to be highly correlated with domestic inflation and especially US risk premia, and whose shocks affect marginal utility in the model specification with the best fit.

1 Introduction

I study bond risk premia in Brazil. Using twelve years of public bond price data, I run regressions of realized excess returns on forward rates, as [Cochrane and Piazzesi \(2005\)](#) do for the US case. Like them, I find that a single linear combination of yields, described by a tent-shaped profile of coefficients, captures the bulk of variation projected by the term structure in returns of bonds with maturities one through ten years. R^2 s often surpass 0.40 (figure [1b](#)). I report test results for the statistical significance of estimated coefficients using both asymptotic and small-sample estimators of their covariance matrices.

Like in Cochrane and Piazzesi's work, the linear combination of forward rates that predicts bond returns leads to a single "return-forecasting" factor. Besides the return of public bonds, I find that such factor also forecasts the return on Brazilian equity stocks ($R^2 \approx 0.18$, figure [1c](#)) and returns on the Brazilian Real/US Dollar carry trade ($R^2 \approx 0.28$, figure [1d](#)). Such predictability holds

*I deeply thank my former professors at Stanford University, Prof. Monika Piazzesi and Prof. John Cochrane, whose brilliant research has heavily inspired the work I document in this article. I also thank Banco Safra for supporting me, as well as Joaquim Levy and Eduardo Yuki for the numerous discussions, and Matheus Wirth for excellent research assistantship.

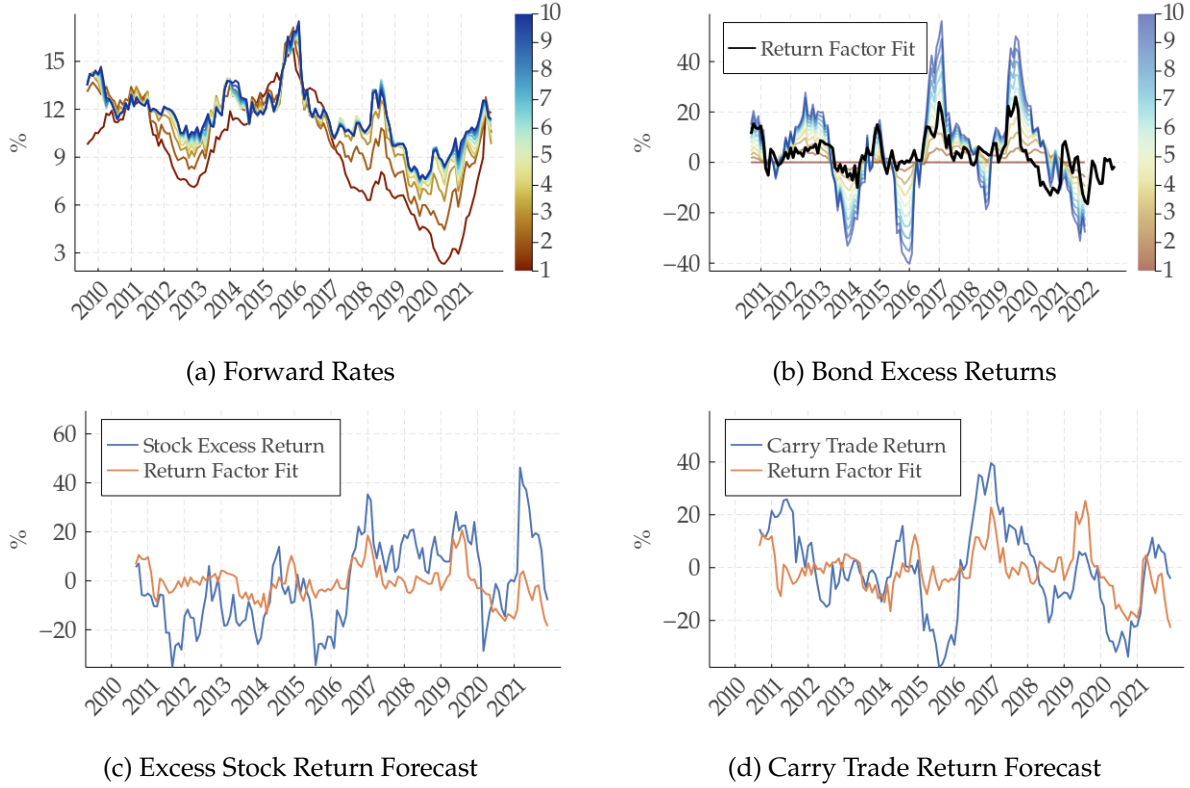


Figure 1: Data and predictability with a bond return factor

even after controlling for variation in Brazilian interest rates and dividend yields, and in US risk premium (proxied by the slope of its term structure, see [Fama and Bliss \(1987\)](#)). In most cases, p -values are lower than 0.01, and are lower than 0.02 in all of them.

To study the interpretation of the return factor, I set it as left-hand side variable on regressions against several domestic and international variables. The return factor is weakly correlated with the MSCI index and to interest rates, but disconnected from other domestic and international variables, such as US risk premia, domestic inflation, the EMBI, industrial production, current account, trade balance, primary surpluses and international commodity prices.

Also in accordance with results for the US, the return factor is not fully captured by traditional level, slope and curvature time series calculated from yields' (I use forward rates') principal components. While these three factors combined explain the average excess return with an R^2 of 0.31, the return factor alone reaches $R^2 = 0.43$. Moreover, in a regression with the four factors on the right-hand side, a χ^2 test fails to reject the null that the level, slope and curve factor coefficients are jointly zero. I conclude that the return factor summarizes the information contained in the term structure that is relevant to bond return predictability and hence risk premium measurement.

To study the dynamics of risk premia in Brazil, I then proceed to estimate an exponential-affine model of the term structure, as [Duffie and Kan \(1996\)](#) and [Ang and Piazzesi \(2003\)](#). I set as state variables the return factor along with term structure principal components (level, slope and curvature).

One can estimate factors' dynamics from time series moments, but, as [Cochrane and Piazzesi](#)

(2009) point out, this approach leads to results that are highly sensitive to model specification.¹ Instead, I follow their procedure and estimate risk-neutral dynamics from the cross section of yields. Under a risk-neutral probability measure, forward rates equal expected future interest: $f_t^n = E_t i_{t+n-1}$. Hence, we directly observe a linear combination of state variables' expected future path, which provides precise information about their dynamics. In symbols, if $i_t = \delta' x_t$ and $E_t x_{t+1} = \phi x_t$ (under the risk-neutral measure), then $f_t^n = \delta' \phi^{n-1} x_t$ and we can estimate ϕ without time series regressions. As the model can produce mean pricing errors as low as 0.059%, near the 0.058% produced by unrestricted OLS (the best linear estimator), I conclude it can precisely reproduce factors' dynamics under the risk-neutral probability measure.

I then recover the true probability measure using market prices of risk. Two assumptions reduce the number of price of risk parameters, from twenty to two. I impose, first, that discount rates vary only due to variation in the return factor, an assumption justified by the findings of the first part of the paper. Second, I restrict the model's discount rate to price shocks to a single factor only. While I try different specifications, a priced *level* factor shock leads to the best model fit. The level factor is highly correlated with domestic inflation ($R^2 \approx 0.40$) and the US term spread ($R^2 \approx 0.54$).²

The estimated dynamics present a notable movement pattern. Variation in the level factor induces variation in the return factor (hence in risk premia) of equal sign. Higher level forecasts higher returns. Under the true probability measure, variation in the return factor then produces variation in the level factor in the *opposite* direction. The system becomes stable: high level \rightarrow high return \rightarrow lower level. Under the risk-neutral measure, the second implication *fails*. The increase in risk premia does *not* bring down the level of yields, and so the system acquires a near-unitary root.³

This pattern, which is robust across multiple specifications, opens the door to interpreting events as the result of the co-movements of the state variables of the model. Positive shocks to inflation or international risk premia, which typically lead to a higher level factor, forecast subsequent surges in Brazilian risk premia, which in turn predict a decline in the level factor. This is exactly the pattern observed in the 2015-16 recession. From the differences between true and risk neutral probabilities, we can also infer that the marginal utility of Brazilian bond investors is sensitive to combinations of structural shocks that provoke prolonged periods of elevated international risk premia, interest rates and inflation. I make this point rigorous in the text.

The article offers two contributions to the existing literature. The first is to investigate if well-studied results about public bonds found in the US case, especially in terms of return predictability and the expectations hypothesis (Fama and Bliss (1987), Campbell and Shiller (1991), Bekaert and Hodrick (2001)), hold in a large, hopefully representative developing country. For example, I show that Fama-Bliss regressions fail to reject the expectations hypothesis, and how the model makes that consistent with predictability. The fact that my return-forecasting factor for nominal Brazilian bonds also predicts returns of equity stocks and currency portfolios extends the scope of the paper's

¹Cochrane and Piazzesi exemplify the issue by showing how a usual VAR model and a co-integrated model of forward rates - two plausible statistical specifications - lead to wildly different forecasts of interest rates.

²Neumeyer and Perri (2005), Calvo et al. (2006) and Morelli et al. (2021) are examples of papers that relate international credit conditions to activity in emerging markets.

³As a consequence of the pattern I explain here, shocks to the level factor are very persistent under the risk-neutral measure. This is a property the model requires to generate the empirical fact that the first principal component explains most variation in forward rates.

contribution to include discount rate variation more broadly.

Studies of predictability in emerging economies are relatively scarce (Akgriray et al. (2016) and Devpura et al. (2021) are recent examples), as many markets lack the required data or liquidity. In the case of Brazil, following the end of hyperinflation in the mid-1990s, sequential administrations stretched the duration of public debt and changed its composition to encompass primarily local-currency bonds. We now have over ten years of market price data, provided by the Brazilian Financial and Capital Markets Association (ANBIMA), to study the questions above. To the best of my knowledge, this is the first research article to consider this dataset.⁴

The second contribution is to measure and characterize bond risk premia in Brazil, and risk premia more broadly given the high predictability I find in the stock and currency markets using the return factor. Knowledge of the expected path of risk premium at a given state of the economy is particularly informative about the marginal utility of the Brazilian investor. On a frictionless market, the first-order condition for the portfolio selection problem is $p_t = E_t(m_{t+1}x_{t+1})$, where p is the asset price, x represents its payoff, and the stochastic discount rate m is the investor's *marginal utility growth rate*. A risk-neutral probability measure π^* changes the actual measure π by "factoring in" the m : $\pi^* = \pi \times m$, state by state. By contrasting the economy's dynamics under π^* and π , we thus highlight the states in which marginal utility change the most, up to a shock orthogonal to payoffs (if $E_t(\epsilon x_{t+1}) = 0$, then $m + \epsilon$ is a discount factor too).

The difference between risk-neutral and actual probability therefore suggests that the marginal utility of Brazilian investors is sensible to prolonged periods of high inflation or international risk premia *that are not reversed by a rise of risk premia at home*. I interpret this finding to be consistent with macroeconomic literature arguing that the business cycle of emerging markets is affected by trend or long-lived shocks (as Aguiar and Gopinath (2006), Aguiar and Gopinath (2007) and Boz et al. (2011)), the effects of which cannot be reversed by a few years of risk premium variation.⁵

Finally, as a by-product of the exercise, I provide a decomposition of Brazilian forward rates into expected interest rates and term premia. Mid and long-term expected interest are volatile and explain a considerable share of the variation in forward rates. Expected interest does *not* quickly converge to the unconditional average. The term premium is countercyclical, but was considerably bigger in the 2015-16 crisis than during the early months of the COVID Pandemic. Also, term premia of bonds with different maturities do not move in lockstep. There is a term structure of risk premia.

2 Data

I use discrete-time language and, later, a discrete-time model. One period unit represents one year. Let p_t^n be the log-price in period t of a zero-coupon bond that pays after n years have passed. Let $f_t^n = p_t^{n-1} - p_t^n$ be the (log) forward rate at which one can lock today a one-period borrowing

⁴Papers studying the Brazilian term structure typically use swap data on the overnight lending rate (the DI rate). Datasets beginning prior to 2009 usually contain data on bonds with maturity up to two years. See, for instance, Tabak and Andrade (2003), Osmani and Tabak (2008), Almeida and Faria (2014), Almeida et al. (2015) and Faria and Almeida (2018).

⁵That the model fit improves when we set the level factor shock to be priced (be part of marginal utility) only increases the case for its economic significance.

$n - 1$ years ahead.⁶ Let $i_t = f_t^1 = -p_t^1$ be the risk-free rate, which I also call the interest rate. Let $rx_{t+1}^n = p_{t+1}^{n-1} - p_t^n - i_t$ be the log of the *ex-post* return on the holding of a n -period bond from period t to $t + 1$, in excess of the risk-free rate. I simply call them excess returns. Also, I define $\bar{rx}_{t+1} = (1/9) \sum_{n=2}^{10} rx_{t+1}^n$ as the average excess return.

The Brazilian Financial and Capital Markets Association (ANBIMA) collects market price data of nominal bonds issued by the Brazilian federal government. Like other countries, at any given point in time outstanding Brazilian bonds do not cover a full range of maturities, and not all bonds are zero-coupon. ANBIMA provides an estimate of the term structure of interest rates through a daily series of estimated load and decay parameters of a fitted Svenson (1994) model. The closest analogue to this estimation approach for the US is studied by Gürkaynak, Sack, and Wright (2007). ANBIMA re-calculates parameters every day to approximate the discount applied to the more liquid bonds outstanding in the Brazilian secondary market.⁷ The Svenson (1994) model contains a total of six parameters. Therefore, the "data" follows from a highly non-linear six-factor model. From now on, I treat the fitted ANBIMA's curve as data, and highlight when this represents a caveat.

I use ANBIMA data to build monthly observations of the prices of bonds with maturities $n = 1, 2, \dots, 10$ years. Price data then leads to a full term structure of forward rates, as well as realized excess returns. It covers the period Sep-2009 to Dec-2021, and each data point corresponds the last business day of the corresponding month. Note that returns are over a *one-year* holding of the bond. I do not consider lower holding periods.

Plots 1a and 1b show the data. Each curve represents a different maturity (in years), with the color bar on the right indicating which is which. The appendix provides the analysis of the unconditional moments estimated from the data.

2.1 Fama-Bliss Regressions

In the context of bond markets, return predictability manifests as a refusal of the *expectations hypothesis*, which one can define as the claim that forward rates reflect only the expected value of future interest. Indeed, the definitions above lead to the following relationship:

$$f_t^n = E_t i_{t+n-1} + \underbrace{E_t(rx_{t+1}^n - rx_{t+1}^{n-1}) + E_t(rx_{t+2}^{n-1} - rx_{t+2}^{n-2}) + \dots + E_t(rx_{t+n-1}^2)}_{\text{Term Premium}_t^n} \quad (1)$$

Decomposition (1) holds *ex-post* (Cochrane and Piazzesi (2009) provide pretty art illustrating it), and it holds *ex-ante* in expected value. It says that a forward rate is given by the sum of the expected interest in the corresponding period and a *term premium*, which sums over time the differences in risk premia of bonds with consecutive maturities. If the expectations hypothesis holds, $f_t^n = E_t i_{t+n-1}$. Applying that to (1) iteratively leads to $E_t rx_{t+1}^n = 0$ for all n . Returns are

⁶I could alternatively work with yield-to-maturities which, in log, equal the average of the corresponding forward rates. I choose to work with the latter, as forward rates have a clearer connection to expected future interest. Using forward rates also leaves results more comparable to the closest literature for the US.

⁷ANBIMA provides market prices of different traded bonds, but not liquidity data, in that replication of the models' parameters is not feasible.

Maturity n	β_n	Small Sample std(β)	Asymptotic χ^2	p -value	R^2
<i>Left-hand variable: Interest Growth</i>					
2	1.41	0.56	7.26	0.01	0.29
3	0.87	0.72	2.38	0.12	0.11
4	1.26	0.78	6.63	0.01	0.24
5	2.14	0.76	21.07	0.00	0.42
<i>Left-hand variable: Excess Return</i>					
2	-0.41	0.56	0.60	0.44	0.03
3	-0.18	0.66	0.05	0.82	0.00
4	-0.02	0.82	0.00	0.98	0.00
5	0.06	1.03	0.00	0.96	0.00

Notes: small-sample standard deviations calculated from bootstrapping residuals from the simplified VAR $f_t^n = \sum_{j=1}^{12} (a_{n,j}^1 f_{t-j}^1 + a_{n,j}^{10} f_{t-j}^{10}) + \epsilon$. I average over 10,000 simulated paths with same length as the data (145 observations). The χ^2 statistic tests $\beta = 0$ using the asymptotic distribution. Spectral density matrix estimated using Newey-West correction with 18 lags.

Table 1: Fama-Bliss Regressions

unpredictable.

Fama and Bliss (1987) regressions provide a straightforward test of the expectations hypothesis. Their empirical model is:

$$i_{t+n-1} - i_t = \alpha_n^i + \beta_n^i (f_t^n - i_t) + \epsilon_{t+n-1}^n \quad (2)$$

$$rx_{t+1}^n = \alpha_n^x + \beta_n^x (f_t^n - i_t) + \epsilon_{t+1}^x. \quad (3)$$

The expectations hypothesis implies $\beta_n^i = 1$ and $\beta_n^x = 0$. Past research has consistently estimated just about the opposite for the US market, $\beta_n^i = 0$ and $\beta_n^x = 1$, a fact attributed to slow adjustment of interest rates.⁸

Table 1 reports results with Brazilian data. I consider forward rates with maturities of two to five years. Since small sample issues are a big concern with only twelve years of data, I calculate standard deviations by bootstrapping residuals from a simplified VAR (see the notes on the table). I also report the χ^2 statistic for the test $\beta_n^i = 0$ or $\beta_n^x = 0$. I calculate it using a spectral density matrix estimated through the Newey-West approach, with 18 months of lags.

Results are the opposite of the usually reported for the US, and appear to support the expecta-

⁸To see the connection, suppose an investor observes the forward spread $f_t^n - i_t$ and decides to purchase the n -year bond. In the following period, suppose that the interest rate stays the same, and that so does the term structure of forward rates, $p_{t+1}^n = p_t^n$. The investor's excess return then is

$$\begin{aligned} rx_{t+1}^n &= p_{t+1}^{n-1} - p_t^n - i_t \\ &= p_t^{n-1} - p_t^n - i_t \\ &= f_t^n - i_t \end{aligned}$$

and $\beta_n^x = 1$ becomes clear.

tions hypothesis. Estimates of β^i are in all but one case within one standard deviation away from one. In the case in which it isn't ($n = 5$), the estimate $\beta_5^i = 2.14$ "overshoots" in the direction of the expectation hypothesis. For $n = 3$ and $n = 4$, the estimated coefficients are less than two standard deviations away from zero, and even using the asymptotic distribution we fail to reject the null at a ten-percent significance test for $n = 3$. Nevertheless, results are economically significant: point estimates close to one and the large R^2 s suggest that forward spreads have predicted interest rate growth.

The bottom part of the table sets excess returns as the predicted variable. All estimated coefficients fall within less than half (small sample) standard deviations away from zero. The χ^2 test fails to reject the null by a long margin, and R^2 s are less than unimpressive.

In all, I conclude that Fama-Bliss regressions fail to reject the expectations hypothesis in the Brazilian market. A potential explanation lies in the high variability of the Brazilian interest rate (see the red line in figure 1a; the Brazilian Central Bank is known in the financial industry for having a rather "nervous" hand). The average absolute interest rate growth in the sample is 2.4%, a large value compared to forward rates, which typically range between 9% and 12%. In the Brazilian case, finding predictability demands that we go further than Fama-Bliss regressions. [I later return to the issue and analyze results using the model.](#)

3 Forecasting Returns

As [Cochrane and Piazzesi \(2005\)](#) do, I run regressions of bond excess annual returns on forward rates.

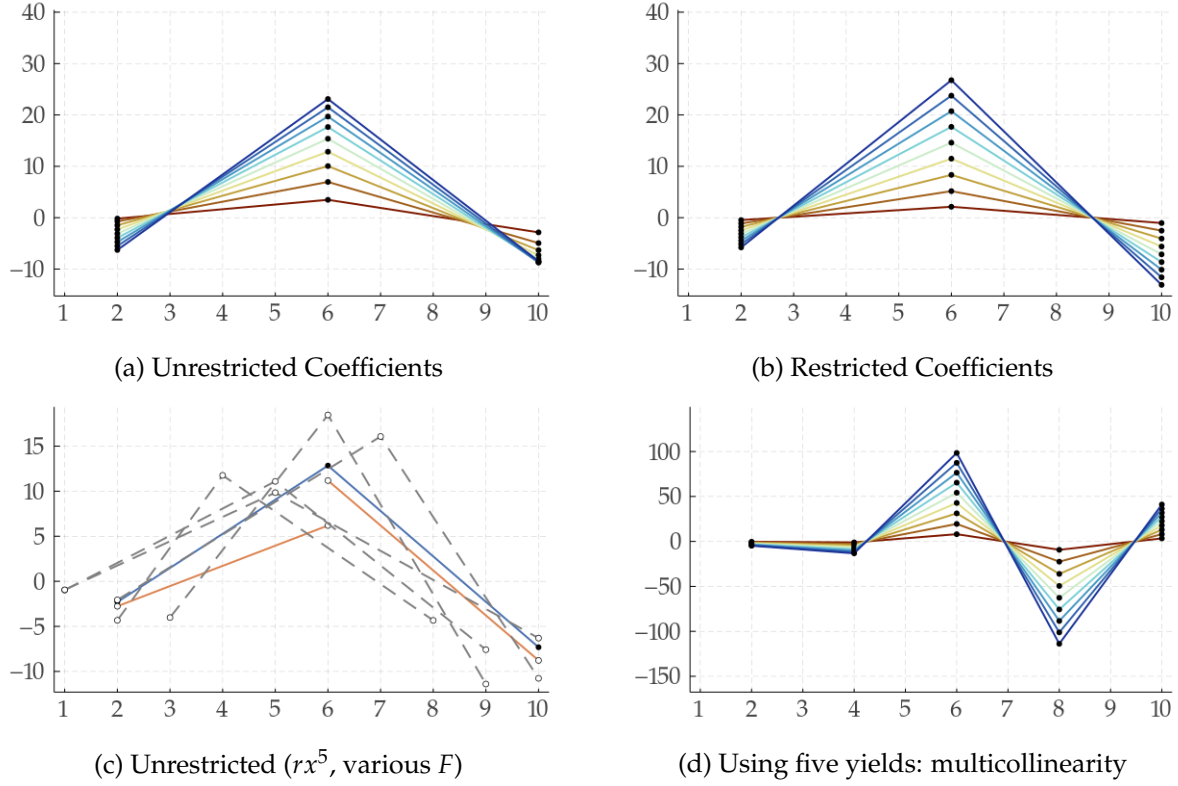
$$rx_{t+1}^n = \alpha^n + \beta_1^n i_t + \beta_2^n f_t^2 + \cdots + \beta_{10}^n f_t^{10} + \varepsilon_{t+1}^n \quad (4)$$

Forward rates are highly correlated, as they follow from a fitted factor model. Its non-linearity means we can still invert f 's covariance matrix, but results scream the issue of multicollinearity.⁹ When I run the regression with all or even a large subset of forward rates on the right-hand side of (4), estimated coefficients display a "W"-shape. I show the case with five forward rates ($n = 2, 4, 6, 8, 10$) in figure 2d. Each curve corresponds to the set of coefficients for a different maturity n , as indicated by the color bar.

The R^2 s of regressions with all forward rates as explanatory variables vary between 0.36 and 0.53. I report them on the appendix, along with the ones for specifications with less forward rates. I do *not* adjust R^2 s for model parameterization. It turns out we can stick to most predictability by keeping only three forward rates on the right-hand side. Forgoing additional terms substantially reduces R^2 s, at least in some regressions.

I find that the combination of short, medium and long-term maturities yields the best overall fit. In my baseline specification, I use $n = 2, 6$ and 10, which leads to R^2 s in the range of 0.24 to 0.48 .

⁹[Cochrane and Piazzesi \(2009\)](#) document the same issue for the US, using fitted-model data from [Gürkaynak et al. \(2007\)](#). Their 2005 paper uses the Fama Bliss dataset (from CRSP), an analogous to which is sadly not yet available to Brazil.



Notes: Panels 2a and 2d display the estimated coefficients of unrestricted regressions $rx_{t+1}^n = \alpha + \beta' f_t + \varepsilon_{t+1}^n$. Each curve corresponds to the coefficients for the regression on a different n , as indicated by the color bars. Panel 2c looks only at the regression of the five-year bond return rx^5 and plots the unrestricted coefficients for various choices of right-hand forward rates. The blue curve is the baseline choice $F = (f^2, f^6, f^{10})$; the red curves use the two pairs (f^2, f^6) and (f^6, f^{10}) . Panel 2b shows the coefficients of the regression with the restriction $\beta = q'\gamma$ (see text).

Figure 2: Return-Forecasting Regression: Coefficient Estimates

Define $F_t = (f_t^2, f_t^6, f_t^{10})'$. The unrestricted model therefore is

$$rx_{t+1} = \alpha + \beta F_t + \varepsilon_{t+1} \quad (5)$$

where rx stacks the returns of bond with maturities two through ten, α (9×1) and β (9×3) are the accordingly stacked coefficients.

Figure 2a display the estimated β . The "tent" shape of the coefficients and the fact that they appear to be scaled versions of each other reminds us of the analogous results found by [Cochrane and Piazzesi \(2005\)](#). As they point out, this profile of estimates suggests that one can keep most return forecasting power using a single factor rather than three. Such factor then captures most (I cannot say "all" since I discard some yield data) time-variation in the expected returns of bonds of *all* maturities.¹⁰ There is nothing special about my choice of right-hand side forward rates: figure 2c shows that similar choices lead to the same "tent"-shaped profiles of coefficients.

¹⁰I assume the time- t information set to be the bonds' price data up to time t . The unrestricted model (5) and its restricted version (6) implicitly assume a Markovian structure, that is, that information prior to time t does not improve return forecasts.

Algebraically, we need to find a 9-entry vector q and a 3-entry vector γ such that $\beta \approx q\gamma'$. Then, the unrestricted model (5) becomes a one-factor, restricted model:

$$rx_{t+1} = \alpha + q(\gamma'F_t) + \varepsilon_{t+1}, \quad (6)$$

in which vector q stores the linear regression coefficient of each excess return on the single factor $\gamma'F_t$.

There are many reasonable ways to compute q and γ and address the fact that they are only defined up to scale. I follow the procedure Cochrane and Piazzesi use in their first article and normalize the *average* coefficient q_n to be one¹¹: $(1/9)\sum_{n=2}^{10} q_n = 1$. This is convenient because it allows me to, first, compute the γ 's by regressing $r\tilde{x}_{t+1}$ on F_t :

$$r\tilde{x}_{t+1} = \bar{\alpha} + \gamma'F_t + \bar{\varepsilon}_{t+1} \quad (7)$$

and, then, estimate q by running OLS on (6) given γ . Figure 2b plots the restricted coefficients, that is, $q\gamma'$. It is visible that the restricted model produces a profile of coefficients that is similar to the unrestricted model.

Table 2 reports additional results. Panel A focuses on the first-stage regression (7). I divide it into two sections. The first section reports estimates and standard deviations in the baseline case. The estimates of γ are consistent with the tent shape. The black line in figure 1b plots the fitted value of the average return rate and illustrates the regression's large $R^2 = 0.43$.

One could worry that predictability follows from pronounced movements in the term structure during the COVID pandemic era, which represents a large shock in a 12-year database. This is not the case. Table 2 reports the R^2 for the regression run after excluding data from 2020 onward. R^2 goes up, not down.

I compute the covariance matrix of parameter estimates using two methods. In the first one ("asymptotic"), I estimate the spectral density matrix using the Newey-West formula with 18 lags and calculate the GMM covariance matrix.¹² The second method ("Small Sample") uses the same procedure explained in the Fama-Bliss regression (see notes to table 1). I compute the sample covariance matrix of estimates from 10,000 simulations of term structure prices, generated by a simplified VAR with bootstrapped residuals.

The table reports the χ^2 statistic for the hypothesis $\gamma = 0$ (no joint significance), which represents the hypothesis of time-invariant risk premia.¹³ The one-percent critical value is 11.34. Using either method to calculate estimates' variance, the test statistic surpasses 20. We can confidently reject the hypothesis of time-invariant risk premium *and* the expectations hypothesis.

Panel B focuses on the second-stage regression. I report in the table the R^2 's from the unrestricted

¹¹In their subsequent paper, Cochrane and Piazzesi (2009) use the dominant eigenvector of the spectral decomposition of the unrestricted β . While the procedure is "information"-efficient and I find very similar results by adopting it, I prefer to set as my baseline the easier-to-understand procedure from their previous article.

¹²In this case, GMM estimates of the covariance matrix of q differ from OLS estimates, as parameter vector γ enters the moment condition $E[\varepsilon_{t+1}^n \times (\gamma'F_t)] = 0$ that pins down q . I approximate the derivatives of moment conditions using forward differentiation.

¹³Accepting the expectations hypothesis test here requires the inclusion of the constant term, which I do not. On the other hand, *rejecting* the time-invariant risk premium does imply rejecting the expectations hypothesis.

Panel A. First Stage Regression: $r\tilde{x}_{t+1} = \bar{a} + \gamma'F_t + \bar{\varepsilon}_{t+1}$						
	γ_2	γ_6	γ_{10}	$\chi^2(3)$	R^2	$R^2 (t < 2020)$
<i>Time-t forward rates</i>						
Estimate	-3.14	14.52	-7.09		0.43	0.58
Std (Asymptotic)	(1.92)	(3.17)	(3.80)	27.18		
Std (Small Sample)	(1.72)	(5.63)	(5.04)	20.30		
<i>One-month lagged forward rates</i>						
Estimate	-2.39	15.16	-8.7		0.40	0.53
Std (Asymptotic)	(2.01)	(3.20)	(3.28)	24.61		
Std (Small Sample)	(1.60)	(5.55)	(5.00)	21.31		
Panel B. Second Stage Regression: $rx_{t+1}^n = \alpha_n + q_n(\gamma'F_t) + \varepsilon_{t+1}^n$						
n	q_n	Asymp. std	SS std	R^2	Unrest. R^2	$\chi^2(3), \beta = q\gamma'$
2	0.15	0.04	0.04	0.18	0.24	5.16
3	0.36	0.05	0.06	0.27	0.30	2.23
4	0.57	0.04	0.05	0.34	0.35	0.94
5	0.79	0.03	0.03	0.39	0.39	0.35
6	1.01	0.02	0.01	0.42	0.42	0.08
7	1.22	0.01	0.01	0.44	0.44	0.00
8	1.43	0.03	0.04	0.46	0.46	0.09
9	1.64	0.05	0.06	0.47	0.47	0.31
10	1.84	0.08	0.08	0.47	0.48	0.65

Notes: results from the restricted model (6). "Small Sample" covariance matrix calculated from 10,000 simulation paths computed from simplified VAR with bootstrapped residuals. See notes to table 1. "Asymptotic" covariance matrix calculated from GMM formulas for the two-step estimation procedure. R^2 s are *not* adjusted for model parameterization. $R^2 (t < 2020)$ indicates R^2 for the subsample ending in December 2019. The χ^2 column shows the test statistic for the hypothesis $\gamma = 0$.

Table 2: Return-Forecasting Regressions

model (5) for comparison. They vary from 0.24 to 0.48. The difference in forecasting power among bonds with different maturities is far more pronounced in the Brazilian case than in the US, as [Cochrane and Piazzesi \(2005\)](#) document. In addition, χ^2 tests of the joint significance of the β_n parameters (equation by equation, not reported) reject the null $\beta_n = 0$ in all regressions. The lowest value I find for the test statistic is 26.5. The table does report the χ^2 test for the restriction $\beta_n = q_n \gamma$. Again, the statistic only tests the joint significance of the equality *in a given equation* n . The test is *not* analogous to [Cochrane and Piazzesi \(2005\)](#), table 6, where the authors test for the joint hypothesis across all parameters *of all equations*. Testing the restriction equation by equation leads to a failure to reject the null $\beta_n = q_n \gamma$ for all n (the 10% critical value of the χ^2 distribution with three degrees of freedom is 6.25).

Focusing on the second-stage regression itself now, loadings of each bonds' return on the single factor q_n increase almost linearly, like in the US case. All estimates are highly significant individually, regardless of how I calculate standard deviations. The table shows that most variation in excess returns explained in a model with three factors is captured by the single return-forecasting factor $\gamma' F_t$. R^2 s vary from 0.18 to 0.47.

Measurement Error. I follow [Cochrane and Piazzesi \(2005\)](#) and test for measurement error bias by running the return-predicting regression (7) with lagged forward rates on the right-hand side. Return-forecasting regressions that use price data on the right-hand side are always subject to having results biased by measurement error. If the price is inaccurately measured to be too high today and the error vanishes tomorrow, we observe a higher *ex-post* return just due to this error. If, however, measurement error is uncorrelated over time, the same return-forecasting regression that uses *lagged* forward rates on the right-hand side should be immune to measurement error bias. We should observe similar estimates.

This is precisely what the second section of Panel A shows. I lag forward rates a month and run regression (7). The model becomes $E_t \tilde{r} x_{t+1} = \bar{\alpha} + \gamma' F_{t-1/12}$. The table reports the same tent-shaped profile of γ , with standard deviation and hypothesis testing conclusions similar to the baseline case, and only slightly lower R^2 . I try the same exercise with lags of two and three months - not reported - and find similar results. I also do not replicate Cochrane and Piazzesi's simulations of an expectations-hypothesis model with measurement errors (figure 4 of their paper). They show that such case would lead to estimates that form "step" functions, not the tent shapes that both studies find. In conclusion, we cannot attribute the main result of return predictability to measurement error.

Scaling. Define the return-forecasting factor to be $z_t = c_0 + c_1 \gamma' F_t$. I choose c_0 so that z_t has a sample average of zero and c_1 so that its sample standard deviation is the same as the interest rate f^1 (about 0.033). Mean/variance normalization of the return factor (which I call z from now on) is useful to interpret the coefficients I estimate next. It does not change its ability to forecast bond returns.

3.1 Equity Stock Returns and Currency Carry Trade

If the return factor successfully captures time variation in the risk premia of Brazilian public bonds, the next thing to ask is whether it also captures variation in the premium of other portfolios. It does. In this section, I regress the excess return of Brazilian equity stocks and returns to the Brazilian Real/ US Dollar carry trade on z_t .

My proxy for Brazilian stock returns r_t^S is the annual (log) price variation of the Ibovespa index, added to log dividends¹⁴. The excess stock return is $rx_t^S = r_t^S - i_t$. Let e_t be the log of the price of one US Dollar in units of Brazilian Reals. The return on the currency carry trade is the difference between Brazilian interest rate (i_t) and US interest rates (i_t^*), minus the depreciation of the Brazilian Real with respect to the US dollar: $rx_t^E = i_t - i_t^* - \Delta e_{t+1}$ (the carry trade portfolio is a zero-price portfolio, hence the "excess return" notation rx).

For the US interest rate, I use the one-year Treasury Bill Secondary Market Rate, downloaded from the St. Louis Fed's FRED database. Ibovespa and exchange rate data I collect from the Brazilian Institute of Geography and Statistics and the dividend yield series from Bloomberg. Figures 1c and 1d plot rx^S and rx^E .

Panel A of table 3 reports the results. All regressions use an intercept term. The numbers in parenthesis are the p -values of the tests of individual significance (below coefficient estimates) or joint significance (under the " χ^2 Tests" columns). In all cases, I calculate the covariance matrix of coefficients' estimates using the Newey-West correction method, with 18 lags.

Regressions of the stock excess return and of the carry trade return share most results. The return factor z forecasts both of them. Coefficients are highly statistically significant, with p -values lower than 1%. They are also *economically* significant. My estimates indicate that, for a change of 1% in z_t , which varies as much as the economy's interest rate, investors expect a 2.2% increase in the stock excess return and a 2.7% increase in the carry trade return.

The model explains a lot of the variation in realized returns as well. R^2 s are respectively 0.18 and 0.28. Cochrane and Piazzesi (2005) report an R^2 of 0.07 for stock return regression in the US case. Figures 1c and 1d plot the fitted values.

The table also reports results for the gross equity stock return r^S and currency depreciation Δe , to check if predictability follows from variation in the risk-free rates rather than variation in the risky components of each return. Again, the return factor maintains its forecasting ability, with $R^2 = 0.26$ and 0.23. Again, statistically and economically significant estimates.

Estimated coefficients appear to have the right sign as well. One can understand an equity share as a perpetuity with dividend risk. The stock return coefficient 2.20 is lower than the 2.25 we find for the average bond return, but becomes 3.22 after controlling for other factors. We can interpret the difference as additional premium for holding a "bond" with stochastic cash flows. As for the carry trade return, its only source of risk is currency variation¹⁵. The positive sign of the coefficient indicates that, at the same time investors demand a higher return from Brazilian stocks

¹⁴Ibovespa is the main index of Brazilian stock prices. I use the formula $r_t^S = \log [(S_{t+1}/S_t)(1 + DP_{t+1})]$, where S is the value of the Ibovespa index and DP is the one-year cumulative dividend yield rate.

¹⁵I do not discuss credit risk. Both the Brazilian and the American governments have most of their sovereign debt issued in local currency, in that "defaults" take place via inflation and, thus, changes in the relative value of each currency. These changes are by definition reflected by the nominal exchange rate.

Panel A. Stock and Carry Trade Returns								
	Right-Hand Side Variables				χ^2 Tests			R^2
	z	i_t	dp_t	ts^{US}	All	Not z (3)	PCs (3)	
Stock Excess Return rx^S	2.20				13.6			0.18
	(0.00)				(0.00)			
	3.22	-1.03	-3.50	-9.52	52.9	28.0	11.1	0.44
	(0.00)	(0.48)	(0.32)	(0.05)	(0.00)	(0.00)	(0.02)	
Stock Return r^S	2.60				20.9		5.2	0.26
	(0.00)				(0.00)		(0.16)	
Carry Trade Return rx^E	2.70				10.2			0.28
	(0.00)				(0.00)			
	2.25	1.37	-3.05	-0.25	18.6	2.5	5.0	0.33
	(0.01)	(0.34)	(0.43)	(0.97)	(0.00)	(0.48)	(0.18)	
Currency Depreciation Δe	-2.31				10.3			0.23
	(0.00)				(0.00)			
	-2.32	-0.38	3.55	0.96	14.8	1.1	3.8	0.26
	(0.01)	(0.78)	(0.35)	(0.90)	(0.01)	(0.79)	(0.28)	
Panel B. Bond Return Predictability.								
	Right-Hand Side Variables				χ^2 Tests			R^2
	z	Level	Slope	Curve	Not z (3)			
Avg Excess Bond Return $r\bar{x}$	2.32							0.43
	(0.00)							
		0.74	2.06	2.58		15.9		0.31
		(0.01)	(0.10)	(0.14)		(0.00)		
	2.49	-0.04	-0.47	0.81		1.0		0.44
	(0.00)	(0.91)	(0.74)	(0.61)		(0.80)		

Notes: p -values indicated in parenthesis. Left-hand side variable indicated on the first column. Estimates covariance matrices calculated using Newey-West correction method with 18 lags. The "All" column shows the χ^2 statistic for the null hypothesis that all (slope) coefficients are equal to zero. The "Not z " column test if the coefficients of all r.h.s variables but z are zero. The "PC" column in Panel A change the set of r.h.s variable to be the return factor z and the principal components (level, slope, curvature; same variables used by Panel B) and test if the estimates of the principal components' coefficients are jointly equal to zero.

Table 3: Return Factor z Forecasting Power

and bonds, they also demand a higher return from holding Brazilian currency ($\Delta e < 0$ means the Real appreciates compared to the US Dollar). This is exactly what one expects from a developing country whose currency does not have the "flight-to-quality" quality (as in [Cho et al. \(2016\)](#)).

Finally, I also run the same regressions using additional controls on the right-hand side. I include the Brazilian interest i_t , the Ibovespa dividend-yield dp_t and the US term spread ts^{US} . The latter I define to be the difference between the yields of the five and one-year T-bills. Dividend yields are known to forecast stock returns in the US¹⁶, and the term spread I include as a proxy for international risk premia¹⁷.

In all cases, the return factor coefficient remains statistically and economically significant. In the case of the carry trade and currency depreciation, the coefficients multiplying the three controls are statistically equal to zero, both individually and jointly (the "Not z" column reports the χ^2 test statistic for null that these coefficients are jointly zero). The return factor drives out the effect of the other variables.

In the case of the stock excess return regression, the coefficient multiplying the US term spread is statistically significant at a 5%-sized test. The loading on the return factor is even higher compared to the univariate model, while that of the other two predicting variables are statistically zero. We gain a lot of predictability too: R^2 jumps to 0.44. A natural conjecture is that the US term spread captures variation in the market price of a risk factor specific to equity cash flows, given that it does not show up on the carry trade return equation.

3.2 Principal Component Factors

[Cochrane and Piazzesi \(2005\)](#) show that the information contained in their return factor cannot be summarized by the information contained in the traditional level, slope and curve factors. The same is true in the Brazilian case. While the first three principal components capture 99.5% of term structure variation, they do not capture the same share of the return factor's variation.

Panel B of table 3 shows that. I again regress the average excess return $\bar{r}x_{t+1}$ on z , but I now include level, slope and curvature factors on the right-hand side:

$$\bar{r}x_{t+1} = \alpha + \beta z_t + \psi_1 \text{level}_t + \psi_2 \text{slope}_t + \psi_3 \text{curvature}_t + \varepsilon_{t+1} \quad (8)$$

The first row of the panel considers only z as explanatory variable. $R^2 = 0.43$ matches that of the first-stage regression in table 2, as it should. The second row considers only the principal components. Combined, they reach $R^2 = 0.31$. I do not adjust R^2 s for model parameterization - they reflect only summed squared errors. On its own, the return factor explains the average return on Brazilian nominal bonds better than the traditional factors combined.

Another key point follows from the last row, which considers the four explanatory variables combined. In that case, coefficient estimates of the principal component factors are individually and jointly statistically equal to zero. The p -value for the χ^2 test is 0.80, an overwhelming failure to reject the null $\psi = 0$. I conclude that, given the return factor z , the three traditional term structure

¹⁶See [Fama and French \(1988\)](#), [Campbell and Shiller \(1988\)](#), [Cochrane \(1992\)](#), and many others since.

¹⁷[Fama and French \(1989\)](#) conclude that the term spread forecasts both stock and bond returns in the US.

factors offer no additional forecasting power to bond returns.

I also test whether these factors improve predictability of the other portfolios considered in Panel A. To do this, I run a regression similar to (8), but replace the dependent variable. Column "PCs (3)" reports the same χ^2 statistic testing $\psi = 0$ along with its p -value. In the case of the carry trade, p -value = 0.18 and we can conclude that the return factor captures the effect of the others factors. In the case of stock returns, however, the level factor adds relevant information to the model, and we cannot reject $\psi = 0$. In all of these cases, the return factor coefficient remains statistically significant at a two-percent sized test.

4 Risk Premia Dynamics

The main point of the last two sections is that the return-forecasting factor z_t contains relevant information regarding the risk premium of different assets in the Brazilian economy. I now estimate a parametric model to characterize its conditional distribution.

4.1 The Exponential-Affine Model

Following [Ang and Piazzesi \(2003\)](#) and [Ang et al. \(2007\)](#), I consider the homoskedastic, discrete-time version of the multifactor exponential-affine model presented by [Duffie and Kan \(1996\)](#). Because the model has already been extensively studied by the existing literature, I spend little time describing the equations. First-time readers are encouraged to check the above references or the appendix, where I provide a full derivation of the model.

I use four factors. The first factor is the return-forecasting factor z . The next three are the first three principal components of the term structure (level, slope and curvature). I normalize all factor to have zero sample mean:

$$X_t = \begin{bmatrix} z_t \\ \text{level}_t \\ \text{slope}_t \\ \text{curve}_t \end{bmatrix} \quad (\text{demeaned})$$

You can find their time series in the appendix. Factors evolve according to

$$X_t = \Phi X_{t-1} + e_t, \quad (9)$$

where $e_t \sim \mathcal{N}(0, \Sigma)$ is independent over time and captures the shocks hitting the economy. Model (9) implies that the populational mean of X is zero. Implicitly, I assume that the sample averages of the four factors coincide with their populational means. I later relax this assumption and find better fit, but results that lead to similar conclusions.

I concluded in the previous section that, given the return-forecasting factor z , the three principal components offer no additional return forecastability. Hence, we would like the model to replicate the empirical regression

$$rx_{t+1}^n = \alpha_n + \beta_n z_t + \epsilon_{n,t+1}. \quad (10)$$

The exponential-affine model allows us to reverse-engineer a stochastic discount factor $\exp \{m_{t+1}\}$ that achieves that. In particular, assume

$$\begin{aligned} m_{t+1} &= -\delta_0 - \delta'_1 X_t - (1/2)\lambda'_t \Sigma \lambda_t - \lambda'_t e_{t+1} \\ \lambda_t &= \lambda_0 + \lambda_1 X_t. \end{aligned} \quad (11)$$

The following proposition summarizes the properties of the models' solution that I use later.

Proposition 1 (Solution to the exponential-affine model). *The solution to the exponential-affine model involves log prices and forward rates that are linear on the factors:*

$$\begin{aligned} p_t^n &= A_n + B'_n X_t, \\ f_t^n &= A_{f,n} + B'_{f,n} X_t. \end{aligned} \quad (12)$$

Coefficients A_f and B_f are such that the interest rate is

$$i_t = \delta_0 + \delta'_1 X_t, \quad (13)$$

and the risk premium for the n -maturity bond is

$$\begin{aligned} E_t r x_{t+1}^n &= -(1/2) \text{var}_t(r x_{t+1}^n) + \text{cov}_t(r x_{t+1}^n, e'_{t+1}) \lambda_t \\ &= [-(1/2) B'_{n-1} \Sigma B_{n-1} + B'_{n-1} \Sigma \lambda_0] + [B'_{n-1} \Sigma \lambda_1] X_t. \end{aligned} \quad (14)$$

Define e_t^* by

$$\begin{aligned} X_t &= \mu^* + \Phi^* X_{t-1} + e_t^*, \quad \text{where } \mu^* = -\Sigma \lambda_0, \\ \Phi^* &= \Phi - \Sigma \lambda_1. \end{aligned} \quad (15)$$

There exists a risk-neutral probability measure under which $e_t^* \sim \mathcal{N}(0, \Sigma)$ and is independent over time. In the solution,

$$f_t^n = -(1/2) B'_{n-1} \Sigma B_{n-1} + E_t^* i_{t+n-1}, \quad (16)$$

where operator E^* takes expected value using the risk-neutral probability measure.

Given μ^* and Φ^* , coefficients A , A_f , B and B_f do not depend on λ_0 , λ_1 , μ or Φ .

Proof. See the appendix. □

Equation (14) describes bond risk premia according to the model. We interpret the covariance term as the amount of "risk" of each bond - how they correlate with existing shocks. Therefore, we interpret λ_t as the market "price" of such risks. For example, $\lambda_t(3)$ (the third element of λ_t) is the price of shocks to the term structure slope factor.

As the second line of (14) shows, the two variance terms are constant over time - the model is conditionally homoskedastic -, and one can thus drop the t subscript. Variation in risk premia $E_t r x_{t+1}$ comes from λ_t , only. So, to replicate (10), all we need is to pick the right λ_0 and λ_1 .

The second part of proposition 1 describes the model's solution in terms of the risk-neutral dynamics. Equation (16) shows that, under the risk-neutral measure, the expectations hypothesis

holds, up to a Jensen's inequality additive constant. Forward rate variation reflects *only* variation in expected interest, not variation in term premia (recall (1)).

4.2 Estimation Procedure and Zero Restrictions

One way to estimate the model is to run OLS on (9), and then compute λ_0 and λ_1 by matching (10) and (14). If we allow the three principal component factors in X_t to predict returns along with z , so that the restrictions imposed by (10) no longer hold, [Adrian et al. \(2013\)](#) show that this procedure prices the term structure with high precision.¹⁸ The downside is that, first, we rely on many free parameters (λ_0 and λ_1 contain a total of twenty parameters in a four-factor model). Second, the estimation is sensitive to model specification - changing (9) leads to wildly different results - and sampling uncertainty.¹⁹

I instead follow [Cochrane and Piazzesi \(2009\)](#) and estimate factors' (risk-neutral) dynamics solely from the cross section of yields. We can then use the model's restrictions to infer the true dynamics.

More specifically, note from (13) and (16) that we can write $f_t^n = \delta_1' \Phi^{*n-1} X_t$ plus a constant. So, given data for f and X , we can estimate Φ^* (and μ^* , which enters the constant). This procedure explores the fact that prices depend only on the risk-neutral dynamics (μ^*, Φ^*) , a point highlighted by the last claim in proposition 1.

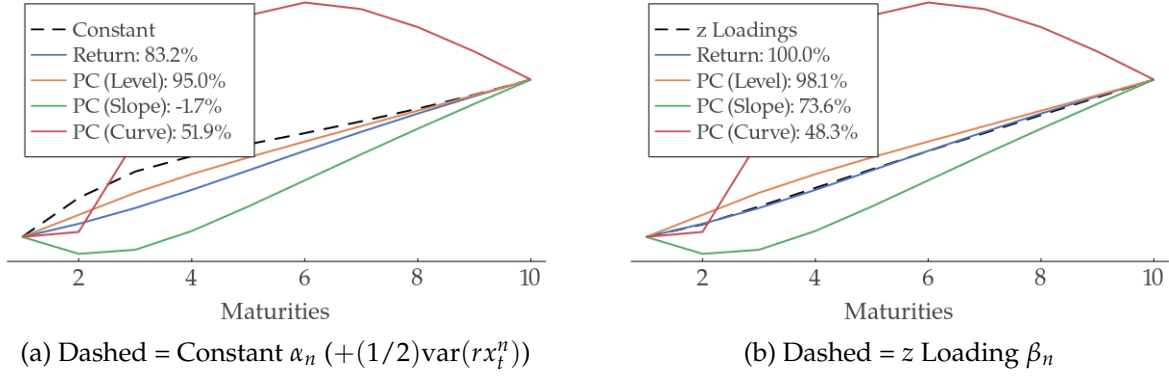
Price of Risk Variation. The step after that is to estimate λ_1 , which then implies the data-generating dynamics via (15). To do it, I match (14) and (10): I force the affine model to match the empirical return-forecasting regression. Its main restriction is that loadings on the level, slope and curvature factors must be zero: *the price of risk varies only due to variation on the return-forecasting factor*. Given the theoretical loadings $[B'_{n-1} \Sigma \lambda_1]$, I satisfy the restriction if the corresponding columns of λ_1 are zero. In that case, (14) becomes $E_t r x_{t+1}^n = [B'_{n-1} \Sigma \lambda_1(1)] z_t$ plus a constant, where $\lambda_1(1)$ is the first column of λ_1 . We reduce the twenty parameters in (λ_0, λ_1) to eight.

Priced Shocks. I take one step further and also assume that only a single shock is priced. In economics language, that is to assume that marginal utility is orthogonal to the remaining three shocks. In the model, it means all but one entry of vector λ_t are equal to zero.

It turns out that we do not need more than one priced factor to reproduce the empirical regression (10), as I show in figure 3. I start by running OLS on (9) and computing residuals \hat{e}_t (the appendix shows the series). For each model factor i , I then plot a solid-line curve that depicts the covariance between its innovations $\hat{e}_{i,t}$ and excess returns on bonds of different maturities $r x_t^n$. They represent the estimated amount of factor- i risk of each bond. The dashed lines represent parameters α (plus the Jensen's inequality term $(1/2)\text{var}(r x^n)$, panel 3a) and β (panel 3b) from equation (10). I estimate both by OLS. The horizontal axis contains bond maturity n . Since $r x_t^1 = 0$,

¹⁸Equation (10) is a restricted version of the more general case $r x_t^n = \alpha_n + b_n X_t + \epsilon_{n,t+1}$.

¹⁹For example, in the case of highly persistent processes, OLS estimates lead to a downward bias on the system's persistence ([Yamamoto and Kunitomo \(1984\)](#)). This is particularly problematic in applications for term structure pricing, since risk-neutral dynamics must have a near-unitary root (the level factor alone explains about 90% of forward rate movements). Indeed, several bias-correction methods have been proposed to address this issue. See [Phillips and Yu \(2005\)](#), [Tang and Chen \(2009\)](#) and [Bauer et al. \(2012\)](#).



Notes: I estimate (9) by OLS and compute residuals \hat{e} . Each panel shows the covariance between excess returns rx_t^n and factor i innovations $\hat{e}_{i,t}$. Each solid curve corresponds to a different factor, and the x -axis contains bond maturity. The dashed curves plot the empirical constant term α (panel 3a, added to $(1/2)\text{var}(rx_t^n)$) and loading β (panel 3b) from the empirical model (10). I scale the curves so that they take the same value for $n = 10$.

Figure 3: Cross-Sectional Amount of Risk *vs* Empirical Model (10)

all curves start at zero. I scale them so the plot looks nice - we are interested only in their shape, not their scale or sign.

Given the estimation strategy of matching (10) and (14), and the assumption that $\lambda_t = \lambda_0 + \lambda_1(1)z_t$, our job is to find linear combinations of the solid-line curves that match the dashed-line ones; λ_0 and $\lambda_1(1)$ are the weights. The figure makes it clear that a single curve will do the job, in both cases. In the legend, I report the R^2 from regressing each curve individually on α (+ Jensen's inequality term) and β . Cross-sectional covariances of excess returns with both return factor shocks and level factor shocks provide, by themselves, an almost perfect fit to the empirical model (14).²⁰ So, I assume throughout that only a single shock is priced. That is to say that only one element of λ_t is different from zero; the same is thus true for λ_0 and $\lambda_1(1)$. I therefore further reduce the number of estimated parameters in (λ_0, λ_1) to two.

Estimation. I estimate $\delta_0, \delta_1, \mu^*, \Phi^*$ by minimizing model pricing errors. Given a choice for priced factor $i \in \{1, 2, 3, 4\}$, note that $\mu^* = \Sigma(i)\lambda_0(i)$, where $\Sigma(i)$ is the i -th row of Σ , and $\lambda_0(i)$ is the i -th element of λ_0 . I therefore search on the space of $\lambda_0(i)$ parameters. Covariance matrix Σ I estimate as $\hat{\Sigma} = \sum_t \hat{e}_t \hat{e}_t' / (T - 1)$.

In all, let $\theta = (\delta_0, \delta_1, \lambda_0(i), \Phi^*)$. The problem is to minimize the mean squared error implied by the model:

$$\text{Min}_{\theta} \left\{ \frac{1}{N} \sum_{n=1}^N \left[\frac{1}{T} \sum_{t=1}^T \left(f_t^n - A_{f,n} - B'_{f,n} X_t \right)^2 \right] \right\}^{\frac{1}{2}}$$

I write the objective function above so as to highlight that this is a method-of-moments estimation. The model contains the stochastic singularity $\eta_{n,t} = f_t^n - A_{f,n} - B'_{f,n} X_t = 0$ which implies $E(\eta_{n,t}^2) = 0$ for all n . The optimization above simply minimizes an equally-weighted combination of these

²⁰In the US case, [Cochrane and Piazzesi \(2009\)](#) show that only covariances with level-factor shocks approximate empirical loadings on the return forecasting factor. In their case, the amount of risk on the other shocks have completely different shapes in the cross-section. Sadly, that is not the case in Brazil.

moment conditions.²¹ It is not quite a GMM procedure since I do not minimize a quadratic form. If that was the case, and assuming an identity weighting matrix, the brackets term of the objective function above would be squared.

I solve the problem using numerical methods. Having estimated θ , we can calculate coefficients B_n . In the model's solution, $E_t r x_{t+1}^n = [B'_{n-1} \hat{\Sigma}(i) \lambda_1(1, i)] z_t$ plus a constant, where $\lambda_1(1, i)$ is the element $(1, i)$ of matrix λ_1 - the last parameter to be estimated. To match the loadings of the empirical return model (14), I run the cross-sectional regression

$$\beta_n = \lambda_1(1, i)(B'_{n-1} \hat{\Sigma}(i)) + \text{error}, \quad (17)$$

in which $B'_{n-1} \hat{\Sigma}(i)$ is the right-hand variable. Having found λ_1 , we can infer the data generating dynamics (μ and Φ) using (15). This completes the estimation.

4.3 Estimation Results

Table 4 contains most results. In Panel A, I compare models and estimation strategies. For each case, the table shows the largest eigenvalue of matrices Φ^* and Φ , the mean squared error in bond pricing and the total number of free parameters.²² Panel B reports the estimated parameters in the baseline specification.

In the baseline specification, I assume four factors in X , as explained above, and that *only level factor shocks are priced*. The table shows that this choice leads to the best model fit, and I also support it under the evidence (presented later) that the level factor is associated to inflation and US risk premium. There is a total of 33 estimated parameters.

The exponential affine model yields a mean squared error of 0.29% and a mean absolute error (not shown in the table) of 0.23%. I compare that to two benchmarks. The first is the simple OLS model $f_{n,t} = a_n + b_n X_t + e_t$, which yields the lowest mean squared errors (0.06%) in the class of linear models at the cost of 50 free parameters (ten equations, five coefficients each), plus 16 if we estimate the state's law of motion (9).

As a second benchmark, I follow a procedure similar to [Adrian et al. \(2013\)](#) to estimate the model. It involves estimating Φ from factor data and (λ_0, λ_1) from cross-sectional regressions of excess returns on sample innovations. The appendix provides details. Unlike unrestricted OLS, this procedure involves active zero restrictions, fewer parameters and a good fit that is almost as good (0.07%).

In our baseline case, the largest eigenvalue of Φ^* is 1.02, which is close to a unitary root. This is an expected result, which we verify in all specifications. Under the risk-neutral measure, there is no term premium (except for a Jensen's inequality term): forward rates equal expected future interest rate (equation (16)). Since forward rates of different maturities tend to move in block - the level principal component captures most variation in the term structure - innovations have to be

²¹Due to the stochastic singularity, maximum likelihood is not feasible - one would have to assume measurement errors in the model.

²²Let $\hat{f}_{n,t}$ be the forward rate predicted by a model. The mean squared error is $\left\{ \frac{1}{N} \sum_n \left[\frac{1}{T} \sum_t (f_{n,t} - \hat{f}_{n,t})^2 \right] \right\}^{\frac{1}{2}}$. That is the objective function I minimize.

Panel A. Stability and Model Fit					
Model	$ \text{eig}(\Phi^*) _\infty$	$ \text{eig}(\Phi) _\infty$	MSE (%)	$[\Phi - \Phi^*]_{2,1}$	# Param
Benchmarks					
OLS			0.06		40 (+16)
VAR	1.04	0.53	0.07	-1.4	55
Baseline	1.02	0.89	0.29	-1.4	33
3 factors	0.99	0.74	0.36	-1.4	21
2 factors	0.99	0.67	0.54	-1.4	12
Free μ	0.99	0.74	0.06	-1.4	37
$t < 2020$	1.00	0.95	0.27	-2.2	33
Return factor priced	1.04	0.92	0.37	-1.4	33
Slope factor priced	1.12	1.59	0.50	-1.2	33
Panel B. Estimated Parameters (Baseline)					
	Constant	Return z	Level	Slope	Curve
Risk-Neutral Dynamics μ^*, Φ^*					
Return z	0	0.62	0.21	-0.01	0.48
Level	0.01	-0.06	0.94	0.48	-0.13
Slope	0	0.04	0.04	0.47	0.84
Curve	0	-0.22	0.12	0.22	0.61
True Dynamics μ, Φ					
Return z	0	0.12	0.21	-0.01	0.48
Level	0	-1.46	0.94	0.48	-0.13
Slope	0	-0.04	0.04	0.46	0.84
Curve	0	-0.22	0.12	0.22	0.61
δ	0.10	0.04	0.46	-0.80	0.34

Notes: Panel A compares different model performance under different estimation methods and underlying assumptions. The $||\text{eig}(\Phi^*)||_\infty$ column indicates the largest eigenvalue matrix Φ^* , and the analogous is true for $||\text{eig}(\Phi)||_\infty$. MSE is the forward rate mean squared error and MAE the mean absolute error; # Param indicates the number of free parameters in each model/estimation strategy. Panel B reports the estimated parameters $\mu, \Phi, \mu^*, \Phi^*, \delta_0$ and δ_1 in the baseline estimation.

Table 4: Estimation Results and Model Dynamics

long-lived so that they affect interest rates far in the future. Hence, the near-unitary root.

Panel B of table 4 reports the estimated parameters in the baseline exercise. By inspecting Φ^* , we see that persistence follows mostly from the level factor, with an autoregressive coefficient of 0.94. The estimate of δ_1 shows that the level factor also impacts interest rate the most (δ_1 's loading on the slope factor -0.80 is about twice the loading on the level factor, but the level factor is almost four times as volatile as the slope factor).

Estimation of λ_1 - its single entry - then yields an estimate of the actual dynamics Φ via (15). Only the first column of λ_1 is non-zero, so only the first columns of Φ^* and Φ differ. The table shows that, in fact, only the first two rows of their first column differ (by more than two decimal digits). Under the true probability measure, the autoregressive coefficient of the return factor declines from 0.62 to 0.12, and its expected effect over next-period level factor goes from -0.06 to -1.46.

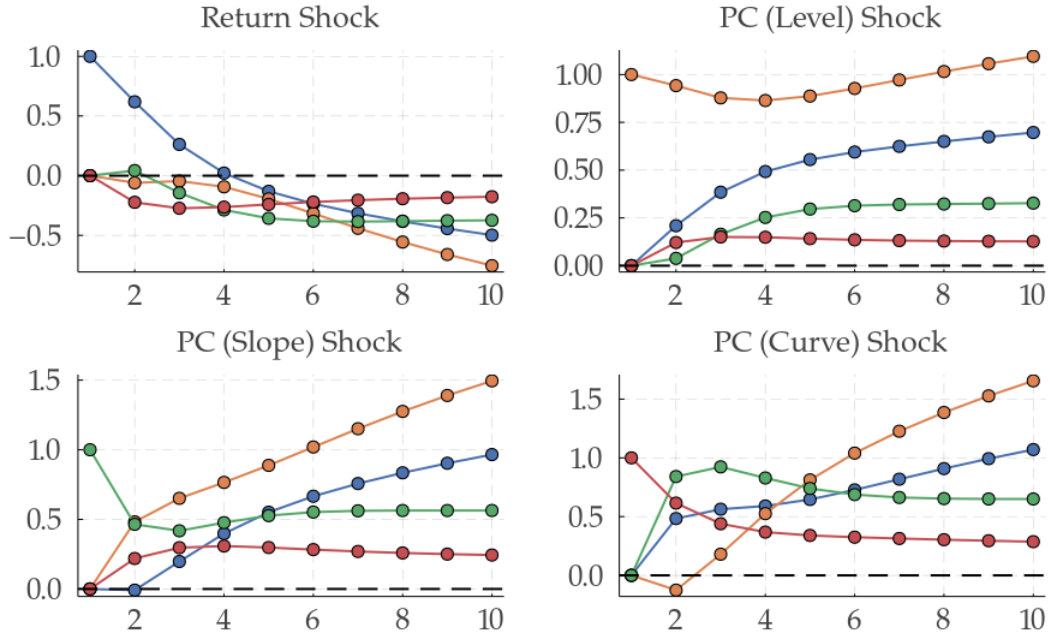
Figures 4 and 5 plot the impulse response functions of the system under the risk-neutral and actual probability measure, respectively. They make it easier to understand the dynamics Φ^* and Φ represent. See the figures' notes for the color legend.

Figure 4 exposes the system's near-unitary and in fact explosive root under the risk-neutral probability measure. An increase in the level factor (top, right panel) leads to an immediate and prolonged increase in all the other factors. The fact that the return factor increases is particularly important. The same observation holds for slope and curve (bottom panels): introducing a shock leads to an increase in all variables in the system. The exception here is the impulse to the return factor (top left panel), which leads to a *small*, yet protracted decline in the level, curve and slope factors. In all cases, movements in the level factor are long-lived: interest rates are thus persistent under the risk-neutral measure.

Dynamics under the true probability measure - figure 5 - have one critical difference: *an increase in the return factor leads to a larger decline in the level factor*. This is where the -0.06 to -1.46 change in the Φ matrices shows up. Large bond risk premia forecasts lower yields for bonds of all maturities. That, of course, leads to changes in the impulse response to the three other shocks as well. Under the risk-neutral measure, they lead factors to move in the same direction, as we saw. Under the true measure, however, the return factor "pushes" the level factor in the opposite direction. *That stabilizes the system's dynamics*. The largest eigenvalue of Φ drops from 1.02 to 0.89.

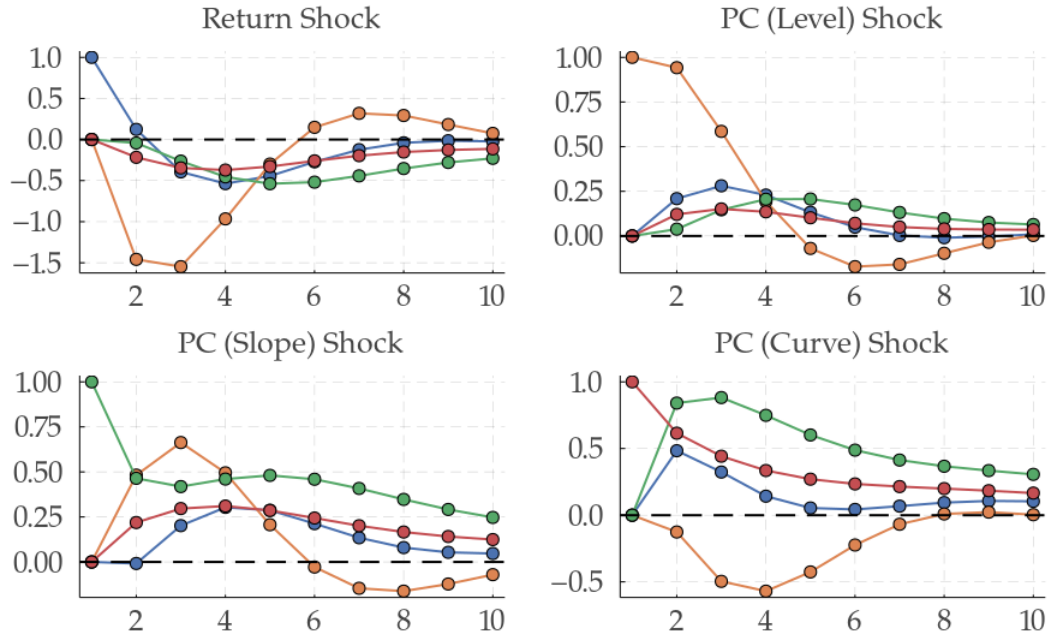
The interaction between return and level is particularly visible in the impulse response to a level shock. As yields increase in period one, it is still true that risk premia increases relative to average for a few periods (about 5 years). But now, such increase causes a decline in yields - the orange curve goes below zero five periods after the initial shock. The interaction goes on: a lower level factor brings down the risk premia to the point that its decay is reversed once again until the system stabilizes.

We can verify this periodic sign reversal of the level factor in the response function to the other shocks too (although somewhat less pronounced in the case of curve shock). This is good news. It means that the model can account for the swings of the Brazilian interest rates verified in sample (red curve in 1a). High interest today forecasts low interest four to five years ahead. Analytically, this feature of the model follows from the negative difference between the entry (2,1) of matrices Φ



Notes: Impulse response function of model dynamics under the risk neutral probability measure. Blue = return factor. Orange = level factor. Green = Slope factor. Red = Curve factor.

Figure 4: Risk-Neutral Dynamics Φ^*



Notes: Impulse response function of model dynamics under the true probability measure. Blue = return factor. Orange = level factor. Green = Slope factor. Red = Curve factor.

Figure 5: Data-Generating Dynamics Φ

and Φ^* , which I report for all specification in Panel A of table 4.

Where does it come from? By assumption, level shocks directly affect marginal utility, and since bond prices go down when the level factor jumps up, we estimate $\lambda_1(2, 1) < 0$. (You can see this from equation (17), $\beta_n > 0$ and $B_{n-1} < 0$ in the second column.) From the second line of (15), $\lambda_1(2, 1) < 0$ leads to $\Phi_{2,1} < \Phi_{2,1}^*$.

Intuitively, we need the sensitivity of marginal utility on level shocks to depend on the return-forecasting factor state - such heteroskedasticity of the stochastic discounting is what allows for time-varying bond premia. When the return factor is positive, marginal utility is more sensitive to level shocks. Risk-neutral pricing internalizes that by increasing the chance of a higher level shock in the following period: $\Phi_{2,1} < \Phi_{2,1}^*$.

To sum up, the dynamics of state variables are characterized by the following key patterns. Under the risk-neutral measure the level factor movements "pull" the return factor in the same direction; shocks are long-lived. Under the actual measure, the level factor "pushes" the level factor to the opposite direction, leads to periodic swings in the interest rate and stabilizes the system.

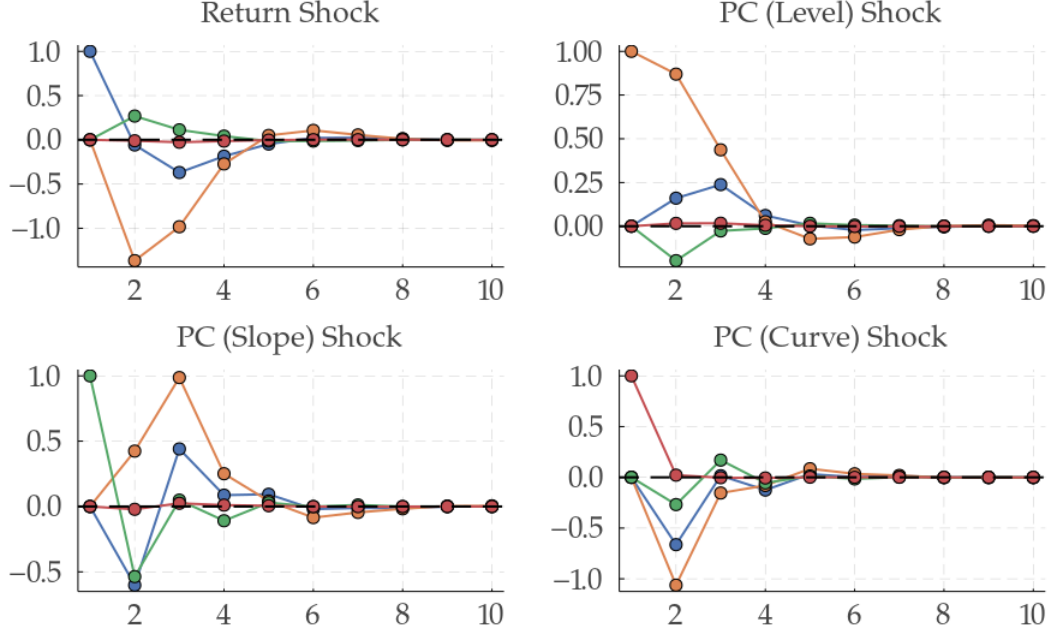
Alternative Specifications. Panel A of table 4 summarizes the results from various different specifications of the model. The "3 factors" and "2 factors" rows correspond to specifications with less than four state variables (I cut the curve factor and then the slope factor). Mean squared errors obviously increase.

In the "free μ " case, I add a constant μ to the law-of-motion of (9), which becomes $X_t = \mu + \Phi X_{t-1} + e_t$, and adjust (15) accordingly: $\mu^* = \mu - \Sigma \lambda_0$. The change relaxes the assumption that the sample average of X equals its unconditional average. At the cost of four additional parameters, the model yields mean squared errors less than two digits smaller than that of OLS, the best linear predictor. This shows that the exponential-affine model *can* lead to small pricing errors.

The "Return factor priced" and "Slope factor priced" rows consider the cases in which not level factor shocks but return factor shocks and slope factor shocks (respectively) are priced. In the former case, model fit does not change by much; in the latter case, it does. The reductions in model fit here are reminiscent of the fact that the cross-sectional covariances between bond returns with the return factor and slope factor shocks do not provide the best fit to the empirical excess return model (10).

With the exception of the slope factor priced model, the different specifications yield a few common results. Risk-neutral dynamics have a root close to unity, while under the true-probability measure dynamics are less persistent (as measured by the largest eigenvalue of Φ). More importantly, our characterization of the dependency of the return and level factors on each other's lagged values is also robust, as evidenced by the difference between entry (2,1) of Φ and Φ^* , negative in all cases.

Figure 6 provides one example. In it, I plot the impulse-response function of the model estimated from the VAR (9) with a constant μ . The VAR estimates a model with lower persistence (0.53 *vs* 0.89 in the baseline), but qualitatively the responses of return and level shocks to each other are similar: a high level factor increases the return factor; a higher return factor lowers the term structure level. This is an encouraging result. The key dynamic property of the model estimated without the use of time series autocovariances also shows up in time series autocovariances.



Notes: Impulse response function of model dynamics under the true probability measure. Blue = return factor. Orange = level factor. Green = Slope factor. Red = Curve factor. IRF estimated from direct estimation of $X_t = \mu + \Phi X_{t-1} + e_t$.

Figure 6: Data-Generating Dynamics Φ - Estimated from VAR

In the appendix, I provide the impulse-response functions to other specifications. In all of them, these two properties are present.

Interpretation. How can one interpret the different behavior of our state variables under the actual and risk-neutral measures?

"Risk-neutral" measure is another name to probability times marginal utility. In factoring in the latter, one can do asset pricing by simply taking expectations of uncertain payoffs. Hence, the economic interpretation of an asset pricing model comes entirely from the discount factor. What do shocks to the discount factor mean?

The exponential-affine model offers a lot of tractability. In a given period $t + 1$, both the actual shock e_{t+1} and the artificial risk-neutral "shock" e_{t+1}^* are Gaussian with the same covariance matrix Σ . Their probability density functions therefore have the same normalizing constant \mathcal{M} ; only their exponential kernel differ. Hence, we can factor in the discount factor m_{t+1} by changing such exponential kernel and put to evidence the density function of the risk-neutral measure:

$$\begin{aligned}
 1 &= E_t \exp\{m_{t+1}\} R_{t+1}(e_{t+1}) = \mathcal{M} \int \exp\{-(1/2)e'\Sigma^{-1}e\} \exp\{-i_t - \lambda_t'\Sigma\lambda_t - \lambda_t e\} R_{t+1}(e) de \\
 &= \mathcal{M} \int \exp\{-(1/2)(e - \phi_t)'\Sigma^{-1}(e - \phi_t)\} \exp\{-i_t\} R_{t+1}(e) de.
 \end{aligned}$$

So, risk-neutral pricing here simply amounts to taking expected value of returns "as if" shocks e_{t+1} had a conditional mean equal to ϕ_t . The last equality then implies

$$\phi_t = -\Sigma\lambda_t,$$

and all we need now is to interpret ϕ_t . To do that, suppose we arrive at $t + 1$ and observe $e_{t+1} = \phi_t$. State variables will then be given by

$$X_{t+1} = \Phi X_t + \phi_t = \Phi X_t - \Sigma\lambda_t = \mu^* + \Phi^* X_t.$$

We have thus a link between the estimated risk-neutral dynamics - to which the IRFs above give a useful interpretation - and the discount factor m . What the equation above tells us is that *the discount factor, or marginal utility, places elevated weight on combinations of shocks ($\approx \phi_t$) that lead our states X to present the risk-neutral dynamics (μ^*, Φ^*)*.

As discussed in the last section, the conditional distribution of the level factor is the main divergence between actual and risk-neutral dynamics of X . Under the risk-neutral dynamics, the level factor increases 0.01 per period (that is the μ^*). Since factors have zero mean, this indicates that, on average, marginal utility increases in the level factor. Portfolios that pay when the level factor experiences a positive shock act as insurance. Conditioning on X_t , we saw that risk-neutral dynamics also predicts a reduced response of the level factor in period $t + 1$ to the current return factor. Marginal utility is thus sensitive to a *lack* of response of the level factor.

All these observations ask for an investigation of which macroeconomic variables are correlated with the level and return factors. To investigate correlation, I report in table 5 the R^2 s of regressions of the form

$$x_t = \alpha + \beta m_t + \epsilon_t$$

where x is one of my factors and m is a macroeconomic variable of choice.²³ Given obvious endogeneity issues, I do not report point estimates. I am interested only in the correlation between variables, not causal or structural relationships.

Yields Decomposition. The exponential-affine model offers a decomposition of the yield and forward-rate curves. I focus on the latter. From (13) we have

$$E_t i_{t+n-1} = \delta_0 + \delta'_1 E_t X_{t+n-1}.$$

From (9), we have expected factors $E_t X_{t+n-1}$. With $E_t i_{t+n-1}$, we can use (1) to infer the term premium. (of course, one could directly calculate the term premium from its formula provided by (1)).

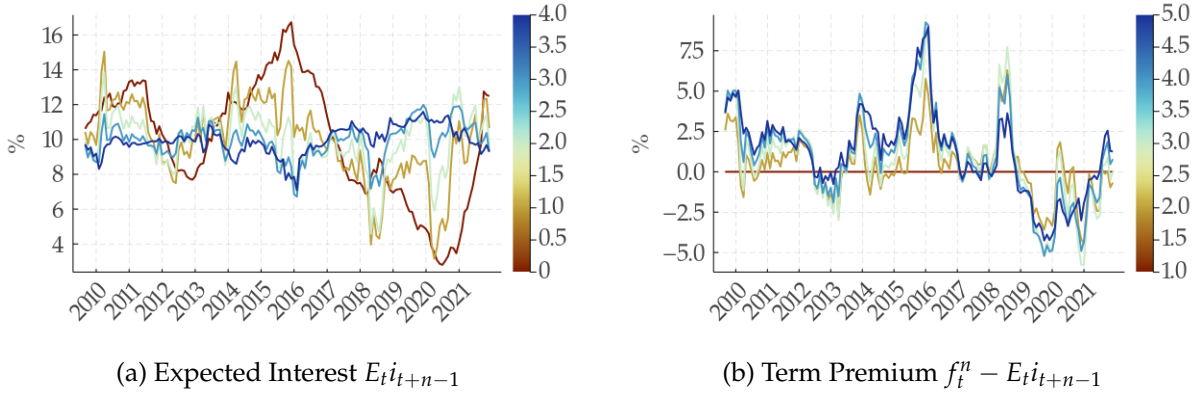
Figure 7 contains the yield curve decomposition. Plot 7a depicts expected interest rates, plot 7b depicts term premia. Added up, they yield the model's forward rates. The format of the graphs is similar to 1a: different colors correspond to different bond maturities, as indicated by the bar

²³Inflation is the annual growth in the consumer price index. Industrial production is the quantum index of industrial production. The primary surplus is the 12-month cumulated public primary surplus deflated by the consumer price index. Trade balance and current account are deflated by the US consumer price index, and are both cumulated in the last twelve months. The proxy for commodities' prices is the Commodity Research Bureau BLS Spot index, taken from Bloomberg. MSCI is a free-float weighted equity index that captures large and mid size stocks across Emerging Market countries. I take from Bloomberg.

Right-Hand Variable	Return Factor	Level Factor	Slope Factor	Curve Factor
<i>International Variables</i>				
US Term Spread	0.09	0.54	0.01	0.09
MSCI	0.20	0.21	0.00	0.07
EMBI (Brazilian bonds)	0.13	0.05	0.05	0.16
Δ Commodities' Prices (CRB)	0.03	0.02	0.05	0.02
<i>Brazilian Variables</i>				
Short-term interest	0.18	0.83	0.17	0.00
Inflation	0.03	0.40	0.11	0.10
Δ Industrial Production	0.05	0.02	0.00	0.13
Trade Balance	0.00	0.17	0.07	0.00
Current Account	0.01	0.02	0.10	0.00
Primary Surplus	0.00	0.08	0.04	0.00

Notes: I run regressions of the form $x_t = a + bm_t + \varepsilon_t$, where x is a model factor and m is a macroeconomic variable. The table reports the R^2 of each regression.

Table 5: R^2 of Regressions of Model Factors into Macro Variables



Notes: I use the baseline estimation of the exponential-affine model to decompose forward rates into expected interest and term premium, as shown by equation (1). Each curve corresponds to a different maturity, and the colors match the terms of the same decomposition across figures. For instance, the blue curves represent $E_t i_{t+4}$ on the left, and Term Premium $_t^5 = f_t^5$ on the right; and $E_t i_{t+4} + \text{Term Premium}_t^5 = f_t^5$.

Figure 7: Decomposition of the Term Structure of Forward Rates

on the right. I choose maturities so that the terms summing up to the same forward rate have the same color.

While we estimate that the economy's dynamics are highly persistent - the largest eigenvalue of Φ is 0.89, shocks take a long time to die off -, the pattern of expected interest rates does not resemble the random walk $i_t = i_{t-1} + e_t$, as [Cochrane and Piazzesi \(2009\)](#) find for the US. High interest rates today are typically associated with lower interest rates three or four years in the future.

Unintuitive as it is, visual inspection of the data does not reject this result. Interest in Brazil (red curve in figure 1a) has clearly moved in waves during the 2010s, with successive up-and-down swings. Forecasting that it will continue to do so is a sensible prediction. It follows from

the combined dynamics of the return and level factors, which I show to be robust across model specifications. Any shock that reduces risk-premia (z goes down) leads to higher yields (and interest rate); higher yields lead to an increase in risk premia, which in turn reduces yields, and so on.

The swings do not deny the conclusion that shocks are long-lived. The fact that the economy takes time to return to its long-term average does not mean we should expect a given shock to have similar sign effects on a certain variable in different time horizons.²⁴ In fact, if I plotted maturities above $t + 4$ (I do not, to avoid making the plots too crowded), we would see that expected future interest would *not* continue to move in the opposite direction of current interest. It would instead fluctuate with up and down movements for a long time until it stabilized in its long-term average of 9.9%.

Plot 7b of term premia also reveals interesting patterns. First, the term premium is non-zero and economically significant. At the height of the 2015-16 crisis, it reaches 8% for some bonds. More importantly, term premia of different maturities do not move in lockstep. The premium does not necessarily increase with maturity always. As equation (1) suggests, the underlying reason is the projected path of excess returns, which varies depending on the state of the models' factors X .²⁵

For instance, the two years that feature the more pronounced surges in term premia (on average) are 2015 and 2018. On the former, term premia increases from maturity $n = 2$ to $n = 5$. On the latter, it *decreases* from $n = 2$ to $n = 5$.

It turns out that, in this example, we can understand the different behavior of term premia only from the joint dynamics of the return and the level factors I highlight above. As I show in the appendix, in 2015 both of them peak in (by then) all-time highs. The year of 2018, on the other hand, features an again high return factor (even higher than it was in 2015), but a level factor that is close to and then lower than the long-term average of zero.

As the impulse-response function figure 5 indicates, in case of a pure return factor shock (top, left plot), agents expect the return factor itself initially to increase, of course, and then decline to negative territory starting in period three. That means that they expect larger excess returns up to period three (remember: z_t predicts *one-year ahead* returns) and then *lower* excess returns. Thus, the terms of the sum that account for the term premia are positive up to maturity three, and become negative for larger ones.²⁶ Hence the increasing term premia up to period three and their decline for larger maturities.

But in 2015 the level factor was also unusually high. The model is linear, so we can try to understand the profile of term premia by summing up the system's responses to a return factor shock and to a level factor shock. The latter is the novelty now. As concluded before (and shown by the top-right graph of figure 5), an above-average level factor induces an increase in the return factor z for a few years in the future. Such increase, combined with the effects of the other two model factors, to a large extent offsets the lower mid-to-long term risk-premium that follows the

²⁴Mathematically, matrix Φ contains a couple of complex eigenvalues that lead to wave movements that characterize the dynamics of X .

²⁵This is what Cochrane and Piazzesi (2009) dub a non-trivial term structure of risk premia.

²⁶The model implies that the betas in (10) are positive and increasing, just like q_n in table 2. Therefore, the terms $E_t(rx_{t+1}^n - rx_{t+1}^{n-1})$ in the definition of the term premium always have the same sign as z .

innovation in the return factor, as explained in the paragraph above. Unlike in 2018, in 2015 agents did not expect risk premium to quickly revert to average. Its drawn-out path accounts for a cross section of term premium that increases in maturity. Precisely what we see in figure 7b.

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	Mean	St Dev	AC(1y)	AC(2y)
	%	%	%	%
f(1)	9.3	3.3	57.8	14.2
f(2)	10.5	2.7	50.7	16.0
f(3)	11.1	2.2	41.1	21.2
f(4)	11.3	2.0	33.5	20.2
f(5)	11.4	1.9	29.2	18.1
f(6)	11.4	1.8	27.0	16.7
f(7)	11.5	1.8	26.1	16.1
f(8)	11.5	1.8	25.6	16.0
f(9)	11.6	1.8	25.2	16.0
f(10)	11.6	1.8	24.9	16.1
S(10-1)	2.3	2.2	27.7	14.2
S(5-1)	2.1	2.1	29.6	13.7
S(10-5)	0.2	0.5	10.3	-9.6
Rx(2)	1.4	2.7	1.0	-11.4
Rx(3)	2.3	5.2	-8.1	-5.7
Rx(4)	2.8	7.5	-16.6	-0.9
Rx(5)	3.2	9.6	-23.1	1.8
Rx(6)	3.5	11.8	-27.7	3.5
Rx(7)	3.7	13.9	-31.1	4.6
Rx(8)	4.0	16.0	-33.6	5.5
Rx(9)	4.2	18.1	-35.5	6.2
Rx(10)	4.5	20.3	-37.0	6.9

Notes: 2010-2021 Brazilian term structure data. All values in percentage units. "f" means forward rate, "S(a-b)" means spreads (f(a)-f(b)), and "Rx" means excess return. AC(1y) means one-year Pearson autocorrelation, same for AC(2y).

Table 6: Table of Sample Moments

Appendix A Data - Further Analysis

A.1 Unconditional Moments

Figures 1a and 1b give us a first impression of the dynamics of yields and returns. However convenient, visual evaluation alone can mask important features of the data.

Tables 6 and 7 present some sample moments of forward rates and excess returns. I also separately report moments for forward *spreads*, defined as differences between forward rates of different maturities. It is of course possible to infer their moments from the moments of the underlying forward rates, but given their importance when building the return factor, I include them here as well. In the tables, spread are depicted by the letter "S": $S(10-1) = f(10)-f(1)$ and so on. Table 6 focuses on each series individual moments (mean, volatility and persistence); table 7 focuses on cross-section correlations. I report all values in percentage units.

Starting with table 6, the term structure of average forward rates is positively sloped: long rates are, on average, larger than short ones. These averages range between about 9 and 12% per year. Short rates, on the other hand, are more volatile. Our estimated standard deviations range between 3.3% for the risk-free bond and 1.8% per year. All forward rates have a positive one-year autocorrelation coefficient, but short-duration

bonds are more persistent, with coefficients above 0.5 in the cases of the one and two-year maturities. These autocorrelations uniformly decline to the 0.15-0.25 range when I look at a two-year horizon and, at a three-year horizon (not reported), they all turn negative.

Note that in the short end of the curve, moments are rapidly changing. Between maturities one and four years, the average rate increases from 9.2 to 11.3%, the standard deviation decreases from 3.3 to 2% and the one-year autocorrelation falls from 0.65 to 0.35. But in the long end of the term structure, moments become considerably more homogeneous across bonds of different maturities. Between maturities four and ten years, the mean rate goes from 11.3 to 11.6%, standard deviations from 2 to 1.9% and autocorrelation coefficients from 0.35 to 0.26.

Moving on to forward spreads, the average 10y-1y spread is 2.4%, with the bulk of this difference coming from the spread in the first five years of the term structure, as one might expect from the remarks made in the previous paragraph. Additionally, the short spread $S(5-1)$ shows an estimated standard deviation between four and five times greater than that calculated for the long spread $S(10-5)$. This observation is useful in the numerical interpretation of the return-forecasting factor. Finally, while $S(5-1)$ presents some persistence, with one and two-year autocorrelation coefficients of 0.36 and 0.16, respectively, $S(10-5)$ appears to quickly revert to its unconditional mean.

Looking now at excess returns - the table ignores $R(1)$ since it equals zero by construction - we see that the "short-term heterogeneity / long-term homogeneity" trait of forward rate moments is, to a large extent, gone. The average excess return increases from the short to the long end of the term structure along with its volatility. Assuming my sample is sufficiently large to approximate populational moments well, estimated means reflect positive unconditional risk premia: holding public bonds provide investors with a return greater than the risk-free rate, on average. This suggests that bonds' payoffs are *risky* in the sense that they covary negatively with marginal utility, a claim we cannot put to the test without a proxy for marginal utility - one of the goals of the paper. Estimated unconditional risk premia range from 1.5% per year for two-year bonds to 5.2% per year for ten-year bonds. In a far looser use of the word *risk*, what we can say from the data is that long-duration bonds are riskier because their return is more volatile. Estimated standard deviations of excess returns increase almost linearly from the two to the ten-year duration bonds, with increments from term to term always between 2 and 2.5%.

Finally, table 6 shows that, while returns from short-duration bonds display almost no persistence, returns from long-duration bonds display *negative* persistence. One-year autocorrelation coefficients become increasingly negative, and reach an economically significant -0.4 value for the ten-year bond. In the case of these bonds with long duration, a good year for investors is usually followed by a bad year. In addition, moving to the two-year autocorrelation, coefficients move back into positive territory, despite having lower absolute values, which suggests that excess return dynamics displays a "wave" shape at an annual frequency. Good years are followed by bad years, and then good years again.

Moving to table 7, we focus now on correlations. Each entry of the table contains the Pearson correlation between the row variable and the column variable. Including all maturities would take a lot of space and add little insight, so I select a subset of them that hopefully represents well the short, medium and long end of the curve. An important clarification: I compare forward rates measured in period t with returns realized in period $t + 1$ (one year later), *not* t . As we saw on table 6, excess returns are negatively correlated over time. So, the comparison between time- t rates and time- t returns yields different results. I stick to the former comparison since our interest is how forward rates relate to the performance of portfolios formed by the time they are observed (the payoffs of which we observe a period later).

Starting with forwards, we see that rates are highly correlated across maturities, as expected. Two additional comments: 1. the closer the maturities, the more correlated are the corresponding rates; 2. the one-year forward - our measure of a risk-free rate - is slightly *less* correlated with the rest of the curve than its counterparts, although coefficient estimates remain pretty high. Similarly, excess returns are highly

	f(1)	f(5)	f(10)	Rx(2)	Rx(5)	Rx(10)	S(10-1)	S(5-1)	S(10-5)
	%	%	%	%	%	%	%	%	%
f(1)	100.0	81.4	79.5	23.6	27.2	30.7	-85.6	-85.0	-17.1
f(2)	93.7	92.5	91.7	20.5	31.5	38.0	-65.6	-65.0	-13.7
f(3)	87.0	98.0	95.5	26.1	41.1	48.5	-52.1	-49.4	-21.0
f(4)	83.1	99.7	96.5	29.5	46.7	54.7	-45.3	-41.6	-24.6
f(5)	81.4	100.0	97.1	29.9	48.2	57.0	-42.2	-38.7	-23.1
f(6)	80.9	99.8	98.0	28.6	47.7	57.1	-40.6	-37.9	-19.0
f(7)	80.7	99.4	98.8	26.8	46.2	56.3	-39.5	-38.0	-13.8
f(8)	80.4	98.8	99.5	24.8	44.5	55.0	-38.6	-38.2	-8.5
f(9)	80.1	98.0	99.9	22.8	42.8	53.6	-37.7	-38.3	-3.6
f(10)	79.5	97.1	100.0	21.1	41.1	52.3	-36.8	-38.3	0.7
S(10-1)	-85.6	-42.2	-36.8	-18.0	-5.6	-1.1	100.0	97.8	26.8
S(5-1)	-85.0	-38.7	-38.3	-10.0	1.2	3.7	97.8	100.0	6.1
S(10-5)	-17.1	-23.1	0.7	-41.8	-35.0	-24.4	26.8	6.1	100.0
Rx(2)	23.6	29.9	21.1	100.0	91.8	82.6	-18.0	-10.0	-41.8
Rx(3)	26.4	39.2	31.0	98.1	97.5	90.8	-13.4	-5.8	-39.6
Rx(4)	26.8	44.6	37.0	94.9	99.6	95.2	-8.7	-1.5	-37.3
Rx(5)	27.2	48.2	41.1	91.8	100.0	97.6	-5.6	1.2	-35.0
Rx(6)	27.8	50.8	44.2	89.3	99.8	98.8	-3.7	2.7	-32.7
Rx(7)	28.5	52.8	46.7	87.1	99.3	99.4	-2.6	3.5	-30.5
Rx(8)	29.2	54.4	48.8	85.4	98.7	99.8	-1.9	3.8	-28.4
Rx(9)	30.0	55.8	50.7	83.9	98.1	100.0	-1.4	3.8	-26.4
Rx(10)	30.7	57.0	52.3	82.6	97.6	100.0	-1.1	3.7	-24.4

Notes: 2010-2021 Brazilian term structure data. All values in percentage units. "f" means forward rate, "S(a-b)" means spreads (f(a)-b(b)), and "Rx" means excess return. Each cell contains the Pearson correlation between the variables indicated in the corresponding row and column. I compare forward rate f_t^n with one-year ahead excess returns rx_{t+1}^n .

Table 7: Table of Sample Moments - Correlations

correlated across maturities, with estimated coefficients declining in the duration difference.

I also observe that forward rates and excess returns are positively correlated: when prices are low compared to their unconditional average, returns tend to be high. The correlation strengthens for long maturity forwards and returns. In fact, forward rate correlations with the excess return of the two-year bond do not go above 0.25 for any maturity.

Finally, we look at forward rate spreads. The 10y-1y spread $f(10)-f(1)$ is negatively correlated with the rates of *all* ten maturities. The term structure of interest rates tends to have a higher slope when yields are low, a common pattern which we can attribute to the fact that long and short duration rates have a strong, positive correlation and that long rates are less volatile:

$$\text{cor}(f^{10}, f^{10} - f^1) \propto \sigma(f^{10}) - \text{cor}(f^{10}, f^1)\sigma(f^1)$$

($\sigma(x)$ means standard deviation of x , \propto means "proportional to"). The table also shows that the bulk of such correlation comes from the dynamics of the short spread, measured in the tables by $f(5)-f(1)$. The long spread, besides displaying reduced volatility and autocorrelation when compared to the other half of the slope, is also less correlated (in absolute terms) with forwards rates. In fact, by looking at the cross-correlation between our spread measures (right column, middle rows), we verify that the dynamics of the "full" slope $S(10-1)$ is very similar to the dynamics of its first half $S(5-1)$. Both display similar volatilities (2.2 and 2.1%) and a correlation coefficient that borders 0.98. On the other hand, I estimate the correlation of either of them with the long spread $S(10-5)$ to be far lower: 0.08 for the short spread, 0.29 for the full slope. Lastly, the short spread (and so, almost as a consequence, the full slope) is positively correlated with excess returns of short-duration bonds. When prices are low compared to current stances of short-term rates, returns tend to be high. Like with forward rates, correlation coefficients increase as we look at the return of bonds with longer maturities. Nevertheless, note that such correlations are somewhat weaker (never above 0.13), especially if we compare them with the coefficients calculated for forward rates of maturities five and ten (often above 0.40).

As for the long spread $S(10-5)$, its relationship with excess returns is not in this case a mere toned down version of the returns/mid-short spread correlation structure. All coefficients are larger in absolute value, negative and decrease (in absolute value) for longer maturities. This feature of the data provides a first indication that different parts of the term structure slope convey different information regarding expected returns. This is also consistent with the "tent"-shaped pattern of OLS coefficients found by [Cochrane and Piazzesi \(2005\)](#) in their return regressions on forward rates. The first half of the tent says that a larger short slope predicts higher returns; the second half says that a larger short slope predicts lower returns. The analysis of unconditional moments for the Brazilian term structure suggests that a similar pattern applies, with the difference being that I look at maturities of up to ten years. Such similarity will show up again when we build the return factor.

A.2 Principal Components

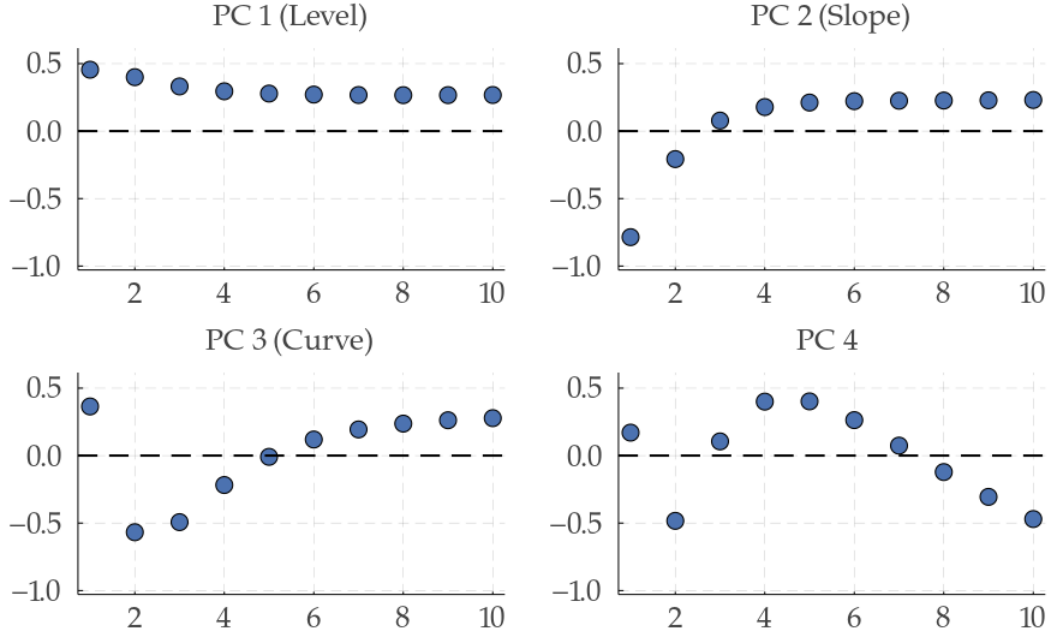
We can build principal component factors that describe the movements of forward rates of different maturities using an eigenvalue decomposition of their covariance matrix ([Litterman and Scheinkman \(1991\)](#)). A well-known result that holds for the US bond market is that the first three principal components explain most (99%+) price variation.

The same is true for the Brazilian term structure, as one can guess from the strong correlation between rates of different maturities reported in table 7. Table 8 reports the cumulative share of variance for each additional principal component I consider. Three components get us over 99% of the total variance in the term structure.

	Cum Var	Std Dev
	%	Annual %
PC 1 (Level)	92.7	6.6
PC 2 (Slope)	98.8	1.7
PC 3 (Curve)	99.5	0.6
PC 4	100.0	0.5
PC 5	100.0	0.1

Notes: I calculate the principal components of the forward rates' data using the eigenvalue decomposition of the sample covariance matrix. The first column reports the cumulative variance of each component. The second column reports their standard deviation. All values reported in percentage units.

Table 8: Principal Components - Cumulative Variance Share



Notes: Horizontal axis indicates the forward maturity. I calculate the principal components of the forward rates' data using the eigenvalue decomposition of the sample covariance matrix. Each plot contains the weights $w^{(i)}$ of each forward rate in the calculation of the indicated principal component, for example $pc_t^{(1)} = w_1^{(1)}f_t^1 + \dots + w_{10}^{(1)}f_t^{10}$. The weights coincide with the loadings of the forward rate on the corresponding principal component, hence the names in parenthesis.

Figure 8: Principal Component Loadings

Figure 8 plots the associated eigenvectors - the weights of each rate when we construct our factors. For example, the top left graph indicates that, to build the first principal component, we sum the series of forward rates of all maturities, each weighted by a positive constant ranging from 0.4 to 0.5. The principal components decomposition uses an orthonormal matrix to convert the data. For this reason, the eigenvectors I plot also correspond to the loadings of each forward rate on the corresponding factor. Focusing again on the top left graph, this means that a 1% increase in the first factor leads to a increase in each forward rate of 0.4 to 0.5%. Because yields in this case change almost uniformly, we call that the *level* factor. Analogously, the next two factors are called *slope* (or steepness) and *curve* factors. I plot the demeaned time series for each of them in figure ??.

The principal components decomposition is useful to justify the adoption of factor models when pricing

the term structure. They also provide natural candidates for factors when theory does not pin them down. Indeed, to the extent that prices summarize the relevant information available to investors and how they react to it, one could assess economic news by measuring their impact on different yields. The downside, of course, is that we lose any "structural" interpretation of new information. All that matters is how the term structure of interest changes, and only in the dimensions we pay attention to (three, if we take three principal components). A new round of monetary tightening by the central bank and news of the latest primary deficit by the government are the exact same thing if both increase rates uniformly. We see the level factor jump and that's it. Similarly, two policy rate cuts of similar magnitude by the central bank account for different shocks if only one of them affects long-term yields.

One can question if it is practical to refer to economic news and events in terms of how they affect "levels" or "slopes" of the term structure. While having structural interpretation to factor movements is certainly nice, perhaps a more important question is whether it is appropriate to brand with the same name ("monetary" or anything other) two shocks that affect prices differently.

Appendix B Full Derivation of the Affine Model

I present the details of the multifactor affine model I estimate in the main text. In stating the model, I follow the same notation as in sections 2 and 3.

I start by defining a vector of factors X_t that evolve according to

$$X_t = \mu + \Phi X_{t-1} + e_t, \quad (\text{B.1})$$

where $e_t \sim \mathcal{N}(0, \Sigma)$ follows a multivariate normal distribution and is independent over time.

The one-period payoff of a bond with maturity n is the price of the same bond in the following period, when it turns into $n - 1$ period bond. Bonds with maturity zero pay one unit of currency. Given a stochastic discount factor M for payoffs denominated in currency units, the price of the zero-coupon bond is given by

$$P_t^n = E_t M_{t+1} P_{t+1}^{n-1}, \quad P_{t+1}^0 = 1$$

or, taking logs and defining $m = \log M$,

$$p_t^n = \log E_t \exp \{m_{t+1} + p_{t+1}^{n-1}\}, \quad p_{t+1}^0 = 0.$$

Hence, a pricing theory amounts to picking m_{t+1} . Before proceeding, it is useful to re-state the pricing equation above in terms of excess returns. To do this, define the (log of the) nominal risk-free rate as usual: $i_t = \log E_t (M_{t+1})^{-1}$. Note this is the same as

$$i_t = -p_t^1. \quad (\text{B.2})$$

So, the pricing equation above implies

$$0 = \log E_t \exp \{m_{t+1} + i_t + r x_{t+1}^n\}, \quad r x_{t+1}^1 = 0 \text{ a.s.} \quad (\text{B.3})$$

("a.s." means almost surely, or for every realization of the shocks that happens with positive probability). I assume the following log discount factor:

$$m_{t+1} = -\delta_0 - \delta_1' X_t - (1/2) \lambda_t' \Sigma \lambda_t - \lambda_t' e_{t+1} \quad (\text{B.4})$$

Heteroskedasticity of m allows for time-varying prices of risk, which is precisely what we need to reproduce time-varying risk premia as observed in the data. Having the innovations to the discount factor come from e_{t+1} rather than X_{t+1} is necessary to generate an affine solution. As we see next, λ_t governs the market price of risk and expected returns. It is linear of the factors:

$$\lambda_t = \lambda_0 + \lambda_1 X_t, \quad (\text{B.5})$$

where λ_1 is square matrix.

Equation (B.2) gives the solution for the interest rate:

$$i_t = \delta_0 + \delta'_1 X_t. \quad (\text{B.6})$$

Replacing the discount (B.4) on the general solution (B.3) yields

$$E_t r x_{t+1}^n = -(1/2) \text{var}_t(r x_{t+1}^n) + \text{cov}_t(r x_{t+1}^n, e'_{t+1}) \lambda_t. \quad (\text{B.7})$$

The covariance term represents the amount of risk of each bond, or its "beta"²⁷, which is why λ_t is referred to as the market price of risk. Given the solution below, the conditional variance and covariance terms are both time invariant, and we can thus drop the t subscripts. All variation in the risk premium $E_t r x_{t+1}^n$ comes from λ_t .

The linear structure implies a solution for equilibrium prices that is linear in the factors. The same is true for forward rates:

$$p_t^n = A_n + B_n' X_t \quad (\text{B.8})$$

$$f_t^n = A_n^f + B_n^{f'} X_t. \quad (\text{B.9})$$

The relationship between forward rates and prices (??) implies $A_n^f = A_{n-1} - A_n$ and $B_n^f = B_{n-1} - B_n$. Using our definition (B.2) of the nominal risk-free rate and the discount factor (B.4), we also find that $-A_1 = A_1^f = \delta_0$ and $-B_1 = B_1^f = \delta_1$. By replacing (??) on the definition of excess returns (??), and that on the pricing equation (B.7), we find expressions that allow for the computation of the solution coefficients recursively:

$$\begin{aligned} A_n^f + B_{n-1}' \mu^* &= -(1/2) B_{n-1}' \Sigma B_{n-1} + \delta_0 \\ B_n' &= -\delta_1' + B_{n-1}' \Phi^* = -\delta_1' (I + \Phi^* + \Phi^{*2} + \dots + \Phi^{*(n-1)}) \\ B_n^{f'} &= \delta_1' \Phi^{*(n-1)}. \end{aligned} \quad (\text{B.10})$$

In the equations above, I define

$$\begin{aligned} \mu^* &= \mu - \Sigma \lambda_0 \\ \Phi^* &= \Phi - \Sigma \lambda_1. \end{aligned} \quad (\text{B.11})$$

These coefficients characterize the *risk-neutral* dynamics of the factors X_t . To see this, replace the computed

²⁷In the beta representation traditionally used in the empirical finance literature, an asset's beta is usually defined as the linear projection coefficient on the corresponding risk factor. Given the homoskedasticity of the innovations in the model, the coefficients emerge by simply left-multiplying λ_t by $\Sigma^{-1} \Sigma$. Then, $\text{cov}(r x^n, e') \Sigma^{-1} = \text{cov}(r x^n, e') E(e e')^{-1}$ becomes the "usual" beta, and $\Sigma \lambda_t$ becomes the market price of risk.

coefficients in (B.10) in (B.9):

$$\begin{aligned}
f_t^n &= -(1/2)B'_{n-1}\Sigma B_{n-1} + \delta_0 - B'_{n-1}\mu^* + \delta'_1\Phi^{*n-1}X_t \\
&= -(1/2)B'_{n-1}\Sigma B_{n-1} + \delta_0 + \delta'_1 \left[(I + \Phi^* + \dots + \Phi^{*n-1})\mu^* + \Phi^{*n-1}X_t \right] \\
&= -(1/2)B'_{n-1}\Sigma B_{n-1} + \delta_0 + \delta'_1 E_t^* X_{t+n-1} \\
&= -(1/2)B'_{n-1}\Sigma B_{n-1} + E_t^* i_{t+n-1}.
\end{aligned}$$

The operator E_t^* above computes expected value using the probability measure under which

$$X_t = \mu^* + \Phi^* X_{t-1} + e_t, \quad e_t \sim \mathcal{N}(0, \Sigma).$$

One can formally define such probability measure through the Radon-Nikodym process $\exp(\chi_t)$, which evolves according to $\chi_{t+1} = \chi_t + m_{t+1} + i_t$. Then, for any random process Y , $E_t e^{\chi_{t+1} - \chi_t} Y_{t+1} = E_t^* Y_{t+1}$. In particular, (B.3) becomes $0 = \log E_t^* \exp rx_{t+1}^n$, which justifies the "risk-neutral" designation of this alternative probability measure, which differs from the data-generating or "true" measure defined in (B.1). The algebra above shows through a different route that, under the risk-neutral probability measure, there is no term-premium: the forward rate coincides with the expected interest (plus a Jensen's inequality term).

Appendix C Estimation by Linear Regressions

Adrian et al. (2013) provide a method of estimating affine models using linear regressions only. I adapt their procedure to my specification, but the general method is the same.

Start by running OLS on the vector autoregression (9). That yields estimates for μ and Φ , as well as sample residuals \hat{e} , which we can use to estimate Σ (this is also how I estimate Σ in the baseline estimation procedure).

I then run the multivariate regression

$$rx_{t+1} = \mathcal{A} + \mathcal{B}\hat{e}_{t+1} + \mathcal{C}X_t + u_t$$

where rx stacks the excess returns of all bonds. The exponential-affine model implies that the covariance of excess returns with innovations is $B\Sigma$, where B is the matrix that stacks B'_{n-1} . Therefore, \mathcal{B} provides an estimate of B .

The last step is to estimate price-of-risk parameters by running cross-sectional regressions. By matching equation (14) with the linear regression above, one can see that

$$\begin{aligned}
\mathcal{A}_n + (1/2)B'_{n-1}\Sigma B_{n-1} &= \lambda'_0(\Sigma B_{n-1}) + \text{error} \\
\mathcal{C}'_n &= \lambda'_1(\Sigma B_{n-1}) + \text{error}
\end{aligned}$$

where \mathcal{A}_n is the n -th entry of vector \mathcal{A} and \mathcal{C}_n is the n -th row of \mathcal{C} . In these regressions, the estimates of (ΣB_{n-1}) are the independent variables. Each bond maturity n is a "data point", which is why I refer to them as a cross-sectional regressions. They estimate λ_0 and λ_1 by projecting expected returns bond-by-bond (the \mathcal{C}_n) to estimated loadings B .

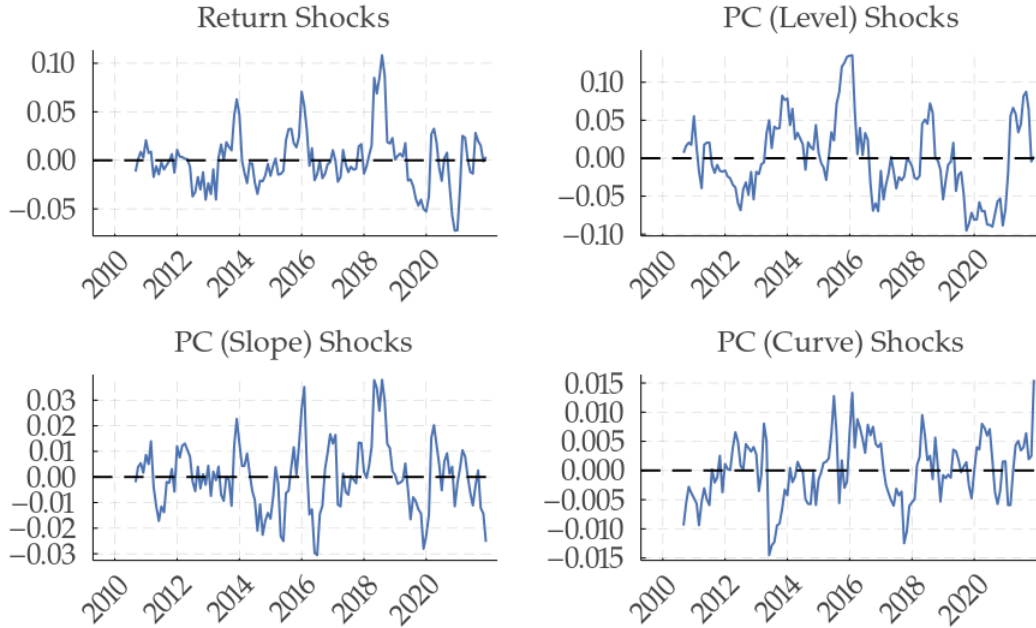
This step of the procedure is the analogous to my cross-sectional regression that selects $\lambda_1(1)$ to match (14) to the empirical excess return model (10).

Having estimated λ_0 and λ_1 , we complete the estimation by calculating risk-neutral parameters μ^* and Φ^* using (15).

RHS Forward Rates	Adjusted R^2 on rx_{t+1}^n regression			
	$n = 2$	$n = 5$	$n = 10$	Average Return
All	0.36	0.47	0.53	0.50
1,2,3,4,5	0.30	0.42	0.50	0.45
2,4,6,8,10	0.27	0.41	0.49	0.44
2,4,6	0.24	0.39	0.48	0.43
2,10	0.05	0.19	0.34	0.25
2,6	0.10	0.32	0.46	0.39
6,10	0.23	0.33	0.37	0.35

The table reports the *unadjusted* R^2 for regression models $rx_{t+1}^n = \alpha + \beta' f_t + \varepsilon_{t+1}^n$. Each row contains results for a different set of right-hand side forward rate maturities.

Table 9: R^2 per model and bond maturity

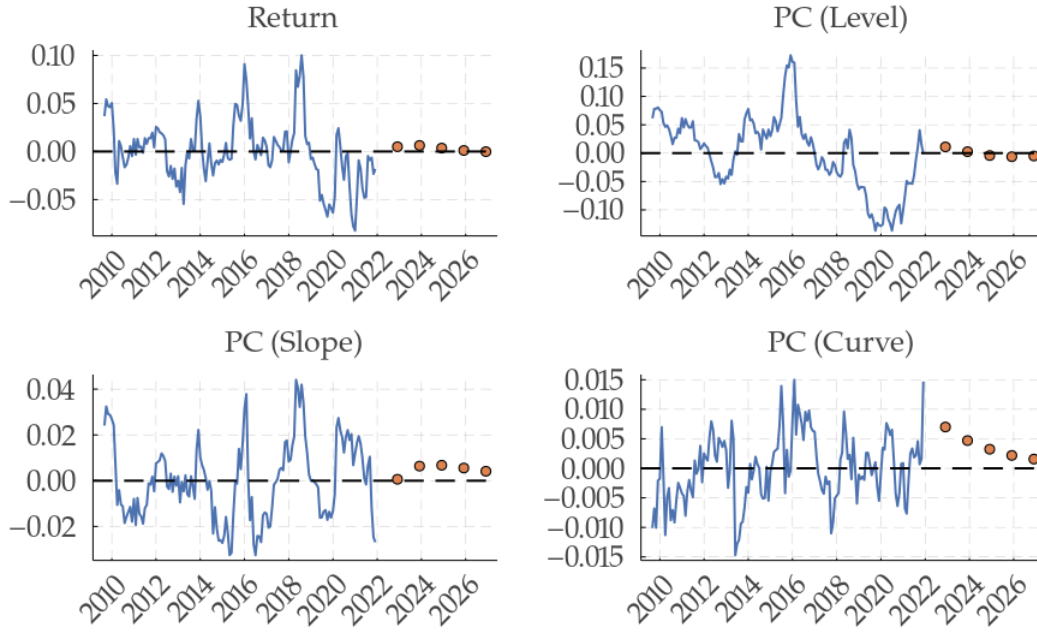


Notes: I estimate (9) by OLS. The figure plots the estimated residuals $\hat{\varepsilon}_t$.

Figure 9: Estimated Factor Shocks

Appendix D Other Results

This section contains additional tables and figures.



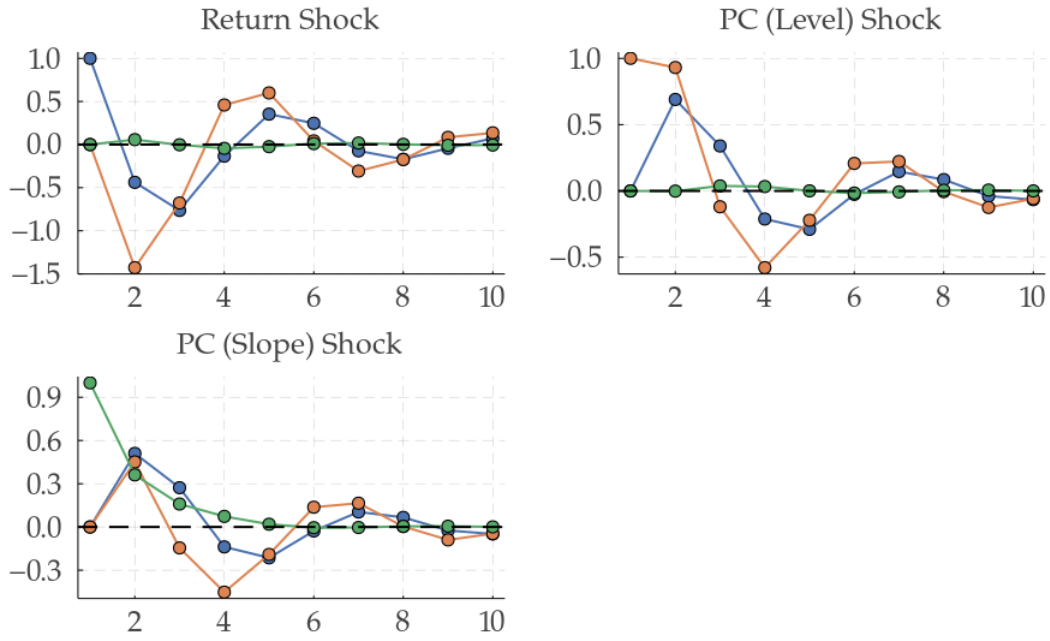
Notes: Each panel plots the time series of the models' factors (blue curves). The orange circles represent the models' one-year ahead forecast using the baseline estimation parameters.

Figure 10: Model Factors and their Forecast

	Return	Level	Slope	Curve
Return	1	0.62	0.70	0.10
Level		1	0.24	-0.05
Slope			1	0.10
Curve				1
Standard Deviation (%)	2.92	5.13	1.33	0.54

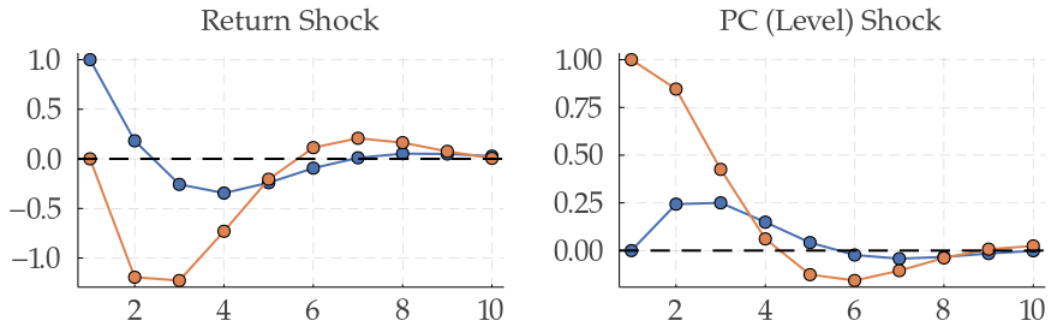
I estimate (9) by OLS and compute residuals $\hat{\epsilon}_t$. The table reports its estimated correlation matrix and, in the last row, standard deviation.

Table 10: Correlation Matrix of Factor Shocks



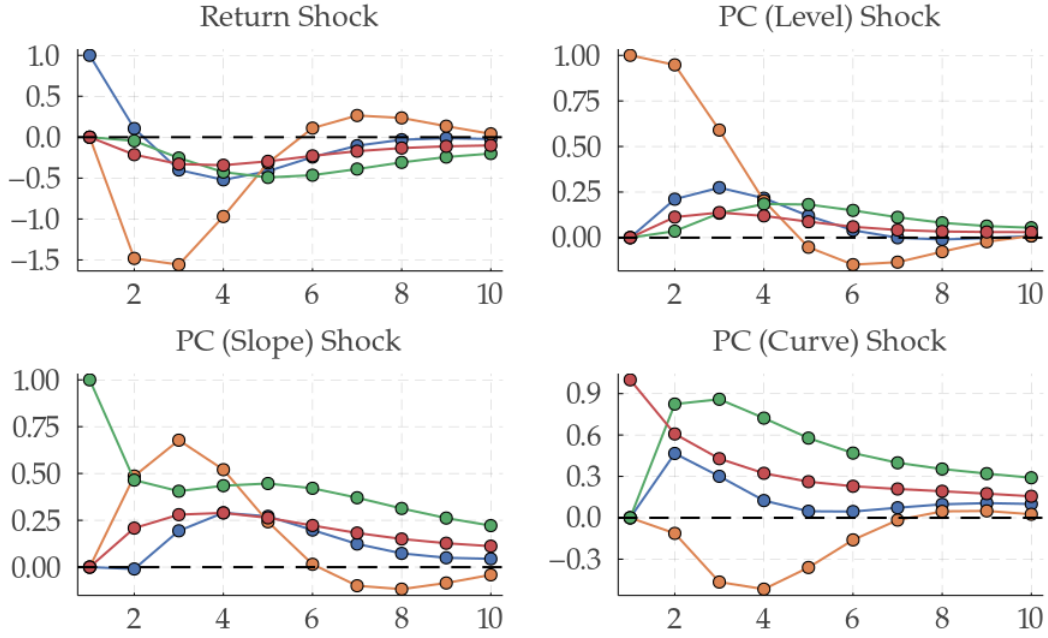
Notes: Impulse response function of model dynamics under the true probability measure. Blue = return factor. Orange = level factor. Green = slope factor. In this specification, I assume X contains three factors only.

Figure 11: Data-Generating Dynamics Φ - 3-Factor Model



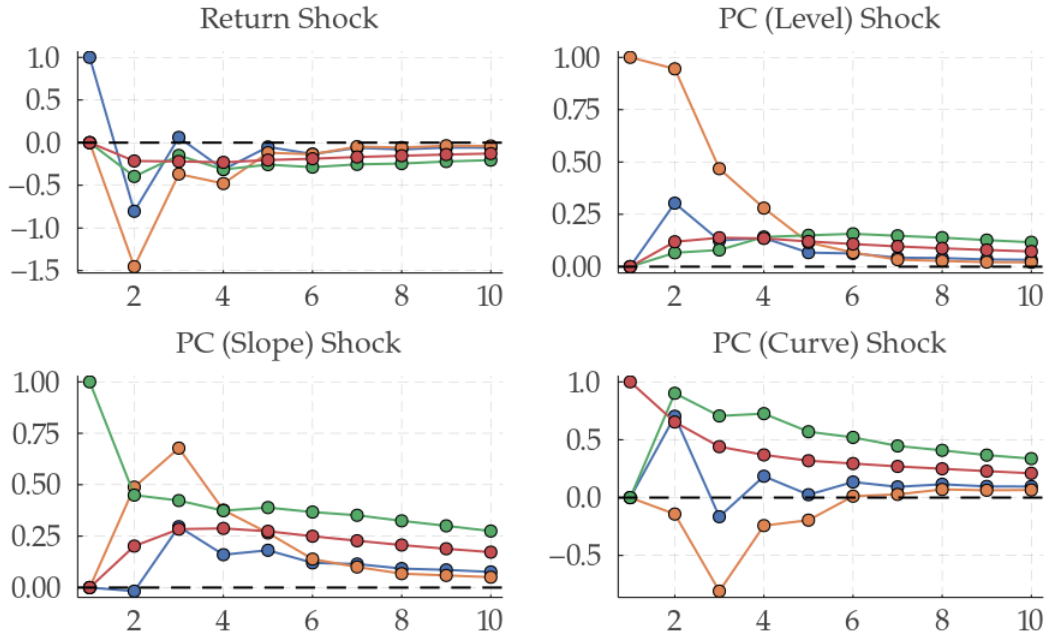
Notes: Impulse response function of model dynamics under the true probability measure. Blue = return factor. Orange = level factor. In this specification, I assume X contains two factors only.

Figure 12: Data-Generating Dynamics Φ - 2-Factor Model



Notes: Impulse response function of model dynamics under the true probability measure. Blue = return factor. Orange = level factor. In this specification, I assume the law of motion of X contains a constant.

Figure 13: Data-Generating Dynamics Φ - Model with Free μ



Notes: Impulse response function of model dynamics under the true probability measure. Blue = return factor. Orange = level factor. In this specification, I assume that only return factor shocks have a non-zero market price of risk (as opposed to level factor shocks in the baseline specification).

Figure 14: Data-Generating Dynamics Φ - Return Factor Priced