1. Introduction - One Period, Nominal Bonds

In this course, we have focused on simplified versions of the government's budget constraint. Working with simplified constraints has allowed us to focus on the theoretical aspects of fiscal policy. However, it is critical that, as an economist, you can model public finances in more realistic environments. This is the objective of this handout. There is no decision-making here, therefore no economics. Our objective is to understand the *accounting* underlying public debt.

The budget constraint we have seen in the models with nominal debt provide a convenient starting point.

$$B_{t-1} = Q_t B_t + P_t S_t + \Delta M_t \tag{1}$$

Recall the interpretation of (1). Each of the B_{t-1} bonds issued in $t-1$ promise the delivery of one unit of currency, which I call "dollar". So B_{t-1} dollars go into circulation in period t. By selling new bonds ($Q_t B_t$ term) and running surpluses ($P_t S_t$ term), the government can remove dollars from circulation. The difference (ΔM_t) increase of decrease the amount of dollars held by the private sector. Notation is the same as always: P_t is the dollar price of the consumption basket and Q_t is the dollar price of bonds.

Because primary surpluses are realistically not stationary, we deflate it by real aggregate output Y_t.[1] Let $s_t = S_t/Y_t$ be the surplus-to-GDP ratio, which I continue to refer to as "surplus".

It is instructive to go over some of the algebra we later apply in a more complex specification of public debt. Start by expressing (1) as share of nominal output:

$$\frac{B_{t-1}}{P_t Y_t} = \frac{Q_t B_t}{P_t Y_t} + s_t + \frac{\Delta M_t}{P_t Y_t} \tag{2}$$

We will call V_t be the nominal, market value of public debt at the *end* of period t. In the case of one-period bonds, V_t coincides with the revenue the

[1] Y_t can be any other variable that plausibly render surpluses stationary. Some authors use a GDP trend instead of realized GDP. This choice avoids contamination of the surplus-to-output series with unwanted volatility from the output series.

government raises when selling new bonds: $V_t = Q_t B_t$. We can also express it as a share of nominal debt: $v_t = V_t / P_t Y_t$.

Divide and multiply the left-hand side of (2) by Q_{t-1}, by Y_{t-1} and by P_{t-1} (you will see why!):

$$\frac{Q_{t-1}B_{t-1}}{P_{t-1}Y_{t-1}} \times \frac{1}{Q_{t-1}} \times \frac{P_{t-1}}{P_t} \times \frac{Y_{t-1}}{Y_t} = \frac{Q_t B_t}{P_t Y_t} + s_t + \frac{\Delta M_t}{P_t Y_t}$$

We can now simplify the expression above by replacing V_t and further defining:

- the implied nominal interest $1 + i_t = 1/Q_{t-1}$,
- the inflation rate $1 + \pi_t = P_t / P_{t-1}$,
- the output growth rate $1 + g_t = Y_t / Y_{t-1}$,
- the real market value of public debt $v_t = V_t$,
- the seignorage-adjusted primary surplus $\hat{s}_t = s_t + \frac{\Delta M_t}{P_t Y_t}$.

We end up with an elegant budget constraint for the government:

$$\frac{1 + i_{t-1}}{(1 + \pi_t)(1 + g_t)}\, v_{t-1} = v_t + \hat{s}_t \tag{3}$$

The left-hand side of (3) represents a volume the government must “pay”. It contains the real market value of public debt outstanding at the end of period $t-1$, evaluated at period-t market prices and expressed as a ratio of real output. The $1 + i_{t-1}$ term converts the market value of bonds in period $t-1$ (Q_{t-1} dollars) to market value in period t (one dollar). Inflation shows up because bonds are not real, *i.e.* not indexed to the price level. Output growth shows up because, for the same stock of bonds, a higher output today relative to yesterday reduces the volume of $t-1$ bonds expressed as a share of period-t output. Higher inflation and higher output growth effectively reduce the amount the government must “pay”.

The right-hand side of (3) represents sources of public "revenue". It contains the market value of debt outstanding at the end of period t, v_t, added to the seignorage-adjusted primary surplus.

2. General Currency and Term Structures

We generalize the model of the previous section in two directions: the currency in which the government issues its bonds and their maturity.

2.1. Multiple Currencies

We start with the case of one-period bonds.

There are $j = 1, \ldots, J$ currencies. One unit of currency of currency j (or simply "one j") is worth $E_{j,t}$ dollars. A public bond indexed to j promises the delivery of one j in the following period. If the bond matures in t, this one j is worth $E_{j,t}$ dollars.

The bond price is $Q_{j,t}$ in currency j, which corresponds to $E_{j,t}Q_{j,t}$ dollars. Therefore, the dollar return on a one-period, j-currency bond is

$$1 + r^1_{j,t} = \frac{E_{j,t}}{E_{j,t-1}Q_{j,t-1}}$$

We can define $1 + i_t = 1/Q_{j,t}$ as the j-interest rate and $1 + d_{j,t} = E_{j,t}/E_{j,t-1}$ as the rate of *dollar depreciation* relative to currency j. Then, we have:

$$1 + r^1_{j,t} = (1 + i_{j,t-1}) \times (1 + d_{j,t}) \tag{4}$$

As an example, the return on a bond indexed to euros equals its promised interest plus the rate of dollar devaluation relative to the euro.

This notation is sufficiently general. It accommodates actual currencies (euro, pounds, etc), as well as nominal and real bonds. The latter cases:

- Nominal bonds: $E_{j,t} = 1$,
- Real bonds: $E_{j,t} = P_t$.

Using (4), we can compute their returns. In the case of nominal bonds, that amounts to the sole interest rate:

$$1 + r^1_{\mathrm{NOM},t} = 1 + i_{t-1}.$$

In the case of real bonds, the depreciation term captures inflation: $1 + d_{\text{REAL},t} = P_t/P_{t-1} = 1 + \pi_t$. Intuitively, you can think of the consumption good as a currency, and that the real bond is indexed to that currency. Indeed, the return on a real bond is the inflation rate (how money loses value relative to the consumption good "currency") plus what we can call the real interest rate:

$$1 + r^1_{\text{REAL},t} = (1 + \pi_t) \times \underbrace{(1 + i_{\text{REAL},t-1})}_{\text{Real Interest}}.$$

2.2. Long-Term Debt

Most government issue long-term bonds, that expire in periods superior to one year. We integrate long-term debt into our budget constraint.

Keeping the currency structure, consider that any bond has a maturity period, which is the period in which the government is due to repay the face value of the bond. The difference between this maturity period and the current period is the bond's maturity or duration. Example, if we are in $t = 2$ and the maturity period of a bond is $t' = 5$, then that bond has a maturity of three periods.

We index maturities with the letter n. Following the notation we established in the previous section, $B^n_{j,t}$ is the volume of outstanding bonds in period t, issued in currency j, with maturity n. If $n = 0$, the government is called to redeem the bonds in exchange of $B^{n=0}_{j,t}$ units of currency j ($E_{j,t}B^{n=0}_{j,t}$ dollars) in t. Otherwise, payment is due in the future period $t + n$.

The government does not issue redeeming bonds, it just repays them. For instance, if it ends period $t-1$ owing $B^{n=1}_{j,t-1} = 2$, then it must repay these two bonds in period t: $B^{n=0}_{j,t+1} = 2$. However, in the general $n > 1$ case, it is *not* true that $B^n_{j,t-1} = B^{n-1}_{j,t}$, since the government can issue and re-purchase new bonds over time. Example: if there are ten outstanding bonds maturing in 2030 today, the government can issue five more of them next year so that it finishes the year owing fifteen 2030 bonds.

We denote a bond's market price $Q^n_{j,t}$ (in units of currency j). Evidently, a bond maturing today is worth one unit of currency j, *i.e.* $Q^{n=0}_{j,n} = 1$.

Generalizing (4), the one-period return on a bond with maturity n is

$$1 + r^n_{j,t} = \frac{Q^{n-1}_{j,t}}{Q^n_{j,t-1}} \times (1 + d_{j,t}). \tag{5}$$

(5) uses the fact that a n-maturity bond today becomes an $n-1$-maturity bond next period. Applying it to $n = 1$ leads to (4).

2.3. Aggregation

We are ready to re-state the government's budget constraint in this more flexible environment.

$$\sum_{j=1}^{J} E_{j,t} B^1_{j,t-1} = \sum_{j=1}^{J} \sum_{n=1}^{\infty} E_{j,t} Q^n_{j,t} \left(B^n_{j,t} - B^{n+1}_{j,t-1} \right) + P_t S_t + \Delta M_t \tag{6}$$

Budget constraint (6) is again stated in dollar terms, and has a similar interpretation to (1): money entering the economy = money removed from circulation + change in household stocks of money. The left-hand side sums up the dollar value of bond redemptions across currency portfolios (recall that $B^{n=1}_{j,t-1} = B^{n=0}_{j,t}$). The right-hand sum presents the terms $P_t S_t$ and ΔM_t, with similar interpretations, and a double sum. The double sum corresponds to the revenue of the new bond sales by the government. The quantity $B^n_{j,t} - B^{n+1}_{j,t-1}$ denotes how the stock of j-bonds change: positive if the government sells new bonds, negative if the government re-purchases them. Each new bond has a dollar price of $E_{j,t} Q^n_{j,t}$, hence the general sum term $E_{j,t} Q^n_{j,t} \left(B^n_{j,t} - B^{n+1}_{j,t-1} \right)$.

We want an expression that generalizes (3). The steps are analogous, but first we group B terms according to their decision periods:

$$\sum_{j=1}^{J} \sum_{n=1}^{\infty} E_{j,t} Q^{n-1}_{j,t} B^n_{j,t-1} = \sum_{j=1}^{J} \sum_{n=1}^{\infty} E_{j,t} Q^n_{j,t} B^n_{j,t} + P_t S_t + \Delta M_t \tag{7}$$

(watch out for the $n-1$ indices in the Q terms of the left-hand sum!)

Now, we just repeat the same steps as before. First, define

$$V_t = \sum_{j=1}^{J} \sum_{n=1}^{\infty} E_{j,t} Q_{j,t}^n$$

as the beginning-of-period market value of public debt, in dollars. In the case of one-period debt, we saw that V_t coincided with the revenue raised by the government for selling bonds. Now, it does not. The revenue raised by the government is the double sum in (6). Like before, we normalize $v_t = V_t/(P_t Y_t)$.

Next, divide both sides of (6) by $P_t Y_t$, then multiple and divide the left-hand side by $Q_{j,t-1}^n$, P_{t-1}, $E_{j,t-1}$ and Y_{t-1}. We get:

$$\sum_{j=1}^{J} \sum_{n=1}^{\infty} \left\{ \frac{E_{j,t-1} Q_{j,t-1}^n B_{j,t-1}^n}{P_{t-1} Y_{t-1}} \times \frac{Q_{j,t}^{n-1}}{Q_{j,t-1}^n} \times \frac{E_{j,t}}{E_{j,t-1}} \times \frac{P_{t-1}}{P_t} \times \frac{Y_{t-1}}{Y_t} \right\} = v_t + \hat{s}_t,$$

or:

$$\sum_{j=1}^{J} \sum_{n=1}^{\infty} \left\{ \frac{E_{j,t-1} Q_{j,t-1}^n B_{j,t-1}^n}{P_{t-1} Y_{t-1}} \times \frac{(1 + r_{j,t}^n)}{(1 + \pi_t)(1 + g_t)} \right\} = v_t + \hat{s}_t,$$

Our goal is to find v terms on both sides of the equation. We cannot pop v_{t-1} on the left-hand side due to the return terms. We therefore define

$$\omega_{j,t}^n = \frac{E_{j,t-1} Q_{j,t-1}^n B_{j,t-1}^n}{V_{t-1}}$$

as the share of market value of debt due to j-currency bonds maturing in n periods. Multiply and divide the left-hand side by V_{t-1} gives the desired result.

$$v_{t-1} \frac{\sum_{j=1}^{J} \sum_{n=1}^{\infty} \omega_{j,t}^n \times (1 + r_{j,t}^n)}{(1 + \pi_t)(1 + g_t)} = v_t + \hat{s}_t. \tag{8}$$

The double sum on the left side of (8) is the weighted-average dollar return of public bonds. If you want a cleaner expression, you can use a new symbol to denote it:

$$1 + \bar{r}_t = \sum_{j=1}^{J} \sum_{n=1}^{\infty} \omega_{j,t}^n \times (1 + r_{j,t}^n).$$

Then:

$$\frac{1 + \bar{r}_t}{(1 + \pi_t)(1 + g_t)} \, v_{t-1} = v_t + \hat{s}_t.$$

We are done. The left-hand side of (8) again denotes the real market value of bonds outstanding at the end of period $t-1$ evaluated at period-t market prices, and expressed as a ratio of real output. The right-hand side adds the market value of public debt to primary surpluses.

Note the similarity with (3). Introducing different currencies and long-term bonds only requires that we accrue existing debt by the average realized dollar return $\bar{r}_t$ rather than the dollar interest rate i_{t-1}.

You can continue to regard (8) as a statement of quantity to be repaid = revenue sources, but keep in mind that this is a heuristic interpretation of the budget constraint. The government does *not* repay the entire stock of outstanding bonds every period. It only repays maturing bonds (the left-hand side of (6)).