Fiscal Policy - Lecture Notes

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Chapter 1

The Basic Two-Period Model

1.1. Environment

The economy is populated by households and a government. They live for two periods, $t = 0$ and $t = 1$, and trade identical consumption goods and public bonds. Public bonds promise their holder one unit of the consumption good in the following period. There is no money in this economy. Agents trade public bonds using consumption goods.

A word on notation: each variable in the model takes a value in period zero and a value in period one, as indicated by their subscript. For example: x_0 and x_1. A process that is a function of time is called a *time series*. When a symbol omits the subscript, it refers to the entire time series vector: $x = (x_0, x_1)$.

1.1.1. The Government

The government demands $g = (g_0, g_1)$ consumption goods (*i.e.*, g_0 in period zero and g_1 in period one). To finance its purchases, it charges lump-sum taxes $\tau = (\tau_0, \tau_1)$ on households. Households cannot avoid paying taxes. The pair g and τ characterize *fiscal policy* in this model.

The government also raises revenue from selling new public bonds. In period zero, the price of one bond is q_0 units of the consumption good. Usually $q_0 < 1$: you pay less than one good in $t = 0$, to get one good in $t = 1$. As such,

$$1 + r_0 = \frac{1}{q_0}$$

is the interest rate implied by the public bond's price. In period one, agents have no incentive to save since the world ends in the following period. Since bonds have no demand, we can set its equilibrium price to zero: $q_1 = 0$.

We make two critical assumptions on government behavior. First, it can *credibly* commit to fully repaying previously issued debt. "Credibly" means that households believe in its commitment, and act accordingly. Second, the government indeed never defaults.

The government brings to period zero a debt of b_{-1} bonds, and must therefore come up with b_{-1} consumption goods to pay bondholders. To that end, it can either sell new bonds b_0 and raise $q_0 b_0$ goods in revenue, or run a *primary surplus*. The primary surplus is defined as the difference between tax proceeds and non-interest spending. In this model, it corresponds to the quantity $\tau_0 - g_0$. The government avoids a default in period zero if

$$q_0 b_0 + \tau_0 - g_0 = b_{-1}. \tag{1.1}$$

The revenue from selling new bonds plus the revenue from taxes in excess of public spending must be enough to redeem old bonds. Since the government will not default, condition (1.1) represents a budget constraint. It restricts the government's choice of how much to tax, how much to spend, and how much to borrow.

Like in period zero, in period one the government again must pay bondholders, which are now due b_0 units of the consumption good. But, in period one, the government cannot sell new bonds, since there is no demand for them (the bond price is zero $q_1 = 0$, so the government would not raise any revenues anyway). Therefore, to pay bondholders, the government must run a primary surplus of b_0 in period one:

$$\tau_1 - g_1 = b_0. \tag{1.2}$$

Expression (1.2) is also a government budget constraint.

1.1.2. Households

The consumption good is non-durable (households can only enjoy them for a single period), and perishable (agents cannot store them). Households value the consumption good in the period they make use of them. The utility function

$$u(c_0) + \beta u(c_1)$$

captures households' preferences over the amount consumed in period zero c_0 and period one c_1. Period utility $u(c)$ is an increasing, strictly concave and twice differentiable function. Parameter $\beta \in (0, 1]$ discounts the flow of future consumption, and therefore captures households' impatience.

Each household receives an endowment of $y = (y_0, y_1)$ consumption goods. You can think of households producing these goods at home; we later model firms, production and labor income more realistically.

We normalize the number of households to one, which avoids the introduction of unnecessary notation. If each household consumes c_0 goods, aggregate consumption will be

$$c_0 \times \text{Number of Households} = c_0 \times 1 = c_0.$$

The same symbol c_0 represents both individual and aggregate consumption. Likewise, (y_0, y_1) represent aggregate production in the economy.

In period zero, each household brings a_{-1} public bonds purchased in the previous period. Since households and the government are the only agents in the model, we restrict the number of bonds initially owned by households to coincide with the number of bonds owed by the government: $a_{-1} = b_{-1}$. Households redeem these a_{-1} bonds for the same number of consumption goods. Add to that their after-tax income $y_0 - \tau_0$ and we find the amount of available goods to each household in period zero. They can use these goods to consume or purchase public bonds from the government. Let a_0 be the household's choice of how many public bonds to purchase. There is no other asset in the economy, so a_0 also represents the household's savings and its net wealth. The following equation is the budget constraint faced by each household in period zero:

$$q_0 a_0 + c_0 \leq a_{-1} + y_0 - \tau_0. \tag{1.3}$$

Equation (1.3) restricts the households' decision of how much to consume and how much to save in period zero. In period one, households redeem a_0 public bonds, and do not demand new ones, as the world ends thereafter. Hence:

$$c_1 \leq a_0 + y_1 - \tau_1. \tag{1.4}$$

Households can borrow too, and the government can lend. While we have referred to b_0 as government "borrowing" and a_0 as household "savings", nothing precludes these variables from being negative (in which case, the household borrows and the government lends).

Suppose households exhaust their available resources, that is, that their budget constraints hold with equality. By equation (1.4), the maximum amount of goods a household can repay from previously acquired debt is $y_1 - \tau_1$ (in that case, the household would consume zero goods in period one, $c_1 = 0$). If the household's debt is larger than $y_1 - \tau_1$, the household defaults. Knowing that, potential lenders (other households or the government) refuse to purchase bonds from (*i.e.*, lend to) a household whose debt exceeds this value. Therefore, the largest debt any household can owe is $y_1 - \tau_1$. We incorporate this *borrowing constraint* in the model by establishing a lower bound $\underline{a}$ on period-zero savings a_0:

$$a_0 \geq \underline{a} = -(y_1 - \tau_1). \tag{1.5}$$

(If you get confused with signs, think of an example; if after-tax income equals 5 goods, then debt cannot be higher than 5, so net wealth cannot be lower than $\underline{a} = -5$.)

Economists often refer to a household's maximum repayable debt as its *natural borrowing limit*. In our model, the natural borrowing limit is $-\underline{a} = y_1 - \tau_1$. Other choices of borrowing limit $-\underline{a}$ are possible, and often more realistic. However, adopting the natural borrowing limit is a convenient starting point to analyze households' allocation decisions, because any choice that involves a positive consumption in period one ($c_1 > 0$) necessarily satisfies it. Consequently, if we prove that period-one consumption is not zero, we can safely ignore the borrowing limit.

Households decide how much to consume $c = (c_0, c_1)$ and how many bonds to purchase (or issue) a_0 taking into account their budget and borrowing constraints (1.3)-(1.5). They take the price of public bonds q_0 as given (*i.e.*, they act *competitively*), and attempt to get as much utility as possible from their choice. Therefore, the choice of how much to consume and save solves the following utility maximization problem:

$$\underset{c \geq 0, a_0}{\text{Max}} \quad u(c_0) + \beta u(c_1) \tag{1.6}$$

$$\text{s.t.} \qquad q_0 a_0 + c_0 \leq a_{-1} + y_0 - \tau_0 \tag{1.3}$$

$$c_1 \leq a_0 + y_1 - \tau_1 \tag{1.4}$$

$$a_0 \geq \underline{a}. \tag{1.5}$$

Optimization problems similar to (1.6) are often referred to as *consumption-savings* problems.

Since u is an increasing, strictly concave function, optimization (1.6) has a single solution.[1] In that solution, budget constraints (1.3) and (1.4) hold with equality - otherwise households could raise consumption and get more utility. Let $c(a_{-1}; q_0, \tau)$ and $a_0(a_{-1}; q_0, \tau)$ be the pair of consumption levels (c_0, c_1) and public bond purchases that solve (1.6). The arguments underscore how households' choices depend on their initial net wealth, the price of public bonds and taxes.

[1] We assume income y and initial wealth b_{-1} are large enough so that the household can choose non-negative amounts of consumption goods.

1.2. Present-Value Budget Constraints

1.2.1. Government and Fiscal Policy Sustainability

Let us return to the government's budget constraints, repeated below for convenience:

$$q_0 b_0 + s_0 = b_{-1} \tag{1.1}$$

$$s_1 = b_0. \tag{1.2}$$

($s = \tau - g$ is the primary surplus sequence). Equations (1.1) and (1.2) are examples of *sequential* budget constraints ("sequential" because we have one of them in each period).

Sequential budget constraints focus on the interaction between surpluses and wealth. But they also indirectly capture the possibilities of *intertemporal allocation* available to the government. For example: if it wants to lower period-zero surpluses by one ($\Delta s_0 = -1$, Δ means "a change in"), it must issue the necessary volume of new bonds $\Delta b_0 = 1/q_0 = 1 + r_0$; and then raise period-one surpluses by $\Delta s_1 = \Delta b_0 = 1/q_0$ to pay the additional debt.

It is often useful to represent the restrictions involving current and future surpluses more directly, with a single expression. Replace (1.2) on (1.1) to get:

$$b_{-1} = s_0 + q_0 s_1. \tag{1.7}$$

Equation (1.7) is the government's *present-value* budget constraint. It immediately shows that $\Delta s_0 = -1$ demands $\Delta s_1 = 1/q_0$.

We say "present-value" because we are converting spending in different points in time to their corresponding value in period zero. Indeed, the value in $t = 0$ of the delivery of X goods in $t = 1$ is $q_0 X$, since any agent can purchase X bonds for that amount, and get the X goods in $t = 1$.[1] In that sense, we can regard q_0 not only as the price of public bonds, but also the price of period-one consumption c_1 relative to period-zero consumption c_0.

We say "budget constraint" because expression (1.7) is a sufficient and necessary condition to ensure that the government does not default. Conveniently, it does not depend on the b_0 term, only on fiscal policy objects τ and g through the surplus terms $s = \tau - g$. In that sense, the present-value budget constraint implies and is implied by fiscal policy *sustainability*.

Let us check this important claim. If the government does not default, then s and b_0 must respect the sequential budget constraints (1.1) and (1.2). Together, they imply (1.7). Thus, no default $\implies$ the present-value budget constraint.

[1] This is a *no-arbitrage* argument: If the value was $A > q_0 X$, you could sell the period-one delivery of X goods for A and purchase the required bonds for $q_0 X$ to make a something-for-nothing profit.

In the opposite direction, suppose we have a surplus process $s = (s_0, s_1)$ that satisfies (1.7). We use the period-zero sequential budget constraint (1.1) to find the necessary volume of bonds the government needs to issue:

$$b_0 = \frac{b_{-1} - s_0}{q_0}.$$

The above b_0 ensures that the government does not default in period zero. Does it default in period one? By assumption, the surplus pair satisfies (1.7). So:

$$b_{-1} = s_0 + q_0 s_1 \implies s_1 = \frac{b_{-1} - s_0}{q_0} = b_0.$$

Since $s_1 = b_0$, period-one sequential budget constraint (1.2) holds. In conclusion, validity of the present-value budget constraint $\implies$ no government default.

1.2.2. Re-Stating Households' Consumption-Savings Problem

Consider now the sequential budget constraints faced by households, expressions (1.3) and (1.4). The conclusions we find above for the government apply somewhat similarly. The sequential budget constraints imply the present-value budget constraint:

$$a_{-1} \geq [c_0 - (y_0 - \tau_0)] + q_0 [c_1 - (y_1 - \tau_1)]. \tag{1.8}$$

Each term in brackets represents the household's expenditure in excess of its after-tax income (you can think of it as the household's own "primary deficit"). The present value of its excess consumption must be lower or equal to the initial wealth a_{-1}. Intuitively, if its exceeds a_{-1}, then households default in period one.

Like in the government's case, a consumption process $c = (c_0, c_1)$ that satisfies the present-value budget constraint (1.8) also satisfies the sequential budget constraints, if we choose the right net wealth a_0. For instance, we can use period-one budget constraint, expressed with equality:

$$a_0 = c_1 - (y_1 - \tau_1). \tag{1.9}$$

The equivalency between restricting households' consumption choice using sequential or present-value budget constraints opens the door to writing the consumption-savings problem

(1.6) in terms of the c only:

$$\operatorname*{Max}_{c\geq 0} \quad u(c_0) + \beta u(c_1) \tag{1.10}$$

$$\text{s.t.} \qquad a_{-1} \geq [c_0 - (y_0 - \tau_0)] + q_0 [c_1 - (y_1 - \tau_1)] \tag{1.8}$$

$$(a_0 =)\ c_1 - (y_1 - \tau_1) \geq \underline{a}. \tag{1.5}$$

(We have used (1.9) to replace a_0 in the borrowing constraint.[1]) The solution $c(a_{-1}; q_0, \tau)$ to problem (1.6) also solves problem (1.10). We can then use (1.9) again to recover the optimal demand for public bonds $a_0(a_{-1}; q_0, \tau)$.

1.3. Ricardian Equivalence

In general terms, *Ricardian equivalence* is the proposition that households' consumption choices are unaffected by the timing of taxation. In this section, we model Ricardian equivalency in our two-period setup and discuss which conditions are key to make it hold. We start with a government that fixes a fiscal policy pair g and $\tau = (\tau_0, \tau_1)$. Fiscal policy is sustainable, therefore the present-value budget constraint (1.7) is satisfied. We can write it as:

$$[\tau_0 + q_0 \tau_1] = b_{-1} + [g_0 + q_0 g_1]\,. \tag{1.11}$$

On the left, the present value of tax proceeds; on the right, the present value of outlays divided between spending and old debt redemption. Households observe the path of due taxes, and plan how much to consume $c(\tau)$ and how much to save $a_0(\tau)$.[2]

Suppose that, still at the beginning of period zero, the government announces a different, *but still sustainable*, path to lump-sum taxes, $\hat{\tau} = (\hat{\tau}_0, \hat{\tau}_1)$. Spending g remains unaltered. How do households revise their consumption plans in response to the government announcement? It turns out that, in the conditions of our two-period model, *they don't*: $c(\tau) = c(\hat{\tau})$. We say that Ricardian equivalence holds.

The key to prove the proposition is to show that different but equally sustainable taxation paths do not change the set of consumption levels affordable by households. Formally, any c that satisfies the constraints of the consumption-savings problem (1.10) under τ will continue to satisfy them under $\hat{\tau}$, and vice-versa.

Let's check that claim. We start with the present-value budget constraint (1.8), which

[1](1.9) is the only level of bond purchases consistent with a consumption choice because the sequential budget constraints hold with equality in the solution of (1.6).

[2]In this section only, I ignore the arguments a_{-1} and q_0 of the optimal solutions for brevity.

holds with equality. We can re-write it as:

$$[c_0 + q_0 c_1] + [\tau_0 + q_0 \tau_1] - [y_0 + q_0 y_1] = a_{-1}.$$

The middle term on the left-hand side is the present value of charged taxes. Since both τ and $\hat{\tau}$ are fiscally sustainable, and since g is unchanged, that quantity must stay constant:

$$[\tau_0 + q_0 \tau_1] = [\hat{\tau}_0 + q_0 \hat{\tau}_1] = b_{-1} + [g_0 + q_0 g_1].$$

Therefore, the household's present-value budget constraint is unchanged.

Next, consider the borrowing constraint (1.5). Since we use the natural borrowing limit, they read:

$$\begin{aligned} c_1 - (y_1 - \tau_1) = \quad & a_0 \geq \underline{a} \quad && = -(y_1 - \tau_1) \\ c_1 - (y_1 - \hat{\tau}_1) = \quad & a_0 \geq \underline{a} \quad && = -(y_1 - \hat{\tau}_1) \end{aligned}$$

Both restrictions above are satisfied whenever $c_1 \geq 0$ (this is how we define the natural borrowing limit!). Hence, the borrowing limit is effectively unchanged.

Since the restrictions of the consumption-savings problem (1.10) remain the same, the optimal level of consumption cannot be different. In conclusion, $c(\tau) = c(\hat{\tau})$.

1.3.1. Interpretation

The central idea behind Ricardian equivalence is the fact that households understand how a one-dollar reduction in charged taxes today (or a standalone one-dollar transfer) must be followed by a one-dollar increase plus interest tomorrow (and vice versa). Being the household, you can save the extra dollar, earn the interest, and duly pay the higher tax tomorrow. No reason to change the groceries list. In that sense, critics of transfer-based programs of fiscal "stimulus" often rely on the Ricardian equivalence result as a theoretical basis for their skepticism. Still, it is critical to understand what the proposition says and what it doesn't.

One could precisely summarize what Ricardian equivalence *does* say as follows:

Household's consumption demand curve does not depend on the *timing* of *lump-sum taxes*.

The two emphasized terms are key. "Timing" means *when*, not *how much*. Ricardian equivalence does not say that households do not respond to different taxation schemes. If the government halves taxes today but promises the same level of taxation in the future, households do use the additional resources to raise consumption. If the government announces higher taxes tomorrow, but no transfers today, then households save some more. (Note however that the government exhausts its resources; thus an increase in overall taxes for instance must lead to an increase in spending g too. See (1.11).) "Lump-sum" means that the proposition excludes taxes that depend on households' actions, like income, consumption and corporate taxes. Unlike these alternative forms of taxation, lump-sum taxes do not change the marginal benefits of these actions; hence, they do not induce changes in household behavior other than because they get wealthier or poorer.

1.3.2. Critical Assumptions

According to the Ricardian proposition, demand for consumption goods $c(\tau)$ is unresponsive to the timing of taxes, but not the demand for bonds $a_0(\tau)$. If the government sends you a 100-dollar check and you do not spend it, your savings account grows by 100 dollars. If the government charges you an additional 100 dollars in taxes, your savings account diminishes by that amount. One critical assumption behind Ricardian equivalence is that, if necessary, households dispose of the necessary credit to sustain their period-zero consumption level. This has been a given in our baseline case of the two-period model: under the natural borrowing limit (1.5), households can always borrow if they can repay. If the government charges 100 dollars more in taxes in $t = 0$, households can borrow an additional 100 dollars (plus interest) as lenders understand taxes will be lower by that amount in $t = 1$. The natural borrowing limit will not bind under the new path of taxes if it didn't under the old one.

However, more restrictive borrowing constraints can bind and thus prevent households from keeping their consumption path unaltered. For instance, a commonly used restriction is the *no-borrowing constraint* $\underline{a} = 0$. In our model, when the borrowing constraint binds, period-zero consumption is given by equation (1.9):

$$c_0 = a_{-1} - q_0\underline{a} + y_0 - \tau_0$$

So if a fiscal policy change $\Delta\tau$ is small enough so that the borrowing constraint continues to bind, $\Delta c_0 = \Delta\tau_0$. In the presence of a binding borrowing constraint, an increase in taxation leads to a reduction in current consumption since households cannot issue more debt to pay for the higher taxes. On the opposite direction, lower taxes (or standalone transfers) might raise consumption. As such, discussions of whether adjustments to fiscal policy will stumble on Ricardian behavior often center around the extent to which households are credit constrained. Obviously, one can only answer that question empirically, on a case-by-case basis.

Also key for Ricardian equivalence to hold is the functioning of public finances, in particular the assumption that fiscal policy is credible and sustainable. In the context of real debt (*i.e.*, public bonds that pay a consumption good), fiscal sustainability is the same as no default. Our model captures best a government that is fully credible to raise enough revenue to eventually repay its debts (e.g. Switzerland). Deficits today lead to surpluses tomorrow. In practice, however, governments do default. Even if they don't, households might *believe* that they can. The credible communication of a fiscal policy plan is just as important to household behavior as the policy path itself. Whenever the government lacks the credibility of debt repayment, lower taxes today do not imply higher taxes tomorrow. Ricardian equivalence fails.

It is easy to take the assumptions of fiscal credibility and sustainability for granted, especially because most modern governments finance themselves primarily through *nominal*, not real debt. Agents redeem nominal debt for money, which is, in most cases, created by the government. Hence, unsustainable fiscal policy paths do not necessarily lead to the dramatic outcome of a government default, but rather to a decline in the value of money (inflation). We come back to that topic later. For now, just note that it is not clear how frequently and to which extent governments can and do promise fully sustainable changes in fiscal policy; and that our use of the expression "fiscal sustainability" in this section is *more restrictive* than the government not defaulting in practice.

Lastly, contrary to our model's assumptions, households are not identical, and tax and transfers are seldom unconditional. The more realistic income, capital and consumption taxes are a sure way to break Ricardian equivalence. Moreover, households with different characteristics are likely to react differently to a change in fiscal policy. We have discussed above the case of credit-constrained households. One might conjecture that older individuals will not be as inclined to save a public transfer in order to pay for a future increase in taxation. Perhaps the same applies to unemployed workers. In all, the lack of household heterogeneity is a major simplification imposed by our model.

1.4. Intertemporal Choice and Equilibrium

We want to characterize the *competitive equilibrium* of our two-period economy. The competitive equilibrium is defined by market prices and quantities that cover two properties. First, agents choose the quantities optimally, taking prices as given. The "taking prices as given" part makes the equilibrium "competitive". Second: all markets clear, which means that quantities optimally supplied equal quantities optimally demanded.

When computing an equilibrium, we fix fiscal policy (g, τ). We will later study how the government can choose fiscal policy to generate the "best" equilibrium possible. For now, we take g and τ as given, assuming that they respect the present-value budget constraint (1.7).

1.4.1. Household Optimality

Consider households' optimal choices, $c(a_{-1}; q_0, \tau)$ and $a_0(a_{-1}; q_0, \tau)$. Because they solve the consumption-savings problem (1.6) (or (1.10)), they must satisfy the first-order optimality condition associated with that problem. In an interior solution (*i.e.*, in a solution with $c_0 > 0$, $c_1 > 0$), that condition is the *Euler equation*

$$q_0 u'(c_0) = \beta u'(c_1). \tag{1.12}$$

We interpret the Euler equation (1.12) as a condition of consumption smoothing. Since the utility function u is increasing and concave, *marginal utility* u' is a positive, but *decreasing* function.[1] Intuitively, consuming more always makes the household "happier", but the amount of extra "happiness" an additional unit of consumption provides declines as it consumes more. Equating marginal utility therefore means balancing value over time. If you are lost in the desert, do not empty the waterskin on the first night.

To prove (1.12) is the first-order condition for optimality, consider the following variational argument. The utility gain of marginally increasing period-one consumption by Δc_1 is $\beta u'(c_1)\Delta c_1$. According to the present-value budget constraint (1.8), to increase period-one consumption by Δc_1, the household must give up $\Delta c_0 = -q_0 \Delta c_1$ units of period-zero consumption.

$$\begin{aligned}
a_{-1} &= [c_0 - (y_0 - \tau_0)] + q_0 \, [c_1 - (y_1 - \tau_1)] \\
\Delta a_{-1} &= \Delta \, [c_0 - (y_0 - \tau_0)] + q_0 \; \Delta \, [c_1 - (y_1 - \tau_1)] \\
0 &= \Delta c_0 + q_0 \Delta c_1
\end{aligned}$$

The utility loss of reducing period-zero consumption is

$$u'(c_0)\Delta c_0 = -q_0 u'(c_0)\Delta c_1.$$

For a choice of c to be optimal, the marginal gain cannot be lower or higher than the marginal loss. Thus, $q_0 u'(c_0)\Delta c_1 = \beta u'(c_1)\Delta c_1$, as we wanted to show.

The Euler equation (1.12) establishes a positive relationship between period-zero and period-one consumption.

$$c_0 \uparrow \implies u'(c_0) \downarrow \implies u'(c_1) \downarrow \implies c_1 \uparrow$$

[1] Technically, marginal utility could be zero even though utility is increasing. Here, I am assuming $u' > 0$.

To find the actual solution $c(a_{-1}; q_0, \tau)$ to the consumption-savings problem, we impose the fact that the present-value budget constraint must hold with equality. We find the pair (c_0, c_1) that satisfies the Euler equation and that guarantees that households exhaust their available resources. Lastly, we can compute the optimal choice of period-zero savings $a_0(a_{-1}; q_0, \tau)$ using the sequential budget constraint (1.9).

1.4.2. The Competitive Equilibrium

In equilibrium, prices adjust so that markets clear. In the consumption goods market, the inelastically supplied quantity of goods y coincides with the government's demand g and households' optimal demand $c(b_{-1}; q_0, \tau)$:

$$c_0(b_{-1}; q_0, \tau) + g_0 = y_0 \tag{1.13}$$

$$c_1(b_{-1}; q_0, \tau) + g_1 = y_1. \tag{1.14}$$

(Recall $a_{-1} = b_{-1}$.) In the bonds market, the volume issued by the government coincides with that demanded by households:

$$a_0(b_{-1}; q_0, \tau) = b_0. \tag{1.15}$$

We now show that if one of these markets clears, the other two will clear as well. First, if the bonds market clears, the market for period-one consumption will also clear. Indeed, from the sequential budget constraints (1.2) and (1.4):

$$c_1 + \tau_1 - y_1 = \quad a_0 = b_0 \quad = \tau_1 - g_1.$$

The terms on the left and right imply (1.14).

Second, if the market for consumption goods clears in period zero, the market for bonds will also clear. We again see this from the sequential budget constraints (1.1) and (1.3). Subtracting the former from the latter:

$$q_0 \underbrace{(a_0 - b_0)}_{\substack{\text{Excess Demand} \\ \text{Bond Market}}} + \underbrace{c_0 + g_0 - y_0}_{\substack{\text{Excess Demand} \\ \text{Goods Market}}} = a_{-1} - b_{-1} = 0.$$

If the excess demand for goods is zero (*i.e.*, if demand = supply), the expression above implies $a_0 = b_0$.

The fact that we only need to clear one market is an application of *Walras' Law*, which states that, in an N-market economy, clearing of the first $N-1$ markets implies the clearing of the last one. Although we have three markets in our model, by now you should be convinced that the market for public bonds is really a market for period-one consumption goods. (This is the rationale behind the present-value budget constraints (1.7) and (1.8); they focus on consumption goods only).

It is convenient that we only need to clear one market, since the only price in the model is the price of public bonds q_0 (obviously this is not a coincidence). To find the equilibrium value of q_0, replace (1.13) and (1.14) in the Euler equation:

$$q_0(y, g) = \frac{1}{1 + r_0(y, g)} = \beta \, \frac{u'(y_1 - g_1)}{u'(y_0 - g_0)}. \tag{1.16}$$

Intuitively, equilibrium bond price $q_0(y, g)$ must provide households the due incentive to allocate consumption intertemporally in a way consistent with the availability of goods. For example, suppose that period-zero endowment y_0 is much lower than period one's y_1. Under which circumstances would households accept to consume so much more in $t = 1$ than in $t = 0$ (so that $u'(c_1)/u'(c_0)$ is low)? According to the Euler equation: when bond prices are too low, or interest rates too high.

The equilibrium bond price (1.16) amplifies the scope of Ricardian equivalence. In the previous section, we saw that households' *demand curve* for goods are unresponsive to the timing of fiscally sustainable taxes. But demand curves are not the same as quantities demanded *in equilibrium*. In principle, the latter could change if bond prices were sensitive to taxes. Expression (1.16) proves this is not the case.

1.4.3. The Fiscal Multiplier

Given a change in public spending Δg_0, economists are often interested in the resulting change in aggregate output Δy_0. The change in aggregate output per unit of public spending $\Delta y_0/\Delta g_0$ is called the *fiscal multiplier*. In the simplified model we study in the section, aggregate output y_0 is exogenous, and unaffected by public spending. The fiscal multiplier is zero. In the following chapters we examine models that assume more elaborate production technologies and therefore allow for non-zero fiscal multipliers.

For now, a few aspects of the fiscal multiplier concept are worth noting. First, economists often limit the definition of fiscal multipliers to *exogenous* changes in public spending. "Exogenous" means that the change does not arise as a feedback response to other variables, but rather as a change in the level of spending *given* other variables.

There is no single fiscal multiplier. Even if we restrict the definition of a fiscal multiplier to encompass exogenous variations in public spending, several factors can influence their effect on the economy. Each possibility leads to a different multiplier. Here are a few examples: is the fiscal shock anticipated? Is it long-lasting? Does the government demand consumption or investment goods? We explore some of these cases in the following chapters.

Lastly, the fiscal multiplier is dual to the crowding-out effect of public spending. That is, the more output grows in response to an increase in public spending, the less private consumption needs to *decline*. You can see this from the market-clearing condition in the goods market (1.13):

$$\frac{\Delta c_0}{\Delta g_0} = \frac{\Delta y_0}{\Delta g_0} - 1$$

When the fiscal multiplier is zero, each additional good purchased by the government reduces private aggregate demand by the same amount. (In this chapter's model we only consider private consumption; we later consider private investment as well.) Based on this idea, economists sometimes claim that expansion of public spending when the economy has no spare capacity (or "slack") is detrimental to households.

Exercises

Exercise 1.1. We study the isoelastic utility function

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad \gamma > 0. \tag{1.17}$$

(a) Apply L'Hôpital's rule to show that when $\gamma \to 1$, the utility function converges to $\log(c)$.

(b) Express the Euler equation (1.12) as

$$\frac{c_1}{c_0} = \left[\beta(1 + r_0)\right]^{\gamma}.$$

The left-hand side is the gross rate of consumption growth $1 + g_1^c$. Use the first-order Taylor approximation of the log function

$$\log(1 + x) \approx x \qquad \text{when } x \approx 0$$

to conclude that

$$\gamma\left[\log\beta + r_0\right] = g_1^c.$$

The equation above show that parameter γ governs the *elasticity of intertemporal substitution*, defined by $\Delta g_1^c / \Delta r_0$.

(c) Explain intuitively why the interest rate is increasing in consumption growth.

Exercise 1.2. This exercise guides you through the complete solution of the consumption-savings problem (1.6), under the isoelastic utility function (1.17) and a general borrowing limit $\underline{a}$ (*i.e.*we no longer assume the natural borrowing limit $-(y_1 - \tau_1)$).

(a) Suppose the household has enough wealth a_{-1} to support positive consumption in period zero. Why can we guarantee positive consumption in *both* periods? Hint: consider the marginal utility of consumption as it approaches zero.

(b) Set up the Lagrangian of the optimization problem (1.6). Compute the first-order conditions to conclude that

$$q_0 u'(c_0) \geq \beta u'(c_1) \qquad (= \text{if } a_0 > \underline{a}).$$

(c) Start by assuming that the borrowing constraint $a_0 \geq \underline{a}$ does not bind. Use the Euler equation to express c_1 as a function of c_0; replace that expression on the present-value budget constraint to find solutions to c_0 and c_1, when the borrowing constraint does not bind.

(d) Replace your solution for c_0 in the period-zero sequential budget constraint (1.3) to find the required public bond position a_0. Does it satisfy the borrowing constraint? If it does, we are done. If it does not, then the borrowing solution binds.

(e) Use the sequential borrowing constraints to find c_0 and c_1 when the borrowing constraint binds.

Exercise 1.3. In this exercise we study the government's present-value budget constraint in a model with T periods.

(a) Suppose the sequential budget constraint

$$q_t b_t + s_t = b - t - 1$$

holds. Show that present-value budget constraint

$$b_{t-1} = \sum_{j=t}^{T} q_{t,j-1} s_j$$

holds, where $q_{t,j} = \prod_{i=t}^{j} q_i$. How do you interpret $q_{t,j}$? What limit condition analogous to $b_1 = 0$ in the two-period model is necessary?

(b) Show that, if the present-value budget constraint holds in every period, the sequential budget constraint holds as well (*i.e.*, the government never defaults).

Exercise 1.4. Prove Walras' Law (equilibrium in the goods market in period zero implies equilibrium in period one) using the two present-value budget constraints (1.7) and (1.8). Assume $q_0 > 0$.

Exercise 1.5. In this example, the government does not demand final goods $g = 0$ and enters period zero with no debt $b_{-1} = 0$. Households' endowment is $y_0 = 5$, $y_1 = 10$, the utility function is $u(c) = \log(c)$ and $\beta = 1$. The government transfers one consumption good to household in period zero, $\tau_0 = -1$.

(a) Find the equilibrium price of bonds and interest rate.

(b) Find the equilibrium consumption in both periods.

(c) What is the fiscally sustainable level of public transfer in period one?

(d) Compute households' savings a_0 at the end of period zero; and verify it is enough to finance their consumption and taxes in the following period.

(e) Consider a different fiscal policy. Instead of a one consumption good transfer, suppose the government enacts a one-period *tax* $\tau_0 = 1$. How do you change your answers to (a), (b), (c) and (d)?

(f) Consider now the existence of a *no-borrowing constraint*. A no-borrowing constraint is a borrowing constraint involving a zero debt limit: $-\underline{a} = 0$. That is, we change equation (1.5) to $a_0 \geq 0$. Consider again the fiscal policy change in τ you found in item (e). At the same bond price as item (a), can the household keep its consumption process unchanged? Does Ricardian equivalence hold?

(g) Under the no-borrowing constraint, is it possible to find an equilibrium with positive bond prices $q_0 > 0$ and period-zero positive taxes $\tau_0 > 0$?

Exercise 1.6. The economy is populated by a unit measure of identical households, subject to the natural borrowing limit. The government announces a new period-zero transfer of one consumption good, but only to half the population. It credibly commits to increase taxation in period one, so that the new fiscal policy remains sustainable. Based on that information, can you say that Ricardian equivalence continues to hold for sure? Can you say that it breaks? Explain.

Exercise 1.7. The government adopts a feedback rule to public spending:

$$g_0 = \theta y_0 + e_0,$$

where θ is a model parameter and e_0 is exogenously determined.

(a) Compute equilibrium output as a function of aggregate consumption c_0 and the shock e_0.

(b) Suppose $\theta > 0$. For an exogenous reason, aggregate output grows by Δc_0. Compute $\Delta y_0 / \Delta g_0$. Your favorite financial media commentator measures $\Delta y_0 / \Delta g_0 > 0$ and, based on his findings, argues that in the future the government should raise public spending in times of low output. Does the model support the commentator's claim?

Chapter 2

Production and Marginal Taxation

Capital k, depreciates at a rate $\delta > 0$. No investment cost.

Labor hours $n \in [0,1]$. Interpret n as the share of available hours devoted for labor activity. Leisure $1 - n$.

Production function $f(k_{t-1}, n_t)$, homogeneous of degree one: $f(\alpha x) = \alpha f(x)$. Aggregate resource constraint:

$$y_t = c_t + g_t + k_t - (1-\delta)k_{t-1} \qquad t = 0, 1 \tag{2.1}$$

The term $k_t - (1-\delta)k_{t-1}$ is the aggregate investment.

Households can purchase government bonds or physical capital. There is no uncertainty, so they choose whichever offers the best after-tax return. Let d be the representative household's net wealth. Market-clearing in the capital market:

$$d_0 = q_0 b_0 + k_0. \tag{2.2}$$

Capital market closes in period one: $d_1 = b_1 = k_1 = 0$.

Capital rental rent r_t. Wage rate w_t. Marginal taxes on consumption $\tau_{c,t}$, labor income $\tau_{n,t}$ and capital income $\tau_{k,t}$. Lump-sum taxes $\tau_{L,t}$. Capital income tax applies on the net returns on both bond and physical capital investments. Depreciation deductible. The government's budget constraints are the following.

$$\begin{aligned} q_0 b_0 + \tau_{c,0} c_0 + \tau_{k,0} r_0 d_{-1} + \tau_{n,0} w_0 n_0 + \tau_{L,0} - g_0 = b_{-1} \\ \tau_{c,1} c_1 + \tau_{k,1} r_1 d_0 + \tau_{n,1} w_1 n_1 + \tau_{L,1} - g_1 = b_0 \end{aligned} \tag{2.3}$$

No-arbitrage in capital market:

$$1 + (1 - \tau_{k,1}) \left(\frac{1}{q_0} - 1 \right) = 1 + (1 - \tau_{k,1}) r_1 \tag{2.4}$$

$1/q_0 = 1 + r_1$ is the real interest rate. In the last chapter, we called r_0 the interest rate; it is now r_1 because the interest rate coincides with the cost of capital rent in period one.

Sequential representation of households' consumption-savings problem:

$$\begin{aligned}
&\underset{c,n,d_0}{\text{Max}} \quad u(c_0) + v(1 - n_0) + \beta \left[u(c_1) + v(1 - n_1)\right] \\
d_0 + (1 + \tau_{c,0})c_0 &\leq \left[1 + (1 - \tau_{k,0})r_0\right] d_{-1} + (1 - \tau_{n,0})w_0 n_0 - \tau_{L,0} \\
(1 + \tau_{c,1})c_1 &\leq \left[1 + (1 - \tau_{k,1})r_1\right] d_0 + (1 - \tau_{n,1})w_1 n_1 - \tau_{L,1} \\
c_0, c_1 &\geq 0 \\
0 \leq n_0, n_1 &\leq 1
\end{aligned} \tag{2.5}$$

First-order conditions. Euler equation:

$$\frac{u'(c_0)}{1 + \tau_{c,0}} = \beta \left[1 + (1 - \tau_{k,1})r_1\right] \frac{u'(c_1)}{1 + \tau_{c,1}} \tag{2.6}$$

Intratemporal condition for optimal supply of labor hours, in an interior solution:

$$w_t \frac{u'(c_t)}{1 + \tau_{c,t}} = \frac{v'(n_t)}{1 - \tau_{n,t}} \qquad t = 0, 1. \tag{2.7}$$

Marginal benefit of working one more hour = marginal cost. Setting $\tau_{n,t} = \tau_{c,t} = 0$ for brevity, consider the effect of a small change in the wage rate $\Delta w > 0$:

$$\underbrace{u'(c)\Delta w}_{\substack{\text{Substitution} \\ \text{Effect, >0}}} + \underbrace{wu''(c)\Delta c}_{\substack{\text{Wealth} \\ \text{Effect, <0}}} = -v'(1 - n)\Delta n$$

Substitution effect: supply more hours of labor because marginal benefit (wage) increases. Wealth effect: reduce supply of labor hours because higher wages leave household wealthier (alternatively: wealthier household purchases more hours of leisure). Effect of $\Delta w > 0$ on labor supply ambiguous.

Firm rents capital and hires labor hours to produce consumption goods. Firm's profit maximization problem:

$$\text{Max}_{k,n} \quad f(k, n) - (r_t + \delta)k - w_t n \qquad t = 0, 1.$$

Since f is homogeneous of degree one, maximized profit equals zero. First-order condition for optimal capital and labor demand:

$$f_k\left(\frac{k_{t-1}}{n_t},1\right)=r_t+\delta, \tag{2.8}$$

$$f_n\left(\frac{k_{t-1}}{n_t},1\right)=w_t, \qquad t=0,1. \tag{2.9}$$

Since f is homogeneous of degree one, derivatives f_k and f_n are homogeneous of degree zero. Both are functions *only* of the capital-labor ratio.

In equilibrium, households and firms act optimally. Labor hours and capital demanded by firms coincide with that supplied by households, in both periods. The initial conditions of the two-period economy satisfy

$$d_{-1}=q_{-1}b_{-1}+k_{-1} \qquad \text{and} \qquad 1+r_0=\frac{1}{q_{-1}}.$$

Therefore, Walras' Law holds. If the market for consumption goods clears in period zero, the capital market will clear; thus the market for goods in period one will clear too.

2.1. Model Implications

- Equivalency between consumption tax τ_c and labor tax τ_n.
- When labor supply is inelastic, constant consumption and labor taxes are not distortionary, like lump sum taxes.
- Capital taxation is distortionary, regardless of labor supply elasticity.
- Public debt crowds out private capital (as long as private savings not infinitely elastic).

Exercises

Exercise 2.1. In this exercise, we focus on the optimal supply of labor by the household in period zero. The properties of labor supply in period one are analogous.

(a) Mind the physical constraint on labor hours: $0 \le n_0 \le 1$. Set up the Lagrangean for the consumption-savings problem (2.5) to find the general first-order condition for the intratemporal choice of labor supply:

$$\begin{aligned} w_0 u'(c_0) &\geq v'(1-n_0) \qquad \text{if } n_0 > 0 \\ w_0 u'(c_0) &\leq v'(1-n_0) \qquad \text{if } n_0 < 1. \end{aligned}$$

(b) For the following items, assume $u(c)$ and $v(1-n)$ are isoelastic:

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad v(1-n) = \frac{c^{1-\frac{1}{\psi}} - 1}{1 - \frac{1}{\psi}} \tag{2.10}$$

Argue that households will not supply their entire labor endowment: $n_0 < 1$.

(c) Find the lowest level of period-zero consumption c_0 compatible with a zero supply of labor hours $n_0 = 0$. Interpret the existence of this lower bound on consumption.

(d) Show that the ψ is the *Frisch elasticity of labor supply*, defined as the change in labor hours supplied given a change in the log of the wage rate, *fixing the marginal value of consumption*:

$$\text{Frisch elasticity} = \left.\frac{\partial n_0}{\partial \log w_0}\right|_{\text{constant } u'}.$$

Hint: use the approximation $\log(1-n) = -n$ when $n \approx 0$. (Note: economists often define Frisch elasticity as the change in *log* hours, to focus on percentual change in labor hours. Here, we define it as a change in n_0 because n_0 already represents the *share* of available hours devoted to labor.)

Exercise 2.2. In this exercise, we study how the volume of taxation affects the equilibrium in the capital market, in the absence of marginal distortions. The government begins period zero with no debt $b_{-1} = 0$. Fiscal policy is characterized by a lump sum tax series $\tau = (\tau_0, \tau_1)$. There is no public spending, and no marginal taxation. Households derive no utility in leisure, and thus supply their entire endowment of working hours $n_0 = n_1 = 1$. The production function available to the representative firm displays perfect elasticity of substitution between capital and labor:

$$y_t = f(k_{t-1}, n_t) = (\bar{r} + \delta)k_{t-1} + \bar{w}n_t,$$

which implies that, in equilibrium $r_t = \bar{r}$ and $w_t = \bar{w}$, for $t = 0, 1$. Households face the natural borrowing limit.

(a) Write the household's consumption-savings problem, using the equilibrium prices. Replace its sequential budget constraints on the Euler equation to find an expression defining the optimal choice of net wealth d_0 in equilibrium.

(b) State the household's present-value budget constraint. Does Ricardin Equivalence hold?

(c) Starting from a given equilibrium, suppose the government raises lump-sum taxes in period zero τ_0 by $\Delta\tau_0 > 0$, without changing τ_1. Use the condition derived in (a) to show that $-\Delta\tau_0 < \Delta d_0 < 0$. Provide an intuition.

(d) Considering the fiscal policy change of (c), compute the change in physical capital Δk_0. What is the effect of a tax increase in period-one output?

Exercise 2.3. Suppose marginal taxes are constant: $\tau_{c,0} = \tau_{c,1}$, $\tau_{k,0} = \tau_{k,1}$. Assuming equilibrium households' consumption is also constant $c_0 = c_1 > 0$. Use the Euler equation (2.6) and the firm's capital demand schedule (2.8) to find the equilibrium interest rate r_1 and the capital labor ratio k_0/n_1. In economic models with with infinite periods, these values are the *steady-state* levels of interest and capital labor ratio. Which forms of taxation affect the steady-state interest rate?

Exercise 2.4. In this exercise, we are interested in representing graphically the equilibrium in the capital markets, in a version of our two-period economy with inelastic labor supply. The functional formats are

$$u(c) = \log(c) \qquad f(k,n) = k^\alpha n^{1-\alpha}.$$

Households derive no utility in leisure, $v(\ell) = 0$, and therefore supply their entire endowment of labor: $n_t = 1$.

You should write your solution code for a general set of parameters, that you can easily change later. In the baseline specification, use $\beta = 0.75$, $\alpha = 0.5$ and $\delta = 0$. For now, we shut down the government: set all taxes, public spending and public debt to zero. The initial conditions for capital and household wealth is : $k_{-1} = d_{-1} = 1$.

Capital labor ratios k_{-1} and k_0 determine prices (w, r) through the firm's first-order conditions (2.8) and (2.9). (k_{-1} and k_0 are capital labor ratios since $n_0 = n_1 = 1$). Initial capital k_{-1} is predetermined, so we focus on k_0. Build an equally-spaced *grid* $\mathcal{K}$ for period-zero physical capital, with twenty points:

$$0.25 = \mathbf{k}_1 < \mathbf{k}_2 < \cdots < \mathbf{k}_{20} = 1.25.$$

For each $k_0 \in \mathcal{K}$, follow the steps below.

(a) Find the associated prices (w, r) using (2.8) and (2.9).

(b) Pick a one thousand-sized grid $\mathcal{D}$ of household net wealth points

$$0.25 = \mathbf{d}_1 < \mathbf{d}_2 < \cdots < \mathbf{d}_{1000} = 1.25.$$

We make $\mathcal{D}$ thinner than $\mathcal{K}$ to make sure that we approximate the optimal choice of household savings with a low error. For each wealth point $d_0 \in \mathcal{D}$, use the sequential budget constraints to find the associated period-zero and period-one consumption, loosely denoted $c_0(d_0)$ and $c_1(d_0)$.

(c) Compute households' optimal savings choice d_0^* as the $\mathcal{D}$ point that maximizes utility:

$$d_0^* = \underset{d_0 \in \mathcal{D}}{\text{Argmax}} \quad u(c_0(d_0)) + \beta u(c_1(d_0)).$$

(Whenever $c_0(d_0) < 0$, discard the candidate choice of d_0.)

(d) Repeat (a)-(c) to all $k_0 \in \mathcal{K}$. You should have a pair of vectors $r_1(k_0)$ and $d_0(k_0)$ containing the period-zero interest and households' savings for each grid point. Do higher capital points k_0 in the grid correspond to lower or higher choices of wealth d_0 by the household? Explain intuitively.

(e) Plot capital demand k_0 and capital supply $d_0(k_0)$ as functions of interest $r_1(d_0)$. Interest should be on the vertical axis of your plot.

(f) How does the equilibrium change if we make housholds more impatient? Repeat (a)-(d) using $\beta = 0.50$, and update your capital equilibrium plot of exercise (e) with the new capital supply curve.

Exercise 2.5. Consider again the environment of Exercise 2.4., but we now add an active government. To keep the exercise simple, the government chooses taxation parameters exogenously, and adjusts public spending to ensure fiscal policy is sustainable. Initially, marginal taxes are fixed at 10%:

$$\tau_c = \tau_n = \tau_k = [0.1 \ 0.1]',$$

and there is no lump-sum taxation, period-zero public spending is $g_0 = 0.3$. The government has no initial debt: $b_{-1} = 0$.

Your mission is to compute the equilibrium of the economy. We adopt an iterative procedure to find the equilibrium capital labor ratio, with each iteration indexed by the symbol i. Given the firms' first-order condition (2.8), searching in the space of capital labor ratios is similar to searching in the space of interest or wage rates. Start by guessing a period-zero capital labor ratio $k_0^{i=0} = 1$.

(a) Given a candidate capital labor ratio k_0^i, follow steps (a)-(c) of the previous exercise to compute households' optimal savings d_0^{*i}. Calculate the government's net debt position b_0^i in period zero, and then the stock of physical capital that clears the capital market:

$$\tilde{k}_0^i = d_0^{*i} - q_0^i b_0^i.$$

If $\tilde{k}_0^i \approx k_0^i$, stop. You have found the solution. Otherwise, you must update the capital labor ratio for the next iteration. Either set $k_0^i = \tilde{k}^i$, or use *damping* to improve numerical stability:

$$k_0^{i+1} = \sigma \tilde{k}_0^i + (1 - \sigma) k_0^i$$

where $\sigma \in (0, 1)$. After finding the equilibrium capital labor ratio, compute equilibrium r, w and c. Compute the level of government spending in period one g_1, and verify that the market for consumption goods clears.

(b) Repeat exercise (a), raising $\tau_{c,1}$ and $\tau_{k,1}$ to 0.2, one at a time. Report how wages, interest and household consumption change, and explain the new results intuitively.

Chapter 3

Income Risk and Public Insurance

3.1. Introducing Risk

Same setup as chapter 1, but now households face *idiosyncratic* income risk in period one.

$$y_1 = \begin{cases} \bar{y}_1 + z & \text{with probability } 1/2 \\ \bar{y}_1 - z & \text{with probability } 1/2 \end{cases}$$

Parameter z introduces *risk*. When $z = 0$, we recover the deterministic case $y_1 = \bar{y}_1$.

The expected value of period-one income is

$$E[y_1] = \frac{1}{2}(\bar{y}_1 + z) + \frac{1}{2}(\bar{y}_1 - z) = \bar{y}_1.$$

Given the existence of a unity measure of households, $\bar{y}_1$ is the aggregate output in period one.

We must adapt utility function to accomodate the existence of uncertainty. Assume *expected utility format*

$$u(c_0) + \beta E[u(c_1)] = u(c_0) + \beta \left[0.5\, u(c_1^H) + 0.5\, u(c_1^L)\right],$$

where c_1^H is consumption in the "high" income state, c_1^L in the "low" income state. When $z = 0$, we recover the original utility function $u(c_0) + \beta u(c_1)$. We say households are (strictly) *risk-averse* when u is (strictly) concave. We assume u to be strictly concave.

How does the introduction of risk changes demand for consumption goods and public bonds? Assume natural borrowing limit, and that $\lim_{c \to 0} u'(c) = \infty$, so that solution to consumption pair is interior. Consider first the original case with deterministic $y_1 = \bar{y}_1$. Let

a_0^D denote public bond demand ("D" for deterministic), and the same for c^D. Optimality requires the Euler equation:

$$q_0 u'(y_0 - q_0 a_0^D) = \beta u'(a_0^D + \bar{y}_1). \tag{3.1}$$

The effect of introducing income risk on households income depends on whether u' is a concave or convex function. That is, it depends on the *third* derivative of the utility function u'''. It is common to assume $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$, suggesting that u' is a *convex* function: $u''' > 0$. This is the case with the common isoelastic utility function.

The consumption-savings problem faced by the household in the presence of income risk:

$$\begin{aligned} &\underset{c,a_0}{\text{Max}} \quad u(c_0) + \beta \left[0.5\, u(c_1^H) + 0.5\, u(c_1^L)\right] \\ \text{s.t.} \quad & q_0 a_0 + c_0 \le y_0 && (3.2) \\ & c_1^H \le a_0 + y_1 + z && (3.3) \\ & c_1^L \le a_0 + y_1 - z && (3.4) \\ & c_0, c_1 \ge 0 \end{aligned}$$

Since $u'(0) = \infty$, households choose positive consumption in both states, and borrowing constraint does not bind. In the interior solution, the Euler equation is:

$$q_0 u'(c_0) = \beta \left[0.5\, u'(c_1^H) + 0.5\, u'(c_1^L)\right] = \beta E[u'(c_1)]. \tag{3.5}$$

Replacing the sequential budget constraints (3.2)-(3.4):

$$q_0 u'(y_0 - q_0 a_0) = \beta E[u'(a_0 + y_1)]$$

When u' is a strictly convex function ($u''' > 0$), households react to the introduction of income risk by *reducing consumption and raising demand for public bonds*. To see this, apply *Jensen's inequality* to the Euler equation:

$$\begin{aligned} q_0 u'(y_0 - q_0 a_0) &= \beta E[u'(a_0 + y_1)] \\ &> \beta u'(a_0 + E[y_1]) \\ &= \beta u'(a_0 + \bar{y}_1) \end{aligned}$$

Compare the inequality above with (3.1). Households react to the introduction of randomness by changing demand for bonds so as to increase period-zero marginal utility, relative to the deterministic case. How come? In the presence of risk, they equate marginal utility in $t = 0$ to *expected marginal utility of consumption* in $t = 1$ ($E[u'(c_1)]$), which is higher than the *marginal utility of expected consumption* ($u'(E[c_1])$). Intuitively, the combination of $u'(c_1^H)$ and $u'(c_1^L)$ is higher than $u'(E[c_1])$ because the value of consumption does not drop as much when c grows as it increases when c declines. Hence, to satisfy the new version of the Euler equation, households reduce consumption in period zero, and increase public bond demand a_0, a behavior called *self-insurance*. Economists also say that households engage in *precautionary savings*.

The introduction of randomness in the income process reduces household welfare, ex-ante:

$$u(c_0^D) + \beta u(c_1^D) \geq u(c_0) + \beta u(\bar{c}_1) \geq u(c_0) + \beta \left[0.5\, u(c_1^H) + 0.5\, u(c_1^L)\right]$$

(In the expression above, c represents optimal consumption in the income risk case.) The first inequality follows from optimality of c^D in the deterministic case; the second inequality follows from concavity of u (Jensen's inequality). Since we assume u to be strictly concave, the expression holds with strict inequality.

Utilitarian government can improve ex-ante welfare by charging 100% income tax in period one, and fully re-distributing proceeds.

3.2. An Environment with Elastic Labor Supply

Introduce elastic labor supply. Households remain identical in period zero, and supply their entire endowment of hours to firms: $n_0 = 1$. In period one, they value leisure, as captured by the utility function:

$$u(c_0) + \beta E\left[u(c_1) + v(1 - n_1)\right].$$

For the remainder of this section, we focus on period one. Period utility in $t = 1$ is $u(c)+v(1-n)$, where n is number of hours devoted to labor. We assume twice differentiable, increasing, concave u and v. Additionally, $\lim_{\ell \to 0} v'(\ell) = \infty$, so households always devote some time for leisure: $n < 1$.

No physical capital. Households provide differentiated labor hours. Each household has an individual (or *idiosyncratic*) level of productivity z, meaning that n hours of its labor corresponds to $z \times n$ *efficiency hours of labor*. Efficiency hours of labor differ from physical hours of labor because they incorporate individual productivity. Random variable z can take S different values: $z_1 < z_2 < \cdots < z_S$, with probability $p_1, p_2, \ldots, p_S$, respectively. Of course, $\sum_s p_s = 1$.

Productivity draws are independent from each other. Therefore, after draws occur, p_1

Symbol	Description
n_1^z	Labor hours supply by household with productivity z
$\bar{n}_1$	Aggregate (efficiency) hours labor
$w_1 z$	Wage rate per hour of labor
w_1	Wage rate per efficiency hour of labor
$h_1 = w_1 \bar{n}_1$	Aggregate labor income
$\text{rev} = \tau h_1$	Public revenue from labor income

Table 3.1: Key Labor Market Variables in Period One

households land state $s = 1$, p_2 land $s = 2$, and so on. This is an application of the law of the large numbers.

We break down production in two layers. A representative intermediary firm hires labor hours from households and builds a homogeneous "aggregate efficiency labor" commodity (or just "aggregate labor", for brevity). The representative firm that produces consumption goods uses aggregate labor as the only production factor.

The intermediary firm aggregates labor using the production function

$$\bar{n}_1 = \int_0^1 z(j)n(j)dj = p_1(z_1 n_1^{z_1}) + \cdots + p_S(z_S n_1^{z_S}) = E\left[z n_1^z\right].$$

In the integral, $z(j)$ is the productivity of household j and $n(j)$ is its labor choice. I also define n_1^z as the working hours choice made by households with productivity z. We characterize their optimal choice later.

The intermediary firm sells the $\bar{n}_1$ aggregate hours at a rate w_1 per hour. Since technology is linear, in equilibrium the wage rate is $w_1 \times z$. Hence, we refer to w_1 as the *wage rate per efficiency hour of labor*. The aggregate labor income h_1 is

$$h_1 = \int_0^1 w_1 z(j)\tilde{n}(j)dj = w_1 \bar{n}_1.$$

Table 3.1 summarizes labor market variables in period one.

Breaking the wage rate between its idiosyncratic and common components is convenient because the number of aggregate hours of labor demanded by the consumption good producer depends only on the latter. Since we focus on taxation, suppose the final good producers converts one hour of aggregate labor into one consumption good. Hence, $w_1 = 1$. (In any case, we continue to write w_1 in the formulas, for clarity of the arguments.) In this setup, all firms are indifferent regarding production scale: their profits equal zero regardless.

3.3. Taxation and Laffer Curve

We are interested in studying if and how the government can use fiscal policy to help insure households against income risk. We start by focusing on a fiscal policy that combines a flat tax τ on labor income and lump-sum transfers R to households, both imposed only in period one. This notation simplifies the more cumbersome $\tau_{n,1}$ and $\tau_{L,1}$ symbols of chapter 2, which we can drop since there are no other taxes.

By the sequential budget constraint, period-one consumption for a household with productivity z is

$$c_1^z = a_0 + (1-\tau)w_1 z n_1^z + R.$$

The first-order condition for labor supply:

$$(1-\tau)\, w_1 z\, u'(c_1^z) \leq v'(1-n_1^z) \qquad (= \text{if } n_1^z > 0) \tag{3.6}$$

Marginal benefit of working +1 hour = marginal cost; otherwise household constrained. Since $v' > 0$, 100% taxation $\tau = 1$ leads to $n_1^z = 0$: households supply no hours of labor.

For the remainder of this section, we fix a_0, w_1 and R, and express optimal labor choice $n_1^z(1-\tau)$ as a function only of the net-of-tax parameter $1-\tau$. Because of substitution and wealth effects, an increase in τ has an ambiguous effect on n_1 (but we know that $n_1 = 0$ when $\tau = 1$).

With the individual labor supply $n_1^z(1-\tau)$, we can compute aggregate labor supply

$$\bar{n}_1(1-\tau) = p_1(z_1 n_1^{z_1}(1-\tau)) + \ldots p_S(z_S n_1^{z_S}(1-\tau))$$

and the aggregate labor income $h_1(1-\tau) = w_1\bar{n}_1(1-\tau)$. By charging a marginal rate τ, the government raises a total revenue of τh_1. Express that as a function of τ:

$$\text{rev}(\tau) = \tau\, h_1(1-\tau) \geq 0.$$

Function $\text{rev}(\tau)$ is known as the *Laffer curve*. Its shape depends largely on the labor supply model at hand. In general:

- $\text{rev}(0) = 0$ ($\tau = 0$, no taxes charged)
- $\text{rev}(1) = 0$ ($n_1 = \bar{n}_1 = 0$, households have no incentive to work).

In the particular case that $h' \geq 0$ and $h'' \leq 0$, the Laffer curve has an inverted-U shape. Figure 3.1 shows an example.

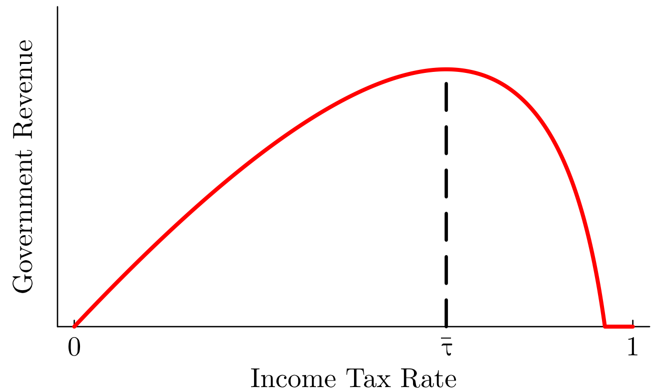


Figure 3.1: Laffer Curve Example

Revenue-maximizing tax rate $\bar{\tau}$ satisfies $\text{rev}'(\bar{\tau}) = 0$:

$$\text{rev}'(\bar{\tau}) = h_1 - \bar{\tau} h_1' = 0, \tag{3.7}$$

where h_1 and h_1' are both evaluated at the point $1 - \bar{\tau}$. Re-writing the expression above yields:

$$\bar{\tau} = \frac{1}{1+e} \qquad \text{where } e = \frac{\partial h(1-\bar{\tau})}{\partial(1-\tau)} \frac{1-\bar{\tau}}{h(1-\bar{\tau})} \tag{3.8}$$

is the elasticity of aggregate labor income to after-tax efficiency wage rate $(1-\tau)w_1$, which we can measure empirically. Higher elasticities associate with lower optimal tax rates.

3.4. Optimal Insurance

To provide insurance against income risk, the government distributes the proceeds from the labor tax charge back to households in the form of a lump-sum transfer R. Each households receives the same transfer: we thus model a *universal basic income* program. If the government uses all available resources, and if the policy is sustainable, $R = \text{rev}(\tau)$; so we can write $R(\tau)$.

We continue to leave a_0 and w_1 fixed. Which tax rate τ maximizes household welfare ex-ante (*i.e.*, prior to the productivity draw)? Utility at the beginning of period one:

$$E\left[u(a_0 - (1-\tau)w_1 z n_1^z + R(\tau)) + v(1 - n_1^z).\right]$$

(Labor supply n_1 evaluated at $(1-\tau)$.) To facilitate notation, let

$$u_z'(\tau) = u'\left(a_0 - (1-\tau)w_1 z n_1^z(1-\tau) + R(\tau)\right)$$

be the period-one marginal utility of a household that draws z. The first-order condition for optimal tax rate τ^* is

$$E\left[u'_z(\tau^*)\right] R'(\tau^*) = E\left[u'_z(\tau^*) z n_1^z (1-\tau^*)\right] > 0.$$

(Note the application of the envelope theorem.) Since $u' > 0$, $R'(\tau^*) = \text{rev}'(\tau^*) > 0$. Compare this condition to (3.7). Isolating optimal taxation:

$$\tau^* = \frac{\lambda}{\lambda + e}, \tag{3.9}$$

where

$$\lambda = -\frac{\text{cov}\left(u'_z, z n_1^z\right)}{\bar{n}_1 E(u'_z)} \in (0,1)$$

measures the degree of consumption inequality after public insurance has been implemented. If the government manages to equalize consumption across households, u' is constant and hence $\lambda = 0$. Otherwise, $\lambda > 0$ since marginal utility decreases in consumption and hence in realized labor income $z n_1^z$.

Optimal taxation increasing in inequality, decreasing in labor supply elasticity.

Exercises

Exercise 3.1. In the context of the model with elastic labor supply and heterogeneous productivity, express aggregate labor income as a function of the labor tax rate $H_1(\tau) \equiv h_1(1-\tau)$. Show that the tax rate $\bar{\tau}$ that maximizes government revenue attains

$$-\frac{\partial H_1(\bar{\tau})}{\partial \tau} \frac{\bar{\tau}}{H_1(\bar{\tau})} = 1.$$

That is, the elasticity of H_1 with respect to the tax rate is equal to one in the revenue-maximizing point. Provide an interpretation.

Exercise 3.2. Consider a model with discrete labor choice and no uncertainty. Households can either supply their whole endowment of hours $n_1 = 1$, or no hours at all $n_1 = 0$. They enter period one with a public bonds, and are offered a wage rate of w. Period-one utility is

$$u(c) - \zeta\ \mathbf{1}_{n=1},$$

where $\zeta > 0$, and $\mathbf{1}_{n=1}$ is an indicator function, which equals one when $n = 1$, and zero otherwise. Function u is increasing.

(a) Start by assuming there is no taxation. Compute the household's labor supply decision rule, as a function of a and w.

(b) Suppose the government charges a flat marginal tax rate $\tau_{n,1}$ on labor income, and uses the revenue for public spending (which households do not value). Re-compute the household's decision rule. Find the expression defining the threshold taxation level $\tau^*_{n,1}$ above which households opt not to work.

(c) Sketch the plot of the Laffer curve.

(d) Suppose $u(c) = \log(c)$. How dows $\tau^*_{n,1}$ depend on households' wealth a? Explain intuitively.

Exercise 3.3. We consider a particular case of GHH preferences (following Greenwood, Hercowitz and Huffman). In period one, the utility function is $u(c + \Phi v(1-n))$, where u is increasing and differentiable, and

$$v(1-n) = \frac{(1-n)^{1-1/\psi}}{1-1/\psi}.$$

(a) Compute the first-order condition for the optimal choice of labor in period one. Since $v'(\ell) \to \infty$ when $\ell \to 0$, $n_1 < 1$: you only need worry about the lower bound on labor choice. Show that when $w \leq \Phi$, the household does not work: $n_1 = 0$.

(b) Use the first-order condition to argue that there is no wealth effect on labor supply.

(c) With the help of a computer, set $\Phi = \psi = 0.5$ to reproduce the Laffer curve in figure 3.1.

Exercise 3.4. *Universal basic income* (UBI) programs propose that every individual receives an unconditional transfer of money, regardless of their earnings and other aspects of tax legislation. Consider a UBI scheme that transfers R consumption goods, and taxes all households at a flat rate of τ. Show that this UB program is economically equivalent to a non-UBI taxation scheme that establishes two income brackets: $h \leq \hat{h} = R/\tau$ and $h \geq \hat{h}$. Find the required tax/transfer function in each income bracket.

Exercise 3.5. Consider the elastic labor supply model of the main text, in which the government inherits no public debt: $b_{-1} = 0$. All households have access to half a unit of the consumption good in period zero ($y_0 = 0.5$) from labor endowment. Households have the utility function

$$u(c_0) + \beta E\left[u(c_1) + v(1-n_1)\right]$$

where $u(x) = v(x) = log(x)$, and $\beta = 0.8$. Productivity z can take two values: $z_1 = 1 + \sigma$ and $z_2 = 1 - \sigma$, each with probability $p = 0.5$.

(a) Initially, the government does not tax or transfer goods. In equilibrium, what is the net wealth a_0 of each household in the beginning of period one? Show that the optimal labor supply is $n_1 = 0.5$, regardless of z. Compute household income as a function of productivity. (To solve this problem, recall that $w_1 = 1$.)

(b) With the help of a computer, plot the equilibrium interest rate as a function of σ (vary σ from 0 to 0.4), and provide an interpretation for your findings.

(c) Let $\sigma = 0.2$. Suppose now that the government introduces a basic income program, funded by a $\tau = 0.2\%$ flat labor income tax. Derive analytically each households' optimal labor supply n_1^z, *given* the government's transfer R. Your first task is to compute the government revenue R from taxing labor income, which depends on household labor supply (which, in turn, depends on R itself).

Write in your code a function that computes optimal labor supply given a lump-sum transfer R. Write up a second function $g(R)$ that uses the first one to calculate the government revenue from taxing households. The equilibrium revenue raised by the government by taxing labor income satisfies the fixed-point problem: $R = g(R)$. Compute R. (Tip: adopt an interative procedure. Guess some R_0; then update your guess using $R_i = g(R_{i-1})$ until R_i is close enough to the fixed point.)

(d) With the equilibrium lump-sum transfer R, compute the equilibrium interest rate, and compare it with the interest rate arising in the absence of taxation. Explain intuitively why results differ. Does public insurance against income risk guarantee a decline in period-zero demand for bonds?

Exercise 3.6. Model with capital.

Chapter 4

Introduction to Finite-Horizon Dynamic Programming

4.1. Dynamic Programming Concepts

4.2. Adding Uncertainty

4.3. Computing Optimal Supply of Labor

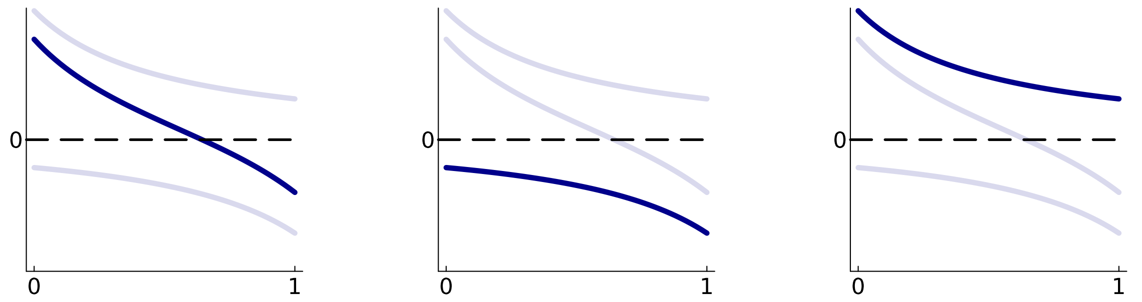
This section provides an algorithm to compute household's optimal supply of labor hours. Define the "net" marginal benefit of increasing working hours

$$h(n) = wu'(wn + z) - v'(1 - n),$$

where w is the after-tax income and a is a term that groups other components of the budget contraint, like bond redemptions, new bond purchases and government transfers. Here, we fix both w and z. When $h(n) > 0$, the marginal benefit (in utility units) of working a little more $wu'(wn + z)$ outweights the marginal cost $v'(1 - n)$ of reducing leisure hours.

We usually assume u and v are concave, which implies that $u'', v'' < 0$ and, thus, $h'(n) < 0$. As you work more, the benefit of increasing labor hours declines - first because leisure becomes scarcer (thus more valuable, v' term) and, second, because consumption grows (thus becomes less valuable, u' term).

Let n^* be the optimal supply of labor hours. We can split the first-order condition for n^* to be optimal in three cases. Case 1: If n^* is an interior solution for the household problem, then $h(n^*) = 0$. Case 2: If $n^* = 0$ and the household is constrained by the fact that it cannot



(a) Interior Solution **(b)** Lower Bound Solution **(c)** Upper Bound Solution

Figure 4.1: Marginal Net Benefit Function h: Solution Cases

work less than zero hours, then $h(n^* = 0) \leq 0$. Case 3: If $n^* = 1$ and the household is constrained by the fact that it cannot work more than all available time, then $h(n^* = 1) \geq 0$. Figure 4.1 depicts three examples of h, each with a solution belonging to a different case.

In practice, we do not know from the beginning which case is right. However, since net marginal benefit always declines in labor hours ($h' < 0$), we know that $h'(n^*) = 0$ can only hold for a single point. We can therefore adopt the following algorithm to numerically (or analytically) compute n^*:

1. If $h(0) \leq 0$, then $n^* = 0$. Stop.

2. If $h(1) \geq 1$, then $n^* = 1$. Stop.

3. Otherwise, search for the zero of h in the interval $(0, 1)$.

If you get to the last step, then you know that $h(0) > 0$ and $h(1) < 0$ (otherwise the algorithm stops in one of the previous steps). In that case, you need to find the zero of function h, that is, the point n^* between zero and one such that $h(n^*) = 0$.

A simple *bisection method* can be applied to find the zero of h. Starting with $n_0 = 0$ and $n_1 = 1$, follow the steps below.

1. Define $n = \frac{n_0 + n_1}{2}$.

2. If $n_0 \approx n_1$ or $h(n) \approx 0$, stop. You have found the zero of h.

3. If $h(n) > 0$, set $n_0 = n$ and go back to step 1.

4. If $h(n) < 0$, set $n_1 = n$ and go back to step 1.

(The bisection method above assumes h is decreasing; if you are interested in finding the zero of an increasing function f, you can imply the steps to $-f$.)

Exercises

Exercise 4.1. Given a decreasing function f, and two points $a < b$, write the code of a function that applies the bisection method described in section 4.3 to find the zero of f between a and b.

(Tip: In the context of iterative procedures that depend on control clauses to end - like the bisection method -, it is good practice to limit the number of iterations the algorithm can perform. Otherwise, typos or unfortunate examples can lead your computer to loop over the iteration endlessly.)

Exercise 4.2. Let

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad \text{and} \qquad v(\ell) = \frac{\ell^{1-\frac{1}{\psi}} - 1}{1 - \frac{1}{\psi}}.$$

When $\gamma = 1$, $u = \log$, and the same is true for ψ and v. Write a code that applies the algorithm described in section 4.3 of this chapter to compute the optimal labor supply choice in the problem

$$\underset{n}{\text{Max}} \quad u(wn + a) + v(1 - n) \qquad \text{s.t.} \quad 0 \leq n \leq 1.$$

Use can use the bisection function you wrote in the previous exercise.

Chapter 5

Overlapping Generations and Pension Systems

5.1. OLG in Infinite Periods

Infinite periods $t = 0, 1, 2, \ldots$. Each period, new generation of households born (unity measure). Households live for two periods, "young" and "senior". Single consumption good. Young households receive an endowment of one unit of the consumption good. No government action.

Let c_s^t be period-s consumption of household born in period t, with $s \in \{t, t+1\}$. Let a_t^t be bond position (households allowed to sell bonds to each other). Linear preferences, no discounting:

$$\begin{aligned} \underset{c^t \geq 0, a_t^t}{\text{Max}} \quad & c_t^t + c_{t+1}^t && (5.1) \\ \text{s.t.} \quad & q_t a_t^t + c_t^t \leq 1 \\ & c_{t+1}^t \leq a_t^t && (5.2) \end{aligned}$$

Finite demand for public bonds only when $q_t = 1$.

Market clearing conditions in period t:

$$\begin{aligned} c_t^t + c_t^{t-1} &= 1 \\ a_t^t &= 0 \end{aligned}$$

In equilibrium, $q_t = 1$ and each household consumes its own endowment when young.

However, this equilibrium is not Pareto optimal. *Problem of infinity.* Alternative allocation:

generation born in t transfers its endowment to generation born in $t-1$. All generations left with the same single consumption good, *except $t=0$ generation, which gets two consumption goods.*

5.2. Three-Period Environment

Three periods: $t = 0, 1, 2$. Two generations: A and B. Each with the same size of one. Generation A lives in periods zero and one, not in period two. Generation B is born in period one, and lives in period two. First period of life: "young". Second period: "senior".

Single consumption good. No capital. Households can only work when young. When senior, they receive an exogenous endowment of e units of the consumption good (home production). Linear production function $f(n) = n$ implies wage rate $w = 1$.

We initially ignore the government. Natural debt limit. Households subject to the natural debt limit. Those of generation A solve the problem

$$\begin{aligned} \underset{c^A \geq 0, a_0^A}{\text{Max}} \quad & u(c_0^A) + v(1 - n_0^A) + \beta u(c_1^A) \\ \text{s.t.} \quad & q_0 a_0^A + c_0^A \leq n_0^A \\ & c_1^A \leq a_0^A + e. \end{aligned} \tag{5.3}$$

Households of generation B solve a consumption-savings problem analogous to (5.3).

The market-clearing conditions are the following:

$$c_0^A = n_0^A \tag{5.4}$$

$$c_1^A + c_1^B = n_1^A + e \tag{5.5}$$

$$c_2^B = e \tag{5.6}$$

In equilibrium, neither generations saves or borrows - bond prices must be such that not trading in the bond market is their optimal choice.

Household heterogeneity embedded in models with overlapping generations provides an easy way to break Ricardian equivalence. The timing of taxes affects individual and aggregate demand because it affects the total income of different households.

5.3. A Pension System Model

Model with a "pay-as-you-go" pension system. Young generation B pays for senior generation A households in period one. Households from generation A face a probability $\rho \in [0, 1]$ of "retiring" in period one. We can use ρ to capture the size of the pension system as well as the retirement age.

Retired seniors receive a lump-sum transfer of ϕ consumption goods. Young households from generation B finance retirement payments through a lump-sum tax τ (we drop subscripts from τ to keep notation light - there are no other taxes). The government runs a balanced budget:

$$\rho\phi = \tau.$$

Generation A utility and consumption-savings problem:

$$\begin{aligned}
&\underset{c^A \geq 0, a_0^A}{\text{Max}} \quad u(c_0^A) + v(1 - n_0^A) + \beta\left[\rho u(\tilde{c}_1^A) + (1-\rho)u(c_1^A)\right] \\
\text{s.t.} \quad & q_0 a_0^A + c_0^A \leq n_0^A \\
& c_1^A \leq a_0^A + e \\
& \tilde{c}_1^A \leq a_0^A + e + \phi
\end{aligned} \tag{5.7}$$

$\tilde{c}_1^A$ represents consumption *if the household retires*. Else, it consumes off its own savings and exogenous endowment. Utility function has *expected utility format*.

Generation B faces conventional consumption savings-problem:

$$\begin{aligned}
&\underset{c^B \geq 0, a_1^B}{\text{Max}} \quad u(c_1^B) + v(1 - n_1^B) + \beta u(c_2^B) \\
\text{s.t.} \quad & q_1 a_1^B + c_1^A \leq n_1^A - \tau_{L,1} \\
& c_2^B \leq a_1^B + e
\end{aligned} \tag{5.8}$$

Market-clearing conditions (5.4)-(5.6) stay the same.

Example: no leisure value $v = 0$. Therefore: $n_0^A = n_1^B = 1$. Euler equations:

$$\begin{aligned}
q_0 u'(1) &= \beta\left[\rho u'(e + \psi) + (1-\rho)u'(e)\right] \\
q_1 u'(1 - \rho\phi) &= \beta u'(e)
\end{aligned}$$

Expansion of the pension system (higher ρ or higher ϕ) reduce the demand for public bonds from households in both generations, as they are left relatively righer when they are older. In equilibrium, bond prices decline, interest rates increase.

Exercises

Exercise 5.1. Consider the basic overlapping-generations model with no government. Continue to assume the linear production function $f(n) = n$, and unity wage rate. Assume $u(c) = v(c) = \log(c)$.

(a) Given β and e, find equilibrium consumption levels and bond prices.

(b) Suppose the government imposes a lump-sum tax of τ_1 consumption goods to households of generation A in period one, and τ_2 to generation B in period two. Both τ_1 and τ_2 can be negative, in which case the government is transfering goods instead taxing them. Assuming the government enters period zero with no debt, write down its sequential and present-value budget constraints.

(c) Assume the government transfers $-\tau_1 > 0$ goods to generation A households. Solve (a) under the new fiscal policy. Provide an intuition as to why the allocation and price vectors differ. Does Ricardian Equivalence hold?

Exercise 5.2. In the context of the two-period unfunded pension system model, consider again the case in which households don't value leisure, $n = 0$. Suppose the government has decided on the size τ of the pension system, but not on parameters ρ and ϕ. You can think that the government is choosing between different eligibility criteria unrelated to economic factors.

(a) Parameters ρ and ϕ must satisfy $\rho\phi = \tau$. How does the choice of ρ affect demand for public bonds by generation B households and interest rate in period one?

(b) How does it affect the demand for public bonds by generation A households and interest rate in period zero? You may assume that u' is a strictly convex function; it satisfies:

$$u'(b) > u'(a) + u''(a) \times (b - a)$$

for $a, b > 0$. Provide an interpretation based on precautionary behavior, as stuied in chapter 3. (Hint: what happens when $\rho = 1$?)

Exercise 5.3. In this numerical exercise, we numerically solve the general equilibrium effects of the introduction of a realistic unfunded pension system. Following the setup of chapter 2, firms produce consumption goods using labor and physical capital through the production function

$$f(k, n) = k^{\alpha} n^{1-\alpha}.$$

Capital depreciates at a rate δ, and the expressions

$$\begin{aligned} r_t + \delta &= \alpha(k_{t-1}/n_t)^{\alpha-1} \\ w_t &= (1-\alpha)(k_{t-1}/n_t)^{\alpha} \end{aligned} \tag{5.9}$$

provide first-order conditions for optimality when firms do not profit. Since households do not work in period two, $r_2 = -\delta$. There is no senior age endowment e.

As in the main text, senior households of generation A have a probability ρ of receiving a pension installment of ϕ consumption goods. The installment are financed by a flat, marginal tax τ on labor income (different from the lump-sum tax of the main text). The government initially has no public debt, and adopts a balanced budget in all periods, which requires $\rho\phi = \tau n_1^B$.

Using the end-of-period notation, generation A households enter period zero with a net wealth of $d_{-1} = k_{-1}$. The market-clearing condition in the capital market is

$$d_t = k_t \qquad t = 0, 1. \tag{5.10}$$

Utility of generation A is similar to that of the text

$$u(c_0^A) + v(1 - n_0^A) + \beta E\left[u(c_1^A)\right],$$

with isoelastic u and v:

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad \text{and} \qquad v(\ell) = \frac{\ell^{1-\frac{1}{\psi}} - 1}{1 - \frac{1}{\psi}}.$$

The utility function of generation B is analogous. For the baseline calibration, use $\alpha = 0.5$, $\beta = 0.8$, $\delta = 0.1$, $\gamma = 0.5$, $\psi = 0.8$, $\rho = 0.5$ and $\phi = 0.1$. The initial stock of physical capital is $k_{-1} = 1$.

(a) Write the consumption-savings problem faced by households of generation A. Consider a grid $\mathcal{D}$ of household net wealth points:

$$0.05 = \mathbf{d}_1 < \mathbf{d}_2 < \cdots < \mathbf{d}_{1000} = 0.6$$

Write a function that takes as given w_0, r_0, r_1, ρ and ϕ, and returns the optimal choice of d_0, c_0^A, $\tilde{c}_1^A$, c_1^A and n_0^A, by households of generation A.

To solve the problem, you need to compute the utility of selecting each candidate net wealth position $d \in \mathcal{D}$, and then choose the point that maximizes it. Hint: to compute the optimal labor supply choice associated with a point d, use the function you wrote in Exercise 4.2. of chapter 4. It needs to solve

$$w_0 u'(w_0 n_0^A + (1 + r_0)d_{-1} - d_0) = v'(1 - n_0^A).$$

(b) Write the consumption-savings problem faced by households of generation B. Using the same grid $\mathcal{D}$, write another function, that takes as given w_1, r_1 and τ, and returns the optimal choice of d_1, c_1^B, c_2^B and n_1^B by households of generation B. The algorithm should be similar to the one you wrote in (a).

(c) You have solved households' consumption-savings problems. Now, you need to find market-clearing prices. By (5.9), wage and interest rates depend only on the capital-labor ratio $kn_t = k_{t-1}/n_t$. So it is easier to search for two market-clearing capital-labor ratios kn_0 and kn_1, then the four prices w_0, r_0, w_1, r_1. You also need to ensure that the pension system is budget-balanced through proper selection of the tax rate τ. We group these variables in a single *solution vector* x:

$$x = \begin{bmatrix} kn_0 \\ kn_1 \\ \tau \end{bmatrix}.$$

Adopt an iterative procedure. Start by guessing a solution vector $x^0 = [1,\ 1,\ 0]$. In iteration i, fix x^i and solve households' problems using (a) and (b). Use the market-clearing condition (5.10) along with optimal labor supply to compute resulting capital-labor ratios:

$$\hat{kn}_0 = \frac{k_{-1}}{n_0^{Ai}}$$
$$\hat{kn}_1 = \frac{d_0^i}{n_1^{Bi}} = \frac{k_0^i}{n_1^{Bi}}$$

(the i superscript indicates the iteration number). Then, use the balanced-budget condition to find sustainable benefits: $\tau = \rho\phi^i / n_1^{Bi}$.

Define $\hat{x} = [\hat{kn}_0\ \hat{kn}_1\ \hat{\tau}]$. If $\hat{x} = x^i$, stop. You have found the solution vector, with equilibrium capital-labor ratios and pension benefits. Otherwise, update the candidate solution vector using damping

$$x^{i+1} = 0.5 \times \hat{x} + 0.5 \times x^i.$$

and move to the next iterations. (Remember to include a maximum number of iterations in your code to avoid an endless loop of the algorithm.)

(d) Repeat your numerical computation, but shut down the pension system: $\phi = 0$. Your solution vector should yield $\tau = 0$. How does shutting down the pension system affect equilibrium aggregate consumption, stock of capital, interest and wages? Explain intuitively.

Chapter 6

Classical Theories of Monetary-Fiscal Interaction