Title: Something with Unexpected Inflation

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1 Introduction

2 Unexpected Inflation Decomposition

2.1 Connection with the Value of Debt

I start by establishing the connection between inflation and the value of debt. Consider an economy with a homogeneous final good which households value. The economy has a government that manages a debt composed by bonds that pay one unit of currency in the next period. Currency is a commodity that only the government can produce, at zero cost. There is nothing special about currency. Households do not value it, and it provides no convenience for trading. Households cannot burn currency.

At the beginning of period t, the face value of debt issued in the previous period is V_{t-1}. The government redeems it for currency, which moves to the hands of households. After redeeming bonds, the government announces that each household i must pay $T_{i,t}$ goods in taxes, payable in currency. It also announces it will sell V_t new bonds and purchase G_t units of the final good at market prices.[1]

Nothing binds the government's choices of $T_{i,t}$ and V_t. Note the difference from the case with bonds payable in goods, in which the government *must* raise tax revenue or issue new debt to redeem old bonds. Additionally, if households accept currency as payment for final goods, nothing binds the government's choice of G_t either.

Let M_t be private holdings of currency at the end of t. As there is no free disposal of currency, the quantity used by the government to redeem $t-1$ bonds and purchase goods is used to either pay taxes, buy new bonds or increase currency holdings:

$$\begin{aligned} V_{t-1} + G_t &= P_t T_t + Q_t V_t + \Delta M_t \\ \implies V_{t-1} &= P_t s_t + Q_t V_t + \Delta M_t \end{aligned} \tag{1}$$

[1] That the government forces households to pay for taxes and new bonds in currency is not necessary for the argument. All else follows if the government accepts payment in goods but stands ready to exchange these goods for currency at market prices.

where T_t are aggregate taxes, $s_t = T_t - G_t$ is the primary surplus, P_t is the final good's price and Q_t is the price of new bonds (I state prices in currency units). Equation (1) provides a low of motion for public debt. It holds for all prices and all choices of money holdings, new debt sales, and primary surplus.[2]

If $P_t = 0$, real public debt V_{t-1}/P_t equals infinity. For now, that is a possiblity.

Suppose $P_t > 0$. Define $\beta_t \equiv Q_t P_{t+1}/P_t$ as the real discount for public bonds, and $\beta_{t,t+k} \equiv \prod_{\tau=t}^{t+k} \beta_\tau$. Since V satisfies (1), it also satisfies

$$\frac{V_{t-1}}{P_t} = \sum_{i=0}^{k} \beta_{t,t+i-1} \left(s_{t+i} + \Delta M_t\right) + \beta_{t,t+k} \frac{V_{t+k}}{P_{t+k+1}} \qquad \text{for any } k \geq 0 \tag{2}$$

regardless of prices and choices. Expressions (1) and (2) do not represent a constraint on the path of surpluses $\{s_t\}$ and bond sales $\{V_t\}$ the government chooses to follow. They merely express future debt given paths for quantities and prices.

So far, all we did was public finances accounting. I now move to economic behavior. If $P_t = 0$, households demand infinite final goods and there is no equilibrium. Therefore $P_t > 0$.

Given a utility function over consumption paths $U(\{c_t\})$, the optimal consumption-savings choice involves two conditions. First: $\beta_{t,t+k}$ = marginal rate of substitution between time-t and time-$t+k$ consumption. Second, the transversality condition $\lim_{k\to\infty} \beta_{t,t+k} V_{t+k}/P_{t+k+1} \leq 0$. Coupled with a no-Ponzi condition that establishes the reverse inequality, it implies

$$\lim_{k\to\infty} \beta_{t,t+k} \frac{V_{t+k}}{P_{t+k+1}} = 0 \text{ at every period } t. \tag{3}$$

If bonds were redeemable in goods, (3) would represent a debt sustainability condition, as it forces the government to eventually raise the resources to pay for past borrowing.

With nominal debt, however, the government has no constraints on its choice of debt and surplus, as already pointed out. Replacing (3) on (2) and taking expectation yields

$$\frac{V_{t-1}}{P_t} = \sum_{i=0}^{\infty} E_t \left[\beta_{t,t+i-1} \left(s_{t+i} + \Delta M_{t+i}\right)\right]. \tag{4}$$

Given the face value of maturing public debt, the relative price of the consumption good is given by the expected β-discounted stream of real primary surpluses. This latter term - the right-hand side of (4) - I call the *real value of public debt.*In the case of nominal debt, (4) is a *valuation equation.* Hence, expression (3) implies not a no-default condition, but a no-bubble condition: market prices should reflect fundamental value.

[2]Strictly speaking, (1) holds for all *feasible* choices of money holdings, that is, choices that respect households' budget constraint. Otherwise, one could point out that, for $B_{t-1} > 0$, $M_t = M_{t-1}$ and $s_t = B_t = 0$ violates (1). That would nevertheless involve households burning up currency.

Now, define the inflation rate $\Pi_t = P_t/P_{t-1}$, and take innovations on both sides

$$\frac{V_{t-1}}{P_{t-1}}\Delta E_t \Pi_t^{-1} = \sum_{i=0}^{\infty} \Delta E_t \left[\beta_{t,t+i-1}\left(s_{t+i} + \Delta M_{t+i}\right)\right]. \tag{5}$$

Any shock can lead the government to change the conditional distribution of future surpluses (in units of goods) backing the stock of public nominal liabilities. The relative price of bonds in terms of goods $1/P_t$ then adjusts to reflect that change, much in the same way that news of future dividends (in currency units) per stock change the relative price of stocks in terms of currency (Cochrane (2005)). Unexpected inflation $\Delta E_t \Pi_t$ follows. Also like stocks, changes in stochastic discounting β also affect fundamental value, and thus affect prices.

Importantly, (4) and (5) do not depend to equilibrium selection mechanisms. Both hold on all models in which (3) holds, inclusing the standard New-Keynesian model.

2.2 Inflation Decomposition in the Simplest Environment

I linearize the law of motion (2).

$$\beta(v_t + s_t) = v_{t-1} + i_t - \pi_t \tag{6}$$

where v_t is *end-of-period* stock of real debt, $i_t = -\log(Q_t)$ and $\pi_t = \log(\Pi_t)$. I assume $\Delta M_t = 0$ (households do not hold currency). Note that v and s are both in levels - I assume them to be stationary for simplicity. Moreover, I linearize around the point $v = 1$, which I take to be the average real debt level.

The interpretation of (6) is the same as before. Previous period debt accrues by the mean real interest $(1/\beta)$ plus its local variation $i_t - \pi_t$. 1% more real interest raises debt by 1% on average because debt is on average one. The government redeems bonds for currency, and then sells new debt v_t and runs a surplus s_t to soak it up.

2.3 Generalizing Public Financing Instruments

2.3.1 Currency Denomination

The debt process considered so far is too unrealistic to be taken to the data. I generalize the financing instruments available to the government in two dimensions: currency denomination and maturity structure. I do not consider the case of debt with stage-contingent nominal payoffs.[3]

Starting here, I recycle part of the notation already established. Fix the case of a country's government. Let P_t be the price of the consumption basket in terms of domestic currency.

[3] Publicly-ran social security and pension systems can be modelled as cases of public debt with state-contigent payoffs or, in a fashion compatible with my framework, as an additional layer of uncertainty in the primary surplus process.

Symbol	Description	Nominal Debt	Real Debt	Dollar-Linked Debt
j	Index Symbol	N	R	D
	Notation	δ, ω	δ_R, ω_R	δ_D, ω_D
P_j	Price per Good	P	1	P_t^{US}
$\mathcal{E}_j$	Nominal Exchange Rate	1	P	Dollar NER
H_j	Real Exchange Rate	1	1	Dollar RER
π_j	Log Variation in Price	π	0	π_t^{US}
Δh_j	Log Real Depreciation	0	0	Δh_t

Notes: P = price of consumption basket in domestic currency. P^{US} = price of consumption basket in US dollars. NER = Nominal Exchange Rate. RER = Real Exchange Rate.

Table 1: Public Debt Denomination

The payoff of public bonds can be indexed to different currencies, enumerated by j. Let $P_{j,t}$ be the price of the consumer price index in units of currency j. Let $Q^n_{j,t}$ be the discount rate for a zero-coupon public bond paying one unit of currency j after n periods. Without loss of generality, all bonds are zero-coupon. Let $\mathcal{E}_{j,t}$ be the price of currency j in units of domestic currency.

The notation is general enough to accomodate currency-linked bonds *per se*, but also inflation-linked securities such as American TIPS. In this case, one can interpret the "currency" as the final goods basket (so that $P_j = 1$ and $\mathcal{E}_j = P_t$). In the empirical exercises that follow, I consider only nominal bonds ($j = N$), inflation-linked (or real) bonds ($j = R$) and US-dollar-denominated bonds ($j = D$). Table 1 shows the value or interpretation of the variables defined above for these three cases.

I do not assume the price index for government's purchases (denoted P_t^S) and levied aggregates (such as income) to be the same as the price index for households' consumption (P_t). While governments do tend to tax private consumption, they also tax and transfer income, and purchase baskets of goods different than that of households. The fiscal effect of variations in the relative price of government's to households' final goods basket must be accounted for.

With these changes, the law of motion of government debt (1) becomes

$$\sum_j \mathcal{E}_{j,t} B^1_{j,t-1} = P^s_t S_t + \sum_j \mathcal{E}_{j,t} \sum_{n=2}^{\infty} Q^{n-1}_{j,t} \left(B^{n-1}_{j,t} - B^n_{j,t-1} \right),$$

where S_t is now the real primary surplus and $B^n_{j,t}$ is the face value of bonds issued in currency j, period t, payable n periods in the future. The term on the left represents the cost of debt in period t; the second term on the right represents proceeds from the selling of new bonds.

Let $\mathcal{V}_{j,t} = \sum_n Q^n_{j,t} B^n_{j,t}$ be the end-of-period market value of nominal debt issued in currency j, $i_{j,t}$ the risk-free rate in bonds issued in currency j and $rx_{j,t} = \mathcal{V}^{-1}_{j,t-1} \sum_{n=1}^{\infty} Q^{n-1}_{j,t} B^n_{j,t-1} - i_{j,t-1}$

the realized excess return on portfolios that mimic the composition of j-currency debt. We can re-write the law of motion in terms of the $\mathcal{V}_j$ and its corresponding returns:

$$\sum_j (1 + rx_{j,t} + i_{j,t-1})\mathcal{E}_{j,t}\mathcal{V}_{j,t-1} = P_t^s S_t + \sum_j \mathcal{E}_{j,t}\mathcal{V}_{j,t}$$

For empirical work, it is ideal to write the law of motion in terms of real and stationary variables. So, let Y_t be real GDP and let $g_t = Y_t/Y_{t-1} - 1$ be its growth rate. Define the real "exchange rate" $H_{j,t} = \mathcal{E}_{j,t}P_{j,t}/P_t$ and its growth rate $\Delta h_{j,t} = H_{j,t}/H_{j,t-1} - 1$. Define the detrended real value of j-indexed debt $V_{j,t} = \mathcal{V}_{j,t}/P_{j,t}Y_t$, the real value of total debt $V_t = \sum_j H_{j,t}V_{j,t}$ and the j-indexed share $\delta_{j,t} = H_{j,t}V_{j,t}/V_t$.

By properly dividing the whole above equation by P_tY_t, and multiplying and dividing the j sum on the left by $P_{j,t-1}$, $P_{j,t}$, Y_{t-1} and $H_{j,t-1}$, we arrive at a final version for the law of motion of public debt:

$$V_{t-1} \sum_j \frac{(1 + rx_{j,t} + i_{j,t-1})(1 + \Delta h_{j,t})}{(1 + \pi_{j,t})(1 + g_{Y,t})} \delta_{j,t-1} = \frac{P_t^s}{P_t} s_t + V_t.$$

The law of motion above generalizes (2) for $k = 1$. During period t, the government must "pay" V_{t-1} plus realized returns and eventual changes to the relative value of currency j.[4]

I linearize the debt law of motion. Let $\beta_j = \exp\{-(rx_j + i_j - \pi_j - g)\}$ be the steady-state real and growth-adjusted discounting for public debt issued in currency j. I linearize around a steady-state - assumed to exist - with $\beta_j = \beta$ for all j and $P^s = P$. This leads to

$$\beta \left(v_t + s_t + s(p_t^s - p_t)\right) = v_{t-1} + v \left[\sum_j \delta_j \left(rx_{j,t} + i_{j,t-1} + \Delta h_{j,t} - \pi_{j,t}\right) - g_t \right], \qquad (7)$$

which generalizes (6). Parameter v is the steady-state level of public debt.

2.3.2 Geometric Term Structure

I assume the government keeps a geometric maturity structure of its debt. The geometric term structure implies a tractable relationship between short-term interest rate and the excess return on public bonds.

The term structure is constant over time, but can vary across the different currency portfolios $\{j\}$ of public debt. Specifically, for the slice of public debt linked to currency j, suppose the outstanding volume of bonds decays at a rate ω_j, so that $B_{j,t}^n = \omega_j B_{j,t}^{n-1}$. Define $Q_{j,t} = \sum_{n=1}^{\infty} Q_{j,t}^n \omega_j^{n-1}$ as the weighted-average price of currency j public bonds. Then, $\mathcal{V}_{j,t} = Q_{j,t}B_{j,t}^1$. The total return on currency-j bonds then is $1 + rx_{j,t} + i_{j,t-1} = (1 + \omega_j Q_{j,t})/Q_{j,t-1}$, which I linearize as

$$rx_{j,t} + i_{j,t-1} = \omega_j \beta_j q_{j,t} - q_{j,t-1} \qquad (8)$$

[4] "Pay" comes in paranthesis here because, unlike in (1), the government does not actually redeem the entire term on the left at period t. It only pays for bonds maturing at t.

where $q = \log Q$ and I use the log approximation of level returns to effectively re-define $rx_j + i_j$.

Equation (8) above defines the excess return on holdings of the j-currency portfolio of public debt. Given a model for the risk premium $E_t rx_{j,t+1}$, it also defines the price of the debt portfolio as a function of short-term interest:

$$\begin{aligned} q_{j,t} &= \omega_j \beta_j E_t q_{j,t+1} - E_t rx_{j,t+1} - i_{j,t} \\ &= -\sum_{i=0}^{\infty} (\omega_j \beta_j)^i E_t \left[rx_{j,t+1+i} + i_{j,t+i} \right]. \end{aligned} \tag{9}$$

The second equation in (9) which clarifies the connection between short-term interest and returns on the market price of debt showing up in (7). Given news of, say, higher interest rates, the discount of public bond increases, and q falls. Equation (8) then prescribes a low excess return on j debt.

3 Empirical Model and Estimation

3.1 Public Finance Model

Governments typically report the par or book value of debt, and this is the data I gather. Because theory is based on the *market* value of debt, some adjustment is necessary. The Dallas Fed provides estimates of the market value of debt for the United States following Cox and Hirschhorn (1983) and Cox (1985).[5] I follow a similar methodology.

Let $\mathcal{V}^b_{j,t}$ be the par value of the j-currency portfolio debt, and let $i^b_{j,t}$ be its average yield-to-maturity by which book values are updated from period to period. The adjustment equation is

$$\mathcal{V}_{j,t} = \mathcal{V}^b_{j,t} \times \frac{\text{market value of bonds}}{\text{book value of bonds}} = \frac{1 + \omega_j Q_{j,t}}{(1 + i^b_{j,t-1}) Q_{j,t-1}} = \frac{1 + rx_{j,t} + i_{j,t-1}}{1 + i^b_{t-1}}.$$

I detrend the $\mathcal{V}$'s, convert to real, sum across portfolios and linearize to arrive at:

$$v_t = v^b_t + \frac{v}{b} \left[\sum_j \delta_j \left(rx_{j,t} + i_{j,t-1} - i^b_{j,t-1} \right) \right]. \tag{10}$$

Estimates of the VAR provide an equation for the law of motion of par-value debt. I use (10) to infer a law-of-motion of market-value debt.

The average interest $i^b_{j,t}$ is not observed, so we cannot estimate an equation for it. Instead,

[5] Web link: https://www.dallasfed.org/research/econdata/govdebt.

I use the following model:

$$i^b_{j,t} = (1-\omega_j)\sum_{k=0}^{\infty}\omega_j^k i_{j,t-k} = (1-\omega)i_{j,t} + \omega_j i^b_{j,t-1} \tag{11}$$

for $j \in \{N, R, D\}$.

3.2 The Bayesian-VAR

3.2.1 Empirical Model

I gather data for a set of twenty-eight economies, and estimate a ten-equation VAR in which the debt law of motion (7) holds by construction. From the ten variables in the VAR, five are observed: the nominal interest (i_t), the inflation rate (π_t), par-value public debt (v^b), the real exchange rate to the dollar (Δh) and GDP growth (g). I select these variables based on (7).

I use annual data. Although quarterly data is available, for most countries it does not go farther than a few years back - particularly developing countries. With a focus on long-term debt sustainability, going further into the past is more important than accounting for quarterly dynamics. Additionaly, we do not have to bother with seasonality adjustments.

Data period range changes from country to country (table), with the longest panel starting in 1970 (United States), and the shortest beginning in 2000 (Russia). For some emerging economies, I arbitrarily trim data to remove periods of hyperinflation, always in the 1990s.

Inflation is the log variation in the consumer price index. The dollar real exchange rate is the nominal exchange rate to the US dollar multiplied by the ratio of US-to-domestic CPI. The nominal interest rate is the log of 1+ interest data. GDP growth is in log too. Public debt data is provided by ratio of GDP by the source, and requires no transformation.[6]

With the exception of exchange rate depreciation, I demean all series. This is to assume that the stationary steady-state (around which I linearize) is given by the sample mean of the series of each country (with the exception of real exchange rate growth, which I assume to be zero).

The functional format of the VAR is

$$x_t = ax_{t-1} + bu_{t-1} + k\, e_t. \tag{12}$$

Both x and u are vectors with ten entries. Five of them are the observed variables enumerated above.

Vector u_t groups the same set of variables as x, but for the United States. I often use the "u" notation to refer to the US case. Because the public debt process of each country has a

[6]Real GDP (constant 2015 prices), GDP deflator, public spending and the nominal exchange rate data come from the United Nations's National Accounts Main Aggregates Database. Consumer price index and and primary surplus data come from the IMF's WEO Database. Public debt (as ratio of GDP) comes Ali Abbas et al. (2011) database, which is kept up-to-date. The sources for interest rate vary from country to country; they are usually the central bank, but also from the IMF's International Financial Statistics database. Appendix **??** provides further details.

dollar component, and hence depends on dollar interest and inflation, u and ε_u enter the regression of all countries.

There are five shocks hitting the domestic economy: $\varepsilon \sim N(0, \Sigma)$, independent over time. The shocks hitting the US economy are $\varepsilon_u \sim N(0, \Sigma_u)$. I group them in $e_t = [\varepsilon_t' \; \varepsilon_{u,t}']'$. Matrix $k_{10\times 10}$ serves to properly reproduce the law of motion of unobserved variables.

The model for US variables is:

$$u_t = a_u u_{t-1} + k_u \, \varepsilon_{u,t} \tag{13}$$

(I use the same notation x to the VAR of all countries and differentiate only in the US case). In (13), k_u is a 10×5 matrix.

3.2.2 Debt Law and Excess Return Adjustments

Cochrane (2022) imposes the law of motion of public debt in the VAR by using primary surplus "data" calculated from it. Primary deficit equals the change in public debt net of realized return. This procedure bypasses the problem of dealing with accounting conventions in primary surplus reporting that, in most cases, make its data inconsistent with that of public debt. Since the OLS estimator is linear, his VAR satisfies the debt law of motion by construction.

In my VAR, the procedure needs to be adjusted. First, I do not measure excess return, or real interest rates. Second, the debt equation depends on US interest and inflation. Hence, the unrestrictive estimation of (12) spuriously projects these two US variables on domestic ones, which is inconsistent with (13). For these reasons, instead of measuring implied surpluses and then estimating the model, I estimate the model and then fill the equation for primary surpluses that ensures the debt law of motion (7) holds. Before doing that, I also need to include the adjustment equation for market-value debt (10) (the estimated equation is for par-value, not market-value debt!) as well as the three definitions of average interest rates (11) required to do it. These five unobserved variables (surplus s_t, market-value debt v_t, and the average interest $i^b_{j,t}$) complete the ten variables of the VAR.

Note that the estimated equation for par-value public debt represents its law of motion *after* replacing the equation determining tax proceeds, or its equilibrium law of motion.

The estimation has three steps.

Step 1. I estimate the VAR

$$\begin{aligned} \tilde{x}_t &= \tilde{a}(L)\tilde{x}_{t-1} + \tilde{b}(L)\tilde{u}_{t-1} + \varepsilon_t \\ \tilde{u}_t &= \tilde{a}_u(L)\tilde{u}_{t-1} + \varepsilon_{u,t} \end{aligned} \tag{14}$$

where $\tilde{x}$ is a vector with all variables in x except tax revenues, and $\tilde{u}$ is defined similarly. Matrix coefficients $\tilde{a}$, $\tilde{b}$ and $\tilde{a}_u$ exclude the row and column corresponding to tax revenue. I proceed under the assumption that the loadings of all variables in the VAR on previous tax revenue equal zero.

Step 2. I assume a constant risk-premium: $E_t rx_{j,t+1} = 0$ for all j. Then, I use the estimates of (14) to compute $E_t i_{j,t+i}$ and apply (9) to compute $q_{j,t}$. Equation (8) then yields expressions for excess return of the form

$$rx_{j,t} = \varphi_j' e_t.$$

In the case of real debt, I use

$$i_{R,t} = i_{N,t} - E_t \pi_{N,t+1} = \zeta(L)' X_t.$$

where $X_t = [x_t' \; u_t']'$ stacks domestic and US variables. In the United States case, $X_t = u_t$. In appendix **??**, I present the formulas for the φ's and $\zeta(L)$.

Step 3. I use the debt law of motion (7) and estimates of φ and ζ to compute the equation for tax revenue and its residual as a function of all the other residuals. This completes the estimation of (12) and (13) which we can then stack into a single system for X_t:

$$X_t = A(L) X_{t-1} + K e_t. \tag{15}$$

More explicitly (and ordering tax revenues at the top of the VAR):

$$\begin{pmatrix} x_t \\ u_t \end{pmatrix} = \begin{bmatrix} a(L) & b(L) \\ 0 & a_u(L) \end{bmatrix} \begin{pmatrix} x_{t-1} \\ u_{t-1} \end{pmatrix} + \begin{bmatrix} & k \\ 0 & k_u \end{bmatrix} \begin{pmatrix} \varepsilon_t \\ \varepsilon_{u,t} \end{pmatrix}$$

$$\text{or yet} \begin{pmatrix} T_t \\ \tilde{x}_t \\ T_{u,t} \\ \tilde{u}_t \end{pmatrix} = \begin{bmatrix} 0 & & (7) & \\ 0 & \tilde{a}(L) & 0 & \tilde{b}(L) \\ 0 & 0 & 0 & (7) \\ 0 & 0 & 0 & \tilde{a}_u(L) \end{bmatrix} \begin{pmatrix} T_{t-1} \\ \tilde{x}_{t-1} \\ T_{u,t-1} \\ \tilde{u}_{t-1} \end{pmatrix} + \begin{bmatrix} & (7) \\ I & 0 \\ 0 & (7) \\ 0 & I \end{bmatrix} \begin{pmatrix} \varepsilon_t \\ \varepsilon_{u,t} \end{pmatrix}.$$

In the last equation, I use symbol (7) to indicate that the coefficients are those implied by the debt law of motion. In appendix **??**, I provide their formulas.

3.3 The Minnesota Prior

4 Empirical Results

5 New-Keynesian Model Benchmarks

6 Robustness

7 Conclusion

References

Ali Abbas, S. M., Belhocine, N., El-Ganainy, A., and Horton, M. (2011). Historical Patterns and Dynamics of Public Debt—Evidence From a New Database. *IMF Economic Review*, 59(4):717–742.

Cochrane, J. H. (2005). Money as stock. *Journal of Monetary Economics*, 52(3):501–528.

Cochrane, J. H. (2022). The fiscal roots of inflation. *Review of Economic Dynamics*, 45:22–40.

Cox, W. M. (1985). The behavior of treasury securities monthly, 1942–1984. *Journal of Monetary Economics*, 16(2):227–250.

Cox, W. M. and Hirschhorn, E. (1983). The market value of U.S. government debt; Monthly, 1942–1980. *Journal of Monetary Economics*, 11(2):261–272.