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Fancy title

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Introduction

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Background

Computer algebra.

Computers have fundamentally two ways to reason over a mathematical expression: **numerical computations**, which are performed using *only numbers* to represent values and **computer algebra** (or **symbolic computations**), which - by contrast - use *both numbers and symbols*.

First, we shall introduce the concept of **floating point number system** used to handle numerical computations.

Definition 2.1 (Normalized-floating point number system)**.** A normalized-floating point number system F is characterized by the 4-tuple of integers β, p, L, U:

- β is called base or radix,
- p precision,
- $[L, U]$ exponent range (with L, U denoting lower and upper bound respectively).

Given a number $x \in \mathbb{R}$, $x \neq 0$ its representation in a floating point number system shall be written out as $fl(x)$ and has the form

$$x = sign(x)\beta^E \sum_{i=0}^{p-1} d_i \beta^{-i}$$

with $L \leq E \leq U$ and the sequence $\{d_i\}$ (which is called mantissa) made up of natural numbers such that $d_0 \neq 0$, $0 \leq d_i \leq \beta - 1$ and d_i eventually different from $\beta - 1$.

Remark. A floating point number system F is discrete and finite: it approximates real numbers with finite numbers.

A de facto standard for computers to work with floating point approximations is IEEE 754 [2], the details of which shall not be discussed.

Definition 2.2 (Machine epsilon)**.** Machine epsilon is the maximum possible absolute relative error in representing a nonzero real number x in a floating point number system

$$\epsilon_{mach} = \max_x \frac{|x - fl(x)|}{|x|}.$$

Example 2.1. Let us define the matrix (made up of both symbols and numbers) M

$$\begin{bmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{bmatrix}.$$

Consider the matrix $\tilde{M}$, having as entries the floating point approximation of those of M

$$\begin{bmatrix} fl(\sqrt{2}) & 1 \\ 2 & fl(\sqrt{2}) \end{bmatrix}.$$

Computing its determinant gives out $2 + 2\epsilon\sqrt{2} + \epsilon^2 - 2 \doteq 2 + 2\epsilon\sqrt{2} - 2 \neq 0$.

Introducing a small change (i.e. an "error") in the input argument may either cause a large or a small change in the result. We shall now introduce the concept of condition numbers.

Definition 2.3 (Condition number)**.** A condition number of a problem measures the sensitivity of the solution to small perturbations in the input data. Given a function f, we define

$$cond(f, x) = \lim_{\epsilon \to 0} \sup_{\|\Delta x\| \leq \epsilon \|x\|} \frac{\left\| f(x + \Delta x) - f(x) \right\|}{\epsilon \left\| f(x) \right\|}.$$

Given a problem, if its condition number is low it is said to be **well-conditioned** (typically $cond(f, x) \sim 1$), while a problem with a high condition number is (said to be) **ill-conditioned** ($cond(f, x) \gg 1$).

Definition 2.4 (Condition number of a matrix)**.** The condition number of a non-singular matrix A is defined as:

$$\kappa(A) = \left\| A^{-1} \right\| \times \|A\|.$$

Let us now investigate what would happen if symbols are allowed by introducing a framework that allows us to work both with numerical and symbolic computations.

Definition 2.5 (Computer algebra system)**.** A computer algebra system (CAS) is a mathematics software package that can perform *both symbolic and numerical mathematical computations.*

A CAS is usually a **REPL** expected to support a few functionalities [3]:

- **Arithmetic**: arithmetic over different fields with arbitrary precision.

- **Linear algebra**: matrix algebra and knowledge of different operations and properties of matrices (i.e. determinants, eigenvalues and eigenvectors).
- **Polynomial manipulation**: factorization over different fields, simplification and partial fraction decomposition of rational functions.
- **Transcendental functions**: support for transcendental functions and their properties.
- **Calculus**: limits, derivatives, integration and expansions of functions.
- **Solving equations**: solving systems of linear equations, computing with radicals solutions of polynomials of degree less than five.
- **Programming language**: users may implement their own algorithms using a programming language.

The CAS chosen for this work is **SageMath** [5], the features and functionalities of which shall not be discussed here.

SageMath is an open source CAS distributed under the terms of the GNU GPLv3 [1].

Hereafter, an example in which symbolic computations are put against numerical (computations) shall be made.

Example 2.2. Take matrix M from Example 2.1:

$$\begin{bmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{bmatrix}.$$

Compare the different results given out when computing its determinant by defining M over the *symbolic ring SR* and the *finite-precision ring CDF*:

```
sage: matrix(SR, [[sqrt(2), 1], [2, sqrt(2)]]).det()
0
sage: matrix(CDF, [[sqrt(2), 1], [2, sqrt(2)]]).det()
-3.14018491736755e-16
```

We can observe that in SR $(\sqrt{2})^2 = 2$ since no approximations are made.

Now, take the polynomial $p(x)$:

$$p(x) = x^6 + 5\,x^5 - 3\,x^4 - 42\,x^3 + 12\,x^2 - x + 1.$$

If an attempt to calculate its roots over SR is made, an exception will be thrown; however, doing this over a finite-precision ring (such as CDF) will work:

```
sage: p = x^6 + 5*x^5 - 3*x^4 -42*x^3 + 12*x^2 - x + 1
sage: p.roots(ring=SR)
    RuntimeError: no explicit roots found
sage: p.roots(ring=CDF)
[(-3.865705050148171 - 1.5654017866113432*I, 1),
(-3.8657050501481702 + 1.5654017866113419*I, 1),
(-0.04843174828928114 - 0.2430512799158686*I, 1),
(-0.048431748289281144 + 0.24305127991586856*I, 1),
(0.38275295887213723 + 7.286537374692244e-17*I, 1),
(2.4455206380027437 - 1.995314986816126e-16*I, 1)]
```

For deeper reasoning about the limits of computer algebra systems, one may refer to Mitic [4].

Eigenvalues, eigenvectors

In the following section, eigenvalues and eigenvectors shall be defined.

Lastly, a result on the condition number of the problem of computing eigenvalues of a matrix shall be given.

Definition 2.6 (Eigenvalue, eigenvector)**.** Given a linear transformation T in a finite-dimensional vector space V over a field F into itself and a nonzero vector $\mathbf{v}$, $\mathbf{v}$ is an eigenvector of T if and only if

$$A\mathbf{u} = \lambda\mathbf{u}$$

with A the matrix representation of T, $\mathbf{u}$ the coordinate vector of $\mathbf{u}$ and λ a scalar in F known as eigenvalue associated with $\mathbf{v}$.

Similarly, we can define a row vector $\mathbf{x}_L$, and a scalar λ_L such that

$$\mathbf{x}_L A = \lambda_L A,$$

which are called left eigenvector and left eigenvalue respectively.

Remark. Note that writing $A\mathbf{u} = \lambda\mathbf{u}$ is equivalent to $(A - \lambda I)\mathbf{u} = 0$.

It follows that the eigenvalues of A are the roots of

$$det(A - \lambda I)$$

which is a polynomial in λ known as the **characteristic polynomial** $ch(A)$.

Definition 2.7 (Eigenspace)**.** Given a square matrix A and its eigenvalue λ, we define the eigenspace of A associated with λ the subspace E_A of all vectors satisfying the equation

$$E_A = \{\mathbf{u} : (A - \lambda I)\mathbf{u} = 0\} = ker(A - \lambda I).$$

Remark. Suppose A is a real square matrix, then the following statements are true:

- the eigenvalues of the left and right eigenvectors of A are the same,
- the left eigenvectors simplify into the transpose of the right eigenvectors of A^T.

Definition 2.8 (Algebraic, geometric multiplicities of eigenvalues)**.** Given a square matrix A and a scalar $\lambda \in \mathbb{C}$: we define the algebraic multiplicity of λ as

$$m_A(\lambda) = max\{k : (\exists s(x) : s(x)(x - \lambda)^k = ch_A(x))\}.$$

The geometric multiplicity of λ is defined as

$$\nu_A(\lambda) = dim(ker(A - \lambda I)).$$

Let us now investigate how introducing perturbations in the representation of a matrix may influence the numerical stability of its eigenvalues (caveat: in the following paragraph, the notation δx shall be used to denote the difference between a symbol x and its floating point approximation $fl(x)$).

Let us define a square matrix A and its eigenvalue $\lambda \in \mathbb{C}$, $\mathbf{x}$, $\mathbf{y}$ the right and left eigenvectors associated with λ.

Consider the perturbed problem

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\lambda}\tilde{\mathbf{x}}$$

with ϵ the machine epsilon, $\tilde{A} = A + \epsilon\delta A$, $\tilde{\mathbf{x}} = \mathbf{x} + \epsilon\delta\mathbf{x}$, $\tilde{\lambda} = \lambda + \epsilon\delta\lambda$.

Differentiating w.r.t. ϵ and multiplying by $\mathbf{y}^T$ on the left side gives

$$\mathbf{y}^T\delta A\mathbf{x} + \mathbf{y}^T A fl(\mathbf{x}) = fl(\lambda)\mathbf{y}^T\mathbf{x} + \mathbf{y}^T\lambda fl(\mathbf{x})$$

and, since $\mathbf{y}$ is the left eigenvector we can rewrite it as

$$\frac{\delta\lambda}{\delta\epsilon} = \frac{\mathbf{y}^T\delta A\mathbf{x}}{\mathbf{y}^T\mathbf{x}}.$$

Assuming the absolute error $\|\delta A\| = 1$ and using the definition of cross product for $\mathbf{y}^T\mathbf{x}$ we get

$$|\delta\lambda| \leq \frac{1}{|cos(\theta_\lambda)|}|\delta\epsilon|.$$

Definition 2.9 (Condition number of an eigenvalue)**.** Given a square matrix A, the eigenvalue $\lambda \in \mathbb{C}$ and θ_λ the angle between the left and right eigenvectors associated with λ, the quantity

$$k_A(\lambda) = \frac{1}{cos(\theta_\lambda)}$$

is called the condition number of the eigenvalue λ.

Jordan canonical form

Definition 2.10 (Generalized eigenvector, generalized eigenspace)**.** Given an $n \times n$ square matrix A and its eigenvalue λ, the vector $\mathbf{x}_m$ such that

$$(A - \lambda I)^m \mathbf{x}_m = 0 \ \wedge \ (A - \lambda I)^m \mathbf{x}_m \neq 0$$

is a generalized eigenvector of rank m associated with λ.

Furthermore, the subspace E_A^m

$$E_A^m(\lambda) = ker((A - \lambda I)^m)$$

is called generalized eigenspace; its dimension

$$\nu_A^m(\lambda) = dim(E_A^m(\lambda)) = n - rank((A - \lambda I)^m)$$

is called the m-th geometric multiplicity of λ in A.

Definition 2.11 (Defective matrix, defective eigenvalue)**.** Given a square $n \times n$ matrix A, if it has less than n distinct eigenvalues then it is called a defective matrix.

Furthermore, we define an eigenvalue λ of such a matrix as a defective eigenvalue if and only if

$$m_A(\lambda) > \nu_A(\lambda).$$

Example 2.3. Consider the matrix M

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

there is only one eigenvalue λ associated with M and $\lambda = 2$; the eigenvector associated with it is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Furthermore, note that λ has algebraic multiplicity $m_A(2) = 2$: it follows that M is a defective matrix and λ is a defective eigenvalue.

Now, we shall compute its generalized eigenvectors.

Note that $dim(ker(A-\lambda I)) = p = 1$, which implies there exist $m-p = 1$ generalized eigenvectors of rank greater than 1. To compute the generalized eigenvector $\mathbf{x}_2$ we solve $(A - \lambda I)\mathbf{x}_2 = \mathbf{x}_1$

$$M - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{20} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Substituting with their values gives us

$$\begin{bmatrix} 2 - 2 * 1 & 1 \\ 0 & 2 - 2 * 1 \end{bmatrix} \begin{bmatrix} x_{20} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving this system we can conclude

$$\mathbf{x}_2 = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

with no restrictions over the value of the scalar t.

Definition 2.12 (Jordan matrix)**.** A diagonal block matrix M is called a Jordan matrix if and only if each block along the diagonal is of the form

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 1 & \lambda \end{bmatrix},$$

and we indicate such a matrix with $diag(J_{\lambda_1,n_1}, ..., J_{\lambda_k,n_k})$ with k the number of diagonal blocks it is made up of.

Each block can be completely described by the tuple (λ, n) as it is an $n \times n$ matrix of zeroes everywhere except for the diagonal, which is filled with λ, and the superdiagonal, with ones.

Theorem 2.1 (Jordan canonical form)**.**

Theorem 2.2 (Stability of the JCF transformation)**.** Given a matrix A and its JCF $A = P^{-1}JP$, the transforming matrix P is highly ill-conditioned whenever A has at least a defective or nearly defective eigenvalue.

Proof. Work in progress... $\square$

Kronecker canonical form

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Conclusions

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