



UNIVERSITÀ DI PISA

Dipartimento di Informatica
Corso di Laurea in Informatica

Fancy title

Relatore
Prof. Federico Poloni

Candidato
Giacomo Trapani

Anno Accademico 2021/2022

Contents

1	Introduction	2
2	Background	6
3	Kronecker canonical form	15
4	Implementation	17
5	Conclusions	19

Introduction

Tempor amet nostrud ex aliquip adipisicing aliqua. Laboris in laboris cillum anim. Mollit ea ut ad sit ut exercitation culpa. Enim reprehenderit Lorem id quis anim. Est commodo pariatur pariatur consequat labore.

Ea incididunt laboris labore aliquip ipsum sunt labore velit labore minim aliquip elit esse. Ad anim et voluptate ea veniam ipsum ea laborum. Quis laboris reprehenderit duis proident culpa Lorem. Ea commodo ea et proident incididunt laborum voluptate. Magna sunt id occaecat sit laboris commodo sint quis occaecat cupidatat qui duis.

Sit fugiat ea dolor est ut ullamco sint dolore irure aliqua. Minim sunt nostrud quis ipsum duis aute consequat aliquip voluptate voluptate ullamco adipisicing. Tempor Lorem commodo sunt nisi proident irure veniam.

Veniam reprehenderit nulla aliquip laboris non culpa velit sint. Aute qui dolore consequat commodo ullamco. Eu elit eiusmod qui enim reprehenderit tempor occaecat ut officia do occaecat dolor anim. Labore occaecat sit do labore minim ut. Magna proident labore magna id ullamco irure aliqua id. In incididunt occaecat ullamco ut eu dolor mollit ad dolore anim.

Ut nostrud qui pariatur nostrud excepteur veniam elit Lorem ut do eiusmod quis duis exercitation. Deserunt occaecat tempor exercitation in minim eu fugiat. Aliqua ex sit qui deserunt dolore consequat esse amet anim occaecat nisi. Officia anim pariatur veniam officia.

Ea ut esse sit non. Ipsum sit cillum cillum cillum ex adipisicing eu sint laboris dolore id. Sit ipsum et in nulla ut eiusmod cillum pariatur quis labore. Et reprehenderit enim quis Lorem exercitation consequat. Aute irure officia anim est elit aute veniam ut nostrud ut quis minim reprehenderit. Duis Lorem esse nostrud duis.

Id proident ad commodo fugiat qui proident dolore dolor adipisicing dolore duis velit. Cillum proident Lorem fugiat ut laborum ipsum officia ex sit. Est nulla velit adipisicing exercitation exercitation aliqua veniam eu do excepteur qui incididunt adipisicing. Ea amet id reprehenderit duis tempor duis irure dolor et labore consequat veniam. Adipisicing id aliqua dolor excepteur elit commodo adipisicing.

Mollit est anim adipisicing excepteur nisi aliquip quis aliquip mollit. Cillum excepteur eu aliqua cupidatat est laborum consectetur. Anim nisi veniam sint nulla magna ad sit nulla velit eu eu culpa nostrud. Ipsum veniam duis eiusmod et aute reprehenderit sunt commodo voluptate do.

Reprehenderit deserunt ex qui sit reprehenderit fugiat pariatur nisi deserunt ut ad do do. Nisi pariatur amet ipsum nisi labore dolor anim enim

veniam. Excepteur ullamco voluptate nisi dolore tempor tempor.

Consequat enim nisi irure non exercitation dolore officia. Est ex sint irure dolor qui nisi. Nostrud sint ex et enim aute consectetur deserunt nisi eu. Velit commodo deserunt occaecat nostrud ullamco voluptate amet consectetur dui dolore sunt culpa sunt aliquip. Lorem sit deserunt commodo sunt.

Eiusmod deserunt cillum fugiat incididunt sit laborum anim ea. Aute sit proident enim sint elit ex. Aute enim amet ad nisi aliqua exercitation dolor reprehenderit dui elit ad. Ullamco qui do est voluptate incididunt nisi qui mollit cupidatat excepteur. Duis do aliqua occaecat ex culpa proident nostrud. Reprehenderit incididunt id adipisicing do dolor. Irure sit irure laboris non ea mollit laborum.

Duis reprehenderit irure ipsum voluptate sint velit. Do nisi laboris aliqua amet voluptate do cupidatat nulla elit amet deserunt ad sint. Laboris proident aute deserunt amet amet veniam eiusmod laboris.

In do do quis in irure commodo ut cillum ad. Ullamco aute do cillum sint in consectetur tempor sit laborum. Aute eiusmod cupidatat est nulla anim deserunt nostrud ullamco occaecat exercitation sit magna anim. Dolor veniam tempor dolore commodo tempor sit quis. Consequat adipisicing aliquip minim labore aliquip ea eiusmod dolor sint veniam sit consectetur reprehenderit. Officia labore Lorem sint velit ipsum laboris id ut dolore sint.

Veniam laborum deserunt in magna non. Eiusmod ex nulla nostrud ut sint ullamco commodo nostrud ea qui. Sunt ad do incididunt culpa fugiat dolor nostrud consectetur non veniam elit officia sunt occaecat. Ullamco ullamco ut ea elit labore. Ullamco irure officia velit pariatur aliquip in enim.

Esse cupidatat est esse nulla ea est pariatur ad velit voluptate dolor sit. Cupidatat eu tempor ad sunt est. Commodum ipsum do enim dui sunt ea voluptate Lorem fugiat deserunt tempor consectetur commodo. Ea minim officia amet sint. Magna exercitation proident officia magna veniam cillum Lorem occaecat esse velit. Aliqua Lorem Lorem eiusmod minim commodo nostrud minim dolor et dolore dui qui aute. Minim nisi consequat eiusmod consequat Lorem laboris consequat non ut laboris quis id dolor dolore.

Eiusmod cillum pariatur est culpa nulla mollit magna voluptate amet in quis id mollit. Ad cupidatat id dolor ad ex adipisicing laborum pariatur nostrud nisi incididunt. Mollit sunt aliqua adipisicing enim ipsum sint eiusmod dolore magna qui reprehenderit proident. Et veniam id nisi dolor occaecat est labore qui sit non enim. Esse nisi voluptate et mollit dolor ad.

Cupidatat mollit do veniam quis quis minim eiusmod sint labore dui

commodo sunt esse nostrud. Culpa excepteur culpa sint ipsum irure cillum deserunt pariatur enim labore. Id velit in irure ex nisi eu ea commodo irure ad ullamco et excepteur. Eu pariatur deserunt duis ullamco qui Lorem dolore reprehenderit velit id excepteur. Non labore eiusmod amet aute consequat magna ut occaecat. Laboris nisi excepteur cillum sit sunt non. Cillum amet voluptate nostrud magna commodo cillum cillum.

Proident veniam enim eiusmod et mollit consectetur incididunt anim. Cillum incididunt quis ad duis excepteur amet est aute commodo velit est incididunt. Ea irure labore anim consectetur officia consequat officia culpa sit minim id commodo. Deserunt qui tempor laborum excepteur nisi magna et pariatur.

Anim adipisicing laboris commodo in qui labore laboris exercitation exercitation exercitation et dolore amet. Laborum qui laboris officia Lorem eu est do. Consequat tempor dolor cillum qui dolore. Sint labore dolor eu ipsum mollit duis excepteur veniam aute anim labore aliqua. Nisi cupidatat ex officia ad.

Laborum reprehenderit sint cupidatat fugiat minim. Consectetur adipisicing elit proident exercitation consectetur exercitation consectetur voluptate veniam sit velit consectetur cupidatat. Velit do cillum consectetur cupidatat labore esse ad adipisicing sunt minim quis elit ad. Id et fugiat enim deserunt.

Commodo velit irure non irure. Fugiat duis nulla enim enim ut consectetur id. Esse proident anim laboris labore amet.

Consectetur nulla ullamco tempor amet nisi sit ad deserunt fugiat mollit. Dolor exercitation et nulla aliqua pariatur occaecat fugiat velit consequat. Aliquip minim enim ut deserunt nostrud. Cupidatat laboris pariatur enim Lorem dolore. Sint eu esse fugiat deserunt nisi cupidatat proident irure consequat esse aliquip culpa adipisicing proident. Duis et aliquip magna ipsum sint pariatur dolore ut aliquip. In enim enim ex est ut ipsum ea nostrud tempor qui.

Cillum quis irure Lorem esse commodo sint velit et eu ea culpa. Eu adipisicing quis laborum ex minim commodo cillum incididunt exercitation excepteur eiusmod anim. Occaecat amet magna irure ea minim nostrud. Qui mollit deserunt irure tempor anim et sit ut fugiat ullamco pariatur non. Irure eiusmod elit commodo aliquip id commodo. Non voluptate culpa laborum Lorem elit tempor consequat quis.

Eiusmod quis eu pariatur eu. Sunt fugiat adipisicing cillum laborum magna fugiat excepteur sit commodo et enim. Cupidatat reprehenderit exercitation aliqua proident eiusmod in sunt esse minim elit non Lorem com-

modo enim. In eu aliqua quis officia Lorem do ut sunt commodo aliqua velit eiusmod culpa. Fugiat est excepteur consequat adipisicing voluptate et reprehenderit. Adipisicing deserunt exercitation tempor mollit qui aliqua est dolore aute.

Exercitation aliqua dolore dolor voluptate pariatur esse ex qui non. Excepteur qui qui nisi do deserunt anim laboris pariatur quis in dolor dolor eu. Consequat aliqua culpa adipisicing occaecat adipisicing velit tempor eiusmod enim officia nulla. Lorem enim et irure esse culpa sunt. In qui adipisicing ad laborum enim nulla ex aliquip consectetur est mollit amet et nisi.

Quis cupidatat labore qui et aliqua ullamco veniam enim qui voluptate. Dolore laboris in fugiat non sint. Aliqua sunt quis dolore nostrud proident ex.

In dolore sint proident duis consequat eu nisi eu labore occaecat culpa minim sint. Aute nulla irure irure deserunt laborum veniam aliquip ad veniam anim pariatur ex elit ipsum. Cupidatat qui aute do mollit. Enim duis nostrud anim ipsum ad consectetur do ea magna ea deserunt sunt. Do irure occaecat amet non ut dolore eu ea pariatur est proident et aliqua pariatur. Cillum laboris dolor aliqua fugiat culpa do. Velit ad proident velit minim quis ipsum dolore nostrud voluptate tempor.

Commodo sunt officia eu in anim culpa. Eiusmod commodo proident et aliqua in ex culpa laborum qui amet. Laboris cupidatat incididunt elit ipsum irure velit quis tempor irure duis. Adipisicing veniam duis proident cupidatat minim. Non ullamco sint incididunt duis qui eu. Sunt eu id sint pariatur incididunt anim enim officia adipisicing in do anim.

Nostrud qui excepteur laboris veniam labore eu sunt deserunt magna ea fugiat aute. Ut ex et labore culpa ea mollit incididunt ullamco Lorem fugiat. Nulla proident fugiat sunt et pariatur ipsum anim nostrud quis nisi qui pariatur. Tempor dolore ipsum tempor aliquip nostrud. Do nostrud esse enim amet qui anim consequat aute. Nisi officia ad sint ut proident proident. Minim labore ut elit id consequat sit ea.

Background

This chapter will serve as prerequisite knowledge throughout the rest of this thesis.

We shall briefly present SageMath, the software system used when implementing the algorithm to be discussed, by introducing computer algebra systems and comparing numerical computations against computer algebra showcasing an example; then, the reader shall familiarize with the concept of condition number as an emphasis on it will be put in every part.

Subsequently, definitions and properties of eigenvalues and eigenvectors shall be concisely introduced.

Lastly, we shall describe the Jordan canonical form of a matrix.

Computer algebra.

Computers have fundamentally two ways to reason about a mathematical expression: **numerical computations**, which are performed using *only numbers* to represent values and **computer algebra** (or **symbolic computations**), which - by contrast - use *both numbers and symbols*.

First, we shall introduce the concept of **floating point number system** used to handle numerical computations.

Definition 2.1 (Normalized-floating point number system). A normalized-floating point number system F is characterized by the 4-tuple of integers β, p, L, U :

- β is called base or radix,
- p precision,
- $[L, U]$ exponent range (with L, U denoting lower and upper bound respectively).

Given a number $x \in \mathbb{R}$, $x \neq 0$ its representation in a floating point number system shall be written out as $fl(x)$ and has the form

$$x = \text{sign}(x)\beta^E \sum_{i=0}^{p-1} d_i \beta^{-i}$$

with $L \leq E \leq U$ and the sequence $\{d_i\}$ (which is called mantissa) made up of natural numbers such that $d_0 \neq 0$, $0 \leq d_i \leq \beta - 1$ and d_i eventually different from $\beta - 1$.

The notation δx shall be used to denote the difference between a symbol x and its floating point approximation $fl(x)$

$$\delta x = x - fl(x).$$

It is important to notice that a floating point number system F is discrete and finite: it approximates real numbers with finite numbers.

A de facto standard for computers to work with floating point approximations is IEEE 754 [4], the details of which shall not be discussed.

Definition 2.2 (Machine epsilon). Machine epsilon is the maximum possible absolute relative error in representing a nonzero real number x in a floating point number system

$$\epsilon_{mach} = \max_x \frac{|x - fl(x)|}{|x|}.$$

Example 2.1. Let us define the matrix (made up of both symbols and numbers) M

$$\begin{bmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{bmatrix}.$$

Consider the matrix \tilde{M} , having as entries the floating point approximation of those of M

$$\begin{bmatrix} fl(\sqrt{2}) & 1 \\ 2 & fl(\sqrt{2}) \end{bmatrix}.$$

Computing its determinant gives out $2 + 2\epsilon\sqrt{2} + \epsilon^2 - 2 \doteq 2 + 2\epsilon\sqrt{2} - 2 \neq 0$.

Introducing a small change (i.e. an “error”) in the input argument may either cause a large or a small change in the result. Now, we shall introduce the concept of condition number.

Definition 2.3 (Condition number). A condition number of a problem measures the sensitivity of the solution to small perturbations in the input data. Given a function f , we define

$$cond(f, x) = \lim_{\epsilon \rightarrow 0} \sup_{\|\Delta x\| \leq \epsilon \|x\|} \frac{\|f(x + \Delta x) - f(x)\|}{\epsilon \|f(x)\|}.$$

Given a problem, if its condition number is low it is said to be **well-**

conditioned (typically $\text{cond}(f, x) \sim 1$), while a problem with a high condition number is (said to be) **ill-conditioned** ($\text{cond}(f, x) \gg 1$).

Let us now consider the problem of solving a linear equation subjected to a perturbation.

Let A be a non-singular matrix and assume we introduce a perturbation in the constant term $\tilde{\mathbf{b}} = \mathbf{b} + \delta\mathbf{b}$. The equation can be written as

$$A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$$

with $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$. We can obtain

$$\tilde{\mathbf{x}} - \mathbf{x} = A^{-1}\tilde{\mathbf{b}} - A^{-1}\mathbf{b} = A^{-1}\delta\mathbf{b}$$

and, by using matrix norms, we can write

$$\|\tilde{\mathbf{x}} - \mathbf{x}\| = \|A^{-1}\delta\mathbf{b}\| \leq \|A^{-1}\| \|\delta\mathbf{b}\|.$$

It is also known that

$$\|\mathbf{b}\| = \|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\|$$

which implies

$$\|\mathbf{x}\| \geq \frac{\|\mathbf{b}\|}{\|A\|}.$$

Tying all this together we can conclude

$$\frac{\|\tilde{\mathbf{x}} - \mathbf{x}\|}{\|\mathbf{x}\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}.$$

Definition 2.4 (Condition number of a matrix). The condition number of a non-singular matrix A is defined as:

$$\kappa(A) = \|A^{-1}\| \|A\|.$$

Now, let us refocus on the topic of math expressions. Let us investigate what would happen if symbols are allowed in computations by introducing a framework that allows us to work with computer algebra.

Definition 2.5 (Computer algebra system). A computer algebra system (CAS) is a mathematics software package that can perform *both symbolic and numerical mathematical computations*.

A CAS is usually a **REPL** expected to support a few functionalities [5]:

- **Arithmetic:** arithmetic over different fields with arbitrary precision.
- **Linear algebra:** matrix algebra and knowledge of different operations and properties of matrices (i.e. determinants, eigenvalues and eigenvectors).
- **Polynomial manipulation:** factorization over different fields, simplification and partial fraction decomposition of rational functions.
- **Transcendental functions:** support for transcendental functions and their properties.
- **Calculus:** limits, derivatives, integration and expansions of functions.
- **Solving equations:** solving systems of linear equations, computing with radicals solutions of polynomials of degree less than five.
- **Programming language:** users may implement their own algorithms using a programming language.

The CAS chosen for this work is **SageMath** [7], the features and functionalities of which shall not be discussed here.

SageMath is an open source CAS distributed under the terms of the GNU GPLv3 [3].

Hereafter, an example in which symbolic computations are put against numerical (computations) shall be made.

Example 2.2. Take matrix M from Example 2.1:

$$\begin{bmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{bmatrix}.$$

Compare the different results given out when computing its determinant by defining M over the *symbolic ring SR* and the *finite-precision ring CDF*:

```
sage: matrix(SR, [[sqrt(2), 1], [2, sqrt(2))]).det()
0
sage: matrix(CDF, [[sqrt(2), 1], [2, sqrt(2))]).det()
-3.14018491736755e-16
```

We can observe that in SR $(\sqrt{2})^2 = 2$ since no approximations are made.

Now, take the polynomial $p(x)$:

$$p(x) = x^6 + 5x^5 - 3x^4 - 42x^3 + 12x^2 - x + 1.$$

If an attempt to calculate its roots over SR is made, an exception will be

thrown; however, doing this over a finite-precision ring (such as CDF) will work:

```
sage: p = x^6 + 5*x^5 - 3*x^4 - 42*x^3 + 12*x^2 - x + 1
sage: p.roots(ring=SR)
      RuntimeError: no explicit roots found
sage: p.roots(ring=CDF)
[(-3.865705050148171 - 1.5654017866113432*I, 1),
 (-3.8657050501481702 + 1.5654017866113419*I, 1),
 (-0.04843174828928114 - 0.2430512799158686*I, 1),
 (-0.048431748289281144 + 0.24305127991586856*I, 1),
 (0.38275295887213723 + 7.286537374692244e-17*I, 1),
 (2.4455206380027437 - 1.995314986816126e-16*I, 1)]
```

What we may conclude from such an example is that numerical analysis is certainly a powerful tool as it allows for computations which could not happen with computer algebra, **but** computer algebra being able to compute an exact answer without any approximation will prove to be useful in our use case.

For deeper reasoning about the limits of computer algebra systems, one may refer to Mitic [6].

Eigenvalues, eigenvectors

In the following section, eigenvalues and eigenvectors shall be defined; a reader may also consult Axler [1] or Strang [8].

Lastly, a result on the condition number of the problem of computing eigenvalues of a matrix shall be given.

Definition 2.6 (Eigenvalue, eigenvector). Given a linear transformation T in a finite-dimensional vector space V over a field F into itself and a nonzero vector \mathbf{v} , \mathbf{v} is an eigenvector of T if and only if

$$A\mathbf{u} = \lambda\mathbf{u}$$

with A the matrix representation of T , \mathbf{u} the coordinate vector of \mathbf{u} and λ a scalar in F known as eigenvalue associated with \mathbf{v} .

Similarly, we can define a row vector \mathbf{x}_L , and a scalar λ_L such that

$$\mathbf{x}_L A = \lambda_L \mathbf{x}_L,$$

which are called **left eigenvector** and **left eigenvalue** respectively.

Remark. Note that writing $A\mathbf{u} = \lambda\mathbf{u}$ is equivalent to $(A - \lambda I)\mathbf{u} = 0$.

It follows that the eigenvalues of A are the roots of

$$\det(A - \lambda I)$$

which is a polynomial in λ known as the **characteristic polynomial** $ch_A(\lambda)$.

Definition 2.7 (Eigenspace). Given a square matrix A and its eigenvalue λ , we define the eigenspace of A associated with λ the subspace E_A of all vectors satisfying the equation

$$E_A = \{\mathbf{u} : (A - \lambda I)\mathbf{u} = 0\} = \ker(A - \lambda I).$$

Definition 2.8 (Algebraic, geometric multiplicities of eigenvalues). Given a square matrix A and a scalar $\lambda \in \mathbb{C}$: we define the algebraic multiplicity of λ as

$$m_A(\lambda) = \max\{k : (\exists s(x) : s(x)(x - \lambda)^k = ch_A(x))\}.$$

The geometric multiplicity of λ is defined as

$$\nu_A(\lambda) = \dim(\ker(A - \lambda I)).$$

Remark. Suppose A is a real square matrix, then the following statements are true:

- the eigenvalues of the left and right eigenvectors of A are the same,
- the left eigenvectors simplify into the transpose of the right eigenvectors of A^T .

Now, let us investigate how introducing perturbations in the representation of a matrix may influence the numerical stability of its eigenvalues.

Let A be a square matrix, $\lambda \in \mathbb{C}$ its eigenvalue, \mathbf{x} , \mathbf{y} the right and left eigenvectors associated with λ . Consider the perturbed problem

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\lambda}\tilde{\mathbf{x}}$$

with ϵ the machine epsilon, $\tilde{A} = A + \epsilon\delta A$, $\tilde{\mathbf{x}} = \mathbf{x} + \epsilon\delta\mathbf{x}$, $\tilde{\lambda} = \lambda + \epsilon\delta\lambda$.

Differentiating w.r.t. ϵ and multiplying by \mathbf{y}^T on the left side gives

$$\mathbf{y}^T \delta A \mathbf{x} + \mathbf{y}^T A \delta \mathbf{x} = \delta \lambda \mathbf{y}^T \mathbf{x} + \lambda \mathbf{y}^T \delta \mathbf{x}$$

and, since \mathbf{y} is the left eigenvector we can rewrite it as

$$\frac{\delta\lambda}{\delta\epsilon} = \frac{\mathbf{y}^T \delta A \mathbf{x}}{\mathbf{y}^T \mathbf{x}}.$$

Assuming $\|\delta A\| = 1$ and using the definition of dot product for $\mathbf{y}^T \mathbf{x}$ we get

$$|\delta\lambda| \leq \frac{1}{|\cos(\theta_\lambda)|} |\delta\epsilon|.$$

Definition 2.9 (Condition number of an eigenvalue). Given a square matrix A , the eigenvalue $\lambda \in \mathbb{C}$ and θ_λ the angle between the left and right eigenvectors associated with λ , the quantity

$$k_A(\lambda) = \frac{1}{\cos(\theta_\lambda)}$$

is called the condition number of the eigenvalue λ .

Jordan canonical form

In the following section, we shall define Jordan matrices and discuss the stability of a transformation of a matrix into its Jordan canonical form.

Definition 2.10 (Jordan matrix). A diagonal block matrix M is called a Jordan matrix if and only if each block along the diagonal is of the form

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 1 & \lambda \end{bmatrix},$$

and we indicate such a matrix with $\text{diag}(J_{\lambda_1, n_1}, \dots, J_{\lambda_k, n_k})$ with k the number of diagonal blocks it is made up of.

Each block can be completely described by the tuple (λ, n) as it is an $n \times n$ matrix of zeroes everywhere except for the diagonal, which is filled with λ , and the superdiagonal, with ones.

Remark. Let V be a vector space defined over a field F and A a matrix defined in V . If $\text{ch}_A(t)$ can be factorized into its linear factors over K , then A

is similar to a Jordan matrix J . We define J the **Jordan canonical form (JCF)** of A .

Definition 2.11 (Defective matrix, defective eigenvalue). Given a square $n \times n$ matrix A , if it has less than n distinct eigenvalues then it is called a defective matrix.

Furthermore, we define an eigenvalue λ of such a matrix as a defective eigenvalue if and only if

$$m_A(\lambda) > \nu_A(\lambda).$$

Now, we shall give a result on the stability of such a transformation the proof of which can be found in other works, such as Datta [2].

Theorem 2.1 (Stability of the JCF transformation). Given a matrix A and its JCF $A = P^{-1}JP$, the transforming matrix P is highly ill-conditioned whenever A has at least a defective or nearly defective eigenvalue.

Lastly, we shall give an example to show the implications of such a theorem by showing the differences in the JCF of a matrix and its perturbed version.

Example 2.3. Consider the $n \times n$ matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 1 & 0 \end{bmatrix}$$

with $\alpha > 0$.

It is evident how A has a defective eigenvalue in $\lambda_A = 0$ and $m_A(0) = n$, $\nu_A(0) = 1$; furthermore, A is already in JCF.

Now, let us switch our focus to B . To compute its eigenvalues, take the characteristic polynomial $ch_B(t) = t^n - \alpha$: it has n distinct roots in

$$t_j = z_n^j \sqrt[n]{\alpha}$$

with $j = 1, \dots, n$, $z_n = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$ and i imaginary unit such that $i^2 = -1$.

To conclude, we shall show the JCF of A and B defined in SR computed by SageMath when $n = 4$.

```

sage: A = matrix(SR, [
    [1 if i == j-1
      else 0 for j in range(4)]
    for i in range(4)
  ])
sage: B = matrix(SR, [
    [1 if i == j-1
      else x if j == 0 and i == 3
      else 0 for j in range(4)]
    for i in range(4)
  ])
sage: A
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[0 0 0 0]
sage: A.jordan_form()
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[0 0 0 0]
sage: B
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
[x 0 0 0]
sage: B.jordan_form()
[ I*x^(1/4) | 0 | 0 | 0]
[-----+-----+-----+-----]
[ 0 | -x^(1/4) | 0 | 0]
[-----+-----+-----+-----]
[ 0 | 0 | -I*x^(1/4) | 0]
[-----+-----+-----+-----]
[ 0 | 0 | 0 | x^(1/4)]

```

Kronecker canonical form

Magna consequat nostrud culpa officia. Reprehenderit tempor et et veniam mollit nostrud. Sint ullamco pariatur ipsum irure excepteur magna non occaecat laborum veniam minim. Nisi pariatur aliquip aute ut aliqua.

Adipisicing labore cupidatat officia labore tempor ex laborum duis est laboris sint cillum consectetur ut. Sunt ut labore est ipsum enim tempor. Deserunt est adipisicing pariatur elit tempor nulla eiusmod ad ipsum Lorem. Mollit occaecat quis esse fugiat deserunt ea. Non aliquip et reprehenderit sint voluptate culpa non quis labore eu laborum enim amet eiusmod.

Consequat amet sunt in nulla occaecat ad veniam proident. Magna sint excepteur laboris excepteur Lorem aliquip. Enim amet elit Lorem aliqua ut elit fugiat. Amet mollit officia excepteur ullamco incididunt minim ipsum consectetur tempor ut do. Dolor non enim fugiat aliquip.

Voluptate dolor do non commodo. Qui qui eu id dolor Lorem excepteur sint cillum esse commodo et. Cupidatat ipsum eiusmod consectetur culpa duis veniam mollit consectetur velit do veniam tempor. Eu nostrud laboris nisi ullamco. Labore fugiat aute nostrud aliquip cillum nulla duis.

Nulla laborum ex est aute incididunt et mollit ea excepteur aliquip cillum tempor ullamco. Exercitation ullamco consectetur aliqua officia quis. Ex amet eiusmod laborum tempor excepteur adipisicing nulla non.

Pariatur Lorem id laboris pariatur nulla officia proident exercitation tempor ut nostrud eu sunt. Ea proident irure dolore cupidatat commodo ad laborum. Eiusmod elit ullamco ad Lorem non laborum consequat minim ipsum. Adipisicing nisi magna incididunt incididunt irure fugiat dolor et Lorem qui dolor pariatur aliqua. Ad adipisicing do proident cupidatat labore magna fugiat nulla esse excepteur culpa. Reprehenderit consequat enim non sunt incididunt eu.

Aute culpa laborum do cillum excepteur laboris consequat. Ullamco Lorem eiusmod ex commodo ea nisi consequat mollit proident amet cupidatat minim ad. Pariatur in cupidatat magna velit fugiat amet enim elit occaecat adipisicing quis sit.

In esse exercitation tempor mollit. Non esse qui elit qui. Sint incididunt quis sit ex est tempor consectetur. Eu velit nisi ullamco consequat Lorem ad veniam sunt sit non aliquip fugiat. Id tempor dolore ex labore deserunt tempor consectetur. Cillum ad anim eu esse Lorem proident et esse anim ex. Magna consectetur voluptate sint aute in cillum dolore occaecat est ullamco.

Culpa fugiat officia velit sunt cillum tempor eu laboris commodo exercitation incididunt. Esse laboris fugiat aute minim culpa irure sint duis.

In laborum duis minim labore in quis minim quis aliqua nostrud tempor excepteur laborum. Elit aliquip aute elit amet est Lorem Lorem quis do dolore ad non tempor. Consequat Lorem aliqua nulla esse id pariatur ea. Incididunt commodo labore tempor excepteur ullamco labore pariatur non ipsum eiusmod.

Do sit cupidatat cupidatat ullamco nisi. Sint aute voluptate nulla dolor mollit. Reprehenderit reprehenderit enim fugiat est nostrud id aute irure id adipisicing nisi. Veniam aute ullamco cillum Lorem labore ex cillum ad labore nulla ad nulla.

Deserunt sint tempor sit cillum. Ex dolor nisi nostrud sint fugiat aliquip pariatur officia eu aliquip proident in ad. Anim cupidatat excepteur ad in duis culpa sint. Sint et dolore irure qui sunt aute.

Tempor commodo non cillum quis amet. Proident proident ea ex exercitation nulla voluptate est eiusmod dolor ut ullamco laborum. Ea id anim consectetur labore minim. Eiusmod culpa id incididunt ut exercitation.

Implementation

Aliqua dolore proident irure aliquip. Ad eiusmod cillum pariatur sunt irure qui irure cupidatat cillum sit. Laboris irure laboris velit dolore. Ipsum eiusmod dolor ex in incididunt cillum amet esse consequat. Culpa dolor reprehenderit consectetur est amet in aliqua sint dolore aliquip.

Ea sint pariatur anim nulla sunt excepteur minim aliqua eiusmod esse eu ad. Enim magna laboris sit tempor. Ea occaecat veniam voluptate commodo duis quis sunt laboris occaecat esse deserunt. Enim in sit eu ut id officia cupidatat.

Dolore duis dolor aute labore deserunt cillum dolore culpa laborum ex fugiat. Ad anim dolore irure veniam ut adipisicing mollit aliquip aute eu nostrud amet. Ullamco consectetur duis quis reprehenderit sunt nulla qui exercitation commodo consectetur. Dolor tempor ad exercitation sint irure eu non ad esse officia cupidatat nulla. Consectetur irure fugiat tempor ea ipsum mollit pariatur occaecat deserunt sit aliqua. Consequat reprehenderit reprehenderit eiusmod enim est nulla.

Velit sit non dolore eu ea cupidatat. Lorem nisi ut labore ipsum pariatur dolore consequat ipsum qui non. Veniam incididunt exercitation adipisicing veniam ipsum mollit excepteur irure voluptate eu culpa ad aute. Do duis est nulla consequat velit pariatur reprehenderit. Ea voluptate occaecat velit tempor.

Occaecat ipsum et duis proident officia ullamco do consequat qui veniam consectetur occaecat. Sint proident ullamco deserunt anim ullamco officia do exercitation sint Lorem nostrud laboris ad tempor. Cillum quis labore occaecat sint incididunt mollit ad aute sunt anim occaecat do nisi. Consequat excepteur exercitation esse cillum do quis occaecat culpa aute pariatur ad est anim.

Eiusmod ut mollit aliquip occaecat tempor cupidatat nulla. Aliqua cillum elit aliquip ad adipisicing magna nulla fugiat sit ullamco irure. Ut dolor Lorem Lorem irure aute elit fugiat nisi non Lorem in mollit laborum.

Consequat laborum consequat eu ex enim eu cillum aliqua nisi. Commodocommodo laboris adipisicing exercitation sit exercitation sit. Eiusmod officia Lorem enim incididunt do mollit enim sunt ea. Eu reprehenderit incididunt eu elit sit eiusmod duis.

Labore laborum duis veniam eu proident ullamco adipisicing irure. Anim deserunt eu exercitation ullamco laboris nostrud tempor ad nisi occaecat Lorem adipisicing laborum ut. Mollit esse aliqua incididunt occaecat commodo ea nostrud. Aliquip magna irure fugiat do occaecat adipisicing conse-

quat duis. Incididunt eiusmod laborum mollit cillum occaecat sint ad aute quis officia mollit nostrud enim pariatur. Cupidatat sit tempor sunt elit minim labore elit commodo. Pariatur nisi tempor laboris minim proident mollit labore excepteur occaecat exercitation ut aute.

Aliqua id esse ut enim ullamco ad et laborum do enim voluptate sit fugiat eiusmod. Et labore sunt nostrud nisi aute anim eu sunt culpa. Dolor nisi elit excepteur duis anim cillum esse amet tempor aute veniam.

Officia pariatur minim anim exercitation minim enim non et laborum. Enim fugiat ullamco eiusmod minim eiusmod in et voluptate quis exercitation consectetur incididunt. Laborum tempor proident laboris ullamco nostrud esse ea amet dolor ut.

Reprehenderit ex dolor reprehenderit reprehenderit ullamco. Dolore eiusmod occaecat eu nulla duis commodo reprehenderit laboris eu. Eu excepteur incididunt anim id aute commodo reprehenderit.

Veniam reprehenderit et cupidatat Lorem irure amet velit elit non consequat sunt ea irure magna. Quis exercitation ea quis cillum officia sint duis. Quis magna eiusmod eu cillum aliquip ex ea proident sit aliqua ea proident cillum. Do ad ut eu ea sint tempor. Cupidatat mollit pariatur id in ex eu elit occaecat reprehenderit aliqua ea consequat mollit labore. Labore sit in deserunt occaecat elit non officia velit incididunt ullamco non amet voluptate.

In fugiat quis non officia elit proident. Cupidatat laboris cupidatat veniam aliqua consectetur laborum non laborum ad. Laborum mollit duis minim nulla. Ad laborum aliquip est ipsum incididunt ad aute.

Magna dolor nisi exercitation pariatur labore. Commodum excepteur elit sit cupidatat magna pariatur eiusmod Lorem. Magna sunt sint exercitation dolore deserunt in exercitation.

Cupidatat magna ea aliqua et nostrud sint veniam nostrud et deserunt voluptate id. Voluptate ex cillum consequat laborum minim fugiat laborum consectetur ullamco adipisicing dolor. Aute ullamco proident fugiat esse occaecat culpa eiusmod irure quis et reprehenderit. Non magna esse ea aliquip cupidatat. Exercitation culpa amet minim elit magna esse et deserunt. Dolore ex do irure id sint voluptate occaecat. Do id do tempor qui incididunt voluptate.

Conclusions

Non aliquip id esse qui consequat do in sit incididunt ipsum. Pariatur aute ut excepteur incididunt qui aute mollit ea. Dolor consequat nostrud nisi duis est esse. In labore pariatur excepteur incididunt. Minim et culpa consequat amet id excepteur amet esse. Cillum ut proident minim esse cillum ea laborum. Commodum ad adipiscing nulla velit irure sunt commodum sunt.

Eiusmod adipiscing incididunt reprehenderit amet dolore veniam aute cupidatat tempor officia id adipiscing. Cupidatat non enim deserunt nisi exercitation fugiat. Exercitation exercitation magna ullamco id adipiscing deserunt irure cupidatat veniam sit reprehenderit non reprehenderit. Ipsum adipiscing anim sint ullamco incididunt pariatur amet consequat nulla dolore qui esse. Veniam ullamco aliquip voluptate est ea cupidatat occaecat id exercitation proident irure non.

Culpa sit ullamco ipsum eiusmod Lorem et. Consequat ex consectetur officia non sint id. Et culpa velit nulla Lorem Lorem adipiscing aute enim cillum officia commodum sint adipiscing. Sit veniam laboris esse magna ipsum aute tempor velit incididunt sint.

Consequat ex aute adipiscing sint pariatur mollit aute eu voluptate reprehenderit dolore laboris sunt ex. Ea nisi laborum nisi excepteur adipiscing cupidatat duis occaecat reprehenderit. Consequat consectetur qui incididunt voluptate culpa. Velit et magna laborum excepteur sint minim proident. Aliquip ipsum non minim qui cupidatat et quis. Non magna occaecat nostrud reprehenderit sint proident ad cupidatat eiusmod elit occaecat enim et nostrud.

Labore reprehenderit amet incididunt irure velit. Ipsum qui reprehenderit dolore adipiscing. Incidunt eiusmod ad do exercitation aute do fugiat elit mollit. Irure exercitation aliqua non minim consequat do adipiscing commodum enim id magna. Quis eu minim culpa eu.

Fugiat ea in pariatur nostrud esse id duis ipsum officia ut. Voluptate tempor est velit pariatur ipsum incididunt mollit consectetur laborum enim laborum dolore dolor. Nostrud culpa ad aliquip magna velit magna ipsum consectetur exercitation dolore dolor. Culpa ullamco aliquip aute deserunt. Proident voluptate cillum adipiscing culpa deserunt eiusmod. Fugiat quis minim sit magna exercitation reprehenderit tempor ullamco velit ipsum laboris.

Duis consectetur fugiat anim ad proident eiusmod mollit cupidatat aute. Occaecat ex minim ad sunt velit ut exercitation eiusmod eiusmod fugiat

culpa tempor quis anim. Officia aute quis anim deserunt laborum dolore elit non fugiat nostrud. Ea irure nisi dolore fugiat. Ea aliqua consectetur ut et nostrud minim. Sint Lorem aute Lorem exercitation.

Duis aliqua deserunt enim cillum nulla ipsum sit anim consectetur tempor reprehenderit. Fugiat et reprehenderit cupidatat nostrud cillum incididunt aliqua reprehenderit laboris laborum deserunt nulla sint. Mollit commodo aliqua magna aliquip.

In consectetur sint culpa incididunt incididunt ipsum proident et consequat velit nisi minim ut incididunt. Ea qui exercitation excepteur excepteur sunt irure ex officia occaecat dolore amet. Proident exercitation laboris anim deserunt nostrud. Minim cillum excepteur Lorem elit esse ea nisi pariatur qui incididunt proident. Sint in velit proident et elit.

Elit in voluptate deserunt est deserunt esse reprehenderit veniam ex esse. Lorem ex dolor id eu duis ut adipisicing sit adipisicing. Aute nulla eu mollit velit est aute fugiat dolore. Ut officia anim ut quis in. Ea velit do ut laborum eu magna est ipsum amet.

Bibliography

- [1] Sheldon Jay Axler. *Linear Algebra Done Right*. Springer, 1997.
- [2] Biswa Nath Datta. In *Numerical Methods for Linear Control Systems*, pages 79–103. Academic Press, San Diego, 2004.
- [3] GNU General Public License, version 3. <http://www.gnu.org/licenses/gpl.html>, June 2007. Last retrieved 2020-01-01.
- [4] IEEE. IEEE-754, Standard for Floating-Point Arithmetic. *IEEE Std 754-2008*, pages 1–58, 01 2008.
- [5] K. Kalorkoti. Introduction to Computer Algebra. <https://www.inf.ed.ac.uk/teaching/courses/ca/notes01.pdf>, January 2019.
- [6] Peter Mitic and Peter G. Thomas. Pitfalls and limitations of computer algebra, 1994.
- [7] W. A. Stein et al. *Sage Mathematics Software (Version x.y.z)*. The Sage Development Team, 2022. <http://www.sagemath.org>.
- [8] Gilbert Strang. *Introduction to Linear Algebra*. Wellesley-Cambridge Press, 2009.