1) Classical MDS

1.1)

- We know that $(*)$ R is orthogonal.
- We will show that: $(**)$ $\forall v \in R^D: \ \|Rv\|_2 = \|v\|_2$

  $\|Rv\|_2 = \sqrt{(Rv)^T Rv} = \sqrt{vR^T Rv} \overset{(*)}{=} \sqrt{v^T v} = \|v\|_2$

- Therefore: $\|Rx_i - Rx_j\|_2 = \|R(x_i - x_j)\|_2 \overset{(**)}{=} \|x_i - x_j\|_2$

2) Metric MDS

2.1)

- We have:
  - $B = C - diag(C1)$
  - $(*)$ $C[i,\ i] \ = \ 0$
- We want to show that B is already centered, its mean is 0 the sum of each row is 0.
- Now, the i-th of B, $Bi \ = [C - diag(C1)]_i$, that is: the C_i vector, with the i item in

  the vector replaced by the sum $-\sum_j C_{ij} = sum(C_i)$.

- So $Bi = C_i$ with $C_{ii} \leftarrow \Sigma C_i$.
  - The sum the i-th of B: $\Sigma B_i = \Sigma C_i - C[i,i] - \sum_j C_{ij} = \Sigma C_i - 0 - \Sigma C_i = 0$
  - The mean of B is also 0.
- B is centered, i.e, $BJ = B$.

2.2)

- We saw in `slide 17` that: $(*)$ $\text{p} = \text{diag}\left(Z^T Z\right)$
- We'll show that: $(**)$ $\langle \Delta_x,\ D_z \rangle = 2\left\langle B,\ Z^T Z\right\rangle$

  $-\langle \Delta_x,\ D_z \rangle = \left\langle C, D_z^{\circ 2}\right\rangle = \left\langle C,\ \boldsymbol{p}\mathbf{1}_N^T - 2Z^T Z + 1_N p^T\right\rangle \overset{(*)}{=} \left\langle C,\ \text{diag}\left(Z^T Z\right)\mathbf{1}_N^T - 2Z^T Z + 1_N \text{diag}\left(Z^T\right.\right.$

  $= \left\langle C,\ 2\text{diag}\left(Z^T Z\right)\mathbf{1}_N^T - 2Z^T Z\right\rangle = 2\left\langle C,\ \text{diag}\left(Z^T Z\right)\mathbf{1}_N^T - Z^T Z\right\rangle$

  $= 2\left(\left\langle C\mathbf{1}_N,\ \text{diag}\left(Z^T Z\right)\right\rangle - \left\langle C,\ Z^T Z\right\rangle\right) = 2\left(\left\langle \text{diag}\left(C\mathbf{1}_N\right),\ Z^T Z\right\rangle - \left\langle C,\ Z^T Z\right\rangle\right)$

  $= 2\left\langle \text{diag}\left(C\mathbf{1}_N\right) - C,\ Z^T Z\right\rangle = \ -2\left\langle B,\ Z^T Z\right\rangle$

- And we get:

  $\|\Delta_x,\ D_z\|_F^2 = \|\Delta_x\|_F^2 + \|D_z\|_F^2 - 2\langle \Delta_x,\ D_z \rangle \overset{(**)}{=} \|\Delta_x\|_F^2 + 2N\text{Tr}\left\{ZJZ^T\right\} - 4\left\langle B,\ Z^T Z\right\rangle = g(Z,Z)$

3) Isomap

3.1)

$$[X\,Y]\,\tilde{J} = [X\,Y]\begin{bmatrix} J_{N_x} & -\frac{1}{N_x}1_{N_x}1_{N_y}^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} XJ_{N_x} & Y - \frac{1}{N_x}X1_{N_x}1_{N_y}^T \end{bmatrix}$$

$$= \begin{bmatrix} X-\mu_x 1_{N_x} & Y-\mu_x 1_{N_y}^T \end{bmatrix} = \begin{bmatrix} \widetilde{X} & \widetilde{Y} \end{bmatrix}$$

3.2)

$$-\frac{1}{2}\,\tilde{J}^T D\tilde{J} = -\frac{1}{2}\begin{bmatrix} J_{Nx} & 0 \\ -\frac{1}{N_x}1_{N_x}1_{N_y}^T & I \end{bmatrix}\begin{bmatrix} D_{xx} & D_{xy} \\ D_{xy}^T & A \end{bmatrix}\begin{bmatrix} J_{Nx} & -\frac{1}{N_x}1_{N_x}1_{N_y}^T \\ 0 & I \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} J_{Nx} & 0 \\ -\frac{1}{N_x}X1_{N_x}1_{N_y}^T & I \end{bmatrix}\begin{bmatrix} D_{xx}J_{Nx} & D_{xy} - \frac{1}{N_x}D_{xx}1_{N_x}1_{N_y}^T \\ D_{xy}^T J_{Nx} & A - \frac{1}{N_x}D_{xy}^T 1_{N_x}1_{N_y}^T \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} J_{Nx}D_{xx}J_{Nx} & J_{Nx}\left(D_{xy} - \frac{1}{N_x}D_{xx}1_{N_x}1_{N_y}^T\right) \\ \left(D_{xy} - \frac{1}{N_x}D_{xx}1_{N_x}1_{N_y}^T\right)^T J_{Nx} & \phi \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2\widetilde{K_{xx}} & -2\widetilde{K_{xy}} \\ -2\widetilde{K_{xy}}^T & \phi \end{bmatrix} = \begin{bmatrix} \widetilde{K_{xx}} & \widetilde{K_{xy}} \\ \widetilde{K_{xy}}^T & \widetilde{A} \end{bmatrix}$$

4) Laplacian Eigenmaps

4.1)

$$\mathrm{Tr}\left\{ZLZ^T\right\} = \sum_{i=1}^{k}\left(ZLZ^T\right)_{ii} = \sum_{i=1}^{k}\left(z_i\,Lz_i^T\right) \overset{Ex\ 3.\ Slide\ 12}{=} \sum_{i=1}^{k}\frac{1}{2}\langle W, D_{z_i}\rangle = \frac{1}{2}\sum_{i=1}^{k}\langle W, D_{z_i}\rangle = \frac{1}{2}\langle W, D_z\rangle$$

4.2)

- Each connected component has an associted Laplacian (`Lecture 9`).

  Therefore, we can write matrix L as a block diganoal matix:

  $$L = \begin{bmatrix} L_1 & \\ & L_2 \end{bmatrix}$$

  Where L_1 corresponds to component V_1, and L_2 corresponds to component V_2.
- We saw in `Ex. 4 slide 13`: $L\mathbf{1} = D\mathbf{1} - W\mathbf{1} = W\mathbf{1} - W\mathbf{1} = \mathbf{0}$.

- Specifically: $L_1 \mathbf{1}_{L_1} = 0$, and $L_2 \mathbf{1}_{L_2} = 0$.
- Using that, Let: $u_1 = (\underbrace{1,\ldots,1}_{\mathbf{1}_{L_1}}, \underbrace{0,\ldots,0}_{0_{L_2}})$ and $u_2 = (\underbrace{0,\ldots,0}_{0_{L_1}}, \underbrace{1,\ldots,1}_{1_{L_2}})$ (note: $u_1 + u_2 = \mathbf{1}$)
- Follow that we have:
  - $\boldsymbol{L} u_1 = 0,\ L u_2 = 0$
  - $\langle u_1, u_2 \rangle = 0$