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WHAT DO MATHEMATICS AND PHYSICS TEACHERS THINK THAT STUDENTS WILL FIND DIFFICULT? A CHALLENGE TO ACCEPTED PRACTICES OF TEACHING

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This paper considers the difference between the development of the concept of vector that is met as force and acceleration in physics, but as translation in mathematics. It studies cognitive problems that arise and tests an approach to the concept of free vector that focuses on the ***effect*** *of the physical action of translation. The result is a more subtle conception of free vector that applies not only to translations but also to a wider range of vector concepts in mathematics and physics.*

> The person who says 'Don't worry me with theory – just give me the facts' is speaking foolishly. A set of facts can be used only in circumstances to which they belong, whereas an appropriate theory enables us to explain, predict and control a great number of particular events in the classes to which it relates. (Skemp, 1971, p. 29.)

INTRODUCTION

The English National Curriculum and the corresponding versions in Scotland and Northern Ireland prescribe a global strategy to raise standards in school. This involves a pragmatic overall framework for teaching each of the major school subjects, and over the years, modifying the details to deal realistically with practical issues in the classroom. In recent years the mathematics curriculum is taking an increasing focus on correcting *misconceptions* that students encounter (Askew, 1995). Here we focus on the concept of vector and consider the misconceptions and the reasons that they occur, in order to formulate a practical theory to improve student understanding of the concept.

The National frameworks for mathematics and physics take very different approaches to the concept of vector. In recent years it has all but vanished from the National Curriculum in mathematics, surviving only in the final higher level of key stage 4; it is not mentioned at all in physics. Instead, the two subjects follow different routes, physics focusing on the concept of *force* and its relationship to motion (including graphical representations of distance, velocity and acceleration), mathematics focusing on the concept of *translation*, which is later used as a foundation for the mathematical concept of vector. In physics A-level when forces are considered in two dimensions, the topic is made more accessible by resolving quantities into horizontal and vertical directions to retain the one-dimensional flavour. Meanwhile, in mathematics, the notion of translation is first described as a column vector, say, $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

representing a shift of 2 units horizontally and 6 vertically. An alternative notation is used representing a translation from A to B as an arrow $\overrightarrow{AB}$ where arrows with the same magnitude and direction are considered equivalent, and then a single symbol $\mathbf{u}$ is used that effectively represents a 'free' vector of given magnitude and direction, irrespective of its particular starting point.

Poynter (in Watson [1], 2002) discovered that, although almost all of her students arriving in the sixth form could deal with calculating the horizontal and vertical components of forces, many had difficulties with other problems involving forces and vectors. Poynter (2004) studied the underlying reasons for such misconceptions and formulated a new approach to address the problem. She found that many of her students initially approached the idea of adding vectors geometrically by moving them 'nose to tail' and yet they were liable to be confused by experiences that they had met in their earlier work in mathematics and physics.

Teachers are already aware of the kinds of problem that students may face. Interviewing teaching colleagues in mathematics and physics however, revealed that there were very different expectations from those teaching the two subjects. We begin by considering those difficulties and then building and testing a new approach to teaching the topic that is intended to address the difficulties.

WHAT IS IT THAT STUDENTS FIND DIFFICULT?

Two physics and two mathematics teachers were interviewed from a sixth-form college that performed very highly in the National tables measuring the improvement of performance in school. They were shown questions given to pupils in a research questionnaire and asked what problems they thought the students might encounter.

Examples of selected questions are shown in figure 1.

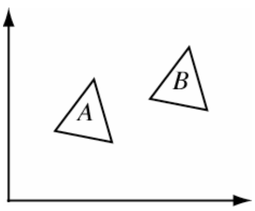
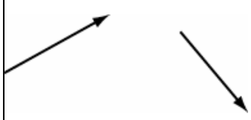
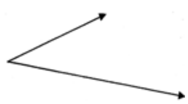
In the picture the triangle has been moved from position *A* to position *B*. How can you represent the translation of the triangle? Can you draw a vector starting at the origin (0,0) which will represent the translation of the triangle from A to B? If so, show it on the drawing.  Can you draw a vector not starting at the origin and not touching either of the triangles which will represent the translation from *A* to *B*? If so, show it on the drawing. (i)	Add the two vectors  (ii)	Add the two vectors  (iii)	Add the two vectors  (iv)

Fig. 1 vector as translation and three examples of vector addition

The four teachers predicted a number of problems that they would expect for each question (i)-(iv) in turn, based on their experience of how students worked in class.

The physics teachers commented on successive questions as follows:

P(i): students may ignore the direction and not place an arrow on the line;
P(ii): they might think that the two vectors should be attached to each other;
P(iii): they are taught to use the parallelogram if the two vectors are connected at one point. When you give them a question with the two tugs, they find it more useful to answer using a parallelogram [...] The triangle is used for resolving vectors […] in Physics […] if it was a displacement;
P(iv): many students do not get the idea that the arrows have to follow ends. Some of them might add the vectors by resolving them in the horizontal and vertical direction and by using matrices. (The teacher refers to the use of column vectors).

The mathematics teachers included the following remarks:

M(i): better students would be expected to be happy with representing translation as an arrow however the weaker students would go across and along. To go from a point on shape *A* to a corresponding point on *B* is another building block;

M(ii): students would be expected to add vectors by placing them 'nose to tail' or by using column vectors. Some students might think that these vectors are fixed in space and will draw two vectors between them to close the polygon;

M(iii): they might put vectors 'nose to tail' without drawing the resultant because sometimes they think that adding vectors means putting one at the end of the other;

M(iv): they might think that the vectors are connected in the wrong way and simply join them up with a third vector.

Some of the students' responses are shown in figure 2.

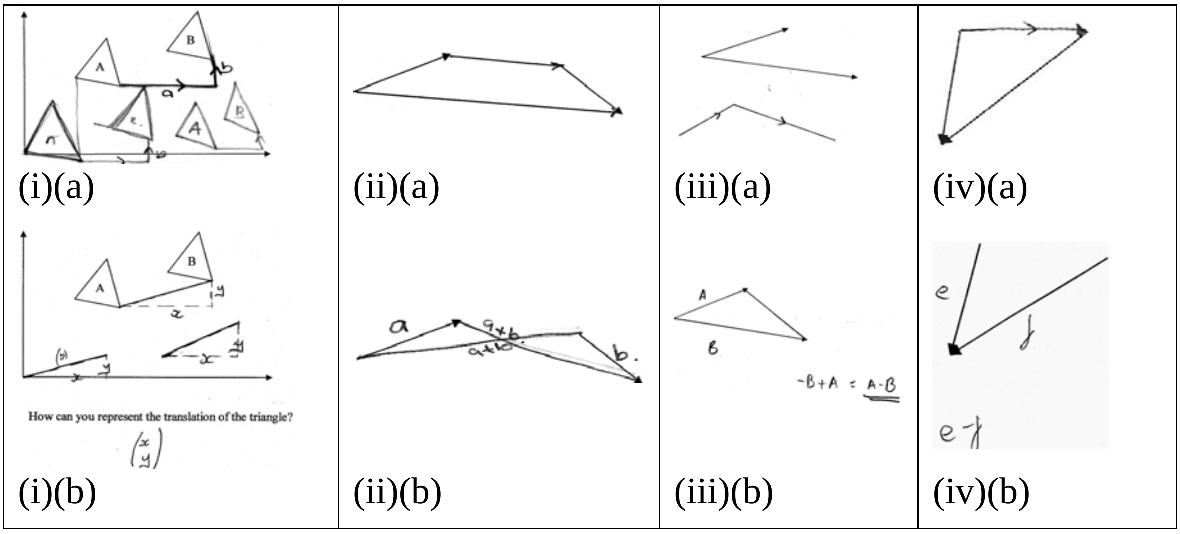


Fig. 2 Examples of student misconceptions

Both responses (i)(a) and (i)(b) show students who draw the figures in different positions using horizontal and vertical components and yet do not have the flexibility to place an arrow starting from any point (as suggested in M(i)), with (i)(b) missing out the arrow (as suggested by P(i)).

Response (ii)(a) shows the student first drawing a line joining the end of the first arrow to the beginning of the second, as if to fill in the middle part of the journey, then adding the line at the bottom, possibly as an equivalent path (M(ii)). Response (ii) shows an arrow '**a+b**' joining the end of vector **a** to the end of **b**, and a corresponding arrow '**b+a**' joining the end of **a** to the end of **b**, as if both represent a 'total path', equivalent to moving along both vectors, and jumping over the gap, consistent with M(ii) and also consistent with the problem noted by P(ii).

Response (iii)(a) draws one arrow followed by the other inaccurately without filling in the third side (as indicated in M(iii)). Response (iii)(b) makes the picture into a journey by moving back along **b** then along **a**, giving the third side denoted by '**–b+a**' (or equivalently by '**a–b**'). In total, only *one* student out of 34 responded using the parallelogram law, contrary to the comment P(iii).

Response (iv)(a) fills in the third side of the triangle as suggested by M(iv), Response (iv)(b) uses the same technique as (iii)(b), this time following down **e** and back along **j** represented as '**e–j**'. This time the student fails to draw the third side. Other solutions (not represented here) do draw approximate horizontal and vertical components to add together (as suggested by P(iv)), but, in general, the errors relate more to the views given by the mathematics teachers than the physics teachers, which is perhaps to be expected in a test given in the context of a mathematics course, prompting responses that students may think are required in mathematics.

WHY DO THE PROBLEMS OCCUR?

The Physics teachers' responses seem to suggest that students will relate vectors to different contexts (using the triangle law for displacements and the parallelogram law for forces). They also suggest that when the students have difficulty, they will work with horizontal and vertical components. In practice these responses occur rarely. Many students respond using the notion of vector as a journey which occurs at an early stage of conceptual development, before the flexible concept of free vector is constructed. This difficulty was intimated in response M(i), that going from any point on A to the corresponding point on B is 'another building block'. The mathematics teachers' responses are sensitive to the problems of students, however they tend to indicate *what* the students might do incorrectly, without saying *why*.

From interviews with selected students it became apparent that their responses do not always represent the whole of their difficulty, and that interpretations relating to the physical context might prevent them from thinking of a solution. Overall therefore we see that all the teachers are sensitive in different ways to the problems that the students encounter. However, their knowledge of the difficulties relate in the main to their episodic memories of what happened in their previous experience. We propose that it would be of help to teachers to develop a practical theory of how students learn that makes sense to both teachers and students.

THEORETICAL FRAMEWORK

Some research studies in science education reveal how students' 'intuitions' arise from working in different contexts and how it effects their problem-solving capabilities. For example, Aguirre and Erickson (1984) talk about "*ten implicit vector characteristics*" involved in "three vector concepts: position; displacement; and velocity" and suggest that students gain "a number of ***intuitions*** about various characteristics" which need to be overcome. On the other hand Jagger (1988) says that, "The problem is in moving from one-dimensional motion to motion in two or more dimensions"; Graham and Berry (1997) suggest that the students "need to overcome this misconception at an early stage," (p 847).

These research studies in science education focus on the misconceptions that occur. They report on the specific *differences* between the contexts that cause problems rather than seek the essential mathematical concepts that are *common* to them. There is also no reference to the mathematical development in which a translation as a journey is reconceptualised flexibly as the concept of free vector.

Skemp (1971) theorized that the way to higher order thinking involves focusing on the essential properties in various context and filtering out the "noise" (the data which is irrelevant to the required abstraction). He also suggests that, "The greater the *noise*, the harder it is to form the concept," (1971, p. 28).

Physics teachers try to reduce the *noise* by teaching students to reduce two-dimensional problems to their components to reduce them back to the one-dimensional context. The meanings are related to physical objects in the 'outside world', which in the Physics curriculum include force, distance, velocity, acceleration. It is assumed in the National Curriculum that these concepts met early in science will be later translated into symbolic form in a mathematical context for students to develop the facility to operate, not just on these concepts, but with anything that resonates with them.

In practice, however, the significant distinctions of meaning in different contexts continue to affect the way that students think. A possible solution might be to attempt to construct the essential mathematical meaning of 'free vector' and then apply this meaning to addition of vectors in different contexts, so that the parallelogram law, the triangle law and the addition of components of vectors are all seen to be different aspects of the same concept.

Skemp suggested a strategy that begins in a particular real-world context with minimal "noise" to focus on essential higher order mathematical concepts. These higher order concepts are then used by "re-embodying the result in the physical realm to give the answer to the original problem," (1971, p. 223). This cycle "reduces noise" ... "and by abstracting only mathematical features it allows us to use a model which we are well practised in working," (1971, p. 223).

A real-world context that is a candidate for building the higher order concept of 'free vector' is already found in the mathematics curriculum. It is the physical notion of ‘translation’. The problem is how to encourage students to construct the flexible concept of ‘free vector’. In a preliminary study, one of the students saw that what matters is not the translation itself, but the *effect* of that translation.

The *effect* of a physical action is not an abstract concept. It can be *seen* and *felt* in an embodied sense. The idea is that, if students had such an embodied sense of the effect of a translation, then they could begin to think of representing it in terms of an arrow with given magnitude and direction. For instance, if the student’s hand was moving a triangle on the table, then the arrow could be taken to show the movement of the tip of a particular finger, or thumb. The particular choice of arrow does not matter. It does not even need to be attached to the object being moved. Any arrow of the given magnitude and direction will give the required effect. The idea is to use the students’ physical experience as a foundation for the building of the concept of free vector and to give an underlying embodied foundation to the symbolism used for vectors within a coherent schema of meaning. For example, the addition of vectors is a simple extension of the idea that the sum of two free vectors is the free vector that has ‘the same effect’ as the combination of one vector followed by another. This would give embodied meaning to the technique of placing vectors ‘nose to tail’ to add them, to see that the sum is any arrow having the same magnitude and direction given by the triangle or parallelogram law, which are now seen as two different ways of looking at the same underlying idea. This flexibility of sensing the meaning of ‘free vector’ would then relate to other simple ideas, such as the idea that addition of free vectors is commutative.

The goal is to create conceptual knowledge with a relational understanding of the concepts rather than procedural knowledge with an instrumental understanding of separate techniques. By founding the ideas on coherent physical actions and by focusing on the notion of ‘effect’, the strategy is to encourage students to reflect on their knowledge and build the notion of free vector as a coherent cognitive unit in a rich schema of relationships, which can then be applied meaningfully in other contexts (Poynter & Tall, 2005). (Figure 3.)

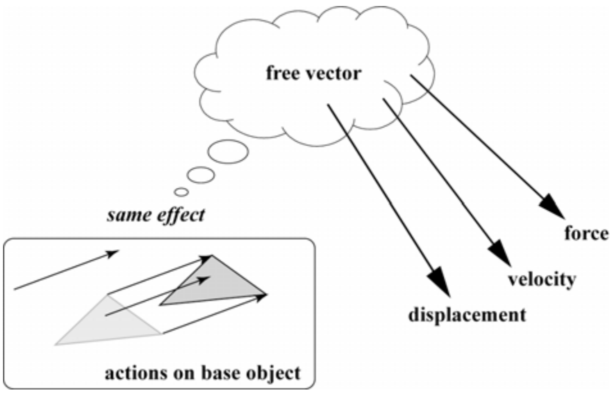


Fig. 3 Focusing on *effect* to build the concept of free vector

EMPIRICAL RESULTS

Poynter (2004) compared the progress of two classes in the same school, Group A taught by the researcher using an embodied approach focused on the effect of the translation, Group B taught in parallel using the standard text-book approach by a comparable teacher. The changes were monitored by a pre-test, post-test and delayed post-test, and a spectrum of students were selected for individual interviews.

The tests investigated the students' progress in shifting in stages, from a one-dimensional conception corresponding to a signed number, to the notion of an arrow as a journey in two dimensions with horizontal and vertical components, to a recognition of shifts having the same magnitude and direction giving the same *effect*, and, ultimately, the use of free vector itself as a manipulable concept (figure 4).

Stage	**Graphical**	**Symbolic**
0	No response	No response
1	Journey in one dimension	A signed number
2	Arrow as a journey from *A* to *B*	Horizontal and vertical components
3	Shifts with same magnitude and direction	Column vector as relative shift
4	Free vector	Vector **u** as a manipulable symbol

Figure 4: Fundamental cycle of concept construction of free vector addition

The students were assigned stages of development on their overall performance on the questionnaire from which (i)-(iv) are taken. This is the highest stage attained on at least two questions judged by two independent markers (Poynter, 2004).

Graphical stage	Group A (N = 17)			Group B (N = 17)		
	Pre-test	Post-test	Delayed Test	Pre-test	Post-test	Delayed Test
4	3	17	17	6	10	13
3	9	0	0	6	4	3
2	5	0	0	1	3	1
1	0	0	0	1	0	0
0	0	0	0	3	0	0

Table 1: Graphical responses to the whole questionnaire

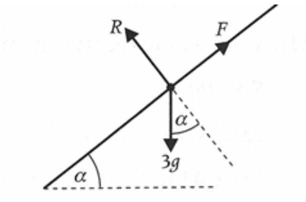
Symbolic stage	Group A (N = 17)			Group B (N = 17)		
	Pre-test	Post-test	Delayed Test	Pre-test	Post-test	Delayed Test
4	6	7	6	4	4	4
3	1	3	1	2	5	1
2	4	0	2	5	2	7
1	5	5	4	5	4	2
0	1	2	4	1	3	3

Table 2: Symbolic responses to the whole questionnaire

The striking change in the graphic response of Group A is clearly apparent and it remains over the longer-term. The symbolic scores, relating to the representation of vectors as coordinates are not significantly different.

The changes in response to the addition of free vectors also affect the performance of students in the use of forces in mechanics.

The improvement in the graphical stages of development dealing with free vectors was accompanied by a corresponding improvement with dealing with mechanics problems. After the course had ended, students from several classes were presented with the problem in figure 5. A similar problem on a pre-test had been solved by only 5 students out of 26 (Watson, 2002). In Group A, *all* the students answered the question. By this time, the students in Group B had been given the experimental treatment in two revision periods and *all* these students except three answered correctly. The three answering incorrectly had all missed the revision periods. In another class who had been given the standard treatment, only *one* out of eleven students correctly answered the question. Meanwhile, in a class taught by another teacher aware of the experimental technique, four out of six gave satisfactory responses. There is therefore evidence that technique is transferable and that it causes long-term improvements not only in the mathematics of free vectors, but in the physics of forces and acceleration.



A particle of mass 3 kg slides down a rough plane at an angle $\alpha = 30°$ to the horizontal. If $\mu = 0.5$ find the acceleration of the mass.

Figure 5: Forces in mechanics

NOTES

1. Anna Watson now publishes under her married name, Anna Poynter.

REFERENCES

Aguirre, J. & Erickson, G. (1984). Students' conceptions about the vector characteristics of three physics concepts. *Journal of Research in Science Teaching*, 21 (5), 439–457.

Askew, M. (1995). *Recent Research in Mathematics Education 5–16*. London: HMSO/OFSTED.

Graham, T., Berry, J. (1997). A hierarchical model of the development of student understanding of force, *International Journal of Mathematics Education in Science & Technology*, 28 (6), 839–853.

Jagger, J. M. (1988). A Report on a Questionnaire to Test Students' Understanding of Mechanics. *Teaching Mathematics and Its Applications*, 7 (1), 35–41.

Poynter, A. (2004). Effect as a pivot between actions and symbols: the case of vector. *Unpublished PhD thesis*, University of Warwick. http://www.annapoynter.net

Poynter, A. & Tall, D. O. (2005, in preparation). Relating theories to practice in the teaching of mathematics. Available from http://www.annapoynter.net

Skemp, R. R. (1971), *The Psychology of Learning Mathematics*, London: Penguin.

Watson, A. (2002). Embodied action, effect, and symbol in mathematical growth. In Anne D. Cockburn & Elena Nardi (Eds), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, 4, 369–376. Norwich: UK.