RELATING THEORIES TO PRACTICE IN THE TEACHING OF MATHEMATICS

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There is nothing as practical as a good theory. (Richard Skemp, 1989, p. 27.)

In this paper we consider the formulation of theory to fit a specific teaching problem. We advocate that a good theory should be practical in the sense that it has application in the classroom to produce long-lasting improvements in ways that make sense to teachers and students. It would also help to simplify the situation by bringing together with essential common elements from other theories rather than aggregating different aspects from distinct theories. Our theoretical perspective therefore listens to the voices being expressed in the classroom, both the teachers and the students, and places these in the wider context of theories of learning mathematics in today's society.

INTRODUCTION

This paper is a contribution to a discussion on "Different theoretical perspectives in research: From Teaching problems to Research Problems". We begin by focusing on a specific teaching problem, consider what theories are available to give insight and use this experience to gain empirical data and develop new theory to solve the particular problem and then to give a broader picture that may be applicable in a much wider arena.

The problem here is the teaching of vectors (although the theory developed applies in a much wider range of contexts). Various solutions have been tried over the years, not only within the specific area of vectors, but also in the approach to mathematics teaching in general. Britain has a pragmatic culture; it identifies the problems and designs techniques to solve them. It is the land of the Industrial Revolution, the land of practical inventors with still more patents than any other country in the world. Its pragmatic solution to teaching vectors is to allow them to be studied practically where they arise in physics as forces, journeys, velocities, accelerations, and only later to study the mathematics of vectors in pure mathematics.

The teaching of vectors has not gone well. It has followed the path of many other topics that students find difficult. The presentation has been made more and more practical and less and less dependent on mathematical theory. In this sense it is similar to other 'difficult' parts of mathematics, including fractions and algebra.

Being a pragmatic nation, in Britain, the teachers are professionals. This means that they take their work seriously, work hard with long hours, with relatively little time in the schedule for analysis and reflection. Our experience (Poynter & Tall, 2005a) of interviewing colleagues show that they are aware that students have difficulties, but their awareness relates more to an episodic memory of what didn't work last year rather than a theory that attempts to explain *why* it went wrong and what strategies might be appropriate to make it go right. Where there are problems, they may develop new strategies the following year in an attempt to improve matters.

As an example we take the case of adding two vectors geometrically. The students are told that a vector depends only on its magnitude and direction and not on the point at which the vector starts. Therefore vectors can be shifted around to start at any point and so, to add two vectors, it is simply a matter of moving the second to start at the point where the first one ends, to give a combined journey along the two vectors, All that is necessary is to draw the arrow from the start point of the first vector to the end point of the second to give the third side of the triangle, which is the sum.

The problem is that many students don't seem to be able to cope with such instructions. Some 'forget' to draw the final side of the triangle to represent the result of the sum, others have difficulties when the vectors are in non-standard positions to start with, such as two vectors pointing into the same point, or two vectors that cross. Some students also find it difficult to cope with the case where two vectors start at the same point, and draw the 'result' of the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as the third side of the triangle, $\overrightarrow{BC}$.

Here we have a specific teaching problem that requires a solution. What theories are available to solve it? The science education theory of 'alternative frameworks' (Driver, 1981) suggests that that the students may have their own individual ways of conceptualising the concept of vector. However, it does not offer a theory of *how* to build a new uniform framework for free vector in a mathematical sense.

SOME EXISTING THEORIES

The embodied theory of Lakoff and his colleagues may offer a solution by encouraging us to consider how the students *embody* the concept of vector. The theory as presented by Lakoff and Núñez (2002) is not a theory of mathematics education; it even avoids mention in the main text of any mathematics education paper that is listed in its bibliography. Instead, it performs a theoretical analysis of sophisticated mathematics concepts (such as real numbers) from a high-level logico-mathematical viewpoint. The *pièce de résistance* in the analysis of Lakoff and Núñez is their analysis of algebra, which, instead of analysing the shift from arithmetic to algebra cognitively, takes an example of an axiomatic system and looks within that for an embodied framework expressed in terms of cognitive science. Our view is that the theory of embodiment is a useful tool, but it is a tool that needs honing for the job of teaching children, rather than abstract analysis of formal mathematical concepts.

The APOS theory of Dubinsky (Asiala et al, 1996) was another candidate for a framework for the research. Vectors represent, among other things, journeys, and journeys are actions that need routinizing and encapsulating as vector concepts.

APOS theory claims that mathematical objects are constructed by reflective abstraction in the sequence **A-P-O-S**, beginning with **A**ctions that are perceived as external, interiorised into internal **P**rocesses, encapsulated as mental **O**bjects developing within a coherent mathematical **S**chema. The theory is claimed to be an overall theory of cognitive development, though in practice its initial applications related mainly to programming in the computer language ISETL using actions, processes and objects that are formulated mainly symbolically. In its early development, visualisation was avoided on the grounds that mathematical objects were formed starting with external actions that could be interiorized as processes that could be encapsulated as objects. The originator of APOS, Ed Dubinsky, is a highly articulate research mathematician who works by logical use of definitions and deductions; as any analyst will tell you, pictures have subtle meanings that invariably suggest ideas that often contain hidden assumptions. So how could we use APOS theory in an English classroom to make sense to the teachers and learners who seek a pragmatic understanding of what they are doing?

We also looked at Skemp's (1976) theory of instrumental and relational understanding. It seemed evident that many students were learning instrumentally how to add vectors without any relational understanding. But what was the relational understanding that was necessary, how do we formulate that? Likewise we looked at theories of procedural and conceptual knowledge (Hiebert & Lefevre, 1986, Hiebert & Carpenter, 1992), for surely the students were learning procedurally and not conceptually. But here again, what is the conceptual structure?

It is apparent that the students begin learning with their own experiences. As they meet different examples of what is eventually desired to become a concept of free vector, they meet various practical examples, including vectors as journeys and vectors as forces. Many theories, such as that of Dienes (1960), suggests that the students must experience *variance* in the different examples and draw out of those examples the essential concepts that link them all and reject the incidental properties that are apparent in some instances but which do not generalise. However, in the case of vector, these incidental properties are coercive and lead to alternative frameworks that are difficult to shift.

We considered other frameworks, for example a framework of intuition and rigour that occurs in Skemp's (1971) intuitive and reflective thinking or in Fischbein's (1987, 1993) tripartite system of intuitive, algorithmic and formal thinking. Our problem in the latter is that these three categories exist as quite separate aspects, as they did in the first design of the English National Curriculum where Concepts and Skills were put under separate headings, before being reunited again in later versions of the Curriculum. But perhaps in this case, APOS theory could help. The version of process-object encapsulation used by Gray and Tall (1994) shows how concepts in school mathematics build from actions such as counting to successively more compressed procedures such as count-all, count-on, count-on-from-larger, to known facts, and the manipulation of known facts as mental objects through deriving new facts from old. Perhaps process-object encapsulation is the clue that links concepts and skills, starting from embodiment and moving to symbolism.

This certainly would fit in broad terms with SOLO taxonomy (Biggs & Collis, 1982) which has successive modes of operation (*sensori-motor*, *ikonic*, *concrete-symbolic*, *formal* and *post formal*) in cognitive development with links to the stage theory of Piaget (*sensori-motor, pre-conceptual, concrete operational* and *formal*) and the Bruner modes of *enactive*, *iconic* and *symbolic*. (There is not a direct match, because the categorisations are performed differently so, for instance, the change from Piaget's concrete operational to formal occurs at an earlier point that the change from concrete-symbolic to formal in the SOLO theory.) Each SOLO mode is contained in the next, and each mode has its own ways of building concepts through a cycle of stages named as unistructural, multistructural, relational and extended abstract. By combining sensori-motor and ikonic as a single mode of conceptual embodiment, then a concrete-symbolic (proceptual) mode, followed by increasing sophistication of formalism, there are broad links between a range of theories. For instance, Pegg (2002) identified a fundamental cycle of concept construction that appears in a range of theories (figure 1).

SOLO	Davis	APOS	Gray & Tall	**Fundamental Cycle**	
Unistructural	VMS† Procedure		Base Object(s)	Base Object(s)	**Schema Building** ↓
		Action	Procedure	Isolated Actions Procedure	
Multistructural			[Multi-Procedure]	Multi-Procedure	
Relational	Process	Process	Process	Process	
Unistructural (Extended Abstract)	Entity	Object Schema	Procept	Entity Schema	

† VMS stands for Visually Moderated Sequence

Figure 1: The fundamental cycle of conceptual construction

Given the underlying commonalities between the sequence of different modes of cognitive development and the fundamental cycle of concept construction in each of the different modes, we can begin to sense an underlying structure of human construction of mathematical concepts. However, any attempt to fit together these theories to form a corporate whole would seem to present insuperable difficulties. The theories may have broad commonalities but they also have subtle distinctions. It would be too much at this point in theory development to expect to have a cognitive theory that unifies all former theories. What would be more helpful is to seek a *practical* theory that resonates with fundamental aspects of these theories and yet also has the potential to make sense to teachers and students in the classroom.

DEVELOPING A GENERAL THEORY TO FIT THE PROBLEM

At this time our separate interests came into play and brought a surprising, and for us, insightful, observation that linked a single incident into the development of a whole theory of learning. The first-named author (Anna Poynter) was convinced that the problem arising from the complications of the examples of physics with their different meanings for journey, force, velocity, acceleration and so on, could be replaced a much simpler framework in mathematics, if only (and this is a big if) the students could focus on the important ideas. The problem was how to give a meaning to the notion of 'free vector' in a *mathematical* sense that could be applied to other contexts in an overall coherent way.

The breakthrough came from a single comment of a student called Joshua. The students were performing a physical activity in which a triangle was being pushed around on a table to emulate the notion of 'action' on an object. Joshua explained that different actions can have the same '*effect*'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors. He also observed that solving problems with velocities or accelerations is mathematically the same.

This single example led to a major theoretical development. In performing an action on objects, initially the action focuses on what to *do*, but abstraction (to coin a phrase of John Mason, 1989) is performed by 'a delicate shift of attention', to the *effect* of that action. Instead of saying that two actions are equivalent in a mathematical sense, one can focus on the embodied idea of having the same effect. At a stroke, this deals with the difficult compression from action to process to object formulated in APOS theory, by focusing attention on shifting from action to effect in the embodied world.

In the case of a translation of an object on a table, what matters is not the path taken, but the change from the initial position to the final position. The change can be seen by focusing on *any* point on the object and seeing where it starts and where it ends. All such movements can be represented by an arrow from start point to end point and all arrows have the same magnitude and direction. In this way *any* arrow with given magnitude and direction can represent the translation, and the addition of two vectors can be performed by placing two such arrows nose to tail and replacing them by the equivalent arrow from the starting point of the first arrow to the end of the second.

This theory of compressing action via process to mental object. by concentrating on the embodied effect of an action on base objects is widely applicable. It is a practical idea that proves of value in the classroom, as well as bringing together a range of established theories developed over the last half century by Piaget, Bruner, Dienes, Biggs & Collis, Fischbein, Skemp, Dubinsky, Lakoff & Núñez and many others.

Our next step is to test the theory empirically in the classroom. As this paper is directed at theory construction, we give a only brief outline of significant results found in Poynter (2004).

EMPIRICAL RESULTS

Poynter (2004) compared the progress of two classes in the same school, Group A taught by the researcher using an embodied approach focusing on the effect of a translation, Group B taught in parallel using the standard text-book approach by a comparable teacher. The changes were monitored by a pre-test, post-test and delayed post-test, and a spectrum of students were selected for individual interviews. The tests investigated the students' progress in developing through a fundamental cycle of concept construction in both graphic and symbolic modes:

Stage	Graphical	Symbolic
0	No response	No response
1	Journey in one dimension	A signed number
2	Arrow as a journey from A to B	Horizontal and vertical components
3	Shifts with same magnitude and direction	Column vector as relative shift
4	Free vector	Vector **u** as a manipulable symbol

Figure 2: Fundamental cycle of concept construction of free vector

Poynter (2004) focused on several aspects of the desired change that could be tested. Here we consider three of them. Poynter hypothesised that students, who encapsulate the process of translation as a free vector, will focus on the *effect* of the action rather than the action itself. This should enable them to add together free vectors geometrically even if the vectors are in 'singular' (non-generic) positions, such as vectors that meet in a point or which cross over each other. It should enable them to use the concept of vector in other contexts, e.g. as journey or force. In the case of a journey, it should allow the student to recognise that the sum of free vectors is commutative. (As a journey, the equation $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{BC}+\overrightarrow{AB}$ does not make sense, because $\overrightarrow{AB}+\overrightarrow{BC}$ traces from A to B to C but, $\overrightarrow{BC}+\overrightarrow{AB}$ first represents a journey from B to C and requires a jump from C to B before continuing. As free vectors, $\mathbf{u}=\overrightarrow{AB}$ and $\mathbf{v}=\overrightarrow{BC}$, we have $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.)

It was hypothesised that experimental students would be more able to:

1. add vectors in singular (non-generic) cases
2. use the concept of vector in other contexts (eg as journey or as force)
3. use the commutative property for addition.

Students were asked to add two vectors in three different examples:

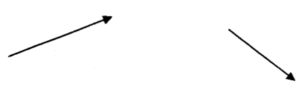	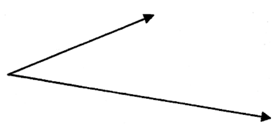	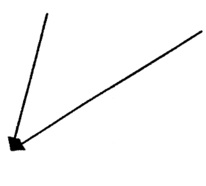	2) In each case add the two vectors together 3) If there is any other way you could have done any of the additions of the two vectors in Q2 show it.
(a)	(b)	(c)	

Figure 3: questions that could be considered singular

When we asked other teachers what they felt students would find difficult, we encountered differences between the responses of a colleague who taught physics and two others who taught mathematics. As mathematicians, we saw part (a) to be in a general position, because it only required the right-hand arrow to be pulled across to the end of the left-hand arrow to add as free vectors; (b) evoked the idea of a parallelogram of forces; (c) was considered singular because it was known to cause problems with some students embodying it as two fingers pressing together to give resultant zero.

All teachers considered part (c) would cause difficulties. However, they differed markedly in their interpretations of parts (a) and (b). The physics teacher considered that the students would see the sum of vectors either as a combination of journeys, one after another, or as a sum of forces. For her, (a) was problematic because it does not fit either model, but (b) would invoke a simple application of the parallelogram law. As an alternative some students might measure and add the separate horizontal and vertical components. The two mathematics teachers considered that students would be more likely to solve the problems by moving the vectors 'nose to tail' with the alternative possibility of measuring and adding components. One of them considered that students might see part (a) as journeys and connect across the gap, and in part (b) might use the triangle law in preference to the parallelogram law. The other sensed that (b) could cause a problem because 'they have to disrupt a diagram' to shift the vectors nose to tail—an implicit acknowledgement of the singular difficulty of the problem—and part (c) would again involve shifting vectors nose to tail although she acknowledged that some students might do this but not draw the resultant (which intimated again that they see the sum as a combination of journeys rather than of free vectors).

The performance on the three questions assigning an overall graphical level to each student is given in Table 1.

Graphical stage	Group A (Experimental)			Group B (Control)		
	Pre-test	Post-test	Delayed	Pre-test	Post-test	Delayed
4	0	1	12	2	0	7
3	1	9	4	1	10	3
2	4	6	1	1	3	2
1	4	1	0	4	1	0
0	8	0	0	9	3	5
TOTAL	17	17	17	17	17	17

Table 1: Graphical responses to the singular questions

Using the t-test on the numbers of students in the stages reveals that there is a significant improvement in the experimental students from pre-test to delayed post-test ($p < 0.01$) but not in the control students.

Similar resultstesting the responses to questions in different contexts and questions involving the commutative law are shown in tables 2 and 3.

Graphical stage	Group A (Experimental)			Group B (Control)		
	Pre-test	Post-test	Delayed	Pre-test	Post-test	Delayed
4	0	0	8	0	0	2
3	0	9	3	2	3	5
2	1	2	2	0	3	3
1	1	5	4	0	2	3
0	15	1	0	15	9	4
TOTAL	17	17	17	17	17	17

Table 2: Graphical responses to questions set in different contexts

The change is again statistically significant from pre-test to delayed post-test ($p<0.01$) using a t-test.

Graphical stage	Group A (Experimental)			Group B (Control)		
	Pre-test	Post-test	Delayed	Pre-test	Post-test	Delayed
TOTAL	0	7	12	4	6	5

Table 3: Responses using the commutative law of addition

In this case the change is from a significant difference in favour of Group B on the pre-test ($p<0.05$ using a χ^2-test) to a significant difference in favour of Group A ($p<0.05$ using a χ^2-test). Further details may be found on the web (Poynter, 2004).

What is clearly important here is not the statistical significance, but the evident changes which can be *seen* not only to improve the situation for Group A from pre-test to post-test, but more importantly to *increase* the level of success by the delayed post-test. There is a clear difference in the long-term effect of the experimental teaching programme.

BROADER THEORETICAL ASPECTS

The theory reveals a parallel between focusing on the effect of embodied actions and the compression of symbolism from procedure to process to object has the potential to be simple to describe and implement with teachers and students. All that is necessary is to have appropriate activities and to mentor the participants to focus on the effects of carefully designed actions. This applies in a variety of areas, not only in representing vectors dually as transformations and as free vectors, but also in other areas where symbols represent a process being encapsulated into a concept. For instance the process of counting is compressed to the concept of number by focusing on the *effect* of counting in terms of the last number spoken in the counting schema. Likewise, the process of sharing and the concept of fraction, in which, say, sharing something into 4 equal parts and taking 3 of them gives the same effect as sharing into 8 equal parts and taking 6. This corresponds symbolically to having equivalent fractions ($^3/_4$ or $^6/_8$). Likewise different algebraic procedures having the same effect gives an alternative way of looking at the idea of equivalent algebraic expressions. Other processes in mathematics, such as the concept of function, also result from a focus on the *effect* of an input-output action, rather than on the particular sequence of actions to carry out the process, revealing the wide range of topics in mathematics that can benefit from this theoretical analysis.

This research into a single classroom problem has therefore stimulated developments in the relationship between embodiment and (proceptual) symbolism as part of a wider general theory of the cognitive development of three worlds of mathematics (embodied, symbolic and formal), (Watson, Spyrou & Tall, 2003, Tall, 2004). This theory, in turn, also builds on earlier work that theorizes three distinct kinds of mathematical object: "One is an *embodied object,* as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. The second is the *symbolic procept* which acts seamlessly to switch from an often unconscious 'process to carry out' using an appropriate algorithm to a 'mental concept to manipulate'. The third is an *axiomatic concept* in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory," (Gray & Tall, 2001).

In this way, looking at how a particular teaching problem benefits from different theories can be fruitful, not only in addressing the teaching problem in a way that makes practical sense to pupils and teachers, but also in analysing and synthesising a range of theories to produce a practical theory.

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[1] Anna Watson is now Anna Poynter, following her recent marriage.