Effect as a pivot between actions and symbols: the case of vector

by

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Declaration

I hereby certify that this thesis is my own work. Portions of the Preliminary Study chapter (Chapter 4) were used in two papers: one submitted to the 26th Annual Conference of the International Group for the Psychology of Mathematics Education (PME) in Norwich, UK, in 2002; a second one submitted to the Mediterranean Journal of Mathematical Education in Cyprus, in 2003. The references to the papers are included in the References of this thesis. I also confirm that this thesis has not been submitted for a degree at any other university.

Abstract

Students' difficulties with vectors in Mechanics, at AS and A level, have been considered in a number of studies to date. Some of the research considers how students' intuitions arise from working in different contexts and how it affects their problem-solving capabilities, others think that pre-Newtonian views affect students' thinking. There are considerations that the idea of vector has different meanings in different contexts and therefore it is not easily transferable from one context to another. There are suggestions in the literature that a qualitative approach to teaching would help students to learn. None of the research studied considers the important idea of focusing on the vector concepts that are *common* to the various contexts, instead they are more concerned with the problems caused by the *differences* between them. Nor do they focus on the compression of a vector as an action into the more flexible idea of a free vector as a single mental object that can be represented by any arrow of given magnitude and direction.

In this thesis an approach is developed to base the students' experience on manipulating physical objects, to focus on the *effect* of a translation rather than on the action itself. The essential idea is to notice that every point on the object translated (and on the hand doing the translation) moves in the same direction and travels the same distance. The *effect* of the translation is therefore represented by *any* arrow of this magnitude and direction, leading to the notion of free vector. From the same viewpoint, the sum of two vectors is simply the single translation having the same effect as the combination of one translation followed by the other.

The main hypothesis is that:

> Teachers can help students develop the notion of a translation as a free vector through focusing on the effects of physical actions, linking graphic and symbolic representations, so that the concept of free vector is constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

This was tested by a comparative study of two classes using both quantitative and qualitative methods. The control group carried on with the normal programme of study while the experimental group was exposed to lessons focusing on of the notion of a free vector as the effect of a transformation. The students' own constructions were supported by activities and discussions in reflective plenary sessions. The results of the study revealed that there were significant changes in the students following the experimental programme, in which they were more likely to conceive of the symbols for vectors as cognitive units operating in a flexible and versatile manner. The quantitative improvement was sustained and increased over a longer period.

Interviews with the teachers revealed differences between the mathematics and physics teachers' perceptions of the students' expected difficulties. Interviews with the students revealed the more successful interviewees referring to the concept of vector as a single cognitive unit across contexts, while the less successful tended to consider the concept of vector operating in different ways as journeys and as forces.

Both quantitative and qualitative data show significant conceptual changes in students following the experimental approach; these changes were more marked over the longer period of time between pre-test and delayed post-test.

Chapter 1

Thesis Overview

'I hear, and I forget; I see, and I remember; I do, and I understand.'
saying of Confucius, quoted in Nuffield (1967a)

'I do and I understand ***how****, I reflect and I understand* ***why****.'*
extended by Anna Poynter (2003)

Prologue

As a teacher of mathematics for many years I became concerned how students seemed to be able to learn to do techniques to score highly on examinations, yet seemed not to be able to apply their knowledge in slightly different situations, or to retain their skills for ready use in subsequent courses. For example, in mechanics my students were highly successful at resolving vectors horizontally and vertically to solve mechanics problems, yet when faced with a rectangular block on a rough inclined plane, most were unable to resolve the force due to gravity down the plane.

Up to this time, I had taught vectors according to the text-books, including physical experiments with objects on a slope, practising all possible calculations and variations. I made sure to warn students about any pitfalls and drilled into them the techniques of answering questions. They seemed to be able to answer every question in the book, until they met something slightly different on the test.

After talking with some of the students and teachers, who seem to have the same problem, I decided to take the matter seriously. I thought to myself: "If only students could concentrate on the simplicity of the mathematical idea instead of the many complications connected to different contexts, then they should be in a better position to solve problems in novel situations." However, although the mathematics seems to be simple for an expert, there seem to be untold pitfalls for the learner.

My quest in this thesis is therefore to understand better ***why*** *students have such difficulties and* ***how*** *they may be encouraged to reflect on their knowledge to focus on the essential ideas in a way that could be used flexibly in new contexts.*

1.1 Introduction

This study is concerned with how students make sense of mathematical ideas and how they symbolise their physical conceptions in meaningful ways. Although it focuses on the notion of vectors, the theory constructed is intended to have a much wider applicability, linking cognitive theory with practical application in the classroom.

Mathematics is usually considered as a highly logical activity but it is created and learnt by human beings who interact with the world and develop higher-order thinking capacities by reflecting on their actions. In practical terms this suggests two distinct stages: encouraging students in 'hands on' specific activities, and then to reflect on the essential elements of these activities to build theories. In theoretical terms the work relates to the cognitive science literature of embodiment and the mathematics education literature which focuses on how processes that we perform can be symbolized so that the symbols can become meaningful ideas that we can manipulate and think about.

The notion of embodiment has rich and varied meanings in the literature of cognitive science, which will be discussed in greater detail in chapter 2. The intention of this thesis is to develop a practical theory to enhance learning by reflecting on physical experience to build sophisticated mathematical concepts.

The concept of the vector is an important and useful notion in science and mathematics. We encounter it as both a geometrical and a symbolic idea. The notion of vector in the English curriculum initially appears mainly in the science syllabus. The first contact with vectors that English students have is in Physics, based on real world experience and observation. Physical experiences, however, occur dynamically in time, while the problems set in Physics and Mathematics are often considered at a specific instant of time. Everyday experience may tell a person what happens eventually but not what happens at an instant, resulting in what I shall call a 'false intuition' that can potentially impair an individual's capacity to think logically and mathematically.

Real experiences with vectors are many and varied (displacement, velocity, acceleration, forces, etc.) and the differences can cause complications, whereas the mathematics of vectors, once it is understood, has a simpler structure that applies to *all* embodiments and therefore potentially has more power. The view I have held for many years, in the case of vectors, is that there are two fundamentally different worlds: the *embodied* world with its various different environments (contexts) that involve sensory experience and visualisations, and the *symbolic* world of mathematics with its use of symbols to represent vectors with their components being written in a column matrix or as **i** and **j** components. There are parallels between these worlds in the way they compress action into process and into concept which we shall discuss in greater detail in chapters 2 and 3. However, in the school environment, the emphasis is on preparing students for assessment and there is always a dilemma of how to approach the teaching of the topic in a crowded curriculum. There often does not seem to be time to show students how these two parallel words work together and enable them to construct the concepts in a meaningful way. It is easy in a crowded syllabus to make wrong assumptions about students' embodied awareness of the topic and enter some of them into the symbolic world too quickly.

In the development of the research in the thesis, a preliminary study helped to build the theoretical hypothesis. I decided to involve the group of students I was teaching, in activities and reflective plenaries in which I focused on how the mathematical notion of vector might be built from the physical experience.

Our specific practical activities involved conceiving transformations as physical movements of a shape on a flat surface and focusing on the idea that different actions can give the same result. For instance, when looking at the hand translating a shape, the end of each finger moves in the same direction by the same amount, thus the arrows from the starting point to the finishing point all have the same magnitude and direction and effectively represent the same translation. By focusing on the *effect* of the translation rather than the physical movement of a particular point gives the student an opportunity to give embodied meaning to the mathematical notion of a *free* vector (with specific magnitude and direction but not starting at any specific point in space).

This approach has the potential of being extended to the mathematical notion of addition of vectors in that the *effect* of two successive shifts is the same as that of the single shift from the initial position to the final position. This in turns leads the concept of the commutativity of vector addition, which only makes sense when we think of *free* vectors detached from any context. (If A, B, C are fixed points, then the journey $\overrightarrow{AB}+\overrightarrow{BC}$ makes sense as a journey from A to B to C, but $\overrightarrow{BC}+\overrightarrow{AB}$ suggests first travelling from B to C, and then seems to require a jump from C to A to perform the second journey from A to B.) By using a combination of practical activity, reflection and plenary discussion, it becomes possible to create an environment in which students can potentially construct links between different ways of looking at the same concept in different contexts. The time devoted to the topic of vectors was the same as in other classes, but the intention was to produce more long-term stability of concepts.

Concepts of vectors are usually formulated in specific contexts, such as displacement and forces in the physical world, where vectors either follow each other (displacement) or come out from one point (forces acting on an object). Free vectors in the mathematical world, on the other hand, are often drawn as separate vectors in space which do not overlap each other. Each context therefore has its own type of general format which will be called a *generic case* in that context. In the context of a journey, a combination of journeys is given by following the first journey by the second, and in the generic case, the first journey ends where the second begins. In the context of several forces acting on a point, a combination of forces is added together symbolically by working out the horizontal and vertical components of each vector and adding them together to present the final answer as a column vector. Graphically, students would be expected to add free vectors ‘nose to tail’ by shifting the start of one vector to the end of the other. However, if they are given example of vectors that cross, or vectors whose ‘noses’ meet at a point, such questions might cause confusion.

These examples are called *singular cases*. In simple terms, a singular case is an example that has incidental properties which are not typical of the general case in the given context. It is my belief that students' ability to handle singular cases will be highly indicative as to whether they have a rich flexible concept rather than a more limited procedural view.

Examinations usually involve generic types of problems and, through extensive practice with past questions, students can be very well prepared to answer such problems. They learn the procedures for the specific situation; however, even students who obtain a top mark in this way in the exam might not be able to answer unfamiliar singular cases. From my experience, when seemingly very capable students come out from an exam and say that a particular question was impossible, by looking at the question afterwards I was often able to classify it as a singular case.

To encourage students' construction of rich connections between different physical and mathematical contexts, I built reflective plenaries into lessons. The intention was to help each student to build a concept of vector as a cognitive unit which encapsulates all the aspects of free vector into a simple single mathematical idea. The goal was to build a long-term conceptual stability of concepts to give students the insight and confidence to tackle singular cases and solve them using understanding at a higher cognitive level.

1.2 The background to the research

I initiated my investigation by looking carefully at the Mathematics and Physics syllabuses and talking to the teachers of both subjects. This seemed appropriate since both subjects use vectors in various topics.

Through discussions with the teachers I found that other teachers shared similar strategies to those that I used, with a positive attitude in trying their best to help students to perform well in exams. Like myself they show awareness of students' difficulties and, in their teaching, do their best to give students the right procedures to answer questions and ways of overcoming the obstacles. They have no time to find the reasons for problems students might encounter but know instinctively which questions they should find easy and which they may find difficult, and what type of mistakes they might make.

My goal was to attempt to find the nature of the students' difficulties, to formulate general ways of teaching that would improve the student's fundamental insight into mathematics and to express this in a language that could be shared both with other teachers and with the students themselves.

In attempting to make headway in such an ambitious task, I sensed that it would be necessary to take account of positive aspects (marked below with a 'smiley' ☺) and negative aspects (marked with a frown ☹)

These aspects were as follows:

☺ In their quest for improving the students' understanding and to give them additional meaning to the ideas they wish the students to learn, teachers often try to give the students a physical experience of the idea by either involving them in doing their own experiments or by teacher demonstration of a physical experiment.

☹ In the case of vectors, experience reveals that this can cause additional problems instead of solving them. When talking to the Physics teachers, it appears that different contexts are treated differently in that subject. For example, vectors as displacements are considered as journeys following each other and, when adding two displacements, the triangular rule of addition would be considered (figure 1.1). On the other hand, vectors as forces are usually presented as acting on an object (presented as particle), and for addition of two forces the parallelogram rule would be considered (figure 1.2).

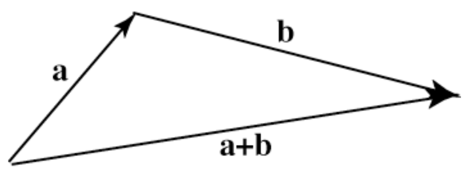


Fig. 1.1 The Triangle Law

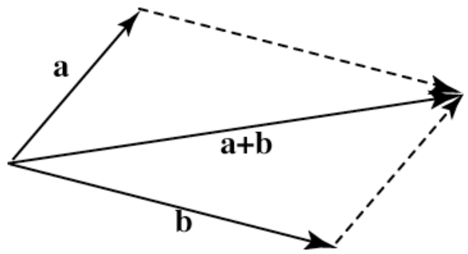


Fig. 1.2. The Parallelogram Law

Alternatively students' would be taught to resolve the vector quantities into horizontal and vertical components, which would be added together. The mathematics teachers involved in this research have the intuitive notion that some questions might be difficult because of the physical representation (singular cases) but generally for them the vector is a cognitive unit which can be applied to different contexts. They do not seem to consider that different contexts have clashing meanings.

☺ In attempting to help the students to build a more flexible conception of the notion of vector that includes the whole structure of embodiment and process-object encapsulation, I was struck by the interpretation formulated by one particular student whom we will call Joshua. He explained that different actions can have the same '*effect*'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors. He also observed that solving problems with velocities or accelerations is mathematically the same: "the only difference is that one is metres per second and another metres per second squared." He was able to operate with the vector as a cognitive unit which can be applied to different contexts.

☺ This idea of 'effect' seemed a possible way of introducing vectors in a mathematical way that focused on the mathematical ideas rather than the different physical contexts that lead to 'false intuitions'. The transformations of objects can be seen as actions on physical objects, but then, by focussing on the *effect* of the transformation, it may be possible to give a meaning of the transformation of the object as a mental concept. Such a concept is already available: it is the physical arrow that represents the magnitude and direction of a translation. In this way, the transformation *as an action* can be related to a vector *as a mental object*. To focus positively on these ideas, I decided to involve the students in physical activities which were then used as the basis for reflection and discussion in reflective plenary sessions, with the teacher helping students to build theoretical ideas based on their own experience.

1.3 The structure of the thesis

The relevant literature is reviewed in chapter 2. As a major focus of the thesis is the transition from physical experience to mathematical symbolism, it is important to look at the literature as to how intuition can affect understanding and how the experience from the physical world can be used to support conceptual development instead of having a negative effect. The chapter also surveys different theories of knowledge and learning, focusing not only the ways in which knowledge is constructed and understood, but also how it is compressed and encapsulated as thinkable symbols in the transition from embodiment to symbolism.

Chapter 3 begins with a study of the curriculum in Mathematics and Physics that is encountered by the students in their earlier studies, focusing on the development of the concept of vector in the text-books that have been used. In the same chapter, I also look at some examples of research conducted in Mechanics and into vector concepts that are relevant to this research.

Chapter 4 reports results from a preliminary investigation which took place before the main research. This enabled me to formulate the central theoretical hypothesis to be tested in the body of the thesis and to design a theory of teaching vectors to use and study in the experimental work with students. The fundamental framework is illustrated in figure 1.3, beginning from (physical) embodiment with actions on objects, to focus on the effects of those actions to form mental concepts.

action (doing)	➔	effect (seeing, reflecting and focusing)	➔	mental concept (symbolising)

Fig. 1.3 compression from action to concept by focusing on the effect

The details of this preliminary investigation and how it led me to the idea of *effect* are discussed in detail in chapter 4. Having focused on the important elements to be considered in the research study, chapter 5 turns to a consideration of the methodology of the research and the specific methods to be used. These are trialled in a Pilot Study described in chapter 6.

The Pilot study focused on carrying out the teaching experiment and designing and testing a questionnaire to assess the comparative effects of the experimental method used in an experimental class, compared with a traditional approach in a parallel control class.

The questionnaire consisted of questions designed to test:

- if students compressed the notion of free vector and the *effect* of actions to see the equivalence of vectors having the same magnitude and direction which could be freely represented anywhere in the plane whether touching the object being shifted or not;
- if students develop a mental concept of vector capable of solving not just generic but also singular cases;
- if students have the flexibility to use a vector as a mathematical mental concept to solve problems independent of the context;
- if they can apply the concept of the commutative law of addition: **u**+**v**=**v**+**u,** which can only be understood by students treating **u** and **v** as 'free vectors'.

The pilot study also included trial interviews to test some of the interview techniques intended to seek greater insight into:

- use and flexibility of language when discussing problems connected with vector addition;
- the focus of attention at any given time (whether it is on actions, or procedures or on the effects of those actions and procedures);
- the way in which different contexts affect their thinking;
- their flexibility in dealing with different modes of operation (graphical/symbolic).

The details of questions to be used in the main study, which were tested in the Pilot study, and the outcome of the Pilot Study are presented in chapter 6.

Chapter 7 presents and analyses the quantitative data collected from the use of the questionnaire in three tests: pre-test, post-test and delayed post test. Chapter 8 presents the qualitative data from interviews with the teachers. Chapter 9 presents the qualitative data from the interviews with selected students. The data collected gives both quantitative and qualitative evidence that is consistent with the main hypothesis that the rich embodied experience in the experimental approach, focusing the students on the notion of effect, helps them to internalise processes into manipulable mental concepts that remain stable through to the delayed post-test.

The final conclusions, including detailed summaries of the research and analysis of results, are presented in chapter 10, together with reflections on the limitations and generalizability of the results, leading to avenues for future research and development.

Chapter 2

Literature Review

2.1 Introduction

This chapter presents the review of the literature that theoretically underpins the thesis. It includes general theories of learning and understanding, but focuses particularly on embodiment on the one hand and symbolism on the other. The relationship between embodiment and symbolism will play a fundamental role in the approach to the teaching of mathematics in general and vectors in particular. It will be used to build a theoretical framework in which meaning for symbolism is constructed from reflection on embodied activities, and to lay out a schema of development to enable the cognitive development to be described and tested in the main study.

2.2 Theories of knowledge and understanding

In working with students, I found that their responses did not seem to fit within a single theoretical position and therefore found it necessary to review a number of different theories to build a theoretical framework to categorise answers that arose in my research. The framework developed is a blend and extension of other theories. In what follows I describe the literature and the theories I have considered, and my reasons for building the theoretical framework used in this thesis.

2.2.1 Intuition

From my experience, different physical encounters of vectors gained in Physics or every-day life can cause complications for students. They answer questions from a 'false physical intuition' point of view. For example, when I asked students in the preliminary study to add two vectors **a** and **b** as shown in fig 2.1, nearly 50% gave a wrong answer **c,** marked with a dotted line.

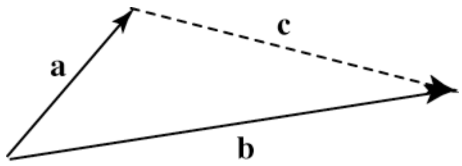


Fig. 2.1 Example of the 'intuitive' addition of two vectors

Although this might seem like a misapplication of the triangle law, from interviews, it appears that several students used a physical experience of two people pulling them in directions of vectors **a** and **b**. There is a stronger pull in the **b** direction and therefore that's where they are going to end up moving. Students in this case seem to forget about the mathematical rules of adding vectors and base their answer on 'physical intuition' which, regrettably, leads them astray. As I have decided to classify such answers as physically intuitive responses I have become interested at the ways that 'intuition' has been formulated in the past.

The early philosophers were interested in intuition as a basic human faculty. In his '*Essay Concerning Human Understanding'* (1690), the English Philosopher, John Locke specifies three degrees of knowledge, which are "intuitive", "demonstrative" and "sensory". In discussing Locke's ideas, Sierpinska (1990) refers to intuitive knowledge as "irresistible and certain". Intuitive knowledge seeks "identity or diversity" because "it is the first act of the mind to perceive its ideas and to perceive their difference and that one is not the other" (Sierpinska, 1990, pp. 28–29).

In his *Critique of Pure Reason* (1781), the philosopher Kant summarizes cognition in the following terms:

> […] all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas. (Kant, 1751, p.404)

Skemp (1971) specifies two modes of functioning of intelligence: intuitive and reflective. He specifies the intuitive mode as being 'aware through our receptors (particularly vision and hearing) of data from the external environment, this data being automatically classified and related to other data by the conceptual structures,' (Skemp, 1971, p.51).

Royce *et al.* (1978), in a review of psychological epistemology, includes intuition as a "distinct cognitive phenomenon, together with perception, thinking and symbolisation."

Fischbein, Tirosh and Melamed (1981) write:

> Accepting intuitively a certain solution or a certain interpretation means to accept it directly without (or prior to) resorting explicitly to a detailed justification. [...] The problem of identifying the natural intuitive biases of the learner is important because they affect – sometimes in a very strong and stable manner – his concepts, his interpretations, his capacity to understand, to solve and memorize in certain area. We are naturally inclined to retain interpretations which suit these natural, intuitive biases, and to forget or to distort those which do not fit them.
>
> (Fischbein *et al.*, 1981, p 491)

They end their article by concluding that:

> Didactical strategies must be devised for shaping improved intuitive interpretations and for overcoming conflicting intuitive biases
>
> (Fischbein *et al.*, 1981, p. 512)

Fischbein (1994) specifies intuition as one of the three components of mathematics as a human activity. The other two components are formal and algorithmic. Theoretically, intuitions may play a facilitating role in the instructional process, but very often, contradictions may appear:

> Intuitions may become obstacles – epistemological obstacles [...] – in the learning, solving, or invention processes.
>
> (Fischbein, 1994, p. 232–234)

Sierpinska (1990) summarises a model of understanding mathematical concepts developed by Herscovics and Bergeron (1989) in which they also look at intuition.

She quotes them:

> Intuition [...] arises from a type of thinking based essentially on visual perception and results in an ability to make rough non-numerical approximations. (Sierpinska, 1990, p. 28)

According to Dewey (1988), and then Piaget, the first stage of concepts are formed from experience of a single object and by building a general category of objects with similar or the same characteristics. The second stage comes from discriminating between properties of characteristic and non-characteristic objects. The third stage consists of "application to explaining new cases with the help of a discovery made in one case," (op. cit., p. 164-165). In my own research, being aware of the possible 'false intuitions' in the second stage, the question arises whether the third stage—if implemented carefully and reflectively—can help to straighten the misconception gained in the second stage.

2.2.2 Instrumental-relational understanding

As I was intending to reintroduce the concept of vector addition to the experimental group of students in a specific context, I decided to look at the theory of the instrumental and relational understanding of Skemp (1976), which was expanded by other researchers, and also at the related theory of procedural-conceptual knowledge introduced by Hiebert & Lefevre (1986).

Skemp (1971, p. 15) describes two types of learning. One he calls habit learning, or rote-memorizing, which is instrumental. The other learning involves understanding, which he calls intelligent learning. Piaget pioneered studying the second type of learning (cognitive processes in children and adults). Skemp indicates that reflective activity "involves becoming aware of one's own concepts and schemas, perceiving their relationships and structures, then manipulating these in various ways," (Skemp, 1971, p 77). He also suggests that "low-order concepts can be formed, and used, without the use of language," (p. 26) however "making an idea conscious seems to be closely connected with associating it with a symbol," (p. 78) and "it is largely by the use of symbols that we achieve voluntary control over our thoughts," (p.78). According to Skemp, symbols help us to "reduce the cognitive strain of keeping the whole of the relevant information accessible," which is very important since "one of the aims of reflection is to become aware of how one's ideas are related, and to integrate them further.

My analysis of observing students for many years, as a mathematics teacher, is that the lack of the requirement for analysis and symbolic accuracy in graphical representations causes many problems when students have to apply their knowledge to questions involving applications of vector quantities in two and three dimensions. The lack of accuracy seems to often stop, observed by me students, relating ideas and integrating them further. Krutetskii (1976) suggested that gifted children have “a tendency to interpret environmental phenomena on the level of logical and mathematical categories, to perceive many phenomena through the prism of logical and mathematical relationships, and distinguish a mathematical aspect when perceiving many phenomena in an activity” (1976, p. 302).

Van Hiele (2002) writes, ‘The theoretical level to which the axioms belong can only be reached by starting from the descriptive level’ (object recognition level) otherwise they have to learn ‘parts of geometry by heart and that means only instrumental understanding’. He also states that ‘Many teachers were very content with such a course of events [...] and there were always pupils who liked mathematics from the very beginning and found their own way to the higher levels. But a great part of the pupils developed a dislike of geometry and after their study was finished forgot practically all of it.’ (van Hiele, 2002, p 34-35). He also warns that, ‘Reflection fails because the pupil only disposes of concepts of the visual level and those concepts do not lead to a result on the descriptive level,’ (van Hiele, 2002, p. 35). The visual level means that shapes are recognised by seeing and not by their properties. He gives as the instrumental example drawing a picture using coordinates or vectors on the squared paper.

2.2.3 Procedural-conceptual knowledge

According to Hiebert and Lefevre (1986), procedural knowledge “is made of two distinct parts. One part is composed of the formal language, or symbolic representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical task.” (p 6). The second one of these seems like

Skemp's instrumental understanding, which indicate step-by-step instructions that prescribe how to complete tasks. Hiebert and Lefevre say, "in general, knowledge of the symbols and syntax of mathematics implies only an awareness of surface features, not a knowledge of meaning," (p. 6). What the authors underline is that conceptual and procedural knowledge have to be linked, otherwise "students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing," (p. 9). Expanding on their ideas of symbols, they say, that symbols represent ideas that can be met in real-world experiences. These ideas can be represented as conceptual knowledge, which provides the referents for symbols.

This would fit with the way I reintroduced the experimental group to the idea of the vector (described in methodology chapter, later on).

> If the procedures are related to the underlying rationale on which they are based, the procedures begin to look reasonable. It is possible to understand how and why the procedures work. [...] procedures that are meaningful, that are understood by their users, are more likely to be recalled. (Hiebert and Lefevre, 1986, p. 10-11)

Therefore, if my technique of reintroducing the experimental group to vectors is correct, the retention should be better and students should be able to perform better than the control group after a break (six months in case of my research).

Basically conceptual knowledge and relational understanding indicate that somebody learnt something with meaning, while procedural knowledge and instrumental understanding indicate that somebody learnt how to solve a problem but not necessarily with meaning.

While the first ideas of Skemp's instrumental and relational understanding placed these two types of understanding into separate classifications, Hiebert and Lefevre (1986) say that

> Not all knowledge can be usefully described as either conceptual or procedural. Some knowledge seems to be a little of both, and some knowledge seems to be neither. (Hiebert and Lefevre, 1986, p. 3)

They write that

> [...] conceptual knowledge is characterized most clearly as knowledge that is rich in relationships (Hiebert and Lefevre, 1986, p. 3)

They also say that

> In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information.
> (Hiebert and Lefevre, 1986, p. 4)

They quote Skemp (1971) when describing “understanding” as “the state of knowledge when new mathematical information is connected to existing knowledge.” (p. 4). The other way they see development of conceptual knowledge is by “the construction of relationships between pieces of information that already have been stored in memory”. They quote Bruner (1961), Ginsburg (1977) and Lawler (1981) as the predecessors of such a theory. They use the term ‘abstract’ as the degree to which a unit of knowledge is tied to a specific context. According to them: “Abstractness increases as knowledge becomes freed from specific contexts,” (p. 5).

This is very relevant to my investigation of students’ responses. I have found from the preliminary investigation that students who performed best in the questionnaire, used in the main study, were those who either connected to a very specific context of a journey or those who saw the vector as a mathematical object. And therefore it seems that they used abstractness to different degrees.

Hiebert and Lefevre write (1986, p. 5) that “some relationships are constructed at a higher, more abstract level than the pieces of information they connect”, which they call a reflective level. They note that it is not always easy to assess where procedural knowledge ends and conceptual starts. I have tried to assess this difference not only through the responses to my questionnaires but also through the interviews. The assessment of the students’ responses is graded in stages of their conceptual development. These stages were developed with the help of the text-book that students study in Year 11. The book introduces vectors in stages from the embodied action of transformation of an object to the idea of the column vector, and arrows having a specific direction and magnitude, through to the idea of vector addition and equivalent vectors to the idea of the free vector. The interviews looked at the development of students' language to express their actions when adding vectors, trying to assess if they express their conceptual or procedural knowledge.

To obtain a deeper insight into the nature of human understanding, it has proved useful to look more closely at the link between intuitions produced by embodiment and the symbolism that is used to represent the processes and concepts.

2.3 Different modes of operation in mathematics

At school students are introduced to vectors in two ways. In Physics, vectors are introduced as journeys or forces, added according to the triangle law or parallelogram law, with different meanings and then represented as two one-dimensional components which are added by adding components. The sixteen-year-old students studying Mechanics in my school, who are taught in this way seemed to cope well with resolving horizontally and vertically and solving problems formulated in this context. However, according to my preliminary study (to be discussed in detail in chapter 4), many of these students have subtle problems with geometric interpretations, particularly with free vectors. The evidence arose in the way they answered certain 'singular' (unusual) questions (shown in figure 1.4b) which do not conform to the general prototypes that are suggested by their earlier experiences. In chapter 1, I theorised that if students are given embodied experiences focusing on the *effect* of transformations rather than the specific actions involved, then they have the potential to construct an embodied conceptualisation of the notion of free vector, and then cope more easily, not only with generic cases, but also with singular cases.

This requires a consideration of the literature that relates how physical experience with the outside world (embodiment) plays its part in the learning process. A major source for these ideas is the work of Lakoff and his colleagues who consider how human embodiment underpins abstract thinking.

I will also be looking at the importance of symbolic representations in the ability to model problems abstracted from the outside world in mathematical terms, and how reflection on mental and physical actions affects the building of coherent conceptual schemas. This involves considering not only how operations are carried out by sequences of step-by-step actions, but also how the effect of these actions can be symbolized and considered as mental entities to think about. A major source for these ideas is the theory of encapsulation of processes as mental objects as formulated by Dubinsky (1991) and Sfard (1991), and in the theory of *procepts* (where a symbol dually evokes either *pro*cess or con*cept*) formulated by Gray and Tall (1994).

2.3.1 Successive stages of cognitive development

Piaget (1985) describes cognitive development of the child in several stages: sensori-motor, pre-conceptual, concrete-operational and formal-operational. To underpin this development, he formulated a three-part theory of abstraction. In the first two stages (sensori-motor and pre-conceptual) a child goes through the process of *empirical* abstraction, when (s)he focuses on physical objects and their properties, noting similarities and differences that are abstracted empirically. In the third, concrete-operational stage, the child focuses on *actions* on objects and the properties of these actions result in what he calls *pseudo-empirical* abstraction. The formal-operational stage is described in his theory in terms of *reflective* abstraction in which 'actions and operations become thematized objects of thought or assimilation' (Piaget, 1985, page 49). He suggests that these stages of development apply to children from birth to about age of 12.

Piaget's ideas of conceptual growth were adapted by many researchers who developed them to apply to any age to formulate how conceptual growth takes place. Bruner (1966), for example, introduced three modes of representation: *enactive*, *iconic* and *symbolic*. He wrote:

> What does it mean to translate experience into a model of the world? Let me suggest there are probably three ways in which human beings accomplish this feat. The first is through action. [...] There is a second system of representation that depends upon visual or sensory organisation and upon the use of summarizing images. [...] we have come to talk about the first form of representation as enactive, the

> second is iconic. [...] Finally, there is a representation in words or language. Its hallmark is that it is symbolic in nature.
> (Bruner, 1966, p. 10–11)

Bruner's enactive mode of operation is based on action and begins in Piaget's sensori-motor stage, to be followed by the ikonic and symbolic modes in the pre-conceptual and concrete-operational stages. In older individuals, all three modes may be available and used as appropriate in different contexts.

Lakoff and Johnson (1980) formulate their idea of conceptual embodiment as follows:

> Our experiences with physical objects (especially our own bodies) provide the basis for an extraordinarily wide variety of ontological metaphors, that is, ways of viewing events, activities, emotions, ideas, etc., as entities and substances. (Lakoff and Johnson, 1980, p. 25)

Lakoff & Núñez (2000) propose that all human ideas are grounded in sensori-motor experience. This involves the use of formulatable cognitive mechanisms by which the abstract is comprehended in terms of the concrete by using a *conceptual metaphor*. They claim that most mathematical thought makes use of conceptual metaphors. (2000, page 5). According to Lakoff & Núñez, human reason crucially depends both on human experience and imagination and therefore categorisation depends partly on human perception and motor activity, and partly on mental imagery.

2.3.2 Construction of meaning

Constructivists see students as active learners, who make sense of the world on the basis of the links between past experience and new information. In doing so, students may need to reconstruct their earlier conceptions to make sense of new information (Driver and Oldham, 1986). This process can occur only when students are dissatisfied with their current conception and feel the need for a new one. According to Posner et al. (1984) they should also consider the new concept as intelligible, plausible, and useful in solving problems.

However, a stumbling block for such a development and reconstruction can be what Lakoff describes as a 'prototype effect'. He (1987) quotes from studies of Rosch that questions the belief that "categories are defined only by properties that all members share," for if that were true, then

> [...] categories should be independent of the peculiarities of any being doing the categorizing; that is, should not involve such matters as human neurophysiology, human body movement, and specific human capacities to perceive, to form mental images, to learn and remember, to organize the things learned, and to communicate efficiently.
> (Rosch, quoted in Lakoff, 1987, p 7)

On the contrary, the research conducted by Rosch and others demonstrated that

> [...] categories, in general, have best examples (called "prototypes") and that all of the specifically human capacities just mentioned do play a role in categorization. (ibid. p.7)

Early stages of mathematics in English primary school, are taught through physical activities using the senses and it is hoped that children will build on this experience to comprehend the nature of mathematical ideas, integrating them with their previous knowledge, and building a new category or concept or, where necessary, rebuilding the previous one. However, there is always the danger that pupils will accept a prototype (an example as representation of the whole category) as the concept. It is therefore important how we introduce our students to a new conceptual idea and to be aware of which context we are going to use for our explanations and discussion. According to Jaworski (1994)

> The pupil might fit the teacher's words into her own experience to get a meaning different from what the teacher tried to convey. Because people interpret words and gestures differently, any attempt to convey knowledge in an absolute sense must be seen as quite likely to fail. A teacher therefore has to find ways of knowing what sense pupils make of the mathematical tasks which they are set, in order to evaluate activities and plan further lessons.
> (Jaworski, 1994, p. 220)

She describes the situation in her article in which

> [...] the activities in which the learners participated and encouraged them to be mathematical, that is to act as mathematicians by mathematising particular situations created by their teacher [...] learners shared perceptions with each other and with the teacher, and their ideas became modified or reinforced as common meanings developed. This enabled learners to become clearer and more confident about what they knew and understood. (Jaworski, 1994, p. 229)

In the case of vectors, the pupils' first introduction in typical English schools occurs in Science lessons, mainly through thinking about forces, which is a highly particular context with implicit properties that act as possible obstacles to the general notion of vector. Many pupils have thinking that is flexible enough to cope with the transition to the general notion of vector, despite the specific peculiarities of this particular embodiment. However, from the initial investigations into the topic of vectors, which will be described in greater detail in chapter 4, it seems that there are many more students for whom the concept of force becomes a prototypical concept of a vector and these pupils have a problem when the construction of the general concept of vector becomes necessary.

2.3.3 An example: the case of fractions

The subtleties required in construction of mathematical concepts can be illustrated by the case of fractions. Many mathematics text-books introduce the idea of the fractions as part of circles (pizzas, pies, cakes). This type of representation is very restrictive, and is only good for the imagination of simple fractions like 1/2, 1/4, 2/3, 5/6, etc. Kerslake's research (1986) shows that

> [...] the only model of a fraction that was widely accepted was that of a geometric 'part of a whole' Not only was it the only universally accepted model of 3/4, but children referred to parts of circles or parts of cakes when trying to explain other problems during the course of the interviews, such as addition of fractions, or whether 2/3 is bigger or smaller than 3/4. (Kerslake, 1986, p. 71)

Because of this representation

> [...] most children found it difficult to think of fractions as numbers, particularly when asked to place them on the number line.
>
> (Kerslake, 1986, p. 71)

After teaching an experimental group with a number line only, she concluded that

> [...] while the geometric 'part of a whole' model may well be useful one in establishing some of the basic ideas about fractions, serious consideration is necessary as to its limitations and to the need for presenting the idea of a fraction in a wider context.
>
> (Kerslake, 1986, p. 96)

A particular conclusion drawn from this research was that "the distinction needs to be drawn between the embodiment and the idea," (p 96).

This experience with fractions shows that a single embodiment of a general concept can inhibit the formation of a more general version of the same concept that has a wider range of application. The same problem seems to be happening in the case of vector. Experiencing a vector in a particular embodiment may lead to the student being able to operate in a limited range of cases that are similar to the students' experiences, but which are too narrow to cope with even slightly unfamiliar examples. These limitations may be revealed by presenting the student with 'singular cases', for instance the case where the resultant is required for two arrows whose heads are at the same point.

2.3.4 Embodiment of mathematical concepts in the physical world

Skemp used the word *embodiment* before it became fashionable in more recent theories of embodied cognition, to describe the way in which a mathematical concept is given a physical meaning that represents the underlying mathematical ideas in a clear and explicit manner. Skemp (1971, p. 176-177) gives an example of embodiment of equivalent fractions arising through the double operation of combining and sharing, which, in mathematical terms, are commutative. Sharing followed by combining gives the same result as combining followed by sharing, in the following sense. He gives example of a fraction 3/8 which we can first share the standard object (a rectangle) into eight parts (figure 2.1)

Fig. 2.2 The first representation of 3/8

then combine three of these parts (shaded); or alternatively combine three of the standard objects and then share the resulting combination into 8 parts (figure 2.2).

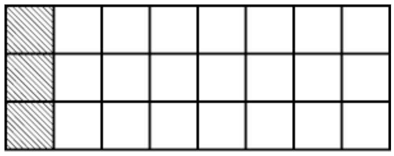


Fig. 2.3 The second representation of 3/8

In both cases we will end up with the three eighths of the standard object.

This is the closest example I have found in literature to what I call 'the same effect' of two different actions. It leads Skemp to the idea of representing sets of equivalent fractions as shown below (figure 2.3):

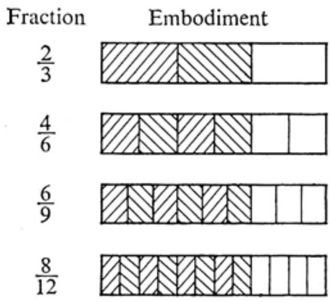


Fig. 2.4 representation of equivalent fractions

This suggests an alternative approach to the learning of fractions. Instead of speaking of the *mathematical* idea of 'equivalent fractions', it may be cognitively more appropriate to look at equivalent fractions as operations that 'have the same effect'.

In the case of vectors, we saw in chapter 1, figure 1.1 that it is possible to add vectors in different ways, and these ways have the same result. By embodying a vector through the action of translating an object on a flat table, we may focus attention on the fact that all the points on the object (or on the hand that moves the object) move in the same direction by the same amount. The shift of any such point can be represented as an arrow from its start point to its end point and any of these arrows is sufficient to represent the translation. As representations, they have the same effect. In this way we can give an embodied underpinning for the notion of free vector by focusing on the effect of a translation.

Lakoff writes:

> "Thought is embodied, that is, the structures used to put together our conceptual ideas grow out of bodily experience and make sense in terms of it; moreover, the core of our conceptual systems is directly grounded in perception, body movement, and experience of a physical and social character." (Lakoff, 1987, p. xiv)

With respect to symbols, however, Lakoff & Nunez (2000) say, "symbols are, just symbols, not ideas," and that the intellectual content of mathematics lies in its ideas for which symbols do the job of characterising their nature and structure. According to this viewpoint, abstract ideas make use of formulatable cognitive mechanisms, such as conceptual metaphors that import modes of reasoning from sensori-motor experiences.

My research is consistent with this statement as students often seem to 'know' the graphical symbol of an arrow representing a vector, without having a fully coherent understanding of ideas that give rise to its intellectual content. Students I have interviewed have had to be helped to attach a mathematical concept to the symbol of an arrow before they can manipulate it successfully in a full range of contexts, particularly in singular instances.

Lakoff & Nunez (2000) quote from cognitive science research, that most of our thought is unconscious and much of mathematical cognition happens at too low a level in the mind to be accessible. We draw conclusions from the world around us without being aware of it. We also have unconscious memory, which gives us implicit rather than explicit understanding. Schacter (1996) writes that the experiences we don't recall often have a detectable and measurable effect on how we behave. The theory of Lakoff & Nunez focuses on these unconscious mechanisms to suggest that understanding of mathematics uses the same cognitive mechanisms that are used for ordinary ideas: basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientation, basic manipulation of objects, iterated actions, and so on (pp. 27, 28).

On this basis it may be possible to reintroduce a concept, that causes a problem in developing understanding, so that, if we use the right experiences in the appropriate context, we may be able to set up the unconscious cognition in a more flexible manner, which will help the students in developing their knowledge. But how do we know when we introduce this idea to students that they build a proper concept and not just rote-learn and forget after a short time? The theory of embodiment suggests that we need to give appropriate experiences to underpin the concepts with bodily activity that integrates and supports the abstract ideas.

Socio-cultural theorists like Lave and Wenger (1991) view gaining knowledge as integration into a community of practice in which social actions are defined. For instance, students might be expected to learn the proper techniques of drawing using set-square, ruler and compasses. However, how does a community of practice pass on its more subtle conceptions that are carried out privately within our minds? Students may learn to perform mathematical manipulation of abstract symbols in accordance with the observed conventions, but there is still the question of the deeper conceptual meanings of the use of symbols to focus on the essential mathematical ideas free (as far as possible) from the coercive effects of specific embodiments.

2.3.5 The transition from embodiment to symbolism

The necessary shift from embodiment to symbolism has been detailed by Skemp (1971):

> First, we learn to manipulate concepts instead of real objects; then, having labelled the concepts, we manipulate the labels instead. Finally, perhaps, we reverse the process by re-attaching the concepts to the symbols and then re-embodying the concept in the real action with real objects from which they were first abstracted. (Skemp, 1971, p 83)

According to him we cannot use mathematics effortlessly unless we detach the symbols from their concepts and we have to be able to manipulate them without attention to their meaning. However he emphasises that this manipulation should be 'automatic' and not 'mechanical'. In automatic manipulation we can easily go back and reattach symbols to their meaning, while in mechanical manipulation the symbols stay meaningless. Skemp also says that:

> In mathematics, what we store is a combination of conceptual structures with associated symbols, and the former would therefore seem to be important for the retention of the whole.
>
> (Skemp, 1971, p. 85)

According to Hiebert and Lefevre (1986), symbols are viewed as cognitive aids, they "help to organize and operate on conceptual knowledge," (p. 15). They even go so far as to say that "The symbols can also produce conceptual knowledge," (p. 15). They further write that: "Being competent in mathematics involves knowing concepts, knowing symbols and procedures, and knowing how they are related,"(p.16).

Hiebert and Carpenter (1992) emphasise the importance of the symbolism to development of understanding and say that knowledge is represented internally, but communicating mathematics requires external representation:

> [...] when the relationships between internal representations of ideas are constructed, they produce networks of knowledge.
>
> (Hiebert and Carpenter, 1992, pp. 66-67)

They also say that students often make inappropriate connections or "represent information as isolated pieces," (p. 76) which cause difficulties in making sense of mathematical situations. Students build on prior knowledge and this may be procedural rather than conceptual, resulting at least in part from years of procedural and instrumental instruction.

Skemp (1979) describes a dynamic process of developing understanding: "to understand a concept, group of concepts, or symbols is to connect with an appropriate schema" (page 148), which puts the above theories into one sentence. However, this still begs the question of *how* the students connect all these bits of information into an appropriate schema.

To be able to conceive of ideas in a coherent form and to link them together requires a way of making this knowledge appropriate for comprehension and making connections. In particular, how do we put together embodied knowledge in a way which allows us to shift from embodiment to symbolism in a way that allows the symbolism to be used flexibly and meaningfully in a range of contexts?

2.4 Concept Images and Compression of Knowledge

Mathematical concepts are highly sophisticated mental constructions. Tall and Vinner (1981) define the *concept image* to be the total cognitive structure associated with a mathematical concept in an individual's mind. The ideas related to the given concept are continually constructed as the individual matures and are changing with new stimuli and experiences. Given such a range of cognitive structure, it is important to understand how the wider aspects of the concept image can be channelled into a thinkable entity that can be manipulated mentally in the mind.

Thurston (1990) described the way in which mathematical ideas start as a collection of disparate ideas which, through use and reflection, are compressed into easily recalled knowledge:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990, p. 847)

On the same note, Crick (1994) states that the brain can make conscious decisions only by suppressing data and focusing on a limited quantity at a time.

Krutetskii (1976) writes:

> Retaining information in generalized and abbreviated form … does not load the brain with surplus information and thus permits it to be retained longer and used more easily. (Krutetskii, 1976, p. 300)

2.4.1 Cognitive Units

Barnard and Tall (1997) introduced the term *cognitive unit* for part of the concept image that can be held consciously in the focus of attention. A cognitive unit can be a symbol or representation or any other aspect related to the particular concept. For example in case of vectors it can be an arrow or a triangle of three vectors showing one side to be the sum of the vectors represented by the other twos. They hypothesise that powerful thinking arises through compressing information into rich cognitive units that can be manipulated in the mind.

> A powerful aspect of reflective thinking is the ability to compress a collection of cognitive units – which may be processes, sentences, objects, properties, sequences of logical deduction etc – into single entity that can be both manipulated as a concept and unpacked as a cognitive schema. (Barnard and Tall, 1997, p. 2)

This is particularly relevant to my own research as I seek ways of helping students to move from a range of experiences with the notion of vector to a central notion of free vector as a cognitive unit in its own right that has coherent meanings across a range of contexts.

2.4.2 Process-object encapsulation

A major theory that builds on the idea of internalising knowledge into thinkable entities is the APOS theory of Dubinsky and his colleagues, which is based on Piaget's epistemology of mathematics (Beth & Piaget, 1966). The acronym APOS stands for Action-Process-Object-Schema:

> An action is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external. It may be a single step response, such as a physical reflex, or an act of recalling some fact from memory. It may also be a multi-step response, by then it has the characteristic that at each step, the next step is triggered by what has come before, rather

> than by the individual's conscious control of the transformation. … When the individual reflects upon an action, he or she may begin to establish conscious control over it. We would then say that the action is interiorized, and it becomes a process …. [Then] actions, processes and objects ... are organized into structures, which we refer to as schemas.
>
> (Cottrill, *et al.*, 1996, p. 171)

This theory is, in part, a theory of compression, from step-by-step actions to processes conceived as a whole that are then conceived as mental objects. In our approach to free vectors, this theory would suggest that individual actions (such as a shift of a triangle on a table) may be considered as a process (the transformation as a whole) and then conceived as an object (a free vector).

Sfard (1991, 1992) describes a similar sequence of compression:

> First there must be a process performed on already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three components of concept development will be called interiorization, condensation, and reification, respectively.(Sfard, 1992, pp. 64–65)

Though ideally the compression from action to process to object is highly desirable, Dubinsky and his colleagues found that college students often were able to move from action to process, but the next stage of producing a mental object was more difficult. (eg. Breidenbach, et al, 1992). They also reviewed their theory to explain that 'the construction of these various conceptions of a particular mathematical idea is more dialectic than a linear sequence' (Dubinsky and McDonald, 2001).

The serious question is therefore how a student can begin to think of a process as a mental object. A process occurs in time, an object is an entity that occurs in space (either real or imagined). Gray and Tall (1994) suggested that the mechanism by which this is done is through the use of a symbol to operate dually as process or concept. Thus the symbol 3+2 is both the process of addition and the concept of sum. They called a symbol that dually represents either *pro*cess or con*cept* a *procept*.

This highlighted the need for a symbol to function in a dual role, such as an arrow for a vector to represent both the process (as a movement from tail to nose) and the object (the arrow itself). However, the fact that an arrow has two distinct interpretations does not mean that students have a genuinely flexible view of vector. It seems that students can learn to operate with vectors as arrows in a limited way without constructing a flexible concept of a free vector. More insight is clearly required.

A clue is found in the description of Sfard: "First there must be a process performed *on already familiar objects*". The process-object encapsulation proposed by both Sfard and Dubinsky starts with actions on objects that already have meaning for the student. Tall and Gray (2001) suggest:

> [...] the theorised encapsulation (or reification) of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon (which we henceforth refer to as *base objects*).
> (Gray & Tall, 2001, p. 266)

This idea links closely to Joshua's notion of *effect*. The compression of knowledge formulated in APOS theory does not begin with the A of 'Action' but with the B of 'Base object'. This gives a 'BAPOS' theory (proposed by Chae, 2002) in which Base objects are operated on by Actions, interiorized as Processes, encapsulated as Objects, within a wider Schema. By focusing on the *effect* of the Actions on the Base Objects, it now becomes possible to *see* the idea that represents the Process as a whole and can be symbolised as an Object. In the case of a translation, the base object is a figure (say a triangle) on a table and the effect is *the shift from the initial to the final position* of the base object without focusing on what happens in between. The effect can be represented by any arrow that has the same magnitude and direction as the shift, and any of these arrows represents the free vector that is the total shift from initial to final position.

This brings us closer to a possible theory of compression of the notion of translation into the concept of free vector. But we still need to seek a way in which this can be encouraged in our students. This takes us back to the fundamental idea of reflective thinking.

2.4.3 Reflection

Driver (1989) says that teaching involves the organisation of the classroom situation in a way which promotes learning outcomes.

Piaget (1985) suggests that one of the strategies to foster conceptual change is, to confront students with discrepant events and invoke a conceptual conflict which forces students to reflect on their conceptions as they try to resolve the conflict. However this can cause problems. Dreyfus, Jungwirth, and Elivitch (1990) found that their more able students react enthusiastically to conceptual conflict, but less successful students try to avoid the conflict or simply do not even recognise it.

As suggested by Barnard and Tall, if a student never builds a cognitive unit out of all the information he manages to assimilate then it would be very difficult for him to manipulate ideas and use them in solving problems presented to him.

Palmer & Flanagan (1995) found out that children develop their own ideas based on their own experiences. These ideas are often quite different from the accepted scientific viewpoints. Gilbert & Watts (1983) call them the "alternative conceptions" and Pines & West (1986) recognise that they significantly interfere with learning. One such concept is the Aristotelian idea that an action of continuous force keeps an object in motion. Sadanand & Kess (1990) found that 82% of senior high school students indicated that a force is required to maintain motion. Clement (1982) found that 75% of a group of university students indicated that there should be a force in the direction of the motion even after one semester of instructions in mechanics.

Kilpatrick (2002) suggests that students might have problems with understanding certain areas because they might not have encountered situations meaningful to them in which mathematics was important to know. Kilpatrick (2002) quotes the USA National Council of Teachers of Mathematics (1991), which specifies that the teacher's role is to orchestrate the discourse so that the students in this class will function as an intellectual community. The teacher should set up a situation and then respond to what the students are saying by building on their observation, seeking clarification, and challenging them to explain and justify. This suggests that reflection is a process which would address these needs. The literature devoted to theories of how learners learn and how teachers teach (for example: Piaget, Skemp, Kilpatrick) have highlighted reflection as a central mechanism in thinking. This links closely with our earlier discussion of the constructivist approach to promoting learning advocated by Jaworski.

2.5 Bringing theories together

We are now moving to a position where the range of theories are bringing forward a general trend moving from initial intuitions from embodiment (which may include 'false intuitions') to a focus on the effects of actions to lead to symbolism. As we saw earlier, this is part of a cognitive development is described by Piaget in his stage theory of sensori-motor, pre-conceptual, concrete-operational and formal operational and by Bruner in his enactive, iconic and symbolic modes.

These were brought together by Biggs and Collis (1982) in their SOLO taxonomy to categorise the Structure of Observed Learning Outcomes. Biggs and Collis proposed five modes of cognitive development: *sensori-motor*, *ikonic*, *concrete-symbolic*, *formal* and *post formal*. They also note that, as each mode becomes available, it remains available alongside the new modes. Thus the introduction of the ikonic mode also includes the sensori-motor mode, which gives a combined embodied mode that encompasses both enactive and iconic (in the sense of Bruner). The concrete-symbolic mode includes the development of arithmetic and algebra and of the symbolic aspects of vectors. The formal modes include the notion of definition and deduction will not concern us here, but were suggested by Biggs and Collis (1982) to take the theory of Piaget beyond secondary education into graduate and postgraduate work.

These modes were consolidated into three by Gray and Tall (2001) in terms of embodiment (enactive and iconic), symbolic, and formal-axiomatic. In considering different types of mathematical concept they wrote:

> For several years [...] we have been homing in on three [...] distinct types of concept in mathematics. One is embodied object, as in geometry and graphs that begin with physical foundation and steadily develop more abstract mental picture through the subtle hierarchical use of language. Another is the symbolic procept which acts seamlessly to switch from a "mental concept to manipulate" to an often unconscious "process to carry out" using an appropriate cognitive algorithm. The third is an axiomatic object in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory. (Gray & Tall, 2001, p.70)

The three levels of object-construction described by Gray & Tall occur in the development of vectors, for instance, an arrow is an embodied object, the notion of the vector as a shift in space or as column vector has the structure of a procept and the axiomatic notion of vector space is an axiomatic object.

The research in this thesis inhabits the first two modes discussed here, the embodied mode which leads to graphical representations of vectors and the symbolic mode. To trace the development through the two modes, I again turned to the SOLO taxonomy where Biggs and Collis suggest that each mode has a common sequence of stages which can be used to test the quality of outcomes observed in tests designed for assessment. The stages are: *pre-structural* where no structure is used; *unistructural*, when student focuses only on a single aspect; *multi-structural,* when student focuses on several separate aspects; *relational* when the student relates different aspects together in a coherent way, and *extended abstract* where the student can see the concept from an overall viewpoint.

Bringing together a range of viewpoints, Pegg and Tall (2003) suggested that the SOLO theory encompasses a 'fundamental cycle' of conceptual development common to a range of distinct theories (figure 2.3).

	Davis	APOS	Gray & Tall	**Fundamental Cycle**	
Unistructural	VMS† Procedure		Base Object(s)	Base Object(s)	**Schema Building** ↓
		Action	Procedure	Isolated Actions Procedure	
Multistructural			[Multi-Procedure]	Multi-Procedure	
Relational	Process	Process	Process	Process	
Unistructural (Extended Abstract)	Entity	Object Schema	Procept	Entity Schema	

† VMS stands for Visually Moderated Sequence

Table 2.1 The fundamental cycle of conceptual construction

In each theory the first stage involves some kind of action on one or more base objects in which the focus of attention can be either on the object, or on the actions. Attention focused on the actions themselves can be consolidated into procedures (or multi-procedures) where there may be different ways (procedures) to carry out the same overall process. With support of symbols, students may at this stage construct a mental object as a cognitive unit which (according to the article) is both a schema within itself and also an entity that is manipulable within a wider schema of activities.

2.5.1 Combining modes

The previous section looked at the fundamental cycle of concept development that happens in a given mode. In our development we wish to see students construct the notion of free vector that relates across different modes. In the SOLO taxonomy, at the concrete symbolic stage, the student will also have available the embodied mode which may be viewed as a combination of enactive and iconic. As we shall see in the later study, some students may prefer to use the symbolic mode, others the embodied graphic mode and some will use a flexible combination of both.

2.5.2 Versatile thinking

Krutetskii (1976) identifies three basic types of mathematical cast of mind: the analytic type (who tends to think in verbal-logical terms), the geometric type (who tends to think in visual-pictorial terms), and the harmonic type (who combines characteristics of the other two)." (1976. p.xiv). He studied 'capable pupils' and discovered that a significant majority of them belonged to the third category. He suggests that such pupils are "quite ingenious in their visual interpretation of abstract relationships, but their visual images and schemes are subordinated to a verbal-logical analysis [...]. They are successful at implementing both an analytic and a pictorial-geometric approach to solving many problems," (Krutetskii, 1976, p. 326).

The distinction between different styles of thinking has long been a focus of attention in the literature. Brumby (1982), for example, noted two different strategies for solving a problem:

> (i) Immediately breaking a problem or task into its component parts, and studying them step by step, as discrete entities, in isolation from each other and their surroundings.
>
> (ii) An overall view, or seeing the topic/task as a whole, integrating and relating its various subcomponents, and seeing them in the context of their surroundings. (Brumby, 1982, p.244)

Her research suggested three distinct groups of students: those who consistently used only serialist/analytic strategies, those who used only global/holistic strategies, and those who used a combination of both, whom she described as *versatile learners*. Overall 42% of her sample maintained a serialist/analytic style, 8% were global/holistic and 50% were versatile.

In his thesis, following Brumby, Thomas (1988) used the term *versatile* to describe the complementary combination of global-holistic thinking and serial-sequential thinking. Subsequently it has been used to describe students who are able to use a variety of techniques in different contexts involving both linear procedural activities and also more flexible conceptual thinking (Blackett, 1990).

In this thesis we will describe students to be *versatile* if they are able to use their knowledge of free vector in a versatile way in solving problems in both the embodied and numeric modes.

2.6 Summary

In this chapter a range of theories of cognitive development have been considered from intuitive beginnings through instrumental and relational understand of procedural and conceptual knowledge. Although philosophers may regard intuition as a basis of all knowledge, it also depends on our human characteristics which can involve false physical intuitions at variance with subsequent theories.

Reviewing the theories of cognitive development proposed by Piaget and Bruner, I used the embodied theory of Lakoff to see the foundation of human development in embodiment (with links to Bruner's enactive and iconic modes) and focus on the transition to the symbolic mode, looking to constructivist theories to help students construct the shift from embodiment to symbolism, in flexible ways, in a variety of contexts. This involves the compression of knowledge from separate pieces of information into thinkable mental cognitive units.

Reviewing theories of Dubinsky, Sfard and Gray & Tall concerning the notion of 'process-object encapsulation', starting from step-by-step actions, interiorised to global processes and encapsulated as objects, we note the perceived difficulty of reconceptualising process as object.

At this point we introduce the notion of 'effect' that arose in discussion with the student Joshua in the Preliminary Study to use an extended BAPOS theory, in which Base Objects have Actions upon them, interiorising to Processes, then Objects, where the encapsulated object is now represented in terms of the 'effect' of the action on the base objects.

Introducing SOLO taxonomy not only incorporates the theories of Piaget and Bruner, but also has a cycle of concept construction that relates to theories of process-object encapsulation, to give a broader theory that can be used not only to describe the development of embodied and symbolic modes of operation, but also to relate them together in a versatile way.

Chapter 3
Towards the Theoretical Framework

3.1 Introduction

This chapter develops the theoretical framework for the empirical research which follows. It begins by describing the nature of the curriculum in the UK and focuses on the experiences that the students in the study will have encountered previously in mathematics and physics.

Early ideas that lead to the notion of vector relate to physical experiences of forces in a single dimension, involving the combination of forces in different directions. In physics this leads to the study of forces resolved in horizontal and vertical components and combining forces by adding their components in each direction. The introduction of vectors in mathematics passes through a sequence in which transformations are re-conceptualised as vectors, which follows a sequence similar to process-object encapsulation. This in turn leads to the possibility of a theoretical framework, which studies how the students cope with successive stages of process-object encapsulation, both graphically as arrows with magnitude and direction and numerically as separate horizontal and vertical components.

However, before embarking on the development of such a framework, we consider relevant research in science education with respect to students' understanding of vectors in Physics and Mechanics. This will lay the groundwork for a preliminary study described in chapter 4 in which the theoretical framework will be further refined before the design of the main studies that follow.

3.2 The school based situation

In the English system, children begin at school in the year when they will become five years old. Compulsory school education is from Year 1 to Year 11, which is the year in which they have their 16th birthday. They spend the Years 1 to 6 in Primary School and then move for five years from Year 7 to 11, in Secondary School (corresponding to the years K-10 in the USA). All children take exams at the end of Year 11, which is called the GCSE examination (General Certificate of Secondary Education). During Years 1 to 11, Mathematics, English and Science are compulsory subjects. Dependent on their results they can then enter Year 12 and 13 (equivalent to the American High School system) where they study 'AS' (Advanced Supplementary) in Year 12 and 'A2' (Advanced) levels in Year 13.

The main educational experience of vectors that students gain at school before studying 'AS' and 'A' levels occur during Physics and Mathematics lessons. It is therefore appropriate to look first at school text-books in the period of compulsory education to see what emphases are made in them and how these may possibly influence the teaching and learning that is taking place. Since students first meet vectors in Physics, initially at age 11, but mainly between the ages of 13 and 16, I therefore looked first at the Physics text books, and then at the Mathematics text books where they meet vectors for the first time in year 11 at age 16.

3.3 Text books analyses

This section looks at the text studied by pupils learning about vectors and aspirations of the authors who wrote the text. It shows how vectors are introduced in Physics and Mathematics.

3.3.1 How and when are vectors introduced in Physics?

In Secondary School (age 11-16) pupils meet the idea of a vector in Physics in the first year (age 11). The approach is very pragmatic and all the vector quantities lie in a line, which is either horizontal or vertical.

For example the first approach is something which, hopefully, most pupils will experience, shown in figure 3.1, taken from Heslop et al. (2000).

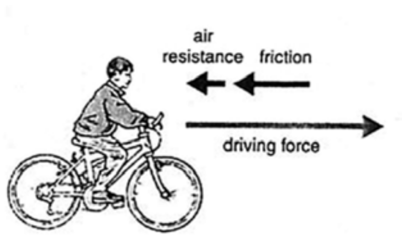


Fig. 3.1 Forces in a horizontal direction

Then pupils are introduced to the idea of the balanced forces acting in a horizontal direction, giving the resultant zero.

Vertical vector quantities are introduced, again as forces. The example, taken from the year 7 book (Heslop et al, 2000), is shown in figure 3.2.

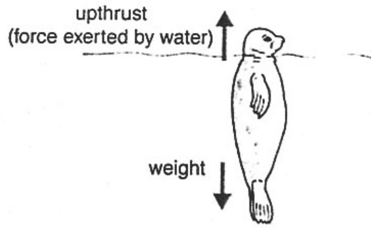


Fig. 3.2 Forces in a vertical direction

When pupils reach the age of 14, they are also introduced in Physics to other vector quantities such as displacement, velocity, acceleration, momentum and pressure. It must be emphasised that all pupils are introduced to the above ideas but differentiation occurs at age of 14 according to the students' level of ability.

In years 10 and 11 (ages 14-16), in an earlier version of the curriculum, students used to be introduced to the vector quantities involving angles, as in figure 3.3, and were then asked to draw a similar diagram for each example, which shows magnitude (size) and direction of the single resultant (overall) force. However this type of question has been removed from the GCSE syllabus (year 11), and is now introduced at AS level (year 12).

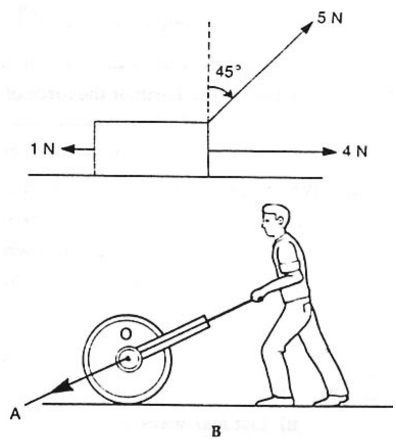


Fig. 3.3 Forces in several directions

Pupils are encouraged to make a very precise drawing of each force, measure its vertical and horizontal component, add them together and draw the resultant force based on these calculations. All these examples give pupils different physical embodiments that are intended to relate to their everyday experience.

In Physics, vectors representing especially forces are resolved as shown in figure 3.4. This uses a perspective for a vector quantity based on polar co-ordinates and suggests a vector starting from a specific point (usually a particle in Physics, or a centre of gravity of a specific object like a car a bicycle or a person), which means that a vector is ‘fixed’ to a specific point. Physics teachers do not use either column vectors or unit vectors and usually the vertical and horizontal components are written separately.

The vertical component when used with forces is usually drawn from the same position as the original force as indicated in figure 3.4.

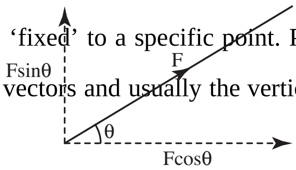


Fig. 3.4 Resolving a force into horizontal and vertical directions

This is implemented due to the belief that a force acts on the particle and therefore both components of the force should also be shown to act on that particle.

Alternatively students are encouraged to draw each vector quantity separately, measure the vertical and horizontal component and then add all the vertical components separately and all the horizontal components separately and then draw the final solution separately and measure the angle their vector quantity makes with the horizontal direction.

The teaching of vectors has been almost completely removed from the GCSE Mathematics syllabus (General Certificate of Secondary Education), which English students follow until they are 15/16. The only students who learn the idea of a vector in mathematics are Higher Course students, who are expected to obtain grades A or B in GCSE and will possibly carry on to study Mathematics at AS and A2 level in Years 12 and 13. Some of them will study Pure Mathematics with Statistics, others Pure Mathematics with Mechanics, and maybe Pure Mathematics with Discrete Mathematics. All of them will meet the idea of a vector in their Pure Mathematics in the second year of the A level study and those studying Mechanics will study them in greater detail.

3.3.2 How and when are vectors introduced in Mathematics?

When pupils are 15/16, those who are capable of achieving a higher grade in mathematics (grade A and B) are introduced to the idea of a vector in their mathematics lessons. Due to the pressure of the syllabus and time, some of them will only have one or two lessons to cover this topic. As it does not appear in the GCSE examination very often, teachers will not consider it as a priority. It may be supposed that people who design the Mathematics syllabus assume that most of the concept of vector will be assimilated by pupils from their Physics lessons.

The Mathematics Higher Course text-book, which students in this research were using in their Mathematics lessons (Pledger, 1996) gives four stages, described below, in developing the idea of vector:

1. the translation which is described as a column vector $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ as shown in figure 3.5.

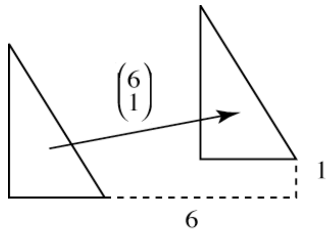


Fig. 3.5 A translation as a column vector

This is the only physical embodiment, that pupils can relate to which is presented in the Mathematical Syllabus.

2. an alternative notion is introduced to described the translation, which is $\overrightarrow{AB}$, where A is the starting point and B is the finishing point (figure 3.6)

Fig. 3.6 A translation as an arrow from one point to another

3. the third way is to describe a translation by using bold type single letters such as **a, b** (underlined when handwritten). In this case translations are simply referred to as **vectors.** The lines with arrows are called **directed segments** and show a unique **length** and **direction** for each of the vectors **a** and **b** (figure 3.7)

Fig. 3.7 Translations as arrows with magnitude and direction

4. The book then introduces the idea that the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ denotes the translation and introduces the idea of the *equivalent* vectors (figure 3.8).

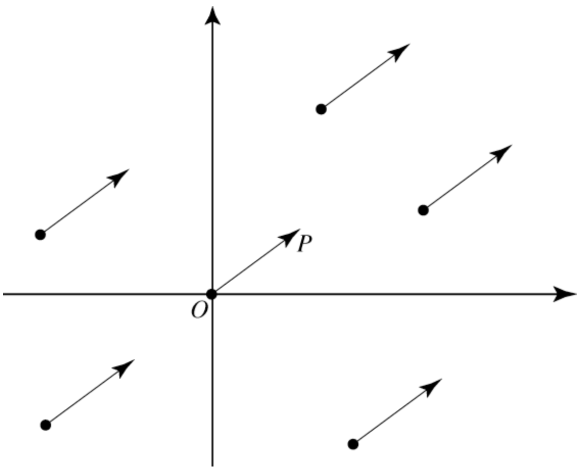


Fig. 3.8 Equivalent vectors and the special concept of position vector

The vector which translates *O* to *P*, $\overrightarrow{OP}$, is a special vector, the *position vecto*r of *P*.

In the first two stages pupils are introduced to a geometric vector with the idea that the movement and location are closely linked. In the first stage (fig. 3.5) the triangle is translated from its original position to a new position and in the second stage (fig. 3.6) , if two points *A* and *B* represent two locations, then the line segment $\overrightarrow{AB}$ represents a movement by the shortest path from *A* to *B*. The arrow shows the direction and the length of the segment represents the distance of the movement.

The fourth stage (figure 3.8) shows that the direction of the vector is represented by each or any of the parallel arrowed lines, which means that the geometrical image of a direction is not just a single line, *but an equivalence class of parallel arrowed lines*, which we call a *free vector*. However by introducing $\overrightarrow{OP}$ as a position vector or a localized vector, the book does not make it clear that if we have the situation as in figure 3.9, we would regard each of the directed line segments $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ as equivalent vectors (equal magnitude and direction). (Skemp, 1971.)

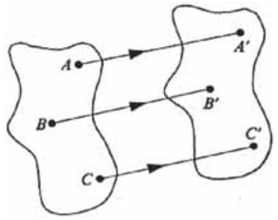


Fig. 3.9 Equivalent vectors representing the same translation

It is very important at this stage that pupils understand this concept very clearly i.e. if we are regarding all movements of the same distance and direction as being the same movement (the same *free vector*), regardless of their differences of starting point we are talking about free vector (as an equivalence class). All the other concepts pupils need to develop to be able to deal with vectors depend on their understanding of this single concept, because free vectors lend themselves to combining operations, following one free vector by another to give the concept of sum. We can always turn a free vector into a position vector, starting at origin (fig.3.8). Generally when we talk about 'vector' in mathematics we talk about a free vector.

This is emphasised in stage 3 of the development in the book.

The book introduces pupils to addition of the vectors by moving vectors parallel to their original position until they are all joined 'nose to tail' (beginning of the next vector joined to the end of the previous one, figure 3.10) as well as to the parallelogram method shown in figure 3.11.

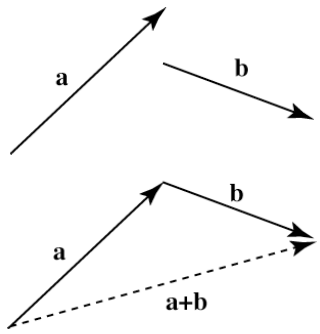


Fig. 3.10 The triangle method of addition

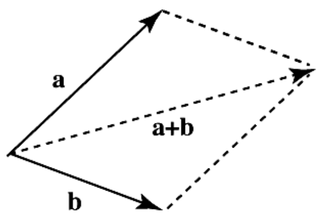


Fig. 3.11 The parallelogram method of addition

The book deals with multiplication by scalars, finding magnitudes of vectors and proving geometrical results with the use of vectors. The emphasis is on the use of free vectors and the algebraic form of *column vector* $\begin{pmatrix} a \\ b \end{pmatrix}$ to aid the calculations.

At this stage, pupils are also exposed in their Maths lessons to examples like the one below (figure 3.12) in which they have to relate vectors in the question to given position vectors.

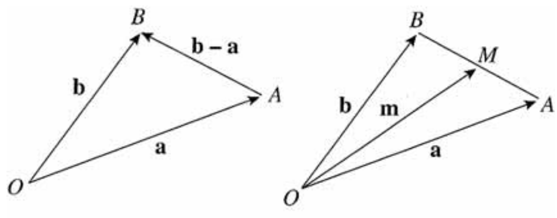


Fig. 3.12 Position vectors in geometry

They are also given examples as shown in figures 3.13 and 3.14 and asked to relate one vector as sum of others.

$\vec{AB} = \mathbf{b} - \mathbf{a}$

$\vec{AC} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$

$\mathbf{c} = \vec{OC}$

$= \vec{OA} + \vec{AC}$

$= \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})$

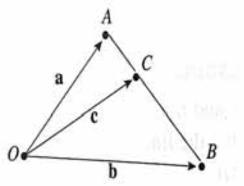


Fig. 3.13 Vector representations of geometrical positions

$\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$ and $\vec{OC} = \mathbf{c}$. N is the midpoint of OB and M is the midpoint of AC.

Express

(a) $\vec{AB}$ in terms of **a** and **b**

(b) $\vec{ON}$ in terms of **b**

(c) $\vec{AC}$ in terms of **a** and **c**

(d) $\vec{AM}$ in terms of **a** and **c**

(e) $\vec{OM}$ in terms of **a** and **c**

(f) $\vec{NM}$ in terms of **a**, **b** and **c**.

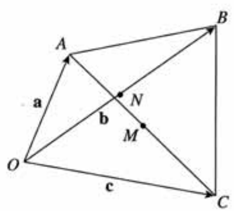


Fig. 3.14 Position vectors in geometrical figures

In the second year of A level Pure Mathematics (Year 13), which is referred to as 'A2', students are introduced to vectors in two and three dimensions. They are encouraged to change from the column vector representation to **i** and **j** representation, where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ can be represented as $3\mathbf{i} + 4\mathbf{j}$ as shown in figure 3.15.

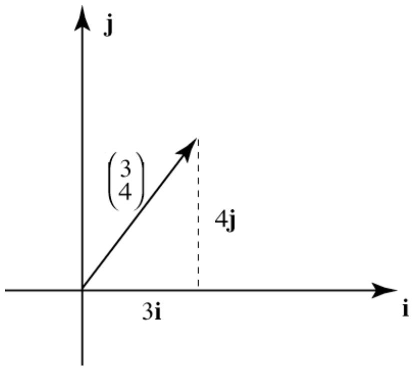


Fig. 3.15 Position vectors in terms of i and j

3.3.3 Linking the text-book sequence to process-object theory

In Physics, in the first four years of the Secondary Education the students meet vector sporadically and only in one dimension. They get to know the graphical symbol of an arrow and learn how to add vector quantities (graphically and numerically) in one dimension. After that, between the ages of 12 and 14 they are introduced to vectors in two dimensions. However, the method of operation on these vectors, for mathematical simplicity, is introduced as only in terms of the components (figures 3.4, 3.5). At this stage, they calculate the *x*-direction component and *y*-direction component for each quantity and deal with two directions separately until the final result, which they represent by drawing first the two shorter sides of the right angled triangle, and then the vector quantity as hypotenuse. This format therefore does not require the students to operate in a full two-dimensional context. The situation is simplified to what may be termed ''two times one dimension' rather than a fully-fledged two dimensional concept.

In Mathematics in year 11, as the first stage of dealing with two-dimensional vectors, teachers often follow the Physics method of solving problems with vectors in two dimensions. However in problems shown in figures 3.12, 3.13 and 3.14, students are required to have some concept of a free vector, which they tend to find, according to the interviews with their teachers, very difficult. Such problems seem to imply a huge conceptual jump.

This suggests an analysis in which the 'two-times-one-dimensional' stage is presented as a preliminary stage to the beginning of the two-dimensional work. Experience suggests that there are certain difficulties in moving from a preliminary stage and passing through successive stages of construction to attain the concept of free vector.

From our analysis, the mathematics text-book is written in a succession of stages that are strongly related to the *process-object-encapsulation* cycle (Dubinsky, 1991). At this first stage, the student is operating on a shape that is being translated in the plane. This shape can be considered as a 'base object' on which the transformation acts. This *action* can be represented by any one of a set of arrows $\overrightarrow{AB}$ of given magnitude and direction starting at some point A and ending at another point B.

At the next stage, the arrow is seen as a single entity, denoted by a single letter, say **u**. Although the move from the symbol $\overrightarrow{AB}$ to the single letter **u** seems small, it is a significant change of perspective. At this *process* stage, what matters is not the specific vector $\overrightarrow{AB}$, but just its magnitude and direction. All vectors of a given magnitude and direction represent the same *free vector*. This idea can be conceived as a mental object. Such mental objects can be added by placing them 'nose to tail'.

At this free vector stage, the addition of two vectors **u+v** gives the same result as **v+u**. By contrast, at the specific action stage, a displacement $\overrightarrow{AB}$ moves from A to B and can be followed by another displacement B to C, so that the combined displacement $\overrightarrow{AB}+\overrightarrow{BC}$ can be achieved by moving from A via B to C. However, the symbol $\overrightarrow{BC}+\overrightarrow{AB}$ has no meaning as a combination of journeys in this sense, for after moving from B to C, a jump would be required to A to continue the second move from *A* to *B*. The move from the original idea of a translation as an action moving from one point to another is therefore quite different from the refined idea of adding free vectors. The construction of free vectors makes the mathematics subtly simpler. At this *object* stage of the APOS cycle, the student is ready to build a flexible *schema* of relationships, including such simple ideas as the commutativity of addition. At this stage the students should be able to solve not only problems like in figures 3.14 and 3.15 but also adapt their knowledge to other situations with which they may not be familiar.

3.4 Relevant examples of research into Mechanics

The research in Mechanics reveals other subtle phenomena that cause problems for students when dealing with vectors in Mechanics. This section considers the results from three specific projects that may have a bearing on the research we are about to undertake. Although the research considered moved in a different direction from our own investigations, it is important to consider the possible conceptions that can arise when students work on vector concepts. The three investigations to be considered are: ‘Students’ Conceptions about the vector characteristics of three physics concepts’ by Aguirre and Erickson (1984); ‘A Report on a Questionnaire Designed to Test Students’ Understanding of Mechanics’ by Jagger (1988); and ‘A hierarchical model of development of student understanding of force’ by Graham and Berry (1997). A brief description of each of these projects is described in sections 3.2.1 – 3.2.3.

3.4.1 Three vector concepts

This study by Aguirre and Erickson (1984) looks at the “extent to which difficulties encountered by students in the area of vectors may be attributed to their failure to comprehend some of the *implicit vector characteristics* and/or their alternative conceptions of these characteristics (alternate to that presupposed by the curriculum materials)” (p. 441). The main aim was to identify “the major constituent elements of the three vector concepts: position; displacement; and velocity; and the relationships among these elements" (p. 441). The analysis resulted in the identification of 10 *implicit vector characteristics* which are given as: reference point for stationary bodies, frame of reference, displacement or change of location, addition of displacement, subtraction of vector position, reference bodies of objects in motion, analysis of component velocities, composition of simultaneous velocities, independence of magnitudes of interacting velocities, simultaneity of component velocities. The two tasks, through which the individual students' ideas were investigated, were set in the context of familiar situations. The clinical interviews were used on a sample of 20 Grade 10 students (equivalent to English Year 11 students) to test their conceptions of vector characteristics. The results suggest that for most of the characteristics

> The largest percentage of students used *inferred rules*, which might be best called a partial description of the phenomena as viewed from a physicist's perspective. [...] when they were asked to predict the resultant magnitude of the velocity and direction of the perceived motion of the boat as it crossed the river, virtually all of the students were aware that the direction of movement of the boat would be in a direction in between those of the two contributing components. [...] But their estimates of this resultant magnitude ranged widely from values in between that of the larger velocity and the arithmetic addition of the two velocities to value in between the two velocities. [...] Other subjects tended to portray this interaction as a type of "fight" between the two components with the component having the greatest magnitude being declared the "winner.
>
> (Aguirre and Erickson, 1984, p. 452)

Another area of problem arising from the investigation of the boat question responses suggested that, "80% of the students think that the magnitude of the velocity component contributed by the boat's motor is changed in some way as a result of the interaction with the current," (p. 452).

The investigation suggested "that students possess a number of ***intuitions*** about various characteristics associated with the rather abstract and difficult topic of vectors. [...] A more detailed analysis of these inferred rules, over a variety of contexts, is required before we will be able to say much about the way in which these

intuitions can assist or inhibit instructional procedures in the area of vector quantities," (p. 453).

As the authors suggested in their conclusion, their methodology of investigating can provide a "framework for further probing of student conceptions in the area of vector quantities," (p. 453).

3.4.2 Understanding of Mechanics

This investigation was conducted by Jagger (1988) with 13 first year honours mathematics undergraduates. Many of them studied mechanics as part of their A-level mathematics and had completed a term's course on vectors in mechanics at the university. "The principal aim was to isolate the particular difficulties in understanding rate of change of velocity," (p 35), and the questions involve vector subtraction in a "pure mathematical" form. Some questions tested the students' notion of force.

In the summary, after analysing questions involving velocity and acceleration, Jagger concludes: "The problem is in moving from one-dimensional motion to motion in two or more dimensions," (p. 38). After analysis of topics related to force and motion she writes, "the pre-Newtonian view that motion implies the existence of a force in the same direction is firmly believed by quite a substantial proportion of these students," (p. 38).

3.4.3 Understanding of force

The students who study mechanics in my school use text-books written by Graham (for example: Mechanics 1, 2000), therefore research done by him is of particular interest to me. I concentrated on one of them: Graham and Berry (1997). It is a continuation of other investigations, carried out by the Centre for Teaching Mathematics at Plymouth University, into the development of students' understanding of mechanics concepts:

> The aim of the investigation was to form a set of levels, each of which would contain questions that demanded a similar level of understanding. [...] A set of criteria was selected which the questions forming the model of the development of understanding should satisfy.
> (Graham and Berry,1997, p. 840)

Some of the conclusions to their investigation were that:

> [...] their understanding of key concepts like gravity are confused, [...] students reverted to considering a constant force to be necessary to maintain the motion. [...] They also have great difficulty identifying forces and expect them to act in the direction of the motion or to be zero if the body under consideration is instantaneously at rest.
> (Graham and Berry, 1997, p.844)

Graham and Berry divided their questions into 3 levels and discovered that students passing only level one questions have sound ideas about the motion in one dimension but, for force in two dimensions, students revert to the misconception that there is a force acting in the direction of the motion. They also found that students passing their level 2 have overcome some aspects of their original misconceptions but reverted to using it in some situations. Their level three students are those who have accepted completely the Newtonian outlook on motion.

They write, that level 1, students'

> [...] difficulties arise because they are unable to identify the forces that are acting in a situation.
> (Graham and Berry, 1997, p. 847)

They suggest that:

> In order to improve students' individual understanding and promote their progression through the levels of the hierarchy they need to overcome this misconception at an early stage. It must be challenged by highlighting the weaknesses of the students' own intuitive ideas. Rectification can then take place by providing alternative explanations that the students can see overcome the weaknesses of their original ideas, explaining satisfactorily the situations used to challenge the students' intuitive ideas.
> (Graham and Berry, 1997, p. 847)

In their analysis only 23% of students have reached level 3. They suggest in their conclusion that a qualitative approach to teaching would help students to identify the forces and they would be able therefore to proceed to dynamic situations with greater confidence.

3.5 Summary of Evidence and Formulation of a Research Framework

The Physics text-books and worksheets show vector quantities always operating on a specific object. Until the end of year 11 they always act and therefore are added in one dimension. Afterwards (in years 12 and 13) when they operate in two dimensions they are resolved, for the sake of simplicity in calculations, into horizontal and vertical components and therefore operated on in what may be described as ‘**two times one-dimension**’, rather than as single entities in two dimensions. The question arises how one can shift students’ attention from working in ‘two times one-dimension’ to a concept of vector in two or more dimensions (Jagger, 1988).

The Mathematics text-book goes through a sequence of activities which seems to move in the direction a *process-object-encapsulation* cycle. However this cycle is not explicit, nor is it explicit in the empirical research described in the previous section which focuses instead on the difference between *displacement*, *free* and *position* vectors.

The research studies quoted also reveal how students’ ‘intuitions’ arise from working in different contexts and how it effects their problem-solving capabilities. For example, Aguirre and Erickson (1984) talk about *“ten implicit vector characteristics”* involved in “three vector concepts: position; displacement; and velocity” and suggest that students gain “a number of ***intuitions*** about various characteristics” which need to be overcome. On the other hand Jagger (1988) says that, “The problem is in moving from one-dimensional motion to motion in two or more dimensions”. Finally Graham and Berry (1997) talk about students’ “need to overcome this misconception at an early stage,” (p 847).

None of these researches consider the important idea of focussing on the vector concepts that are *common* to the various contexts, instead they are more concerned with the problems caused by the *differences* between them. Nor do they focus on the compression of a vector as an action into the more flexible idea of a free vector as a single mental object that can be represented by any arrow of given magnitude and direction.

As we shall see in the data of the preliminary study to be discussed in the next chapter (already published in Watson, 2002), when students meet the separate notions of displacement and force in distinct contexts, they are more likely to use the triangle law for displacement and, although encouraged in Physics lessons to use the parallelogram law for forces, they rarely do so. Indeed we find that many students use the triangle law with forces in an inappropriate way (see figure 4.16 in chapter 4) that leads to serious misconceptions. By building a coherent notion of free vector using translations, it may be hoped that the students will see the triangle law and parallelogram law not as separate rules in different contexts, but as two different ways of representing the same underlying idea. This will be investigated in greater detail in the delayed post-test analysis and the interviews in the Main Study.

From the analysis of the text-books and discussion with teachers, it was concluded that students meet the notion of vector in different contexts with subtle differences in embodiments. For instance vectors may be encountered as displacements sensed as physical journeys from one place to another, or as forces acting at particular points. In the addition of displacements, one journey followed by another is naturally interpreted using the triangle law, but the addition of forces operating at a point is more naturally represented by the parallelogram rule. In the mathematical curriculum, according the reviewed text-books, the notion of vector is first introduced as a translation in the plane and dealt with as a column matrix in mathematics, or as the separate horizontal and vertical components in physics. Both versions are linked to a picture of the vector as the hypotenuse of a right-angled triangle with components as horizontal and vertical sides. In turn this links more easily to the triangle law than to the parallelogram law.

Looking at the literature in comparison with the experience of the teachers and the text-books, it became apparent that students might be confused by trying to gain a concept of vector from many different contexts, each having different incidental properties. Aguirre and Erickson (1984) found that students fail to comprehend some of the *implicit vector characteristics* when learning from so many examples. In terms of process-object encapsulation, it does not seem that many students can move from operating on *base objects* (physical bodies, mathematical shapes) to building a *cognitive unit* from these implicit vector characteristics in the form of free vector, which in turn they could use to operate in any chosen context. A number of issues had to be determined:

- in what ways students turn the implicit properties of vectors in various contexts into misconceptions which trigger false intuitive thinking;
- what made some students able to think logically and use symbols appropriately;
- how can we may change students approach of concentrating on actions to concentrating on the effects of these actions;
- how we may help students build their vector concept into a cognitive unit which can be used easily in any context (translation, velocity, acceleration, forces, etc.);
- how can we help students use a vector as a mathematical symbol which conforms to mathematical laws of equivalence, commutativity, etc.

3.5.1 Theoretical framework perspective

According to Skemp (1971), the way to higher order thinking is through focusing on the essential properties in a given context and to filter out the "noise" (the data which is irrelevant to the required abstraction). The parts of the problem which are relevant to the solution of the problem are to be *abstracted* from the 'outside world' and manipulated in the 'mathematical world'. Later, the reverse process happens "of re-embodying the result in the physical realm to give the answer to the original problem" (1971, p. 223). This cycle, according to Skemp "reduces noise" ... "and by abstracting only mathematical features it allows us to use a model which we are well practised in working," (1971, p. 223). He also says that: "The greater the *noise*, the harder it is to form the concept," (1971, p. 28).

Physics teachers try to reduce the *noise* by teaching students to work in two times one dimensional way, which is well-practised (described in chapter 4.4.4). In the case of vectors, composition of vectors by different ways of adding them and also decomposition of vectors in order to be able to apply symbolic ways of calculation is very important, just as in fractions it is very important to be able to apply the rule of equivalence to mixed numbers and improper fractions in order to be able to multiply and divide them. In vectors, to start with, the meaning is related to physical objects in the 'outside world' (translation, velocity, acceleration, forces, etc.), but then pupils are expected to develop a concept, translated into symbols, which they could operate on in a mathematical context. Eventually they should have a facility to operate, not just on those concepts, but with anything that resonates with them. The ability to work with the mathematical ideas, without the need to evoke the physical object gives the student power in solving more subtle problems. The problem is how to encourage this abstraction to occur in practice.

3.5.2 The idea of 'effect'

A major contribution to my theoretical framework occurred in a classroom discussion with a student I will call Joshua, who solved all the questions given to him in the preliminary study. During the interview Joshua explained that different actions can have the same 'effect'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation, He said "this is the same and it corresponds to the sum of the two vectors." He therefore interpreted the physical situation as mathematical, seeing the addition of two vectors having the same effect (mathematically) as the resultant vector arising from that addition.

He showed that by focusing on the *effect* rather than the specific actions involved, it was possible to get to the heart of several highly sophisticated concepts.

This concept seemed very powerful as it could be visualised as useful in other areas of mathematics. For instance, in algebra, $2(x + 4)$ and $2x + 8$ involve a different sequence of actions that have the same effect, leading to the notion of equivalent expressions.

In the case of vectors this idea could be presented graphically as in figure 3.16.

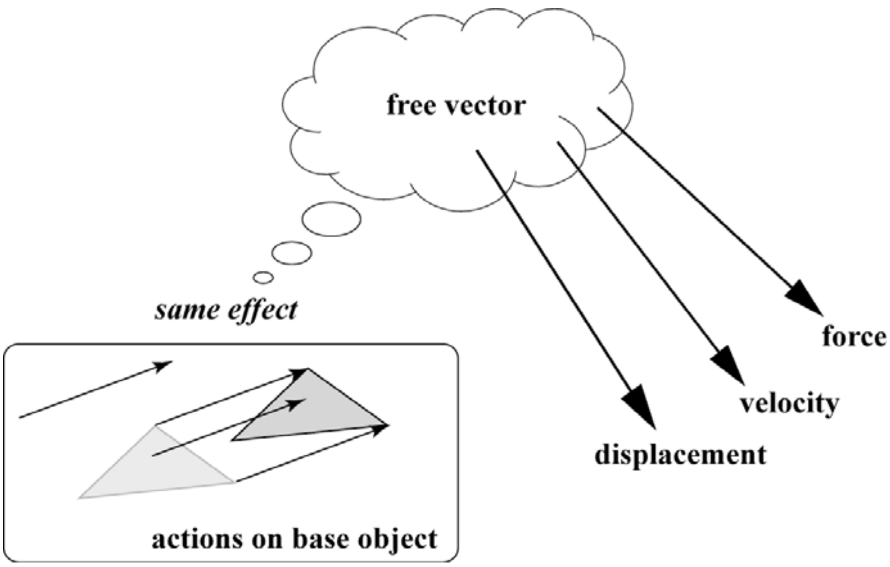


Fig. 3.16 Focusing on *effect*

The *effect* of a physical action is not an abstract concept. It can be *seen* and *felt* in an embodied sense. My idea was that, if students had such an embodied sense of the effect of a translation, then they could begin to think of representing it in terms of an arrow with given magnitude and direction. For instance, if the student's hand was moving a triangle on the table, then the arrow could be taken to show the movement of the tip of a particular finger, or thumb. The particular choice of arrow did not matter. What does matter to give the required effect is the magnitude and direction of the arrow. My idea was to use the students' physical experience as a foundation for the building of the concept of free vector and to give an underlying embodied foundation to the symbolism used for vectors building a coherent schema of meaning. For example, the addition of vectors is a simple extension of the idea that the sum of two free vectors is the free vector that has 'the same effect' as the combination of one vector followed by another. This would give embodied meaning to the technique of placing vectors 'nose to tail' to add them and would provide a foundation that showed that the triangle law and parallelogram law are just two different ways of seeing the same underlying concept, leading on to simple ideas, such as the idea that the addition of vectors is commutative.

The goal is to create conceptual knowledge with a relational understanding of the concepts rather than procedural knowledge with an instrumental understanding of separate techniques. By founding the ideas on coherent physical actions and by focusing on the notion of 'effect', the strategy is to encourage students to reflect on their knowledge and build the notion of free vector as a coherent cognitive unit in a rich schema of relationships.

This approach is also a natural extension of the foundational ideas of Piaget. It took researchers some time to realise that the important Piagetian idea of activity does not necessarily mean a physical one. As Piaget puts it: "The most authentic research activity may take place in the spheres of reflection, of the most advanced abstraction, and the verbal manipulations…," (Piaget, 1970, p. 68).

Following the literature reviewed in chapter 2, I decided to frame my work in a broad context of research including *embodied cognition* of Lakoff & Nunez (2000) — which situates the foundations of learning in real world activity, as does the embodiment of Skemp (1971) — and the encapsulation of a mathematical process into a mathematical concept through reflective abstraction, found in the work of Dubinsky (1991), Sfard (1991) and Gray & Tall (1994).

The different stages noticed in the study, which is also related to the way the text book is written, are shown below in figures 3.17 and 3.18.

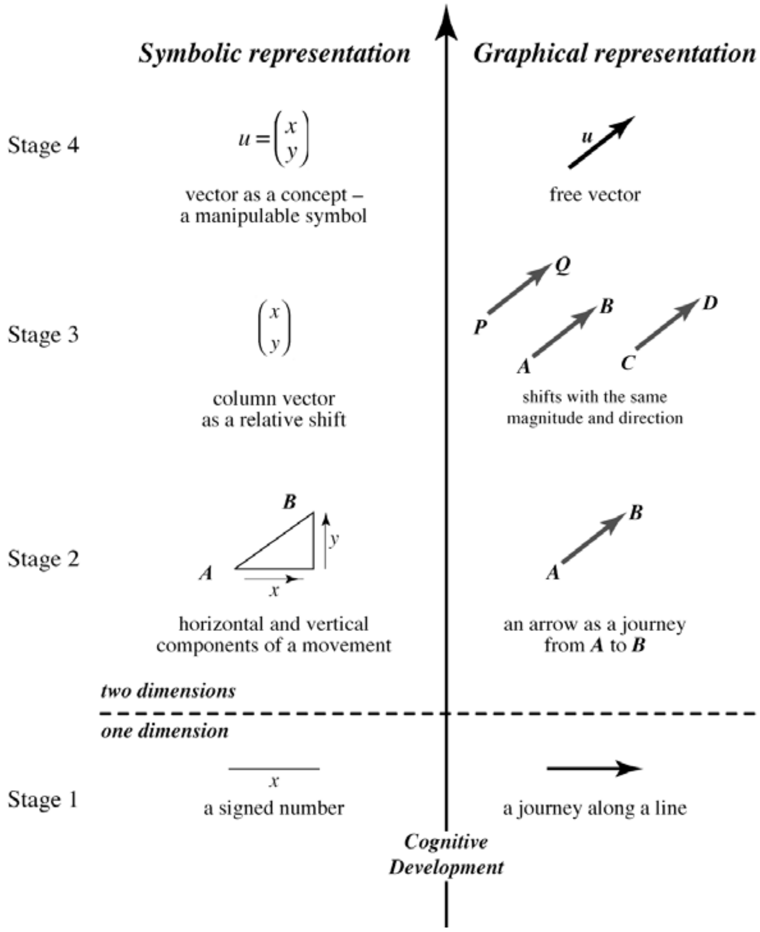


Fig. 3.17 Cognitive development of vector

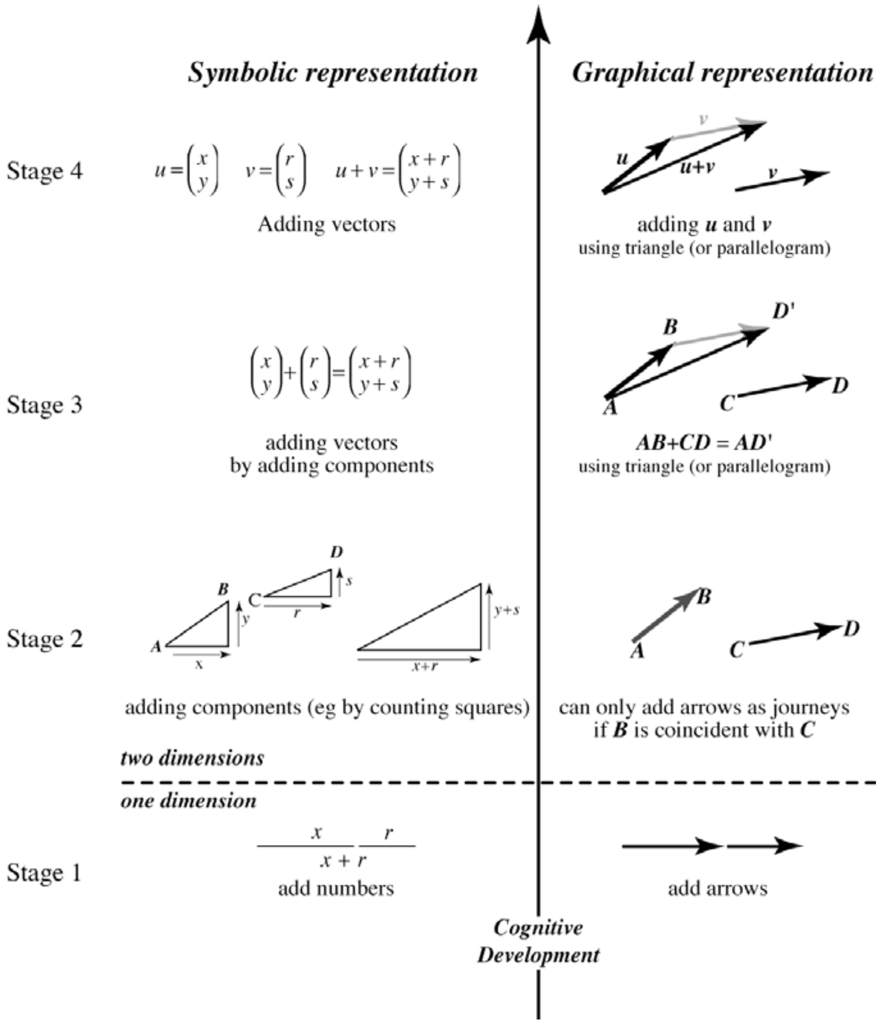


Fig. 3.18 Cognitive development of vector addition

Both of the figures show the concept-building ladder passing through stages of encapsulation. At the bottom of the ladder is the first stage, when students can deal with vectors only in one dimension and the next three levels of development in two dimensions indicate an increasing growth of encapsulation from procedure to encapsulated concept. Stage 0 is reserved for students who use their physical instinct instead of the knowledge of the vectors to answer the questions.

A further level of classification will be used that correlates the separate measures of student performance between the symbolic and graphical modes to give an overall picture of the student's development. This will use a framework that relates to the literature considered in chapter two. It has its origins in the work of Bruner, who expanded Piaget's ideas and applied them to a person's cognitive growth at any stage of life. He distinguished between three modes of mental representation: *enactive*, *iconic* and *symbolic*. He considered that these representations grow in sequence. His enactive representation begins in Piaget's sensori-motor stage and the iconic mode emerges in the pre-conceptual stage with the symbolic mode arising through language and the symbolism of mathematics. However, Bruner saw that, as each mode becomes available, all three modes are available to the individual at any age.

My interest is in teenagers who have all three modes available and, for convenience, the enactive and iconic mode of physical action and visual perception are seen to relate to physical translation and graphic representation, as opposed to the symbolic representation of vectors as column matrices and single letter symbols satisfying familiar mathematical rules, such as $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

In chapter 2, I noted how the SOLO taxonomy of Biggs and Collis (1982) incorporates both Piaget's and Bruner's idea to provide a Structure of Observed Learning Outcomes in assessing students' progress. There are five SOLO taxonomy modes of cognitive development: *sensori-motor*; *ikonic*; *concrete-symbolic*; *formal*; and *post-formal*. In particular, according to Biggs and Collis, each of these modes builds on the previous ones, so the ikonic mode incorporates the earlier sensori-motor mode, and the concrete-symbolic mode builds on these two. This fits closely with the development I am proposing in which the embodied activities refer to a combination of sensori-motor and ikonic leading to graphic representations, and the symbolic developments build on these activities.

In each mode, Biggs and Collis see the cognitive development through a sequence that they term *pre-structural, uni-structural, multi-structural, relational,* and *extended abstract*. The theoretical framework used here takes note of the analysis of Pegg and Tall (table 2.1) which suggests that the learning of any mathematical concept follows a fundamental cycle of compression related to this sequence of development in SOLO taxonomy, the APOS theory of Dubinsky, and the procedure-process-object theory of Gray and Tall (1994). In the current research study, the cycle of construction of the concept of free vector passes from pre-conceptions, via step-by-step actions (unistructural), different actions (multistructural) having the same effect (relational) to free vectors as entities (extended abstract) in both graphic and symbolic problems. These stages are as given in figures 3.17 and 3.18.

The general cycle of development underlying these stages (table 2.1) was then considered to develop a description of the stages appropriate for this study. Having simplified the SOLO taxonomy to focus essentially on embodied foundations that are represented by two modes of representation—graphic and numeric—I sought to develop an overall classification that united the developments in the two modes together.

This began with stage 0, in which students responded essentially in terms of physical intuition without any clear evidence of mathematical activity. Such a response in both graphic and numeric modes was classified as ***physical intuitive***. The next identifiable level occurs in a way that focuses on mainly symbolic or mainly graphical representations at lower stages of cognitive development. I took the decision to assign performances that attained level 1 in one of the modes but failed to reach level 2 in the other as being ***uni-modal***. This was subdivided into ***lower uni-modal*** if the activities in the higher scoring mode were at stage 1 or 2 and ***higher uni-modal*** if at stage 3 or 4. If both modes reached level 2, then the performance was categorized as ***multi-skilled***. Performances reaching at least level 3 in both modes are classified as ***versatile*** and those who attain level 4 in both modes are termed ***fully integrated***.

The following summary of this classification shows the broad correspondence with SOLO cycles (in brackets);

- **Physical Intuitive** (pre-structural) applies to students who do not abstract enough information from their physical experience to build a proper mathematical model from it and who stop using learnt procedures in unfamiliar situations.

- **Uni-modal and Higher uni-modal focused** (unistructural) applies to students can work in one mode only.

- **Multi-skilled focused** (multi-structural) applies to students who can switch between the modes dependent on the question they are asked.

- **Versatile (**relational) applies to students who use a variety of modes to answer the same question and in different physical contexts..

- **Fully integrated** (extended abstract) describes students who show a compressed concept of vector addition and show that they concentrate on an outcome rather than procedures leading to it (have the idea of the *same effect*).

The assignment of these categories is a pragmatic activity based on a careful analysis of the responses based on the theorized stages of development. Table 3.1 below shows the assignment of categories for the concept of vector in relation to the combination of symbolic development (laid out horizontally) and graphical development (vertically).

graphical mode							
	free vector (u)	stage 4	higher uni-modal	higher uni-modal	multi-skilled	versatile	fully integrated
	shifts with the same magnitude and direction	stage 3	higher uni-modal	higher uni-modal	multi-skilled	versatile	versatile
	an arrow as a journey from *A* to *B*	stage 2	uni-modal	uni-modal	multi-skilled	multi-skilled	multi-skilled
	a journey along a line	stage 1	uni-modal	uni-modal	uni-modal	higher uni-modal	higher uni-modal
	intuitive responses	stage 0	intuitive	uni-modal	uni-modal	higher uni-modal	higher uni-modal
			stage 0	stage 1	stage 2	stage 3	stage 4
			no response	x a signed number	horizontal and vertical components of a movement	$\begin{pmatrix}x\\y\end{pmatrix}$ column vector as a relative shift	$u=\begin{pmatrix}x\\y\end{pmatrix}$ vector as a concept D a manipulable symbol
			symbolic mode				

Table 3.1 Development of the vector concept, combining graphic and symbolic

Table 3.2 shows the corresponding assignment of categories for the concept of vector addition. This again relates to the combination of symbolic development (laid out horizontally) and graphical development (vertically).

graphical mode							
	u, v, $u+v$; adding u and v using triangle (or parallelogram)	stage 4	higher uni-modal	higher uni-modal	multi-skilled	versatile	fully integrated
	$AB+CD=AD'$ using triangle (or parallelogram)	stage 3	higher uni-modal	higher uni-modal	multi-skilled	versatile	versatile
	can only add arrows as journeys if B is coincident with C	stage 2	uni-modal	uni-modal	multi-skilled	multi-skilled	multi-skilled
	add arrows	stage 1	uni-modal	uni-modal	uni-modal	higher uni-modal	higher uni-modal
	intuitive response	stage 0	intuitive	uni-modal	uni-modal	higher uni-modal	higher uni-modal
			stage 0	stage 1	stage 2	stage 3	stage 4
			intuitive response	x, r, $x+r$; add numbers	adding components (eg by counting squares)	$\binom{x}{y}+\binom{r}{s}=\binom{x+r}{y+s}$ adding vectors by adding components	$u=\binom{x}{y}$ $v=\binom{r}{s}$ $u+v=\binom{x+r}{y+s}$ Adding vectors
			symbolic mode				

Table 3.2 Development of vector addition, combining graphic and symbolic

The lowest categories in each table show students operating on basic objects. The highest categories of each of the tables show that students are able to compress their knowledge to operate with vectors as cognitive units in any situation.

Students might be able to operate in one mode (symbolic or graphical) only and achieve a high stage at that level (higher uni-modal) or they can operate in both modes using them to reinforce their answer to a particular question or use a flexible choice of different modes in different questions (versatile). Students who can use both modes comfortably to any type of situation will recognise the commutative law of addition of free vectors. This is the highest stage of cognitive development for the purpose of this research, which does not include developments into formal mathematics based on axiomatic definitions and formal proof.

Chapter 4

Preliminary Investigations

4.1 Introduction

In this chapter I discuss preliminary investigations into students' difficulties in my classroom and in consultation with other teachers and interviews with students. Topics of importance include:

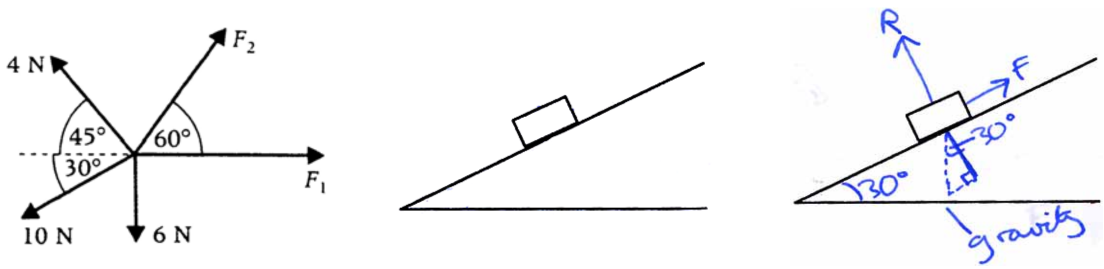
- my investigation of the way vectors are presented in Physics and Mathematics classes at the secondary level;
- my observations, as a teacher, of the problems students have in dealing with vectors in Mechanics (from mathematical and physical points of view) and Pure Mathematics, together with discussions with other teachers to check if they have different experiences from mine;
- the three researches described in section 3.2 (Aguirre and Erickson, 1984; Jagger, 1988; and Graham and Berry, 1997);
- the theoretical framework gained from cognitive science literature of embodiment, and the mathematics education literature focused on the use of symbols representing both process and concept;
- the development of a method of assessment of cognitive development stages to formulate a framework to interpret students' responses.

As it was not obvious at which stage a problem was occurring, and because the claims from other researches needed to be tested, the preliminary investigation began by investigating a question from the Mechanics text-book which, from experience, students found difficult to solve. Students were given the question and then, after analysing the range of responses, some students were interviewed to investigate how they went about solving the problem. Due to the claims of the researches described in chapter 3, students were also questioned on Newton's three laws, to check if this had any bearing on their responses. This investigation is described in detail in Watson (2002). Some of the results presented in that paper are shown below.

4.2 Preliminary empirical investigation

The research has been conducted in a Comprehensive School with good average results with the Sixth Form Centre fifth in the national tables comparing schools in terms of the Value Added (the increase of the level which students achieve in the centrally controlled National Curriculum).

Given a problem solvable by using horizontal and vertical components such as figure 4.1 (a), 25 out of 26 students were able to solve it. However, given a more complex physical problem such as that in figure 4.1 (b), asking the student to mark the forces involved with an object on a rough sloping plane, only 4 out of 26 students were successful. In interviews, it transpired that several students, who used the triangle law to draw a picture as in figure 4.1 (c), used the triangle of forces to mark the components; because the force parallel to the plane is drawn well below the object, it did not seem to be acting *on* it and was ignored.



(a): find F_1, F_2 (b): describe & mark forces (c): forces as marked

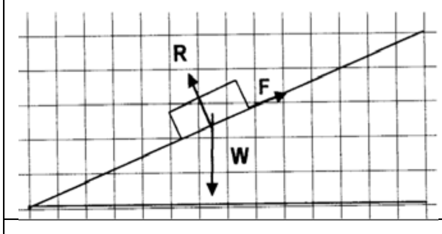
Fig. 4.1 Two questions on forces (a slope)

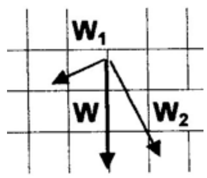
`Five students—who gave varied responses, from one not answering the question at all to the one giving a correct answer—were interviewed. These interviews indicated that even students who did not answer the question knew that the object will slide if there is a resultant force, acting on it. The problem was that, according to their analysis of their own drawings, the resultant force was acting in the in a wrong way — up the slope.

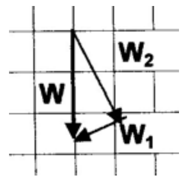
Most students, as in drawing 4.1c, resolved the weight in parallel and perpendicular directions (two components of a vector), drawing the parallel component as part of triangle with the relevant component well away from the surface. The two components were calculated correctly by the majority of students, showing that they are trigonometrically competent. It seemed that they could not proceed any further because although, they knew from their correct *intuition* that the body will either stay where it is (if the forces are in equilibrium) or will slide down (if there is a resultant force), they could not find a force which performs the expected action.

The procedure of drawing the components of weight were well-learnt but not understood. The lack of arrows (correct use of symbols) on the weight and its components could be to blame but a more likely source of difficulty was the fact that the parallel component did not seem to operate directly on the object. The students were able to explain in the interview that the perpendicular component balanced the reaction force *R* and therefore “the object will not sink into the surface or fly off it”. However, it seemed that the only evident force parallel to the plane was the frictional force *F*.

To investigate further the reasons underlying the original problem in figure 4.1 (c), a question was given to students showing a body on an inclined plane, as in figure 4.2(a). Figures 4.2(b) and 4.2(c) were said to represent the ways in which two students James and Chris split the weight W into components W_1 and W_2.







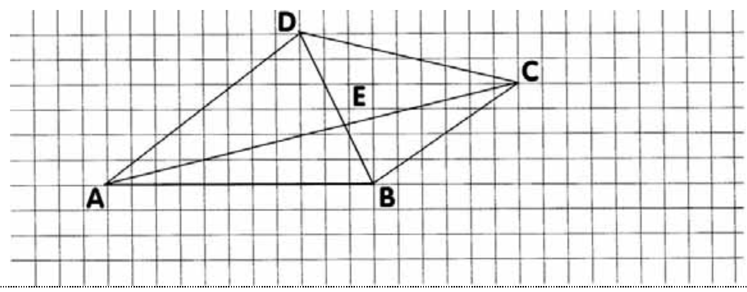
(a): a body on an inclined plane	(b): James’s picture	(c): Chris’s picture

Fig. 4.2 Preliminary study questions

The students were asked: “Are either or both of James and Chris right?” The question was given in this specific way to take the pressure off the students so that, rather than giving their own answer, they were asked to comment on somebody else’s responses.

The 23 students beginning the course in year 11 gave a variety of responses, 11 said both were right, 4 chose fig 4.2 (b), 1 chose fig 4.2 (c) and 6 said neither. These results indicated that although half of the students seemed familiar with the equivalence of the triangle and parallelogram laws by saying that (b) and (c) were both right, the other half responded differently. Since 25 out of 26 students could solve a problem presented in figure 4.1 (a), they seemed to be familiar with vertical and horizontal components. It might have been possible that the context in which the question was asked caused the problem, which may have occurred with the 5 students who had chosen only one of (b) or (c) in figure 4.2.

To test the student's ability to deal with vectors graphically, without any physical context being involved, they were given the question shown in figure 4.3 which was of a type they encountered in Year 10. Part (i) is a natural triangle problem with the vector *AB* followed by *BC*. Part (ii) could be solved either with the parallelogram or the triangle law, however students had to draw the additional lines, which they had not been expected to do in their text book exercises. Part (iii) is more subtle. If they were aware of the commutative law of addition of vectors they could add them as $\overrightarrow{CA}+\overrightarrow{AB}$, however if they saw the addition as 'journeys' this would not make sense to them. On the other hand they could have treated the vector as free and move them 'nose to tail'. A third option was to answer numerically.



Show clearly a vector equivalent to:

(*i*) $\overrightarrow{AB}+\overrightarrow{BC}$

(*ii*) $\overrightarrow{AB}+\overrightarrow{AD}$

(*iii*) $\overrightarrow{AB}+\overrightarrow{CA}$

Fig. 4.3 Testing the visual sum of two vectors

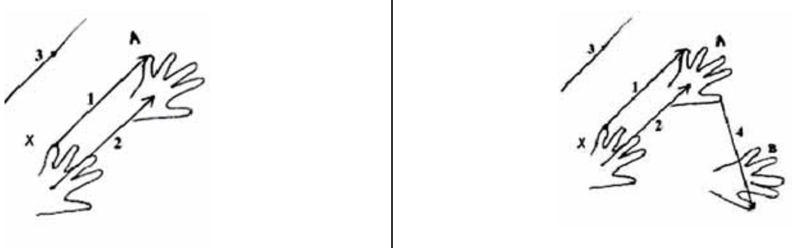
In the test, *all* the students were easily able to cope with the first sum $\overrightarrow{AB}+\overrightarrow{BC}$. However, parts (ii) and (iii) were more problematic and only 3 students out of 23 managed to answer at least one of these questions; all of these who responded correctly solved the problem numerically. This suggested that the students did not grasp the idea of a free vector as a cognitive unit that can be operated on in any context; they were only able to cope with either a simple mathematical problem, as in figure 4.3 (i) or simple physical problem, as in figure 4.1(a).

The research literature discussed in chapter 2 suggests that students should be able to construct a meaning from the experience from the physical world (Piaget, 1985; Bruner, 1966; Lakoff and Johnson, 1999; Lakoff & Nunez, 2000). However concern is also expressed about the "prototype effect," (Rosch, in Lakoff, 1987) and "interpreting words and gestures differently," (Jaworski, 1994).

With all these factors in mind I decided, as suggested by Jaworski (1994) to perform "the activities in which learners participated and encourage them to be mathematical, that is to act as mathematicians by mathematising particular situations created by their teacher" and by including group work and reflective plenaries to encourage learners to "share perceptions with each other and with the teacher", and therefore to make sure that "their ideas became modified or reinforced as common meaning developed."

Two groups were chosen in Year 12, specified as experimental and control, where the experimental group was taught using physical activities and reflective plenaries, which the control group were taught by following the text-book. The students in both groups were tested again and assessed according to the same method as before. A selection of students from each group was also interviewed.

In dealing with the specific vector problems, the students in the experimental group were encouraged to participate actively by shifting a hand placed on the paper and draw the vectors which could represent the translation as shown in figure 4.4(a); then a second translation represented from a different finger as shown in figure 4.4(b). Then the students were encouraged in plenaries to discuss different vector representations of the translation and the way the resultant movement can be represented using vectors in their drawings.



(a) (b)

Fig. 4.4 Embodied action

After two weeks the students were again given the questions presented in figures 4.2 and 4.3. When we considered those students who were able to solve all three problems, we obtained the data in tables 4.1

Year 12	Embodied (N=7)	Standard (N=16)
All 3 correct	5	1
Other	2	15

Table 4.1 Effect of embodied approach in reflective plenaries

Those following an embodied approach had more success answering the questions.

Interviews with six selected students, three from each group, confirmed that students following a standard course had problems adding two vectors that did not follow on one after the other, especially in cases where they were joined head to head. In the latter case, two out of three students thought that two vectors pointing to the same point would have resultant zero, because they would cancel out.

4.3 Summary to preliminary empirical investigations

The study so far has revealed the complexity of the meaning of vectors as forces and as displacements and the subtle meanings that are inferred in differing contexts. Studies in science education have attempted to build a classification of misconceptions without clearly identifying the underlying problems. Our approach is to develop a pragmatic method that will work in the classroom. One aspect is the use of conceptual plenaries, which are already becoming part of the formally defined curriculum in England. The other is to continue to develop a theory that links physical embodiments to mathematical concepts via a strategy that focuses on the effects of actions. Our experience shows that such an approach can be beneficial in the short-term and we are continuing our practical and theoretical developments over the longer term.

As a next stage of preliminary investigation I gave selected students modified questions, based on the above research. However after analysing the results and interviews, I decided to look more at students understanding of an idea of a vector in its different contexts, rather than only the Newtonian problems students are faced with in mechanics.

Some of the questions in that stage of investigation were set on the squared paper as in figure 4.3. After looking at the results of the test and the interviews this idea was dismissed as students simply counted squares to add the components of the vectors and did not show any conceptual thinking.

The preliminary study also seemed to show some evidence for the work of Dubinsky (1991) and Sfard (1991) of *process-object encapsulation* and the theory of Gray & Tall (1994) that students use such symbols both as processes to *do* mathematics and as concepts to *think about*. However there was evidence that many students do not seem be able to use the concept of equivalent vector or free vector in every context. It is as if, for some students, the complication that occurs in a specific context triggers 'false intuitive' reasoning and removes the ability of logical/mathematical thinking. However, when a given problem is presented in an easier way or they are reminded during the interview about the theory (for example of addition) their power using procedures is often correctly recalled. The problem seems to be complicated by the fact that the students are more concerned with remembering to carry out a given procedure rather than reflecting on its total effect. In terms of Dubinsky's theory, the students seem to be focusing more on the *action* stage (of externally taught sequences of steps) than the *process* stage (where the process is interiorised as a whole).

In the teaching experiment I decided to focus students' attention on the underlying mathematical concepts that I believe to be theoretically simpler even though students often find them challenging. My approach was based on the embodiment of the ideas initially as physical actions and then to focus on thinking of the actions as processes that are symbolised and considered as thinkable objects as expressed in APOS theory. However, although this theory starts with *actions*, the starting actions must act on already known objects. In the case of vector as a transformation in the plane, the action operates on figures in the plane that are translated. My research therefore begins with the 'base objects' that the initial actions act upon, with the initial learning strategy based on how the actions transform the objects. In the case of vectors as translations, a base object might be a triangle on a flat table and the actions may be the translations that shift the triangle from one position to another. The essential problem, which has proved problematic in many settings in the literature (eg Cottrill et al, 1996, p.187), is how to achieve the full development from the initial focus on the actions to the final encapsulation of the ideas as mental objects.

4.4 Relating empirical evidence to theoretical framework

By comparing the students' responses to the questions posed in the preliminary investigation it may be concluded that students reach different stages presented in the mathematics text-book and, dependent on the stage achieved, they can solve questions of varied difficulty. They also often seem to have a preferable mode of operation (graphical or symbolic). They might be at a different stage of development in understanding the vector concept than in understanding the idea of vector addition.

The examples of the way these levels should be understood in terms of students' responses and the way they were awarded will be considered in detail in the data analysis in chapter 7. Many researchers indicate that is it easier to show what students cannot do rather than what they think and imagine (see, for example, Sfard, 1991). To complement the quantitative data obtained from the questionnaires, the assignment of stages will be triangulated with qualitative methods arising from interviews with staff and students.

The preliminary study shows that there is a difference to students' development when they are exposed to the experimental lessons in which the emphasis was directed to compressing the embodied actions into process by focusing on the notion of *effect* (if two actions have the same effect then they are considered as giving the same process). It was therefore decided that for the Pilot and the Main studies one group of students, which we will call the experimental group will be involved in lessons in which they will move a hand across the paper as well as push objects across the paper with a hand and focus on the effects of these actions. In the follow-up plenaries, the students will discuss the idea that two physical actions of movement, from point *A* to *B* and then from *B* to *C* (one following the other), are mathematically equivalent to the physical action of single movement from *A* to *C*.

The students will be encouraged in plenaries following the embodied exercises to reflect:

- that the physical action in the embodied world can be modelled mathematically as a symbolic procedure, and on the effect of that procedure;
- that the same mathematical meaning underlies different physical contexts (particularly journeys and forces);
- and appreciate that the mathematical process conducted through different modes of operation (symbolic/graphical) gives the same effect even though the representations may be different.

The researcher hypothesises that the notion of 'effect' is an important stepping-stone in a cognitive development that links the concepts in the embodied, symbolic, (and later the formal) worlds of mathematics. It was conjectured that this will correspond to the cognitive compression of mental processes into thinkable objects in which processes become concepts, and in which the symbols will allow the students to use their knowledge equally successfully in different contexts. (For instance, the notion should later lead to the notion of equivalence relation in the formal world.)

The teaching experiment will be aimed at students giving meaning to concepts in the embodied world, and then sharing their experiences with their teacher (as mentor) who will guide them to express their ideas to each other in ways that enable the embodied concept to be converted in meaningful and flexible ways into symbolic and formal ideas.

In the experimental stage, the rigorous pattern of the Numeracy Strategy will been used which specifies that each lesson should have three stages: starter, core, and plenary. During the starter activity, the teacher sets the scene with the whole class for the main part of the lesson. During the core part, the students work in groups or on individual tasks, and the final plenary reflects on the ideas met in the lesson and makes connections between them. In years 12 and 13 this pattern is usually followed only to a limited extent.

The hypothesis of the researcher that this approach should help students move to a higher levels of cognitive development and retain the conceptual awareness, will be tested through the three tests, staged at intervals: one before the experimental lessons, another soon after and the third after half a year. The experimental group's responses will be compared with responses of students in another group which will not participate in the experimental lessons and which we will call the control group. The interviews conducted after the first and second test are intended to clear any uncertainties about students' test responses and show if students in the experimental group will use more mathematically based language compared with the control group students, independent of the context they will work in.

Chapter 5
Methods and Methodology

5.1 Introduction

5.1.1 Method

According to Cohen, Manion and Morrison (2000) **method** means “the range of approaches used in educational research to gather data which are to be used as a basis for inference and interpretation, for explanation and prediction” (p. 44). They also say that “the aim of methodology is to help us to understand, in the broadest possible terms, not the products of scientific inquiry but the process itself” (p. 45).

This inquiry started with the teachers’ discussion on problems students were encountering while dealing with vectors in Pure Mathematics, mathematical and physical Mechanics. The area of Mechanics has been chosen and exploratory research was conducted. The results of this research helped in developing the preliminary research. This in turn helped in development of the pilot study and then the main study. Cohen, Manion and Morrison (2000) quote Merton and Kendall who, as long ago as 1946, argue that one should try to find a balance between the quantitative and qualitative data and concern oneself “with the combination of both which makes use of the most valuable features of each” (p. 45).

In order to strike the balance this research method draws upon qualitative and quantitative data. The quantitative data, which includes: pre-test, post-test and delayed post test gives an indication to problems which were then scrutinized more thoroughly by collecting the qualitative data gathered during lesson observation and interviews.

The focus of the tests (further described in chapter 6.5.1) has been on finding students’ ability:

- of using vectors to solve problems in different mathematical (graphical, coordinates) and physical contexts (displacement, forces);

- to describe their understanding of vectors and vector quantities;
- to think flexibly (using different mode to add vectors: numerical, graphical);
- to think logically instead of depending on their instincts, in particular, solving singular cases;
- to understand that different procedures can produce equivalent outcomes (the same effect).

5.1.2 Methodology

The methodology is partly influenced by the Piagetian view that a consistently made error to a given problem reflects the child's cognitive structure. The errors therefore, recorded from the tests, have been used to give the direction for the interview questioning. The analysis is based upon the students' perspective and their way of functioning with respect to the task rather than upon the logic of the task, which should provide insight into the student's cognition.

Piaget developed the clinical interviewing procedure which later has been developed and used to achieve the analysis by Ginsburg 1981, Swanson et al. 1981. However, this research uses a semi-clinical interviewing technique in which every interview had a common starting point (their responses to the tests). Thereafter the questions followed what they said or what needed to be clarified. The questions have been based only on the topic of vectors as this is the topic through which the research has been done.

The research has been done with two groups (control and experimental), consisting of a total of 34 students (17 in each group). The research was conducted during year 12. All of the students, from the same school, were studying Pure Mathematics with Mechanics. Both groups were using the same text books but had different teachers. The experimental approach has been introduced to the group we will call group A, who were taught by the researcher. The experimental approach was introduced after the pre-test for a period of three lessons. Its aim has been to:

- o help students in reappraisal of the roles of physical action and symbolic manipulation which relates actions having the same effect to mathematically equivalent concepts;
- o encourage students to share ideas and to help them reassess and refine their knowledge;
- o relate different aspects of mathematical theory to different aspects of mathematical applications in Physics.

The experimental approach has been conducted to test researcher's hypothesis that introducing students to the specific embodied experience followed by purposeful plenaries will help these students:

- o to build a better conceptual base to the topic of vectors;
- o sustain the knowledge for a longer period of time;

Graham (the author of the Mechanics books) and Berry, theorized that:

> "Rectification can then take place by providing alternative explanations that the students can see overcome the weaknesses of their original ideas, explaining satisfactorily the situations used to challenge the students' intuitive ideas." (1997, p. 847)

I have adapted this way of challenging students' ideas during plenary discussions and on an individual basis.

As in the Kerslake (1977) experiment, I have based the experimental teaching phase on good teaching practice and 'cognitive instruction' defined by Belmont and Butterfield (1977). "In this model the child's thought processes and the use made of the instruction are monitored as the treatment progresses. The experimenter must observe as directly as possible how the child is thinking while performing a criterion task, having identified the nature of successful reasoning on that task. The important feature of this model is that the researcher's task is to help the child to build up a particular cognitive framework," (Kerslake, 1977, p. 6).

The experimental lessons have been video-recorded in order to observe as much as possible students' reasoning while performing the tasks and to establish if the experimental procedures and plenaries with the whole class help students to establish a particular cognitive framework.

The ethics of research arose from the fact that the students were under 18 years of age and the informed consent has been obtained from the parents of the students who were to be interviewed. As the teacher was conducting the research it was considered important to study Mason's (1994) considerations at the ways of validating such semi-action research.

Mason (1994) distinguishes between three research perspectives: *intraspection* (observing oneself), "*extraspection* (looking from outside), and *interspection* (sharing and negotiating observations with others)" (p. 13). He argues that: "research from the inside can be every bit as systematic and disciplined as traditional (extraspective) research; researching from the inside provides a much-needed balance to traditional research," (p. 2). He also emphasises that methodology of the research done from inside has to be "supported by a consistent epistemology, and that norms for justification and validation of conjectures and assertions are maintained and developed," (p.11) and suggests that "inner research depends on constantly re-validating distinctions and frameworks with colleagues," (p. 11). He writes that: "Distinctions and frameworks which arise from inner research have a domain of validity consisting of the perceiver-researcher, and the situations in which the distinction comes to mind and is found to be informative," (p. 12). According to him, "Inner research recognises that classifications are evidence of sensitivities which are in flux and may alter and develop over time," (p. 14). From my observation some things do change but others such as students' difficulties stay the same. Students who come to study Pure Mathematics with Mechanics come already with their 'conceptual baggage' gained in the previous years. The inner researcher by doing research during more than one year can seek classifications that are robust and transferable from year to year. Inner researchers can also validate their investigations by studying work done by others and look for confirmation of classifications which are being used.

As it has been impossible to run an extraspective research in the full meaning of that word, the research had to be semi-extraspective. During the four years of research it was inevitable that the researcher and other teachers at the school would be affected by the findings at each stage. This was even more probable as the first stage of findings were published during the course of the research (Watson, 2002). The research has therefore become *semi-action research*.

The intraspection perspective plays a very important role when validating data or looking at preliminary studies in comparison with the main study. Mason writes "the educator has noticed, thereby bringing awareness into consciousness and enabling it to inform future practice," (p. 14) which means that conducting research is bound to have an influence on the researcher and therefore on the educator if it is the same person (*intraspection*). However, the researcher/educator has to discuss the findings with his/her colleagues to be able to conduct the research and therefore they are also influenced by the initial assumptions. The tests conducted in the control groups also had some influence on the possible importance of specific ideas on interchanged with other teachers (*interspection*). So to some extent the balance was sustained between the teaching approaches of the experimental group and the control group.

According to Mason, "Outer research seeks classifications that are robust and transferable from researcher to researcher. Taking into consideration Mason's advice on methodology, frequent interviews with colleagues were conducted throughout the research. They served to understand the way students were introduced to vectors in Physics and Mathematics, problems they encountered during studying the topic and teachers' predictions on the outcomes of the tests. The tests were moderated due to comments made by other teachers in order to rationalise the language used, and to establish the questions which would provide valuable data. Talking to the teachers showed the need for asking questions in many different ways and in different contexts. This has given them a better opportunity to show their understanding of a vector and vector addition. The literature involved in the research into vectors and fractions which show many similarities in problems they cause to learners has been studied. The results of this have been presented in chapters 2 and 3.

The influence of teaching through visual methods and a reflective style, specifically concentrated on the equivalence relationships, has been examined to identify possible changes in students' conceptual development.

5.2 Research Design

5.2.1 Sample

The quantitative study involved 17 students in the experimental group A and 17 students in the control group B both from the same school and the same year but taught by different teachers. The group of students who were investigated can be called a *non-probability sample* as it "derives from the researchers targeting a particular group, in the full knowledge" that it might not "represent the wider population," (Cohen, Manion and Morrison, 2000, p. 102).

Purposive sampling has been conducted to choose the students for the interviews. "In purposive sampling, researchers handpick the cases to be included in the sample on the basis of their judgement of their typicality. In this way, they build up a sample that is satisfactory to their specific needs," (Cohen, Manion and Morrison, 2000, p. 103). In this research the sample for the interviews has been chosen on the basis of the error that has been recurring and needed to be investigated further or the students who have answered in a specific way needed to be questioned about their knowledge of different methods.

5.2.2 Triangulation

According to Cohen, Manion and Morrison, "triangular techniques in the social sciences attempt to map out, or explain more fully, the richness and complexity of human behaviour by studying it from more than one standpoint and, in so doing, by making use of both quantitative and qualitative data," (2000, p. 112). This implies a *multi-method approach* to a problem, which attempts to ensure the validation of the research. Denzin (1970) talks also about other types of 'methodological triangulation': *time triangulation, space triangulation, combined levels of triangulation, investigator triangulation and methodological triangulation.*

Time triangulation has been achieved by longitudinal studies which were based on collecting data from the same group of students at different points in the time sequence. The same test three times was given before the course, straight after and the last one, half a year after the course. The time spans throughout the whole year to make sure students do not remember the test too clearly from one time to another. From the experience it seemed that the time span in students' busy life has been enough to make them view the test as new each time.

Theoretical triangulation has been achieved by drawing on alternative theories. The data was analysed and interpreted within the theoretical framework described in the previous chapters. Results of the pre- and post-course questionnaires, together with school-set tests were used to select students for the follow-up interviews.

The students' responses were triangulated with the intentions of the authors of the school text books and the interviews with the teachers on their preferences.

The methodological triangulation was attempted by collecting data through research questionnaires, school tests, exam results, interviews. To support the data triangulation, the results were collected from students of all the abilities and taught by different teachers. There were also efforts made to integrate the quantitative and qualitative techniques to validate the results.

5.2.3 Variables

The term variable is going to be used in an educational context and it will describe the aspects which might vary. For example variables can include methods of answering questions, concept images or levels of performance.

Inner research as Mason (1994) suggests "depends on constantly re-validating distinctions and frameworks with colleagues" (p. 11). Therefore one should seek to obtain results "which are invariant over specific domain of potential variation, and to develop means for validating that invariance" (p. 11). The conclusions should be 'robust over time'.

The research has developed during 4 years of the researcher building knowledge. The researcher published any new knowledge so that it did not affect just the teaching done by the researcher but the teacher of the control group as well. The researcher made an effort not to change the overall way of teaching apart from the ways agreed by the department. Apart from the time of the introduction of the experimental lessons, teachers of both groups adopted a similar style of teaching. All students taking part in the research were from the same Sixth Form Centre. The teaching in the centre is based on the philosophy that students should be actively engaged in doing mathematics and physics rather than just be 'taught' by the teacher. The students follow a curriculum which is set by the exam board. They are all taught topics in a specific sequence guided by the text books used in the centre.

After establishing an effort made to keep invariance there is a need to establish variables and establish those that might have the significant effect on how students think and those which might hopefully not have affected the research too much.

The variables which are going to be considered are:

- *prior variables* which consist of factors dependent of the students' background like they competencies, concept images already developed through previous experiences, their attitudes and cognitive preferences;
- *intervening variable*s which involve dynamic relationship in the classroom, social influences, students' attitude to the tests and the interviews, students' dedication to the course and teachers attitudes and teaching styles;
- *dependent variables* are how students' can think flexibly, recognise the role of context and ability of assessing their own cognitive progress;
- *consequent variables* which are students' success with the course and long-term changes in flexibility of thinking and conceptual understanding.

There will be other consequent variables which involve change in the researcher's style of teaching, the questioning techniques and even understanding of the topic but there are beyond the scope of this study.

5.2.3.1 Prior variables

The preliminary study and the pre-test helped to establish prior variables (students' existing concept images before undertaking the main study). The students had different teachers of Mathematics and Physics in the previous years and therefore different experiences, often dependent on the teachers' strengths. According to discussions with different teachers some of them made an effort to stretch students beyond the requirements of the syllabus, others took teaching vectors more seriously than others. Students were in the 'slowest' or 'fastest' groups which made the difference to the time the teacher could spend on the topic of vectors. All these variations have affected the way students learnt or assimilated their knowledge from the previous years teaching. This prior experience of students formed their cognitive units, concept images and schemas.

5.2.3.2 Intervening variables

In recent years the number of subjects students have had to study in year 12 has increased from 3 to 4 and they can drop one of these subjects in year 13. Some of them undertake mathematics only for one year to help them get to their chosen university. Students are now allowed to retake the exams as many times as they want, so they can enter year 12 with varied commitments to their study of mathematics, independent learning and abilities to reflect.

It is impossible to fully investigate if there are differences between teaching experiences two groups of students get in their classrooms. The control and the experimental group are also in three different physics groups but not split up in the same way as the maths groups due to other subject options they have decided to study in year 12. As the year 12 physics involves substantial teaching about vector quantities, students would have had varied experiences coming from that direction. There is also a small number of students who did not study physics (usually about 20%).

5.2.3.3 Dependent variables

This variable is tested through the post-test and the delayed post-test. It is hoped that it would depend on the students' experiences during the first two months of study. The students' from both groups were expected to make a substantial progress in their understanding of the concept of vector and of vector addition, especially through graphical representation as this is part of the Mechanics course during that period of time. Both groups post-test results were expected to be similar, however, due to the 'special treatment' of the experimental group, those students were expected to sustain the knowledge for a longer period of time. The students from the experimental group were given a facility to build a better conceptual base for the topic of vectors and so their results were expected to be better in the delayed post-test.

5.2.3.4 Consequent variables

This variable indicates students' future success when dealing with the vector problems, long-term changes in mathematical ability to think flexibly and be able to solve singular problems. This is tested by the delayed questionnaire, which shows the students' long-term ability to solve problems in vectors.

5.2.4 Qualitative and quantitative data collection instruments

The data collection instruments used in the main study include: pre- , post- and delayed tests; follow up interviews conducted after administrating pre- and post-questionnaires.; lesson observations; and interviews with the teachers.

5.2.4.1 Questionnaire

The questionnaire has been designed to give quantitative data. It was initially tested in the pilot study with the previous year of students. According to the students' responses, each answer was given the stage of the cognitive development separately in the graphical mode and in the symbolic mode, as described in figures 4.5 and 4.6 in chapter 4. Allocation of the stages was tested with other teachers.

5.2.4.2 Interviews

This qualitative data has been gathered with the help of the interviews with the staff before the course, interviews with the sample of the students conducted during the year, and lesson observation conducted during the time of the teaching experiment.

According to Cohen, Manion and Morrison (2000) in the interviews the greatest sources of bias are "the characteristics of the interviewer, the characteristics of the respondent, and the substantive content of the questions," (p. 121). During four years of this research, the interviewer kept practicing the interview techniques with many students. These interviews have been video-recorded and discussed with other researchers to enhance the reliability. The coding of responses has also been discussed with other researchers as well as with my research supervisors. Oppenheim (1992: 96-7) suggests several causes of bias in interviewing such as biased sampling, poor rapport, wording of the questions, poor prompting and biased probing, poor use and management of supporting materials, alterations to the sequence of questions, inconsistent coding of responses, selective or interpreted recording of data/transcripts and poor handling of difficult interviews. These were all considered when trying to improve the interviewing techniques.

The sampling has been limited by the students who refused to be interviewed (when letters were sent to parents to agree for their son or daughter to be interviewed, some refused for that to happen).

Poor prompting and biased probing has undergone improvement stages. The wording of the questions has been reviewed to limit the misunderstanding on the part of the respondent of what is being asked. There did not seem to be any problems with the rapport between interviewer and the interviewees.

This pre-course data collection (interviews with the teachers) served to check students' background. Different teachers have varied styles of teaching and use different resources. The interviews helped to establish teachers' expectations of students' knowledge of vectors. This prior variable has been considered in designing questionnaires and interview techniques. During the interviews the teachers were also consulted on the suitability of different questions in the tests and of the problems their students have encountered in Physics and Mathematics lessons where vectors were concerned.

5.2.4.3 Lesson observation

The students in the experimental group have been observed during the time of the teaching experiment. They were encouraged to comment while performing the embodied exercises in vector addition and when doing vector translations. The students who worked at the board were filmed as well as the group work and the plenary discussions. The control group was filmed during two of the lessons at the same time, to see if the students had any embodied experiences during their lessons.

5.3 Teaching experiment and plenaries

The objectives of the teaching experiment and the plenaries were to:

- help students in reappraisal of the roles of physical action and symbolic manipulation which relates actions having the same effect to mathematically equivalent concepts;
- encourage students to share ideas and to help them reassess and refine their knowledge;
- relate different aspects of mathematical theory to different aspects of mathematical applications in Physics.

The idea which was hoped to lead students from the action (embodied world) to the symbols with which they could operate (proceptual world) in their conceptual understanding of the free vector is presented in figure 5.1.

The experimental lessons were only conducted with the experimental group (group A).

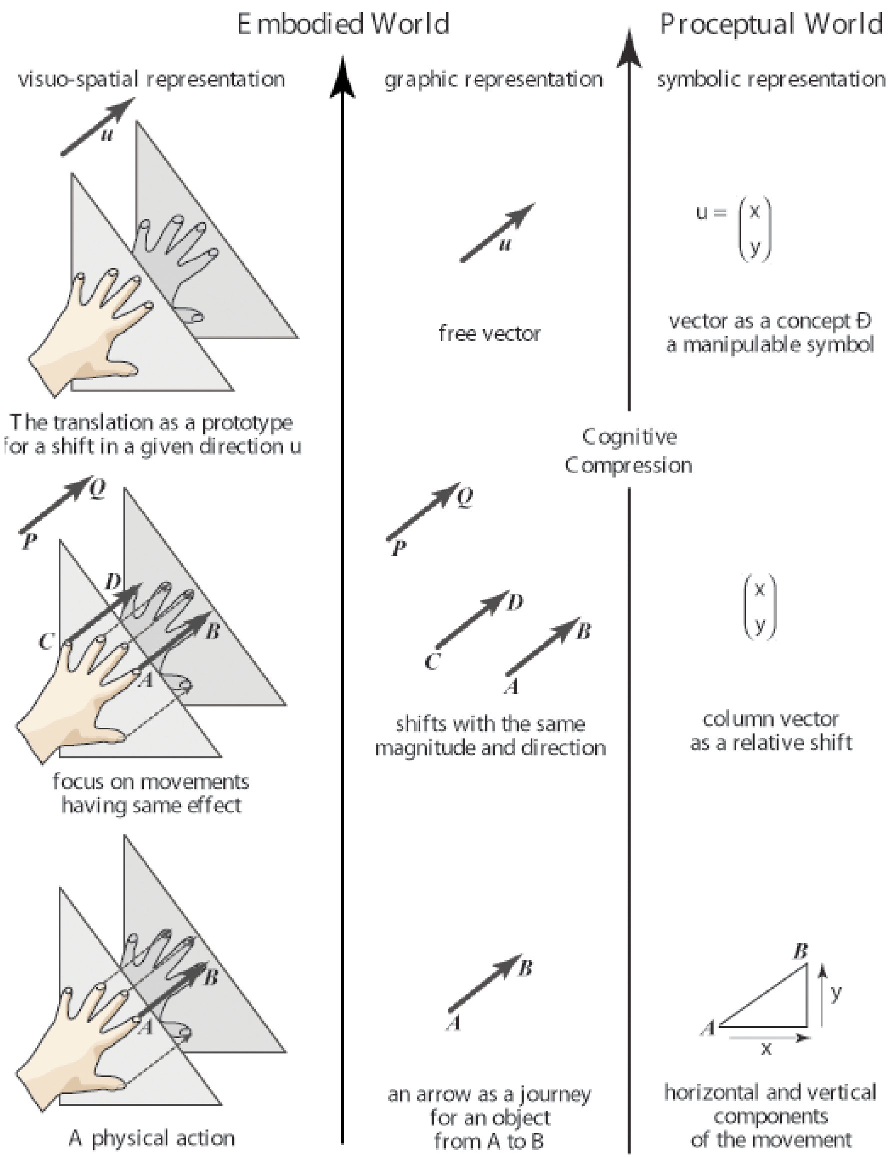


Fig. 5.1 Cognitive compression of the vector concept

First a volunteer student was asked to come to the white board, to put his hand on the board and draw around it. The student was then asked to move the hand across the board and draw around it again (movement from X position to A position, fig 5.2).

All students in group A were to discuss the way the translation could be presented. Some of the answers are shown in figure 5.2 (arrows 1, 2 and 3).

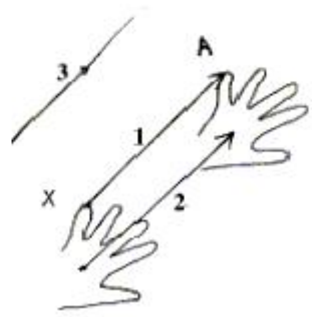


Fig. 5.2 Action of translation—experimental lesson

Another student came to the board put the hand in position A and translated it into position B (fig. 5.3).

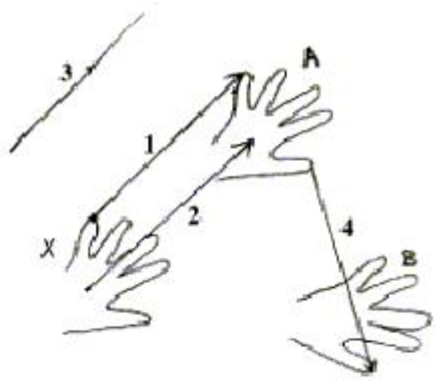


Fig. 5.3 Action with two translations—experimental lesson

The students were asked to discuss the way of representing the second translation on the drawing. The students agreed on the answer shown as the arrow 4 (fig. 5.3).

They were then asked to show the overall translation and agreed on the answer 5 shown in figure 5.4.

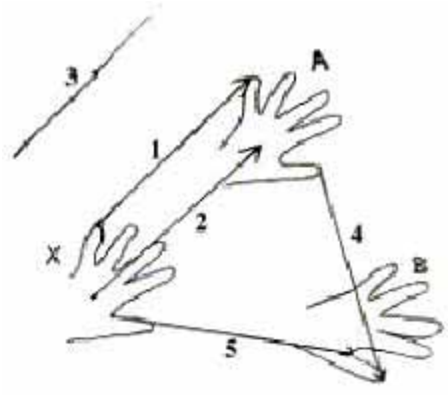


Fig. 5.4 Overall translation—experimental lesson.

The students were then invited to discuss how the answer can be shown mathematically. The preferred method to start was to calculate the horizontal and vertical movements (the way they are encouraged to do it in Physics) and add them together showing equal values for the vector 5. However being asked for more methods, some students came up with the idea of writing vectors in a symbolic form of a column vector and adding them together (fig 5.5), showing equivalence to vector 5. Eventually two students came with the idea of adding vectors graphically. They were encouraged first to work on the idea in small groups and then as the class discussion. Although students all knew the rule of moving vectors 'nose to tail' in order to add them, this was not easily remembered in the above situation.

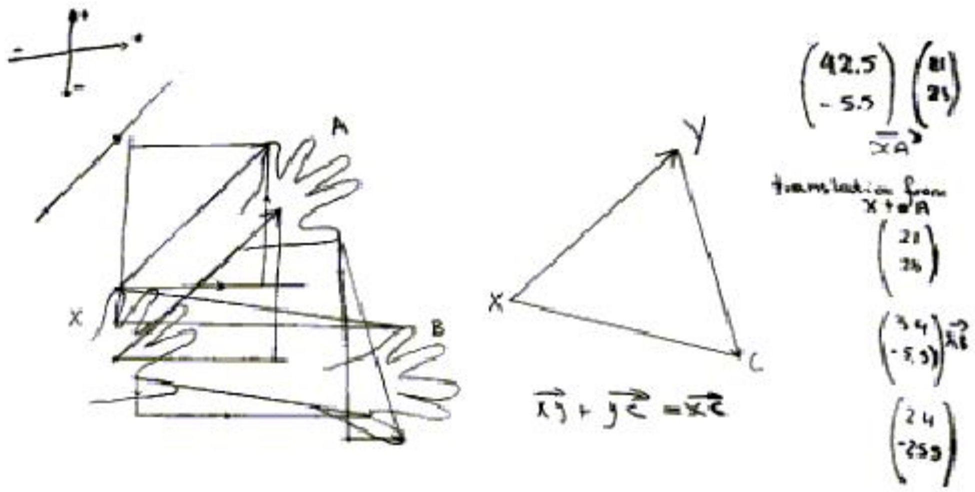


Fig. 5.5 Action of addition of two translations—experimental lesson.

The students were asked during the next lesson to repeat the exercise but with a triangular shape which they pushed with their hand, during which they had another type of embodied experience.

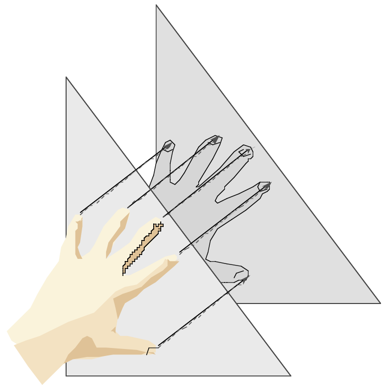


Fig. 5.6 Moving object—experimental lesson.

The students were asked first to draw a shape and cut it out, one of the students was asked to put his shape on the board and translate it (movement without rotation, fig. 5.6). Then the exercise followed the same steps as in the previous experimental lesson, but with the triangle. Not all the students were able to recall experiences gained from the previous lesson and there was still a lot of discussion in plenaries of the geometrical way of vector addition.

During the third lesson the students were asked to add forces acting on the objects, showing the addition on the paper with the precision enhanced by using a ruler and a set square to give them an embodied sense of the operation. They had to move the set square along the ruler with their hands to draw precise parallel lines and then measure the magnitude and repeat it with the precision on the translated vector.

5.4 Summary

The research methodology and the rationale for various ways of collecting data and for the way the teaching experiment has been conducted were described in this chapter. The data collection consisted of three major components: questionnaires, interviews and observation of students during the teaching experiment. Additional data has been conducted through research of the textbooks the students use, interviews with teachers and collecting students' results from external exams and internal tests. These additional components also provided means of triangulation and in establishing the prior variables. Changes to those variables were documented using pre-test, post-test, delayed post-test and interviews. The quantitative and qualitative components of the main study were designed to address the three main research interests:

1. discover what makes some students link the real world activities with mathematical symbolism in a meaningful way;
2. discover what prevents some students in making these links;
3. to increase students understanding of these links through specific teaching techniques of which the main part are the plenary sessions and make them more flexible in their thinking.

Pre-, post- and delayed post- tests were designed to examine the above points and to provide data showing if the teaching experiment had any effect on students learning and their flexibility of thought. The class observation and interviews were conducted in order to add a qualitative component to the first two points of interest and provide the insight into reasons for students' difficulties and strengths.

Chapter 6

Pilot Study

6.1 Introduction

The pilot study tested:

- the design of questions to be used in the main study;
- the method of collecting and analysing both quantitative and qualitative data;
- whether the experimental lessons given to students have the anticipated impact on their results from the test.

However in the pilot study the test was only conducted twice, before the experimental lessons and after. In the main study the test was conducted three times and the focus was mainly on the difference between the pre-test and the delayed post-test because the long-term retention of knowledge was important in this analysis.

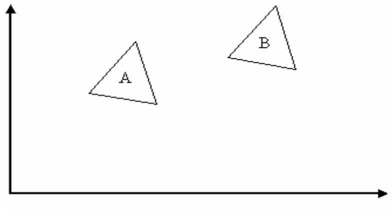
6.2 Design of the questions

The first two parts of the chapter show test questions and their intentions the next part shows how the stages of the cognitive development were awarded to students' responses; the last part shows the results of the pilot pre-test and post-test.

6.2.1 Cognitive development of vector

Figure 6.1 shows the diagram for the first question in the test.

1) In the picture the triangle has been translated from position *A* to position *B* as shown below:



(a) How can you represent the translation of the triangle?

(b) Can you draw a vector starting at the origin (0,0) which will represent the translation of the triangle from A to B? If so, show it on the drawing.

(c) Can you draw a vector not starting at the origin and not touching either of the triangles which will represent the translation from A to B? If so show it on the drawing.

Fig. 6.1 Test question 1

This question tested students' cognitive stage of vector awareness. The first part of question (a) expected students to represent a vector representing the movement of an object, either in a symbolic way, by representing vertical and horizontal components, or graphically as an arrow from one point on the object in position *A* to a corresponding point on the object at position *B*.

Students learn in their course that among all the equivalent vectors there is one which starts at the origin which is called the 'position vector'. Therefore, to test if they have developed so far in their cognitive understanding, the second part (b) expected a student to draw an equivalent vector to the one in part (a) but starting at the origin. The students who has continued on the cognitive development ladder to construct the idea of a free vector, are expected to understand that an equivalent vector can be represented anywhere on the page. To test this, part (c) of the question asked them to draw a third vector representing the translation but not touching the triangles or the origin.

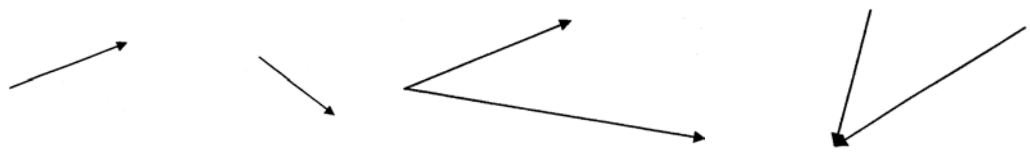
The first part gave students an opportunity to answer graphically or symbolically ("represent") but the second and third part asked them for the graphical representation ("draw"). Therefore symbolically this question may not encourage the higher stages of the cognitive development but the real purpose of this question is to see if students have a concept of a free vector, specifically if they can draw an arrow which is not on the triangle. For instance, students who conceive of the vector as a physical 'push' on the triangle might sense that for a vector to cause movement, it must actually touch the object being moved. They may believe this quite separately from the possibility that they are able to reproduce the learned response to draw a position vector at the origin. This question, despite the expectation of testing mainly the graphical mode was very important as it tests students' development from acting on the base object to the process of drawing equivalent vectors, to the concept of free vector.

6.2.2 Cognitive development of vector addition

The cognitive development of vector was further analysed in questions asking for vector addition.

The question presented in figure 6.2 asked students to add two vectors in the three situations shown below (a, b and c):

Question 2: In each case below add the two vectors:



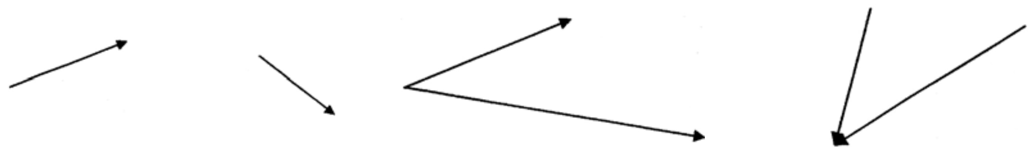
(a) (b) (c)

Fig. 6.2 Test question 2: adding vectors as arrows

This question includes three examples that have certain specific properties that are in some sense unusual. Part (a) is a prototypical example in mathematics when discussing addition of free vectors that are to be placed 'nose to tail'; the vectors are not touching or overlapping, so that either one can be shifted until its tail coincides with the nose of the other. However, from a physics viewpoint, this is not typical in the context of forces where the students are used to having the forces all acting on a single point. Part (b) is typical of two forces acting on a point, and from this viewpoint, it might evoke the use of the parallelogram law. However, as we have seen in the literature and in figure 2.1, it might evoke other meanings, such as two competing forces tugging at a point, or two sides of a triangle, leading to a misuse of the triangle law. Part (c) is considered to be a singular example that has not been discussed in Physics or Mathematics lessons. The manner in which the two vectors meet nose to nose may lead to misconceptions, such as the idea of two forces pressing on each other and perhaps cancelling each other out.

Question 3 (figure 6.3) is designed to test students' versatility by asking them for another way of adding vectors in question 2. They may respond by performing the sum in the same mode (perhaps nose to tail in figure (a) with the vectors in a different order) or in different modes (responding geometrically on one occasion and numerically on the other).

> **Question 3: If there is any other way you could have done any of the additions of the two vectors in Q2 show it here:**



(a) (b) (c)

Fig. 6.3 Test question 3: adding vectors in another way

The next question is designed to test the students' understanding of free vectors in a more general case.

> **Question 4: Add the three vectors shown below:**

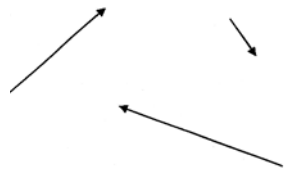


Fig. 6.4 Test question 4: Add three vectors

If they can add vectors in question 2 (a) or 3 (a), and also can add three vectors in question 4. (figure 6.4), then they are more likely to have some understanding of vector addition, even if they did not answer parts (b) and (c) in figures 6.2 and 6.3.

Students who learnt procedurally and not conceptually might only understand that adding two vectors will graphically be in a form of a triangle and not have a concept that whenever you add any number of vectors by shifting them 'nose to tail' you will get the resultant, which has the same *effect* as adding them all together (as shown later in figure 6.24(a)).

On the other hand they might realise the procedure of shifting vectors 'nose to tail' but not getting a triangle, they may not know what to do next (as shown later in figure 6.24(b)).

The next two questions are set up in two different physical contexts and are opened-ended:

Question 5: **Draw a representation of three forces and add them together.**

Question 6: **Draw a representation of two displacements and add them together.**

Students have an option, for example, to draw forces acting all in one direction or only in two, or all three forces acting from one point and then add them as 'free vectors'. However if students are attached to the physical situation they might draw the forces, but unless they act in one dimension or maximum two (where they can use numerical methods) they may find them difficult to add.

In the case of displacements they might draw two vectors following each other and add them, however if they are confident with a concept of free vectors they might draw them separately and add them together.

The two questions above test how students operate when faced with vector addition in different physical contexts, and is included to see if it makes a difference to the stage at which they respond to vector addition.

The next question (figure 6.5) tests if the students can recognise answers in the midst of the drawn lines.

Using the drawing below, or otherwise, add:

(a) $\overrightarrow{AB} + \overrightarrow{BD}$

(b) $\overrightarrow{DA} + \overrightarrow{ED}$

(c) $\overrightarrow{AB} + \overrightarrow{AE}$

(d) $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DE}$

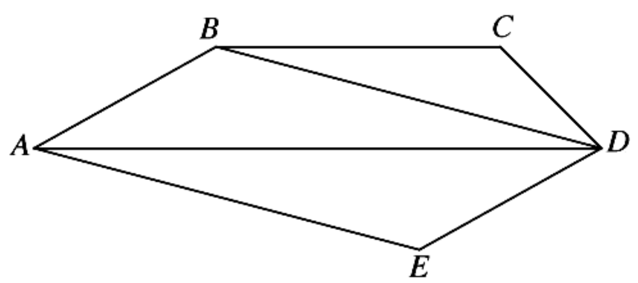


Fig. 6.5 Test question 7: adding vectors in a drawing

Questions like these are familiar to students from their Mathematics course in the previous year (Year 11). Part (b) can be answered in two ways, either using the commutativity law that $\overrightarrow{DA}+\overrightarrow{ED}=\overrightarrow{ED}+\overrightarrow{DA}$ which, in turn equals $\overrightarrow{EA}$, or by seeing that $\overrightarrow{AB}$ is parallel to $\overrightarrow{ED}$ and has the same magnitude, and using the idea of equivalent free vectors to give the answer $\overrightarrow{DB}$. Because of the involvement of the commutative law of addition part (b) of the question has been categorised as a 'singular' case.

The last set of questions (figure 6.6) has been assigned as a 'singular' case. The students were not familiar with the answers not being part of the diagram and it was considered that in order to answer them, they had to be familiar and confident with the idea of free vector.

Using the drawing below, or otherwise, add:

(a) $\overrightarrow{AD}$ and $\overrightarrow{CD}$

(b) $\overrightarrow{AD}$ and $\overrightarrow{BC}$

(c) $\overrightarrow{AC}$ and $\overrightarrow{BD}$

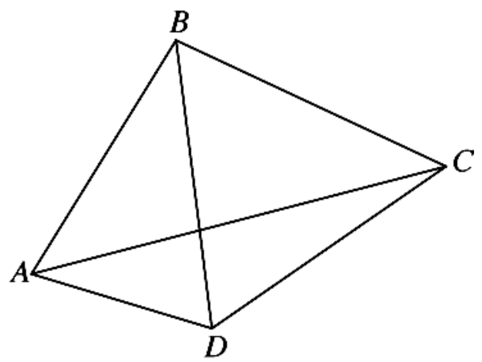


Fig. 6.6 Test question 8: more sophisticated addition in a drawing

Part (a) of question 8 is singular for two reasons: one because the answer is not part of the diagram (which was always the case in the students' earlier experience); but also because two vectors meet at one point (as in questions 2c and 3c). Part (c) of question 8 is also singular for two reasons, first that the result is not part of the diagram and the second that two vector cross each other. Students who can answer all questions in figure 6.6 are therefore considered to be at the top of the ladder indicating their cognitive development in at least one of the two modes (graphical or symbolic).

The pilot study tests the final design of questions and the way the responses were going to be evaluated in the main study. In the following two sections it can be seen how the responses to the questions will be analysed.

6.3 Method of collecting quantitative data

In the main study the test was given to students 3 times, before the course, straight after the course (which included the experimental lessons in case of experimental group) and the delayed post-test given to students half a year after the course. However in the pilot study the test was given only twice, before the course and straight after the course which also involved two experimental lessons for one group out of two taking part.

The quantitative data analysis focused on the stages attained by students in the graphical and symbolic modes of operation, as formulated in chapter 4, figure 4.6. In practice, the students often responded in ways that required careful analysis to place them in appropriate stages. This was done with help of another teacher from the same school who taught Pure Mathematics with Mechanics for many years but was not involved at any stage with the students under investigation. The researcher and the teacher independently allocated stages to a sample of nine varied responses from students to the test. These allocations were discussed and final versions established. Thereafter the rest of the responses were allocated stages of cognitive development according to the agreed format.

The principle was to give the highest stage for each question, consistent with the response, if there was no graphical or symbolic response at all, then stage 0 was awarded, even though the stage 0 is also for the intuitive responses. In both cases it meant that student did not reach the first stage of the cognitive development ladder. The more precise interpretation of such results had to be tested through the interviews.

It has been also decided that in the symbolic mode the answers given in letters or numbers will be treated the same. It might be debated if those two responses show different cognitive level of development, but it will not be part of this study. The student had to satisfy a certain stage at least twice (in two different questions or parts of the question) before being awarded the stage in the general case (taking into consideration all the test questions) but only once in specific questions analyses ('singular' cases, different contexts cases).

As could be seen from the questions described in part 1 of this chapter, some were more suitable for graphical mode responses than others and question 1 (b) and (c) specifically asked for the graphical responses, which could have influenced some students that this was expected mode for the rest of the test. However the overall expectation was that students may answer in the mode they are more familiar with in most of the questions and perhaps show the other ability either when their favoured mode is not possible, or if they are asked to do it differently. A sample of students answering in only one mode throughout the test was interviewed to check if in fact only one mode was familiar to them.

For the reasons stated in the previous paragraph, when making a judgement for overall stages of development, the final analysis was performed in two ways. The first way used all the responses given by the student to prescribe an overall stage of development. There were 17 questions in total and the student had to achieve their highest stage twice to be given it. If student answered, for example, once at stage 4 and once or more at stage 3 then stage 3 was given. In the case where a student answered once at stage 4 and once or more at stage 2 then stage 3 was given. In general the rule is to take the two highest stages awarded, calculate the average, and round it down to the nearest whole number.

The second way involved focusing on all the questions which contributed to a specific aspect of study. There were two cases considered: the 'singular' cases (four questions) and the questions testing different physical contexts (displacements and forces) of which there were two. Because of the smaller number of such questions, students had to gain the stage only once to be prescribed that overall stage in the aspect under consideration.

The pilot study only looks at the overall stages gained by students, however the main study also will analyse the specific type of questions as the changes occurring in the students' responses show the more precise insight into their development.

The stages gained by each student in both graphical and symbolic modes were then plotted on a scatter graph. The scatter graph was divided into 25 regions and these regions were given categories, developed in chapter 4, as shown in table 6.1.

This begins with stage 0, in which students responded essentially only in terms of physical intuition without any clear evidence of mathematical activity. Such a response in both graphic and numeric modes was classified as ***physical intuitive***. The next identifiable level occurs in a way that focuses on mainly symbolic or mainly graphical representations at lower stages of cognitive development. I took the decision to assign performances that attained level 1 in one of the modes but failed to reach level 2 in the other as being ***uni-modal***. This was subdivided into ***lower uni-modal*** if the activities in the higher scoring mode was at stage 1 or 2 and ***higher uni-modal*** if at stage 3 or 4. If both modes reached level 2, then the performance was categorized as ***multi-skilled***. Performances reaching at least level 3 in both modes are classified as ***versatile*** and those who attain level 4 in both modes are termed ***fully integrated***.

graphical mode	stage 4	higher uni-modal	higher uni-modal	multi-skilled	versatile	fully integrated
	stage 3	higher uni-modal	higher uni-modal	multi-skilled	versatile	versatile
	stage 2	uni-modal	uni-modal	multi-skilled	multi-skilled	multi-skilled
	stage 1	uni-modal	uni-modal	uni-modal	higher uni-modal	higher uni-modal
	stage 0	intuitive	uni-modal	uni-modal	higher uni-modal	higher uni-modal
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Table 6.1 Table of the second stage of the categorisation

The data in the main study is going to be presented in the form of the table above but instead of names of the categories there will be indication of how many students in each group responded in those categories. The sign 'A' will indicate a student from group A and the sign 'B' will indicate a student from group B. The χ^2 test will show if there is a significant difference between the control and the experimental group by comparing the number of students in two regions marked in a thicker line: one including intuitive and uni-modal categories; another including higher uni-modal, multi-skilled, versatile, and fully integrated.

6.3.1 Quantitative Data Analysis of Understanding the Symbol of a Vector

Figures 6.7 and 6.8 (taken from figure 4.6) show how the stages will be allocated to students' test responses as far as their cognitive development in understanding the concept of vector is concerned graphically and symbolically.

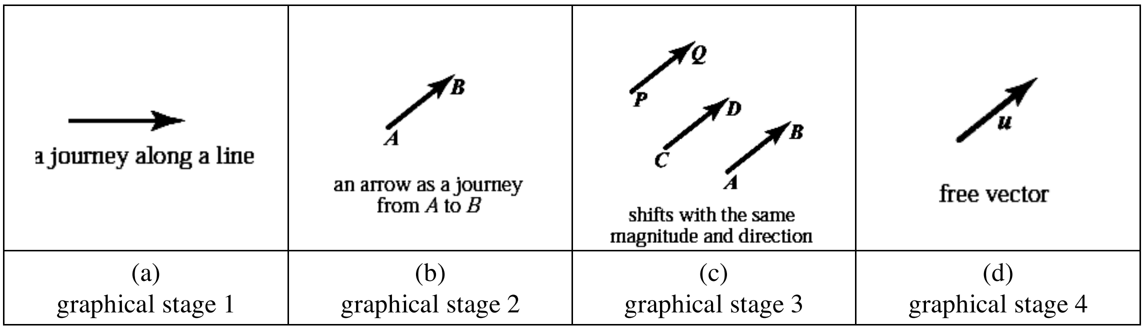


Fig. 6.7 Four stages of cognitive development of vector in the graphical mode

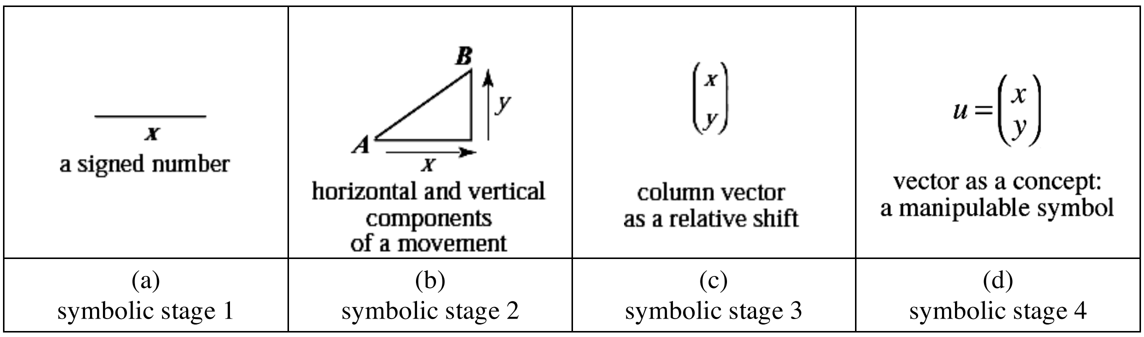


Fig. 6.8 Four stages of cognitive development of vector in the symbolic mode

In addition, the relationship between the stages of development in the symbolic and graphic responses will be categorised using the corresponding cycles shown in figure 4.8, which were formulated in chapter 4 in the following terms:

physical-intuitive: signifies the performance of those students who do not have any specific understanding of the graphical or symbolic representation of a vector;

uni-modal: applies to the students who can operate in basically only at stages 1 and 2 in both modes (symbolic or graphical);

higher uni-modal applies to the students who can operate in basically only one mode (symbolic or graphical) at stages 3 or 4 but only at stage 0 or one at the other mode;

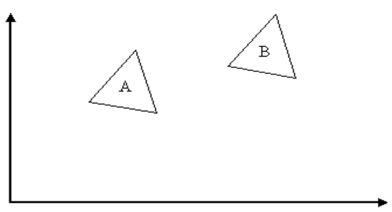
multi-skilled: students who show that they can use both modes of vector representation but do not use them flexibly (the context affects the level of their responses);

versatile: students who use both modes of operation flexibly whatever the context.

fully integrated: relates to the students who recognise the concept of free vector and see it as a mathematical manipulable symbol whatever the context and using the appropriate mode of representation (graphical/symbolic: numerical and algebraic).

The first question is repeated in figure 6.9, so that it can be compared with the responses in figure 6.10. The preliminary study indicated that students often understand the *position vector* (vector starting at the origin) as a movement of an object. Similarly students often showed a translation as an arrow from a specific point on an object to the corresponding point on its translated image.

1) In the picture the triangle has been translated from position *A* to position *B* as shown below:



(a) How can you represent the translation of the triangle?
(b) Can you draw a vector starting at the origin (0,0) which will represent the translation of the triangle from A to B? If so, show it on the drawing.
(c)Can you draw a vector not starting at the origin and not touching either of the triangles which will represent the translation from A to B? If so show it on the drawing.

Fig. 6.9 Test question 1

The questionnaire reveals only the responses written at the time and do not necessarily reveal whether the students have possibly broader levels of flexibility available to them beyond the written answers. This possibility will be considered in the qualitative analysis.

Figure 6.10 show six examples of students' responses.

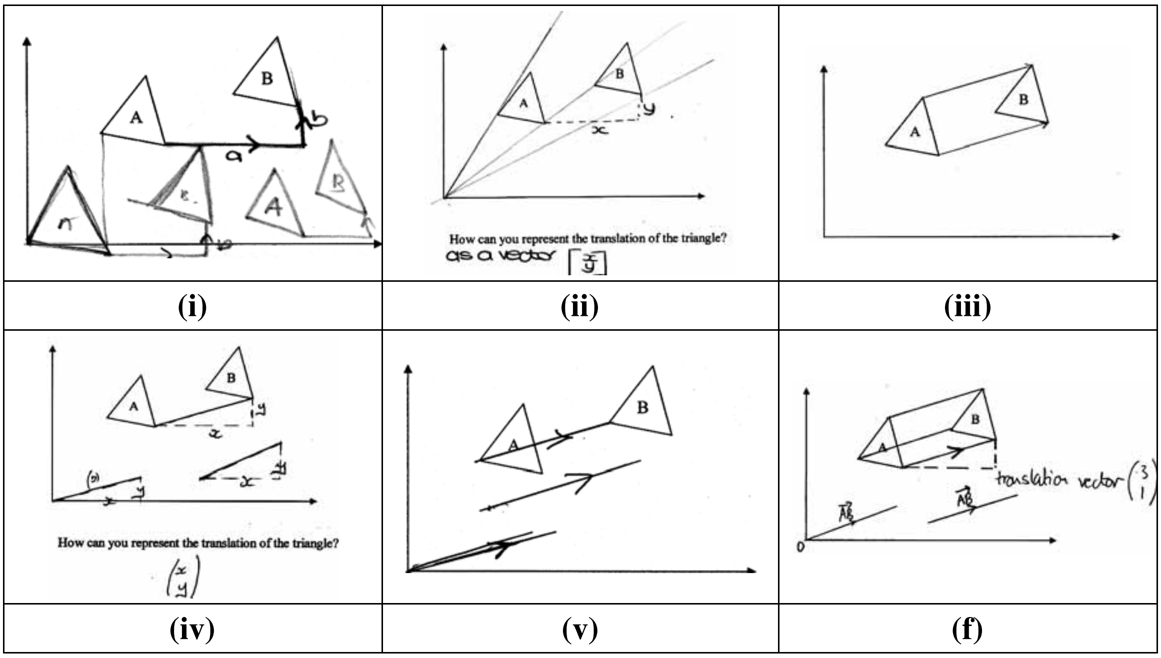


Fig. 6.10 Examples of students' responses to test question 1

The categorisation of students' responses, at different stages of the cognitive development of a vector concept, according to the examples shown in figure 6.4, is discussed below.

The student who responded as shown in figure 6.10 (i) was categorised to be at the stage 0 of the graphical representation but at stage 2 of the symbolic representation of vector. (S)he presented the translation symbolically as horizontal and vertical components but graphically only translated an object without showing the action as an arrow from one point to another. This student is also at the stage of action on an object and does not use the symbol of a vector (an arrow) to indicate the translation. (S)he does not realise the equivalence of vectors but only the equivalence of movements.

The student who responded as shown in figure 6.10 (ii) was categorised to be at stage 0 for the graphical representation as there are no arrows on the drawing or even an indication of moving lines parallel to each other. However the student was given stage 3 for the symbolic representation as (s)he not only showed the translation as the horizontal and vertical movement but also as a column vector, as a relative shift. This student's written response shows no awareness of the notion of vector in a graphical form as even the *x* and *y* components have no arrows on them.

The student who responded as shown in figure 6.10 (iii) responded at stage 3 for the graphical representation as the two arrows represent a 'journey' of the object from a specific point to another specific point and a shift with the same magnitude and direction. However the student did not respond symbolically and from the principles established earlier was given stage 0 in that mode.

The student who responded as shown in figure 6.10 (iv) was categorised to be at stage 3 of the graphical representation and at stage 3 for the symbolic representation. Although the translation is only represented as a line (stage 1) the student shows the concept of 'the same magnitude' and to some extent 'the same direction' by placing *x* and *y* in the same order and revealing some indication of the direction.

The student who responded as shown in figure 6.10 (v) responded at stage 4 for the graphical representation as (s)he drew the notion of free vector, not attached to the object or any specific point, however, (s)he did not respond symbolically and was given stage 0.

The student who responded as shown in figure 6.10 (vi) was categorised to be at stage 4 for the graphical representation as the notion of free vector is indicated graphically and stage 3 for the symbolic representation as (s)he showed a column vector as a relative shift.

Some students did not give any symbolic response to question in figure 6.9 and the data about their cognitive development in the symbolic mode had to be collected from other questions. From the preliminary study it seems that the changes in the symbolic mode are not statistically significant after the experimental lessons, mainly due to the fact that the underlying data is not clear. The interviews tend to reveal more information; it is here that more insight appears, though not for all students.

The second example of the questions given to students is shown in figure 6.11.

2 (b) add the two vectors together

3 (b) If there is any other way you could have done any of the additions of the two vectors in Q2 show it.

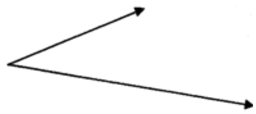
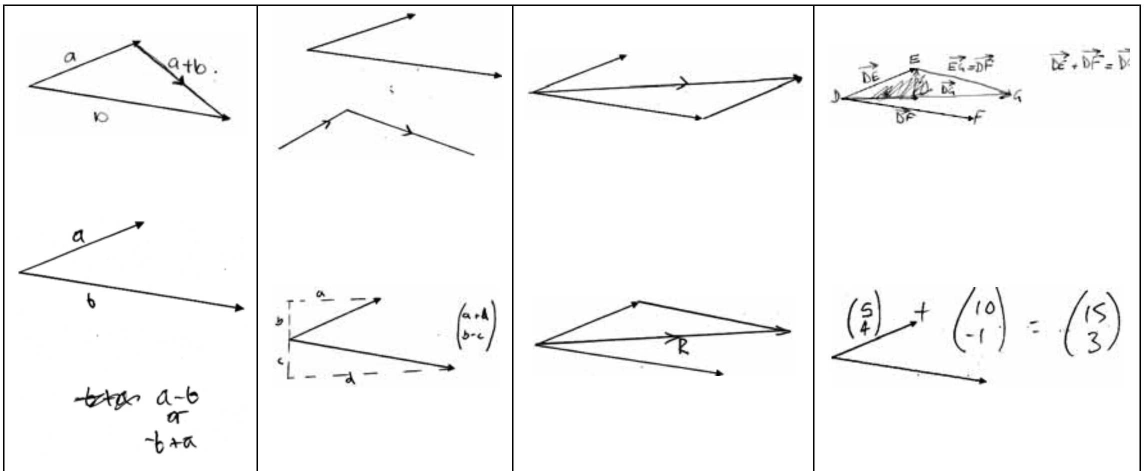


Fig. 6.11 Questions 2(b) and 3(b).

Figure 6.12 shows the examples of the responses of four selected students. The top picture shows the response to question 2(b) and the bottom picture response of the same student to 3(b).



(i) (ii) (iii) (iv)

Fig. 6.12 Examples of four students' responses to questions 2(b), 3(b)

The response in part (i) of figure 6.12 indicates that in a graphical mode a student might interpret vectors as a journey along the line as vectors follow each other in the top answer, described as stage 1. On the other hand, on the basis of the preliminary and pilot studies, some students might be treating **a** and **b** as the position vectors of some point *A* and another point *B* and draw a displacement vector from *A* to *B*, which would indicate stage 2. On this occasion the student wrote first the expression **-b+a** below which he crossed and changed to **a-b**. This would imply that the student was considering a journey, (along **b** in the reverse direction, then along **a**)and therefore the first interpretation was assumed to be the more likely, and the graphical response was categorised at stage 1. The symbolic response was categorised as stage 1 because student only assigned letters to the vectors and did not try to manipulate symbols in any meaningful way.

In part (ii) the student shows that (s)he can shift vectors with the same magnitude and direction but does not show the resultant. The student therefore does not recognise the full idea of the same *effect*. The student was allocated stage 3 for this graphical response. The symbolic response which the student gave to 3(b) question was awarded stage 3, as this indicated the column vector as a representation of the relative shift.

The student in part (iii) of figure 6.12 was awarded stage 4 for the graphical response. This student shows not only an understanding of the concept of free vector but also the concept of the commutative law of addition.

Part (iv) of figure 6.12 shows a student who was allocated stage 4 for the graphical response. This student not only can shift vectors with the same magnitude and direction but also can add them showing concept of *effect*. This student has been also awarded stage 4 for the symbolic representation as (s)he uses a column vector as a manipulable symbol.

The questions discussed above use the concept of vectors formulated in general situations but for someone who can only think about the vectors as symbols related to the physical world, they could be interpreted, for example, as a displacement in case of question 1 (figure 6.9) and forces in case of question 2 and 3 (figure 6.11). These types of question we call *generic cases* (chapter 1, p. 4). However the students are given examples of vectors whose 'noses' meet at a point or where vectors cross we call them *singular cases* (questions which might cause confusion from the physical/intuition point of view). It therefore seemed important to show how responses to such questions were awarded with stages.

The example of questions which were categorised as 'singular' cases are shown in figures 6.13 and 6.15 and are discussed next.

Add the two vectors

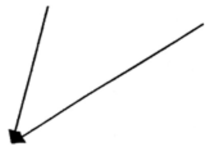


Fig. 6.13 Singular question

Figure 6.13 shows two vectors which meet at one point. According to the teachers and the preliminary tests this example goes against many students' intuition.

The example of the way different stages were awarded to students' responses is shown in figure 6.14. Parts (i)-(iii) show graphical responses and parts (iv)-(vi) show symbolic responses.

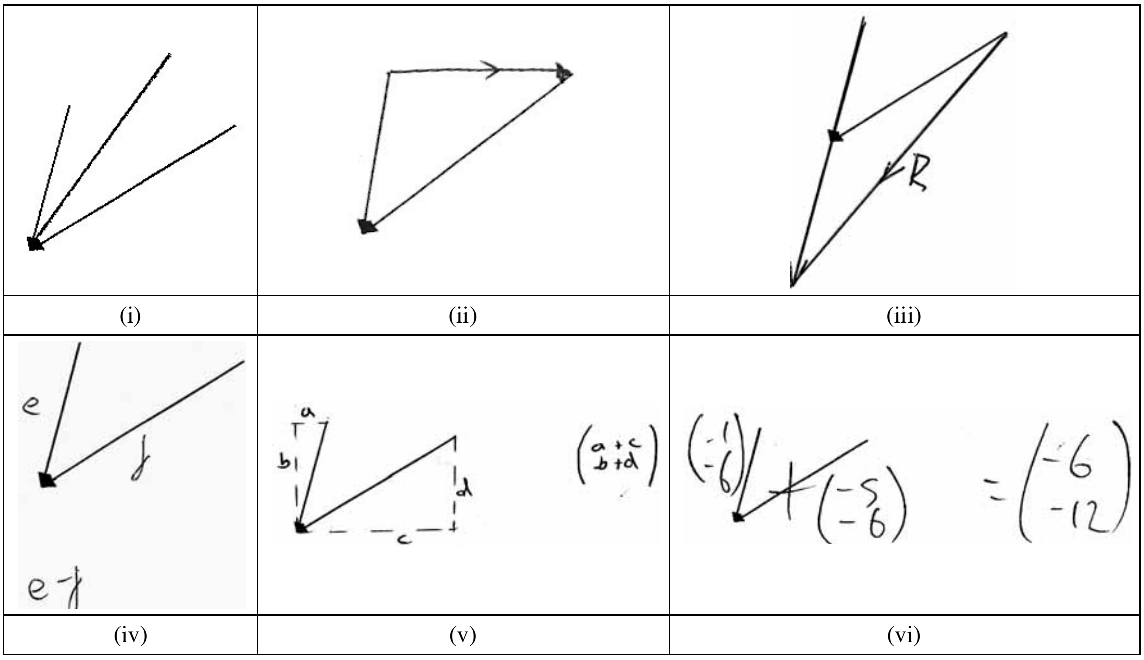


Fig. 6.14 Allocation of stages to the responses to the singular case in 2(c), 3(c)

Part (i) shows what seems to be an intuitive response. The student seems to be aware in which more or less direction the resultant should be, however neither the magnitude nor direction of the resultant are correct and therefore stage 0 was given.

In part (ii) It is not clear if students shows some sort of intuitive response or (s)he is trying to 'close a triangle' from the beginning of one vector to another. As different interpretations give the highest stage 2 for the graphical response then according to the general principle adopted for the ambiguous cases, stage 2 was given to the response.

Stage 4 was given for the response in part (iii) as student shifted one vector with the same magnitude and direction and showed the resultant.

Part (iv-vi) responses were given stage 0 for the graphical response.

Part (iv) was given stage 1 for the symbolic response as the student simply put a signed letter to the arrow.

Parts (v) and (vi) are similar in the final response although it can be debated if using letters or numbers shows a difference in the stage of development. However as decided at the beginning of this section of the chapter, it is not part of the analysis. There is however slight difference in the way two responses are presented in part (v) student shows column vector as a relative shift with horizontal and vertical components being added, however in part (vi) it is evident that two vectors were added to show the answer and therefore higher stage was given to part (vi) – stage 4, than to part (v) – stage 3.

Another 'singular' case set of questions is shown in figure 6.14.

Using the drawing below, or otherwise, add:

(a) $\overrightarrow{AD}$ and $\overrightarrow{CD}$

(b) $\overrightarrow{AD}$ and $\overrightarrow{BC}$

(c) $\overrightarrow{AC}$ and $\overrightarrow{BD}$

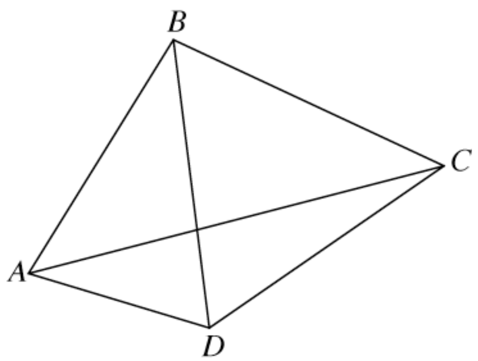


Fig. 6.15 Singular question

The question shown in figure 6.15 is set up differently from the questions students met in the previous year in their text-book. The questions they were dealing before had all answers as part of the diagram. In this case none of the answers fit that pattern. Part (a) of this question has also two vectors meeting at one point and part (c) has two vectors crossing each other.

Figure 6.16 shows a selection of students' responses to the above and allocation of stages to these responses.

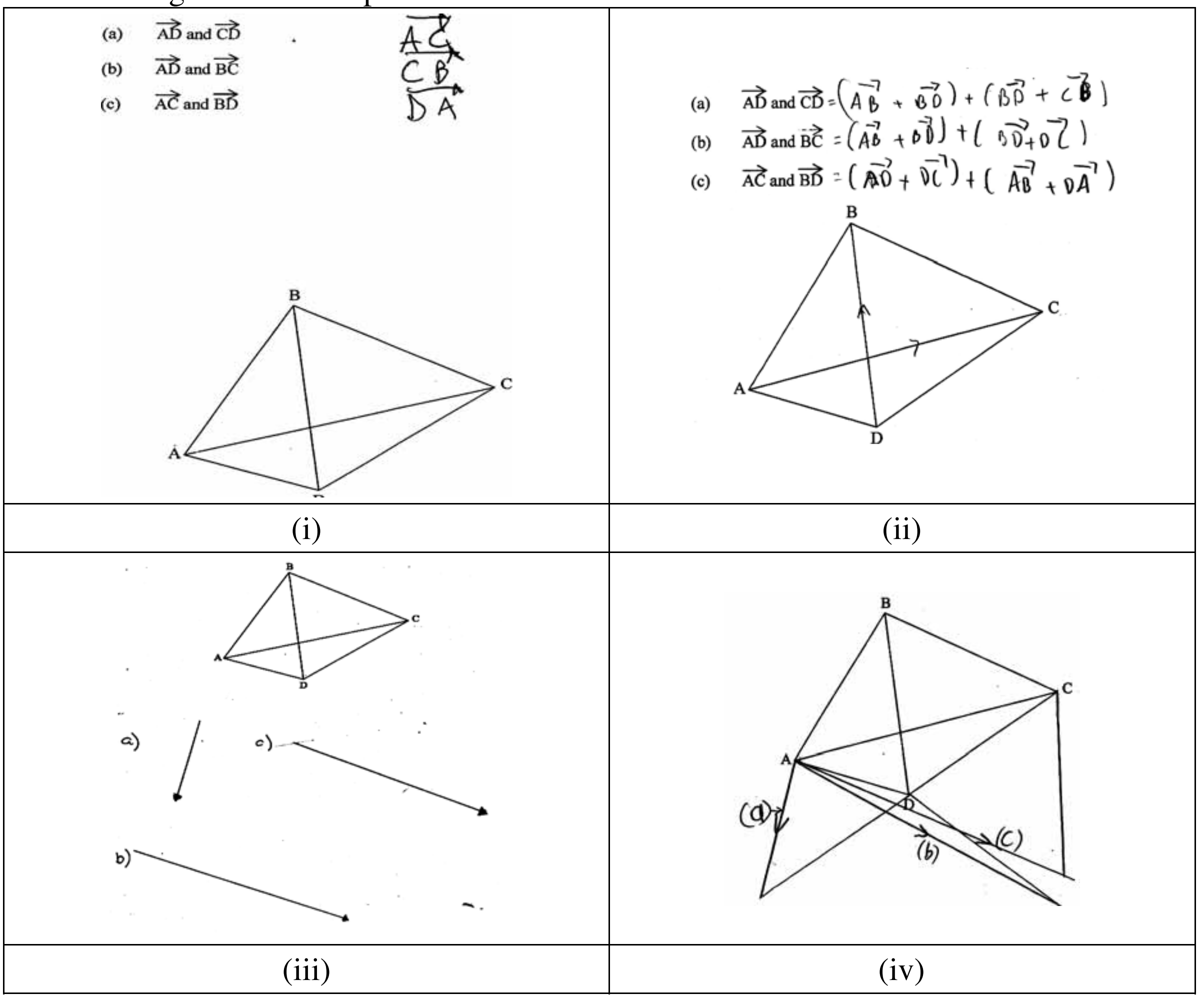


Fig. 6.16 Examples of students' responses to singular questions.

In figure 6.16 part (i), the student responded similarly to the response in figure 6.14 part (ii) and therefore stage 2 in graphical mode was awarded. The student gave a vector response (with an arrow above the letters) to the answers and therefore stage 1 was given for the symbolic response.

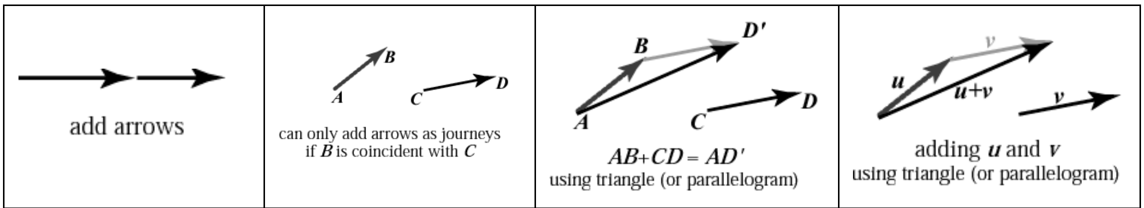
In figure 6.16 part (ii) the student drew the arrows along the lines AC and DB and therefore stage 2 (describing an arrow as a journey from one point to another) was awarded. As in part (i), the student gave a vector response to the arrows and therefore stage 1 was given in the symbolic mode.

In figure 6.16 (iii) the graphical responses were given stage 4 as the student shows the correct resultants as free vectors. As there is no evidence of symbolic addition, stage 0 was assigned to the symbolic mode.

In figure 6.16 (iv), in the graphical mode the student shifted vectors with the same magnitude and direction and also manipulated them to perform the correct addition, therefore stage 4 was given despite the lack of arrows on equivalent vectors. However there is no indication of any symbolic use of vectors and therefore stage 0 was given for that mode.

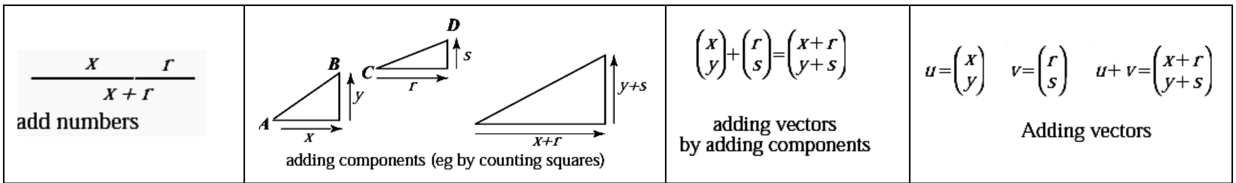
6.3.2 Quantitative Data Analysis of Understanding Vector Addition.

The figures 6.17 and 6.18 show the theory developed in chapter 4, how the stages were going to be allocated to students' test responses as far as their cognitive development in understanding vector addition is concerned,



graphical stage 1	graphical stage 2	graphical stage 3	graphical stage 4

Fig. 6.17 Stages of cognitive development of vector addition in the graphical mode



symbolic stage 1	symbolic stage 2	symbolic stage 3	symbolic stage 4

Fig. 6.18 Stages of cognitive development of vector addition in the symbolic mode

These stages as in case of concept of vector can be plotted on the scatter graph as shown in table 6.1.

The second level of categories come from the scatter graph presented in the table 6.1 depends on the stages students were awarded initially in the tests. The description of the categories give some idea of the student that fits into them.

physical-intuitive: signifies the performance of those students who do not have any specific understanding of vector addition in a graphical mode and at the same time do not present addition symbolically;

uni-modal: applies to the students who can operate in only at stage 1 or 2 in either mode (symbolic or graphical;

higher uni-modal applies to the students who can operate in basically only one mode (symbolic or graphical) at stages 3 or 4 but only at stage 0 or 1 at the other mode;

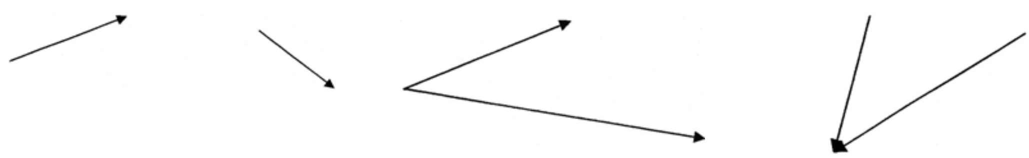
multi-skilled: students who show that they can use both modes in vector addition but do not use them flexibly (the context affects the level of their responses);

versatile: students who use both modes of operation flexibly whatever the context.

fully integrated: relates to the students who recognise the concept of free vector in vector addition whatever the context and using the appropriate mode of representation (graphical/symbolic: numerical and algebraic).

The students' responses to 7 different sets of questions on addition of vectors were considered in this part of the analysis. The first two questions (repeated from figures 6.2 and 6.3) are shown in figures 6.19 and 6.20 and the analysis of a sample of students' responses follows.

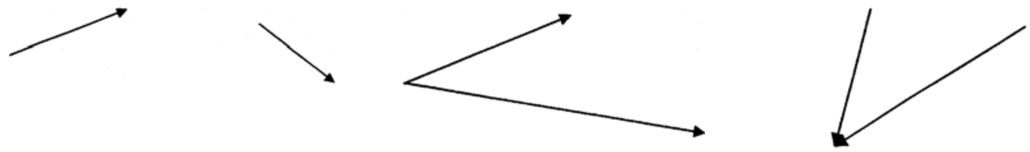
Question 2: In each case below add the two vectors:



(a) (b) (c)

Fig. 6.19 Test question 2

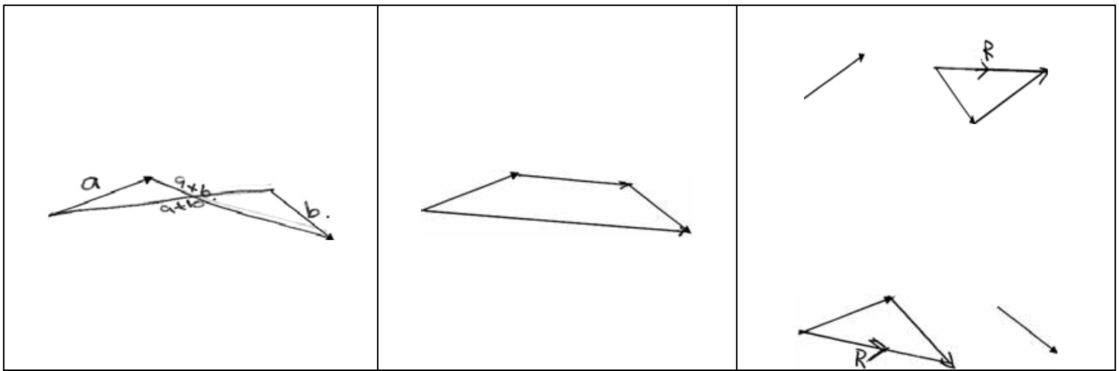
Question 3: If there is any other way you could have done any of the additions of the two vectors in Q2 show it here:



(a) **(b)** **(c)**

Fig. 6.20 Test question 3

The examples of the graphical responses, to questions 2 (a) and 3 (a) are presented in figure 6.21.



(i)	(ii)	(iii)

Fig. 6.21 Example of graphical responses to questions 2 (a) and 3 (a)

The response in part (i) of figure 6.21 shows no graphical rules of addition applied at all, and therefore the student was given stage 0 (however in the symbolic mode the student was given stage 1 as the letters representing vectors were added).

In part (ii) the student added an additional vector from the nose of the first vector to the tail of the second, but then seemed to go on to add all these three vectors together to complete the polygon. He could be only given stage 0, as he created his own continuity of journey by inserting the extra arrow.

Part (iii) of figure 6.21 show responses from the same student to questions 2 (a) and 3 (a) respectively. The student seems to have knowledge of the commutative law of addition and therefore is assumed to realise the concept of free vector and is given stage 4.

Figure 6.22 shows two examples of typical symbolic responses to questions 2 (a) and 3 (a) (shown in figures 6.19a and 6.20a).

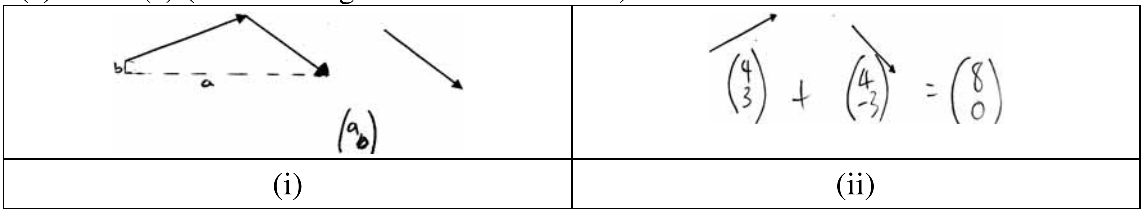


Fig. 6.22 Examples of graphical responses to questions 2(a) and 3 (a)

The examples shown in figure 6.22 (i) was given stage 3 in the graphical mode as the student added arrows as a journey, and stage 3 in the symbolic mode as the student shows the resultant in the form of vertical and horizontal components only. The response in figure 6.22 (ii) was given stage 3 as student added vectors by adding components.

The examples of the graphical responses, to questions 2 (b) and 3 (b) are presented in figure 6.23.

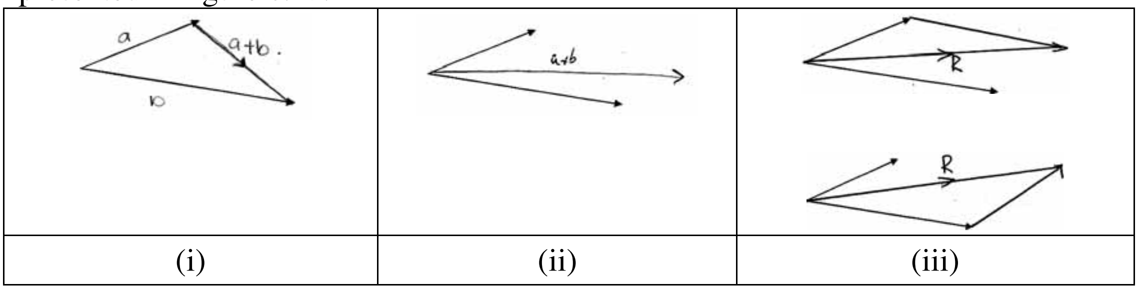


Fig. 6.23 Example of graphical responses to questions 2 (b) and 3 (b)

The response in figure 6.23 (i) was given stage 0 because student put the symbol **a+b** on the arrow drawn as the answer. If (s)he had not done this, it may be considered that the student assumes that the arrow is the resultant of addition of the other two vectors. However by writing **a+b**, the student seems to indicate that this is a resultant. From the experience in the preliminary study this happens when a student thinks that the resultant will go in the direction of the longer arrow (force), which is an intuitive response.

The response in figure 6.23 (ii) seems as if the student used the parallelogram rule but only approximately, however, according to the principle of giving the highest mark, stage 4 was given. The responses in figure 6.23 (iii) come from the same student and were given stage 4. This student seems to have the concept of free vector and uses the commutative law of addition in the graphical mode.

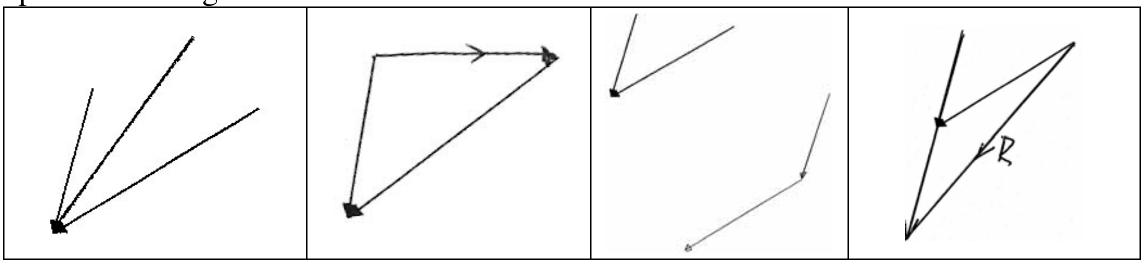
Figure 6.24 shows two examples of typical symbolic responses to questions 2 (b) and 3 (b) (shown in figures 6.19a and 6.20a).

(i)	(ii)

Fig. 6.24 Examples of symbolic responses to questions 2(b) and 3 (b)

Both responses shown in figure 6.24 were given stage 3 in the symbolic mode. Additionally response shown in figure 6.24 part (ii) was also given stage 3 in the graphical mode, as student seem to be using the triangle addition of the components.

The examples of the graphical responses, to questions 2 (c) and 3 (c) are presented in figure 6.25.



(i)	(ii)	(iii)	(iv)

Fig. 6.25 Example of graphical responses to questions 2 (c) and 3 (c)

The responses in figure 6.25 refer to the *singular case* presented in figures 6.19 (c) and 6.20 (c). In part (i) the arrow is too short for the parallelogram law to have been used and the direction is only approximate, that is why the intuitive response has been given in the form of stage 0. The response in part (ii) was also given stage 0. In figure 6.25 part (iii) the student joins the vectors 'nose to tail' but does not add them which is considered to be stage 2 (adding arrows as a journey). The response in part (iv) was given stage 3 as a single answer, however, if the same student were to show an understanding of the commutative law in question 2 (c) and 3 (c), then stage 4 would be given.

Figure 6.26 shows two examples of typical symbolic responses to questions 2 (c) and 3 (c) (shown in figures 6.19a and 6.20a).

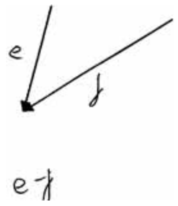
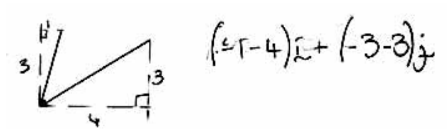
	
(i)	(ii)

Fig. 6.26 Examples of graphical responses to questions 2(c) and 3 (c)

The first response (figure 6.26 (i)) was given stage 0 as student as the student created the continuity by changing the sign of one of the vectors and therefore thinks of a journey. This seems to be a symbolic equivalence to the graphical answer shown in figure 6.25 (ii).

The second response in figure 6.26 (ii), was given stage 3 as the resultant is shown in a vector form obtained by adding the components.

The preliminary study gave some indication that students might be graphically adding two vectors procedurally in a way that the triangle has to be obtained and might have tried to apply this procedure to addition of three vectors trying to make a triangle, or being unable to draw the resultant if the three vectors did not make a triangle. Question 4 shown in figure 6.27 was given to students for that reason.

Question 4: Add the three vectors shown below:

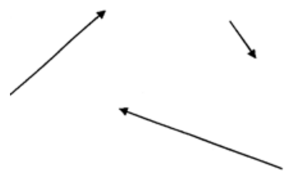


Fig. 6.27 Test question 4: Add three vectors

The examples of the responses to question 4 are shown in figure 6.28.

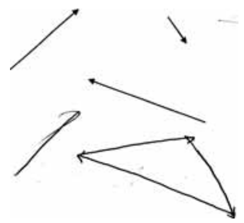
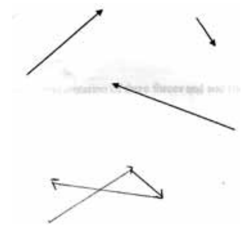
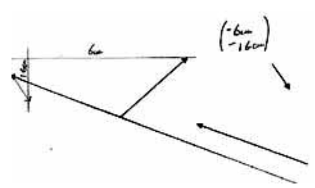
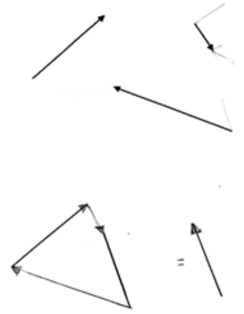
	
(i)	(ii)
	
(iii)	(iv)

Fig. 6.28 Response to question 4: Add three vectors

Part (i) of figure 6.28 shows a response described earlier in which the student might recall the idea of two vectors being added using the triangular law and attempt to force the three vectors displayed to make it look like a triangle. The student stretched and shrank some of the arrows and distorted angles so that the three sketched vectors make a triangle. This response was given stage 0 in the graphical mode; it was also given stage 0 in the symbolic mode as no symbols were used.

Part (ii) was given stage 2 in the graphical mode as the student placed arrows together 'nose to tail' like a journey but did not add them. Again stage 0 was given for the symbolic response or rather lack of it.

Part (iii) response was given stage 3 for the graphical response as the student shifted the vectors 'nose to tail', and stage 3 in the symbolic mode as the student shows the resultant in the form of the horizontal and vertical components.

Part (iv) was awarded stage 4 in the graphical mode and stage 0 in the symbolic mode.

The student not only added 3 vectors but also shows the resultant again as a free vector.

The different physical contexts questions are also going to be analysed separately in the main study.

Question 5: **Draw a representation of three forces and add them together.**

Question 6: **Draw a representation of two displacements and add them together.**

The examples of the responses to question 5 questions are shown in figure 6.29.

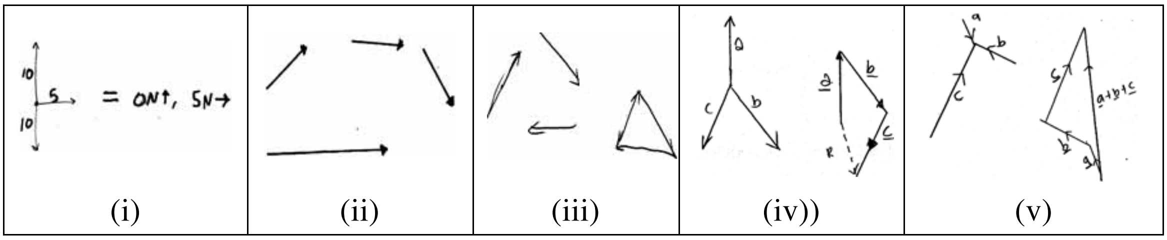


Fig. 6.29 Examples of responses to question set in the context of forces.

The response in part (i) was given stage 0 in the graphical mode and stage 2 in the symbolic mode. The student drew a very simple example and added components.

The response in part (ii) was given stage 2 in the graphical mode. The correct resultant is drawn separately without any indication of how it was obtained. Stage 0 was given for the symbolic mode (although student probably used the numerical addition to be able to draw the correct resultant, there is no indication of that in the test). Only the follow-up interview showed that the student added vectors numerically, by measuring the vertical and horizontal components, adding them together and drawing the answer graphically. This is a case, therefore, where what the student actually wrote in the test did not fully indicate his or her capacity. It is for this reason that the overall responses look for performance at the highest level shown by a student in all the questions, rather than average performances where individual cases may be given stage 0, merely because the student did not to use that mode explicitly.

Part (iii) was given stage 3 in the graphical mode and stage 0 in the symbolic mode. As the student drew his/her own vectors, with a very approximate drawing, it was possible that the resultant was 0. The follow up interview revealed that the student had the misconception that the three vectors should make a triangle.

Parts (iv) and (v) were both given stage 4 in the graphical mode. However part (iv) was given stage 0 in the symbolic mode, while part (v) was given stage 1 in that mode as the student added written letters. It was interesting to see that the student in figure 6.29 (v) worked in such a way that the response became 'singular' (with all three vectors directed into a single point).

The examples of responses to question 6 are shown in figure 6.30.

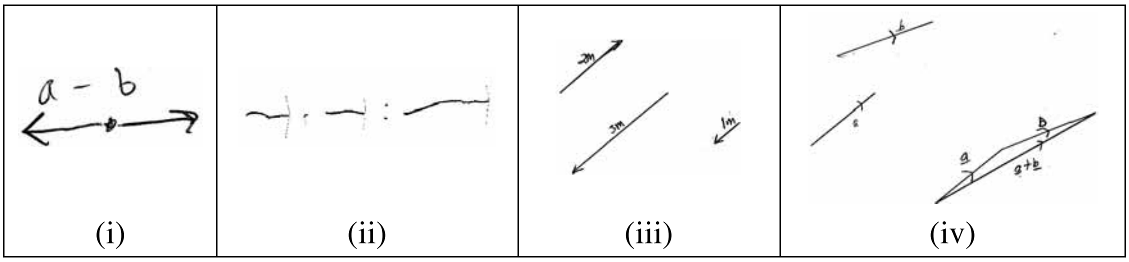


Fig. 6.30 Examples of responses to question set in the context of displacements.

The response in part (i) was given stage 0 in the graphical mode and stage 1 in the numerical mode as the student wrote the symbol **a–b**.

The response in part (ii) was given stage 1 in the graphical mode as the student added arrows in one dimension, and stage 0 for the symbolic mode.

The response in part (iii) was given 1 in the graphical response and stage 1 in the graphical mode as it seems that student just added signed numbers in one dimension.

The response (iv) was given stage 4 in the graphical mode and stage 1 in the symbolic mode.

6.4 Method of collecting qualitative data

Mason (1996) suggests that the sampling on the basis of chosen categories relevant to the research questions and one's theoretical position is called a *theoretical sampling*. My theory looked at students flexibility of thinking and the initial investigation suggests that students who present their work visually are often more flexible at this stage of their study and therefore I decided to choose students for the interviews on this basis.

During the interviews I considered different categories of questions described by Ainley (1988). It seems that the category described as testing questions (to find out if the subject knows the answer) and directing questions (provoking the subject to think further about a problem) are the most appropriate for this research.

The students were first asked how they attempted different questions and if they know any other way they could have answered (to find out if the subject knows the answer) and then some directing questions asking them where and how they used vectors in the past, if they know any rules for vector addition and what symbols they are familiar with (provoking the subject to think further about a problem).

As an additional qualitative data sample, the Mathematics and Physics teachers were interviewed about how they think student learn vectors and how vectors are taught in their subjects. This enabled further triangulation between what the students did and what their teachers expected them to do. The main study will also include interviews with teachers on how they think that the students will respond to give more information for triangulation purposes.

In addition, in the main study, the experimental lessons will be recorded to observe the students' development more closely.

6.5 Quantitative Analysis of the results

The students' results of the qualitative and quantitative analysis show a shift of the experimental class to being more 'graphical' than before and towards being more flexible. The vital delayed post-test missing from the pilot study was included in the main study. It was conducted half a year after the course to investigate long-term retention of ideas.

The analysis of the pilot study (published in part in Watson, Spyrou & Tall, 2002) indicates that a small number of students arrive in Year 12 to begin their A-level studies with an already-developed concept of free vector, and ability to apply it in all the above cases. However, the experimental lessons involving action on moving objects and reflective plenaries discussing free vectors moved many other students faster through the stages of the cognitive development of vector addition than the students who were not given this opportunity.

It is important to emphasise that the general principle of allocation of stages to students' responses had some effect on the analysis. Some cases are straightforward and the stages could be allocated straight from the theory designed in chapter 4. However, in ambiguous cases the students are given the highest category consistent with the response. As an example, some students, especially in the control group, lacked precision in their drawings to the extent that it was not obvious if they have the concept of addition or not. However they were given the benefit of the doubt and the highest stage consistent with the precise answer was awarded. The effect of this principle is that any bias in the recorded changes tends to benefit the control group rather than the experimental group, thus not falsely enhancing the effects of the experimental treatment.

The students' responses were divided into graphical responses and numerical responses as shown in chapter 4 in figures 4.6 and 4.7. Each response was given a level of development. The students' graphical and symbolic responses in the pre-test were plotted on the scatter graph as shown in figure 6.31.

The scatter-graph below (figure 6.31) shows the distribution of the stages given to students in both: experimental group A and control group B in the pre-test.

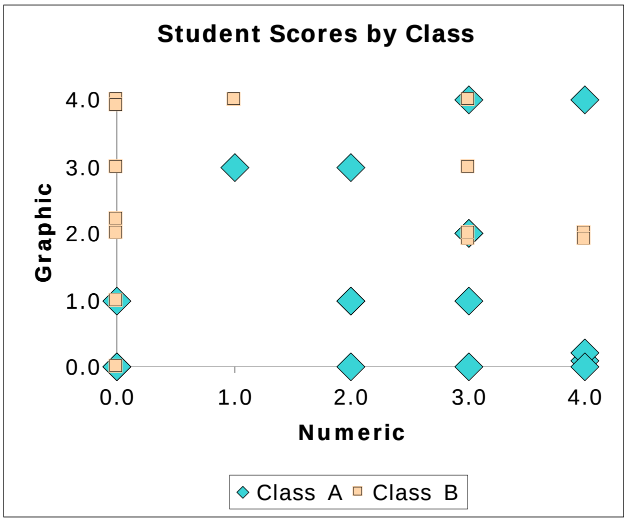


Fig. 6.31 Results of pilot pre-test.

The experimental group A is marked with rhombuses and the control group B is marked with squares. If we look at the 'squares' of group B, it can be noted that most of them are close to the vertical axis representing graphical responses. On the other hand more of the ‘rhombuses’ of group A seem to be closer to the horizontal axis, indicating numerical/symbolic responses. This difference was confirmed by statistical analysis. By comparison of the means and standard deviations, group B is more graphically biased than group A in the pre-test. The means and standard deviations of the two groups’ graphical scores are μ_A=0.7, s_A=0.6 and μ_B=1.5, s_B=1.1. Using the t-distribution, there is statistical evidence ($t=2.84$, $p<0.05$) to suggest that group B is more graphically biased.

It was hypothesised that, through providing students in group A with the embodied experience translated into symbolism, we could move more of them into right top corner of the scatter graph. Both experimental and control treatments involved a substantial experience of graphical representations and addition of forces as vector quantities, so both groups would be expected to change in this direction. A t-test conducted on the improvement of responses in the graphic mode shows that the changes were as follows:

Group A: $t=5.9$ significant at $p<0.0005$;

Group B: $t=2.4$ significant at $p<0.025$.

Both groups therefore made statistically significant improvements, but the changes in group A were greater than those in group B. In part, this may be attributed to the better final results of group A, but there is also a contribution to the difference which occurs because group B was already more graphically orientated in the pre-test.

When looking at the stages 3 and 4 of the cognitive development, both groups started with about 65% of students in the pre-test responding at those stages in one or both modes of operation. However in the post test 95% of student in the experimental group responded at stage 3 or 4 in either mode and at the same time 72% of students responded at stage 3 or 4. This gives χ^2 =3.16, which is significant at $p\leq 0.1$.We can see in the scatter-graph in figure 6.32 that only one student in group A responded at the stage lower than 3 in the graphical mode while 5 students in group B answered at the stage below 3 in the graphical mode.

The changes in symbolic responses are not significant. However, there was a greater difficulty in assigning stages as what the students write does not always represent what they are capable of doing. There is a great need therefore for in-dept interviews to study this aspect. This will play a major role in the main study.

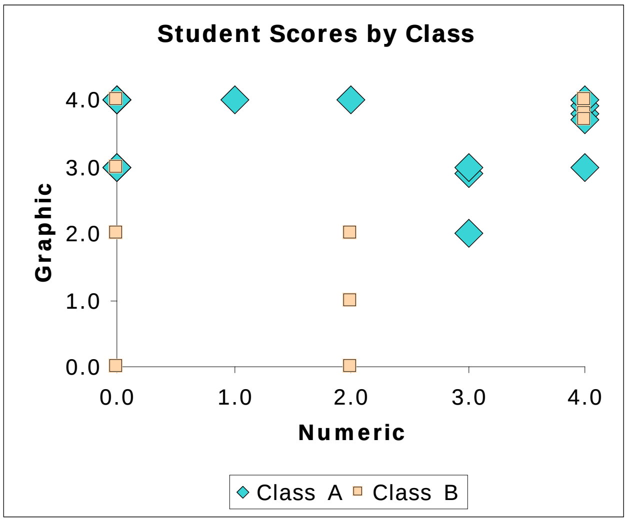


Fig. 6.32 Results of pilot post-test

The scatter graph in figure 6.32 shows that in group A, most students moved up to the third or fourth stage of the cognitive development in the graphical mode and more students moved to the top-right corner. Meanwhile, the results of group B split into two main groups, one of which occupies the top right corner, with others who continue to cling to the vertical graphic axis with zero numeric score.

From the initial interviews there was an indication that students who are more ‘graphical’ are also more flexible and think more conceptually. For instance, they use graphical methods in a more efficient way in the questions given in the test, while students who are more ‘numeric’ or symbolic, tend to use the symbolic method procedurally and without flexibility. This would suggest that before the experiment starts, the Experimental Group A are more numeric and probably less flexible than the Control Group B.

6.6 Conclusions

The tests proved adequate information about students' development and is going to be used in the main study. It also showed that the change in the experimental group was significantly different to the change in the control group. This difference shows straight after the course has been completed. The main study hopes to prove long-term concept stability in the experimental group and therefore the difference should become greater in the delayed post-test.

The method of collecting data showed that all of the responses could be classified although the classification of stage 0 is not always completely clear. However it was felt that five stages including the zero stage gave a fair overall indication of the student development. What matters in this study is the movement through the cognitive stages to the higher levels, and this is the main focus of attention rather than a deeper study of the pre-conceptual development.

It was difficult at times to give students a higher stage based on the response from the test, knowing that the preliminary interviews show the possibility of a lower stage. The moderation of the stages caused also some difficulty as it was a tendency to assume what student might have wanted to say instead of keeping strictly to the work shown. When the consensus could not be achieved between two teachers giving stages the higher stage was adopted. In order to maintain a consistency in assigning stages, the same method will be used in the main study.

It was also decided that although the post-test questionnaire is going to be applied in the classes following the experiment, the main focus of attention will be on the delayed post-test which will be given to students after their holidays and 11 months after the experimental lessons (at the beginning of year 13). This should test if the changes caused by the experimental lessons can be sustained for a longer period of time.

Chapter 7
Main Study: Quantitative Data Analysis

7.1 Introduction

The emphasis of the experimental lessons was directed to compressing the embodied actions into process by focusing on the notion of *effect* (if two actions have the same effect then they are considered as giving the same process).

Reflective plenaries were introduced for the Experimental Group in the Teaching Experiment to concentrate on the effect of different procedures.

The intention was to test the following hypothesis:

> **Main Hypothesis**: Teachers can help students develop the notion of a translation as a free vector through focusing on the effects of physical actions, linking graphic and symbolic representations, so that the concept of free vector is constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

The intention of the teaching was to help the students appreciate the equivalence of '**free vectors'** with **the same magnitude and direction** and the flexible use of equivalent vectors for vector addition. The testing of this hypothesis was performed by designing a questionnaire as a tool to test the changes in the stages of cognitive development (as shown in chapter 3, figures 3.16 and 3.17) and discussed in detail in chapter 6. The main hypothesis infers:

> **Hypothesis 1**: Students, who were involved in experimental lessons, are expected to rise through the cognitive stages further than students who are not exposed to the experimental lessons.
>
> **Hypothesis 2**: Students who were helped in building a concept of a free vector are expected to be more able to:
>
> **(a)** add vectors in singular cases, not just generic ones;
> **(b)** use free vectors independent of the context;

(**c**) realise that the commutative law applies to vector addition.

Hypothesis 3: Students who can concentrate on the *effect* of actions rather than actions themselves are more likely to build the concept of free vector as a cognitive unit, which can be used by students after a longer period of time and not only just after the experiment.

For this reason, the main comparison of the data will be done between the pre-test and the delayed post-test.

This chapter tests the main hypothesis through the outcome of the analysis conducted at three different stages of the research according to the Methods and Methodology developed through chapters 4, 5 and 6. This chapter will present the quantitative analysis of the data from the questionnaire, which will then triangulated with the qualitative data from both teachers and students (chapters 8, 9).

Hypothesis 1 will be tested using the data related to the students overall performance on the questionnaire with respect to the concept of vector and vector addition. Hypothesis 2 will be tested using questions specifically focused on (a) singular questions, (b) questions in different contexts (c) questions that may be solved using the commutative law. Hypothesis 3 will be tested quantitatively by focusing on the same data over the period from pre-test to immediate post-test through to delayed post-test. It will later be tested qualitatively by analysing the responses of students in the interviews reported in chapter 9.

7.2 Quantitative Data Analysis of Understanding the Concept of Vector and Vector Addition

The quantitative analysis arises from the data collected in 3 tests conducted before the course (pre-test, T1), straight after the course (post-test, T2) and half a year after the course (delayed post-test, T3) with two groups of students: Group A (experimental) and Group B (control). Both groups had 17 students each.

The tests, given at different times of the year, are considered to be indicators of the students' cognitive development stage. Therefore the change of that stage from one test to the next is considered as an indication of the students' cognitive development.

The general/overall distribution of students between different stages of cognitive development in understanding of vector and vector addition can be viewed in parts 7.1.1 and 7.1.2 of this chapter. This analysis considers students' understanding of vector and vector addition at three stages of their development into studying Mechanics, without looking at how they can apply their knowledge in questions involving singular cases or different contexts cases. The later parts of the chapter show the distinctions between students' responses to singular questions (section 7.1.3), to questions set in two different physical contexts (displacement and forces) (section 7.1.4), and to questions that may use commutative law of vector addition in the solution process. The data was built from the students' responses using the methods discussed in detail in chapter 6.

7.2.1. The General Case: Understanding the Symbol of Vector

Tables 7.1 and 7.2 show the number of students at different stages of the cognitive development ladder captured in the three tests. Table 7.1 shows the categorisation of graphical responses and 7.2 the categorisation of symbolic responses. There were 17 questions and sub-questions in total and the student had to achieve their highest stage twice to be given it (as described in detail in chapter 6.2).

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	3	17	17	6	10	13
3	9	0	0	6	4	3
2	5	0	0	1	3	1
1	0	0	0	1	0	0
0	0	0	0	3	0	0
TOTAL	17	17	17	17	17	17

Table 7.1 Graphical responses to test questions

From the table 7.1 it would appear that there were more students responding at stage 4 in group B than in group A at the beginning of the year 12 (T1), however, there were also 3 students in group B responding only at stage 0. At the same time, all students in group A reached stage 4 in the post test (T2) and retained their knowledge until the delayed post-test (T3). Meanwhile, in Group B, only 10 out of 17 students reached stage 4 in the post-test (T2) and that number increased to 13 in the delayed post-test (T3).

The two-tail t-test performed for each group on students' changes in the stage of the graphical cognitive development between test 1 and test 3 shows:

$t=5.37$ which is highly significant ($p<0.01$) for group A and

$t=3.83$ which is significant ($p<0.01$) for group B.

This indicates that both groups have improved their responses of the graphical representation of the vector.

The next table (table 7.2) shows the results on the basis of the symbolic representation responses.

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	6	7	6	4	4	4
3	1	3	1	2	5	1
2	4	0	2	5	2	7
1	5	5	4	5	4	2
0	1	2	4	1	3	3
TOTAL	17	17	17	17	17	17

Table 7.2 Symbolic responses to test questions.

From table 7.2 we can observe that the changes between the pre-test and delayed post-tests are not substantial and the t-test performed on student's changes in the stage of symbolic cognitive development proved not significant.

If we look at the scatter graphs in figures 7.1-7.3, we can confirm that there are no significant differences between the experimental group A and the control group B at the three stages and that both groups developed their understanding of the symbol of vector between the pre-test and the delayed post-test.

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4		BB	A BBB	A	A B
stage 3	A	AAA B	A B	BB	AAA BB
stage 2		AA	AA		A B
stage 1		B			
stage 0	B	B	B		A

Fig. 7.1 Scatter graph of responses to all pre-test questions (vector)

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4	A BB	AAAAA B	AA BBBBB		AAAAAAAA BBBB
stage 3		B	B	A	
stage 2	BB	B			
stage 1					
stage 0					

Fig. 7.2 Scatter graph of responses to all post-test questions (vector)

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4	AAAA B	AAAA B	AA BBBBBB	A B	AAAAAA BBBB
stage 3	B	B	B		
stage 2	B				
stage 1					
stage 0					

Fig. 7.3 Scatter graph of responses to all delayed post-test questions (vector)

Using the classification given in table 6.1 (chapter 6), figure 7.1 shows that in the pre-test T1 there were 2 students from group A categorised as ***uni-modal*** and 4 classified as graphically-orientated ***higher uni-modal***; 5 students were classified as ***multi-skilled***; 5 students were already in the ***versatile*** or ***fully-integrated*** categories. At the same time, in group B, one student was in the intuitive category; 3 in the ***uni-modal*** category; 3 in the graphically orientated ***higher uni-modal***; 5 students were in the ***multi-skilled*** category and 5 were already ***versatile*** or ***fully-integrated.*** Both groups were therefore similar in their cognitive development of vector and flexibility in using the graphical or symbolic modes of operation.

The χ^2-test could not be used as the expected numbers are too low. However, it can be seen that their general development of understanding the symbolic and graphical representation of vector remained similar even at the time of the delayed post-test, with the graphical categories all high and the numerical categories spread out over the full range.

7.2.2. The General Case: Understanding Vector Addition

Both sets of students had considerable experience with the concept of vector in Mechanics in dealing with forces. It is therefore not surprising that they have improved in both groups in understanding of the symbol of vector. However, how well they understand the symbol in order to be able to manipulate it was tested through the questions asking them to add vectors. The analysis of addition of vectors is considered in this section.

Tables 7.3 and 7.4 below show the numbers of students responding at different stages of the cognitive development of the concept of vector addition in a graphical and symbolic mode of representation.

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	3	16	2	4	9
3	2	13	1	9	9	4
2	10	1	0	1	1	4
1	0	0	0	2	3	0
0	5	0	0	3	0	0
TOTAL	17	17	17	17	17	17

Table 7.3 Graphical responses to the test questions

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	3	4	0	0	1
3	6	3	1	6	2	1
2	2	2	0	3	5	4
1	6	5	5	7	3	6
0	3	4	7	1	7	5
TOTAL	17	17	17	17	17	17

Table 7.4 Symbolic responses to the test questions

The data from the pre-test T1 shows that, in the graphical mode, only 2 out of 17 students in group A responded in the two highest stages (3 and 4), compared to 11 out of 17 students in group B. However, in the symbolic mode, the two groups had very similar distributions, each with 6 students at stage 3 and none at stage 4.

In the delayed post-test T3, in the graphical mode, the number of students in group A responding at stages 3 and 4 increased from 2 to 17, while in group B the numbers of students stayed nearly the same (a small increase from 11 to 13).

The significance of the changes can be determined using the two-tail t-test. The t-test taken for the graphical changes between the test T1 and T3 shows:

$t=3.83$ which is significant ($p<0.01$) for group A and

$t=0.348$ which is not significant for group B.

The two-tail t-test conducted on the symbolic responses show that the changes between pre-test and delayed post-test in not significant for either group.

Between the time of the pre-test and the post-test, both groups did a lot of work in their Mechanics lessons on addition of forces presented in a graphical way and it should be noted that at the stage of the post-test, both groups seem to be at a similar stage of their cognitive development of vector addition. However it is very noticeable that students in group A achieved long-term concept stability: 16 out of 17 students responded at stage 4 in the graphical mode in test T3. Group B also improved, and 9 out of 17 reached stage 4.

The scatter graphs in figures 7.4-7.6 show the students' development through the second stage of categorisation: from ***intuitive*** and ***uni-modal*** to ***higher uni-modal***, ***multi-skilled***, ***versatile*** and ***fully-integrated.***

graphical mode						
	stage 4		**BB**			
	stage 3	**A**	**BB**	**BBBB**	**A BB**	**B**
	stage 2	**AAA**	**AAAAA B**	**A**	**A**	
	stage 1		**B**		**B**	
	stage 0	**AAA B**	**BB**	**A**	**A**	
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.4 Scatter graph of responses to all pre-test questions on addition

graphical mode						
	stage 4	**AA BB**	**A**	**BB**		
	stage 3	**A BBB**	**AAAA B**	**AA BBB**	**AAA BB**	**AAA**
	stage 2	**A B**				
	stage 1	**B**	**BB**			
	stage 0					
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.5 Scatter graph of responses to all post-test questions on addition

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4	AAAAAAA B	AAAA BBBB	BBBB	A	AAAA
stage 3	BB	A		B	B
stage 2	BB	BB			
stage 1					
stage 0					

Fig. 7.6 Scatter graph of responses to all delayed post-test questions on addition

In the pre-test T1, in group B, four students began in the higher uni-modal (graphical mode) category, four in multi-skilled (graphical mode) and 3 in the versatile category, while, at the same time, in group A only two students were in any of those categories.

However by the time of the delayed post-test T3, the picture has changed substantially as far as group A is concern. In group B there are still four students in the uni-modal category while in group A, 16 out of 17 students responded at stage 4 (mainly graphically) and 4 of those are in the fully-integrated category.

These results are the evidence for hypotheses 1 and 3. The students who were involved in the experimental lessons rose through the cognitive stages further than students who were not exposed to the experimental lesson and their conceptual understanding worked after a longer period of time and not just after the experiment.

7.2.3. Singular Cases: Understanding Vector Addition

Hypothesis 2(a) states that the difference should show when looking at students' flexibility tackling singular questions. As we already know from section 7.1.1, there may be little difference in the overall spectrum of understanding of the concept of vector in the two groups, so we only analyse whether the singular questions cause a difference to the ways in which students carry out vector addition.

Tables 7.5 and 7.6 give a summary of students' responses to the singular cases.

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	1	12	2	0	7
3	1	9	4	1	10	3
2	4	6	1	1	3	2
1	4	1	0	4	1	0
0	8	0	0	9	3	5
TOTAL	17	17	17	17	17	17

Table 7.5 Graphical responses to the singular questions

The t-tests performed on students' changes in the stage of the graphical cognitive development between the pre-test and the delayed post-test show:

t=3.13 which is significant ($p<0.01$) for group A and

t=1.3 which is not significant for group B.

This supports hypothesis 2(a) that there is a statistically significant improvement in group A but not in group B.

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	0	4	0	0	1
3	0	4	1	2	2	1
2	2	4	0	2	2	2
1	5	3	4	4	3	7
0	10	6	8	9	10	6
TOTAL	17	17	17	17	17	17

Table 7.6 Symbolic responses to the singular questions.

The t-tests performed on students' changes in their stage of symbolic cognitive development between pre-test and delayed post-test shows that the changes are not significant for either group.

The data in these tables reveals that prior to the experimental study, in the pre-test T1, only one student in group A was able to respond to singular questions at stages 3 or 4 in the graphical mode and no student replied in these stages in the symbolic mode. At the same time, in group B, three students were able to respond in graphical mode and two in symbolic mode at stages 3 and 4.

This changed substantially in the graphical mode by the time of the delayed post-test. In group A, 16 out of 17 students responded at stages 3 and 4 (of whom 12 were at stage 4), while in group B, 10 students were able to respond at those stages (with 7 at stage 4).

It must be emphasized that the post-test was carried out straight after the mechanics and physics courses dealt with forces in vector forms and after the students in group A had their experimental lessons, while the delayed post-test was carried out half a year after that time. The immediate post-test does not show any significant differences between the groups, however significant changes occur later, in the delayed post-test, which indicates a long-term stability of conceptual growth in group A.

The scatter graphs in figures 7.7-7.9 show the students' development through the second stage of categorisation: from ***intuitive*** and ***uni-modal*** to ***higher uni-modal***, ***multi-skilled***, ***versatile*** and ***fully-integrated.***

graphical mode						
	stage 4		**BB**			
	stage 3	**A**	**BB**	**BBBB**	**A** **BB**	**B**
	stage 2	**AAA**	**AAAAA** **B**	**A**	**A**	
	stage 1		**B**		**B**	
	stage 0	**AAA** **B**	**BB**	**A**	**A**	
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.7 Scatter graphs of responses to singular pre-test questions

graphical mode					
stage 4			A		
stage 3	AAAA BBBBBB	AA BB	AA BB	A	
stage 2	AA B	A	A	AA BB	
stage 1	B			A	
stage 0	BB	B			
	stage 0	stage 1	stage 2	stage 3	stage 4
	symbolic mode				

Fig. 7.8 Scatter graphs of responses to singular post-test questions

graphical mode					
stage 4	AAAAAA BB	AA BBB	B	A	AAA B
stage 3	A B	AA B	B		A
stage 2	A	BB			
stage 1					
stage 0	BBB	B		B	
	stage 0	stage 1	stage 2	stage 3	stage 4
	symbolic mode				

Fig. 7.9 Scatter graphs of responses to singular pre-test questions

The testing of the difference between the proportions of the students in the *intuitive/ uni-skilled* area shows significant difference in favour of group A (χ^2 = 2.97 significant at p<0.01). In the pre-test 12 students out of 17 from group A but only 5 out of 17 student from group B were in the *intuitive/uni-skilled* area. However in the delayed post-test only 1 student in group A was in this area compared with 6 students in group B. This supports hypotheses 2(a) and 3, in that group A students' conceptual knowledge of vector addition was more firm by the time of the delayed post-test and they could apply it more flexibly, even in the singular cases. On the other hand, in comparison with group A, a greater number of students in group B had a limited procedural view of vector addition as they could only answer generic questions and had problems with singular examples.

7.2.4. Different contexts: Understanding Vector Addition

This section considers the students' responses to the questions set in two different contexts. The intention is to check whether there is a significant difference between the improvement in marks of the experimental and control groups in their solution of problems in different contexts.

Hypothesis 2(b) states that the difference should show when looking at students' flexibility in tackling different contexts. The analyses in this part of the chapter show what happens in the case of responses to questions set in two different contexts (forces and displacements). Tables 7.7 and 7.8 present students' responses to the questions in the test set in two different contexts.

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	0	8	0	0	2
3	0	9	3	2	3	5
2	1	2	2	0	3	3
1	1	5	4	0	2	3
0	15	1	0	15	9	4
TOTAL	17	17	17	17	17	17

Table 7.7 Graphical responses to questions set in different contexts

The t-tests performed on students' changes in the stage of the graphical cognitive development between the pre-test and the delayed post-test show:

$t=8.71$, which is highly significant ($p<0.01$) for group A and

$t=2.17$, which is significant ($p<0.05$) for group B.

This supports hypothesis 2(b) that Group A made a more significant overall improvement in their stages of cognitive development than Group B.

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	0	0	0	0	0
3	0	1	1	2	1	0
2	2	1	0	6	4	4
1	2	1	1	2	4	8
0	13	14	15	7	8	5
TOTAL	17	17	17	17	17	17

Table 7.8 Symbolic responses to questions set in different contexts

The t-test performed on students' changes in the stage of the symbolic cognitive development between the pre-test and the delayed post-test was insignificant for both groups.

From the data in Table 7.7 it can be seen that in the delayed post-test, 11 out of 17 students in group A answered the questions at stage 3 or 4 of the graphical mode. Taking the two stages together, the number of students has not changed. However, if we just look at the stage 4, the numbers changes from 0 in the pre-test to 8 in the delayed post-test. At the same time in the delayed post-test, 7 students out of 17 in group B managed to answer the questions at the stage 3 or 4, but only two students responded at stage 4.

Table 7.8 shows that group A students were less inclined to respond symbolically in all three tests than group B. In addition, the scatter graphs below (figures 7.10-7.12) show that, in pre-test T1, the students in both groups did not show any signs of flexibility. In the delayed post-test T3, the students' answers are at higher cognitive stages than in the previous tests and their answers are mainly in the graphical higher uni-modal category. A t-test showed no significance in either of mode of operation. To make a more subtle analysis, it was decided in the second type of categorisation to look at the two different contexts separately.

Both groups worked on the topic of forces for the same number of lessons and covered the same questions from the textbook and therefore the results should be similar at all stages if the experimental lessons had no consequence on group A students' cognitive development. The three figures below (figure 7.10 – 7.12) show the results of students responses to the question set in the context of forces.

graphical mode					
stage 4					
stage 3	**BBB**				
stage 2	**AA** **BB**		**B**	**BB**	
stage 1	**A**			**BB**	
stage 0	**AAAAAAAAAAA** **BBBB**	**A** **B**	**A** **B**	**AA** **B**	
	stage 0	stage 1	stage 2	stage 3	stage 4
	symbolic mode				

Fig. 7.10 Scatter graph of responses in the context of forces, pre-test

graphical mode					
stage 4	**AAAAAA** **BB**	**AAA** **B**			**A**
stage 3	**A** **B**	**A**			**B**
stage 2	**A** **BB**	**B**			
stage 1	**A**	**A**		**A**	
stage 0	**AA** **BBB**		**BBBBB**		**B**
	stage 0	stage 1	stage 2	stage 3	stage 4
	symbolic mode				

Fig. 7.11 Scatter graph of responses in the context of forces, post-test

graphical mode					
stage 4	**AAAAAA** **BBBB**	**AAAAA** **B**			**A**
stage 3	**AAA** **BB**				
stage 2	**A**				
stage 1	**A** **BB**		**BBBB**		
stage 0			**BBB**		**B**
	stage 0	stage 1	stage 2	stage 3	stage 4
	symbolic mode				

Fig. 7.12 Scatter graph of responses in the context of forces, delayed post-test

The students operating at the combined lower stages (0-2) of the cognitive development in the graphical and symbolic modes fall into the *intuitive/uni-modal* area of the chart. The students operating at higher stages of the cognitive development (3-4), in either of the modes, and those operating at stage 2 in both modes, fall into the *higher uni-modal, multi-skilled, versatile and fully integrated* area of the chart. The χ^2-test compared the differences between both groups in each area:

> pre-test results (χ^2 = 5.25 significant at $p<0.05$) showed a significant difference between the two groups **in favour of group B** being in the higher area of the graph;
>
> post-test results (χ^2 = 2.95 not significant at $p<0.05$) shows that there was no significant difference between the groups;
>
> delayed post-test results (χ^2 = 4.84 significant at $p<0.05$) shows an even greater difference between the two groups **in favour of group A** being in the higher area of the graph.

These results indicate that group A, in comparison with group B, gained conceptually from the experimental lessons in the context of vector as force, and sustained their knowledge between the post-test and the delayed post-test. The difference between the groups changed from Group B being significantly higher in the pre-test to Group A being significantly higher in the delayed post-test. It is relevant that there was no significant difference between the groups in the immediate post-test. The gain is long term rather than short-term.

The first meeting of the concept of vector in the Mathematics Syllabus happens in the context of translation — displacement in physical terms. The students in both groups should have therefore had a similar competence at the beginning of year 12. The experimental lessons (which focused on translations) should have a positive effect on group A students in their cognitive development of vector addition and therefore the difference between the groups should be significant in the post-test and the delayed post-test.

The three figures below (figures 7.13-7.15) show the results of students responses to the question set in the context of displacement.

graphical mode	stage 4		BB			
	stage 3					
	stage 2		A			
	stage 1	A	A	BB		
	stage 0	AAAAAAAAAAAAA BBBBBBB	A BB	BBBB		
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.13 Scatter graph of responses in the context of displacements, pre-test

graphical mode	stage 4	AAAAAA B	A B			A B
	stage 3					
	stage 2	B				
	stage 1	A B	AA BB	B		A
	stage 0	AAA BBBBB	AA B	BB		B
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.14 Scatter graph of responses in the context of displacements, post-test

graphical mode	stage 4	AAAAA BBB	AAA			A
	stage 3	AAA B				
	stage 2	A BB		B		
	stage 1	AA B	B			A
	stage 0	A BBBBB	B	BB		
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.15 Scatter graphs of response in the context of displacements, delayed post-test

It can be seen from the scatter graphs above that in the post-test there was a tendency for both groups to respond symbolically to the question on displacement. Most of the responses are clustered around the symbolic axis but at the low stages. In the post-test this changed and the students moved more towards graphical responses. By the delayed post-test the tendency to give only graphical responses increased even further and most students are clustered around the graphical axis. Only one student (from group A) answered in the fully integrated category and only one responded at the highest symbolic level, with a low graphic score.

The χ^2-test on the difference between students in two groups gave the following results:

> pre-test results (χ^2 = 0.53, not significant), no significant difference between the two groups;
>
> post-test results (χ^2 = 1.99 not significant), no significant difference between the two groups;
>
> delayed post-test (χ^2 = 5.78, significant at $p<0.05$) shows a significant difference between the two groups in favour of group A.

These results support Hypothesis 2(b), showing a significant improvement long-term in favour of group A, with 13 out of 17 students benefiting from the experimental lessons so that they could use a vector as a mathematical mental concept to solve problems in different contexts. The results also show a long-term improvement which supports hypothesis 3.

7.2.5 The commutative law in vector addition

Hypothesis 3 also states that the students who were helped in building a concept of a free vector can realise that the commutative law applies to the vector addition.

The students had the opportunity to use commutative law of vector addition in four questions in the test. The numbers of students in both groups using this opportunity in three tests can be seen in table 7.9 below:

	Group A	Group B
Pre-test	0	4
Post-test	7	6
Delayed Post-test	12	5

Table 7.9 Responses using the commutative law of addition

There is little change in the results of group B. However, in the pre-test, group A students did not use the commutative law at all, but by the time of the delayed test, 12 out of 17 students used it, which is 70% of students in comparison with 29% in group B. As the understanding of the commutative law is related to the use of vectors as free vectors, this is consistent with the interpretation that 70% of students in group A have a concept of free vectors, compared to only 29% of students in group B.

The χ^2-test run on the difference between students in two groups gave the following results:

pre-test results ($\chi^2 = 4.53$, significant at $p<0.05$) shows a significant difference between the two groups in favour of group B;

post-test results ($\chi^2 = 0.5$, not significant) shows no significant difference between the two groups

delayed post-test ($\chi^2 = 5.76$, highly significant at $p<0.05$) shows a significant difference between the two groups in favour of group A.

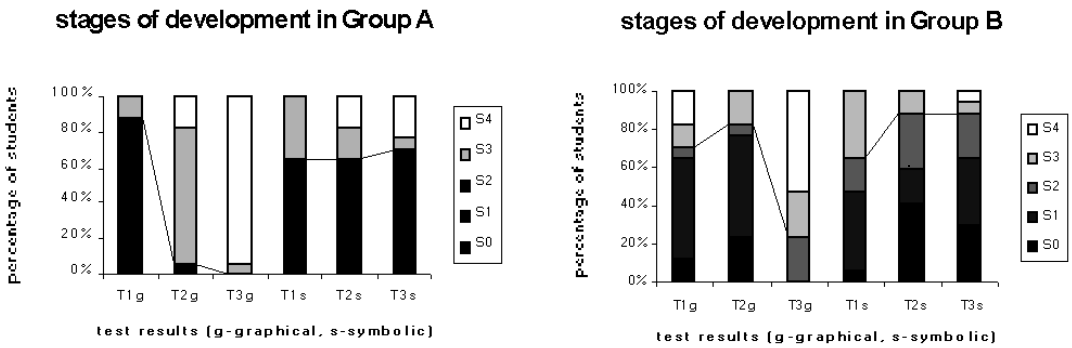
Hypothesis 2(c) is therefore also confirmed.

Therefore hypotheses 2(a), 2(b) and 2(c) are all supported, with significant improvements by group A over group B in handling singular cases, questions in different contexts, and the use of the commutative law. This analysis also gives quantitative support for Hypothesis 3, in that all three cases showed a significant long-term improvement in the performance of group A on the delayed post-test. It would seem that the embodied experiences may have given deep cognitive support that allowed the concept to continue maturing over a long period of time.

7.3 Summary of the results

The results so far show a significant improvement in the performances of group A in the graphical mode, but little significant improvement in the numerical performance. In this section we review the data from each test in turn to see if the evidence reveals any further evidence of differences between the two groups.

The results of the study, revealing differences between the experimental and control groups can be represented by the graphs in figures 7.16 and 7.17. The two graphs in figure 9.7 show the students' cognitive development through 4 stages in vector addition at three successive points in the year (T1, T2 and T3), with Group A on the left and Group B on the right. T1g, T2g and T3g represent the graphical results in the three tests, and T1s, T2s and T3s represent the symbolic results.



(a) (b)

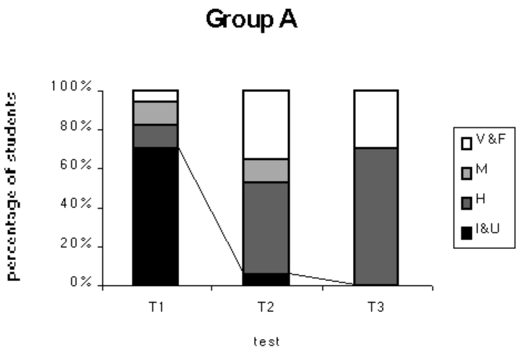
Fig. 7.16 Comparative General Developments of Groups A and B

The stages in each column are represented vertically in successive shades from stage 0 (black) at the bottom to stage 4 (white) at the top. The lines joining successive columns show the changing levels of the point between the two higher stages (3 and 4) and the three lower stages (0, 1, 2).

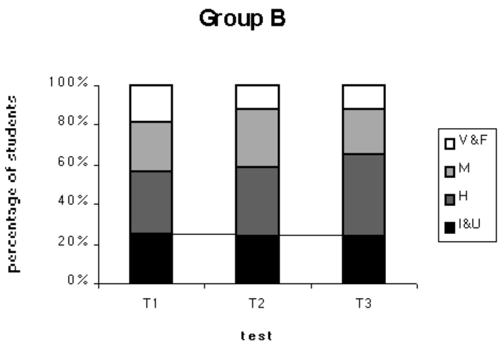
The percentages scoring in the two higher stages of performance (stage 3 and 4) in the graphical development of Group A increase from around 12% in the first test to 94% in the third test. The corresponding graphical results in Group B increased from around 65% up to 76%. Thus, in graphical development, Group A started *below* Group B, yet ended up *above* them.

In the numerical development, group A started with about 35% of students in the higher stages and finish at about 29%. At the same time group B also started at about 35% of students at the higher stages of development and finish with about 12%. In this case the changes are not statistically different, but there is a tendency for high-level numeric responses to decrease during the teaching.

Figure 7.17 combines the graphic and numeric information based on the categories developed in figure 3.1 of chapter 3 to show the percentages of students belonging to each category (I: *intuitive*, U: *uni-modal*, H: *higher uni-modal*, M: *multi-skilled*, V: *versatile* and F: *fully integrated*). In the figure the *intuitive* and *uni-modal* categories are integrated into a single category, as are the *versatile* and *fully integrated* categories.



(a) Experimental group A



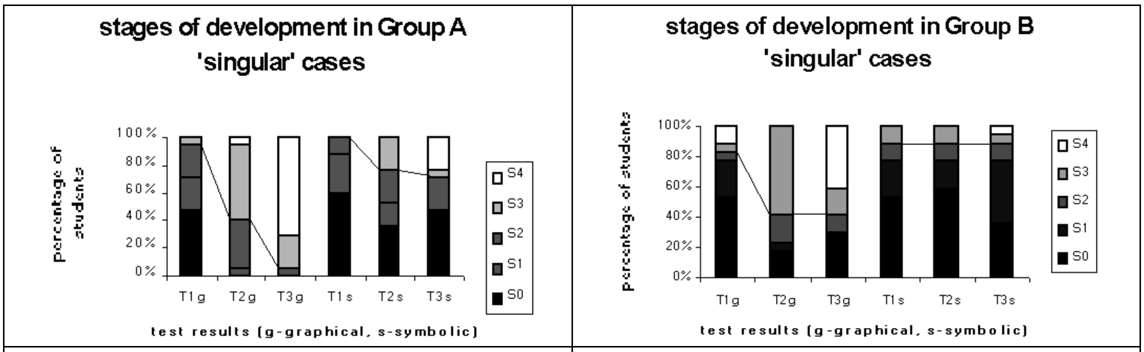
(b) Control Group B

Fig. 7.17 Development of students through combined categories

In the pre-test (T1), group A appeared to be more intuitive/uni-modal (~70%) in comparison with group B (~23%), and therefore at a lower stage of cognitive development. However, by the time of the delayed post-test (T3), no students in group A remained intuitive/uni-modal. At the same time in group B, at the time of the pre-test, a lower percentage of students were intuitive/uni-modal (~23%), however, this number did not change throughout the year.

These results confirm the hypotheses stated at the opening of the chapter. The main hypothesis that the experimental treatment focusing on 'effect' would be more likely to lead to the notion of a free vector used in a versatile manner is supported by statistical data that Group A rise further through the cognitive stages (hypothesis 1) and that these gains are retained over the longer term (hypothesis 3).

The differences between both groups in case of the singular questions can be seen in figures 7.18 and 7.19.



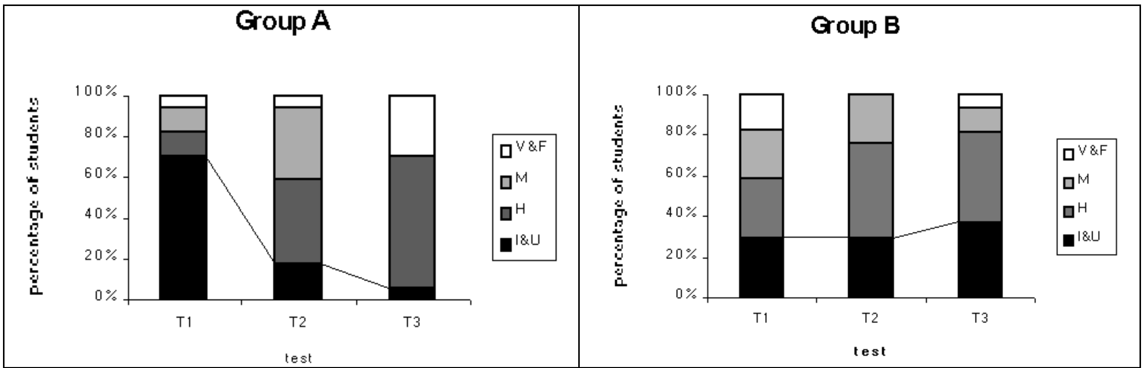
(a) (b)

Fig.7.18 Comparative development of Groups A and B (singular cases)

The percentages scoring in the two higher stages of performance (stage 3 and 4) in the graphical development of Group A increased from about 6% in the first test to about 94% in the third test. The corresponding graphical results in Group B increased from about 18% to about 59%. Thus, in graphical development, Group A again started *below* Group B, yet ended *above* them.

In the symbolic representation, Group A increased their performance (at stages 3 and 4) from about 6% to about 30%, while at the same time Group B stayed consistently at about 12%.

The differences between the two groups can also be highlighted when the responses are combined into another set of categories, presented in figure 7.19.



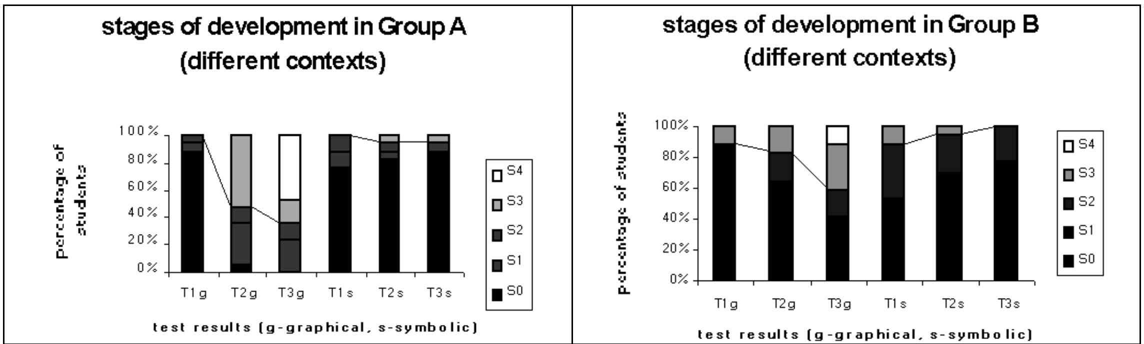
(a) (b)

Fig. 7.19 Development of students through combined categories (singular cases)

In the pre-test (T1) Group A appeared to be more intuitive/uni-modal (~70%) in comparison with group B (~30%), and therefore Group A was at a lower stage of cognitive development. However at the time of the delayed post-test (T3), in Group A, only small number (~6%) of students, remained intuitive/uni-modal. At the same time in Group B the number of the intuitive/uni-modal students increased slightly (~35%).

These results confirm further the hypotheses stated at the opening of the chapter. The main hypothesis that the experimental treatment focusing on ‘effect’ would be more likely to lead to the notion of a free vector used in a versatile manner, not just in generic cases but also in singular cases is supported by statistical data that Group A rise further through the cognitive stages (hypothesis 1, and hypothesis 2(c)) and that these gains are retained over the longer term (hypothesis 3).

There are also differences in case of the questions set in different contexts, which can be observed in figures 7.20 and 7.21.



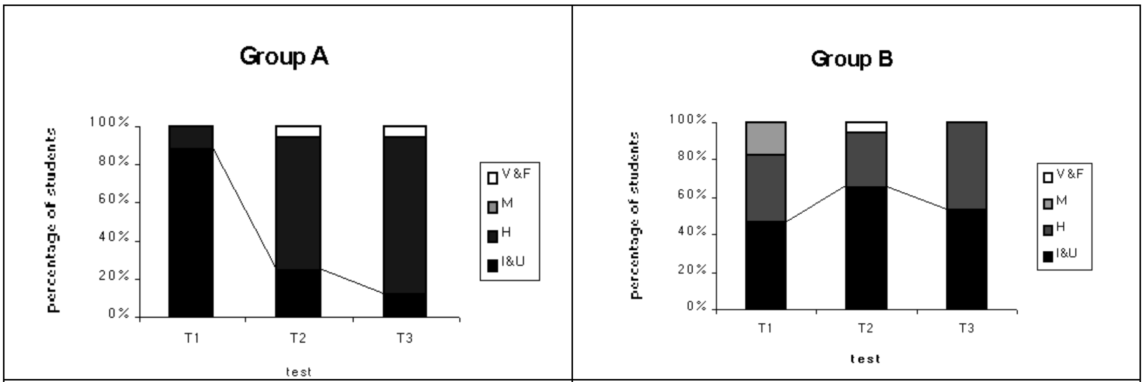
(a) (b)

Fig.7.20 Comparative developments of Groups A and B (different contexts)

The percentages scoring in the two higher stages of performance (stage 3 and 4) in the graphical development of Group A increased from about 0% in the first test to about 65% in the third test. The corresponding graphical results in Group B increased from about 12% to about 41%. Thus, in graphical development, Group A started *below* Group B, and again ended up *above* them.

In the symbolic representations, there very insignificant changes in both groups (figure 7.20 (b)).

The differences between the two groups can also be noted when the responses were combined into another set of categories, presented in figure 7.21. The changes are only shown in case of context of forces as both groups did an equal amount of work in that context.



(a)	(b)

Fig. 7.21 Development of students through combined categories (context of forces)

In the pre-test (T1) Group A appeared to be more intuitive/uni-modal (~88%) in comparison with group B (~47%), and therefore Group A was at a lower stage of cognitive development. However at the time of the delayed post-test (T3), in Group A, only small number (~11%) of students, remained intuitive/uni-modal while in Group B the number of the intuitive/uni-modal students increased slightly (~53%).

7.4 Summary

The quantitative data analysis reveals statistical support for the hypotheses stated in the opening of the chapter. The improvements occurred mainly in the graphical mode, with no statistically significant changes in the numerical mode. In particular, the students in group A showed little evidence of moving to the fully integrated area and responded mainly in the graphical mode. The reason for this may be that, since the questions were easier to answer in the graphical mode, and experimental students gained confidence in operating in this mode, they chose this means of response. This reason cannot be confirmed by the written evidence alone, but it will be tested in interviews with a sample of students (discussed later in chapter 9).

The difference between the two groups is apparent in the responses to the questions involving singular examples (hypothesis 2(a)) and problems set in different contexts (hypothesis 2(b)). The evidence of the performance of the group as a whole is consistent with the hypothesis that the experimental group students are more flexible in adapting their knowledge to different circumstances even after a longer period of time. The evidence is also consistent with the hypothesis that students in group A will construct the notion of free vector to a greater extent than group B, as they show greater ability in applying the concept of the commutative law to vector addition.

In my experience there seems to be a common belief between teachers that students forget quickly (from one year to the next) and they have to be 'taught again'. The analysis shows that this did not happen to so great an extent with the experimental students, and most of students who were taught to concentrate on the *effect* of actions gained the concept and retained it into the next school year.

Not all students in the experimental group reached the higher stages of. Some may possibly benefit from more experience of concentrating on the *effect* of actions before they gain the benefit of such an exercise.

It is notable that, despite the distinction made in chapter between the triangle law and the parallelogram law, where the first was seen as more natural for combining journeys and the latter for combining forces at a point, in all three tests, only *one* student used the parallelogram law of addition. The use of triangles dominated the graphical mode and the symbolic mode deals with components individually in a way that also does not involve the parallelogram law. Therefore, apart from noting that the parallelogram law was rarely used, no comparison between the use of the two laws was possible from the written responses.

It was also evident from the post-test and the delayed post-test that the students from group A sketched with more understanding of equivalent vectors having the same direction and magnitude and were less likely to have the misconception that the addition of three given vectors required the vectors to be in the form a triangle. In the post-test and the delayed post-test, the main difference between the sketches of groups A and B is that many students in group A, especially in questions with different physical contexts, moved vectors around as ‘free vectors’, meaning that they treat the questions from a mathematical point of view, while that type of response was rare in group B.

Further triangulation is required in the form of comments of the teachers to gain insight into their views of how the students may perform, and, more particularly, into how students talk about their work. This triangulation will be performed in the next two chapters. In particular, the interviews with the students will be framed to give insight into how the students talk about the concepts and whether the more successful do have a different way of thinking of the concept of vector as a cognitive unit—a single entity with different uses in different contexts—or as a number of different concepts (force, journey, etc) which have very different properties.

The evidence of the use of vectors in different contexts already shows that the experimental students are likely to have a more coherent overall view of the notion of vector that can be applied in different contexts. The evidence of the handling of singular examples shows a greater degree of flexibility in using the notion of free vector. The greater use of the commutative law (which works for free vectors, but not for journeys) also shows that they are more likely to be operating fluently with free vectors.

In almost all respects (particularly in the use of the graphical mode), the quantitative evidence supports the three hypotheses, 1, 2 and 3, which together give quantitative support for the main hypothesis stated at the beginning of the chapter that:

> Teachers can help students develop a notion of a translation as free vector through building on physical experiences, leading to graphic and symbolic representations, with the notion of free vector being constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

Chapter 8

Main Study: Qualitative Data Analysis

Interviews with the teachers

8.1 Introduction

This chapter focuses on the qualitative issues through individual interviews with teachers.

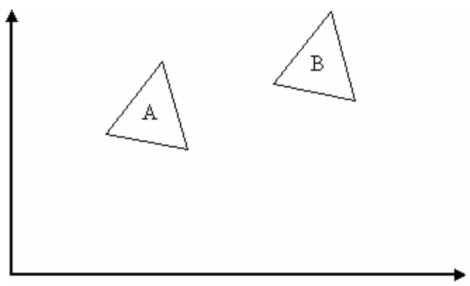
The interviews with one Physics teacher and two Mathematics teachers served the purpose of finding differences in the teaching and expectations of teachers in Physics and Mathematics. They were conducted during the period of the Main Study and were based on the test questions.

The three teachers were shown the questions and asked what they thought the students' responses would be. The Physics teacher was coded as P and the Mathematics teachers as M1 and M2. All three teachers are females.

8.2 Interview with the teachers

The selection of the test questions are shown in figures 8.1 to 8.5 and a sample of the typical teachers' responses are quoted below each figure.

1) In the picture the triangle has been translated from position A to position B as shown below:



(a) How can you represent the translation of the triangle? Show it on the picture

(b) Can you draw a vector starting at the origin (0,0) which will represent the translation of the triangle from A to B? If so, show it on the drawing

(c) Can you draw a vector not starting at the origin and not touching either of the triangles which will represent the translation from A to If so show it on the drawing

Fig. 8.1 Test question 1: Represent translation

P: "They will choose a specific point for (a). [...] in (b) they should be alright, but they will be confused in (c). [...] Sometimes they ignore the direction and don't place an arrow on the line."

M1: "I would expect them to go across and along. To go from a point on shape A to a corresponding point on B is another building block. [...] I am not sure if at the beginning they connect both together."

M2: "Many will be happy with representation of the translation. Only a few will think of the horizontal and vertical."

Summary:

The Physics teacher seems to think that the students would have no problem with an equivalent vector starting at the origin, but would have a problem with a vector starting elsewhere off the triangle. She did not mention horizontal and vertical components, in contrast to both Mathematics teachers. Teacher M1 comments that from using the horizontal and vertical components to considering a vector from point on A to a point on B is "another building block", whereas teacher M2 thought they will be "happy" with the translation and only "a few" would use horizontal and vertical components.

Comment:

At first, the Physics teacher's response seemed unusual to me, however, if one views the instruction as asking students to move a triangle in a physical way, this cannot be done unless one places one's hand on the triangle itself. Hence the physics teacher's response in part (c) may be based on real world activity, whereas in part (b) it is based on the familiar task of drawing a position vector from the origin in a manner the students will have met in class. In this rather subtle way, 'real world' experiences might therefore interfere with the mathematical notion free vector. She also expects some students to "ignore direction" and therefore not realise what the vector represents. She implies that they do not realise the meaning of the graphical symbol of vector. From the preliminary interviews with the physics teachers it seems that they use the numerical methods to calculate resultant vectors and yet the idea of representing the translation with the horizontal and vertical components came from the teachers of mathematics. One of the Mathematics teachers showed awareness that using an arrow instead of just horizontal and vertical components is another stage in the cognitive development: “To go from a point on shape A to a corresponding point on B is another building block”. So the teachers expressed there might be problem with the symbol of an arrow, flexibility of using that symbol flexibly as a mental concept and although they expressed that even if students know the equivalence of vectors, they might not have compressed that knowledge into the concept of free vector.

The next set of questions that the teachers were asked to comment upon (figure 8.2), involved addition of two vectors, where each example is ‘singular’ in some way (see section 6.1.2)

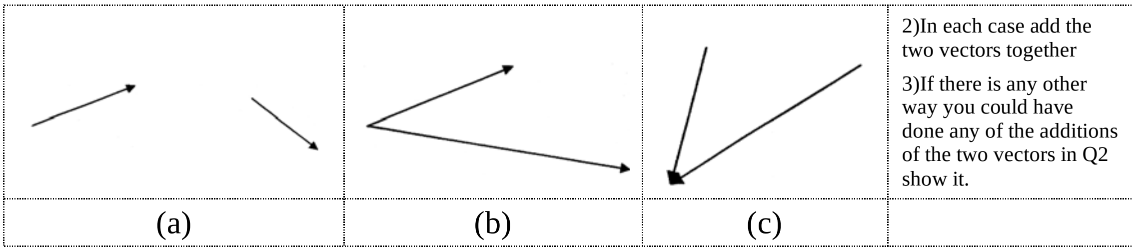


Fig. 8.2 Test questions 2 and 3: Add two vectors

P: “This is confusing, especially part (a). They might think that two vectors should be attached to each other. [...] In (b) they will add them using the parallelogram. [...] They are taught to use the parallelogram if the two vectors are connected at one point. When you give them a question with the two tugs, they find it more useful to answer using a parallelogram. They can see that it is pulled in a specific direction.” (Interviewer: “What about the triangle?”) “The triangle is used for resolving.[...] I would use the triangle with them if it was a displacement.” (Interviewer: “What about part (c)?”) “Confusing, they do not get the idea that the arrows have to follow ends. Some might resolve using matrices.” (The teacher refers to the use of column vectors). (Interviewer: “What about question in part (c)?”). “They might do it horizontally and vertically.”

M1: “They will do them by placing ‘nose to tail’ or will use column vectors. For some, these vectors might seem fixed in space and they will draw another two vectors between them to close the polygon. In part (b) they might think that they are connected in the wrong way and simply join them up with a third vector. Later on in the year they should use a triangle or maybe a parallelogram. They would think of the translation as a displacement. Parts (b) and (c) are two different visual images. They should be able to answer part (c) by the end of the course. They might use a triangle rather then a parallelogram. [...] Part three should make them think, and maybe even amend their answers to question two.”

M2: “They might be able to answer part (a) if they have the idea of moving them ‘nose to tail’. In (b) they have to disrupt a diagram, so only some might do the parallel displacement of the bottom vector and use the triangle rule. I don’t think that they will think of a parallelogram. In (c) if they answer they will definitely think of translating a vector. [...] Question three should make them think that there are different ways of doing things. If they already used a triangle rule one way they might use it the other way. [...] They might put vectors ‘nose to tail’ without drawing the resultant because sometimes they are taught to do this and expect them to fill the gap, not realising they will not know it should be filled.”

Summary:

The Physics teacher refers to real-life situations that occur in teaching, when for example 2 forces (“the two tugs”) are acting on an object, or two journeys that follow each other. The teacher thought that the students would use the parallelogram law of addition in case (b). She thought that the students would be confused about part (a) as it has no real-life significance. This suggests that, from a physical point of view, part (a) is a ‘singular’ case. She thought that students would find part (c) an unusual situation and would resort to numerical methods such as column vectors (horizontal and vertical components). She was aware that some students have difficulties moving vectors ‘nose to tail’.

The Mathematics teachers thought that the students would answer differently before the course and after the course and allowed therefore for a conceptual development. They both realised that if students have the idea of using the ‘nose to tail’ technique of adding vectors then they should be able to solve all three parts. However one teacher (M1) thought that students might think of the vectors as ‘fixed in space’ and therefore have a problem with the questions unless they use the numerical method. The other teacher (M2) expressed the idea of vectors being ‘fixed in space’ as diagrams which the students might not want to “disturb”. The first teacher thought that the students might use the triangle law of addition in part (b) or maybe a parallelogram law, but the second teacher thought that it is unlikely that students will use the parallelogram. They realised that parts (b) and (c) present “two different visual images” and that part (c) being ‘more difficult’ should be solvable to the students by the end of the course. One teacher mentions that the students might place vectors ‘nose to tail’ without showing the resultant. The second teacher mentions that the students who are familiar with the technique of ‘nose to tail’ but not addition, might instead feel that the addition means placing vectors next to each other (like a journey).

Comment:

The Physics teacher thought about these questions in a physical way: “two tugs”, “use of the triangle if it was a displacement.” She also indicated that the parallelogram rule is for forces and the triangle rule for displacement. It might be that Physics teachers do not use the ‘nose to tail’ technique when adding vectors. She also treated question (a) where vectors are separated as a ‘singular’ case. It was neither a journey nor forces. She was not considering vectors in a general mathematical context, focusing only on what the diagram could mean physically. When we consider that students learn vectors in physics first, then it would seem realistic to consider that it would be very difficult for them to conceptualise a vector (arrow) as a mathematical symbol without referring in their mind to some physical situation. This is the reason why the experimental lessons were conducted by starting students working on a physical object in a flexible physical context to allow them to realise the mathematical implications of a free vector.

The Mathematics teachers mentioned two ways of solving the questions. Implicitly, according to the teachers, some students have knowledge of moving vectors 'nose to tail' and others see them as fixed in space and will either do the addition of components numerically or use some other method based on partial knowledge. The mathematics teachers used the meaning of the parallelogram and triangular laws of addition interchangeably supported by phrases such as "translating a vector", "parallel displacement" and "close the polygon." In their language they considered the questions partly as set in the mathematical context, but with hints of a physical context by saying "translation as a displacement" and "fixed in space".

All 3 teachers saw that the arrows coming to one point in part (c) represent a 'singular case' (as something that students would have not met before).

The teachers also realised that some questions are 'harder' which the researcher considered to be at the higher conceptual level. In particular, teacher M2 used the term "disrupt" to represent the movement of vectors in (b), which indicates that she regarded it as being significantly different from the case students usually meet.

This use of the word "disrupt", which the teacher M2 mentions on several occasions interests me very much, as it seems to give the diagram a physical meaning. It is as if the teacher herself embodies the diagram with a physical meaning. Given the many subtle ways in which physical meaning interfere with the mathematical meanings, I found the phenomenon very significant.

The overall impression from the teachers was that some questions are 'singular' as students would not have met such situations before. The physical implication of the question could have affected the way it is responded to by students. In Watson, Spyrou and Tall (2002) we started by regarding questions (a) and (c) as more suitable for use of the triangle law of addition, while question (b) as more suitable for the parallelogram law. The teachers also mention both laws in some of their responses. However when we triangulate this which the students' test responses only one student used the parallelogram law of addition.

The next question, (Figure 8.3) asked the students to add 3 free vectors. The teachers' responses to the possible answers that students might give to the question shown in are listed below.

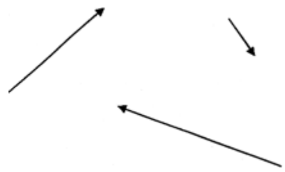


Fig. 8.3 Test question 4: Add three vectors

P: "We never do anything like this, so I am not sure."

M1: "I think that they will be tempted to join them together. The problem is that they need mathematical equipment to do this and they are not used to using it. Pupils had to do Technical Drawing years ago but these skills are not required any more. They might have a problem connecting them if they happen to cross each other. It might seem wrong."

M2: "They will choose the longest and then join them in the order of size. They probably will do it below, so they do not cross."

Summary:

The Physics teacher said that the students had no experience with this type of question and therefore their responses will be unknown to her. The Mathematics teachers thought that the students might "join them together" meaning place them 'nose to tail'. However one teacher was concerned about the students' lack of skills for such an activity, due to lack of technical drawing skills; the other was concerned about students' confusion if the vectors cross.

Comment:

The Physics teacher indicated that the context matters and if the question cannot be used in the physical context it is not valid from her subject point of view. Actually the question could be placed in the physical context which was used in the experimental lesson. It could have been for example three displacements each taken from a different point on the object, or three forces placed at three different points on one object, but since in her subject such questions are not being considered this did not come into consideration. It can be concluded that in Physics the students are not taught the idea of the free vector which can be used in any required context.

One of the Mathematics teachers (M1) implied that the question would be difficult to answer without the use of mathematical equipment, and that students are no longer taught drawing skills. The other teacher (M2) was concerned about the difficulty that might occur when the drawing leads to two vectors crossing. This is consistent with the idea that such a case is a 'singular' and not generic because it contains specific properties that need not occur in the general case. (Such a case happens again in a later question discussed with the teachers (in figure 8.6c). Both teachers seem to concentrate concentrated on how students can use the procedure of adding vectors and what might prevent them using it.

The next 2 questions the teachers commented on are shown in figure 8.4. The questions were placed in the specific physical situation and restricted the number of vectors students were supposed to use, but otherwise were open-ended.

Draw a representation of three forces and add them together.	Draw a representation of two displacements and add them together.
(a)	(b)

Fig. 8.4 Questions set in two different contexts

P: "In the first one they would draw forces acting at one point, all in line, at least to start with, and later they might draw an object and draw forces acting vertically and horizontally. In the second one they will draw one journey followed by the other."

M1: “After the course they should draw forces acting on a particle, but then it is not easy to add. [...] Bit like question 2(a) [Figure 8.2b]. For displacements they would draw one after the other. You cannot displace from the same point twice.”

M2: “They would draw them from one point and then they would have to disturb the diagram to add them. With forces they would think of something acting on a particle. I think in a question with displacement they would have to draw a shape to displace. Most students would have difficulty to do this question without something to refer to.”

Summary:

All three teachers agreed that the students would use three forces acting at one point in part (a). The discrepancy occurred in their anticipation of the way the students would add the forces together. The Physics teacher gave a simple solution of drawing (the components of) the forces only horizontally and vertically.

All three teachers distinguished between the two different contexts and anticipate that students would show the difference between the two ways of representation. One Mathematics teacher (M2) suggested that students will find difficulty in representing the displacement without the object to act on: “in a question with displacement they would have to draw a shape to displace” and also that students might find difficulty adding vectors which are connected in the question, again using the expression “they would have to *disturb* the diagram”.

Comment:

Since a lot of mathematics has been cut out from the Physics syllabus at all stages of studying of this subject, a solution showing forces as vectors in 2-Dimensions only, as anticipated by the Physics teacher, would have been the most likely response by the students. However she thought that the concept of journey will be stronger in terms of vectors following each other. The mathematics teacher M1 agreed about the concept of journey with the Physics teacher, however teacher M2 was concerned that students need an object to act on. The teacher M1, by saying "you cannot displace from the same point twice", intimated that if a displacement is measured from the origin, this is saying that a *second* displacement must be drawn from where the first ends, you cannot go back and displace the first point twice.

Both Mathematics teachers agreed with the assumption of the Physics teacher about drawing forces acting on an object and therefore being drawn from one point. However neither of them realised the way students are taught to add forces in Physics and thought and therefore anticipated problems with addition. They seemed to know instinctively that students have problems with dealing with 'free vectors' geometrically in 'singular' type of cases. Thus M1 said what the student would try to do, which in the context of forces means drawing them from a point, (which may give them difficulties) but for the second part (displacements) she said that they would place them one after the other. The teacher M2 expressed the opinion that students need an object to perform an action on, which means she anticipated a very physical attitude to addition of displacements.

All three teachers suggested different approaches from the students according to the context they would operate in. One teacher suggested that in the physical context students would only be able to follow the procedures if they first had an object to act on, and therefore implied that students operate differently with vectors in different physical contexts.

The last two questions (figures 8.5 and 8.6) are very different from the previous questions.

Using the drawing below, or otherwise, add:

(a) $\overrightarrow{AB}+\overrightarrow{BD}$

(b) $\overrightarrow{DA}+\overrightarrow{ED}$

(c) $\overrightarrow{AB}+\overrightarrow{AE}$

(d) $\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}$

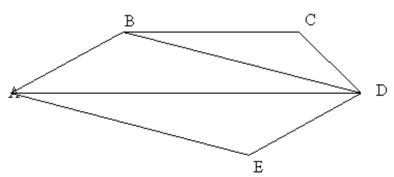


Fig. 8.5 Question on vector addition set in the diagram

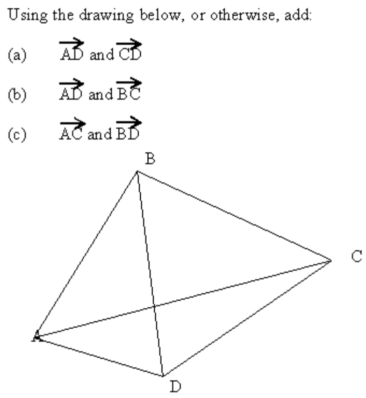


Fig. 8.6 Singular question on vector addition set in the diagram

The students had to pick their information from ‘busy’ diagrams and recognise the specific symbolism for the displacement vector (starting at a specific point and ending at another specific point). The questions in figure 8.5 should be familiar to students because it is of a type encountered in the previous year’s course. However the questions in figure 8.6 would not be familiar because they require the students to draw additional lines not in the figure. The teachers’ responses are shown below.

P: “We never do anything like this, so I do not know” (Interviewer: “Any thoughts”) “Some of these [points to questions shown in figure 8.6] are impossible [...] the answer does not fit on the drawing.”

(Interviewer: ”They do not need to be a part of the drawing”). “It is not clear, I think the students would assume that it should be part of the drawing.”

M1: “Most students should do the first one (Figure 8.5). This is a GCSE material (Year 11). Part (b) seems more difficult but the students who want to do ‘A’ levels are used to keep going so it does not matter if the most difficult question is in the middle. [...] The second one (Figure 8.6) is very difficult. It is a mental jump to draw a new line. They would expect the answer to be in a diagram. This might go beyond their understanding. It would require a confidence to answer. [...] Only those who understand about vectors will be able to do this question.”

M2: "Looking at the first one (Figure 8.5) they will put arrows on the lines when trying to add the vectors. The arrows would help them to think if they are going the right way. [...] Students would assume that *ABDE* is a parallelogram and use the opposite side in part (b) to make the second vector follow the first. It is back to a journey. They should be able to answer these questions. The second one (Figure 8.6) is much more difficult. [...] They might feel uncomfortable moving out of the drawing and even when drawing outside they might try to compare it to the sides already present in the diagram."

Summary:

The lack of context mattered to the Physics teacher. She did not want to commit herself to speculating on students' responses in an area which was unfamiliar to her. However, when asked for some sort of comment, she responded that the second set of questions is more difficult as students have to draw the additional lines as their responses. She therefore recognised that the questions presented in figure 8.6 are 'harder'.

The Mathematics teachers thought that only part (b) of the question presented in figure 8.5 could cause a problem. The question asks students to add vectors which are written in a reverse order than the way they follow each other. They also anticipated students would have a problem when answering the set of questions presented in figure 8.6.

Comment:

All three teachers recognised that the question presented in figure 8.6 could cause students problems and in fact this was a 'singular question' as the students never met anything like this before.

The Mathematics teachers also recognised that part (b) of the questions presented in figure 8.5 is different. In this part students could have used the commutative law of addition where $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. The teachers however did not consider students using that law and thought that they might use the rule of the equivalent vectors. The teacher, M1, implied that dealing with free vectors is another stage in the cognitive development: “It is a mental jump to draw a new line”.

8.3 General Summary

The main hypothesis in chapter 7 states that:

> Teachers can help students develop the notion of a translation as a free vector through focusing on the effects of physical actions, linking graphic and symbolic representations, so that the concept of free vector is constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

If we triangulate the teachers’ interviews and the students’ responses to the tests, we may conclude that teachers are very aware of most mistakes that students might make.

For instance, one of the Mathematics teachers comments, “They might put vectors ‘nose to tail’ without drawing the resultant, because sometimes they are taught to do this.” From the results of the quantitative tests, it is clear that a number of students added two vectors $\overrightarrow{AB}+\overrightarrow{BC}$ by simply drawing them ‘nose to tail’ but did not draw the resultant $\overrightarrow{AC}$. The theoretical framework would suggest that the student may see the addition of two vectors as a journey from *A* to *C* via *B* and not as the higher level concept of free vector. However, this was not a concern expressed by any teacher. It may be that in coping with a crowded syllabus, the quest is often to get the students to respond correctly rather than seek for subtle reasons why they may make mistakes.

If students have no experience of the idea of the free vector in placing vectors ‘nose to tail’ then they are less likely to make the connection of an arrow being a symbol for a free vector that can be used the same way regardless of the context.

The Science teacher said, “... sometimes they ignore the direction and not place an arrow on the line”, which means that she is used to students misinterpreting a graphical symbol of a vector which could cause them difficulties in coping with the concept of free vector in any context that does not have a physical meaning. She did not expect students to make connections between the actions and symbols.

The second hypothesis in chapter 7 states that:

> Students who were helped in building a concept of a free vector can add vectors in 'singular' cases, not just generic ones; they can also use free vectors independent of the context the addition is set in and realise that the commutative law applies to the vector addition.

If we again triangulate the students' test responses with the teachers' comments we can see that the teachers were aware that some questions are more difficult than the others (singular cases) but the views from the Physics teacher and the Mathematics teachers differed substantially.

The Physics teacher expected students to have problems with questions where vectors were drawn separately (figures 8.2 (a) and 8.3). She did not expect the student to use the idea of the free vector at all and every time the question could not be connected to the *obvious* physical context, she treated it as a 'singular' case.

Generally the Mathematics teachers expected students to use the idea of 'free vectors' at the end of year 12, but did not realise that the concept was essentially required by the end of their year 11 teaching; the text book introduces the concept of vector (through the implied APOS theory) by moving from action on objects through equivalent vectors to the concept of free vector. However they spoke in a way which indicated a subtle sense of the steps in the development: For instance, one teacher said, "I would expect them to go across and along. To go from a point on shape A to a corresponding point on B is another building block". This suggests an awareness of the conceptual change that is required to move from one stage to the next. When the need of drawing the equivalent vector arose, one teacher said: "It is a mental jump to draw a new line." In the case of the 'singular' question in figure 8.2 (c) one of the teachers responded: "This might go beyond their understanding" and the other teacher said: "They might feel uncomfortable."

These comments all show the sensitive realisation by the teachers of the possible areas were students may have difficulty. It is matched by the difficulties that are experienced by many students prior to the course and by some of the control students after the course. The success of the experimental group suggests that it may be a help to other teachers to be aware of the simple idea of ‘focusing on the effect of an action’ which may help students form the idea of free vector in a more versatile and confident way.

Chapter 9
Main Study: Qualitative Data Analysis
Interviews with the students

9.1 Introduction

This chapter focuses on the qualitative issues through individual interviews with the students. The interviews were intended to gain a greater insight into:

- students' use and flexibility of language when discussing problems connected with vector addition;
- students' focus of attention at any given time (whether it is on actions, or procedures or on the effects of those actions and procedures);
- the way in which different contexts affect their thinking;
- their flexibility in dealing with different modes of operation (graphical/symbolic).

The interviews were also intended to test if the students were placed, on the basis of the test analysis, in the right categories according to the theoretical framework developed in chapter 3. It was intended that interviewed students should be selected from a spread of different categories (uni-modal, higher uni-modal, multi-skilled, versatile & fully integrated) as well as from both groups (experimental and control).

There were two sets of interviews: after the pre-test and after the post-test. Different students were interviewed each time. The extracts from the interviews with students are presented in the two sections below (9.1, 9.2).

9.2 The interviews following the pre-test

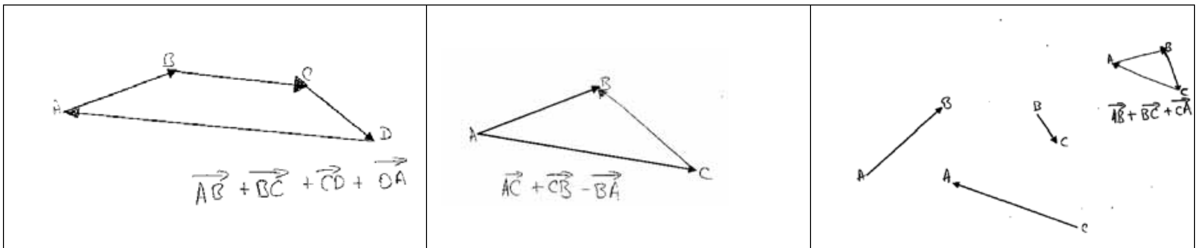
The pre-test interviews were conducted with four student coded as:

> S1: answered in an '*intuitive*' way;
>
> S2: answered mainly *symbolically* and has been classified as *higher uni-modal*;
>
> S3: answered mainly *graphically* and has been classified as *higher uni-modal*;
>
> S4: classified as belonging to the highest *fully integrated* category.

The students S1 and S2 were from the experimental group (A) and students S3 and S4 were from the control group (B).

9.2.1 Student S1

The examples of the student's S1 responses are shown in figure 9.1.



(a) (b) (c)

Fig. 9.1 Student S1: examples of responses to the pre-test

Figure 9.1 parts (a) and (b) are the student's pre-test (T1) responses to the questions shown in chapter 6, figure 6.2 (a) and (b), while 9.1 part (c) is the response to the questions asking for the addition of the three given vectors (figure 6.4). In the first question (figure 9.1 (a)) the student was asked to add two vectors, which he named $\overrightarrow{AB}$ and $\overrightarrow{CD}$. The student filled gaps between vectors with extra vectors ($\overrightarrow{BC}$ and $\overrightarrow{DA}$). In the second question (figure 9.1 (b)) the student was asked to add two vectors, which he named $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The vector $\overrightarrow{CB}$ was drawn to close the gap. In the third question (figure 9.1 (c)) the student was given three vectors to add, which he named $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$. He ignored the magnitude of the vectors in this question and just considered the approximate directions to form a triangle. This is why the student was categorised as ***Physical Intuitive.*** Parts of the interview are presented below (I – stands for the interviewer).

I: Look at your answers in the first part of question 2 can you explain it to me? (fig. 9.1a).

S1: In this I drew them clockwise.

I: What is the result of the addition?

S1: Sum.

I: Which one is your sum?

S1: A there to D [points].

I: So what would this one mean? [The interviewer points to BC.]

S1: This one just joins these two vectors so they can be added together.

I: So what were you looking for?

S1: Continuity.

I: What about the next part? (fig. 9.1b).

S1: I was trying to do them the same way?

I: How did you do them the same way?

S1: These two (points to B and C) go in separate directions.

I: Did it worry you in any way?

S1: No

I: Can we look at your answer to Q4? (fig. 9.1c). How did you add the vectors?

S1: I connected all the vectors together so it will be easier to add them all together.

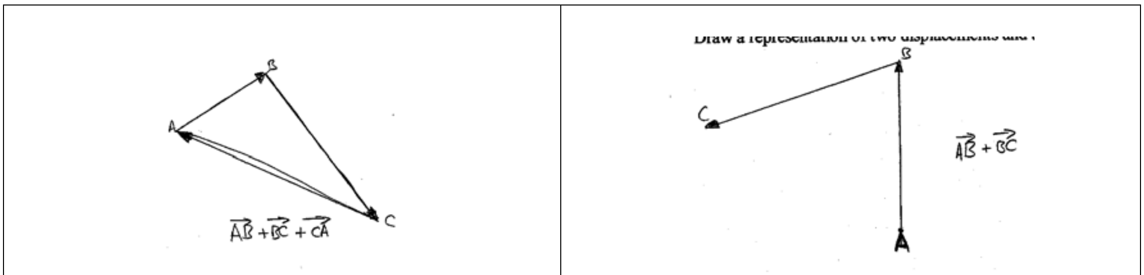
I: Didn't worry you that they have different length on your drawing

S1: I did not draw them to scale but to different scale just to give a general idea of how to add them.

I: Didn't you think of doing some measurements when adding the three vectors?

S1: No

The responses the student gave to the questions in the two different contexts are shown in figure 9.2.



(a) (b)

Fig. 9.2 Student S1: examples of responses to the pre-test

I: When you were asked to add three forces together, you drew a triangle [fig. 9.2a] but when you were adding two displacements you did not [fig. 9.2b], can you explain it?

S1: I think I forgot a line joining beginning to the end (point from A to C, figure 9.2 (b)).

I: What would this line represent?

S1: It would be a displacement from the first point to the last point (points from A to B)

I: Can you explain it a bit more?

S1: Distance, instead of going from A to B and B to C you can go shorter distance from A to C.

I: In the other question, if all three forces form a triangle does it mean anything physically?

S1: Not clearly, no.

I: Could there be a situation when they do not meet like that, when there is a gap left?

S1: Didn't come back to the starting point? What would it mean if they didn't?

I: Would it make any difference if they did not meet?

S1: Then you would not be able to add them all together.

I: Can you draw for me an example of forces acting on an object?

S1: If you have a particle, you would have the gravity, the resistance and if you were pushing it from one side you would have a force acting this way [student draws two vertical forces and one horizontal].

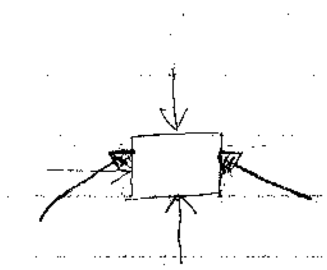


Fig. 9.3 Student S1: example of response to the pre-test

I: Can you add them for me

S1: You add the y components, you would multiply the gravity by the mass and this has to be equal to the force up.

I: Why do they have to be the same?

S1: Because if it remains still in the horizontal angle then it is not going downwards or going upwards

I: If you had two additional forces on your object (interviewer draws two forces) and the object would not be moving, how could it work?

S1: You could work it up by using a force and then using an angle and use cosine to work out what this component is and what this component is and work it out (student waves his pen in the horizontal direction below each of the forces, but without the particular direction).

I: Could you do it graphically?

S1: Well, [student starts by drawing horizontal components and vertical components].

Summary

The student S1 does not show the flexibility of language when discussing the problem connected with vectors. Although he knows what it means to join vectors ‘nose to tail’ he has no awareness of the notion of equivalent vectors and vector addition. He does not fully understand the symbolic representation in a graphical sense. The student uses the word “continuity” to mean that one vector follows immediately after another, without a gap. He attached labels to the graphical symbols of vectors (the way it is taught in the Year 11 text book) and shows addition in that way but without showing the result — the *effect* of that addition.

When adding vectors, the student focuses on the idea of continuity in two ways: either by adding extra vectors to fill in the spaces, or joining them together one after another, but without showing the resultant. He used a ruler to draw all his answers except the one shown in figure 9.10 part (c) as this could possibly defy his own theory about vector addition. He confirms it by saying that if the vectors did not join then "Then you would not be able to add them all together." He does not connect the physical effect of the addition in a more general way, as a total effect. Although he draws the arrows in figures 9.1 (a) and 9.2 (a), when he describes what is happening he indicates the correct direction and he seems to have some concept of ***the same effect*** in the embodied sense of a journey: "Distance, instead of going from *A* to *B* and *B* to *C* you can go the shorter distance from *A* to *C*".

When he adds vectors, in any context he looks for the continuity. He can operate in one directional environment as far as forces are concerned. He indicates it by saying that, for the object not to move, the vectors should be of the same magnitude but in the opposite directions (his explanation to his own drawing in figure 9.3). He also thinks that the forces should close the loop. When asked what would happen if after putting forces graphically together there was a gap left, he seems dismayed and answers: "Didn't come back to the starting point? What would it mean if they didn't? [...] Then you would not be able to add them all together." His preferable mode of operation when he thinks of forces, under different angles than vertical, seems to be symbolic. He suggested adding the horizontal and vertical components for the vectors drawn (in figure 9.3) under different angles. However in the context of displacement he thinks of a journey which follows the vectors as they are placed one after another, in physical terms. He does not consider the addition of two displacements giving the total effect but simply as getting from the first point to the last one in a shorter distance.

From the above discussion with him it can be concluded that he is partly in a ***physical intuitive*** class and partly at stage 1 of the graphical mode and stage 2 of the symbolic mode. So maybe he could be classified as ***uni-modal*** however the dividing line is not clear.

Comment

The student's understanding seems to be at a very low stage in the graphical mode: putting vectors 'nose to tail' to give continuity, as if it is a journey of which one part has to start where the previous finishes and working with forces in one direction. However he has no awareness of the idea of equivalent vectors or free vectors. This student seems to be graphically locked in either a context of journey or forces but only in one direction in dealing with vectors. He might have answered better in the symbolic mode if everything was set on grids. However, he sensed from the type of questions asked that the graphical method was the preferable one but he was not confident with it.

9.2.2 Student S2

The second student (S2) responded, in the pre-test, mainly using the symbolic mode. The examples of her pre-test responses are presented in figure 9.4. The relevant part of the interview is shown below the figure 9.4.

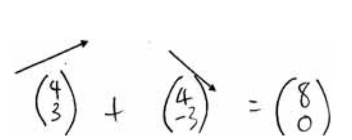
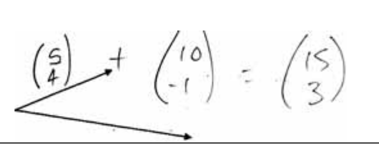
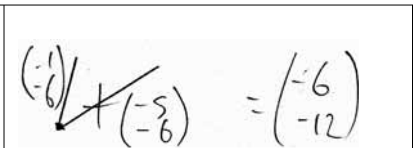
		
(a)	(b)	(c)

Fig. 9.4 Student S2: examples of responses to the pre-test

I: What do you think the question is asking you to do?

S2: I thought you want an actual number.

I: Why do you think I wanted a number?

S2: Because of the way I was taught, we were taught to put them on the grids.

I: So what is your technique in a graphical mode?

S2: I put them nose to tail [places them 'nose to tail without any precision, figure 9.5].

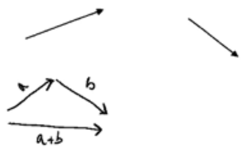


Fig. 9.5 Student S2: response to the interview question

I: What about the last question?

S2: [Student connects them again nose to tail on a separate drawing, using a ruler, figure 9.6].

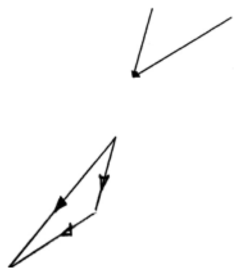


Fig. 9.6 Student S2: the interview response to the singular question

I: The next question asks to do it in a different way. How would you do that?

S2: You can work out all the vertical and horizontal stuff, I am not sure.

I: The numbers you have used in your answer, what meaning did they have for your resultant?

S2: I think I just used centimetres.

I: Why centimetres?

S2: I did not think it really mattered what the scale was.

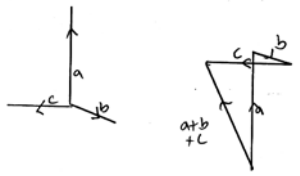
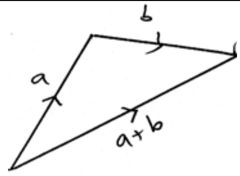
	
(a) pre-test response	(b) interview response

Fig. 9.7 Student S2: responses to the different contexts pre-test questions

I: What about the next question? (referring to the question set in the context of forces; the test response in figure 9.7, part a).

S2: I am used to forces going out.

I: What about the next question, you left it blank?

S2: I guess I just didn't really know what you meant by displacement. Are displacements the same as vectors?

I: So doesn't Physics talk about displacement?

S2: Oh, it is like movement? Isn't it? in a certain direction, but isn't it what a vector is? [...] So would you like me to do just the same thing again?

I: Would you like to answer it now?

S2: [student draws two vectors following each other and the resultant as shown in figure 9.7 (b)].

I: When you started answering you first thought was to use numbers. Why do you think that the number answer was your first choice?

S2: Because if you just draw there are no numbers involved and you should have some numbers in the answer. Drawing the picture doesn't really answer the question.

Summary

The language the student S2 uses lacks flexibility when different contexts are concerned. She is not sure about the concept of vector in case of displacements but seem to be quite confident with the idea of vector to represent forces.

She is able to focus on the effect of the procedures which she shows when adding vectors even in the singular cases (figure 9.6) as well as in case of adding forces (figure 9.7, part a). When she realises that displacement is "movement" and therefore can be represented by a vector she also can use addition in the second physical context.

She proved in the interview to have flexibility in dealing with different modes of operation, which was not so clear from the test, as she answered only one question graphically. She explained that she used numbers because this is what she though was expected in vector questions. She is not sure about the scale as only met problems presented on grids till now. She was also under impression that "drawing the picture doesn't really answer the question".

It would seem that the student S2 could belong in the '*versatile*' category and not the *higher multi-skilled category*.

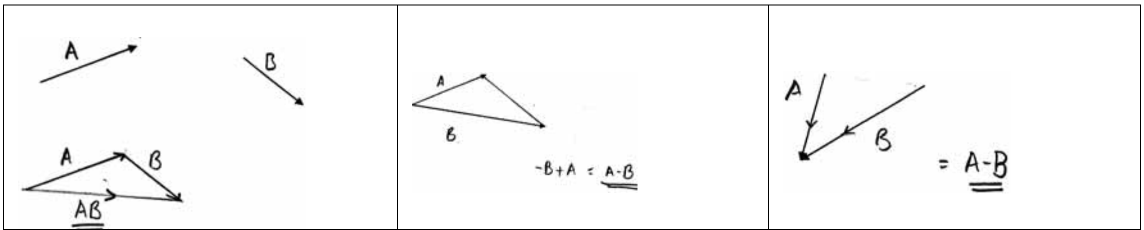
Comment

The student's answer raises an interesting point of what is considered to be mathematical response. The student said "Drawing the picture doesn't really answer the question." The implication is that graphical responses are not valued in Mathematics despite that the text book from the previous year included that stage of development in teaching vectors. The student did not seem to be aware that the numbers which she used did not have any significant value. In fact, as an answer, they were meaningless in both a mathematical and a physical sense. However the student was aware of the graphical responses she could give and gave a vague answer to the generic type of question (figure 9.5) and a much more precise response to the singular question (figure 9.6), as if the singular question demanded more thought from her.

It could be suggested from all of the student's responses that she understands vectors as a mathematical concept that can be used in the same way in any physical context. The student also does not mention the parallelogram law of addition and the only rule she mentions is 'nose to tail' movement, which is the necessary part of the triangle law of addition although this was never explicitly mentioned. She also does not use the commutative law of addition in graphical or symbolic mode.

9.2.3 Student S3

The third student (S3) answered most questions graphically. His pre-test responses to three of the questions are shown in figure 9.8.



(a)	(b)	(c)

Fig. 9.8 Student S3: examples of responses to the pre-test

The response in 9.8 (c) is equivalent to the graphical response as student indicates that to add vectors as journeys the direction of vector B would have to be changed and indicates it with '-B'. However the student actually completes the addition only in the question shown in figure 9.8 part (a). The relevant parts of the interview are shown below.

I: could you look at your answers and give me some idea of your thinking at that time?

S3: I thought it was asking me to put them into a triangle and then join up (fig. 9.8a).

I: What about the other two? (fig. 9.8 b and c)

S3: I did not know how to do it.

I: Was it anything you did in the past which made you answer in this way?

S3: I didn't know how to do it so I just guessed.

I: Is there any other way you could have answered any of these questions?

S3: I cannot think of any other way.

I: In question 5 you were asked to draw a representation of three forces and add them together and in question 6 you are asked to draw a representation of two displacements and add them together. Can you explain why you answered them in this particular way? (The student's answers are shown in figure 9.9.)

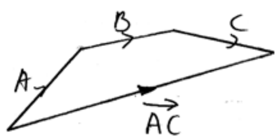
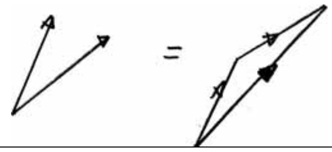
	
(a) response to Q5	(b) response to Q6

Fig. 9.9 Student S3: responses to different contexts questions in the pre-test

S3: I though that if they are displacements they have to come out from the same point (figure 9.8 (b)). [...] I thought that a displacement has to always start at the origin, hasn't it?

I: And what about forces?

S3: I just drew any three vectors?

I: No specific reason?

S3: No

I:: Why did you write $\overrightarrow{AC}$ here (the interviewer points to the resultant in figure 9.9 part (a)).

S3: I cannot remember. I think that's because it starts at A and end with the point on C. Should it be ABC?

Summary

The language student uses lacks clarity. For example when adding two vectors students says: "I thought it was asking me to put them into a triangle and then join up". What he probably means was to join the vectors and close the triangle. His explanation seems to indicate some learnt procedure and not the answer he thought out. He also mixes the names of vectors: displacement with position vector. Also the student's language of notation lacks precision. He does not realise that in symbol $\overrightarrow{AC}$ the *A* and *C* initially refer to specific points in space.

The student seem to concentrate on the procedure :" I thought it was asking me to put them into a triangle and then join up", which he thought is expected of him. It is not clear from the student responses if he focuses also on the effect of his actions. He was able to answer his own addition of two vectors (figure 9.9 part (b)) while he could not answer the same question set for him in a general context, as shown in figure 9.8 part (b). This could indicate that the context affects this student's thinking.

The student S3 does not show the flexibility in dealing with different modes. At no stage did he indicate that he could answer the questions symbolically/numerically, even when prompted. However this could just indicate that the student realised that the graphical mode was the most efficient way of answering the questions.

The student used a ruler to show all his responses, but at no time he used commutative law of addition or the parallelogram law of addition and only mentioned, in a vague way, the triangle as a way of adding. The student S3 seems therefore to be *high uni-modal focused* on the basis of some responses to the test.

Comment

One of the interviewed teachers suggested during the interview that students might feel that vectors drawn in this way are "fixed in space" and just "connected in the wrong way and simply join them up with a third vector." This might be a case with this student when the questions are set in the purely mathematical context. The student therefore lacks flexibility and versatility of using his knowledge. It has been not embedded properly nor turned into a cognitive unit. The student also mixes the notation of the free vectors, for example **u** and **v**, which can be written as a sum **u + v** with the notation of the displacement vector from A to B written as $\overrightarrow{AB}$ and from B to C written as $\overrightarrow{BC}$, which added together would give $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

9.2.4 Student S4

The next student (S4) has been categorised as ***fully integrated***. He could vary his answers from the graphical mode to symbolic and did not have any problems with different contexts or with the singular questions. Below are the examples of his test responses (figure 9.10) to the test.

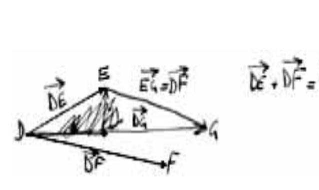
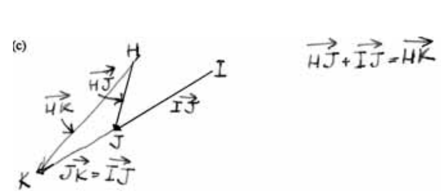
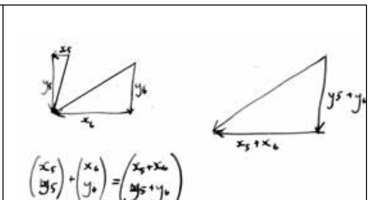
		
(a)	(b)	(c)

Fig. 9.10 Student S4: examples of responses to the pre-test

I: What do you think the question is asking you to do? (refers to question answered in figure 9.10 part (a)).

S4: Find the resultant vector when there are two given together so if you put the end to end it will be an overall translation.

I: So what about the next one? (refers to question answered in figure 9.10 part (b)).

S4: The same. You add them together.

I: What do you think question 3 is asking you to do?

[Pause]

S4: If you display each vector into two perpendicular directions, and then add the two horizontal and the two vertical.

I: Did your previous teacher teach you this method?

S4: I don’t think so, It makes sense, I must have got it from somewhere.

I: Thank you very much. Which method seems easier to you, the first one or the second one?

S4: If I was given values for the vectors and if they were given on the graph or squared paper I would find this one easier. [He points to figure 9.10 part (c)).

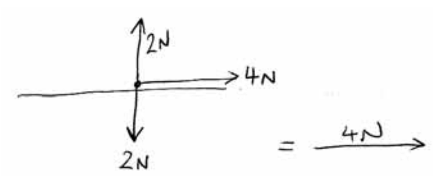
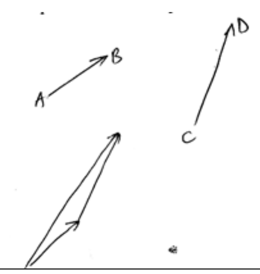
	
(a)	(b)

Fig. 9.11 Student S4: responses to different contexts questions in the pre-test

I: Did they show you anything like this in Physics? (the interviewer points to the students’ answers in figure 9.11 (a)).

S4: No I do not think [...] Yes we’ve done vectors as forces, so we would have used Newton’s as the vectors. Stuff like … [He draws the example of three forces acting from one point with the values in Newton’s next to them and angles in between, figure 9.12]

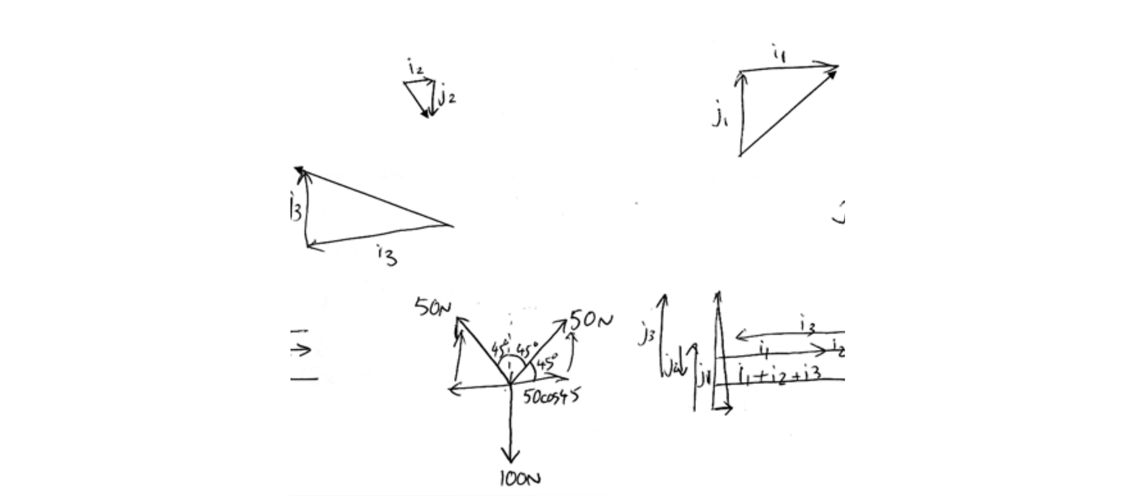


Fig. 9.12 Student S4: response to the interview question

I: How would you find the resultant?

S4: I would split them, yes, I would split into horizontal and vertical forces [draws horizontal and vertical component on each force] and add them.

I: Would you use the angles?

S4: Yes, this would be 45, [points] and this would be a hypotenuse [points to the force] then this would be cos 45° and so on.

I: What about the next one? [referring to the question shown in part (b) of figure 9.11].

S4: Oh, this is displacement.

I: Is that natural for you to draw the displacements separately?

S4: I just drew them like this because they are drawn separately in question 2.

I: But Question 2 says vectors not displacements?

S4: Well a vector looks more like a displacement. Displacements are obviously vectors.

I: So are forces vectors?

S4: Well they are. It is an arrow there (points to his answer with forces, to one of the vertical forces) but nothing actually moves.

Summary

The student S4 uses the language of vectors more flexibly than the previous 3 students. He uses phrases like: "if you put the end to end it will be an overall translation"; and "you display each vector into two perpendicular directions, and then add the two horizontal and the two vertical" (the word 'components' is missing from the last sentence). He also said: "I would split into horizontal and vertical forces" and does not maybe realise that they are only components of the forces and not two different forces.

From his statement about the "overall translation" we can also assume that he thinks about the effect of actions. He also considers forces acting on an object, without object actually moving although this is not quite correct according to his drawing in figure 9.12 as there will be a resultant force which would cause the object to accelerate.

The student S4 seems to think of a vector in the same way whatever the context. This could be implied from his verbal responses: "I just drew them like this because they are drawn separately in question 2"; "Displacements are obviously vectors; and in reply to a question "So are forces vectors?" he responded "Well they are. It is an arrow there". The student seem to understand (judging from his responses) the concept of vector as something representing a quantity which has a magnitude and a direction.

The student S4 seem to be able to operate in both modes (graphical or symbolic), in both generic and singular type of questions (figure 9.10 parts (b) and (c)). The student did not use the ruler to draw or measure but his drawings are fairly precise approximations and it is clear that he understands the idea of equivalent and free vectors. He does not use the commutative law of addition or the parallelogram law anywhere in his responses. The student was placed in the *fully integrated* category on the basis of the test and there is no evidence to change this categorisation.

Comment

The student seems to operate with ease on vectors using two different modes (graphical and symbolic) of operations, and in different contexts. He is aware of the same ***effect*** in a mathematical sense and does not try to use the vocabulary of a specific physical context when dealing with general situation: “Find the resultant vector when there are two given together so if you put them end to end it will be an overall translation.” This student can deal with the singular cases (figure 9.10 (b) and (c)) and seems to adapt the mode of answering according to whichever is more suitable. It could be also concluded from some of the responses that student gained his knowledge and understanding of vectors on basis of one context: “Well a vector looks more like a displacement,” and built this into a cognitive unit which he uses in other contextual situations. When asked if forces are vectors, he answers: “Well they are, it is an arrow there but nothing actually moves.” The student is aware of the triangular law of addition but does not mention the parallelogram law.

9.3 The interviews following the post-test

The interviews were conducted with the student coded as:

S5: from group B and classified as belonging to the *uni-modal* category;

S6: from group B and classified as belonging to the *higher uni-modal* category;

S7: from group B and classified as belonging to the *versatile* category.

S8: from group A and classified as belonging to the *higher uni-modal* category;

S9: from group A and classified as belonging to the *versatile* category;

There were no students in group A, at that stage, left in the *uni-modal* category.

9.3.1 Student S5

The examples of student's S5 responses are presented in figure 9.13. Part (i) shows two responses, the top one to question 2 (a) and the bottom one to question 3 (a). Similarly part (ii) shows responses to questions 2 (b) and 3 (b) and part (iii) shows responses to questions 2 (c) and 3 (c). The relevant parts of the interview with the student S6 are shown below the figure 9.13.

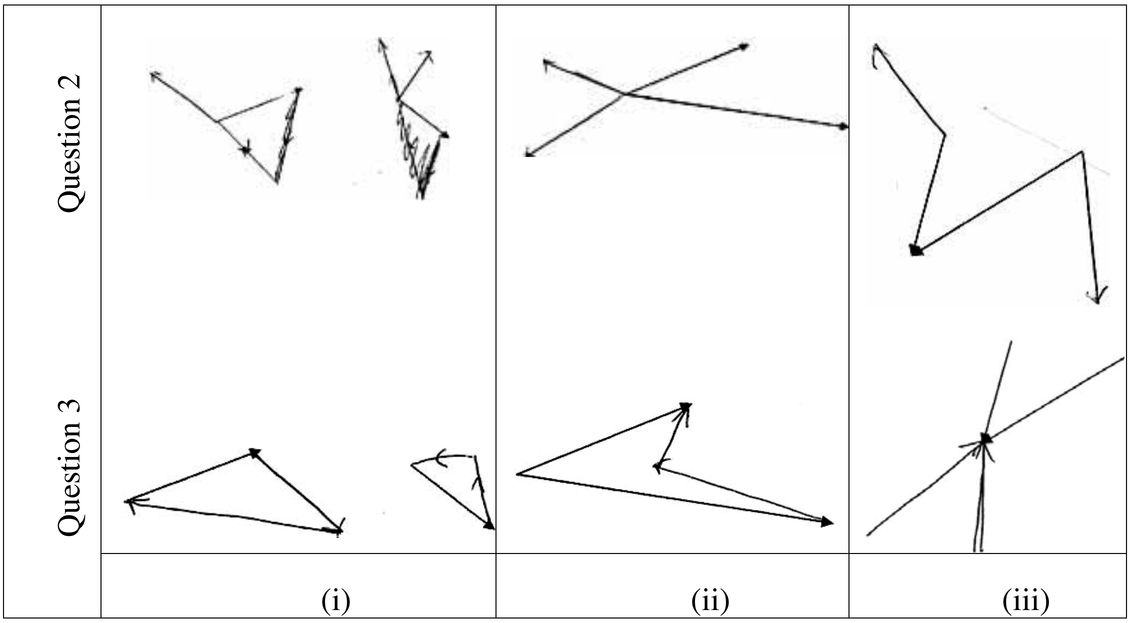
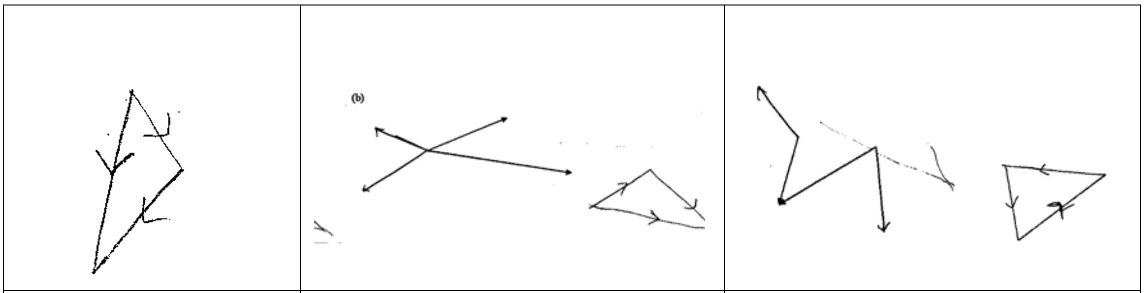


Fig. 9.13 Student S5: responses to questions 2 & 3 in the post-test

I: How did you answer question 2?

S5: I misunderstood the questions and I was adding another two vectors from the end of the one already there.

I: Can you answer the question as you understand it now?



(a)	(b)	(c)

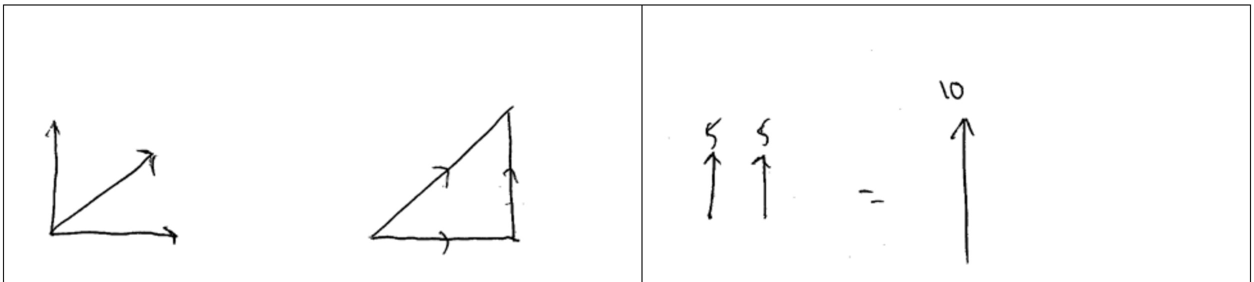
Fig. 9.14 Student S5: corrected responses to question 2 during interview

S5: [The student draws his answers as shown in figure 9.14.]

I: So what is the rule for adding vectors?

S5: When one ends, starts the other one, only I could not do it here [points to his answer in figure 9.14 (c)] because they meet in one point, but here I could because they start at one point [points to his answer in figure 9.14 (b)].

The student's S5 responses to the two questions, set in two different contexts, are shown in figure 9.15. Part (a) shows student's response when being asked to draw three different forces and add them together and part (b) when being asked to draw two displacements and add them together.



(a) - forces	(b) - displacements

Fig. 9.15 Student S5: responses to different contexts questions in the post-test

I: How did you answer questions 5 and 6?

S5: I have seen questions like this before. [He points to his answer in figure 9.15a.]

I: Why did you draw answer in Q6 in this specific way? (referring to figure 9.15 (b))

S5: Because then they go in the same direction and I don't have to use sine or cosine, just add the two forces.

Summary

The student seems to say that he misunderstood the instruction "add the two vectors".

He also referred in both questions set in two different contexts to vectors as forces. When describing the rule for vector addition he said: "When one ends starts the other one" and never explicitly mentioned the 'nose to tail' rule.

He mentions two procedures he knows for the addition of vectors: one of them—in the graphical mode—seems to relate to him to vectors set in a general context, which he also tried to use in the case of adding forces (figure 9.15 (a)); however in case of adding two displacements, he mentioned a different procedure, "they go in the same direction and I don't have to use sine or cosine."

The student's interview answers, shown in figure 9.14 parts (a) and (b), lack the precision but show the correct concept of the vector addition, however in part (c) the resultant has the wrong direction. He is aware of the procedure of putting vectors 'nose to tail' when adding them graphically: "When one ends start the other one." However he could not add vectors in the singular case even after realising that adding vectors meant finding the resultant, but did not have the same problem with the question is generic for forces: "I could not do it because they meet in one point, but here I could because they start at one point."

He added forces (figure 9.15 (a)) into a closed triangle, without the resultant, even although his proper magnitudes did not agree with his assumption and therefore he does not seem to understand the equivalence of vectors. In the case of displacements he used a 1-dimensional situation for simplicity as he did not want to use "sine or cosine". He never used or mentioned use of the column vectors or other symbolic methods although by mentioning the trigonometrical ratios he is obviously aware of other methods of operation on vectors. He does not seem to be flexible in dealing with different modes of operation.

Comment

The responses of student S5, written as well as verbal, show his awareness of the idea of the '*same effect'* only at a very basic level. He did not use the flexibility of the language when discussing problems of addition of vectors and his knowledge is limited to the generic cases, which means that he has a limited procedural view. He lacked precision in his drawings, and therefore had no awareness of operating on equivalent or free vectors, which might have influenced his development of conceptual ideas. He did not have symbolic knowledge to fall back on either. He seem to be a good example of what the lack of teaching proper techniques of drawing equivalent vectors can cause and how it can prevent building the concept of vector into a cognitive unit. This student does not seem to be aware of the parallelogram rule of addition and is only aware of the rule of joining the beginning of one with the end of the other and thinks that 3 forces should make a triangle so he does not understand the idea of the resultant force. He also does not give any indication of awareness of the commutative law of addition.

After the interview the student was still classified in the ***uni-modal*** category. The student answers only graphically but at a lower stage of the cognitive development.

9.3.2 Student S6

The responses of student S6 to questions 2 and 3 are presented in figure 9.16. Part (i) shows two responses, the top one to question 2 (a) and the bottom one to question 3 (a). Similarly part (ii) shows responses to questions 2 (b) and 3 (b) and part (iii) shows responses to questions 2 (c) and 3 (c). The relevant parts of the interview with the student S6 are shown below the figure 9.16. In each case he draws the resultant as a line of dashes.

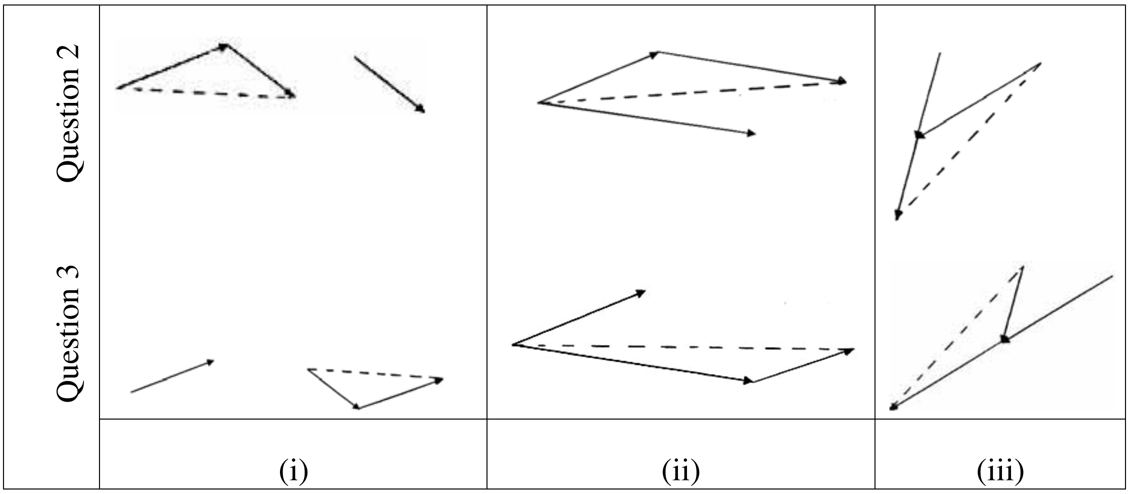


Fig. 9.16 Student S6: post-test responses to questions 2 & 3

I: Can you talk me through your answers to question 2?

S6: I can do it in two ways [indicates with the pencil translating vectors to the end of one another in two ways].

I: what about b and c?

S6: The same [indicates with the pencil translating vectors to the end of one another in two ways].

I: What about question 3?

S6: Simply moved vectors in a different order.

I: Did you think at all about answering in a different way.

S6: I could draw horizontal and vertical but it would difficult in this case.

I: If you go back to the question 2. How did you choose the direction of the arrow for the resultant vector?

S6: It is simply the direction of the two arrows together.

The next figure (9.17) shows the responses from student S6 to the two questions set in two different contexts.

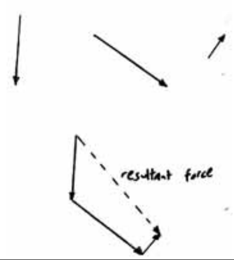
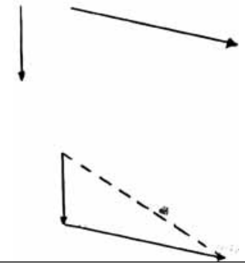
	
(a)	(b)

Fig. 9.17 Student S6: post-test responses to the different contexts questions

I: In this question 7 (figure 9.17 part a) you have to represent 3 forces and add the together, how did you do it?

S6: It is the same as adding three vectors together.

I: Why did you specifically draw vectors like this? Is this all right for forces?

S6: Well, these are just representations of forces.

I: What about question 8?

S6: First you move in this direction [points to one of the separate vectors] then in this direction [points to the other separate vector] then you add them together.

I: It does not worry you that they are not connected?

S6: We can transfer vectors anywhere.

Summary

The student S6 indicates, in the language he is using, that he treats vectors in the same way whatever the context. However looking at his graphical responses he does not put arrows on the resultant vector, even when prompted, anywhere apart from the question set in the context of forces (figure 9.17 (a)). He implies the commutative law of vector addition when he says: “Simply moved vectors in a different order”. He also implies that he realises some idea about the same *effect* by saying “It is simply the direction of the two arrows together”. However, he only mentions the direction and not the magnitude. He realises that vectors are only the “representations of forces” and displacements and indicates that he has a concept of equivalent and free vectors by saying: “We can transfer vectors anywhere”.

The student S6 seems to be focusing on the effect of the procedures and treats vectors in the same way whatever the context. He also implied (“I could draw horizontal and vertical, but it would difficult in this case") that he is aware of the symbolic mode of operation and did not use it as, according to him, it is not proper in this case. This could mean that he has flexibility in dealing with different modes of operations and uses the most appropriate one for the question.

Comment

The student S6 seem to have very good grasp of a vector as a mathematical tool for solving problems in mathematical and physical problems. He is very competent in his use of the equivalent vectors and free vectors and realises that the commutative law of addition applies to addition of vectors. What is not clear is how he understands the idea of the same *effect*. He seems to understand it in the context of forces. His resultant force has a direction but the rest of the questions seem to lack that part of the answer. It might be possible that his understanding of the addition is limited to following the arrows from the beginning to the end, after they are placed 'nose to tail'. Further questioning was not possible due to the lack of time.

The student S6 maybe could be reclassified as versatile, but the classification is unclear as he never explains why he does not place the arrows on his resultant vectors in response to the addition.

9.3.3 Student S7

The next interview was conducted with another student from group B who was coded as S7. This student was also classified to belong to the ***multi-skilled*** category according to his responses to the post-test and observation of how he was attempting the test questions (he was measuring horizontal components and angles of the vectors, adding them together and then drawing the resultant vector, which is indicated in oblongs on his responses). Some of his answers to the post-test are shown in figure 9.18. Part (i) shows two responses, the top one to question 2 (a) and the bottom one to question 3 (a). Similarly part (ii) shows responses to questions 2 (b) and 3 (b) and part

(iii) shows responses to questions 2 (c) and 3 (c). The relevant parts of the interview with the student S6 are shown below the figure 9.18.

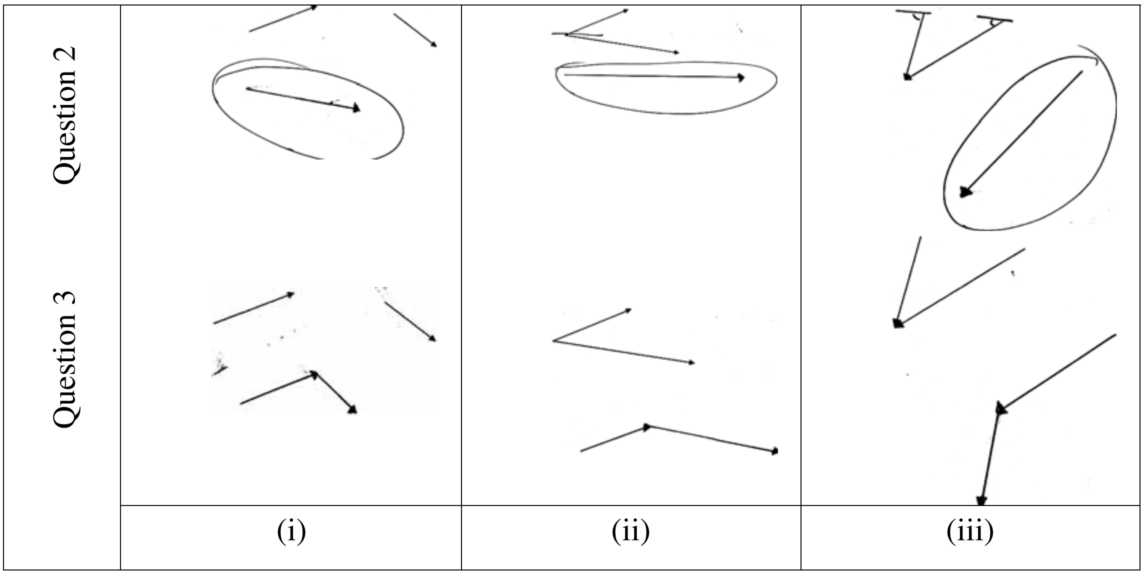


Fig. 9.18 Student S7: responses to the pre-test questions 2 & 3

I: How did you answer question 2?

S7: I used a ruler and compasses [he means protractor] to measure these vectors and then drew them here [points to the answers in oblongs].

I: Can you show me what you did?

S7: I very roughly took an angle [uses protractor to measure the angles from the horizontal direction] so it is twenty and then the other one here, which is about 40 and drew them together and measured the answer...and....[shows the resultant vector].

I: What about question 3?

S7: The problem with this question ... I did not know what you meant by "add the two vectors" so I assumed it was put them together as arrows.

I: What do you understand by the addition?

S7: Join them by end to tail and draw the arrow between them.

I: When we look at your answers to the next question you actually put them as we say ‘nose to tail’ but you did not draw an arrow?

S7: Well I did not quite know what add means so I just join them together so it shows the direction.

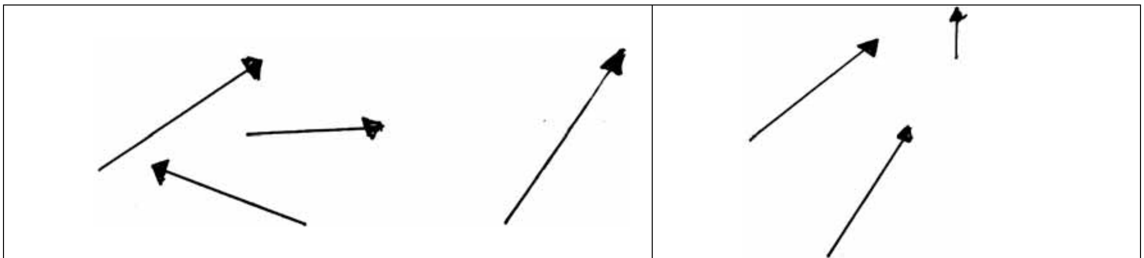
I: So do you understand addition of vectors as putting them together?

S7: Yes I understand the addition as showing the total movement.

I: So did you show the total movement in the previous question?

S7: Yes, and here I am showing the total movement but in two separate parts. If I was told to put vectors together I would draw the resultant force or whatever movement it was and the other way I just thought I would show it the other way.

Figure 9.19 shows the student’s S7 responses to two different contexts questions: part (a)- displacement; part (b)-forces.



(a) (b)

Fig. 9.19 Student S7: responses to the questions set in different contexts

I: What did you do in the question 7 and 8?

S7: Yes, the same thing.

I: Can you explain, why did you answer both questions in the same way?

S7: Apart from that there is an extra force in 7, they are exactly the same.

I: Why do you think they are the same and yet there is a different physical situation?

S7: How do you mean a different physical situation?

I: The first refers to three forces and the second to two displacement

S7: I honestly did not read it like this. Ah,…… The forces ……..are not necessarily vectors I don't think, they are movements. Whereas displacements are distance from a point, displacements and vectors are different things, the displacement is a distance from a point, which will be still the same. Sorry I will start again. These are vectors (points to the question on forces) and have magnitude and direction, whereas these are just movements from a certain point, it is just a difference, a movement, which is this one.

Summary

The student's S7 use of language when discussing problems connected with vector addition is confused, especially when he talks about addition of vectors representing the physical quantities (forces and displacements). His last response indicates it very clearly. He answered all his questions not thinking about the contexts they were set in but when this was brought to his attention he was confused.

His focuses on vectors as an action of movement ("If I was told to put vectors together I would draw the resultant force or whatever movement it was"), even when he has to deal with forces he think of movement ("The forces ……..are not necessarily vectors I don't think, they are movements").

He implies that the addition can be shown in two different ways: one when he shows the resultant "I understand the addition as showing the total movement"; and another as a journey following the route, without showing the resultant "I am showing the total movement but in two separate parts".

The context he is working in makes a difference to the way he thinks: "These are vectors and have magnitude and direction, whereas these are just movements from a certain point, it is just a difference, a movement.

He can use two modes of operation, however only at a lower stage of the cognitive development (journey) in the graphical mode.

The student was classified as *multi-skilled*; maybe he could be classified as *higher uni-modal* in the symbolic way, but this is also arguable.

Comment

The student S7 displaced great confusion in his understanding of vectors and vector addition. He has answered some further questions in a way which could indicate that he could be classified as *higher uni-modal* on the basis of all of his test responses but not on the responses shown above. His language lacks flexibility. He did not develop a concept of vector as a cognitive unit.

9.3.4 Student S8

Some students in group A were inconsistent in their graphical solutions and one of them, S8, was interviewed to find out how serious the problem was. The student was classified as ***multi-skilled***. For example, as shown in figure 9.20, the student did not draw the resultant in some general addition questions. As previously, some of his answers to the post-test are shown in figure 9.20. Part (i) shows two responses, the top one to question 2 (a) and the bottom one to question 3 (a). Similarly part (ii) shows responses to questions 2 (b) and 3 (b) and part (iii) shows responses to questions 2 (c) and 3 (c). The relevant parts of the interview with the student S6 are shown below the figure 9.20.

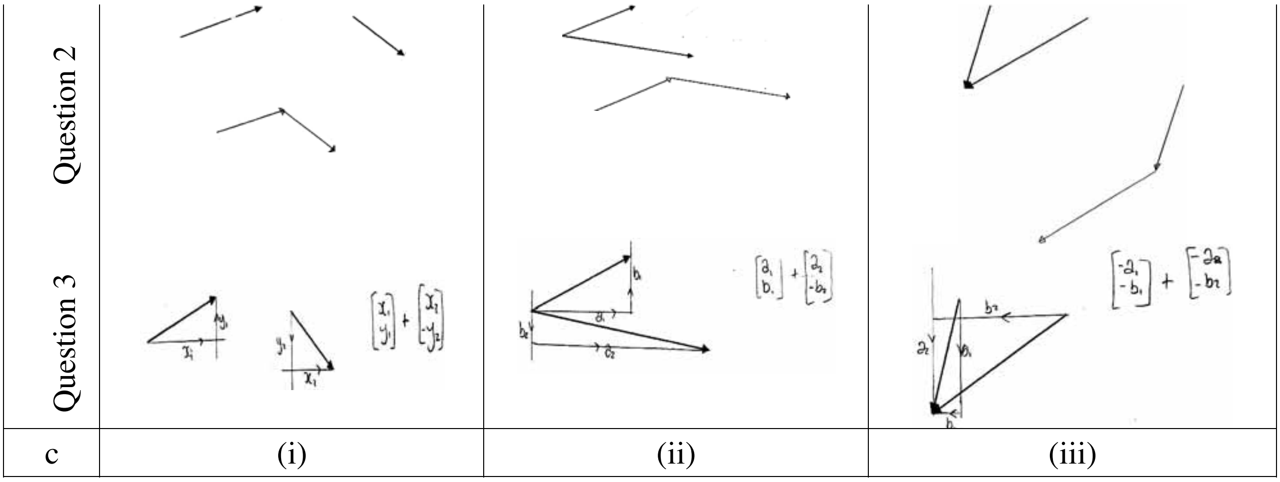


Fig. 9.20 Student S8: post-test responses to questions 2 & 3

I: When you were answering question 2 did you give me the full answer?

S8: As I was meant to. I could do it differently now. I could draw an arrow from here to here [showed the correct resultants with his hand].

I: So that would be your alternative?

S8: Yes.

I: Do you remember why you did not draw the arrows?

S8: I must have read the questions the wrong way?

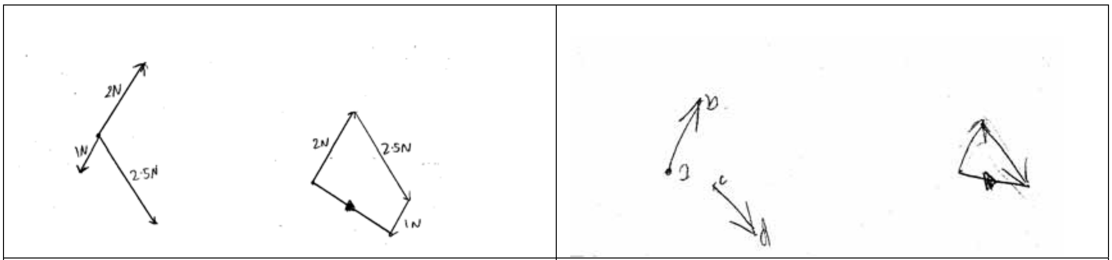
I: Can you explain a bit more?

S8: If it said show the resultant I would have drawn an arrow going from that point to that point [shows correctly with the pen]. When you added vectors you do it in a different way.

I: In the next question you showed two column vectors but you did not give the final answer, why?

S8: It is the same, because it shows the direction.

In the next part of the interview the student was asked to complete the questions on vectors in different context that he missed out in the test, therefore figure 9.21 (a) shows the questions completed during the interview. The next question (figure 9.21b) was done from the beginning during the interview.



(a) (b)

Fig. 9.21 Student S8: responses to the post-test questions set in different contexts

I: In question 7, if I had asked for the resultant, how would you draw it? (Fig. 9.21a).

S8: Should I draw it?

I: Yes please.

S8: [Student drew the correct resultant.]

I: When I asked the next question you did not answer, was there any reason for it?

S8: I don't know.

I: If I asked you to do it now, how can you do it?

S8: [student's work in Fig. 9.21b].

I: When you drew the forces you drew them from one point, the displacements you drew separately, can you explain why?

S8: I kind of relate forces acting on a point, whilst a displacement I don't know, I would see it as some type of instrument.

I: How do you see displacement?

S8: Moving an object…..across a distance.

I: If you would have to make a story what story could you make?

S8: If someone would walk from *A* to *B* and then from *C* to *D,* how far would they walk, I don’t know [points to his triangle] I don’t know. If they were together [draws them following each other] it would be easier to explain. I don’t know why I drew them separately.

Summary

This student in the same was as some of the previous students connects ‘adding vectors’ with placing them ‘nose to tail’ and nothing else, but when asked to give the resultant seems not to have a problem. He also mentions that adding vectors is showing a direction, which could imply the total effect of the addition.

The vectors set in a context of displacement, seem to make a difference to the way he was thinking, but only when he actually is prompted to think of that context in a precise way. His first impulse was to just treat displacements as any vectors. However when dealing with the context of forces he was he was very precise of the way he was thinking: “I kind of relate forces acting on a point”.

He indicated that, as long as the question asks for the resultant and not addition, he can manipulate vectors in both modes of operation, even in case of the singular questions. He therefore remains in the *multi-skilled* category.

Comment

The student, coded S8, seems to be confused in the language of vector addition. If the vectors are journeys in his mind then he might be thinking that the addition means showing whole ‘journey’ from the start to the end (figure 9.20). However this is not consistent with his other responses (figure 9.21). This student is from the experimental group and drew the displacements separately in the test (figure 9.21b) but could not remember why. So although he might have remembered from the experimental lessons that the displacement can be drawn separately he did not built a cognitive unit of a free vector from that work and therefore moved only partly towards understanding. Maybe he would have benefited from revisiting the idea a few times.

This can be triangulated with the teachers' comments. One of the teachers said that the adding vectors is putting them 'nose to tail' but because the answer is 'obvious' some of them omit showing it, which could have been one reason for omitting indicating the 'total effect'. However, in case of the student S8, the omission could be due to lack of understanding of the language rather that not realising the concept of the total effect. He seems to be connecting addition with the procedure of placing vectors 'nose to tail', and showing the resultant with the total effect of 'addition'.

9.3.5 Student S9

The student coded S9 was chosen as she was one out of six students in group A, who, in the post-test analysis, was classified into the *versatile* category. The interview meant to check whether the student is indeed flexible in her thinking and whether she just uses the procedures or whether she has a conceptual understanding of vector addition. Figure 9.22 shows the student's responses to question 2 & 3, which asked to add vectors in two different ways. Part (i) shows two responses, the top one to question 2 (a) and the bottom one to question 3 (a). Similarly part (ii) shows responses to questions 2 (b) and 3 (b) and part (iii) shows responses to questions 2 (c) and 3 (c). The relevant parts of the interview with the student S6 are shown below the figure 9.22.

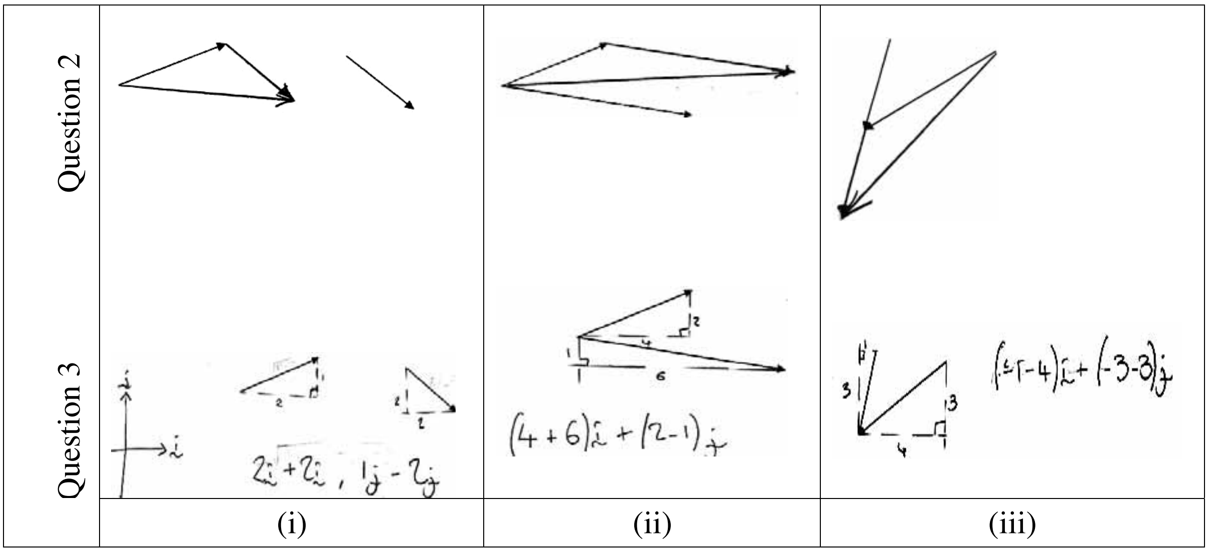


Fig. 9.22 Student S9: responses to the post-test questions 2 and 3

I: How did you answer question 2?

S9: I was sliding the vectors so one is at the end of the other one, so that they are nose to tail, and then drew a resultant from the beginning of one to the end of the second one (shows correctly for all three with her finger)

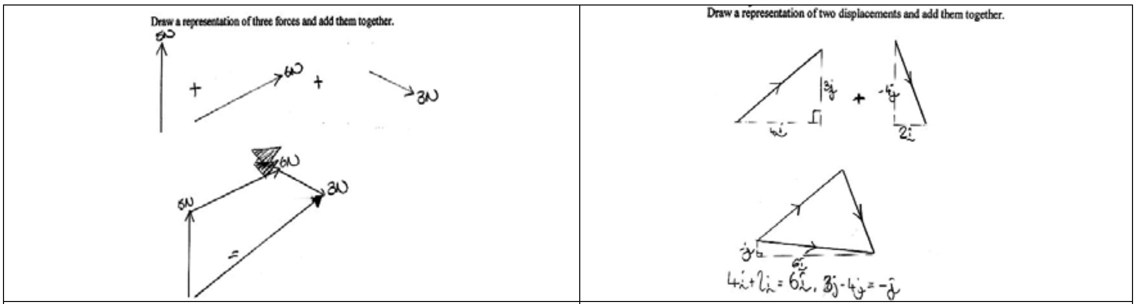
I: So what about the next question 3?

S9: I worked out the length and direction and put them together. I did them in i and j directions and added them together.

I: Did you do questions like 2 (c) before?

S9: I've never done questions like this before, so I was making my own way of doing it.

The responses from student S9 to the two questions set in two different contexts are shown in figure 9.19. Part (a) shows the student's response when being asked to draw three different forces and add them together and part (b) when being asked to draw two displacements and add them together.



(a) (b)

Fig. 9.19 Student S9: responses to the post-test questions set in different contexts

I: Could you look at questions 5 and 6? (figure 9.19 a and b)

S9: They are the same. You could do them in i and j directions and add them together or you could draw them so they are nose to tail and draw the resultant.

Summary:

The interview indicated that the student S9 reached the highest stage of cognitive development in both graphical and numerical mode. He gives enactive responses like: “slide the vector”, “put them together”, and “I drew them so the tail of one is at the beginning of that one” and after reflecting on his ‘actions’ could this through the next stage of the development The student is not confused about the language of addition: “I did them in i and j directions and added them together” or the total effect: “then I drew a resultant from the beginning of one to the end of the second one”. She was not confused by the two different contexts and said: “They are the same. You could do them in i and j directions and add them together, or you could draw them so they are nose to tail and draw the resultant.” By the same statement the student implies its own flexibility in using either modes of operation. It seems from this student’s responses that she can use the procedures very well and uses mathematical concept of vector in different contexts. In further questions in the test the student also used the commutative law of addition.

Comment:

The student S9 showed the ability to operate at the highest cognitive stage of development in both modes of operations. She showed flexibility in dealing with singular cases "I've never done questions like this before, so was making my own way of doing it." When asked to comment about two questions placed in different contexts his response was: "They are the same." which indicates that she used the mathematical concept of vector independent of the context. She used the triangle method of addition but did not mention or imply the parallelogram law. The student is still considered to be at least in the ***versatile*** category or maybe even *fully integrated*.

9.4 Summary from the interviews

By performing the triangulation between the responses from the teachers and the responses from the students we can see that, although the mathematics teachers anticipated students' perception of vector addition as thinking of compiling journeys, the students at higher levels of cognitive development did not have this problem. They developed a concept of vector as a cognitive unit which they could use flexibly.

The students working at the lower levels of the cognitive development had problems/misconceptions in their responses to both the generic and singular questions which were anticipated by the mathematics teachers. For example the student S1 answered the question in figure 9.1 (a) as if the vectors were "fixed in space." The teachers also anticipated that students might not show the resultant as "they might feel the addition means placing vectors one after another," and indeed student S1 (figure 9.2 (b)), student S7 (figure 9.18) and student S8 (figure 9.20) had that problem.

They also thought that students might have more problems with singular questions and we can see this happening in case of the student S3 who answered the simpler generic question (figure 9.8 (a)) but had a problem with more complicated generic question and the singular question (figure 9.8 (b) and (c)). The same situation occurred with student S5 (figure 9.13).

The Mathematics teachers were also saying that questions which might evoke physical implications may cause problems; this occurred with students at the lower stages of the conceptual ladder. This was especially clear in the questions set in two different contexts. Students S1, S3 before the course and S5 afterwards showed clearly that their thinking was changing dependent on the context and also with questions which might imply a physical context (two vector starting at one point, as in questions 2 (b) and 2 (c)).

On the other hand the problems anticipated by the Physics teacher that students will not be able to add vectors drawn separately occurred only in case of the student S1 and only in one question (figure 9.1 (a)). However her prediction that the students might not place arrows on the resultant vectors proved correct in many cases and one of them occurred with the student S6 (figure 9.16), who did not correct this omission even when prompted in the interview.

It seems from the interviews that the students could not communicate their problems as clearly as the teachers in anticipating the problems and therefore the interviews may not, in some cases, show the students' misconceptions or strengths clearly enough.

Generally all of the students interviewed concentrated on specific procedures, however the students working on the higher levels of the cognitive development generally made better connections between those procedures. Although different contexts affected most of their thinking, the higher level students seem to have been using a vector as a tool to solve problems of addition, showing the awareness of both aspects of a vector: magnitude and direction. The lower level students seemed to have concentrated only on one of those aspects (mainly direction) and ignored the other one (magnitude). This gives support to hypothesis 3 that the students operating at the higher levels conceive the concept of vector as a cognitive unit, as an entity in itself which can be used in different contexts.

In chapter 7, the question arose as to whether students were revealing their full ability in the separate graphic and numeric modes on the questionnaire. The students who answered questions mainly symbolically but at a higher level, as seen in the interview with the student S2, proved to be *multi-skilled* and capable of answering questions graphically at a higher level. However this does not necessarily follow with students who responded only graphically at a lower level. Students S1, S3 and S5 did not show this ability, even when prompted to respond symbolically.

There is an apparent difference in the language which the students operating at different levels of cognitive development use to describe their responses to the test questions. This difference occurred in the interviews following the pre-test as well as in the interviews following the post-test.

The lower level students use phrases like: "I connected all the vectors together so it will be easier to add them all together," (S1); I did not draw them to scale but to different scale just to give a general idea of how to add them," (S1); "you can go shorter distance from A to B," (S1); " I thought it was asking me to put them in a triangle and then join up," (S3); "I cannot think of any other way," (S3); "When one ends starts the other one," (S5); "Well, I did not quite know what add means so I just join them together so it shows the direction," (S7); "... displacements and vectors are different things...,"(S7).

The higher level students use phrases like: "I put them nose to tail," (S2); "... if you put the them end to end it will be an overall translation," (S4); "If you display each vector in two perpendicular directions, and then add two horizontal and the two vertical," (S4); "Displacements are obviously vectors," (S4); "simply moved vectors in a different order," (S6); "Well, they are just representations of forces," (S6); "I was sliding the vectors so one is at the end of the other one, so that they are nose to tail, and then drew a resultant from the beginning of one to the end of the second one," (S9); "They are the same. You could do them in **i** and **j** directions and add them together or you could draw them so they are nose to tail and draw the resultant," (S9).

From the responses we may conclude that the higher-level students were more likely to develop the concept of vector as a cognitive unit, while the lower-level students were not. This gives evidence to support hypothesis 3 from a qualitative viewpoint, giving fuller information to underline the quantitative support in chapter 7.

The interviews were consistent with the overall theoretical framework, revealing new detail. For example, when students connect vectors ‘nose to tail’, they do not use the idea of free vectors, only the procedures of joining different journeys together to show a total journey. They therefore do not add vectors in the mathematical sense. Another important aspect is that these students do not consider the parallelogram law when adding two vectors and that the triangle law has an overpowering importance with some of them to such an extent that they do not use equivalent vectors.

9.5 Overall triangulation between the interviews and the quantitative data

The interviews in this chapter revealed a consistency between the teachers’ views of the kind of difficulties that the students would have with the questions and the responses of the students. There is a clear difference in the views from physics and mathematics, where the first focuses on meaningful real-life examples which, nevertheless cause difficulties with the concept of free vector, and the latter focuses on the development of the concept within successive years of the syllabus.

The interviews in this chapter were consistent with the idea that the quantitative study satisfactorily represented the students’ performance, with the exception that some graphical questions may not show the full range of symbolic thinking that was available to the student as this was not directly required. This confirmed the decision to measure the higher stages attained by the students rather than taking a numerical average of student performance across the whole set of questions.

The language of the students at different levels revealed graphically that students who were succeeding at the higher levels regarded the notion of free vector as a coherent single concept that had meaning across different contexts whereas the students who were less successful tended to apply different procedures in different contexts. The fluent and flexible way in which the more successful students operated with the concept of free vector is consistent with hypothesis 3 and gives support the general theoretical framework. The research hypotheses formulated in chapter seven are supported by the evidence, both quantitative and qualitative.

Chapter 10
Summary and plans for the future research

10.1 Introduction

This research was designed to test if the cycle of process-object encapsulation to form the concept of free vector can be enhanced by concentrating on *effect*. The intention was to build on students' intuitions to get 'real' understanding of the vector concept and to encapsulate it as a symbol of a free vector which can be operated on as a cognitive unit.

The sample of literature from science and mathematics education showed how complicated physical intuitions can be (Aguirre & Erickson, 1984, Jagger, 1988, Graham and Berry, 1997, Dubinsky, 1991). Aguirre and Erickson found various *vector characteristics* used in different contexts and discovered that most students used partial descriptions, mainly based on intuitions related to these characteristics when describing and dealing with different physical phenomena. Their research concentrated on students' conceptions in different areas of vector quantities and suggested further investigation on the same basis. Jagger's (1988) research also concentrated on studying difficulties students had with vectors in different physical contexts. She found that the change from one dimension to two dimensions proved to be a significant problem as well as lack of understanding of the Newton's laws of motion. Graham & Berry (1997) similarly concluded that students seem to have problems with different physical concepts and Newtonian laws, which acted as an obstacle to their use of mathematics. They suggested as a remedy an approach that challenges students' 'intuitive ideas'. On the other hand, the research of Dubinsky (1991) from a mathematical viewpoint showed that the cycle of process-object encapsulation is difficult to complete, with students often reaching only the process level and failing to conceptualise the process as a mental object.

Skemp (1976) suggests that the mathematical idea should be built, not by working with several different contexts at once, but by focusing on one particular context to develop the mathematical concepts in a way that can then be applied to other contexts. In the case of constructing the mathematical concept of vector, the science education literature shows ample evidence of a range of 'false intuitions' that may arise. In choosing a specific context to work in, I chose to start, not with forces or journeys, but with the idea of physical transformations used in the mathematics text-book.

The main goal for my research is to seek a solution that enables students to reach a level where a free vector is encapsulated as a flexible mental object. The proposed solution, tested in this research, is to begin in the single context of a vector as a transformation, to focus on the *effect* of the transformation, to provide students with a focus for the construction of the concept of free vector.

To encourage students to construct meaning for themselves, in a way that is consistent with the mathematical theory of vectors, the lessons began with physical activities in which students performed the action of translating a triangle on a table. The triangle functioned as a 'base object' on which the translations acted and, by focusing on the effect of the translation, students could gain experience that any arrow of a given magnitude and direction could be used to represent a translation of that magnitude and direction. The concept of an arrow as a free vector was then made the focus of attention and the addition of 'free vectors' by moving them 'nose to tail', giving a result that has the ***same effect*** as the action of following one vector by the other. The activities looked at different ways in which the vectors could be added (for example using the triangle method or the parallelogram method) to see their equivalence.

The students' own construction of the notion of free vector was supported by activities and discussions in reflective plenary sessions. The idea of the reflective plenaries has arisen from work of Barbara Jaworski (1993) who implied that after activities in which students participate, the teacher should create the situation in which (s)he can enable them to construct meaningful concepts. This proved also to be advantageous as it linked with the idea of using plenaries in the English National Curriculum. In these plenaries, students were encouraged to build a meaningful concept of free vector as encapsulated object that they could operate on in different contexts, mathematical as well as physical.

The Preliminary Investigations helped to build the main hypothesis formulated for testing in the main study:

> **Main Hypothesis**: Teachers can help students develop the notion of a translation as a free vector through focusing on the effects of physical actions, linking graphic and symbolic representations, so that the concept of free vector is constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

This was developed from my instinctive feeling that if students were able to concentrate on the simplicity of mathematical ideas instead of the many complications connected to different contexts, they then would be in a better position to solve problems occurring in those contexts.

The goal of the research was to find a strategy that would enable students to concentrate on the simplicity of the mathematical idea of vector instead of considering difficulties and variations in different contexts using vector quantities.

After a review of relevant research (chapter 2), the research framework to be used at the outset was outlined in chapter 3.

The empirical research consisted of three stages: an initial exploration of ideas that seemed relevant in a preliminary classroom study (chapter 4); the methodology and methods to be used (chapter 5); a pilot study to test out the teaching experiment and the design and analysis of the questionnaire to produce refined hypotheses and methodology for the main study (chapter 6); the hypotheses were tested through the analysis of the results of the questionnaire in a pre-test, post-test and delayed post-test (chapter 7), to be triangulated with interviews with teachers (chapter 8); and interviews with the students (chapter 9).

10.2 Theoretical framework

The strategy evolved from the Preliminary Investigations was to work in an environment which enables students to have the potential of focusing on the essential properties. To be able to work in such situations and to move from activities to essential mathematical concepts, a fundamental focus on specific ideas has to occur, which should lead to the essential compression of knowledge. This was encouraged in two ways:

- by embodying actions and focusing on the *effect* of these actions;
- by assigning a symbol to the effect to enable it to be conceptualised as a single idea — a cognitive unit.

It was hoped that the power of this essential idea can be related to other contexts where the focus is now on the essential properties rather then the incidental details that previously caused difficulties.

In the case of vector this was done through the translation of an object on a flat table; the students were encouraged to not concentrate on the movement of the object or some particular point on that object but on the *effect* of the movement.

The movement of the particular point on the object from A to B can be represented by a particular arrow to which we can assigned a symbol $\overrightarrow{AB}$, however the essential idea is the effect of the movement which can be represented by *any equivalent arrow* having the same magnitude and direction. In this way it becomes possible to imagine that these equivalent vectors operate as a single entity that represents the more subtle concept of free vector. A bonus is that the combined effect of one free vector followed by another can be represented by placing arrows representing the vectors ‘nose to tail’ to give the sum as the single free vector that has ‘the same effect’. If we can free ourselves from the physical contexts, such vectors can be joined together in any order to give a unique result. In particular, the triangle law and the parallelogram law are two different ways of seeing *the same idea* and can be used interchangeably. The theoretical framework was design to test the hypothesis claiming that if students participated in the experimental lessons and meaningful discussions, they should be able to use vector flexibly, as mathematical symbol, and retain their knowledge for a longer period of time.

The hypotheses, discussed in detail in section 10.2, were tested through three tests. The comment on the results of the tests and interviews will be discussed in section 10.3.

10.3 Themes of the testing

Three parts of the main hypothesis, described already in chapter 7, were developed and tested. These were:

> **Hypothesis 1**: Students, who were involved in experimental lessons, are expected to rise through the cognitive stages further than students who are not exposed to the experimental lessons.
>
> **Hypothesis 2**: Students who were helped in building a concept of a free vector are expected to be more able to:
> **(a)** add vectors in singular cases, not just generic ones;
> **(b)** use free vectors independent of the context;
> **(c)** realise that the commutative law applies to vector addition.
>
> **Hypothesis 3**: Students who can concentrate on the *effect* of actions rather than actions themselves are more likely to build the concept of free vector as a cognitive unit, which can be used by students after a longer period of time and not only just after the experiment.

The interviews, as described in chapter 9, were intended to gain a greater insight into:

- students' use and flexibility of language when discussing problems connected with vector addition;

- students' focus of attention at any given time (whether it is on actions, or procedures or on the effects of those actions and procedures);
- the way in which different contexts affect their thinking;
- their flexibility in dealing with different modes of operation (graphical/symbolic).

10.4 Testing Hypothesis

The three hypotheses were tested three times: in the Preliminary Investigations, which helped to build the methodology; the pilot study which tested the methodology; and the main study, which proved that there were significant positive changes in the experimental group, compared to no significant changes in the control group. All three studies indicated positive change in students who have undergone the experimental lessons. During the main study, two groups of students were tested three times throughout year 12. The first test (pre-test) was conducted at the beginning of the year. The second test (post-test) was conducted one month after the part of the Mechanics course involving addition of forces finished and two months after the pre-test. The third test (delayed post-test) was conducted a year after the pre-test, when students came back from their summer holidays. They were analysed using methods developed in chapter 4 and detailed in chapters 5, 6 and 7.

The significance of the changes in the stages of the cognitive development that were achieved by students, was determined using the two-tail t-test. The other comparison was done using the scatter graphs and the chi-squared test. This test looked at the comparison of the proportions of students in two different areas of the graph: lower lever area included *intuitive* and *uni-modal* categories; and the higher level area included *higher uni-modal*, *multi-skilled*, *versatile* and *fully integrated* categories. The t-test taken for the graphical changes between the post-test T1 and the delayed post-test T3 show highly significant changes for Group A ($t=3.83$ at $p<0.01$) and no significant changes for Group B. The changes for the symbolic mode were not significant for either group. When we triangulate the overall responses to all three tests and teachers' comments together with the students' interview responses, we can see that only some students at the beginning of the year even considered answering the questions graphically. The students seemed to realise that the test (without grids) was answered more efficiently graphically and attempted mainly to do so even if their graphical competence was not adequate to do so. The symbolic answers were given mainly as alternative responses but rarely as main responses. This is in all probability the reason why in all the tests the changes in the symbolic responses for both groups were not significant.

These results provide evidence for hypotheses 1 and 3. The students who were involved in the experimental lessons rose through the cognitive stages further than students who were not exposed to the experimental lessons and their conceptual understanding worked after a longer period of time and not just immediately after the experiment.

Differences occurred in the case of singular (hypothesis 2 (a)) questions where, for example, students from Group A were able to cope better with two vectors meeting at one point. The t-tests performed on students' changes in the stages of the graphical cognitive development between the pre-test and the delayed post-test show that Group A underwent highly significant positive changes ($t=3.13$ at $p<0.01$) while the changes in Group B were not significant. These results support further hypothesis 3 that Group A students' conceptual knowledge of vector addition was more firm by the time of the delayed post-test and they could apply it more flexibly, even in the singular cases. The chi-squared test showed there was no significant difference between the two Groups in the delayed post-test. However if we consider that there was a significant difference with $\chi^2 = 4.24$ ($p<0.05$) in the pre-test in favour of Group B and in the delayed post test χ^2 changed to 2.32 in favour of Group A, we can see that the positive change has occurred in favour of Group A. In fact Group B has not changed and only Group A has.

The highly significant positive changes also occurred in case of Group A when responding to the questions set in two different contexts ($t = 8.71$ at $p<0.01$) The Group B also improved but less significantly ($t = 2.17$ at $p<0.05$). The chi-squared test also shows a significant difference which favoured Group B in the pre-test ($\chi^2 = 5.24$ at $p<0.05$) to the significant difference which this time favoured Group A in the delayed post-test ($\chi^2 = 4.84$ at $p<0.05$).

This shows that, on the whole, Group A made much more significant improvement than Group B in their stages of cognitive development as far as the singular questions and the different contexts questions are concerned. From the students' post-test and the delayed post-test responses it also became evident that students in Group A treated these questions in a more 'mathematical' way. The substantial number of them used their knowledge of free vectors in addition with confidence.

These results support hypothesis 2 (a), that the experimental Group A, in comparison with control Group B, gained conceptually from the experimental lessons in the context of vector as force and sustained their knowledge between the post-test and the delayed post-test. The difference between the groups changed from Group B being significantly higher in the pre-test to Group A being significantly higher in the delayed post-test. It is relevant that there was no significant difference between the groups in the post-test, which emphasises the long-term effect of the experimental treatment.

In addition to the results from the quantitative analysis the interviews also showed that there is an apparent difference in the language which the students operating at different levels of cognitive development use to describe their responses to the test questions (chapter 9). The analyses indicate that students operating at lower cognitive levels use procedures without using the concept of the free vector, while the students operating at higher cognitive levels developed the concept of vector as a cognitive unit. This gives the additional qualitative support to hypothesis 3.

For example one of the high attaining students from Group A, when asked how he tackled the singular case, in which two vectors met at one point, he said: "I was sliding the vectors so one is at the end of the other, so that they are nose to tail, and then drew a resultant. [...] I worked out the length and direction, did them in i and j direction and added them together." When asked how he answered two questions set in different contexts (forces and displacement), he answered: "They are the same. You could do them in the **i** and **j** direction and add the together, or you could draw them so they are nose to tail and draw the resultant." When asked how he approached the singular question at the end of the test, which many students found difficult even to start, he responded: "I didn't know how to do this and this was like a second thought [...] I was making my own way of doing it." His answers show that he has built a cognitive unit, which he was confident to apply to an unfamiliar situation. He also implied that every time he looked at the *result* of the addition, which meant that he concentrated on the *effect* of the addition, in both symbolic and graphical mode.

On the other hand another student, this time from Group B, who just put two vectors together but did not draw the resultant, when asked how he went about answering the question responded: "I did not know what you meant by 'add the two vectors', so I assumed it was put them together as arrows." He obviously concentrated on the procedure of addition and not on the *effect* of it. When asked to explain a bit more what he understood by addition, he answered: "I understand the addition as showing the total movement." When asked how he tackled two questions, one asking him to draw and add three forces and another to draw and add two displacements, he answered: "Apart from the fact that there is an extra force in the first one, they are exactly the same." He concentrated in both situations as if they applied to forces and answered them in that context, but not as free vectors in a mathematical context, in the manner that the previously described student did. When asked if he noticed the contexts are different he said: "Ah […] the forces are not necessarily vectors I don't think they are movements." He related to vectors as movements but since he knew that forces acting on an object do not have to cause a movement, he therefore did not think that forces are vectors. He did not build a notion of vector into a cognitive unit. To him it was a different symbol when used in the different contexts of forces and journeys. There was no indication in the test that he knew that the addition of vectors is commutative.

10.5 Summary of testing theory

The quantitative analysis in Chapter 7, firmly confirm the main hypothesis by providing statistical evidence to support hypothesis 1, 2 and 3. The first part of the qualitative analyses (chapter 8) shows that the teachers clearly understand the kind of mistakes that the students might make, particularly the two mathematics teachers. The physics teacher differs, probably because she was looking at how students would respond in the Physics context, while the Mathematics teachers were considering how students would respond in Mechanics context. Students might try to adapt their responses to the subject they have to operate in, as they might try to respond in they way they think the teacher wants them to respond.

Certain things came clear which the original theory did not consider explicitly, for example when we look at the way some students add two vectors together ($\overrightarrow{AB}$ and $\overrightarrow{BC}$), they move the beginning of the vector $\overrightarrow{BC}$ to the end of the vector $\overrightarrow{AB}$ and leave it. They treat this addition as showing the journey from A to C via B and therefore as far as they are concerned the task is completed. They see it as showing a journey and not as vector addition using equivalent vectors. To triangulate this with the teachers' comments: one teacher says that "they forget to put the arrow in," however it is possible that they do not forget but they just are not at the level where they understand the purpose of putting the arrow in. If they are told to remember it may not help them to understand the concept but might help them to get the good marks in the exam.

The theory developed in this thesis, on the other hand, suggests the alternative to warning students what to do or not to do. It says that if we involve them in specific physical actions on vectors and mentor them in reflecting on the effect of these actions (correspond to the idea of free vector), then the students will have some personal experience, from which they will be able to sense what it means to move the free vectors around. If they afterwards see the vectors on the paper they are more likely to be able to imagine moving them around. The other students might just have the experience of being told to place them one after another and place another arrow from the beginning of the first one to the end of the second one, without developing a concept of a free vector. These two approaches to teaching might have the effect of how long students remember how to solve vector problems. From the test results it shows that the embodied approach followed by reflecting on actions helps the teaching to have more permanent effect on what students remember. It may very well be that the non-verbal action of physical movement and the sensory and visual effect of this movement is more deeply entrenched in their personal psyche. Thus, over the long term, it fits more naturally with their thinking processes and is enhanced as time goes by. It may also be that, being non-verbal, the students find that they can 'do' the operation naturally and successfully, and yet, when interviewed, they may not be fully able to verbalise what they are doing.

Chapter 8 shows also another discrepancy with the theoretical framework developed in the Preliminary Investigations, which involves use of the parallelogram law of addition. The theoretical framework suggested initially that forces were added using the parallelogram rule of addition and displacements were added using the triangular rule, although it might be true, in this case, the students seem to deal so much with the individual vectors that parallelogram law simply did not occur to them.

Chapter 9 confirmed that the categorisation of students developed in chapter 4 was satisfactory and did not need have to be modified. However the interviews show the wide difference between the language used by students working at the lower and higher cognitive levels. The students working at the higher cognitive level use language, which suggest they deal with some kind cognitive unit. They may not be eloquent in the way they express it but the language they use is more powerful than the language which the students working at the lower stages use. The experimental lessons and reflection on actions were intended to move students to thinking about vector as a cognitive unit which in turn would allow them to be more flexible when using it.

10.6 Limitation of the study

The question arises as to whether the change is due to the teacher or the method. The study was done in one classroom in which I participated myself. It would be interesting to see if it could be repeated in another classroom with another teacher.

There were also practical limitations. Many students who would have been interesting to study further and could have given an interesting insight into some answers were not available for interviews. The groups were not well balanced as they started with different levels of the cognitive development. Three quarters of each group also studied Physics and it was difficult to assess the influence which the teaching of Physics had on the students' changes.

10.7 Directions for future research

As far as present research is concerned, the way of teaching students by focusing on the 'effect' of actions needs to be established in a school and tested with a wider range of students. If the premise is true that the use of non-verbal physical actions improves the students' sense of meaning, then, given the different views expressed by the teachers, it is important to discuss this aspect with them in a way that helps them too to gain an insight into the process. It is also important to discuss with them the language used in lessons and its meaning, to refine it and to improve the clarity of communication with all students at different stages of development. These suggestions should be the object of future research.

In general there is need for more research of the theory relating to embodiment and the symbolic compression. Some researchers (e.g. Pinto, 1998) have found that some students construct their ideas from their personal concept images while others do so from formal definitions and the structure of formal theorems. In the present research it was noted that some students, at the beginning of the course, were already at the highest stages of cognitive development of the concept of vector and had built the cognitive units themselves from the theory given in the earlier education. These students were successful without having any exposure to the embodied approach in the experimental Group. This suggests that, although an embodied approach may be useful to give overall statistical improvements in the class as a unit, there needs to be continued research into the needs of students who may think in different ways.

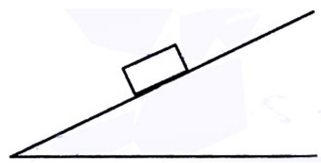
The notion of ‘effect’ of actions on base objects has applications in the construction of mathematical concepts encapsulated from processes. For example, the idea of two different actions having the same effect arises in a wide range of areas that are often interpreted in terms of an equivalence. For instance, equivalent fractions are different sharing procedures with the same effect, equivalent algebraic expressions are different procedures of evaluation with the same effect, and so on. A major line of research is to investigate the use of the focus on ‘effect’ in giving cognitive meaning to such mathematical concepts.

10.8 Reflecting on the effect of the study

In this thesis an approach was developed to make the transition from thinking of embodiments to manipulating symbols through the pivotal notion of *effect*. The results of the study revealed that there were significant changes in the class of students who followed this programme, in which they were more likely to conceive of the symbols for vectors as cognitive units that they could manipulate in a flexible and versatile manner. It is hoped that this fundamentally simple idea will be of use in improving the practical way in which teachers teach and students learn, not only in considering vectors but in every context where the effect of mathematical actions are represented by manipulable symbols.

Epilogue

Having completed this research I found it of value to return to the source of my original inspiration. The opening of this thesis referred to my increasing concern that students seemed to be able to learn to perform techniques to score highly on examinations, yet seem not to be able to apply their knowledge to slightly different situations, nor to retain their skills for ready use in subsequent courses. One particular question seemed to symbolize this problem that was used in the Preliminary Investigations, but did not feature in the main study. As the writing of this thesis came to a close, I decided to revisit the problem to see if my theoretical approach had made any long-term difference, not only to the concept of free vector, but also to the application of the ideas in other contexts such as mechanics.

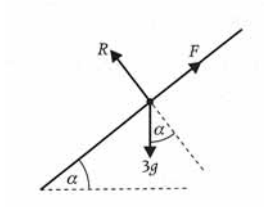


(draw the forces and resolve them parallel and perpendicular to the plane)

Fig. 10.1 Question on forces (a slope)

Most of our students could successfully resolve forces horizontally and vertically and solve problems using this technique, but they had serious problems in drawing the forces involved when a rectangular block was placed on an inclined plane (figure 10.1).

The analysis in this thesis suggests that students who approached vectors from an embodied viewpoint, focusing on the effect of translations to construct the notion of free vector in a meaningful way, would be able to build a mathematical concept that they could use in other contexts. I decided to give a variant of the original problem to several different groups of students, a year and a half after the students involved in the research had finished their course on vectors and half a year after their exams (figure 10.2).



A particle of mass 3 kg slides down a rough plane at an angle α = 30° to the horizontal. If $\mu = 0.5$ find the acceleration of the mass.

Fig. 10.2 Revisiting the original problem

The question was given to four groups in all. Two were the groups who had been involved in the Main Study:

> ***Group A** who had been given the experimental treatment in Year 12 and were now at the end of 13,*
>
> ***Group B** who had been given the standard treatment in Year 12, but subsequently had revised the work with me in Year 13 including two plenary sessions.*

Two other Groups were also included from Year 12:

> ***Group C**, in year 12 who were taught by a teacher who had been interviewed as part of the research and had shown interest in the ideas I had used and had adopted the techniques in her own teaching it.*
>
> ***Group D,** in Year 12, taught by a teacher who was not involved with the research.*

Every student in Group A answered correctly**. **All students in Group B except three answered correctly**. The three who answered incorrectly made the error of omitting the parallel component of weight (as in happened in the Preliminary Investigations). When I checked my register, I realised that **all three who made errors were absent for the experimental revision lessons.

In Group C**, taught by a teacher aware of the experimental technique, **four out of six students answered correctly** while two missed the parallel component of weight in their calculations. This supports the idea that the method may be used successfully by other teachers. However, **in Group D, only one out of eleven students answered

correctly *while the rest of them missed the parallel component of weight in their calculations.*

The data is gratifying. It shows that the class of students who were taught in the standard way continued to have difficulty with the resolution of forces with only one out of the whole class responding correctly. Meanwhile, almost all of the students who used the focus on effect, even for a short time, conceptualized the forces correctly several months after the lessons were given. They had not only conceptualized the idea flexibly, they had retained the ideas after the passage of time.

Appendix

Main Test Quantitative Data for Individual Students

Students' cognitive development of vector																	
T1: Pre-Test graphical responses																	
Student		**1**	**2a**	**2b**	**2c**	**3a**	**3b**	**3c**	**4**	**5**	**6**	**7a**	**7b**	**7c**	**8a**	**8b**	**8c**
1	A	3															
2	A	3	4	4	4	1	1	1	1	1	1	1	0	0			
3	A	3										3	3				
4	A	3	3	0	0							1			1	1	1
5	A	2									1	2	0	0	0	0	
6	A	2										2	2	0	0	0	0
7	A	2							3		1						
8	A	3										2					
9	A																
10	A		2	2	2												
11	A					4	4	4		0		2	0	0	0	0	
12	A											2	4	4	0	0	0
13	A	2	4	4	4				4	4		2	0	0			
14	A	3										2	0	0	0	0	0
15	A	3	4									2	4	0	0	4	4
16	A																
17	A																
1	B	2	4	1	1							2	4	1			
2	B		0	0	0							2	3	4	0	0	3
3	B	1										2	1	4	0	0	
4	B	0	4	0	0	3	0	0	3	1	1						
5	B		4	4	4							2	3	4	2		
6	B		0	0	0	0	0		0	0							
7	B	2	0									2	0	0	0	0	0
8	B	1	1	1								2	4	0	0	0	0
9	B	1*	4	4	4					2		2	3	4	0	4	4
10	B											2	3	4			
11	B					0	0	0				2	3	3	0	0	0
12	B	0	0	1	1	0	1	1	0			2	0	0	0	0	0
13	B	0	0	0	0	0	0	0	0	0	0						
14	B	1		4	4				3	1	1	2	3	3	0		
15	B	0							0								
16	B	3	4	4	4	4	4	4	4	4	4	2	0	4			
17	B	1	4	4	4	4	4	4	4	4	4	2	4	4	3	3	3

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