1. (15 points) Give an algebraic proof that a straight line in the world projects onto a straight line in the image. In particular

(a) Write the parametric equation of a line in three-space.

$x = x_0 + x_1t$

$y = y_0 + y_1t$

$z = z_0 + z_1t$

(b) Use the simplest form of the perspective projection camera from the start of the Lecture 5 notes to project points on the line into the image coordinate system. This will give you equations for the pixel locations x and y in terms of t. Note that t will be in the denominator.

Simplest form of perspective projection:

$u = fx/z$ $\qquad$ $v = fy/z$

After substitution:

$u = f(x_0 + x_1t) / z_0 + z_1t$

$v = f(y_0 + y_1t\ ) / z_0 + z_1t$

(c) Combine the two equations to remove t and rearrange the result to show that it is in fact a line. You should get the implicit form of the line.

Remove t:

$u = f(x_0 + x_1t) / z_0 + z_1t$

$u(z_0 + z_1t) = f(x_0 + x_1t)$

$uz_0 + uz_1t = fx_0 + fx_1t)$

$uz_1t - fx_1t = fx_0 - uz_0$

$t(uz_1 - fx_1) = fx_0 - uz_0$

$t = (fx_0 - uz_0) / (uz_1 - fx_1)$ Replace x for y and u for v => $t = (fy_0 - vz_0) / (vz_1 - fy_1)$

Set two equations equal to each other:

$(fx_0 - uz_0) / (uz_1 - fx_1) = (fy_0 - vz_0) / (vz_1 - fy_1)$

Cross Multiply:

$(fx_0 - uz_0)\ (vz_1 - fy_1) = (fy_0 - vz_0)\ \ (uz_1 - fx_1)$

$fx_0vz_1 - f^2x_0y_1 - uz_0vz_1+uz_0fy_1 = fy_0\ uz_1 - f^2y_0x_1- vz_0uz_1 + vz_0fx_1$

Group U and V and simplify:

$u(z_0fy_1 - fy_0z_1\ ) + v(fx_0z_1 - z_0fx_1\ )= f^2(x_0y_1-y_0x_1\ )$

Implicit Form:

$u(z_0fy_1 - fy_0z_1) + v(fx_0z_1 - z_0fx_1) - f^2(x_0y_1 - y_0x_1) = 0$

(d) Finally, under what circumstances is the line a point? Show this algebraically.

The line is a point when all x,y,z values result in the same u and v values, meaning the line results in a single point. So regardless of t values in the initial projection equations:
$f(x_0 + x_1t) / z_0 + z_1t = \text{constant}$
$(x_0 + x_1t) = c(z_0 + z_1t)$
$(x_0 + x_1t) = cz_0 + cz_1t$

Bring t term together:
$x_1t - cz_1t = cz_0 - x_0 = \text{constant}$
$t(x_1 - cz_1) = c_1$
$x_1 - cz_1 = 0$
$c = x_1/z_1$ apply the same logic for v => $c = y_1/z_1$

$cz_0 - x_0$ must equal zero to maintain value regardless of t value
$(x_1/z_1)z_0 - x_0 = 0$
$z_0x_1/z_1 = x_0$
$z_0x_1 = x_0z_1$ apply the same logic for v => $z_0y_1 = y_0z_1$

If $z_0x_1 = x_0z_1$ and $z_0y_1 = y_0z_1$ then (u,v) will map on to the same point