Tutorial: Image Filtering using PDEs

Note

This tutorial aims to process images with the help of partial differential equations.

The different processes will be applied on the following MR image Fig. 1.

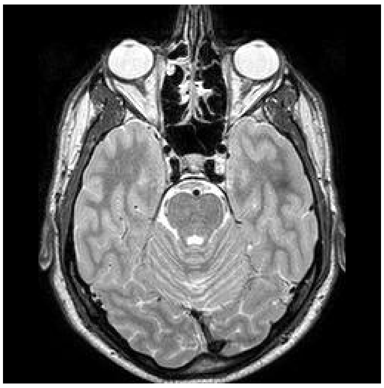

Figure 1: MRI image of a human brain.

Notations

The different operators used here are:

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} \tag{1}$$

$$\operatorname{div} \overrightarrow{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \tag{2}$$

$$\vec{\operatorname{grad}}\, u(x,y) = \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial y}{\partial y} \end{pmatrix} \tag{3}$$

The Laplacian operator is denoted $\Delta u = \nabla^2 u = \operatorname{div} \overrightarrow{\operatorname{grad}}\, u$. A numerical approximation can be used, but is not recommanded in this tutorial.

1 Linear diffusion

The heat equation is defined as follows:

$$\left\{\begin{array}{rcl} \frac{\partial u}{\partial t}(x,y,t) & = & \mathrm{div}(\nabla u(x,y,t)) \\ & = & \frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t) \\ u(x,y,0) & = & f(x,y) \end{array}\right. \tag{4}$$

If $f(x,y)$ is an image (with $(x,y) \in D \subset \mathbb{R}^2$, D is the spatial support), this defines a filtering method that is equivalent to the convolution of f by a Gaussian function [2]. This equation can be solved by a finite difference numerical method.

1.1 Numerical scheme

The following notations are employed. N, S, E, W stand for north, south, east and west.

$$\begin{array}{rcl} +\delta^N u & = & \frac{u(x,y+h)-u(x,y)}{h} \\ -\delta^S u & = & \frac{u(x,y-h)-u(x,y)}{h} \\ -\delta^E u & = & \frac{u(x-h,y)-u(x,y)}{h} \\ +\delta^W u & = & \frac{u(x+h,y)-u(x,y)}{h} \end{array} \tag{5}$$

Thus, the numerical scheme is (h has value 1):

$$\frac{u^{t+1} - u^t}{\delta t} = \frac{1}{h}\left\{\delta^N u - \delta^S u + \delta^E u - \delta^W u\right\} \tag{6}$$

?

1. Code the numerical scheme. It should use two parameters: the number of iterations n, and the step time δt.
2. Filter the original image by varying the parameters of the discrete diffusion process.
3. Comment the filtering results. Compare the results with the Gaussian filter.

2 Nonlinear diffusion

Image filtering by nonlinear diffusion reduces noise in a controled way. The diffusion coefficient is locally adapted, becoming negligible as object boundaries are approached.
The heat equation is replaced by the following PDE:

$$\left\{\begin{array}{rcl} \frac{\partial u}{\partial t}(x,y,t) & = & \mathrm{div}\left(c(\|\nabla u(x,y,t)\|) \cdot \nabla u(x,y,t)\right) \\ u(x,y,0) & = & f(x,y) \end{array}\right. \tag{7}$$

with c the diffusion fonction satisfying the following properties:

- $c(0) = 1$,
- $\lim\limits_{s \to +\infty} sc(s) = 0$,
- $\forall s > 0, c'(s) \leq 0$.

Perona and Malik have proposed 2 diffusion coefficients [4, 1]:

- $c_1 : \exp\left(-\left(\frac{s}{\alpha}\right)^2\right)$,
- $c_2 : \dfrac{1}{1 + \left(\frac{s}{\alpha}\right)^2}$.

2.1 Numerical scheme

The numerical scheme is the following:

$$\begin{aligned}\frac{u^{t+1} - u^t}{\delta t} = \frac{1}{h^2} \{ & c(|\delta^N u|) \cdot \delta^N u + c(|\delta^S u|) \cdot \delta^S u \\ & + c(|\delta^E u|) \cdot \delta^E u + c(|\delta^W u|) \cdot \delta^W u \}\end{aligned} \tag{8}$$

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1. Code the numerical scheme
2. Filter the original image by varying the parameters of the discrete nonlinear diffusion process.
3. Comment and compare the filtering results with the previous scheme.

3 Degenerate diffusion

The following degenerate PDEs have been shown to be equivalent to the morphological operators of dilation and erosion (using a disk as structuring element, see tutorial on mathematical morphology):

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t}(x, y, t) & = & +\|\nabla u(x, y, t)\| \\ u(x, y, 0) & = & f(x, y) \end{array} \right. \tag{9}$$

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t}(x, y, t) & = & -\|\nabla u(x, y, t)\| \\ u(x, y, 0) & = & f(x, y) \end{array} \right. \tag{10}$$

3.1 Numerical schemes

The previous numerical scheme is easy to find, but the results present large differences with the dilation and erosion operators (shocks due to discontinuities in the original image). This is why the following numerical scheme is preferred [3]:

For the erosion:

$$\frac{u^{t+1} - u^t}{\delta t} = \frac{1}{h^2} \sqrt{\begin{aligned} &max^2(0, -\delta^N u) + min^2(0, -\delta^S u) \dots \\ &\qquad + max^2(0, -\delta^W u) + min^2(0, -\delta^E u) \end{aligned}} \tag{11}$$

For the dilation:

$$\frac{u^{t+1} - u^t}{\delta t} = \frac{1}{h^2} \sqrt{\begin{aligned} &min^2(0, -\delta^N u) + max^2(0, -\delta^S u) \dots \\ &\qquad + min^2(0, -\delta^W u) + max^2(0, -\delta^E u) \end{aligned}} \tag{12}$$

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1. Code the numerical scheme.
2. Filter the original image while varying the parameters of the discrete degenerate diffusion process.
3. Comment the morphological filtering results and compare the numerical scheme to the approach with operational windows.

References

[1] Francine Catt, Pierre-Louis Lions, Jean-Michel Morel, and Tomeu Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM Journal on Numerical Analysis*, 29(1):182–193, 1992. 3

[2] JanJ. Koenderink. The structure of images. *Biological Cybernetics*, 50(5):363–370, 1984. 2

[3] P. Maragos and M. Akmal Butt. Partial differential equations in image analysis: continuous modeling, discrete processing. In *Image Processing, 1996. Proceedings., International Conference on*, volume 3, pages 61–64 vol.3, Sep 1996. 4

[4] P. Perona and J. Malik. Scale-space and edge detection using anisotropic diffusion. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 12(7):629–639, Jul 1990. 3