Tutorial: Integral Geometry

This tutorial aims to characterize objects by measurements from integral geometry.

The different processes will be applied on the following synthetic image:

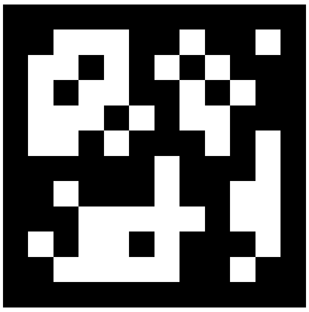
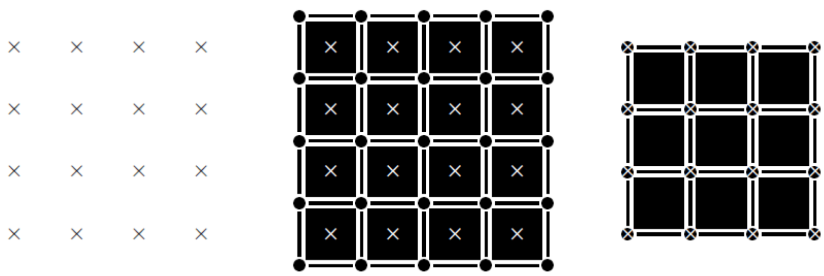


Figure 1: X

1 Cell configuration

The spatial support of an image can be covered by cells associated with pixels. A cell (square of interpixel distance size) is composed of 1 face, 4 edges and 4 vertices. Either a cell is centered in a pixel (intrapixel cell) or a cell is constructed by connecting pixels (interpixel cell). The following figure shows the two possible representations of an image with 16 pixels:



(a) Image. (b) Intrapixel cells. (c) Interpixel cells.

Figure 2: Pixels representation.

We respectively denote f^{intra} (resp. f^{inter}), e^{intra} (resp. e^{inter}) and v^{intra} (resp. v^{inter}) the number of faces, edges and vertices for the intrapixel (resp. interpixel) cell configuration

	0 0 0 0	0 0 0 1	0 1 0 0	0 1 0 1	0 0 1 0	0 0 1 1	0 1 1 0	0 1 1 1
α	0	1	2	3	4	5	6	7
	1 0 0 0	1 0 0 1	1 1 0 0	1 1 0 1	1 0 1 0	1 0 1 1	1 1 1 0	1 1 1 1
α	8	9	10	11	12	13	14	15

Table 1: Neighborhood configurations.

Using these two configurations, the measurements from integral geometry (area A, perimeter P, Euler number χ_8 or χ_4) can be computed as:

$$\begin{aligned} A &= f^{intra} = v^{inter} && (1) \\ P &= -4f^{intra} + 2e^{intra} && (2) \\ \chi_8 &= v^{intra} - e^{intra} + f^{intra} && (3) \\ \chi_4 &= v^{inter} - e^{inter} + f^{inter} && (4) \end{aligned}$$

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1. Count manually the number of faces, edges and vertices of the image X for the two configurations (Intra- and Inter-pixel) of Fig. 1.

2. Deduce the measurements from integral geometry (Eq. 1-4).

2 Neighborhood configuration

In order to efficiently calculate the number of vertices, edges and faces of the object, the various neighborhood configurations (of size 2x2 pixels) of the original binary image X are firstly determined. Each pixel corresponds to a neighborhood configuration α. Thus, sixteen configurations are possible, presented in Tab. 1.

Thereafter, each configuration contributes to a known number of vertices, edges and faces (Tab. 2). To determine the neighborhood configurations of all the pixels, an efficient algorithm effective consists in convolving the image X by a mask F, whose values are powers of two, and whose origin is the top-left pixel:

$$F = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$$

The resulting image is $X * F$. Notice that this image X has no pixel touching the borders, your code should ensure this (for example by padding the array with zeros). In this way, the histogram h of $X * F$ gives the distribution of the neighborhood configurations from image

X. And each configuration contributes to a known number of vertices, edges and faces:

$$v = \sum_{\alpha=0}^{15} v_\alpha h(\alpha) \tag{5}$$

$$e = \sum_{\alpha=0}^{15} e_\alpha h(\alpha) \tag{6}$$

$$f = \sum_{\alpha=0}^{15} f_\alpha h(\alpha) \tag{7}$$

The following table gives the values of v_α, e_α and f_α for each cell configuration:

intrapixel cells																
α	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_α	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
e_α	0	2	1	2	1	2	2	2	0	2	1	2	1	2	2	2
v_α	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
interpixel cells																
α	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_α	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
e_α	0	0	0	1	0	1	0	2	0	0	0	1	0	1	0	2
v_α	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 2: Contributions of the neighborhood configurations to the computation of v, e and f

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1. Compute the distribution of the neighborhood configurations from image X.
2. Deduce the number of vertices, edges and faces for each cell representation and compare these values with the previous (manually computed) results.

3 Crofton perimeter

The Crofton perimeter could be computed form the number of intercepts in different random directions. In discrete case, only the directions $0, \pi/4, \pi/2$ and $3\pi/4$ are considered; they are selected according to the desired connexity.

The number of intercepts are denoted $i_0, i_{\pi/4}, i_{\pi/2}$ and $i_{3\pi/4}$ for the orientation angles $0, \pi/4, \pi/2$ and $3\pi/4$ respectively.

In this way, the Crofton perimeter (in 4 and 8 connexity) is defined in discrete case as:

$$P_4 = \frac{\pi}{2}\left(i_0 + i_{\pi/2}\right) \tag{8}$$

$$P_8 = \frac{\pi}{4}\left(i_0 + \frac{i_{\pi/4}}{\sqrt{2}} + i_{\pi/2} + \frac{i_{3\pi/4}}{\sqrt{2}}\right) \tag{9}$$

These perimeter measurements can be computed from the neighborhood configurations of the original image:

$$P_4 = \sum_{\alpha=0}^{15} P_\alpha^4 h(\alpha) \tag{10}$$

$$P_8 = \sum_{\alpha=0}^{15} P_\alpha^8 h(\alpha) \tag{11}$$

with the following weights P_α^4 and P_α^8 of these linear combinations (Tab. 3):

α	0	1	2	3	4	5	6	7
P_α^4	0	$\frac{\pi}{2}$	0	0	0	$\frac{\pi}{2}$	0	0
P_α^8	0	$\frac{\pi}{4}\left(1+\frac{1}{\sqrt{2}}\right)$	$\frac{\pi}{4\sqrt{2}}$	$\frac{\pi}{2\sqrt{2}}$	0	$\frac{\pi}{4}\left(1+\frac{1}{\sqrt{2}}\right)$	0	$\frac{\pi}{4\sqrt{2}}$
α	8	9	10	11	12	13	14	15
P_α^4	$\frac{\pi}{2}$	π	0	0	$\frac{\pi}{2}$	π	0	0
P_α^8	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4\sqrt{2}}$	$\frac{\pi}{4\sqrt{2}}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	0	0

Table 3: Weights for the computation of the Crofton perimeter

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Evaluate the perimeters P_4 and P_8 with the previous formula on Fig.1