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ITERATIONS OF A NON-LINEAR TRANSFORMATION FOR ENHANCEMENT OF DIGITAL IMAGES

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Abstract—A novel non-linear transformation for sharpening digitized gray-level images is defined. The transformation replaces the gray level value at a point by either the minimum or the maximum of gray levels in its neighborhood, the choice depending on which one is closer in value to the original gray level. It is shown that after a finite number of iterations, the image stabilizes to one in which every point is either a local maximum or else a local minimum. The ability of several iterations of the transformation to restore white–black crispness in a fuzzy image is shown both analytically and by example.

Image enhancement Functions on graphs Line drawing reconstruction

INTRODUCTION

To take advantage of the data handling versatility of computers for image processing, it is necessary to digitize pictures. This is generally done by the electronic equivalent of placing a rectangular grid over the picture and assigning to each of the rectangles one of a finite number of gray levels. The requirement to digitize pictures arises in scanning electron microscopy, fingerprint analysis, optical character recognition, X-ray and fluoroscope diagnosis, planetary exploration, assembly line automation, and possibly other areas. Since the desire for increased resolution or more facile recognition is fairly general among picture digitizers, almost all digitized pictures will, prior to subsequent processing, be considered fuzzy and in need of sharpening. As a consequence, a considerable effort has been expended to achieve image sharpening and edge enhancement. The chief linear techniques are based on correlation and on Fourier and Walsh transforms and notions of filtering in two dimensions.[1] Particularly for edge enhancement the Laplacian provides a useful transformation. To locate the edges in a fuzzy picture, the calculation of the gradient,[2] a non-linear operation, has led to good results. For a comprehensive treatment of image enhancement techniques, see the special issue on image enhancement of Pattern Recognition[3] or the review of the subject by Huang.[4]

In our work on optical character recognition, we have discovered a transformation that is different from any of those referred to previously. It consists simply of replacing the gray value at a point by the lightest or darkest gray shade in its neighborhood, choosing whichever of the two is closer in value. The transformation is effective in sharpening, especially after several iterations. Since it acts only on the gray shades of neighboring points, it has the practical advantage that it can be implemented in digital electronics by using a minimal amount of data storage. The calculation of the new value, of course, involves no more than comparisons.

In Section 1 we set down some definitions which place the transformation in a general setting. Since it turned out that we could gain both clarity and generality, we did not limit the notion of neighborhood to one of the versions that is suitable for the discussion of digitized plane images. Instead, a neighborhood of a point is simply a finite set of points which includes the point in question. In addition the relation of being a neighbor is required to be symmetric. The ideas here are broad enough to include the case of a digitized scalar field in any finite number of dimensions. The definitions do not limit applications to regular lattices but include the case where the number of points in a neighborhood may vary from point to point.

In Section 2 we show that after a finite number of iterations of the transformation the resulting picture no longer changes if it is again subjected to the sharpening transformation. In Section 3, we examine the properties of the limit of iterations.

Section 4 presents an analytical test of the sharpening transformation. We start with black and white figures. Then we subject them to the smoothing effect of a unimodal symmetric line-spread function which results in a picture whose edges are blurred both in position and contrast. The first test picture we analyze is a straight edge on one side of which the picture is black and on the other side of which it is white. We prove that the limit of sharpening iterations restores the original picture perfectly. Next we examine a black stripe on a white background. If the width of the stripe is larger than the "shoulder" of the line-spread function, the picture is restored perfectly both in shading and position. If the width of the stripe is less than that of the "shoulder", then the stripe is restored with respect to shading, but the iterations yield a wider stripe than the original.

An analysis of two black stripes separated by a white stripe shows that after processing, the white between the stripes is grayer and narrower than in the original picture.

Finally, in Section 5 we describe the digitizing of some pictures and the result of our attempt at restoration.

1. MATHEMATICAL SETTING

We shall now make precise the terms to be used. Instead of speaking of a digitized picture, we shall refer to a real valued function F defined on a finite set X. With each point x of X, we shall associate a set $N(x)$, the neighborhood of x, satisfying the following properties.

(a) $x \in N(x)$

(b) $y \in N(x) \Rightarrow x \in N(y)$.

The pair (X, N) is what is commonly referred to as a graph. If $A \subset X$, we introduce two concepts: the frontier of A, $FR(A)$, and the interior of A, $INT(A)$.

$$FR(A) = \{x \in A | N(x) \cap \bar{A} \neq \phi\}$$
$$INT(A) = A \cap \overline{FR(A)}.$$

If $A \subset X$, then in the above definitions $\bar{A}$ denotes the complement of A. If F is a function on X, we associate with it two other functions, the local maximum function $\bar{F}$ and the local minimum function $\underline{F}$.

$$\bar{F}(x) = \max\,[F(y), y \in N(x)]$$
$$\underline{F}(x) = \min\,[F(y), y \in N(x)]$$

We are now in a position to define the sharpening transformation S.

$$(SF)(x) = \begin{cases} \bar{F}(x) & \text{if } \bar{F}(x) - F(x) \leqslant F(x) - \underline{F}(x) \\ \underline{F}(x) & \text{otherwise.} \end{cases}$$

We write $S^0F = F$ and for any integer $n, S^{n+1}F = S(S^nF)$.

2. ITERATION

If one application of the sharpening transformation S does some good, one might hope and expect that several applications in succession will improve the picture even more.* It seems reasonable that all good things come to an end. So it is with repeated applications of the sharpening transformation. In fact, we shall prove the pointwise convergence of the sequence S^nF.

Theorem 1

For each $x \in X$,

$$\lim_{k \to \infty} (S^kF)(x) = P(x).$$

Proof. Note that, since X has a finite number of points, if the limit exists, it is reached in a finite number of steps. $P(x)$ represents the limiting function. We shall prove the theorem by induction on the number $|X|$ of points in X. The statement is trivially true for $|X| = 1$.

If $|X| = 2$ and $X = \{x,y\}$, then if $F(x) = F(y)$, $SF(x) = F(x) = \underline{F}(x) = \bar{F}(y) = \underline{F}(y) = (SF)(y) = F(x) = F(y)$ and the statement is true. Suppose therefore that $F(x) < F(y)$. Then

$$(SF)(x) = \underline{F}(x) = F(x)$$

and

$$(SF)(y) = \bar{F}(y) = F(y)$$

so that $SF = F$ and consequently the statement of the theorem is true for $|X| = 2$. Now suppose that the statement of the theorem holds for all sets X such that $|X| \leqslant m$. Now consider a set X for which $|X| = m + 1$. The conclusion of convergence is obvious if F is constant. Therefore suppose F is not constant and let U be the maximum and L the minimum values of F on X. Let $M(F)$ be the set of points in X where F achieves its maximum value.

$$M(F) = (x \in X | F(x) = U).$$

We have

$$M(F) \subset M(SF) \subseteq \ldots \subseteq X.$$

Since X is finite,

$$\lim_{n \to \infty} M(S^nF) = \mathscr{M},$$

and there exists an integer N_1 such that for all non-negative integers k

$$M(S^{N_1 + k}F) = \mathscr{M}.$$

Let

$$Y = X \cap \bar{\mathscr{M}}.$$

For $x \in FR(Y)$,

$$S^{N_1}F(x) \geqslant S^{N_1+1}F(x) \geqslant \ldots \geqslant L$$

Clearly,

$$\lim_{n \to \infty} (S^nF)(x) \text{ exists}$$

and is achieved for finite index N_2. Let us denote this limit by $m(x)$.

Let $N_3 = \max\,(N_1, N_2)$ and $G = S^{N_3}F$. Let the restriction of G to the domain Y be called $G_Y, N_Y(x) = N(x) \cap Y$, and S_Y the restriction of S to functions on Y with neighborhood N_Y. Since $|Y| \leqslant m$, the induction hypothesis applies and

$$\lim_{k \to \infty} S_Y^k F \text{ exists.}$$

Thus, it remains to show that

$$\lim_{n \to \infty} S^nF = \lim_{n \to \infty} S_Y^n F \text{ on } Y.$$

(It is obvious that S^nF converges on $X - Y = m$).

To this end we demonstrate that for $x \in Y$ and all non-negative integers l

$$S_Y^l G_Y(x) = S^l G(x) \tag{1}$$

The proof proceeds by induction. For $l = 0$, $S_Y^0 G_Y(x) = G_Y(x) = G(x) = S^0G(x)$ by definition. Let $l = 1$. Suppose, first, that $x \in INT(Y)$. Then $N_Y(x) = N(x), G_Y(x) = G(x)$, and thus

$$\bar{G}_Y(x) = \bar{G}(x)$$

and

$$\underline{G}_Y(x) = \underline{G}(x).$$

It follows that

$$S_Y G_Y(x) = SG(x).$$

If $x \in FR(Y)$, then $N_Y(x) = N(x) \cap \bar{\mathscr{M}}$.

But $G_Y(x) = G(x) = \underline{G}(x) = \underline{G}_Y(x) = m(x)$ since omission of points of $\mathscr{M}$ does not affect the fact that x is a local minimum of G in $N(x)$ and G_Y in $N_Y(x)$.

Therefore, in this case also

$$(S_Y G_Y)(x) = (SG)(x).$$

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* Bell[5] has described the advantages of repeated applications of a sharpening method using the Laplacean.

Suppose that statement (1) is true for $l = n$. Let $l = n + 1$.

$$(S_Y^{n+1}G_Y)(x) = S_Y(S_Y^n G_Y)(x).$$

By the induction hypothesis

$$= S_Y(S^n G)(x).$$

Since the points of the frontier are also local minima for $S^n G$, the proof for $l = 1$ applies by replacing G by $S^n G$.

If $x \in INT(Y), N(x) = N_Y(x)$ and

$$S_Y^{n+1} G_Y)(x) = S_Y(S^n G)(x)$$
$$= S(S^n G)(x) = (S^{n+1}G)(x).$$

Consequently on all of Y.

$$S_Y^l G_Y = S^l G.$$

On noting that for $x \in \mathcal{M}$, and for every integer l, $(S^l F)(x) = F(x) = U$ the conclusion of the theorem follows.

3. PROPERTIES OF THE LIMIT

To describe a picture that is the end result of the sharpening iterations, we introduce two additional notions.

$x \in X$ is a local maximum of F if $F(x) = \bar{F}(x)$ and a local minimum if $F(x) = \underline{F}(x)$. A description of the limit is contained in the following.

Theorem 2

If $\lim S^n F = P$, then every point $x \in X$ is either a local maximum or a local minimum of P.

Proof. $SP = P$ and therefore either $P(x) = \bar{P}(x)$ or else $P(x) = \underline{P}(x)$.

4. AN ANALYTICAL TEST

In order to test the efficacy of our procedure in sharpening images we shall construct an analytic model. Let us suppose that the image we wish to sharpen is the result of passing a black and white picture through a lens and electronic filter which have caused it to become fuzzy. To retain simplicity in the analysis, we shall deal only with one-dimensional pictures, i.e. those in which variations in gray shades occur in only one direction.

Let G be a function of one variable that represents the original picture. The fuzzy version is given by

$$F_G(x) = \int_{-\infty}^{\infty} h(x - y)G(y)\,dy.$$

Let us assume a symmetrical "lens", i.e.

$$h(x) = h(-x)$$

with finite aperture, i.e.

$$h(x) = 0 \qquad x < -a \text{ and } x > a.$$

Moreover, $h(x) \geqslant 0$.

The simplest picture that can be examined consists of a black half-plane on a white background. We take the value of white to be 0 and of black to be 1. The picture is then described by the unit step function:

$$U(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0. \end{cases}$$

We have, on replacing G by U in the convolution integral,

$$F_U(x) = \begin{cases} 0 & x < -a \\ \int_{-a}^{x} h(u)\,du & -a \leqslant x \leqslant a \\ \int_{-a}^{a} h(u)\,du & x > a. \end{cases}$$

Suppose that h is unimodal with a single maximum and F_U^* is a sampled version of F_U, then P defined by

$$\lim_{n\to\infty} S^n F_U^* = P$$

is a perfect sampled reconstruction of U with uncertainty regarding the edge location being equal to at most one sampling interval.

Now let us look at what happens to a black stripe of width w on a white background. We represent the white stripe by

$$W(x) = U(x) - U(x - w).$$

It follows from the linearity of the convolution operation that

$$F_W(x) = F_U(x) - F_U(x - w).$$

Note that for any twice differentiable function F which is sampled at regularly spaced points to produce F^*, we have:

If F' is increasing at x, i.e. $F''(x) > 0$

$$(SF^*)(x) < F^*(x), \quad (2)$$

and

If F' is decreasing at x, i.e. $F''(x) < 0$

$$(SF^*)(x) > F^*(x). \quad (3)$$

We show first that if $w > a$, then the reconstruction is perfect (i.e. except for sampling error), that is

$$\lim_{k\to\infty} S^k F_W^* = W.$$

It suffices to show that (1) for $x < 0$ the derivative of F_w is increasing while for $x > 0$ the derivative decreases until F_W reaches 0 at $x = w/2$ and (2) for $w/2 < x < w$ the derivative is decreasing while for $x > w$ it increases to 0 at $x = a + w$. We have

$$F_W'(x) = h(x) - h(x - w)$$

and

$$F_W''(x) = h'(x) - h'(x - w).$$

For $x < 0$, $h'(x - w) = 0$ and $h'(x) > 0$.

Hence $F_W''(x) > 0$. For $x = 0$, $F_W''(x) = h'(x) - h'(x - w) = 0 - 0 = 0$; and for $a > x > 0$, $h'(x) < 0$, $h'(x - w) > 0$. Hence $F_W''(x) < 0$. At $x = w/2$, $F'(x) = 0$ and $F_W''(w/2) = h'(w/2) = h'(-w/2) < 0$.

The rest of the assertion results from the fact that F_W is symmetric about $w/2$.

Next, suppose that $w < a$. We consider the following intervals:

A. $-a < x < -a + w$
B. $0 \leqslant x \leqslant w$
C. $a < x < a + w$.

We have

$$F_W''(x) = h'(x) - h'(x - w).$$

In interval A, $h(x - w) = h'(x - w) = 0$. Therefore $F_W''(x) = h'(x) > 0$, F' is increasing and $SF^*(x) < F^*(x)$.

In interval B, $x - w \leqslant 0$. It follows that $h'(x - w) \geqslant 0$ and $h'(x) \leqslant 0$. Therefore $F''_W(x) < 0$, F'_W is decreasing and $SF^*(x) > F^*(x)$.

In interval C, $h(x) = h'(x) = 0$. So that $F'_W(x) = -h'(x - w) > 0$ and $SF^*(x) < F^*(x)$.

Consequently there exist $x_1(w - a < x_1 < 0)$ and $x_2(w < x_2 < a)$ such that $F''_W(x_1) = F''_W(x_2) = 0$. Applying the sharpening algorithm to the sampled version F^*_W or F_W gives the result

$$\lim_{k \to \infty} S^k F^*_W = U(x - x_1) - U(x - x_2).$$

A stripe has thus been reconstructed. Its width w_1, however, exceeds that of the original stripe. In fact,

$$w < w_1 < 2a - w.$$

The analysis for the case of two black stripes separated by a white stripe is carried out by using similar considerations. The result is that if $F_W(w/2 + c/2) < 1/2\ F_W(w/2)$, where c is the separation of the two stripes, the two stripes are resolvable, and the closer they move together the grayer and narrower the white interstice between them becomes. The iteration of the sharpening transformation succeeds in reducing the number of light levels to three: white, a dark shade of gray for the black stripe, and a lighter shade of gray representing the originally white stripe.

5. NUMERICAL EXAMPLES

Our apparatus for producing digitized images is shown schematically in Fig. 1. It consists of a rotating drum with the image mounted on it. Reflected light from the image is projected through a lens onto a self-scanning diode array (International Photomatrix Limited No. 7128) having 128 diodes of which we select 64 under computer control. The signal is amplified, sampled and digitized by an 8-bit analog to digital converter. Using direct memory access, the image is stored in 4 K of core of a General Automation Inc. SPC 16, 16-bit mini-computer. The iteration of the transformation S is carried out until no more changes are observed and the resulting gray shades are printed in a regular array on a line printer. The pictures were produced by photographing the screen of an 8-level CRT display. In this connection, the 8-bit light intensities have been reduced to 8 gray-levels. The neighborhoods were chosen cruciform except on the frontier of the rectangular field. In the interior:

$$N(x,y) = \{(x,y); (x + 1,y); (x - 1,y); (x,y + 1); (x,y - 1)\}.$$

On the right edge, for example,

$$N(x,y) = \{(x,y); (x - 1,y); (x,y + 1); (x,y - 1)\}$$

and at the lower right corner

$$N(x,y) = \{(x,y); (x - 1,y); (x,y + 1)\}.$$

It was found, as expected, that smaller neighborhoods resulted in better sharpening.

For a picture of 27 × 33 points, 20–50 iterations have been necessary for complete convergence although sharpening is generally quite adequate for the important edges of the image after only 2 iterations.

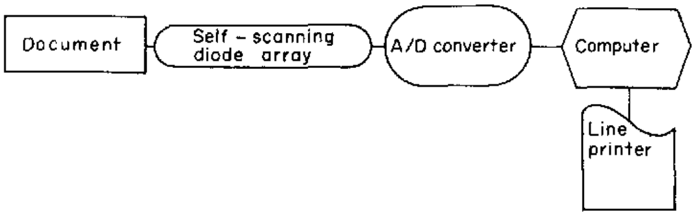

Fig. 1. System schematic.

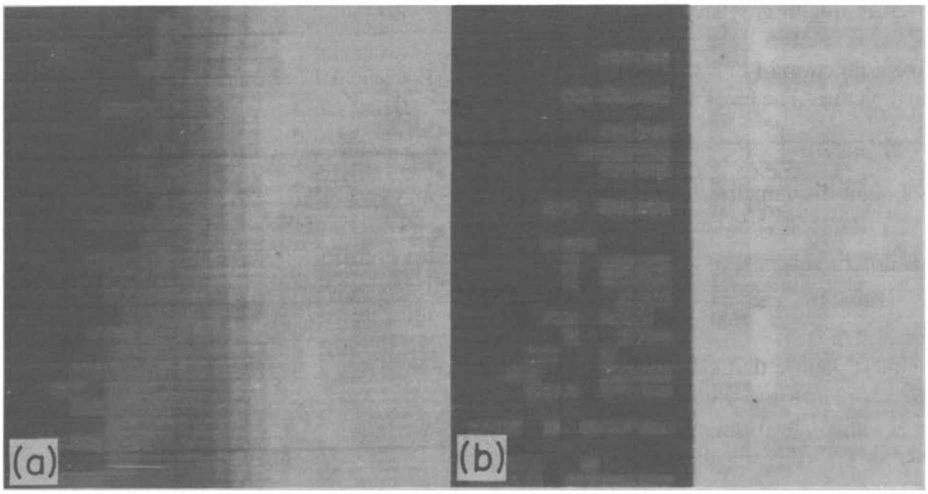

Fig. 2.

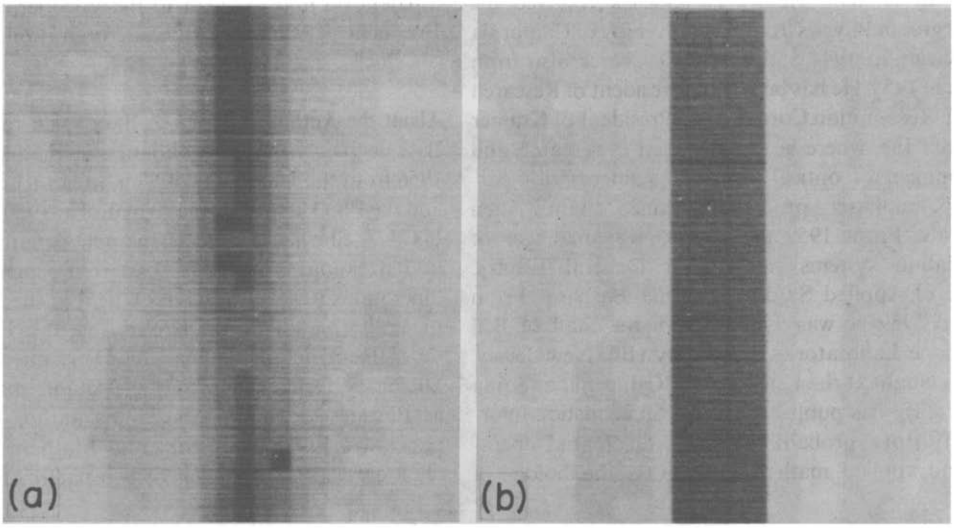

Fig. 3.

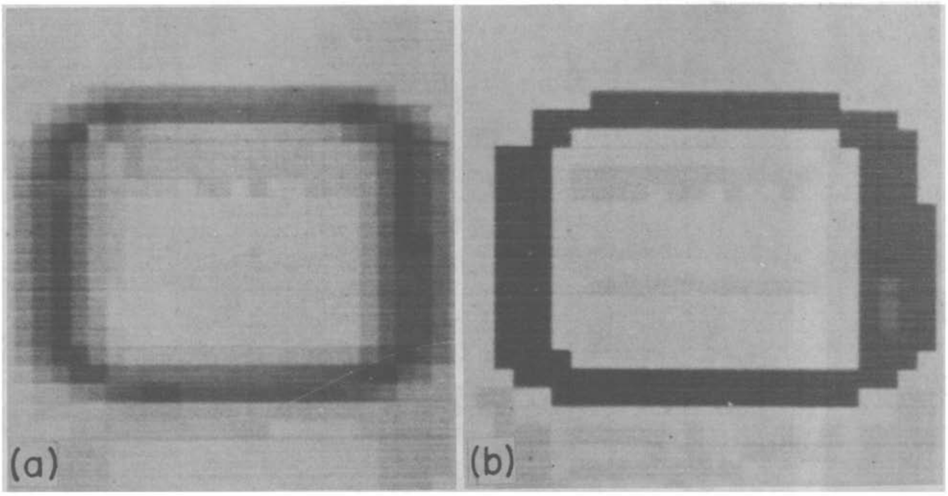

Fig. 4.

Figure 2(a) shows the blurred image of a black half-plane; Fig. 2(b) shows the half-plane after convergence of the sharpening process (47 iterations). Figure 3(a) shows a black stripe on a white background before sharpening; Fig. 3(b) shows the same picture after sharpening (27 iterations). Figures 4(a) and 4(b) show the character zero before and after 27 iterations of the S transform respectively.

6. SUMMARY

A transformation S has been defined which has the property that it tends to sharpen a fuzzy digitized picture. It is particularly useful in handling line drawings or printed symbols. Its virtue lies in the combination of effectiveness and simplicity. It was shown that iterations of the transformation converge to a picture where every point is either a local maximum or a local minimum. The effectiveness of the transformation was tested both analytically and by applying it to digitized pictures.

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