Computational Plasticity

Chapter 4 – High accurate integrations in computational plasticity

Li-Wei Liu, Ph.D.

Department of Civil Engineering

National Taiwan University

Overview

- The exponential map
- Group theory & Lie group
- Augmented spaces
- Group preserving schemes
- Summary

The exponential map

Solutions of ODEs

- An ordinary differential system, $\frac{d}{dt}\mathbf{x} = \mathbf{f}(t, \mathbf{x})$ is said to be autonomous if the time variable does not appear explicitly in $\mathbf{f}$, i.e. $\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x})$.
- If a system of ordinary differential equations is the linear system with constant coefficients and no time-dependent forcing, this system is called time-invariant and can be written as

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x},$$

and has the unique solution

$$\mathbf{x}(t) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0),$$

where $\exp(\mathbf{A})$ denotes the matrix exponential of the matrix $\mathbf{A}$.

Matrix exponential

- The matrix exponential has the following properties:

$$\exp(\mathbf{A}t)\mathbf{A} = \mathbf{A}\exp(\mathbf{A}t), \forall t \in \mathbf{R},$$
$$\exp(\mathbf{A}(t+s)) = \exp(\mathbf{A}t)\exp(\mathbf{A}s), \forall t, s, \in \mathbb{R},$$
$$\det(\exp(\mathbf{A}t)) = \exp(\mathrm{tr}(\mathbf{A})t)\,.$$

- If $\mathbf{M}_1$ and $\mathbf{M}_2$ are commute, $\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_1$, then $\exp(\mathbf{M}_1 + \mathbf{M}_2) = \exp(\mathbf{M}_1)\exp(\mathbf{M}_2)$.
- If $\mathbf{M}_1$ and $\mathbf{M}_2$ are square matrices, the

$$\exp\left(\begin{bmatrix}\mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2\end{bmatrix}\right) = \begin{bmatrix}\exp(\mathbf{M}_1) & \mathbf{0} \\ \mathbf{0} & \exp(\mathbf{M}_2)\end{bmatrix}.$$

- If $\mathbf{Y}$ is nonsingular, then $\exp\left(\mathbf{Y}^{-1}\mathbf{M}\mathbf{Y}\right) = \mathbf{Y}^{-1}\exp\left(\mathbf{M}\right)\mathbf{Y}$.
- If $\lambda_i \in \mathbb{C},\ i = 1,2,3,\cdots n$, then $\exp(\text{diag}(\lambda_1, \cdots, \lambda_n)) = \text{diag}(\exp(\lambda_1), \cdots, \exp(\lambda_n))$.
- If $\mathbf{A}$ is hermitian, $\exp(\mathbf{A})$ is hermitian positive definite.
- If $\mathbf{A}$ is skew-hermitian, $\exp(\mathbf{A})$ is unitary.

Hand-in

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$$

If $\mathbf{x}(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, please calculate $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ where

$t_{i+1} = t_i + \Delta t$ by using the exponential map method. Please

compare the $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ with the result from the the

forward Euler method.

Group theory & Lie group

Hand-in

There is a matrix

$$\mathbf{G} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ **and**

$\theta = \pi/2$. **Please calculate**

$\mathbf{G}^2, \mathbf{G}^3, \mathbf{G}^4.$

Groups

A set $G = \{g_1, g_2, \cdots\}$ is said to form a group if there is an operation "$\circ$", called group multiplication, such that the following four axioms are satisfied.

- Closure: If $g_1, g_2 \in G$ then $g_1 \circ g_2 \in G$.
- Associativity: $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k), \forall g_i, g_j, g_k \in G$.
- Identity: there is group element, called identity element I, with the following property.

$g_i \circ I = g_i = I \circ g_i, \forall g_i \in G.$

- Inverse: every element g_i has an inverse element, called g_i^{-1},

$g_i \circ g_i^{-1} = I = g_i^{-1} \circ g_i, \forall g_i, g_i^{-1} \in G.$

Hand-in

There is a matrix

$$\mathbf{G} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ **and**

$\theta = \pi/2$. **Do the matrices**

$\mathbf{G}, \mathbf{G}^2, \mathbf{G}^3, \mathbf{G}^4$ **form a group?**

Lie group

Marius Sophus Lie
(1842-1899)

Roughly speaking, a Lie group is an infinite group whose element can be parametrized smoothly and analytically. Lie groups are beautiful, import, and useful because they have on foot in each of the two great divisions of mathematics --- algebra and geometry.

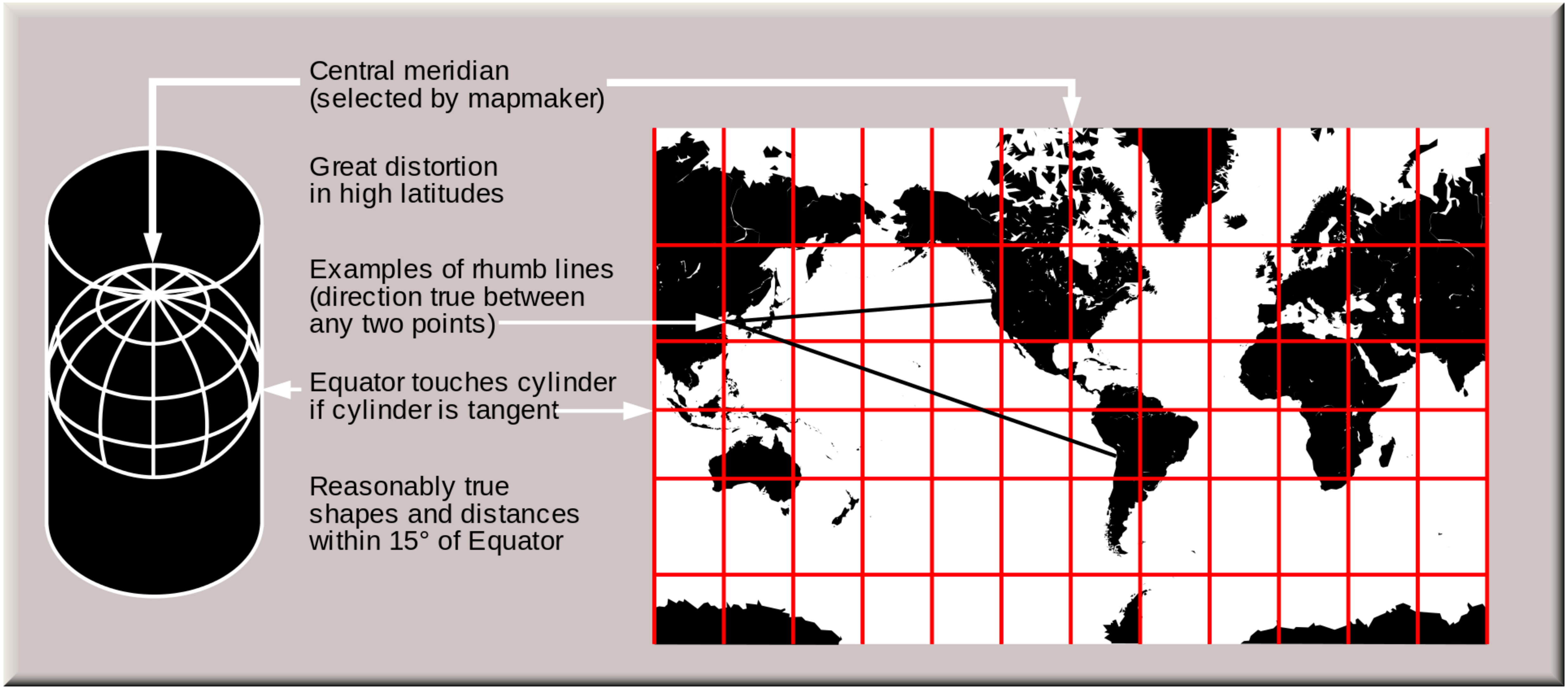
Manifold

Definition: An n-dimensional manifold M^n consists of

- a topological space T which includes a collection of open sets P_i (a topology) that cover T, i.e. $T = \cup_i P_i$.
- a collection of charts ϕ_i with $\phi_i(P_i) = V_i \subset \mathbb{R}^n$ that each ϕ_i is a homoeomorphism of P_i to V_i.
- (smoothness conditions) The homoeomorphism $\phi_i \circ \phi_j^{-1} : \phi_j(P_i \cap P_j) \to \phi_i(P_i \cap P_j)$ of open sets in $\mathbb{R}^n$ to open sets in $\mathbb{R}^n$ are one-to-on, invertible, and differentiable.

Manifolds

Mercator projection



Lie group

Definition: A Lie group consists of a manifold M^n that parameterizes the group elements $g(x), x \in M^n$ and the operation defined by $g(x) \circ g(y) = g(z)$ where the coordinate $z \in M^n$ depends on the coordinates $x, y \in M^n$ through a function $z = \phi(x, y)$. There are two topological axioms for a Lie group

- Smoothness of group composition map: The group composition map $z = \phi(x, y)$ defined by $g(x) \circ g(y) = g(y)$ is differentiable.

- Smoothness of the group inverse map: The group inversion map $y = \psi(x)$, defined by $g^{-1}(x) = g(y)$, is differentiable.

Hand-in

Please calculate $\mathbf{G}^T\mathbf{g}\mathbf{G}$ where

$$\mathbf{G} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ and}$$

$$\mathbf{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Lie groups preserve a metric

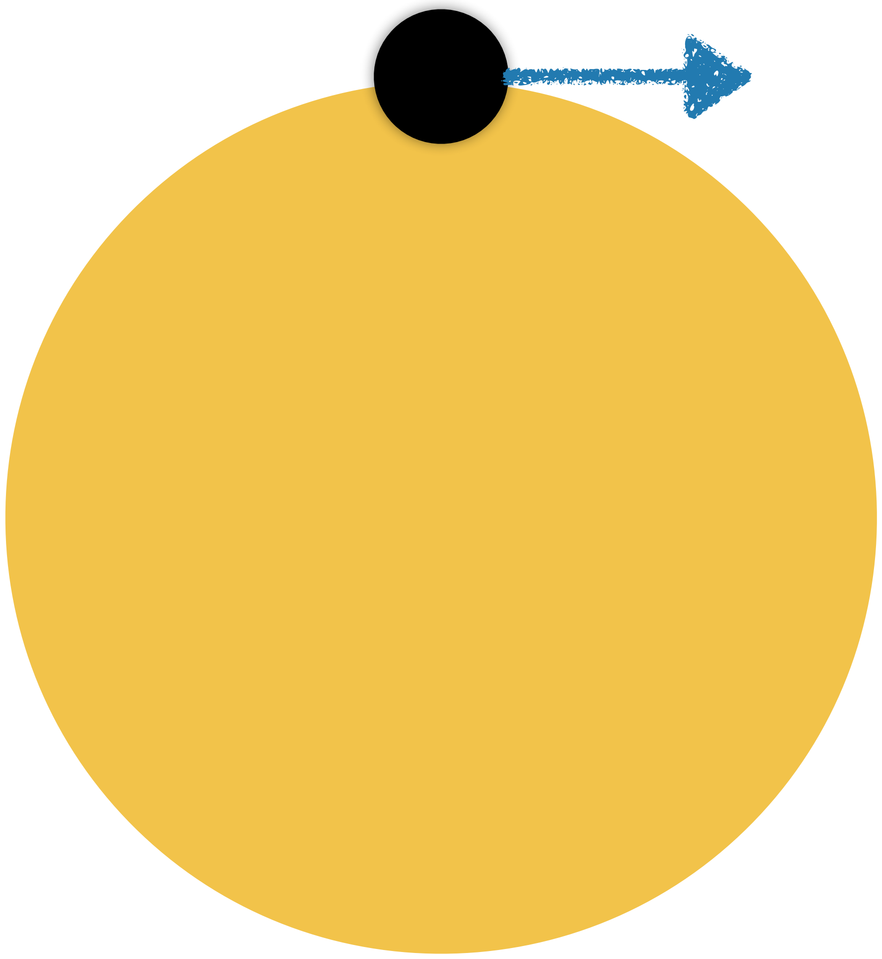
$$\mathbf{G}^{\star}\mathbf{g}\mathbf{G} = \mathbf{g}$$

- Compact metric-preserving groups

If $\mathbf{g} = \mathbf{I}_n$, then the set of $\mathbf{G} \in \mathbb{R}^{n\times n}$ forms the orthogonal group $\mathrm{O}(n)$, the set of $\mathbf{G} \in \mathbb{C}^{n\times n}$ forms the unitary group $\mathrm{U}(n)$, and the set of $\mathbf{G} \in \mathbb{H}^{n\times n}$ forms the symplectic group $\mathrm{Sp}(n)$.

Lie groups preserve a metric

Circular motion



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

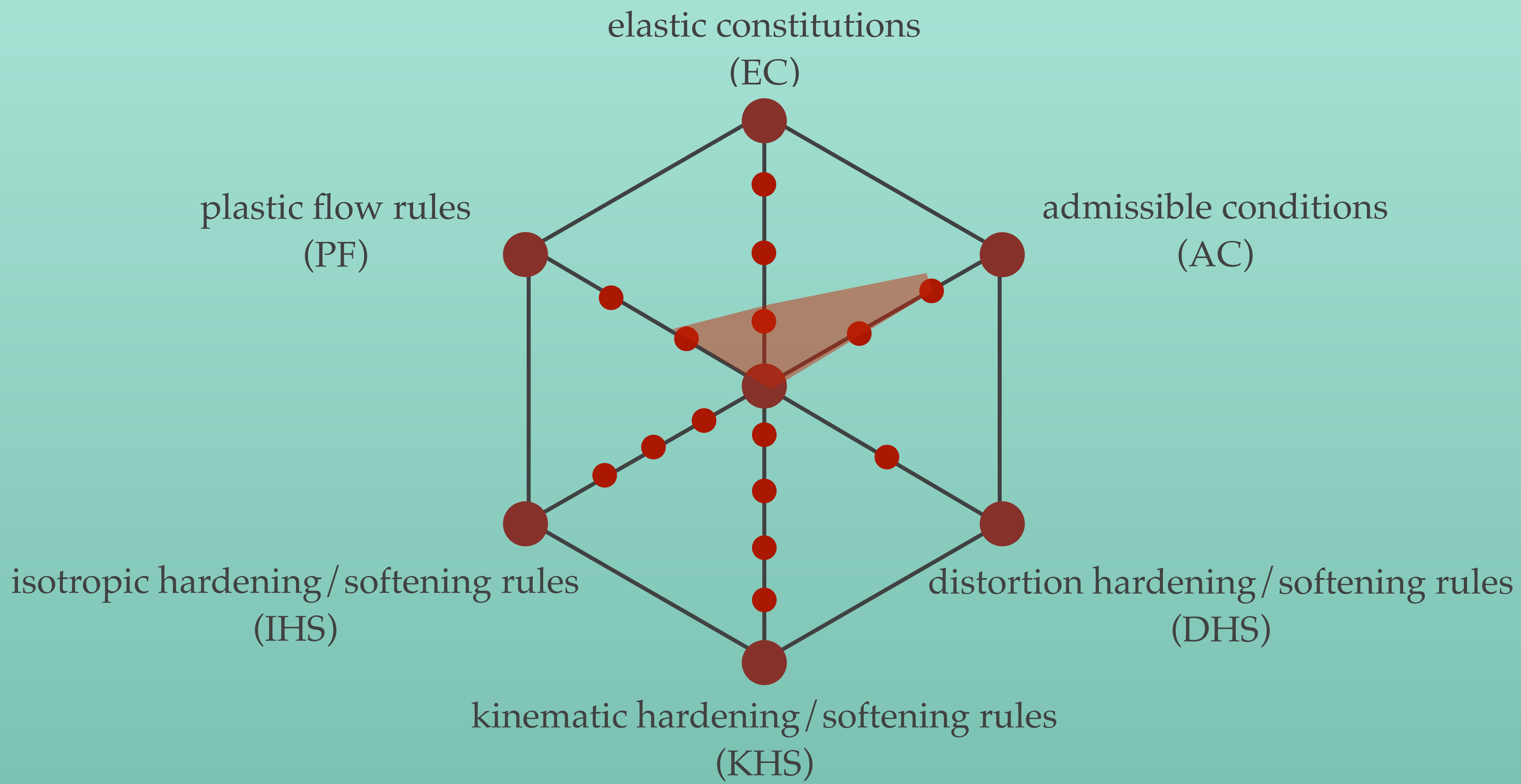
$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$$

$$\exp\left(\mathbf{A}t\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\mathbf{x}^T\mathbf{g}\mathbf{x} = x_1^2 + x_2^2 = 1$$

$$\exp(\mathbf{A}t)^T\mathbf{g}\exp(\mathbf{A}t) = \mathbf{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Augmented spaces



Models of perfect elastoplasticity

Perfectly elastoplastic models–generalized model 1

Model of perfect elastoplasticity

$$\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},$$
$$\mathbf{Q} = k_e \mathbf{q}^{\mathrm{e}},$$
$$\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}} \dot{\lambda},$$
$$f = \|\mathbf{Q}\| - Q_y \leq 0,$$
$$\dot{\lambda} \geq 0,$$
$$f \dot{\lambda} = 0.$$

On-off switching

$$\dot{\lambda} = \begin{cases} \dfrac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y} & \text{if} \quad \|\mathbf{Q}\| - Q_y = 0 \quad \text{and} \quad \mathbf{Q}^T \dot{\mathbf{q}} > 0 \quad \text{ON phase} \\ 0 & \text{if} \quad \|\mathbf{Q}\| - Q_y < 0 \quad \text{or} \quad \mathbf{Q}^T \dot{\mathbf{q}} \leq 0 \quad \text{OFF phase} \end{cases}$$

A two-phase dynamical system

$$\dot{\mathbf{Q}} = k_e \dot{\mathbf{q}} \quad \text{if} \quad \|\mathbf{Q}\| - Q_y < 0 \quad \text{or} \quad \mathbf{Q}^T \dot{\mathbf{q}} \leq 0$$
$$\dot{\mathbf{Q}} = -k_e \frac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}} \quad \text{if} \quad \|\mathbf{Q}\| - Q_y = 0 \quad \text{and} \quad \mathbf{Q}^T \dot{\mathbf{q}} > 0$$

Augmented stress space

- Plastic phase

$$\dot{\mathbf{Q}} = -k_e\dot{\lambda}\frac{\mathbf{Q}}{Q_y} + k_e\dot{\mathbf{q}}$$

$$\|\mathbf{Q}\| - Q_y = 0$$

Integration factor

$$\Longrightarrow \exp(\frac{\lambda}{q_y})\dot{\mathbf{Q}} = -\exp(\frac{\lambda}{q_y})\dot{\lambda}\frac{\mathbf{Q}}{q_y} + k_e\exp(\frac{\lambda}{q_y})\dot{\mathbf{q}} \qquad q_y = \frac{Q_y}{k_e}$$

$$\Longrightarrow \frac{d}{dt}\left(\exp(\frac{\lambda}{q_y})\mathbf{Q}\right) = k_e\exp(\frac{\lambda}{q_y})\dot{\mathbf{q}}$$

$$\Longrightarrow \int_\tau^t \frac{d}{dt}\left(\exp(\frac{\lambda}{q_y})\mathbf{Q}\right)dt = k_e\int_\tau^t \exp(\frac{\lambda}{q_y})\dot{\mathbf{q}}dt$$

nonlinear

Breakthrough

$$X_0\frac{\dot{\mathbf{Q}}}{Q_y} + \dot{X}_0\frac{\mathbf{Q}}{Q_y} = X_0\frac{\dot{\mathbf{q}}}{q_y}, \qquad X_0 = \exp(\frac{\lambda}{q_y})$$

Augmented generalized stress $\mathbf{X} = \begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix}$

H.-K. Hong and C.-S. Liu. Internal symmetry in the constitutive model of perfect elastoplasticity. International *Journal of Non-Linear Mechanics*, 35, 447-466, 2000.

Two-phase augmented system

- The plastic phase

$$\{\|\mathbf{Q}\| = Q_y \ \text{ and } \ \mathbf{Q}^T\dot{\mathbf{q}} > 0\} \Leftrightarrow \{\dot{\lambda} = \frac{\mathbf{Q}^T\dot{\mathbf{q}}}{Q_y} > 0\} \Leftrightarrow \{\dot{\lambda} > 0\}$$

$$\frac{d}{dt}\begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \frac{\dot{\mathbf{q}}}{q_y} \\ \frac{\dot{\mathbf{q}}^T}{q_y} & 0 \end{bmatrix} \begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix}$$

- The elastic phase

$$\{\|\mathbf{Q}\| < Q_y \ \text{ or } \ \mathbf{Q}^T\dot{\mathbf{q}} \le 0\} \Leftrightarrow \{\dot{\lambda} = 0\}$$

$$\frac{d}{dt}\begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \frac{\dot{\mathbf{q}}}{q_y} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix}$$

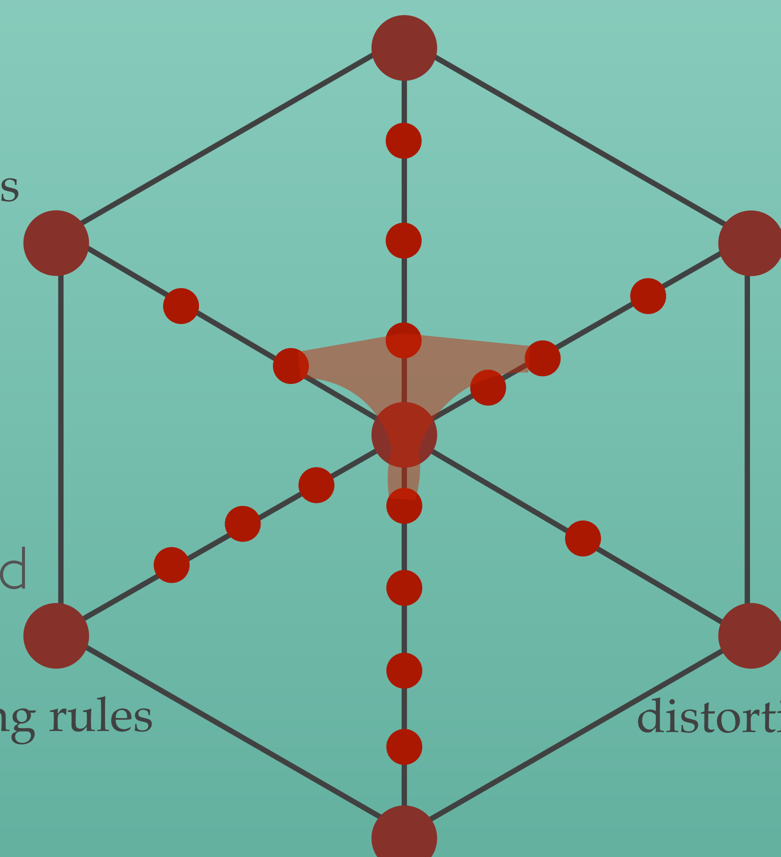
Elastoplastic models with kinematic hardening

- elastic constitutions

iso+linear — aniso+linear — iso+nonlinear — aniso+nonlinear

- stress admissible conditions

linear — quadratic — quadratic+linear — high order (cubic)



- plastic flow rules

associated — generalized associated — non-associated

- kinematic hardening

non — Prager — A-F — O-W — A-K

A Model With Bilinear Elastoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p, \qquad \mathbf{q}, \mathbf{q}^e, \mathbf{q}^p \in \mathbb{R}^n,$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, \qquad \mathbf{Q}, \mathbf{Q}_a, \mathbf{Q}_b \in \mathbb{R}^n,$$

$$\mathbf{Q} = k_e \mathbf{q}^e, \qquad k_e > 0, Q_y > 0, k_p > 0$$

$$\mathbf{Q}_b = k_p \mathbf{q}^p, \qquad \text{given } k_e, Q_y, k_p \in \mathbb{R},$$

$$Q_y \dot{\mathbf{q}}^p = \mathbf{Q}_a \dot{\lambda}, \qquad \dot{()} = d()/dt,$$

$$f\dot{\lambda} = 0,$$

$$f = \| \mathbf{Q}_a \| - Q_y \leq 0, \qquad \| \mathbf{Q}_a \| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a},$$

$$\dot{\lambda} \geq 0,$$

Two-phase dynamical system

$$\frac{d}{dt}\begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\dot{\lambda}}{Q_\mathrm{y}} \\ 0 & -\frac{k_p \dot{\lambda}}{Q_\mathrm{y}} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{Q}}/k_e \\ \dot{\mathbf{Q}} \end{bmatrix} \text{ if } f = 0 \text{ and } \mathscr{S} > 0, \text{ plastic phase, where } \dot{\lambda} = \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{k_p Q_y}$$

$$\frac{d}{dt}\begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{Q}}/k_e \\ \dot{\mathbf{Q}} \end{bmatrix} \text{ if } f < 0 \text{ or } \mathscr{S} \leq 0, \text{ elastic phase,}$$

Augmented space

$$\frac{d}{dt}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_\mathrm{a}\end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\dot{\lambda}}{Q_\mathrm{y}} \\ 0 & -\dfrac{k_p\dot{\lambda}}{Q_\mathrm{y}}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_\mathrm{a}\end{bmatrix} + \begin{bmatrix}\dot{\mathbf{Q}}/k_e\\ \dot{\mathbf{Q}}\end{bmatrix}$$

$$\frac{\dot{\mathbf{Q}}}{\mathbf{Q}_y}X_0 = \frac{d}{dt}\left(\frac{\mathbf{Q}_a}{Q_y}X_0\right) \longrightarrow \dot{\mathbf{X}}_s = \frac{\dot{\mathbf{Q}}}{Q_y}X_0 .$$

$$\frac{\dot{\mathbf{Q}}^T}{Q_y}\frac{\mathbf{Q}_a}{Q_y}X_0 = \frac{k_p}{Q_y}\dot{\lambda}X_0 \longrightarrow \frac{\dot{\mathbf{Q}}^T}{Q_y}\mathbf{X}_s = \dot{X}_0$$

$$\mathbf{X} = \begin{bmatrix}\mathbf{X}_s\\ X_0\end{bmatrix} = \begin{bmatrix}\dfrac{\mathbf{Q}_a}{Q_y}X_0\\ \exp\left(\dfrac{k_p\lambda}{Q_y}\right)\end{bmatrix}$$

$$\dot{\mathbf{X}} = \mathbf{A}_{\text{on}}\,\mathbf{X} \text{ i.e. } \begin{bmatrix}\dot{\mathbf{X}}_s\\ \dot{X}_0\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \dfrac{\dot{\mathbf{Q}}}{Q_y}\\ \dfrac{\dot{\mathbf{Q}}^T}{Q_y} & 0\end{bmatrix}\begin{bmatrix}\mathbf{X}_s\\ X_0\end{bmatrix}$$

Group preserving integrations

Theoretical fundation

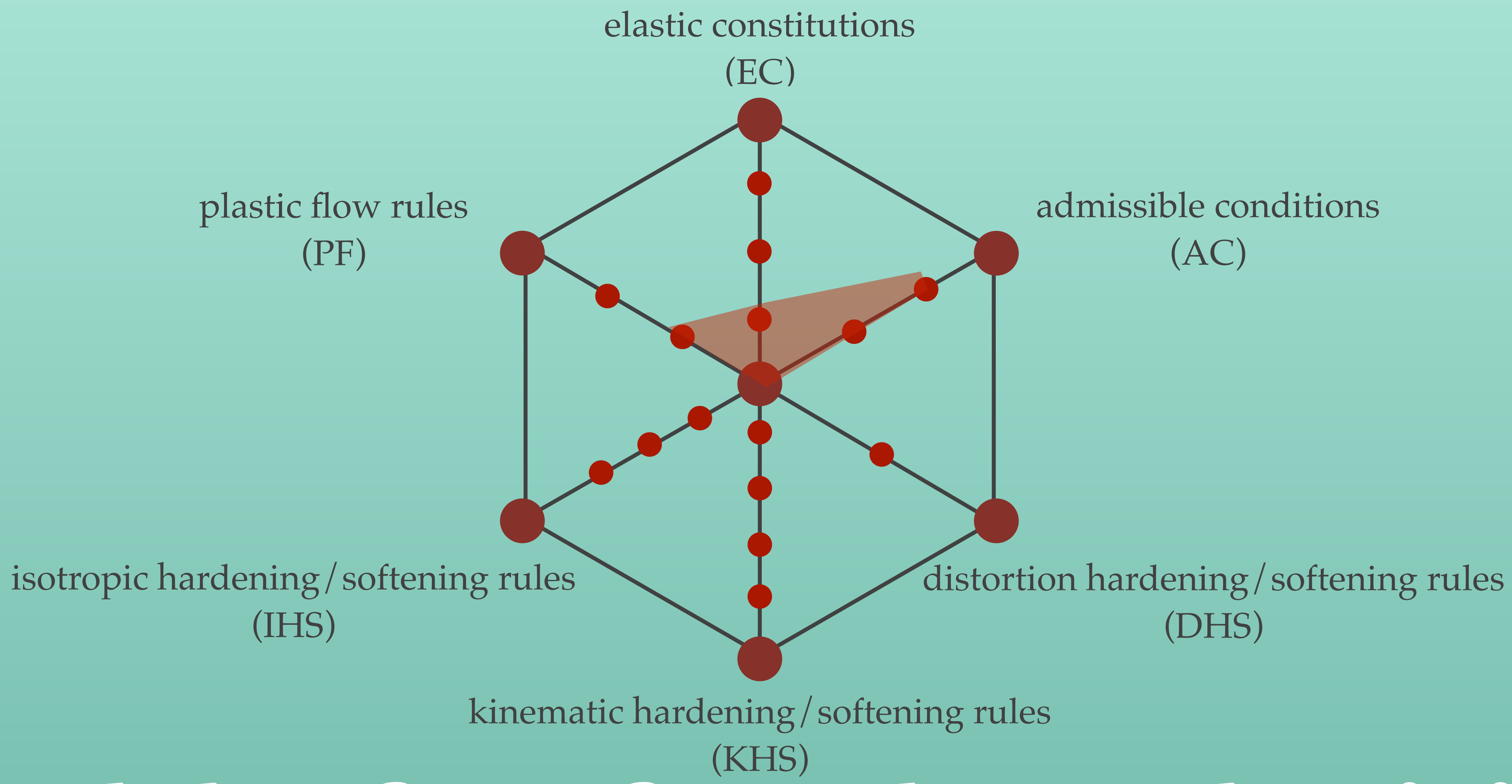
If a dynamic system is written as follows:

$$\frac{d}{dt}\mathbf{X} = \mathbf{A}\mathbf{X},$$

where $\mathbf{A}^T\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}$ and its fundamental solution $\mathbf{G}(t)$, where

$$\mathbf{X}(t) = \left[\mathbf{G}(t)\mathbf{G}^{-1}(t_i)\right]\mathbf{X}(t_i),$$
$$\mathbf{G}(0) = \mathbf{I},$$

satisfies $\mathbf{G}^T(t)\mathbf{g}\mathbf{G}(t) = \mathbf{g}$. please show that the state $\mathbf{X}(t)$ obeys the condition $\mathbf{X}^T(t)\mathbf{g}\mathbf{X}(t) = 0, \forall t$.



Models of perfect elastoplasticity

Perfectly elastoplastic models–generalized model 1

Group preserving integrations

- Two-phase augmented dynamical system

$$\frac{d}{dt}\begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \frac{\dot{\mathbf{q}}}{q_y} \\ \frac{\dot{\mathbf{q}}^T}{q_y} & 0 \end{bmatrix} \begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix} \quad \{\|\mathbf{Q}\| = Q_y \text{ and } \mathbf{Q}^T\dot{\mathbf{q}} > 0\} \Leftrightarrow \{\dot{\lambda} = \frac{\mathbf{Q}^T\dot{\mathbf{q}}}{Q_y} > 0\} \Leftrightarrow \{\dot{\lambda} > 0\}$$

$$\frac{d}{dt}\begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \frac{\dot{\mathbf{q}}}{q_y} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \frac{X_0}{Q_y}\mathbf{Q} \\ X_0 \end{bmatrix} \quad \{\|\mathbf{Q}\| < Q_y \text{ or } \mathbf{Q}^T\dot{\mathbf{q}} \le 0\} \Leftrightarrow \{\dot{\lambda} = 0\}$$

Plastic phase

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$$

$$f = \mathbf{X}^T\mathbf{g}\mathbf{X} = 0$$

$$\mathbf{A} = \frac{1}{q_y}\begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T & 0 \end{bmatrix}$$

Fundamental solution $\mathbf{G}(t)$

$$\mathbf{X}(t) = \left[\mathbf{G}(t)\mathbf{G}^{-1}(t_i)\right]\mathbf{X}(t_i)$$

$$\mathbf{G}(0) = \mathbf{I}_6$$

Internal symmetry

$$\mathbf{A}^T\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}_{m\times m}$$ Lie algebra so(n,1)

$$\mathbf{G}^T(t)\mathbf{g}\mathbf{G}(t) = \mathbf{g}$$ Lie group $SO_o(n,1)$

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_{n\times n} & \mathbf{0} \\ \mathbf{0} & -1 \end{bmatrix}$$

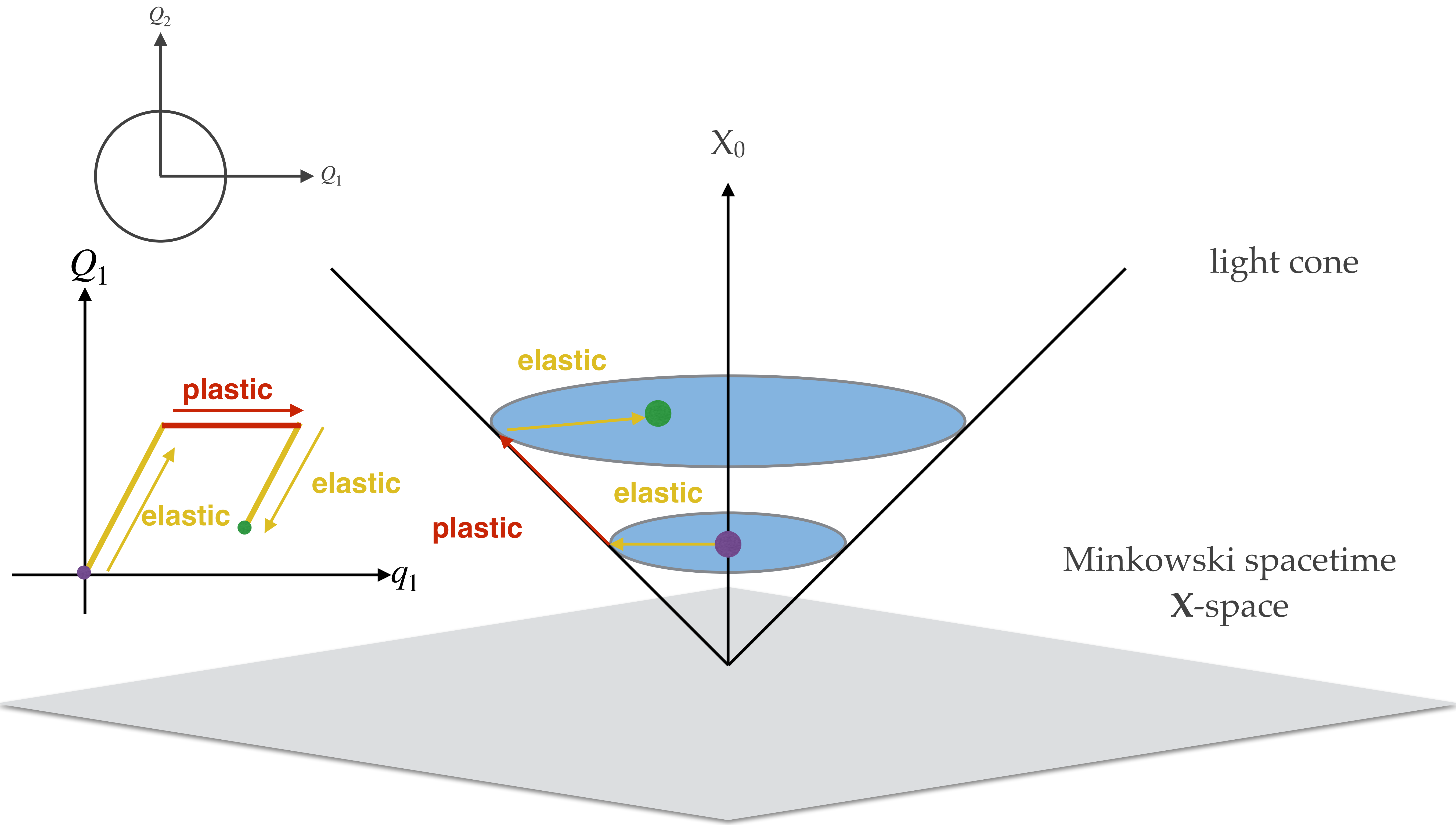
proper orthochronous Lorentz group ($m = n + 1$)

$\mathbf{X}^T(t)\mathbf{g}\mathbf{X}(t) = 0, \forall t \in$ plastic phase

updating stress automatically on the yield surface

ACMT NSV

Geometry of augmented space



Fundamental solutions

$$\mathbf{X}(t_{i+1}) = \mathbf{G}(\Delta t)\mathbf{X}(t_i)$$

- **Exponential map:**

$$\mathbf{G}(t) = \exp\left(\mathbf{A}t\right) = \begin{bmatrix} \mathbf{I}_n + \dfrac{a-1}{\|\dot{\mathbf{q}}\|^2}\dot{\mathbf{q}}\dot{\mathbf{q}}^T & \dfrac{b}{\|\dot{\mathbf{q}}\|}\dot{\mathbf{q}} \\ \dfrac{b}{\|\dot{\mathbf{q}}\|}\dot{\mathbf{q}}^T & a \end{bmatrix}$$

where $a = \cosh(t\|\dot{\mathbf{q}}\|/q_y), b = \sinh(t\|\dot{\mathbf{q}}\|/q_y)$

- **Cayley transformation:**

$$\mathbf{G}(t) = \left(\mathbf{I} - \frac{t}{2}\mathbf{A}\right)^{-1}\left(\mathbf{I} + \frac{t}{2}\mathbf{A}\right)$$

Hand-in

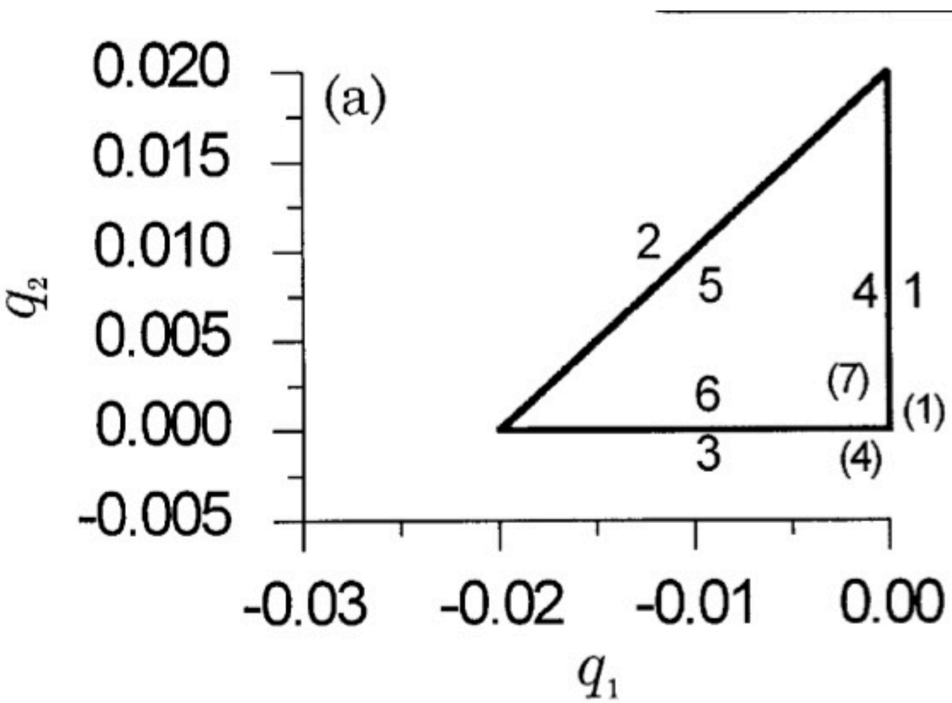
Please calculate $\mathbf{Q}(t_1)$ and $\mathbf{Q}(t_2)$, where $t_{i+1} = t_i + \Delta t, i = 0,1,$ by the group preserving integration where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^2$, $k_e =$ 10 GPa, $Q_y =$ 4000 MPa, $\mathbf{Q}(t_0) = \begin{bmatrix} Q_y \\ 0 \end{bmatrix}$, and $\dot{\mathbf{q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Please compare the $\mathbf{Q}(t_1)$ and $\mathbf{Q}(t_2)$ with the result from the the substepping integration.

Coding

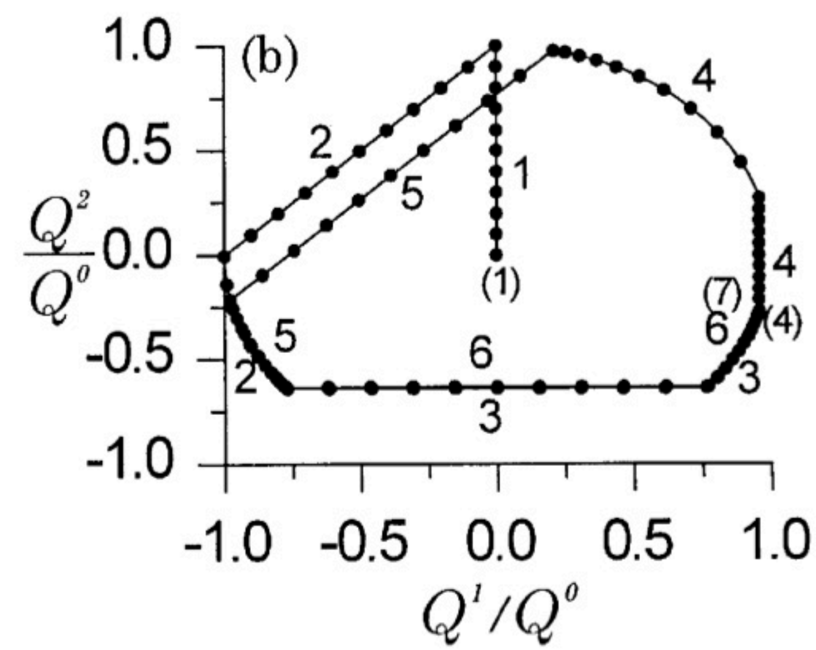
Please code plastic module to compute the $\mathbf{Q}(t_i)$ response with varying timestep Δt by the group preserving integration where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^2, k_e = 10$ GPa, $Q_y = 4000$ MPa, $\mathbf{Q}(t_0) = \begin{bmatrix} \frac{Q_y}{k_e} \\ 0 \end{bmatrix}$, and $\dot{\mathbf{q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

A Model With Perfect Elastoplasticity

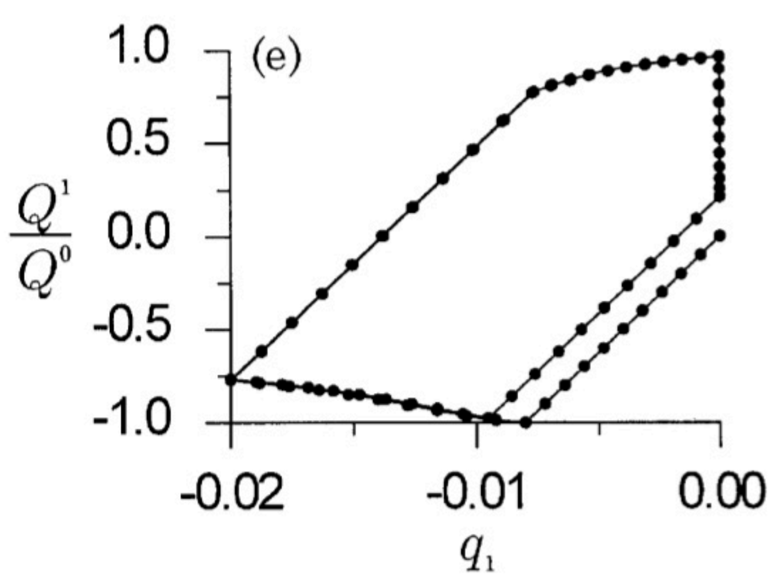
Input of the generalized strain path

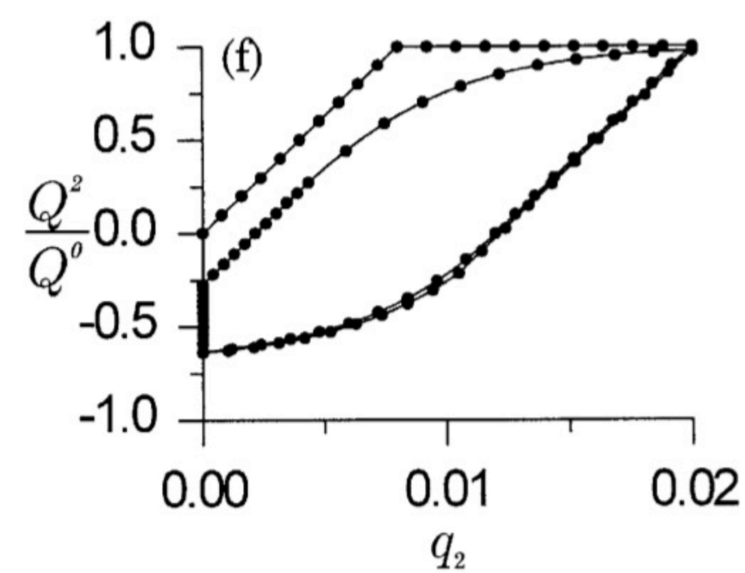


Output of the generalized strain path



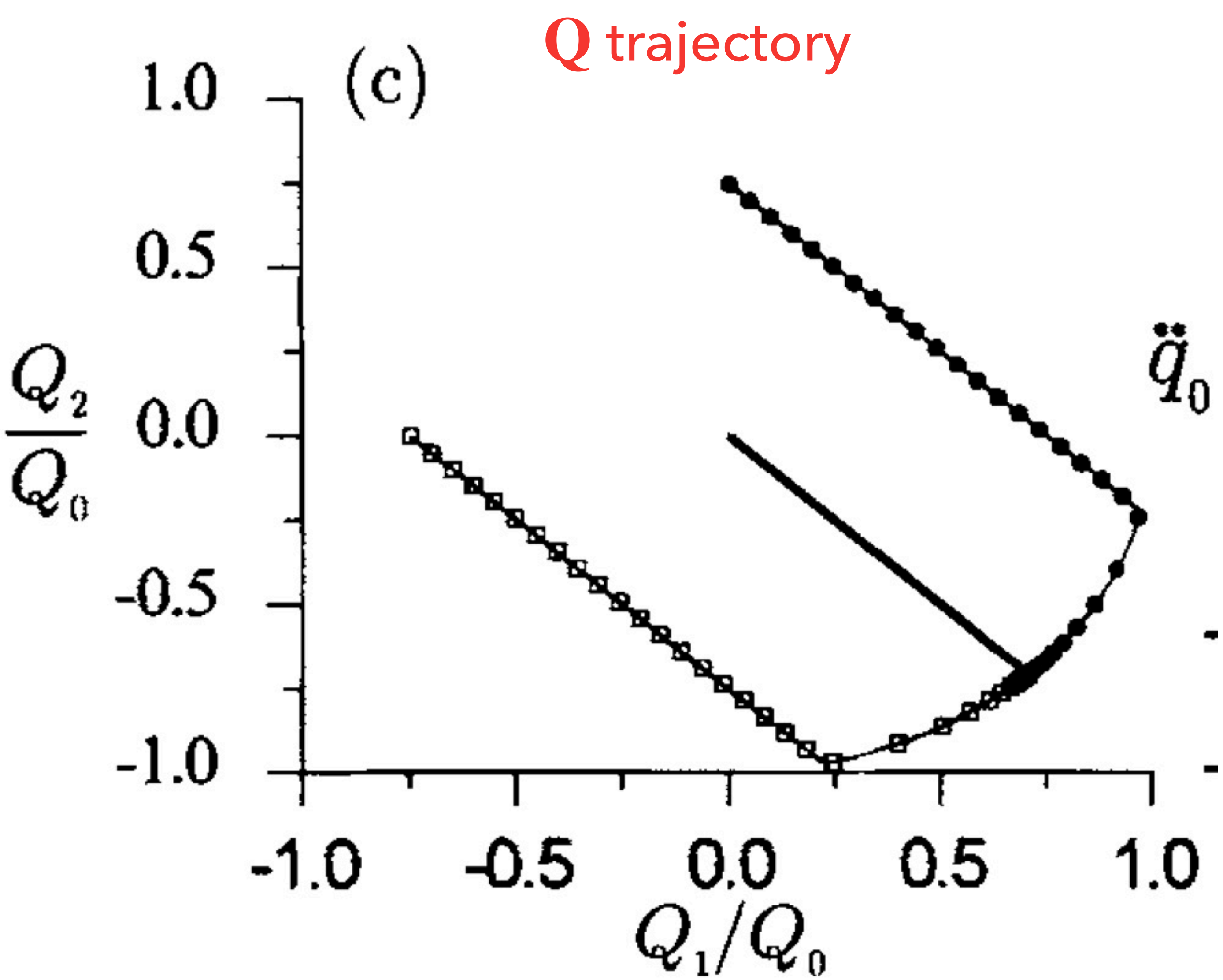
Stress-Strain curve

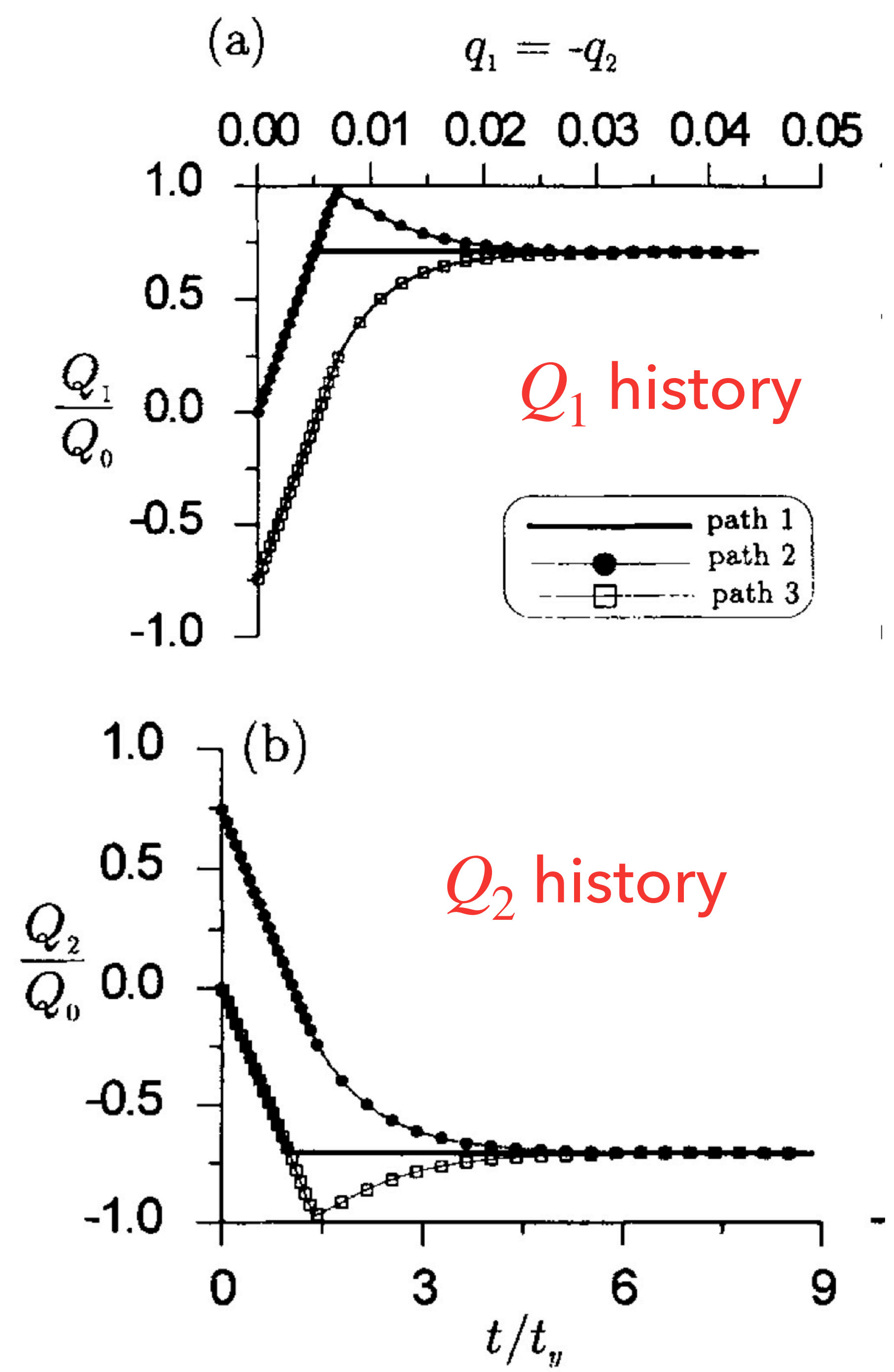




Hong H.K., Liu C.S. Internal symmetry in the constitutive model of perfect elastoplasticity. International Journal of Non-Linear Mechanics, 2000; 35: 447–466.

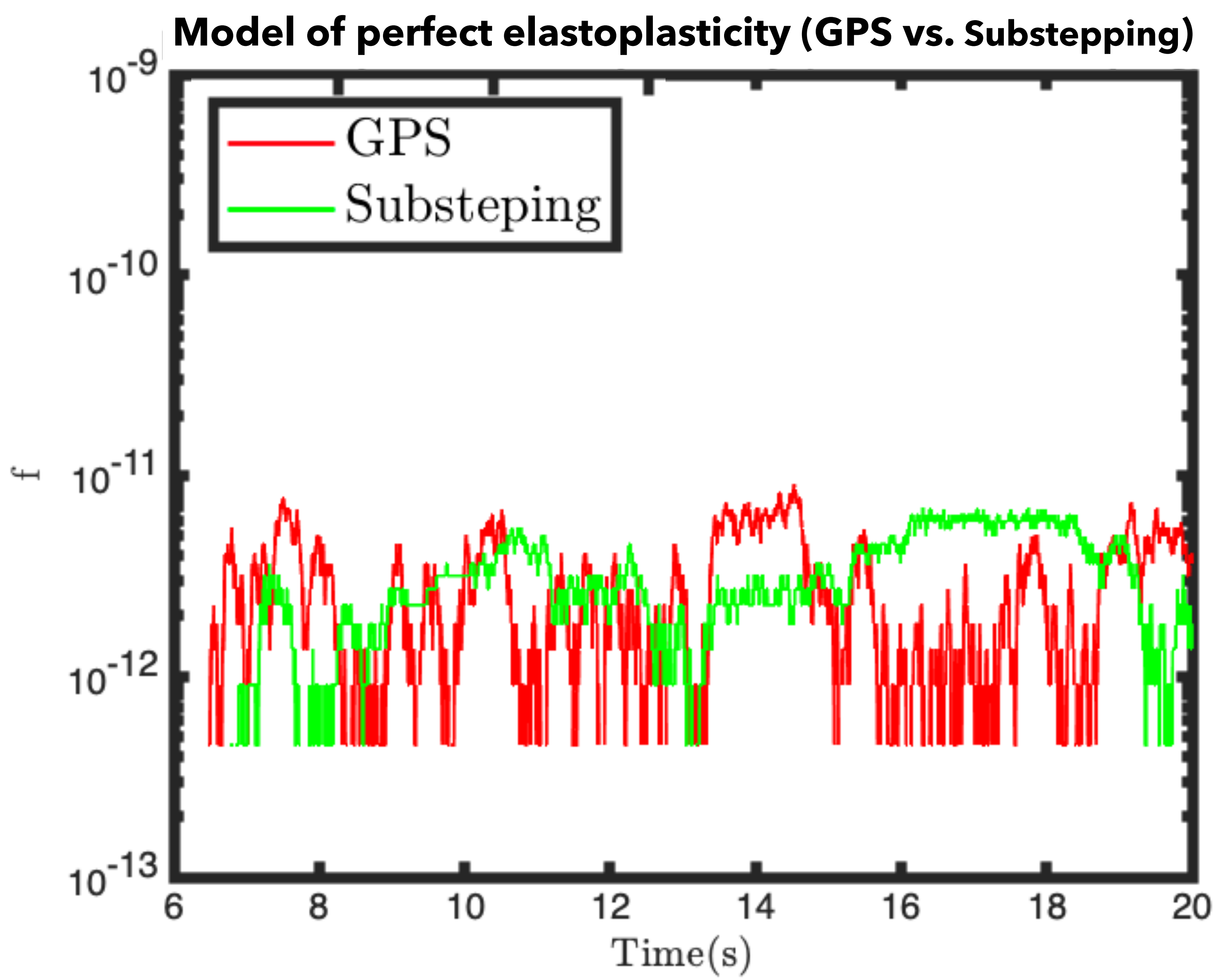
Influence of initial conditions

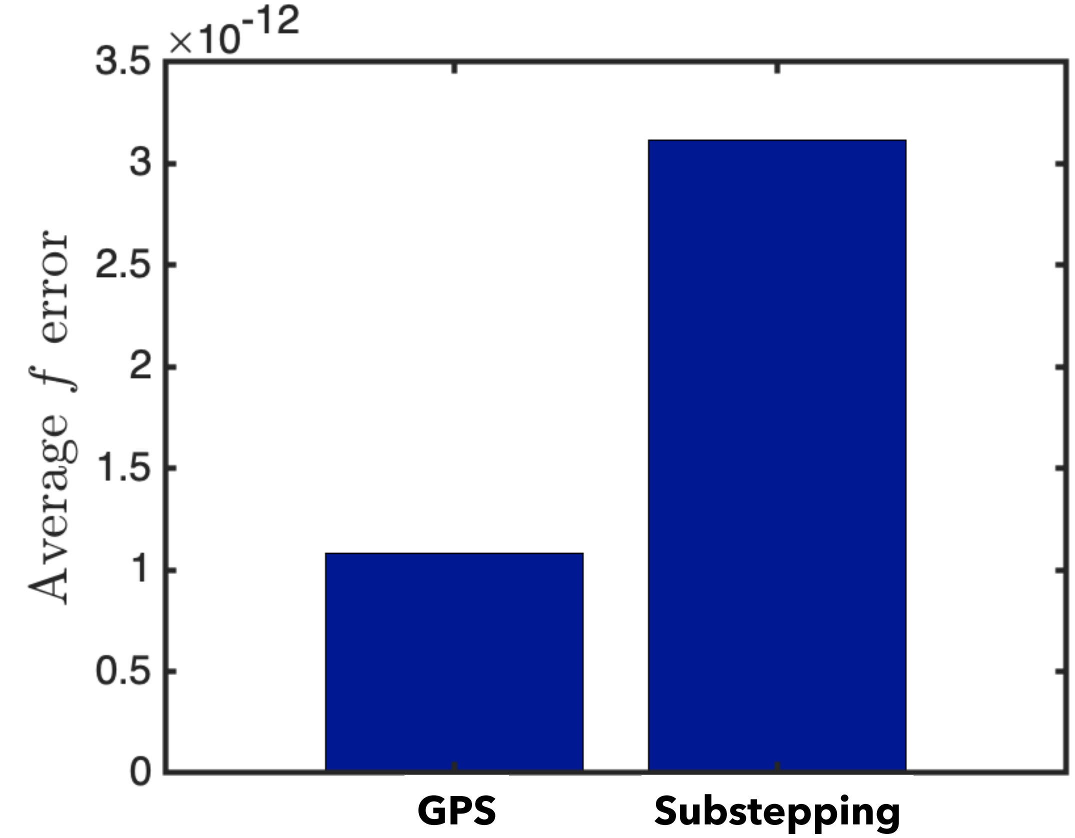




Hong H.K., Liu C.S. On behavior of perfect elastoplasticity under rectilinear paths. International Journal of Solids and Structures ,1998; 35 :3539–3571.

Model of perfect elastoplasticity





Elastoplastic models with kinematic hardening

- elastic constitutions

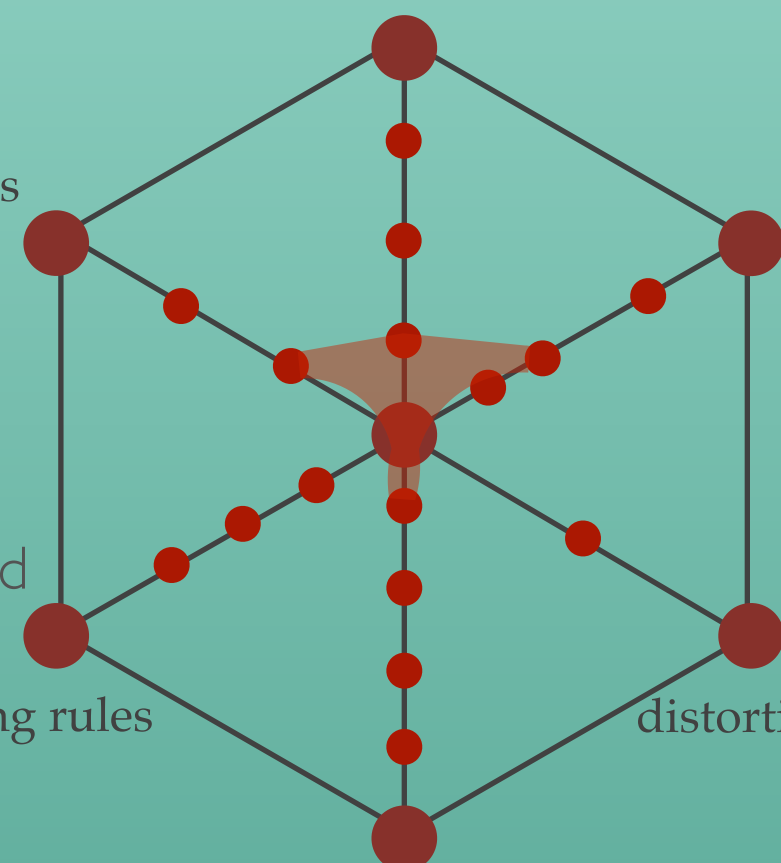
aniso+linear aniso+nonlinear

iso+linear iso+nonlinear

- stress admissible conditions

linear quadratic+linear

quadratic high order (cubic)



- plastic flow rules

generalized associated

associated non-associated

- kinematic hardening

Prager O-W

non A-F A-K

A Model With Bilinear Elastoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p, \qquad \mathbf{q}, \mathbf{q}^e, \mathbf{q}^p \in \mathbb{R}^n,$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, \qquad \mathbf{Q}, \mathbf{Q}_a, \mathbf{Q}_b \in \mathbb{R}^n,$$

$$\mathbf{Q} = k_e \mathbf{q}^e, \qquad k_e > 0, Q_y > 0, k_p > 0$$

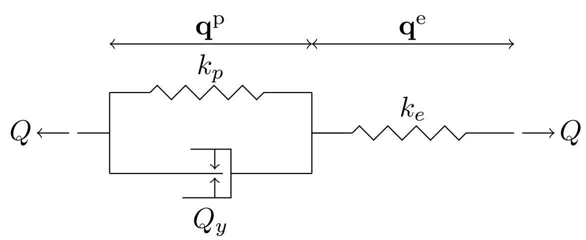
$$\mathbf{Q}_b = k_p \mathbf{q}^p, \qquad \text{given } k_e, Q_y, k_p \in \mathbb{R},$$

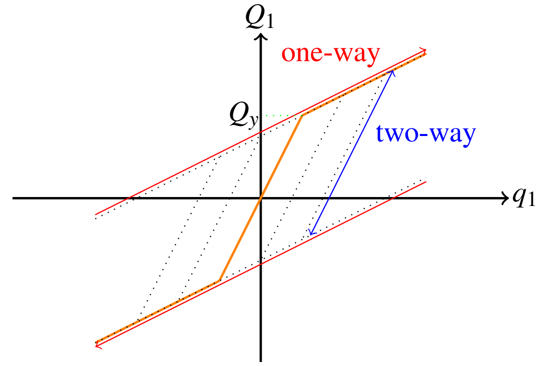
$$Q_y \dot{\mathbf{q}}^p = \mathbf{Q}_a \dot{\lambda}, \qquad \dot{()} = d()/dt,$$

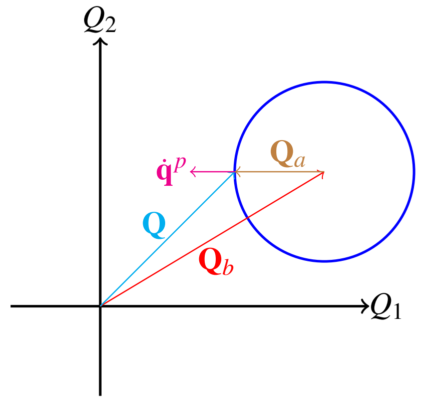
$$f\dot{\lambda} = 0,$$

$$f = \| \mathbf{Q}_a \| - Q_y \leq 0, \qquad \| \mathbf{Q}_a \| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a},$$

$$\dot{\lambda} \geq 0,$$







Two-phase dynamical system

$$\frac{d}{dt}\begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\dot{\lambda}}{Q_\mathrm{y}} \\ 0 & -\frac{k_p \dot{\lambda}}{Q_\mathrm{y}} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{Q}}/k_e \\ \dot{\mathbf{Q}} \end{bmatrix} \text{ if } f = 0 \text{ and } \mathcal{S} > 0\text{, plastic phase, where } \dot{\lambda} = \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{k_p Q_y}$$

$$\frac{d}{dt}\begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{Q}}/k_e \\ \dot{\mathbf{Q}} \end{bmatrix} \text{ if } f < 0 \text{ or } \mathcal{S} \leq 0\text{, elastic phase,}$$

Group preserving integration

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_s \\ X_0 \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{Q}_a}{Q_y} X_0 \\ \exp\left(\frac{k_p \lambda}{Q_y}\right) \end{bmatrix}$$

$$\frac{\dot{\mathbf{Q}}}{\mathbf{Q}_y} X_0 = \frac{d}{dt}\left(\frac{\mathbf{Q}_a}{Q_y} X_0\right) \longrightarrow \dot{\mathbf{X}}_s = \frac{\dot{\mathbf{Q}}}{Q_y} X_0 \,.$$

$$\frac{\dot{\mathbf{Q}}^T}{Q_y} \frac{\mathbf{Q}_a}{Q_y} X_0 = \frac{k_p}{Q_y} \dot{\lambda} X_0 \longrightarrow \frac{\dot{\mathbf{Q}}^T}{Q_y} \mathbf{X}_s = \dot{X}_0$$

$$\dot{\mathbf{X}} = \mathbf{A}_{\text{on}}\, \mathbf{X} \text{ i.e. } \begin{bmatrix} \dot{\mathbf{X}}_s \\ \dot{X}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \frac{\dot{\mathbf{Q}}}{Q_y} \\ \frac{\dot{\mathbf{Q}}^T}{Q_y} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_s \\ X_0 \end{bmatrix}$$

$$\mathbf{X}(t_{i+1}) = \mathbf{G}_{\text{on}}\, \mathbf{X}\left(t_i\right)$$

$$\mathbf{G}_{\mathbf{on}} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\dot{\mathbf{Q}}\|^2} \dot{\mathbf{Q}} \dot{\mathbf{Q}}^T & \frac{b\mathbf{Q}}{\|\dot{\mathbf{Q}}\|} \\ \frac{b\dot{\mathbf{Q}}^T}{\|\dot{\mathbf{Q}}\|} & a \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{X}_s(t_{i+1}) \\ X_0(t_{i+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\dot{\mathbf{Q}}\|^2} \dot{\mathbf{Q}} \dot{\mathbf{Q}}^T & \frac{b\dot{\mathbf{Q}}}{\|\dot{\mathbf{Q}}\|} \\ \frac{b\dot{\mathbf{Q}}^T}{\|\dot{\mathbf{Q}}\|} & a \end{bmatrix} \begin{bmatrix} \mathbf{X}_s\left(t_i\right) \\ X_0\left(t_i\right) \end{bmatrix}$$

$$a = \cosh\left(\frac{\|\mathbf{Q}\|}{Q_y}\left(\Delta_t\right)\right) \text{ and } b = \sinh\left(\frac{\|\mathbf{Q}\|}{Q_y}\left(\Delta_t\right)\right)$$

Obtain the $\mathbf{Q_a}$

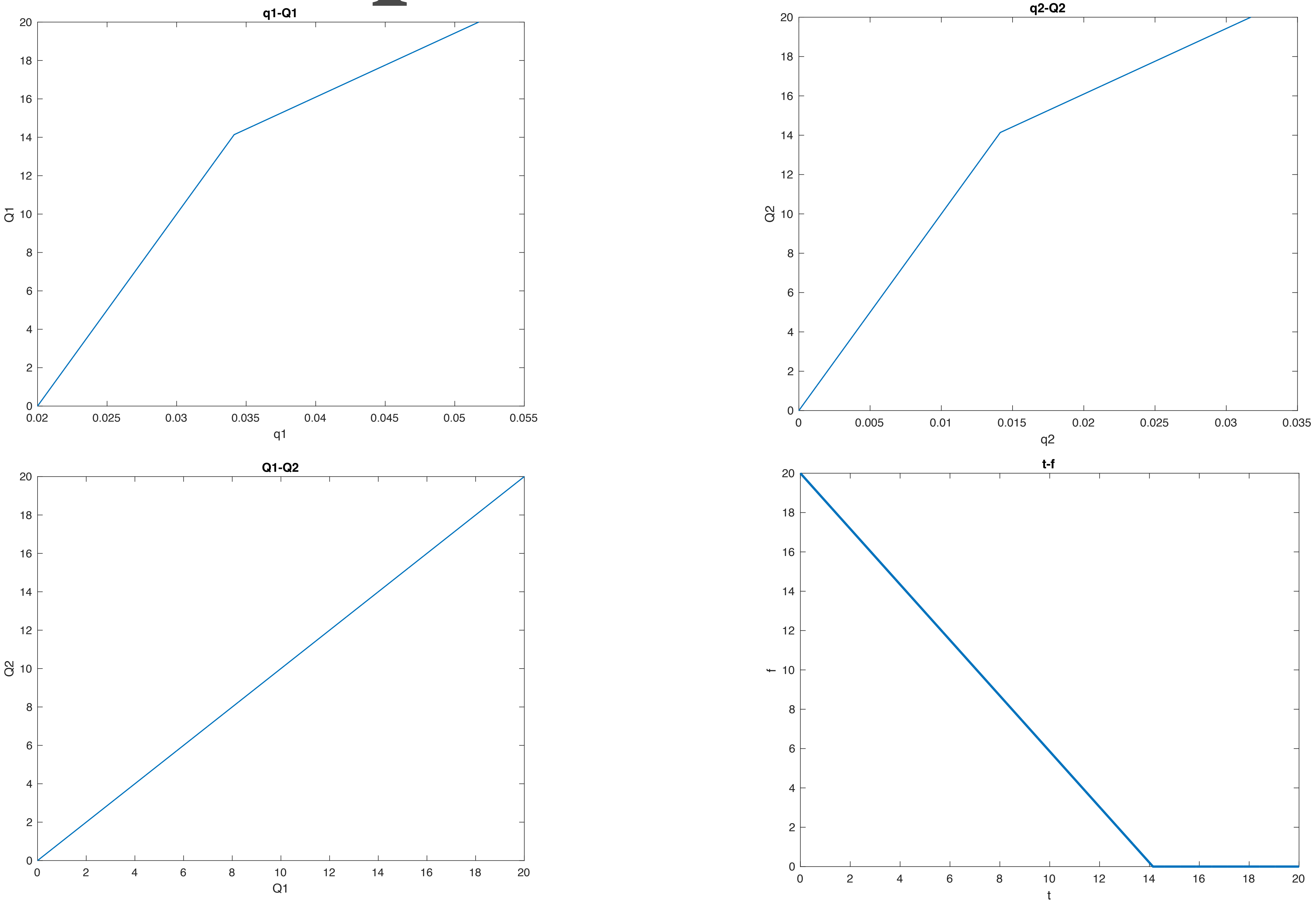
Hand-in

Please calculate $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$, where $t_{i+1} = t_i + \Delta t, i = 0,1,$ by the group preserving integration where $\mathbf{Q}, \mathbf{Q}_a, \mathbf{q} \in \mathbb{R}^2$, $k_e = 1000$, $Q_y = 20$, $k_p = 500$, $\mathbf{q}(t_0) = \begin{bmatrix} Q_y/k_e \\ 0 \end{bmatrix}$, and $\dot{\mathbf{Q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Please compare the $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$ with the result from the the substepping integration.

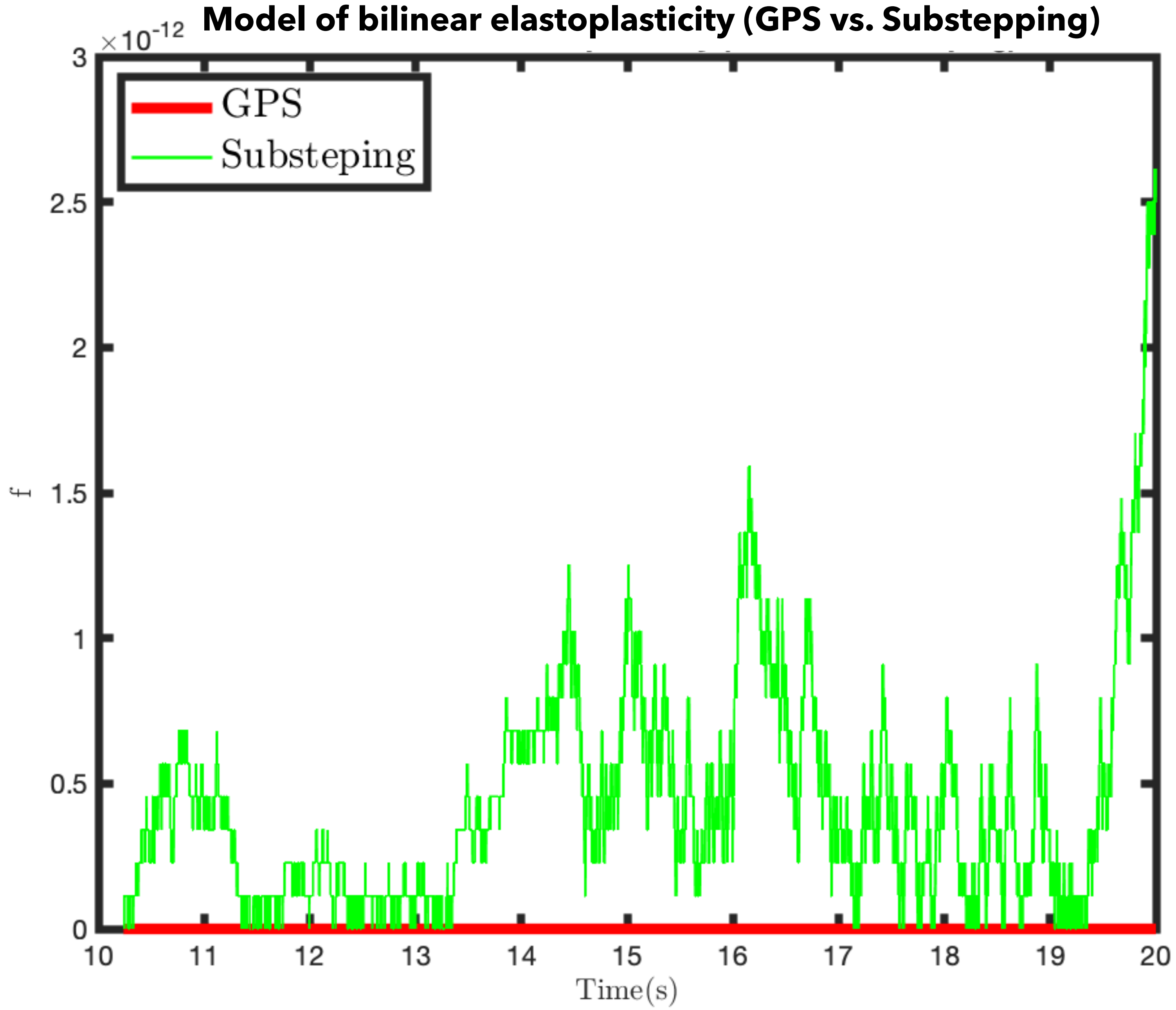
Coding

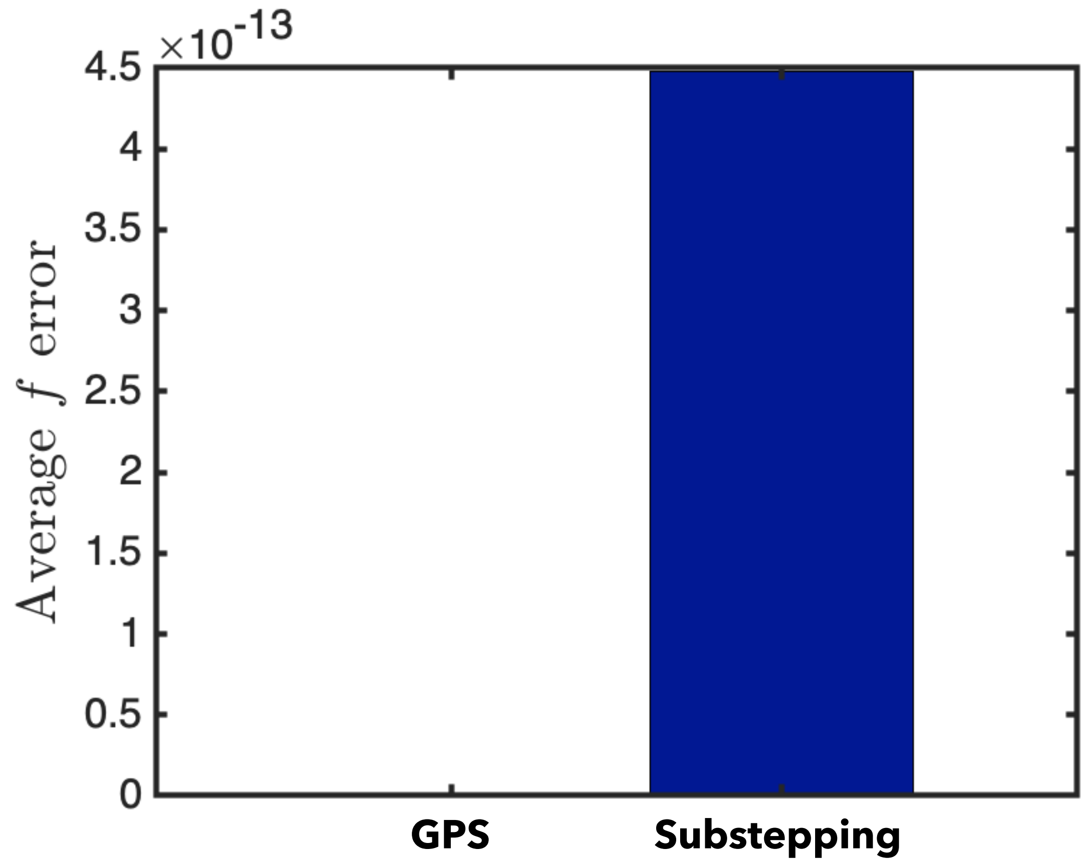
Please code plastic module to compute the $\mathbf{q}(t_i)$ response by the group preserving integration where $\mathbf{Q}, \mathbf{Q}_a, \mathbf{q} \in \mathbb{R}^2$, $k_e = 1000$, $Q_y = 20$, $k_p = 500$, $\mathbf{q}(t_0) = \begin{bmatrix} Q_y/k_e \\ 0 \end{bmatrix}$, and $\dot{\mathbf{Q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Computational results



Model of bilinear elastoplasticity





Codes

https://www.dropbox.com/scl/fo/
moyseuk7u4hdvxl4a1ea1/h?
rlkey=84kdqajokztl9wkmf68xwiaik&dl=0

Manual of codes

https://www.dropbox.com/scl/fi/
mu32gxpt73pgjk2rb97d5/
ACMT_short_course_2023.pdf?
rlkey=q67ghlili8yrn76ssju4h8mqw&dl=0

Structure preserving integrations for physics

Structure preserving integrations

- Symmetric Runge-Kutta methods
- Symplectic Euler methods
- Symplectic Rugne-Kutta methods
- Variational integrations
- Poisson integrations
- Multi-symplectic integrations

Summary

Review

Plastic model

1. elastic-plastic decomposition
2. elastic constitutions
3. plastic flow rules
4. hardening / softening rules
   (isotropic, kinematic, and distortion)
5. stress admissible conditions
6. non-negative conditions
7. alternative conditions

Elastic phase
ODEs or AEs

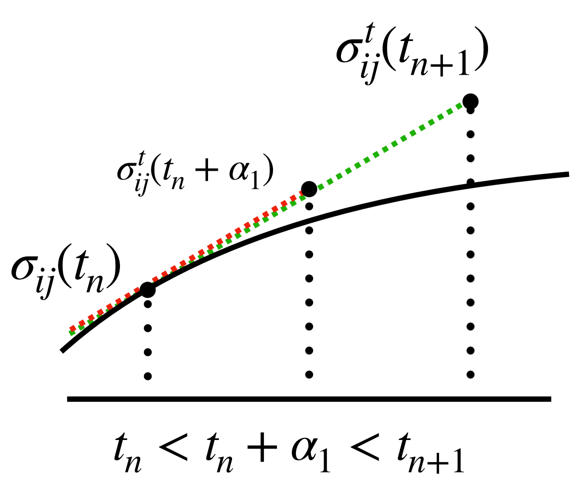
Plastic phase
ODEs & AEs

Elastic module
ODEs or AEs

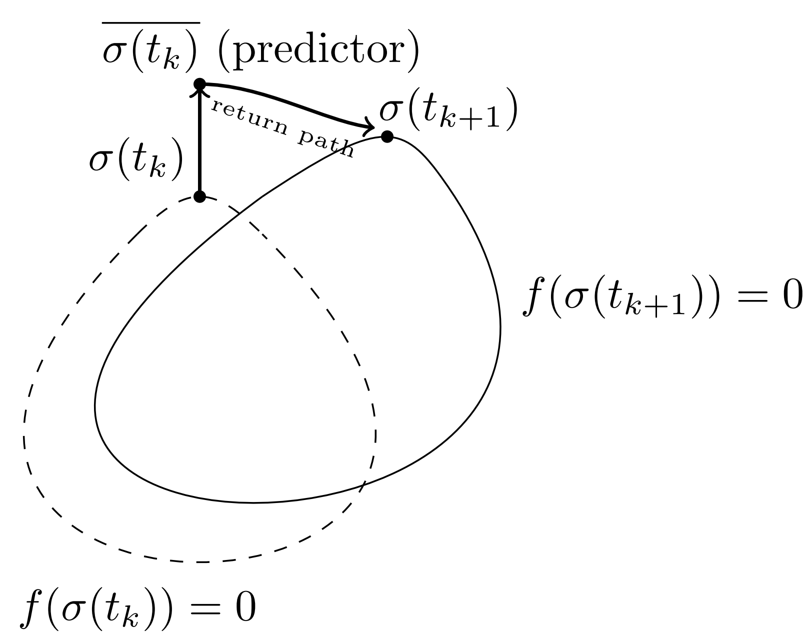
Pullback module

Plastic module
ODEs & AEs

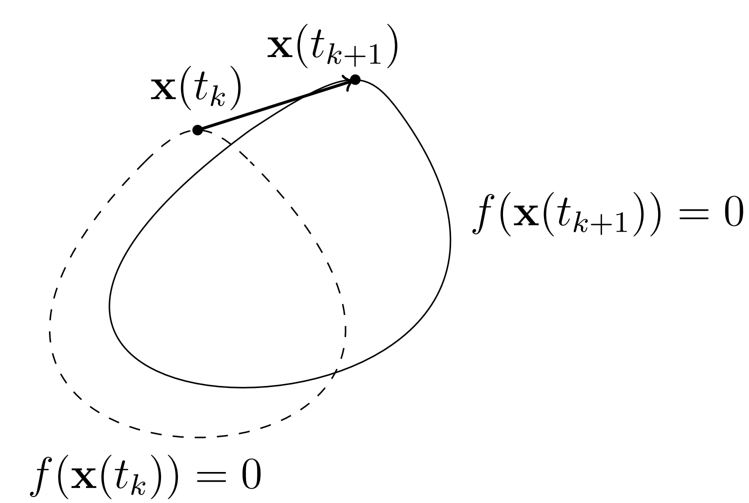
Sub-stepping integrations



Return-mapping integrations



Group preserving integrations



Thanks for your attention