Computational Plasticity

Chapter 2 – First step in computational plasticity

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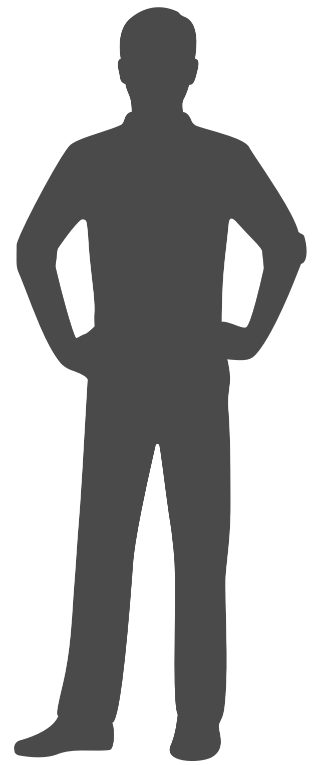
Overview

- Euler methods
- Yield condition and pullback
- Substepping integrations



Solid mechanics

Strain-controlled problems

q-controlled problems

Q-controlled problems

Structural mechanics

Force-controlled problems

Q-controlled problems

Euler methods

The family of Euler's methods

- The forward Euler scheme (the explicit Euler scheme)

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \Delta t \mathbf{f}(t_k, \mathbf{X}_k)$$

- The backward Euler scheme (the implicit Euler scheme)

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \Delta t \mathbf{f}(t_{k+1}, \mathbf{X}_{k+1})$$

- The θ-method
- $\mathbf{X}_{k+1} = \mathbf{X}_k + \Delta t \left[\theta \mathbf{f}(t_k, \mathbf{X}_k) + (1-\theta)\mathbf{f}(t_{k+1}, \mathbf{X}_{k+1})\right], \quad \theta \in [0,1]$

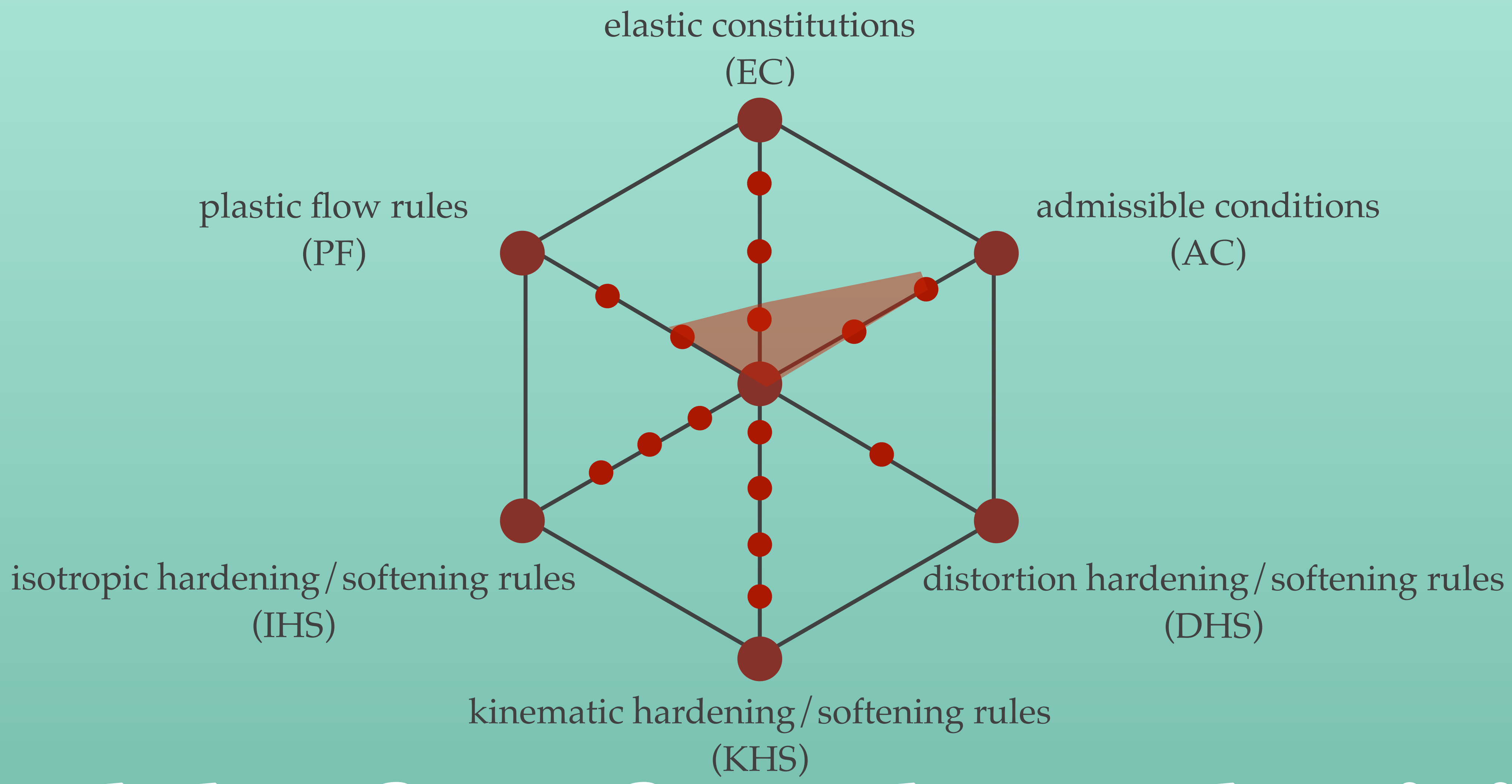
Hand-in

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$$

If $\mathbf{x}(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, please calculate $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ where

$t_{i+1} = t_i + \Delta t$ by using the forward Euler method.

Yield condition and pullback



Models of perfect elastoplasticity

Perfectly elastoplastic models–generalized model 1

Generalized models of perfect elastoplasticity

Elastic-plastic decomposition $\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}}$

Elastic constitutions $\mathbf{Q} = k_e \mathbf{q}^{\mathrm{e}}$

Associated plastic flow rules $\dot{\mathbf{q}}^{\mathrm{p}} = \dfrac{\partial f}{\partial \mathbf{Q}} \dot{\lambda} \Longleftrightarrow d\mathbf{q}^{\mathrm{p}} = \dfrac{\partial f}{\partial \mathbf{Q}} d\lambda\,.$ (incremental form)

Admissible conditions $f = \|\mathbf{Q}\| - Q_y \leq 0$

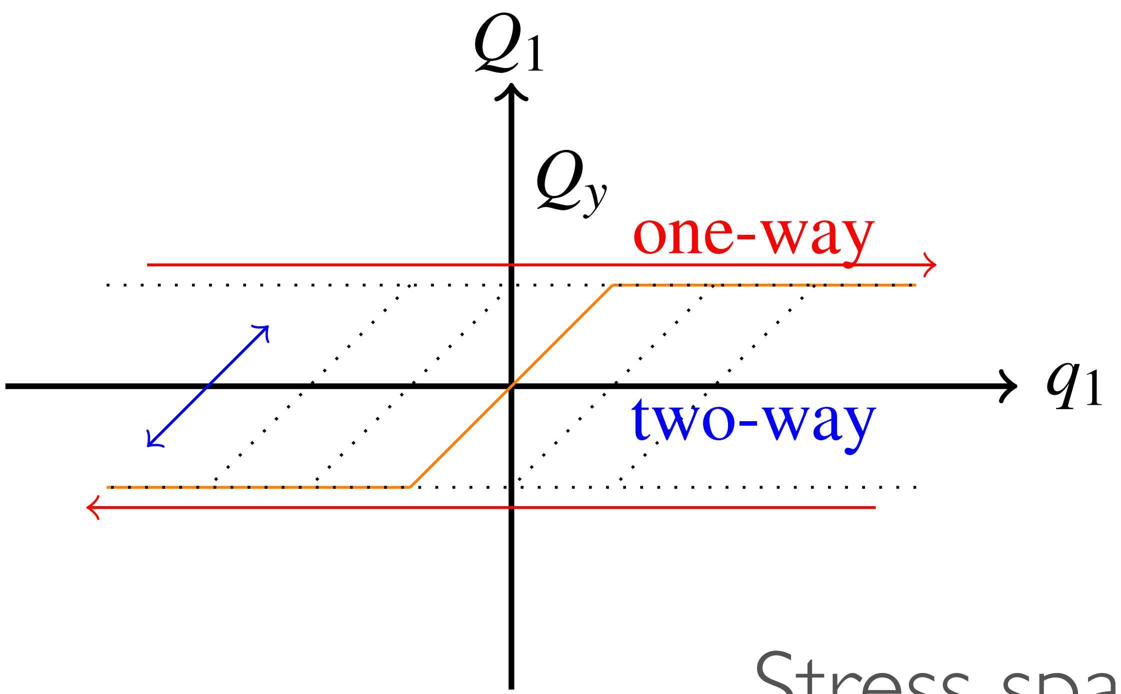
Non-negative dissipation $\dot{\lambda} \geq 0 \Longleftrightarrow d\lambda \geq 0$

Alternative $f\dot{\lambda} = 0 \Longleftrightarrow f d\lambda = 0$

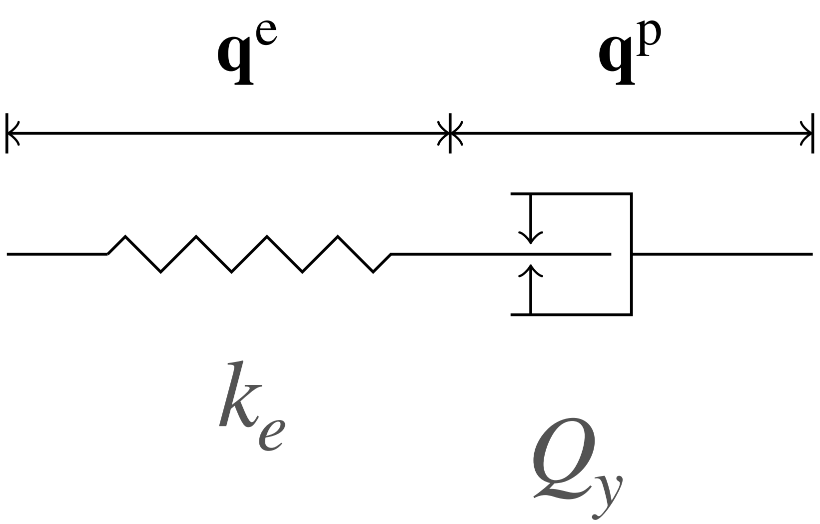
Shift-Invariant ! time-independent plasticity

Generalized models of perfect elastoplasticity

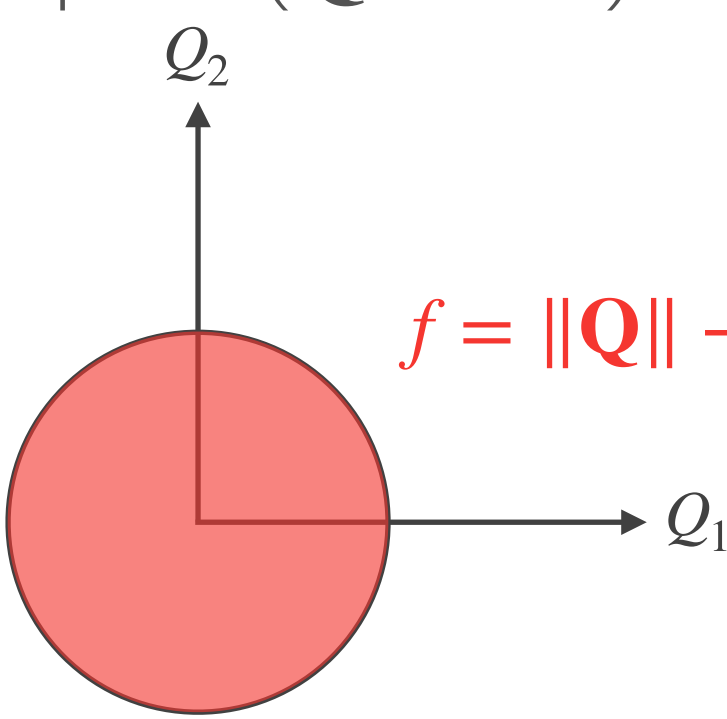
stress-strain curve



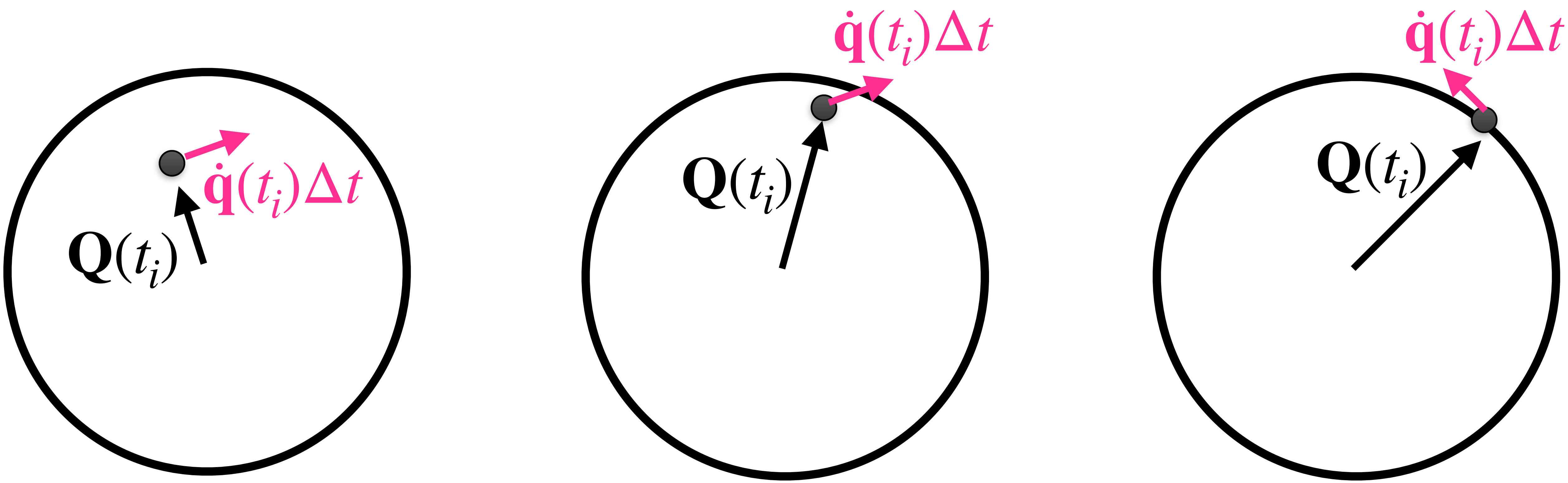
mechanical element



Stress space ($\mathbf{Q} \in \mathbb{R}^2$)



Three types of calculations



Elastic module
$\dot{\mathbf{Q}} = k_e \dot{\mathbf{q}}$

Pullback module

Plastic module

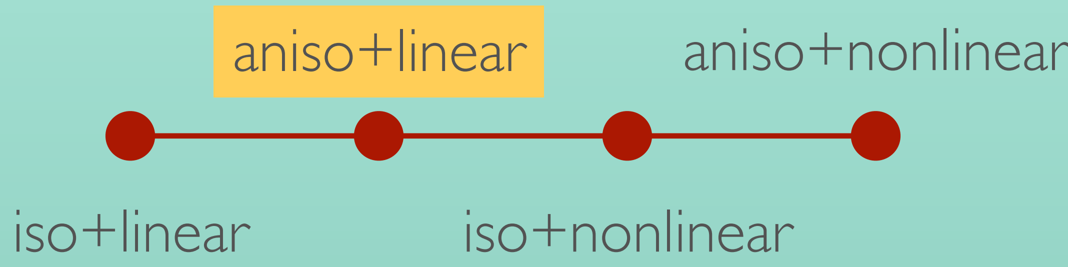
Hand-in

For the case of $f = \|\mathbf{Q}(t_k)\| - Q_y < 0$ and

$f = \|\mathbf{Q}(t_{k+1})\| - Q_y > 0$, please find the time increment τ where

$f = \|\mathbf{Q}(t_k + \tau)\| - Q_y = 0$ if $\mathbf{Q}(t_k + \tau) = \mathbf{Q}(t_k) + k_e \dot{\mathbf{q}}(t_k)\tau$.

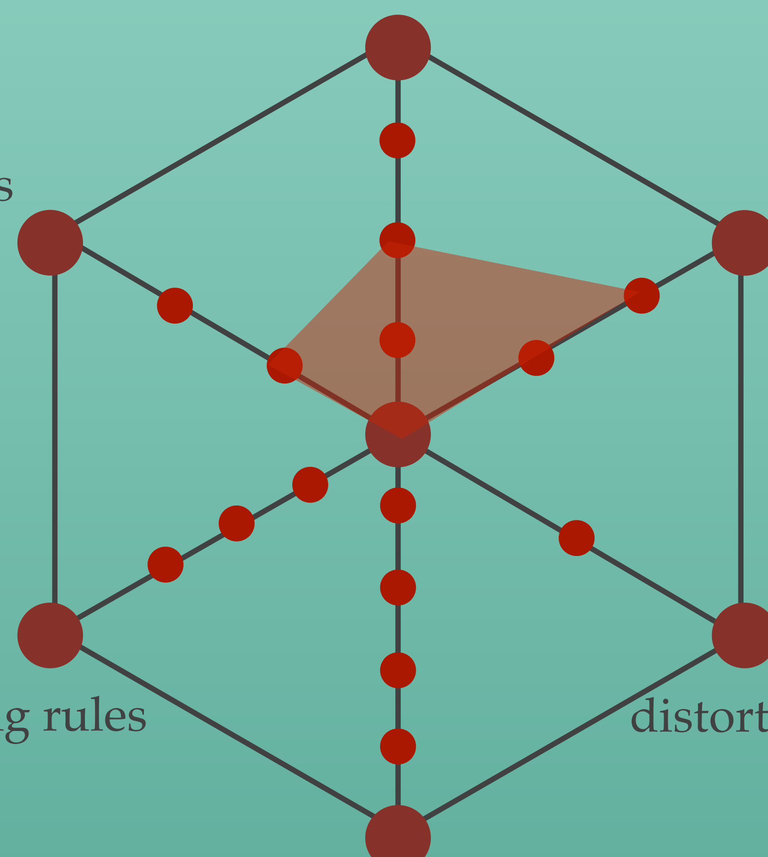
- elastic constitutions



- stress admissible condition

Perfectly elastoplastic models

–Generalized model 2

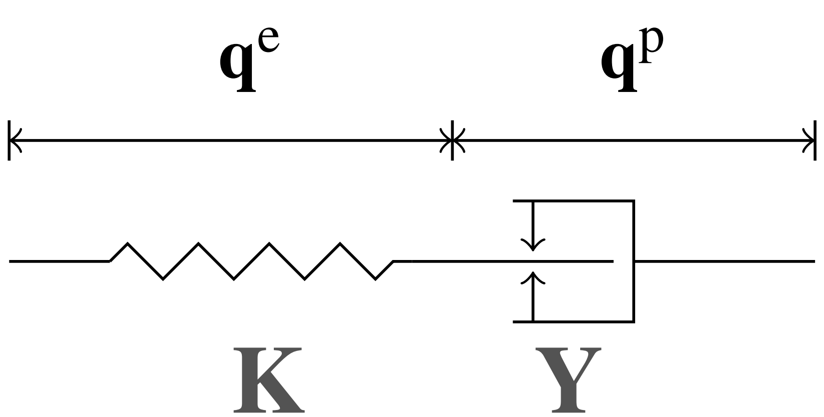


Generalized models of perfect elasticity with quadratic yield surface

- Mathematical formulation

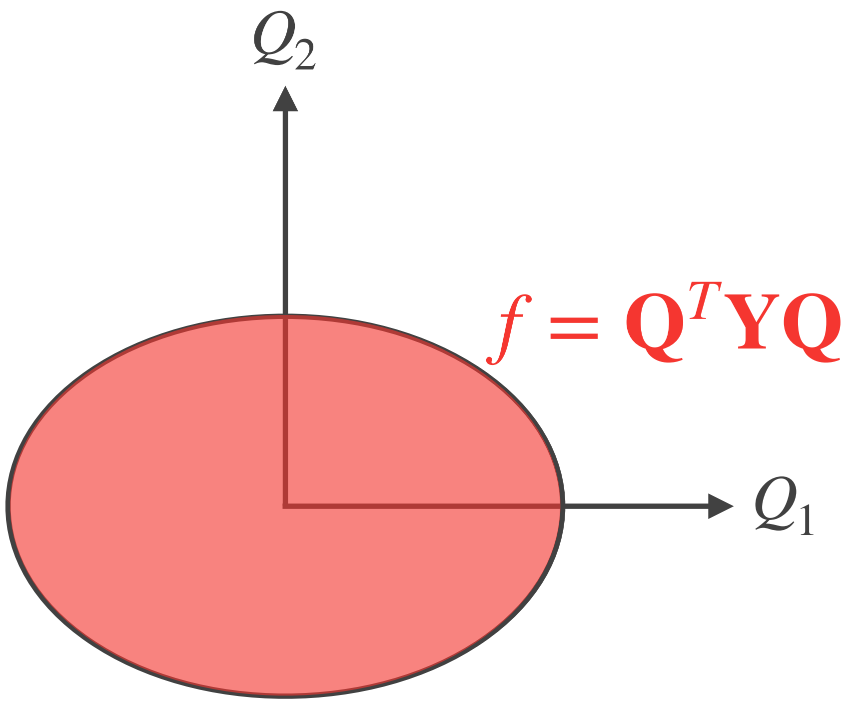
$$\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},$$
$$\mathbf{Q} = \mathbf{K}\mathbf{q}^{\mathrm{e}},$$
$$\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}}\dot{\lambda},$$
$$f = \frac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} - 1 \leq 0,$$
$$\dot{\lambda} \geq 0,$$
$$f\dot{\lambda} = 0.$$

- Mechanical element



$\mathbf{K}, \mathbf{Y} \in \mathbb{R}^{N \times N}$

Stress space ($\mathbf{Q} \in \mathbb{R}^2$)



Hand-in

For the case of $f = \mathbf{Q}^T(t_k)\mathbf{Y}\mathbf{Q}(t_k) - 1 < 0$ and

$f = \mathbf{Q}^T(t_{k+1})\mathbf{Y}\mathbf{Q}(t_{k+1}) - 1 > 0$, please find the time increment τ

where $f = \mathbf{Q}^T(t_k + \tau)\mathbf{Y}\mathbf{Q}(t_k + \tau) - 1 = 0$ if

$\mathbf{Q}(t_k + \tau) = \mathbf{Q}(t_k) + \mathbf{K}\dot{\mathbf{q}}(t_k)\tau.$

Elastoplastic models with kinematic hardening

- elastic constitutions

aniso+linear aniso+nonlinear

iso+linear iso+nonlinear

- stress admissible conditions

linear quadratic+linear

quadratic high order (cubic)

- plastic flow rules

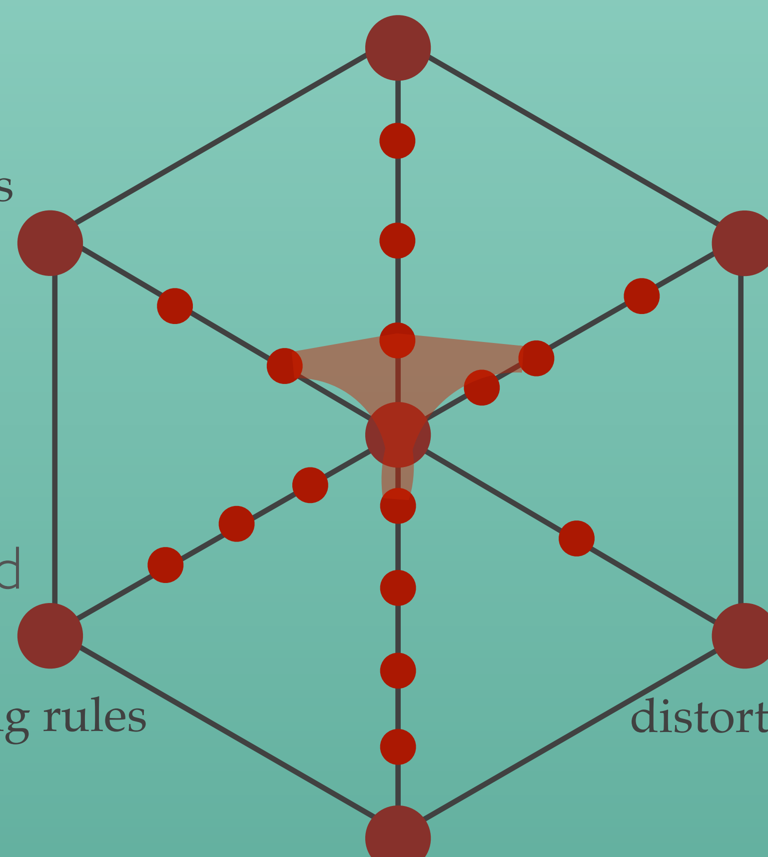
generalized associated

associated non-associated

- kinematic hardening

Prager O-W

non A-F A-K



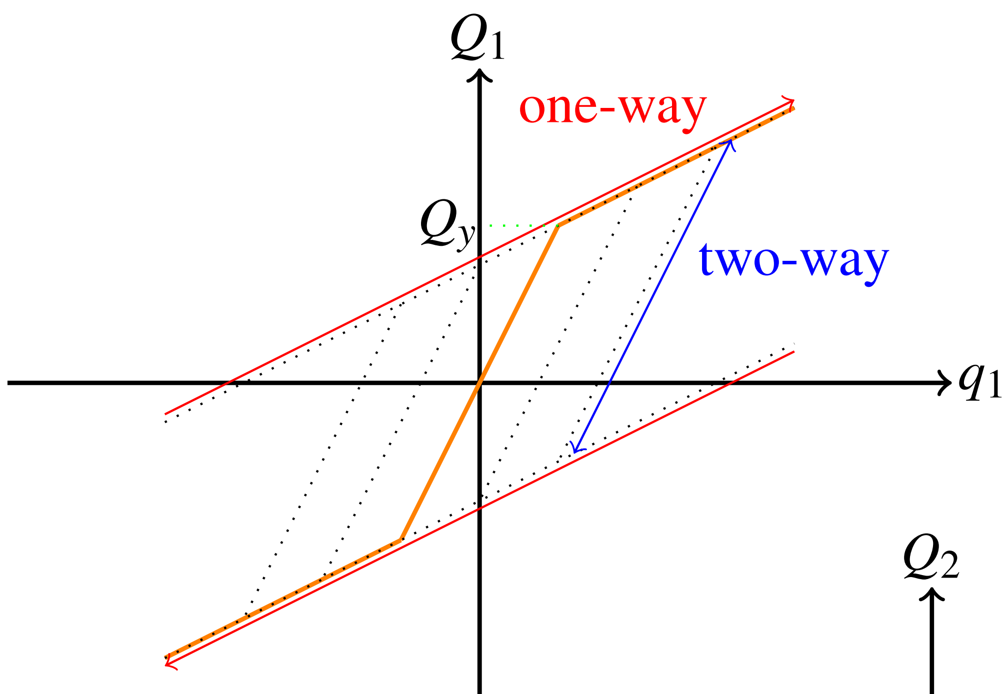
A bilinear model elastoplasticity with single yield surface

Model I

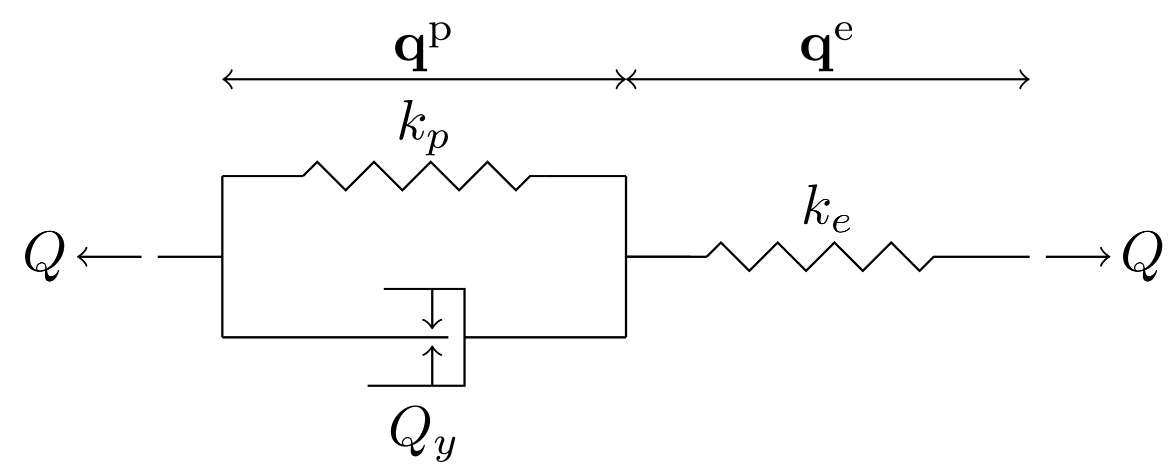
Elastic-plastic decomposition	$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p,$
Active-back decomposition	$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b,$
Elastic constitutions	$\mathbf{Q} = k_e \mathbf{q}^e,$
Associated plastic flow rules	$\dot{\mathbf{q}}^p = \dot{\lambda} \frac{\partial f}{\partial \mathbf{Q}}, \iff d\mathbf{q}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}} d\lambda,$
Prager's kinematic hardening rule	$\dot{\mathbf{Q}}_b = k_p \dot{\mathbf{q}}^p,$
Admissible conditions	$f = \lVert \mathbf{Q}_a \rVert^2 - Q_y^2 \leq 0,$
Non-negative dissipation	$\dot{\lambda} \geq 0,$
Alternative	$f \dot{\lambda} = 0.$

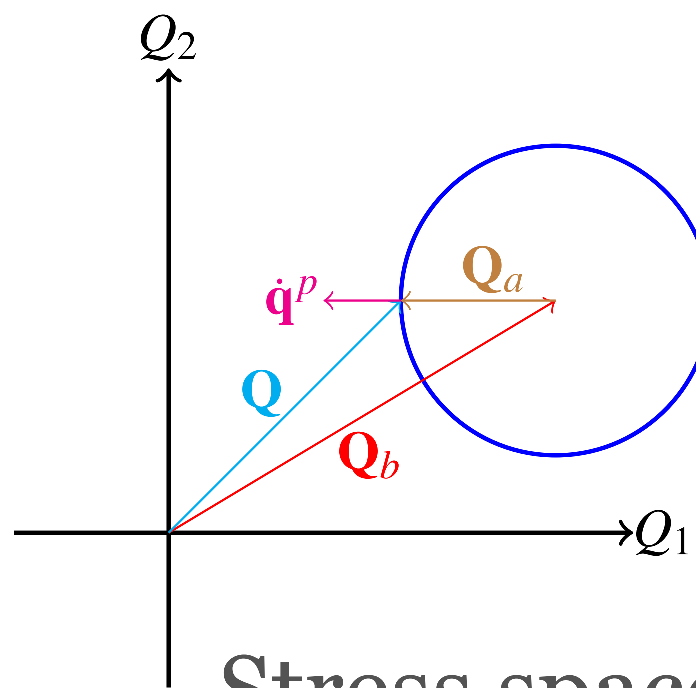
Generalized models of bilinear elastoplasticity

Stress-strain curve



Mechanical element





Stress space

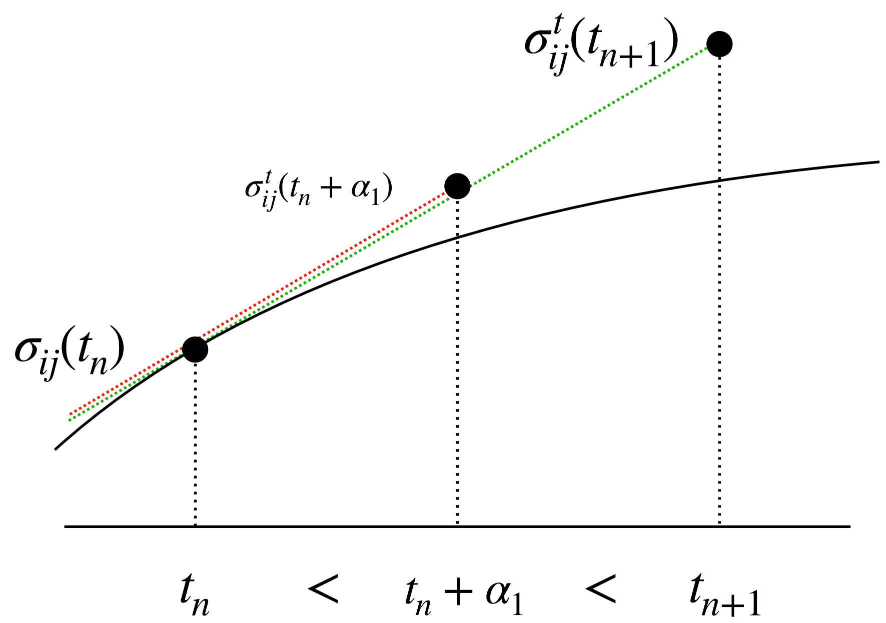
$$\mathbf{Q}^T\dot{\mathbf{q}}^p < 0$$

Dissipation

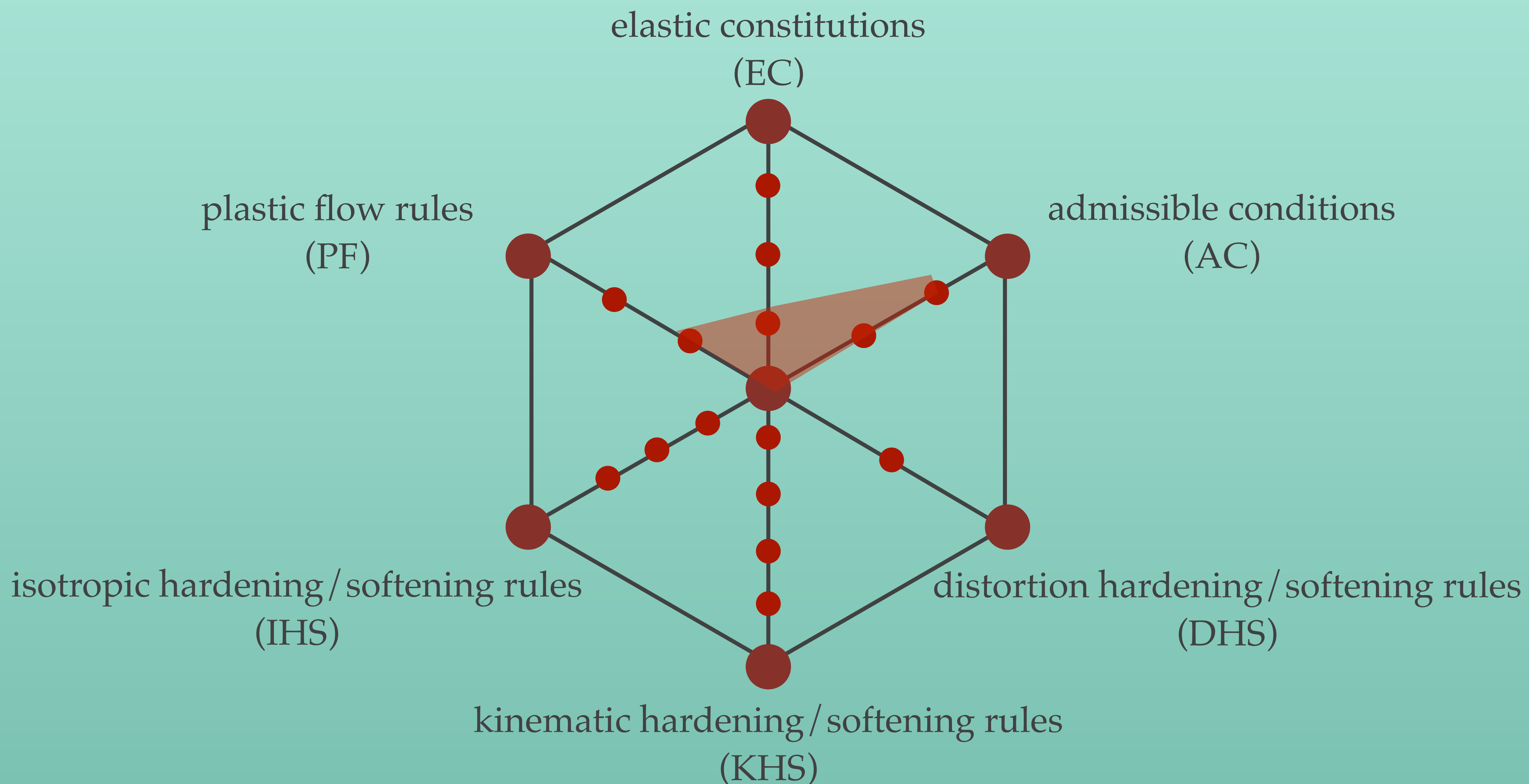
$$\mathscr{D} = \mathbf{Q}_a^T\dot{\mathbf{q}}^p > 0$$

Substepping integrations

Schematic diagram



- Sloan (1987)
- Polat & Dokainish (1989)
- Potts & Ganendra (1989)
- Sloan et al. (2011)
- Kaizhong (2007)



Models of perfect elastoplasticity

Perfectly elastoplastic models–generalized model 1

Model of perfect elastoplasticity

$$\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},$$

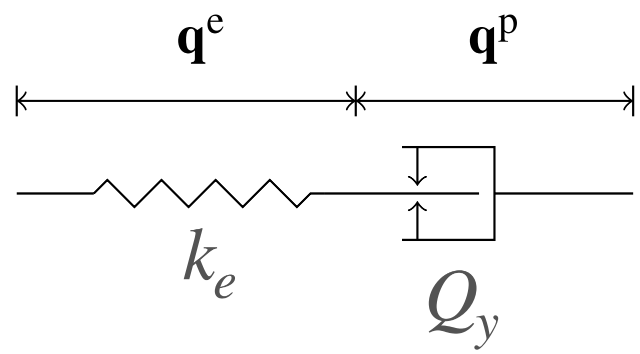
$$\mathbf{Q} = k_e \mathbf{q}^{\mathrm{e}},$$

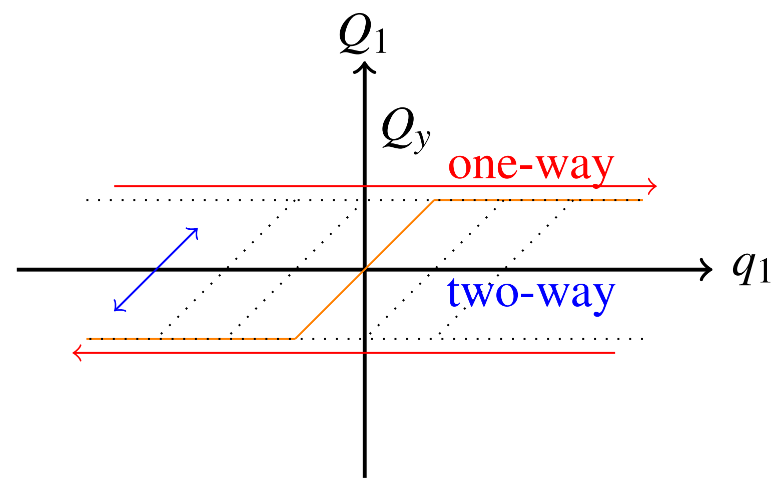
$$\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}} \dot{\lambda},$$

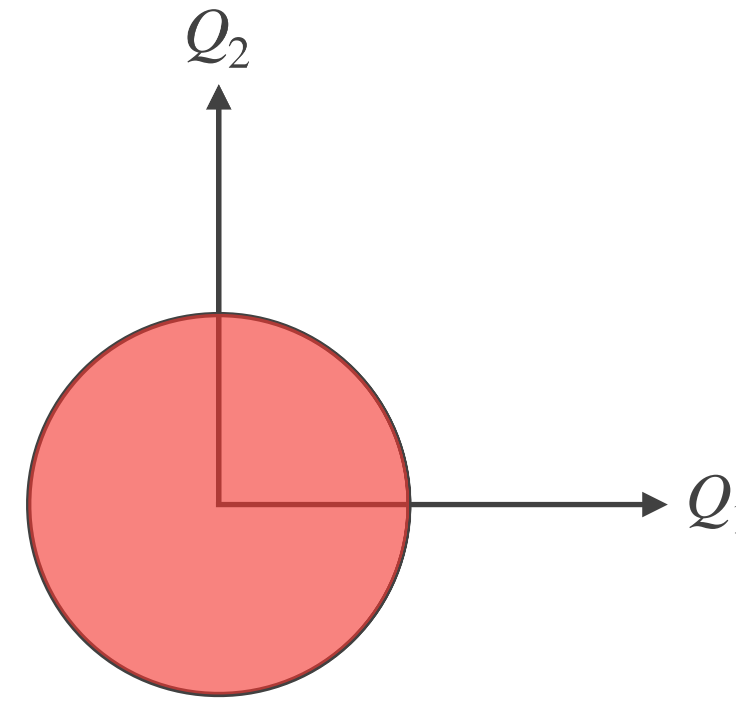
$$f = \|\mathbf{Q}\| - Q_y \leq 0,$$

$$\dot{\lambda} \geq 0,$$

$$f \dot{\lambda} = 0.$$







$f = \|\mathbf{Q}\| - Q_y \leq 0$

Two-phase dynamical system

$$\begin{cases} \dot{\mathbf{Q}} = -k_e \dfrac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}}, & \text{if } f = 0 \text{ and } \mathcal{S} > 0, \quad \text{plastic phase,} \\ \dot{\mathbf{Q}} = k_e \dot{\mathbf{q}}, & \text{if } f < 0 \text{ or } \mathcal{S} \leq 0, \quad \text{elastic phase.} \end{cases}$$

Hand-in

Please calculate $\mathbf{Q}(t_1)$ and $\mathbf{Q}(t_2)$, where $t_{i+1} = t_i + \Delta t, i = 0,1,$ by the forward Euler method to discretize the governing equation of the plastic phase

$$\dot{\mathbf{Q}} = -k_e \frac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}}$$

where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^2$, $k_e =$ 10 GPa, $Q_y =$ 4000 MPa, $\mathbf{Q}(t_0) = \begin{bmatrix} Q_y \\ 0 \end{bmatrix}$, and $\dot{\mathbf{q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

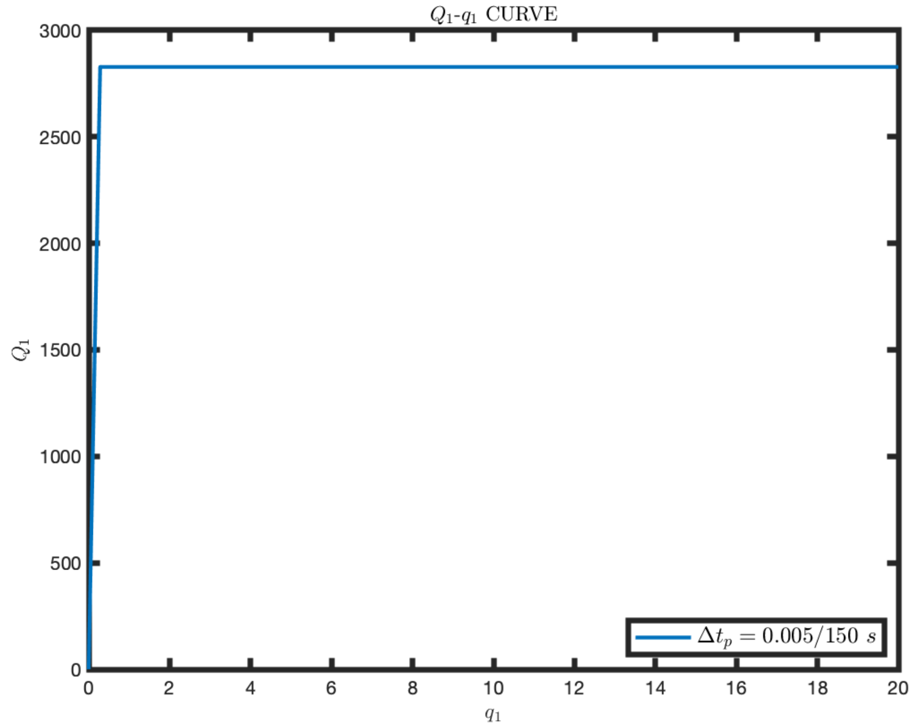
Coding

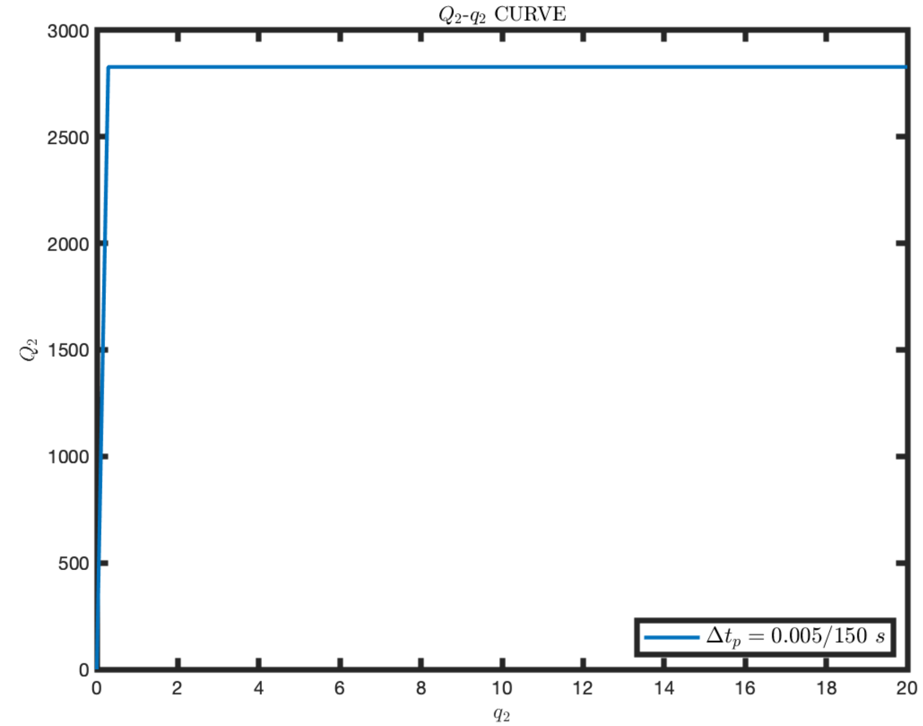
Please code plastic module to compute the $\mathbf{Q}(t_i)$ response with varying timestep Δt by the forward Euler method to discretize the governing equation of the plastic phase

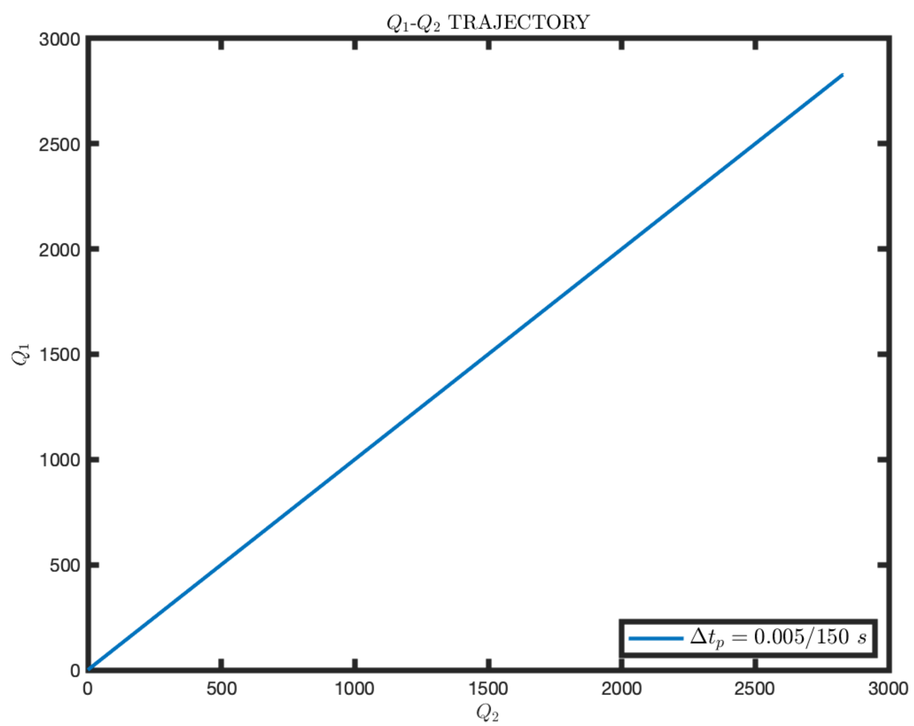
$$\dot{\mathbf{Q}} = -k_e \frac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}}$$

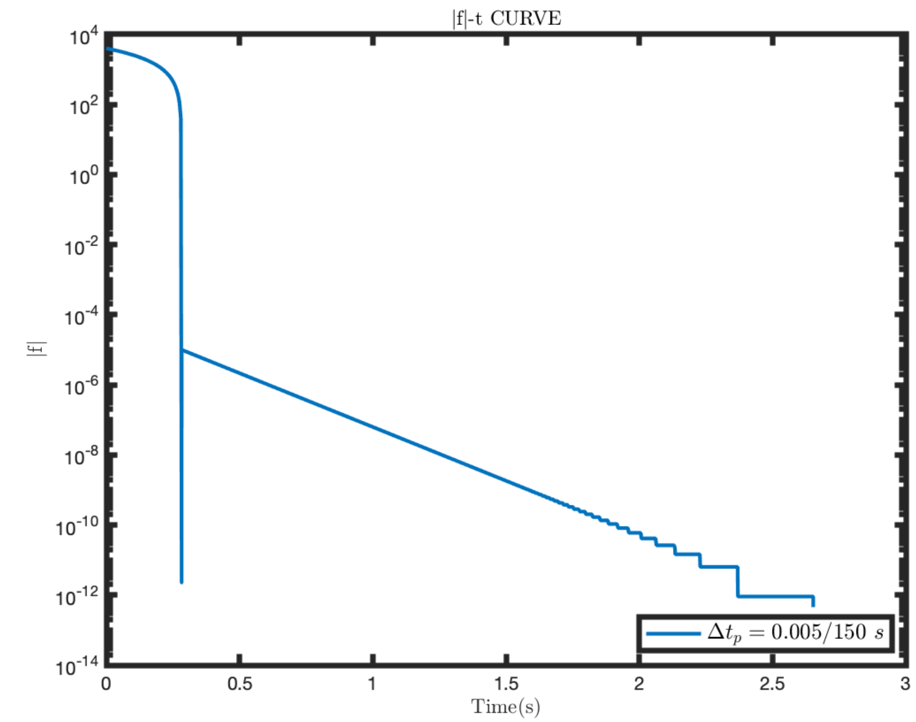
where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^2$, $k_e = 10$ GPa, $Q_y = 4000$ MPa, $\mathbf{Q}(t_0) = \begin{bmatrix} Q_y \\ 0 \end{bmatrix}$, and $\dot{\mathbf{q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Computational results

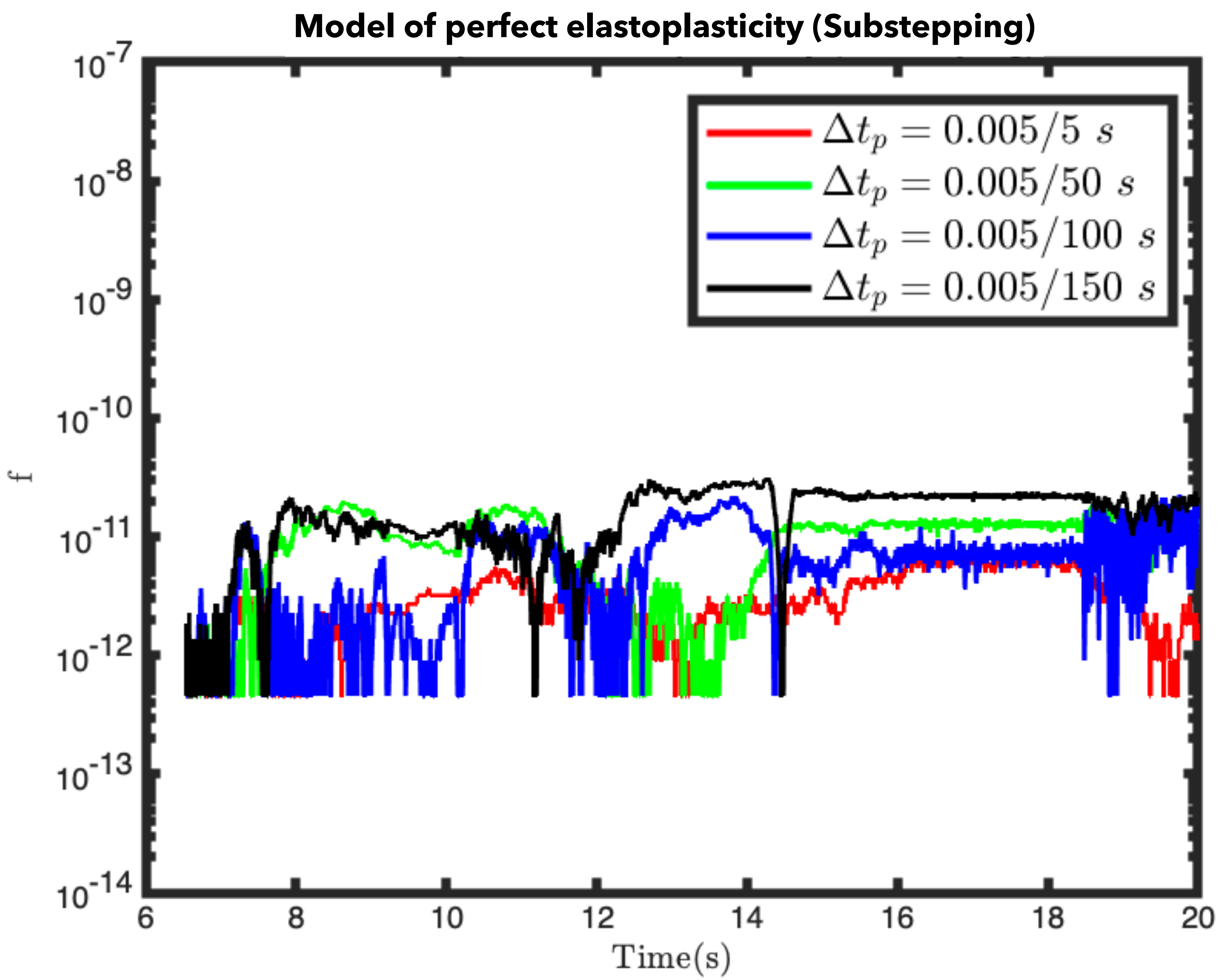


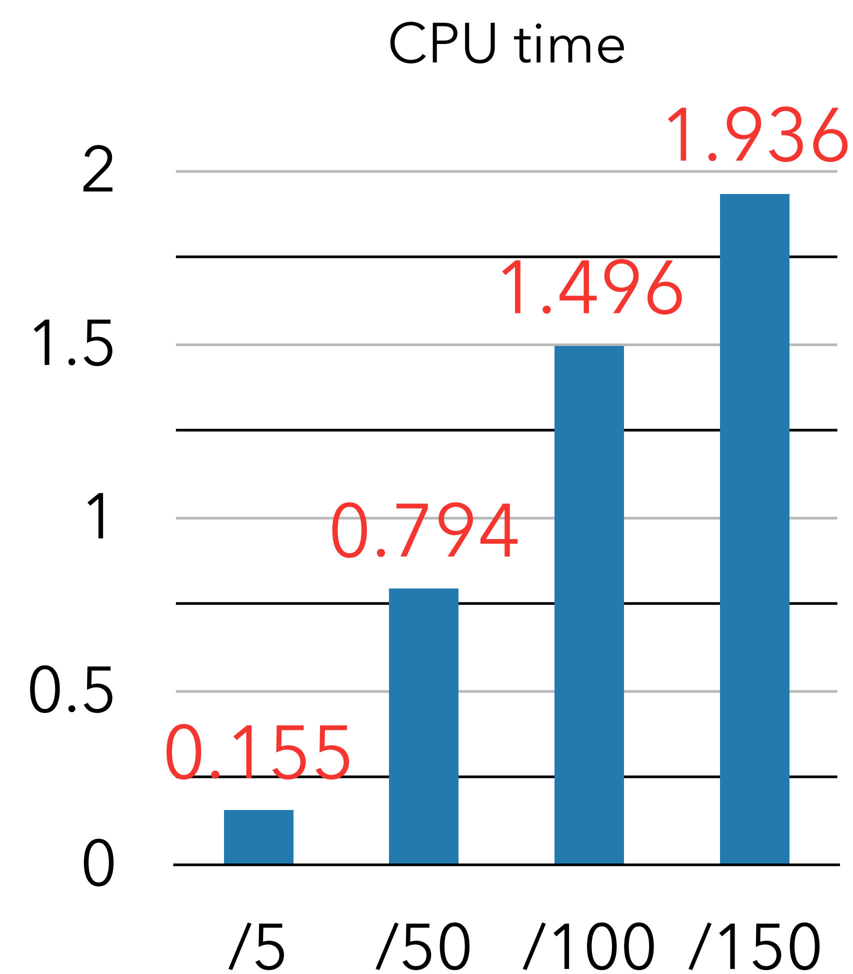






Model of perfect elastoplasticity





Elastoplastic models with kinematic hardening

- elastic constitutions

aniso+linear | aniso+nonlinear

iso+linear | iso+nonlinear

- stress admissible conditions

linear | quadratic+linear

quadratic | high order (cubic)

- plastic flow rules

generalized associated

associated | non-associated

- kinematic hardening

Prager | O-W

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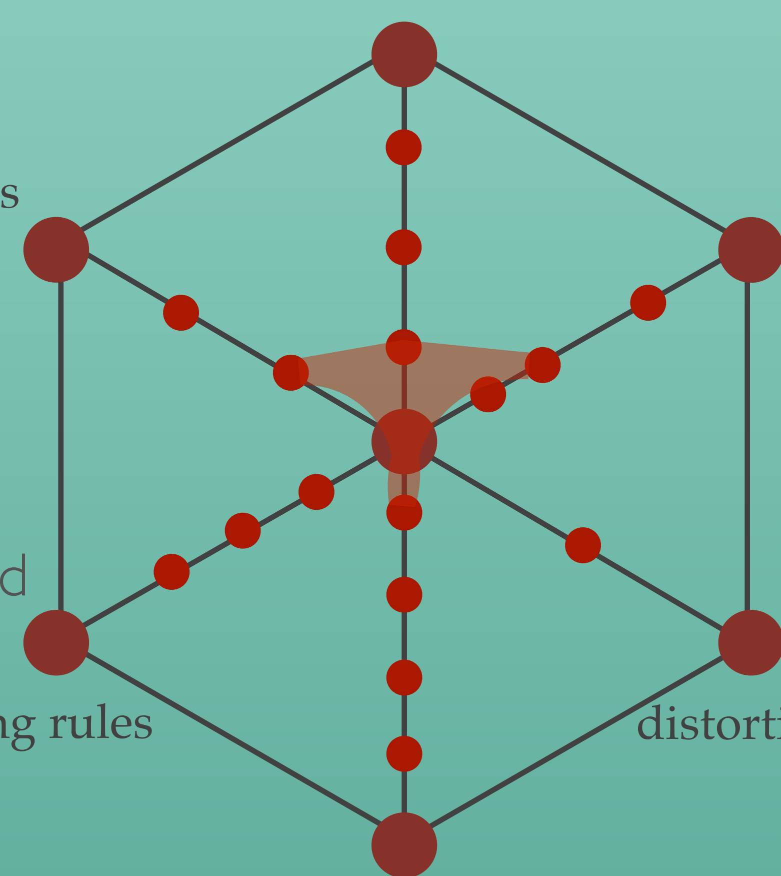


A Model With Bilinear Elastoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p, \qquad \mathbf{q}, \mathbf{q}^e, \mathbf{q}^p \in \mathbb{R}^n,$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, \qquad \mathbf{Q}, \mathbf{Q}_a, \mathbf{Q}_b \in \mathbb{R}^n,$$

$$\mathbf{Q} = k_e \mathbf{q}^e, \qquad k_e > 0, Q_y > 0, k_p > 0$$

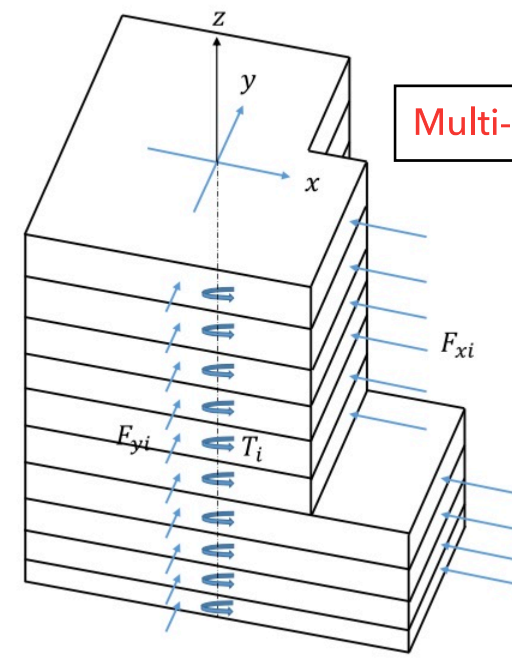
$$\mathbf{Q}_b = k_p \mathbf{q}^p, \qquad \text{given } k_e, Q_y, k_p \in \mathbb{R},$$

$$Q_y \dot{\mathbf{q}}^p = \mathbf{Q}_a \dot{\lambda}, \qquad \dot{()} = d()/dt,$$

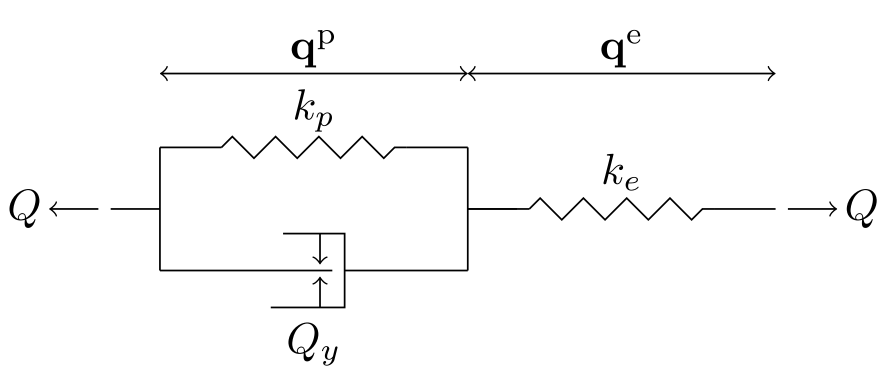
$$f \dot{\lambda} = 0,$$

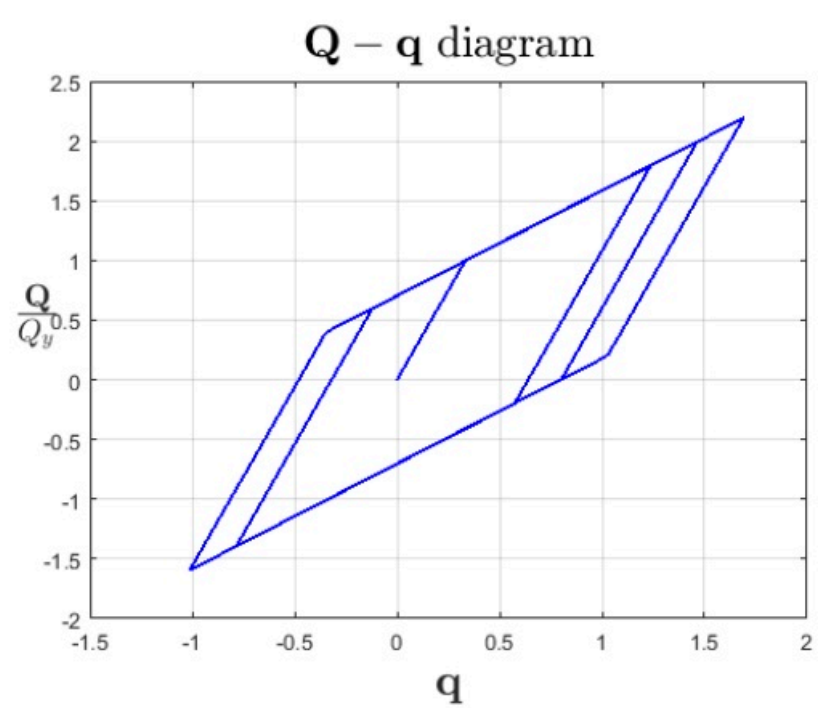
$$f = \| \mathbf{Q}_a \| - Q_y \le 0, \qquad \| \mathbf{Q}_a \| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a},$$

$$\dot{\lambda} \ge 0,$$



Multi-story building structure





Hong-Ki Hong, Li-Wei Liu, Ya-Po Shiao, Cheng-Jih Chang, Building structure with elastoplastic bilinear model under multi-dimensional earthquake forces, *Journal of Mechanics*, Volume 38, 2022, Pages 598–609.

A Model With Bilinear Elastoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p, \qquad \mathbf{q}, \mathbf{q}^e, \mathbf{q}^p \in \mathbb{R}^n,$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, \qquad \mathbf{Q}, \mathbf{Q}_a, \mathbf{Q}_b \in \mathbb{R}^n,$$

$$\mathbf{Q} = k_e \mathbf{q}^e, \qquad k_e > 0, Q_y > 0, k_p > 0$$

$$\mathbf{Q}_b = k_p \mathbf{q}^p, \qquad \text{given } k_e, Q_y, k_p \in \mathbb{R},$$

$$Q_y \dot{\mathbf{q}}^p = \mathbf{Q}_a \dot{\lambda}, \qquad \dot{()} = d()/dt,$$

$$f\dot{\lambda} = 0,$$

$$f = \| \mathbf{Q}_a \| - Q_y \leq 0, \qquad \| \mathbf{Q}_a \| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a},$$

$$\dot{\lambda} \geq 0,$$

Two-phase dynamical system

$$\frac{d}{dt}\begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\dot{\lambda}}{Q_y} \\ 0 & -\frac{k_p \dot{\lambda}}{Q_y} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} + \begin{bmatrix} \dot{\mathbf{Q}}/k_e \\ \dot{\mathbf{Q}} \end{bmatrix} \text{ if } f = 0 \text{ and } \mathcal{S} > 0\text{, plastic phase, where } \dot{\lambda} = \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{k_p Q_y}$$

$$\frac{d}{dt}\begin{bmatrix} \mathbf{q} \\ \mathbf{Q}_\mathrm{a} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{Q}}/k_e \\ \dot{\mathbf{Q}} \end{bmatrix} \text{ if } f < 0 \text{ or } \mathcal{S} \leq 0\text{, elastic phase,}$$

Hand-in

Please calculate $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$, where $t_{i+1} = t_i + \Delta t, i = 0,1,$ by the forward Euler method to discretize the governing equation of the plastic phase

$$\frac{d}{dt}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_\mathrm{a}\end{bmatrix} = \begin{bmatrix}0 & \dfrac{\dot{\lambda}}{Q_\mathrm{y}}\\ 0 & -\dfrac{k_p\dot{\lambda}}{Q_\mathrm{y}}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_\mathrm{a}\end{bmatrix} + \begin{bmatrix}\dot{\mathbf{Q}}/k_e\\ \dot{\mathbf{Q}}\end{bmatrix}$$

where $\dot{\lambda} = \dfrac{\mathbf{Q}_a^T\dot{\mathbf{Q}}}{k_pQ_y}$, $\mathbf{Q}, \mathbf{Q}_a, \mathbf{q} \in \mathbb{R}^2$, k_e = 1000, Q_y = 20, k_p = 500, $\mathbf{q}(t_0) = \begin{bmatrix}Q_y/k_e\\ 0\end{bmatrix}$, and $\dot{\mathbf{Q}} = \begin{bmatrix}1\\ 1\end{bmatrix}$.

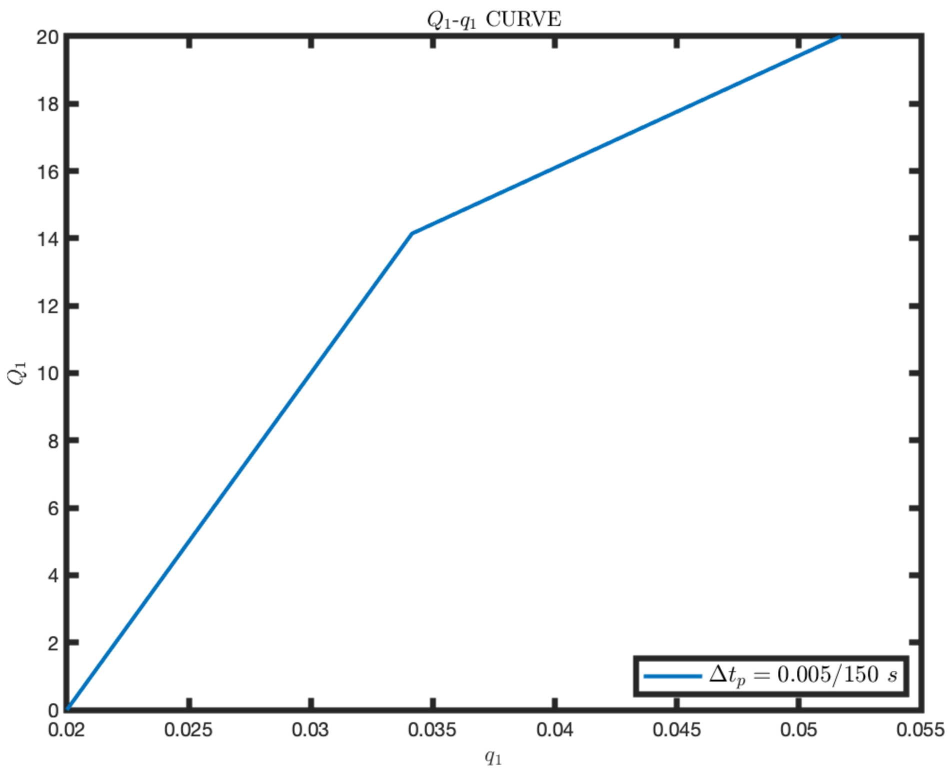
Coding

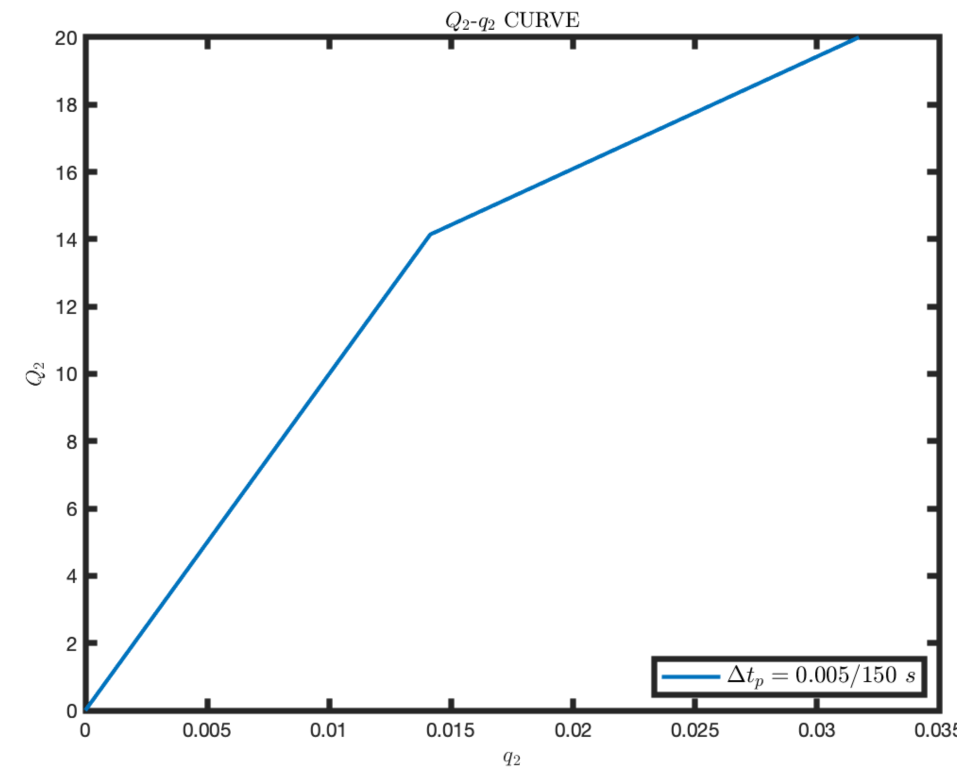
Please code plastic module to compute the $\mathbf{q}(t_i)$ response with varying timestep Δt by the forward Euler method to discretize the governing equation of the plastic phase

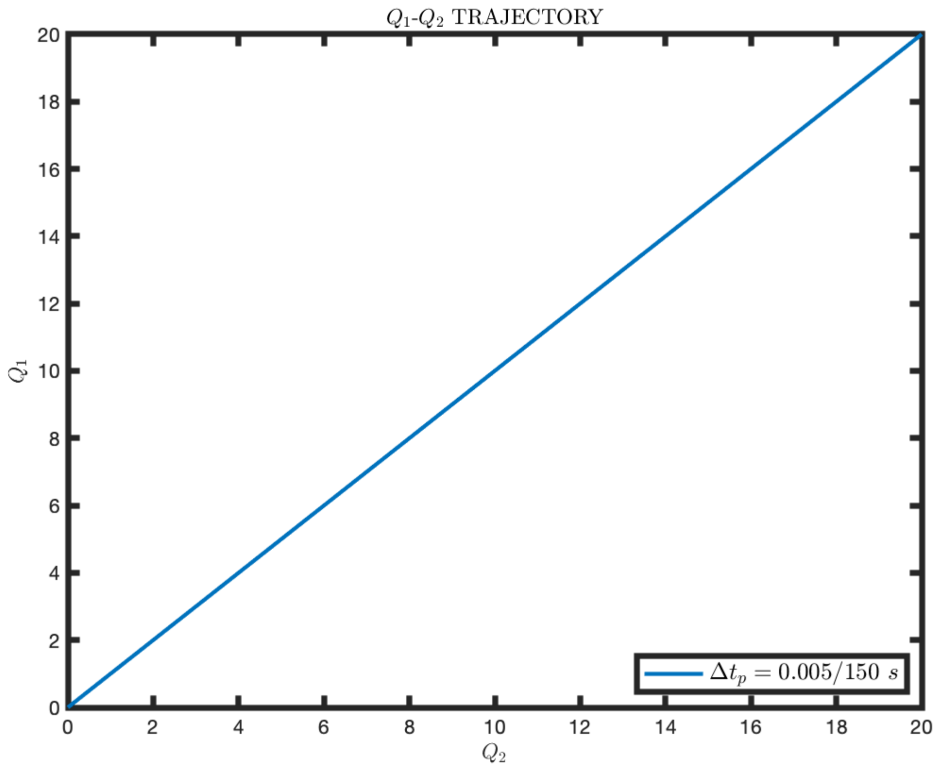
$$\frac{d}{dt}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_{\mathrm{a}}\end{bmatrix}=\begin{bmatrix}0 & \frac{\dot{\lambda}}{Q_{\mathrm{y}}}\\ 0 & -\frac{k_p\dot{\lambda}}{Q_{\mathrm{y}}}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_{\mathrm{a}}\end{bmatrix}+\begin{bmatrix}\dot{\mathbf{Q}}/k_e\\ \dot{\mathbf{Q}}\end{bmatrix}$$

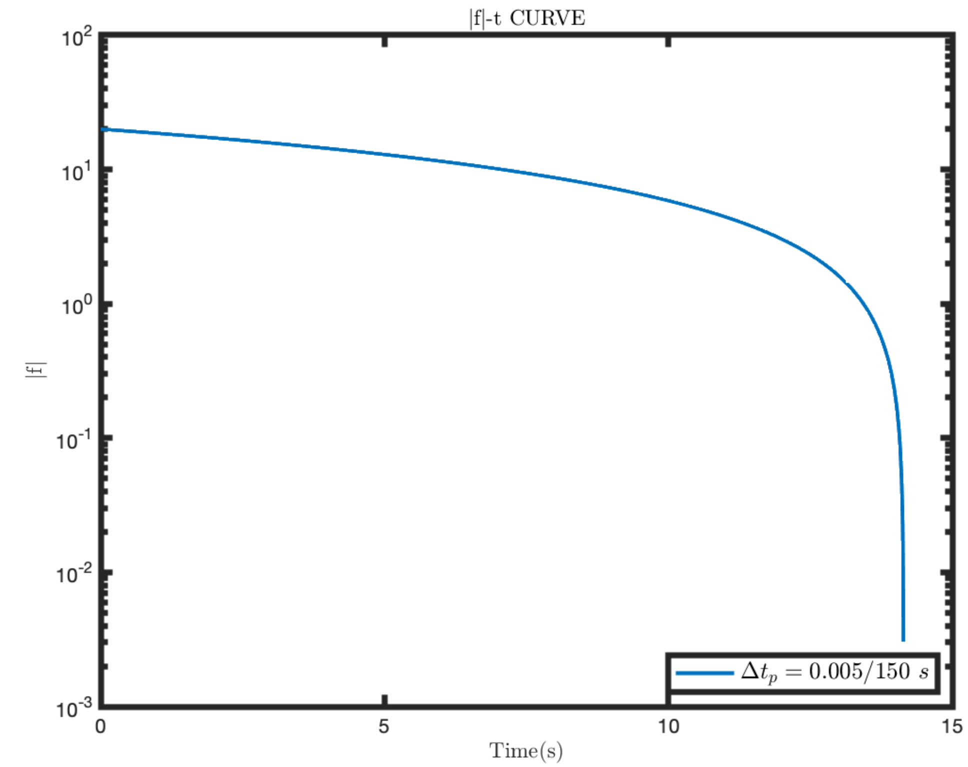
where $\dot{\lambda}=\dfrac{\mathbf{Q}_a^T\dot{\mathbf{Q}}}{k_pQ_y}$, $\mathbf{Q}, \mathbf{Q}_a, \mathbf{q} \in \mathbb{R}^2$, $k_e = 1000$, $Q_y = 20$, $k_p = 500$, $\mathbf{q}(t_0)=\begin{bmatrix}Q_y/k_e\\ 0\end{bmatrix}$, and $\dot{\mathbf{Q}}=\begin{bmatrix}1\\ 1\end{bmatrix}$.

Computational results

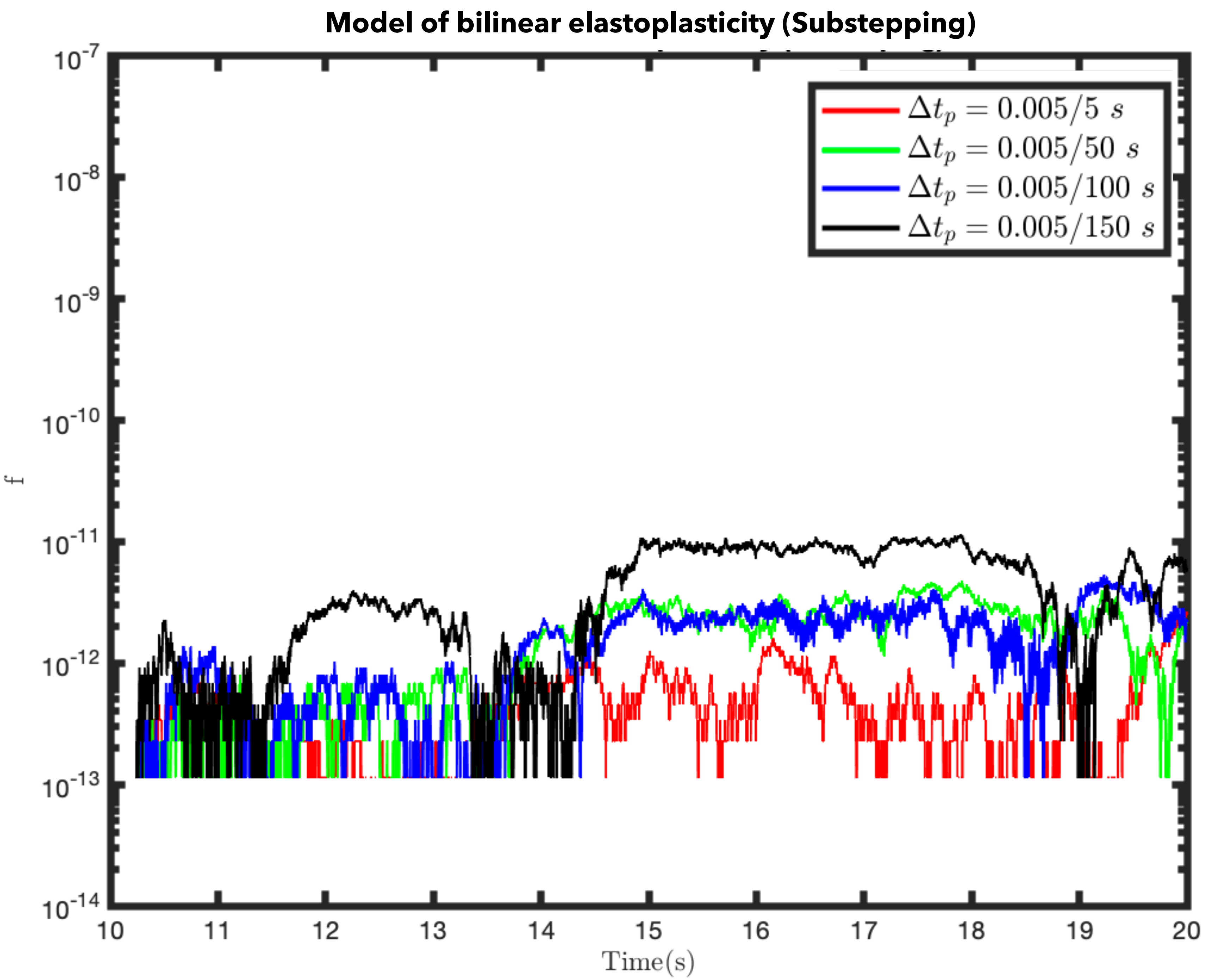








Model of bilinear elastoplasticity



Elastoplastic models with isotropic-kinematic hardening

- elastic constitutions

aniso+linear
aniso+nonlinear
iso+linear
iso+nonlinear

- stress admissible conditions

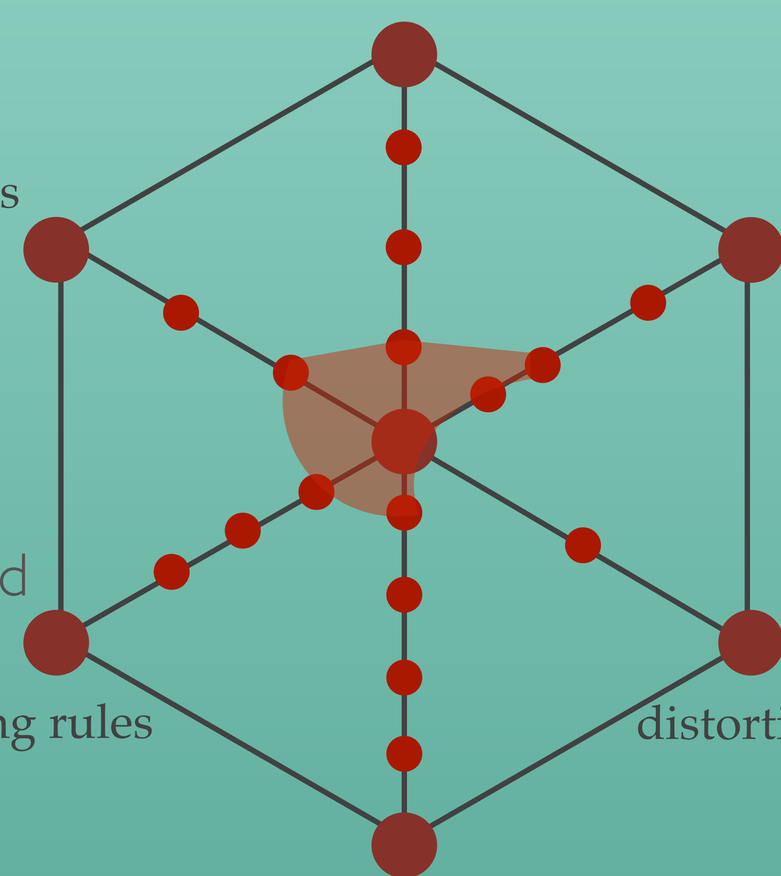
linear
quadratic+linear
quadratic
high order (cubic)

- plastic flow rules

generalized associated
associated
non-associated

- kinematic hardening

Prager
O-W
non
A-F
A-K



Model of bilinear isotropic-kinematic Hardening elastoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p,$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b,$$

$$\mathbf{Q} = \mathbf{K}_e \mathbf{q}^e,$$

$$\mathbf{Q}_b = k_p \mathbf{q}^p, \qquad \text{given } k_e, R_\infty, r, k_p \in \mathbb{R},$$

$$R_\infty \dot{\mathbf{q}}^p = \mathbf{Q}_a \dot{\lambda}, \qquad \dot{()} = d()/dt,$$

$$f\dot{\lambda} = 0, \qquad \lambda \in \mathbb{R},$$

$$f = \|\mathbf{Q}_a\| - R(\lambda) \leq 0, \qquad \|\mathbf{Q}_a\| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a},$$

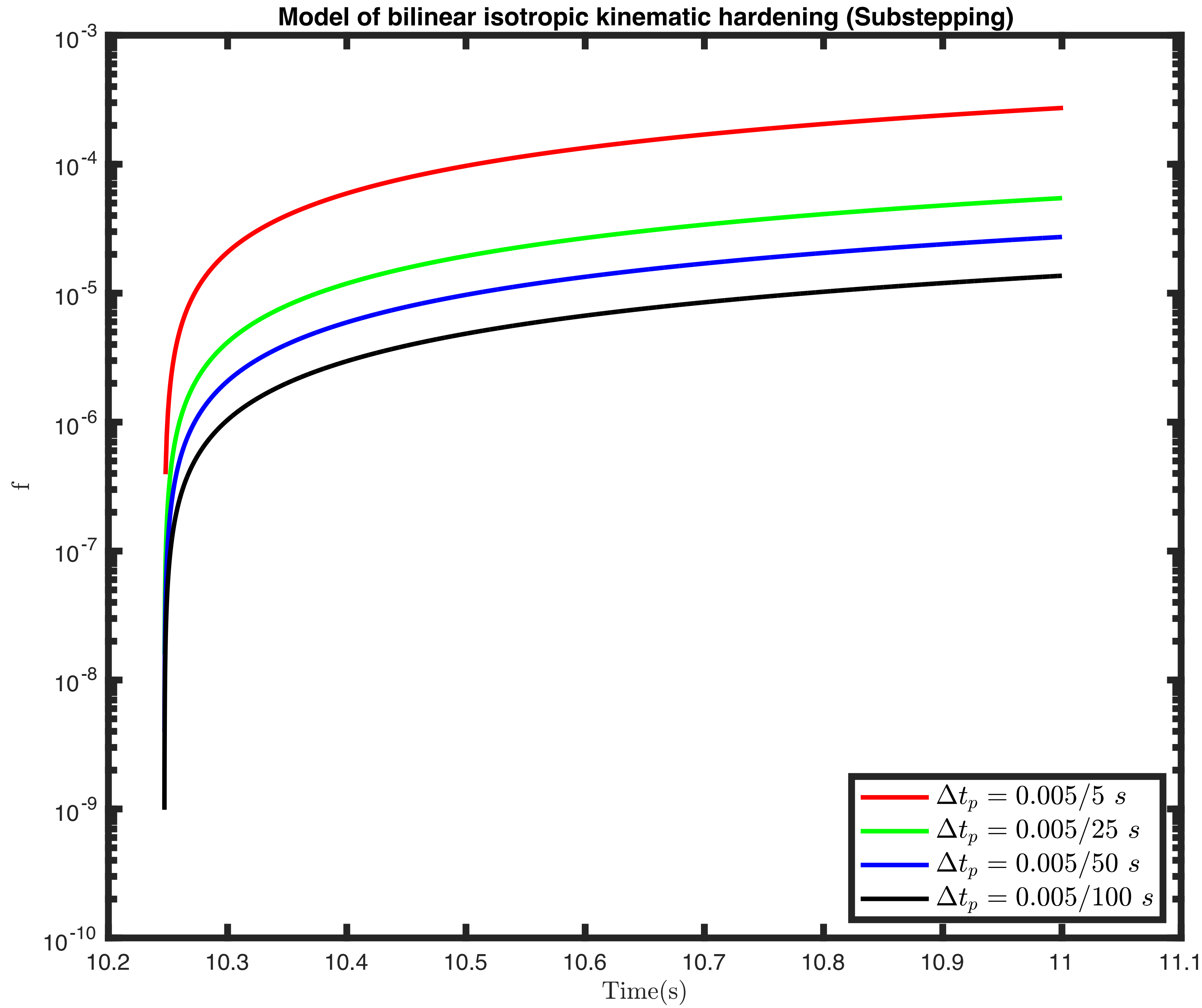
$$\dot{\lambda} \geq 0,$$

$$R(\lambda) = R_\infty \sqrt{1 - r\exp\left(\frac{-2\lambda}{\lambda_u}\right)}, \qquad \lambda_u = \frac{R_\infty}{k_p}, R_\infty = R(\infty)$$

On-Off switch

$$\dot{\lambda} = \begin{cases} \lambda_u \dfrac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{R_\infty^2} > 0 & \text{if } f = 0 \text{ and } \mathbf{Q}_a^T \dot{\mathbf{Q}} > 0 \quad \text{on-phase} \\ 0 & \text{if } f < 0 \text{ or } \mathbf{Q}_a^T \dot{\mathbf{Q}} \leq 0 \quad \text{off-phase} \end{cases}$$

Model of bilinear isotropic-kinematic Hardening elastoplasticity



Numerical integrations for ODEs

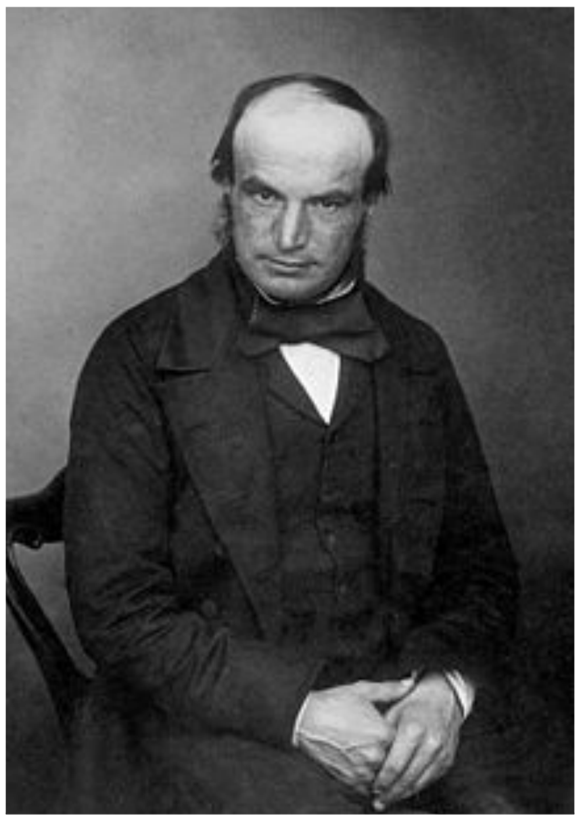
One-step schemes

$$\mathbf{X}(t_{k+1}) = \mathbf{X}(t_k) + h\mathbf{S}(t_k, \mathbf{X}(t_k), h)$$

$$t_{k+1} = t_k + h$$

- The Euler schemes
- θ-schemes
- The Runge-Kutta schemes

Linear multistep methods



John Couch Adams FRS
(1819-1892)

Urbain Le Verrier
(1811-1877)

- Discoverers of Neptune
- Orbit of Uranus
- Existence and position of Neptune
- Mathematics
- Urbain Le Verrier–"he discovered a star with the tip of his pen, without any instruments other than the strength of his calculations alone"

$$\sum_{j=1}^{q} \alpha_j \mathbf{X}_{i+j} = h \sum_{j=0}^{q} \beta_j \mathbf{f}(t_{i+j}, \mathbf{X}_{i+j}), \quad \alpha_1 \neq 0, \ \ |\alpha_0| + |\beta_0| \neq 0.$$

- Adams-Bashforth methods
- Adams-Moulton methods
- The first backward differentiation methods

Thanks for your attention