Computational Plasticity

ACMT short course 2024

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Contents

Chapter 1

Models of perfect elastoplasticity

1.1 Theoretical foundations

The model of perfect elatoplasticity is written as follows:

$$\begin{aligned}
&\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},\\
&\mathbf{Q} = k_e \mathbf{q}^{\mathrm{e}},\\
&\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}}\dot{\lambda},\\
&f = \|\mathbf{Q}\| - Q_y \leq 0,\\
&\dot{\lambda} \geq 0,\\
&f\dot{\lambda} = 0.
\end{aligned} \tag{1.1}$$

In this model, the generalized stress is denoted by $\mathbf{Q}$; the generalized strain by $\mathbf{q}$; the generalized elastic strain by $\mathbf{q}^{\mathrm{e}}$; the generalized plastic strain by $\mathbf{q}^{\mathrm{p}}$; and the generalized equivalent plastic strain by λ. Besides above-mentioned variables, the material constants of the model are the generalized elastic modulus k_e and the generalized yielding stress Q_y.

In this model, the rate of generalized equivalent plastic strain can be obtained as

$$\dot{\lambda} = \begin{cases} \frac{\mathbf{Q}^T\dot{\mathbf{q}}}{Q_y} & \text{if } \|\mathbf{Q}\| - Q_y = 0 \quad \text{and} \quad \mathcal{S} > 0 \quad \text{plastic phase} \\ 0 & \text{if } \|\mathbf{Q}\| - Q_y < 0 \quad \text{or} \quad \mathcal{S} \leq 0 \quad \text{elastic phase} \end{cases} \tag{1.2}$$

where the straining condition $\mathcal{S} := \mathbf{Q}^T\dot{\mathbf{q}} > 0$. According to the previous derivation, we obtain the two-phase representation of model (1.1) as follows:

$$\begin{cases} \dot{\mathbf{Q}} = -k_e\frac{\mathbf{Q}^T\dot{\mathbf{q}}}{Q_y^2}\mathbf{Q} + k_e\dot{\mathbf{q}}, & \text{if} \quad f = 0 \quad \text{and} \quad \mathcal{S} > 0, \quad \text{plastic phase,} \\ \dot{\mathbf{Q}} = k_e\dot{\mathbf{q}}, & \text{if} \quad f < 0 \quad \text{or} \quad \mathcal{S} \leq 0, \quad \text{elastic phase.} \end{cases} \tag{1.3}$$

1.2 Numerical integrations—the substeping method

After the introduction of the proposed model in Eq. (1.1) and its two-phase representation in Eq. (1.3), we start our journey to develop its numerical integration. Obviously, the governing equation of the model on the elastic phase is linear ordinary differential equations. Therefore, we merely need to focus on the development of numerical integrations on the plastic phase.

1.2.1 Algorithms

The algorithm for the computation of bilinear elastoplastic models can be classified into two modules:

- Elastic module: It serves the computation of elastic response under the time increment Δt.

- Plastic module: It serves the computation of plastic response under the time increment Δt.

In the elastic module, one can obtain $\dot{\mathbf{q}}$ according to Eq. (1.1.2) with $\dot{\lambda} = 0$ as follows:

$$\dot{\mathbf{Q}} = k_e \dot{\mathbf{q}} \tag{1.4}$$

In the plastic module:

$$\dot{\mathbf{Q}} = -k_e \frac{\mathbf{Q}^{\mathbf{T}} \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}}$$

(i) Setting parameters

$\mathrm{k_e}$	elastic stiffness
$\mathrm{Q_y}$	yield strength
$\triangle \mathrm{t_e}$	elastic time step
$\triangle \mathrm{t_p}$	plastic time step

(ii) Setting initial values

$\mathbf{q} = \mathbf{0}$	initial total strain
$\dot{\mathbf{q}} = \mathbf{0}$	initial total plstic atrain rate
$\mathbf{q}^p = \mathbf{0}$	initial plastic strain
$\mathbf{Q} = \mathbf{0}$	initial total stress
$\mathrm{S} = 0$	initial straining condition
$\mathrm{Iflage} = 0$	initial switching index

(iii) input data

$$\mathbf{q}$$

(iv) analysis

- If the elastic-plastic index Iflage=[0] , calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with elastic-phase.
  - ∗ Check whether the admissible conditions are satisfied.
    - · If it is not satisfied
      1.Calculate the switching time response $\dot{\mathbf{Q}}(\mathbf{t_{on}})$ and $\mathbf{q}(\mathbf{t_{on}})$.
      3.Calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with plastic-phase.
- If the elastic-plastic index Iflage=[1]
  - ∗ Check straining conditions $s = \dot{\lambda} = \mathbf{Q}^T \dot{\mathbf{q}}$.
    - · If s>0
      1.Use plastic time step $\triangle \mathrm{t_p}$ to calculate plastic-phase.
      2.Calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with plastic-phase.
    - · If s<0
      1.calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with elastic-phase.
      2.Check whether the admissible conditions are satisfied.
      If it is not satisfied
        A.Calculate the switching time response $\dot{\mathbf{q}}(\mathbf{t_s})$ and $\mathbf{Q}(\mathbf{t_{on}})$.
        C.Calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with plastic-phase.
      If it is satisfied Iflage=0

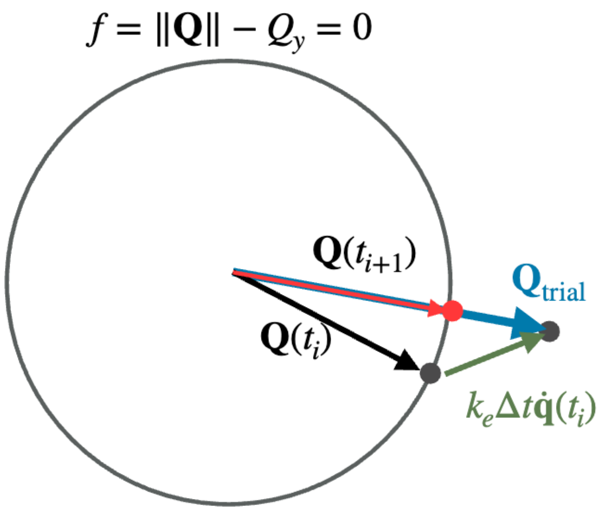

Figure 1.1: Perfect model—radial return algorithm

1.3 Numerical integrations—the return-mapping method

1.3.1 Algorithms

In the elastic module, one can obtain $\dot{\mathbf{q}}$ according to Eq. (3.16) with $\dot{\lambda} = 0$ as follows:

$$\dot{\mathbf{Q}} = \mathbf{K}_e \dot{\mathbf{q}} \tag{1.5}$$

In the plastic module,we predict $\mathbf{Q}_{\text{trail}}$:

$$\mathbf{Q}_{\text{trail}} = \mathbf{Q}(t_i) + (\mathbf{K}\frac{\mathbf{Q}^T(t_i)\dot{\mathbf{q}}(t_i)}{\mathbf{Q}_y^2}\mathbf{Q}(t_i) - \mathbf{K}\dot{\mathbf{q}}(t_i))\Delta t \tag{1.6}$$

then we need to correct it to fit the yield function. Fig. 1.1

$$\mathbf{n}_{\text{trail}} = \frac{\mathbf{Q}_{\text{trail}}}{\|\mathbf{Q}_{\text{trail}}\|} \tag{1.7}$$

then

$$\mathbf{Q}(t_{i+1}) = \mathrm{Q}_y \mathbf{n}_{\text{trail}} \tag{1.8}$$

1.4 Numerical integrations—the GPS method

1.4.1 Algorithms

The algorithm for the computation of elastoplastic models can be classified into three modules and one timespace conversion:

- Elastic module: It serves the computation of elastic response under the time increment Δt.
- Plastic module: It serves the computation of plastic response under the time increment Δt.
- Pull-back module: It serves the computation of the elastic-plastic switching time t_s.
- The group-preserving scheme (GPS) : It serves the computation of the active floor shear force $\mathbf{Q}$ under plastic phase.

In the elastic module, one can obtain $\dot{\mathbf{q}}$ according to Eq. (3.16) with $\dot{\lambda} = 0$ as follows:

$$\dot{\mathbf{Q}} = k_e \dot{\mathbf{q}} \tag{1.9}$$

In the plastic module, first of all we convert $\mathbf{Q}$ space to Minkowski spacetime according to GPS method and obtain the $\mathbf{Q}$ as follows:

$$\dot{\mathbf{X}} = \mathbf{A}_{\text{on}} \mathbf{X} \text{ i.e. } \begin{bmatrix} \dot{\mathbf{X}}_s \\ \dot{X}_0 \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \frac{\mathbf{Q}}{Q_y} X_0 \\ \exp\left(\frac{\lambda}{q_y}\right) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \frac{\dot{\mathbf{q}}}{q_y} \\ \frac{\mathbf{q}^T}{q_y} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_s \\ X_0 \end{bmatrix} \tag{1.10}$$

$$\begin{bmatrix} \mathbf{X}_s(t_{k+1}) \\ X_0(t_{k+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\dot{\mathbf{q}}\|^2} \dot{\mathbf{q}} \dot{\mathbf{q}}^T & \frac{b \dot{\mathbf{q}}}{\|\dot{\mathbf{q}}\|} \\ \frac{b \dot{\mathbf{q}}^T}{\|\dot{\mathbf{q}}\|} & a \end{bmatrix} \begin{bmatrix} \mathbf{X}_s\,(t_k) \\ X_0\,(t_k) \end{bmatrix} \tag{1.11}$$

and then,one can obtain $\dot{\mathbf{q}}$ according to Eq. (**??**) with $\dot{\lambda} = \frac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y}$ as follows:

$$\dot{\mathbf{Q}} = -k_e \frac{\mathbf{Q}^{\mathbf{T}} \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}}$$

(i) Setting parameters

$\mathrm{k_e}$	elastic stiffness
$\mathrm{Q_y}$	yield strength

(ii) Setting initial values

$\mathbf{q} = \mathbf{0}$	initial value of total strain
$\dot{\mathbf{q}} = \mathbf{0}$	initial value of total strain rate
$\mathbf{q}^p = \mathbf{0}$	initial value of plastic strain
$\mathbf{Q} = \mathbf{0}$	initial value of total stress
$\mathrm{s} = 0$	initial straining condition
$\mathrm{Iflage} = 0$	initial switching index

(iii) input data

$$\mathbf{q}$$

(iv) analysis

- If the elastic-plastic index Iflage=[0] , calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with elastic-phase.
  - Check whether the admissible conditions are satisfied.
    - If it is not satisfied
      1.Calculate the switching time response $\dot{\mathbf{Q}}(\mathbf{t_s})$ with pull-back.
      2.Use GPI method to calculate .
      3.Calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with plastic-phase.
- If the elastic-plastic index Iflage=[1]
  - Check straining conditions $s = \dot{\lambda} = \frac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y}$.
    - If s>0
      1.Use GPI method to calculate .
      2.Calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with plastic-phase.
    - If s<0
      1.calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with elastic-phase.
      2.Check whether the admissible conditions are satisfied.
      If it is not satisfied
      A.Calculate the switching time response $\dot{\mathbf{Q}}(\mathbf{t_s})$ with pull-back.
      B.Use GPI method to calculate .
      C.Calculate the next time response $\dot{\mathbf{Q}}(t_{i+1})$ with plastic-phase.
      If it is satisfied Iflage=0

Chapter 2

Ansiotropic models of perfect elastoplasticity

2.1 Theoretical foundations

The model of perfect elatoplasticity with hoffman yield criterion written as follows:

$$\begin{aligned}
&\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p \\
&\mathbf{Q} = \mathbf{K}\mathbf{q}^e \\
&\dot{\mathbf{q}}^p = \dot{\lambda}\frac{\partial f}{\partial \mathbf{Q}} \\
&f = \frac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} + \mathbf{P}\mathbf{Q}r - \frac{1}{2}r^2 \leq 0, \\
&\dot{\lambda} \geq 0, \\
&f\dot{\lambda} = 0.
\end{aligned} \tag{2.1}$$

In this model, the generalized stress is denoted by $\mathbf{Q}$; the generalized strain by $\mathbf{q}$; the generalized elastic strain by $\mathbf{q}^{\mathrm{e}}$; the generalized plastic strain by $\mathbf{q}^{\mathrm{p}}$; and the generalized equivalent plastic strain by λ. Besides above-mentioned variables, the material constants of the model are the generalized elastic modulus $\mathbf{K}$ and the generalized yielding stress r.

If the stress $\mathbf{Q}$ satisfies the yield condition $f = 0$, we have

$$\frac{d}{dt}f = \frac{\partial f}{\partial \mathbf{Q}}\dot{\mathbf{Q}} + \frac{\partial f}{\partial r}\dot{r} = 0.$$

The yield condition also implies the value of $\dot{\lambda}$ is non-negative, hence above equation and Eq. (2.1)enable us to obtain the rate of equivalent plastic strain

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \mathbf{Q}}\mathbf{K}\dot{\mathbf{q}}}{\frac{\partial f}{\partial \mathbf{Q}}\mathbf{K}\frac{\partial f}{\partial \mathbf{Q}}}, \tag{2.2}$$

if the straining condition $\mathcal{S} := \frac{\partial f}{\partial \mathbf{Q}}\mathbf{K}\dot{\mathbf{q}} > 0$ is satisfied. According to the previous derivation, we obtain the two-phase representation as follows:

$$\begin{cases}
\text{plastic phase: } \dot{\mathbf{Q}} = -\dot{\lambda}\mathbf{K}\frac{\partial f}{\partial \mathbf{Q}} + \mathbf{K}\dot{\mathbf{p}}, & \text{if} \quad f = 0 \quad \text{and} \quad \mathcal{S} > 0, \\
\text{elastic phase: } \dot{\mathbf{Q}} = \mathbf{K}\dot{\mathbf{q}}, & \text{if} \quad f < 0 \quad \text{or} \quad \mathcal{S} \leq 0.
\end{cases} \tag{2.3}$$

2.2 Numerical integrations—the return-mapping method

2.2.1 Algorithms

In this integration, the stress on the plastic phase is updated by the formulation as follows:

$$\mathbf{Q}(t_{k+1}) = \mathbf{Q}(t_k) + \left(\mathbf{K}^{-1} + \Delta\lambda\mathbf{Y}\right)^{-1}\left(\mathbf{K}^{-1}\mathbf{Q}(t_k) + \Delta\mathbf{q} - \Delta\lambda\mathbf{P}\right), \tag{2.4}$$

where $\Delta\mathbf{q}$ is the given strain increment and $\Delta\lambda$ is the equivalent plastic strain increment. In order to find a correct stress $\mathbf{Q}(t_{k+1})$ which is located on the yield surface, one needs to solve the nonlinear algebraic equation $f(\mathbf{Q}(t_{k+1})) = 0$ of the increment $\Delta\lambda$ where

$$f(\mathbf{Q}(t_{k+1})) = f(\mathbf{Q}(t_k) + \left(\mathbf{K}^{-1} + \Delta\lambda\mathbf{Y}\right)^{-1}\left(\mathbf{K}^{-1}\mathbf{Q}(t_k) + \Delta\mathbf{q} - \Delta\lambda\mathbf{P}\right)). \tag{2.5}$$

According to the Newton' method, one iterates to obtain

$$\Delta\lambda^{i+1} = \Delta\lambda^i - \left.\frac{f}{\frac{\partial f}{\partial\Delta\lambda}}\right|_{\Delta\lambda^i}, \tag{2.6}$$

where

$$\frac{\partial f}{\partial\Delta\lambda} = -\left(\mathbf{Y}\mathbf{Q}(t_{k+1}) + \mathbf{P}\right)\left(\mathbf{K}^{-1} + \Delta\lambda\mathbf{Y}\right)^{-1}\left(\mathbf{Y}\mathbf{Q}(t_{k+1}) + \mathbf{P}\right). \tag{2.7}$$

Chapter 3

Models of bilinear elastoplasticity

3.1 Theoretical foundations

$$\begin{aligned}
&\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p, && \mathbf{q}, \mathbf{q}^e, \mathbf{q}^p \in \mathbb{R}^n, && (3.1)\\
&\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, && \mathbf{Q}, \mathbf{Q}_a, \mathbf{Q}_b \in \mathbb{R}^n, && (3.2)\\
&\mathbf{Q} = k_e \mathbf{q}^e, && k_e > 0, Q_y > 0, k_p > 0 && (3.3)\\
&\mathbf{Q}_b = k_p \mathbf{q}^p, && \text{given } k_e, Q_y, k_p \in \mathbb{R}, && (3.4)\\
&Q_y \dot{\mathbf{q}}^p = \mathbf{Q}_a \dot{\lambda}, && \dot{()} = d()/dt, && (3.5)\\
&f \dot{\lambda} = 0, && && (3.6)\\
&f = \|\mathbf{Q}_a\| - Q_y \leq 0, && \|\mathbf{Q}_a\| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a}, && (3.7)\\
&\dot{\lambda} \geq 0, && && (3.8)
\end{aligned}$$

Taking the time derivative of both sides of (3.1):

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}^e + \dot{\mathbf{q}}^p \tag{3.9}$$

Substituting (3.3) and (3.5) into (3.9):

$$\dot{\mathbf{q}} = \frac{1}{k^e} \dot{\mathbf{Q}} + \frac{\mathbf{Q}_a \dot{\lambda}}{Q_y} \tag{3.10}$$

The stress input bilinear model:

$$\begin{aligned}
&\dot{\mathbf{q}} = \frac{1}{k_e} \dot{\mathbf{Q}} + \frac{\mathbf{Q}_a \dot{\lambda}}{Q_y}, && && (3.11)\\
&\frac{\dot{\mathbf{Q}}_a}{Q_y} + \frac{k_p \dot{\lambda}}{Q_y} \frac{\mathbf{Q}_a}{Q_y} = \frac{\dot{\mathbf{Q}}}{Q_y} && && (3.12)\\
&f \dot{\lambda} = 0, && && (3.13)\\
&f = \|\mathbf{Q}_a\| - Q_y \leq 0, && \|\mathbf{Q}_a\| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a}, && (3.14)\\
&\dot{\lambda} \geq 0, && && (3.15)
\end{aligned}$$

$$\dot{\lambda} = \begin{cases} \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{k_p Q_y} > 0 & \text{if } f = 0 \text{ and } \mathbf{Q}_a^T \dot{\mathbf{Q}} > 0 \quad \text{on-phase} \\ 0 & \text{if } f < 0 \text{ or } \mathbf{Q}_a^T \dot{\mathbf{Q}} \leq 0 \quad \text{off-phase} \end{cases} \tag{3.16}$$

3.2 Numerical integrations—the substeping method

3.2.1 Algorithms

The algorithm for the computation of bilinear elastoplastic models can be classified into two modules:

- Elastic module: It serves the computation of elastic response under the time increment Δt.

- Plastic module: It serves the computation of plastic response under the time increment Δt.

In the elastic module, one can obtain $\dot{\mathbf{q}}$ according to Eq. (3.16) with $\dot{\lambda} = 0$ as follows:

$$\dot{\mathbf{q}} = \frac{1}{k_e}\dot{\mathbf{Q}} \tag{3.17}$$

In the plastic module:

$$\frac{d}{dt}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_{\mathrm{a}}\end{bmatrix} = \begin{bmatrix}0 & \frac{\dot{\lambda}}{Q_{\mathrm{y}}}\\ 0 & -\frac{k_p\dot{\lambda}}{Q_{\mathrm{y}}}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_{\mathrm{a}}\end{bmatrix} + \begin{bmatrix}\dot{\mathbf{Q}}/k_e\\ \dot{\mathbf{Q}}\end{bmatrix}$$

(i) Setting parameters

$\mathrm{k_e}$	floor elastic stiffness of building structure
$\mathrm{k_p}$	floor plastic stiffness of building structure
$\mathrm{Q_y}$	floor yield strength of building structure
$\triangle \mathrm{t_e}$	elastic time step
$\triangle \mathrm{t_p}$	plastic time step

(ii) Setting initial values

$\mathbf{q} = \mathbf{0}$	initial total floor displacement
$\dot{\mathbf{q}} = \mathbf{0}$	initial total floor velocity
$\mathbf{q}^p = \mathbf{0}$	initial plastic floor displacement
$\mathbf{Q} = \mathbf{0}$	initial total floor shear force
$\mathbf{Q}_a = \mathbf{0}$	initial active floor shear force
$\mathbf{Q}_b = \mathbf{0}$	initial back floor shear force
$\mathrm{S} = 0$	initial straining condition
Iflage $= 0$	initial switching index

(iii) input data

$$\mathbf{Q} = a_g$$

(iv) analysis

- If the elastic-plastic index Iflage=[0] , calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with elastic-phase.
  - ∗ Check whether the admissible conditions are satisfied.
    - · If it is not satisfied
      1.Calculate the switching time response $\dot{\mathbf{q}}(\mathbf{t_{on}})$ and $\mathbf{Q}(\mathbf{t_{on}})$.
      3.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
- If the elastic-plastic index Iflage=[1]
  - ∗ Check straining conditions $s = \dot{\lambda} = \mathbf{Q}_a^T \dot{\mathbf{Q}}$.
    - · If s>0
      1.Use plastic time step $\triangle \mathrm{t_p}$ to calculate plastic-phase.
      2.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
    - · If s<0
      1.calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with elastic-phase.
      2.Check whether the admissible conditions are satisfied.
      If it is not satisfied
      A.Calculate the switching time response $\dot{\mathbf{q}}(\mathbf{t_s})$ and $\mathbf{Q}(\mathbf{t_{on}})$.
      C.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
      If it is satisfied Iflage=0

3.3 Numerical integrations—the GPS method

3.3.1 Algorithms

The algorithm for the computation of elastoplastic models can be classified into three modules and one timespace conversion:

- Elastic module: It serves the computation of elastic response under the time increment Δt.
- Plastic module: It serves the computation of plastic response under the time increment Δt.
- Pull-back module: It serves the computation of the elastic-plastic switching time t_s.
- The group-preserving scheme (GPS) : It serves the computation of the active floor shear force $\mathbf{Q}_a$ under plastic phase.

In the elastic module, one can obtain $\dot{\mathbf{q}}$ according to Eq. (3.16) with $\dot{\lambda} = 0$ as follows:

$$\dot{\mathbf{q}} = \frac{1}{k_e}\dot{\mathbf{Q}} \tag{3.18}$$

In the plastic module, first of all we convert $\mathbf{Q_a}$ space to Minkowski spacetime according to GPS method and obtain the $\mathbf{Q_a}$ as follows:

$$\dot{\mathbf{X}} = \mathbf{A}_{\text{on}}\mathbf{X} \text{ i.e. } \begin{bmatrix} \dot{\mathbf{X}}_s \\ \dot{X}_0 \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{Q}_a}{Q_y} X_0 \\ \exp\left(\frac{k_p \lambda}{Q_y}\right) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \frac{\dot{\mathbf{Q}}}{Q_y} \\ \frac{\dot{\mathbf{Q}}^T}{Q_y} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_s \\ X_0 \end{bmatrix} \tag{3.19}$$

$$\begin{bmatrix} \mathbf{X}_s(t_{k+1}) \\ X_0(t_{k+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n + \frac{a-1}{\|\dot{\mathbf{Q}}\|^2}\dot{\mathbf{Q}}\dot{\mathbf{Q}}^T & \frac{b\dot{\mathbf{Q}}}{\|\dot{\mathbf{Q}}\|} \\ \frac{b\dot{\mathbf{Q}}^T}{\|\dot{\mathbf{Q}}\|} & a \end{bmatrix} \begin{bmatrix} \mathbf{X}_s\,(t_k) \\ X_0\,(t_k) \end{bmatrix} \tag{3.20}$$

and then,one can obtain $\dot{\mathbf{q}}$ according to Eq. (3.16) with $\dot{\lambda} = \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{k_p Q_y}$ as follows:

$$\dot{\mathbf{q}} = \frac{1}{k_e}\dot{\mathbf{Q}} + \frac{\mathbf{Q}_a(\frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{k_p Q_y})}{Q_y} \tag{3.21}$$

(i) Setting parameters

$\mathrm{k_e}$	floor elastic stiffness of building structure
$\mathrm{k_p}$	floor plastic stiffness of building structure
$\mathrm{Q_y}$	floor yield strength of building structure

(ii) Setting initial values

$\mathbf{q} = \mathbf{0}$	initial total floor displacement
$\dot{\mathbf{q}} = \mathbf{0}$	initial total floor velocity
$\mathbf{q}^p = \mathbf{0}$	initial plastic floor displacement
$\mathbf{Q} = \mathbf{0}$	initial total floor shear force
$\mathbf{Q}_a = \mathbf{0}$	initial active floor shear force
$\mathbf{Q}_b = \mathbf{0}$	initial back floor shear force
$\mathrm{s} = 0$	initial straining condition
$\mathrm{Iflage} = 0$	initial switching index

(iii) input data

$$\mathbf{Q}$$

(iv) analysis

- If the elastic-plastic index Iflage=[0] , calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with elastic-phase.
  - ∗ Check whether the admissible conditions are satisfied.
    - · If it is not satisfied
      1.Calculate the switching time response $\dot{\mathbf{q}}(\mathbf{t_s})$ with pull-back.
      2.Use GPI method to calculate .
      3.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
- If the elastic-plastic index Iflage=[1]
  - ∗ Check straining conditions $s = \dot{\lambda} = \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{k_p Q_y}$.
    - · If s>0
      1.Use GPI method to calculate .
      2.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
    - · If s<0
      1.calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with elastic-phase.
      2.Check whether the admissible conditions are satisfied.
      If it is not satisfied
        A.Calculate the switching time response $\dot{\mathbf{q}}(\mathbf{t_s})$ with pull-back.
        B.Use GPI method to calculate .
        C.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
      If it is satisfied Iflage=0

Chapter 4

Models of bilinear isotropic-kinematic hardening elastoplasticity

4.1 Theoretical foundations

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p, \qquad \mathbf{q}, \mathbf{q}^e, \mathbf{q}^p \in \mathbb{R}^n, \tag{4.1}$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, \qquad \mathbf{Q}, \mathbf{Q}_a, \mathbf{Q}_b \in \mathbb{R}^n, \tag{4.2}$$

$$\mathbf{Q} = k_e \mathbf{q}^e, \qquad k_e > 0, R_\infty > 0, k_p > 0 \tag{4.3}$$

$$\mathbf{Q}_b = k_p \mathbf{q}^p, \qquad \text{given } k_e, R_\infty, r, k_p \in \mathbb{R}, \tag{4.4}$$

$$R_\infty \dot{\mathbf{q}}^p = \mathbf{Q}_a \dot{\lambda}, \qquad \dot{()} = d()/dt, \tag{4.5}$$

$$f\dot{\lambda} = 0, \qquad \lambda \in \mathbb{R}, \tag{4.6}$$

$$f = \|\mathbf{Q}_a\| - R(\lambda) \leq 0, \qquad \|\mathbf{Q}_a\| = \sqrt{\mathbf{Q}_a^T \mathbf{Q}_a}, \tag{4.7}$$

$$\dot{\lambda} \geq 0, \tag{4.8}$$

$$R(\lambda) = R_\infty \sqrt{1 - \text{rexp}\left(\frac{-2\lambda}{\lambda_u}\right)}, \qquad \lambda_u = \frac{R_\infty}{k_p}, R_\infty = R(\infty) \tag{4.9}$$

$$\dot{\lambda} = \begin{cases} \lambda_u \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{R_\infty^2} > 0 & \text{if } f = 0 \text{ and } \mathbf{Q}_a^T \dot{\mathbf{Q}} > 0 \quad \text{on-phase} \\ 0 & \text{if } f < 0 \text{ or } \mathbf{Q}_a^T \dot{\mathbf{Q}} \leq 0 \quad \text{off-phase} \end{cases} \tag{4.10}$$

4.2 Numerical integrations—the substeping method

4.2.1 Algorithms

$$\dot{\lambda} = \begin{cases} \lambda_u \frac{\mathbf{Q}_a^T \dot{\mathbf{Q}}}{R_\infty^2} > 0 & \text{if } f = 0 \text{ and } \mathbf{Q}_a^T \dot{\mathbf{Q}} > 0 \quad \text{on-phase} \\ 0 & \text{if } f < 0 \text{ or } \mathbf{Q}_a^T \dot{\mathbf{Q}} \leq 0 \quad \text{off-phase} \end{cases} \tag{4.11}$$

Taking the time derivative of both sides of (4.1):

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}^e + \dot{\mathbf{q}}^p \tag{4.12}$$

Substituting (4.3) and (4.5) into (4.12):

$$\dot{\mathbf{q}} = \frac{1}{k_e}\dot{\mathbf{Q}} + \frac{\mathbf{Q}_a \dot{\lambda}}{R_\infty} \tag{4.13}$$

The stress input bilinear isotropic-kinematic hardening model:

$$\dot{\mathbf{q}} = \frac{1}{k^e}\dot{\mathbf{Q}} + \frac{\mathbf{Q}_a\dot{\lambda}}{R_\infty}, \tag{4.14}$$

$$\frac{\dot{\mathbf{Q}}_a}{Q_y} + \frac{k_p\dot{\lambda}}{R_\infty}\frac{\mathbf{Q}_a}{Q_y} = \frac{\dot{\mathbf{Q}}}{Q_y} \tag{4.15}$$

$$f\dot{\lambda} = 0, \tag{4.16}$$

$$f = \|\mathbf{Q}_a\| - R(\lambda) \leq 0, \qquad \|\mathbf{Q}_a\| = \sqrt{\mathbf{Q}_a^T\mathbf{Q}_a}, \tag{4.17}$$

$$\dot{\lambda} \geq 0, \tag{4.18}$$

$$R(\lambda) = R_\infty\sqrt{1 - \mathrm{rexp}\left(\frac{-2\lambda}{\lambda_u}\right)}, \qquad \lambda_u = \frac{R_\infty}{k_p}, R_\infty = R(\infty) \tag{4.19}$$

The algorithm for the computation of bilinear elastoplastic models can be classified into two modules:

- Elastic module: It serves the computation of elastic response under the time increment Δt.

- Plastic module: It serves the computation of plastic response under the time increment Δt.

In the elastic module, one can obtain $\dot{\mathbf{q}}$ according to Eq. (3.16) with $\dot{\lambda} = 0$ as follows:

$$\dot{\mathbf{q}} = \frac{1}{k_e}\dot{\mathbf{Q}} \tag{4.20}$$

In the plastic module:

$$\frac{d}{dt}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_\mathrm{a}\end{bmatrix} = \begin{bmatrix}0 & \frac{\dot{\lambda}}{R_\infty}\\ 0 & -\frac{k_p\dot{\lambda}}{R_\infty}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{Q}_\mathrm{a}\end{bmatrix} + \begin{bmatrix}\frac{\dot{\mathbf{Q}}}{k_e}\\ \dot{\mathbf{Q}}\end{bmatrix}$$

(i) Setting parameters

$\mathrm{k_e}$	floor elastic stiffness of building structure
$\mathrm{k_p}$	floor plastic stiffness of building structure
$\mathrm{Q_y}$	floor yield strength of building structure
$\triangle \mathrm{t_e}$	elastic time step
$\triangle \mathrm{t_p}$	plastic time step

(ii) Setting initial values

$\mathbf{q} = \mathbf{0}$	initial total floor displacement
$\dot{\mathbf{q}} = \mathbf{0}$	initial total floor velocity
$\mathbf{q}^p = \mathbf{0}$	initial plastic floor displacement
$\mathbf{Q} = \mathbf{0}$	initial total floor shear force
$\mathbf{Q}_a = \mathbf{0}$	initial active floor shear force
$\mathbf{Q}_b = \mathbf{0}$	initial back floor shear force
$\mathrm{S} = 0$	initial straining condition
$\mathrm{Iflage} = 0$	initial switching index

(iii) input data

$$\mathbf{Q} = a_g$$

(iv) analysis

- If the elastic-plastic index Iflage=[0] , calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with elastic-phase.
  * Check whether the admissible conditions are satisfied.
    · If it is not satisfied
      1.Calculate the switching time response $\dot{\mathbf{q}}(\mathbf{t_{on}})$ and $\mathbf{Q}(\mathbf{t_{on}})$.
      3.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
- If the elastic-plastic index Iflage=[1]
  * Check straining conditions $s = \dot{\lambda} = \mathbf{Q}_a^T \dot{\mathbf{Q}}$.
    · If s>0
      1.Use plastic time step to calculate plastic-phase.
      2.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
    · If s<0
      1.calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with elastic-phase.
      2.Check whether the admissible conditions are satisfied.
      If it is not satisfied
        A.Calculate the switching time response $\dot{\mathbf{q}}(\mathbf{t_s})$ and $\mathbf{Q}(\mathbf{t_{on}})$.
        C.Calculate the next time response $\dot{\mathbf{q}}(t_{i+1})$ with plastic-phase.
      If it is satisfied Iflage=0

Chapter 5

Models of bilinear isotropic-kinematic-distortional hardening elastoplasticity

The model of bilinear isotropic-kinematic-distortional hardening is written as follows:

$$
\begin{aligned}
&\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p, \\
&\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, \\
&\mathbf{Q} = \mathbf{K}_e \mathbf{q}^e, \\
&\dot{\mathbf{Q}}_b = \dot{\lambda} \left\| \frac{\partial f}{\partial \mathbf{Q}} \right\| a_1 (\mathbf{n} - a_2 \mathbf{Q}_b), \\
&\dot{\mathbf{q}}^p = \dot{\lambda} \frac{\partial f}{\partial \mathbf{Q}}, \\
&f \dot{\lambda} = 0, \\
&f = \frac{3}{2}(1 - c \frac{\mathbf{Q}_b^{\mathbf{T}} \mathbf{P}_d (\mathbf{Q} - \mathbf{Q}_b)}{\sqrt{(\mathbf{Q} - \mathbf{Q}_b)^{\mathbf{T}} \mathbf{P}_d (\mathbf{Q} - \mathbf{Q}_b)}}) (\mathbf{Q} - \mathbf{Q}_b)^{\mathbf{T}} \mathbf{P}_d (\mathbf{Q} - \mathbf{Q}_b) - \tau_y^2, \\
&\dot{\lambda} \geq 0, \\
&\dot{\tau}_y = \frac{1}{2} \lambda k_1 (1 - k_2 \tau_y)
\end{aligned}
\tag{5.1}
$$

In this model, the generalized stress is denoted by $\mathbf{Q}$; the generalized strain by $\mathbf{q}$; the generalized elastic strain by $\mathbf{q}^{\mathrm{e}}$; the generalized plastic strain by $\mathbf{q}^{\mathrm{p}}$; and the generalized equivalent plastic strain by λ. Besides above-mentioned variables, the material constants of the model are the generalized elastic modulus $\mathbf{K}_e$ and the generalized yielding stress τ_y.

This formulation exactly expresses the non-reduced variables of the model including the external variables $\mathbf{Q}$ and $\mathbf{q}$ and the internal variables λ. According to Eq. (5.2), if the stress point $\mathbf{Q}$ located inside the yield surface *i.e.* $f < 0$, then $\lambda = 0$ and model (5.1) belongs to the elastic phase.

$$
\lambda = \begin{cases} \frac{\frac{\partial f}{\partial \mathbf{Q}} : \mathbf{K} \dot{\mathbf{q}}}{\mathbf{K}_p + \frac{\partial f}{\partial \mathbf{Q}} : \mathbf{K} \frac{\partial f}{\partial \mathbf{Q}}} & \text{if } f = 0 \quad \text{and } \frac{\partial f}{\partial \mathbf{Q}} : \mathbf{K} \dot{\mathbf{q}} > 0 \quad \text{on-phase} \\ 0 & \text{if } f < 0 \quad \text{or } \frac{\partial f}{\partial \mathbf{Q}} : \mathbf{K} \dot{\mathbf{q}} \leq 0 \quad \text{off-phase} \end{cases}
\tag{5.2}
$$

where

$$
\mathbf{K}_p = \frac{1}{2} \kappa_1 (1 - \kappa_2 k) - \frac{\partial f}{\partial \mathbf{Q}_b} : \left(\frac{\partial f}{\partial \mathbf{Q}} - a_2 \left\| \frac{\partial f}{\partial \mathbf{Q}} \right\| \mathbf{Q}_b \right) a_1
\tag{5.3}
$$

if the straining condition $\mathcal{S} := \frac{\partial f}{\partial \mathbf{Q}} : \mathbf{K} \dot{\mathbf{q}} > 0$ is satisfied. According to the previous derivation, we obtain the two-phase representation as follows:

$$
\begin{aligned}
&\text{plastic-phase:} \begin{cases} \dot{\mathbf{Q}} = -\dot{\lambda}\mathbf{K}\frac{\partial f}{\partial \mathbf{Q}} + \mathbf{K}\dot{\mathbf{q}} \\[2ex] \dot{\mathbf{Q}}_b = \dot{\lambda}\left\|\frac{\partial f}{\partial \mathbf{Q}}\right\| a_1(\mathbf{n} - a_2\mathbf{Q}_b) & \text{if} \quad f = 0 \quad \text{and} \quad \mathcal{S} > 0, \\[2ex] \dot{\tau}_y = \frac{1}{2}\lambda k_1(1 - k_2\tau_y) \end{cases} \\
&\text{elastic-phase:} \begin{cases} \dot{\mathbf{Q}} = \mathbf{K}\dot{\mathbf{q}}, \\ \dot{\mathbf{Q}}_b = 0, & \text{if} \quad f < 0 \quad \text{or} \quad \mathcal{S} \leq 0. \\ \dot{\tau}_y = 0, \end{cases}
\end{aligned}
\tag{5.4}
$$

5.1 Return mapping method

5.1.1 Algorithms

The algorithm for the computation of elastoplastic models can be classified into three modules and one timespace conversion:

- Elastic module: It serves the computation of elastic response under the time increment Δt.
- Plastic module: It serves the computation of plastic response under the time increment Δt.
- Pull-back module: It serves the computation of the elastic-plastic switching time t_s.

In the elastic module, one can obtain $\dot{\mathbf{q}}$ according to Eq. (5.2) with $\dot{\lambda} = 0$ as follows:

$$\dot{\mathbf{Q}} = \mathbf{K}_e\dot{\mathbf{q}} \tag{5.5}$$

In the plastic module:

$$\mathbf{Q}_{t+1} = \mathbf{Q}_t + \mathbf{Q}^{trial} - \lambda\mathbf{K}\frac{\partial f}{\partial \mathbf{Q}} \tag{5.6}$$

where

$$\mathbf{Q}^{trial} = \mathbf{K}\Delta\mathbf{q} \tag{5.7}$$

(i) Setting parameters

$\mathbf{K}$	stiffness tensor
$\mathbf{K}_p$	plastic modulus
$\mathbf{Q}_b$	backstress tensor
τ_{y}	yield stress
c	distortional model parameter
$\mathrm{a}_1, \mathrm{a}_2$	material parameters
$\mathrm{k}_1, \mathrm{k}_2$	material parameters

(ii) Setting initial values

$\mathbf{q} = \mathbf{0}$	initial strain
$\mathbf{Q} = \mathbf{0}$	initial stress
$\mathbf{Q}_b = \mathbf{0}$	initial backstress
Iflage $= 0$	initial switching index

(iii) input data

$$\mathbf{q}$$

(iv) analysis

- If the elastic-plastic index Iflage=[0] , calculate the next time response $\mathbf{Q}_{t+1}$ with elastic-phase.
  - If $f(\mathbf{Q}_{t+1}) \leq 0$

$$\mathbf{Q}_{t+1} = \mathbf{Q}_t + \mathbf{Q}^{trial} \tag{5.8}$$

  - If $f(\mathbf{Q}_{t+1}) > 0$

    1.Calculate the switching time response $\mathbf{Q}_{t_{on}}$ with pull-back.
    2.Update hardening and distortional parameters.

$$\mathbf{Q}_b(t+1) = \mathbf{Q}_b(t) + \dot{\lambda} \left\| \frac{\partial f}{\partial Q} \right\| a_1(\mathbf{n} - a_2 \mathbf{Q}_b(t))$$
$$\tau(t+1) = \tau(t) + \frac{1}{2} \lambda k_1 (1 - k_2 \tau(t))$$

    3.Use return mapping method to calculate.

$$\mathbf{Q}^{\mathrm{fin}}_{t+1} = \mathbf{Q}^{\mathrm{i}}_t - f \cdot \mathbf{n}(\mathbf{n} : \frac{\partial f}{\partial \mathbf{Q}})^{-1}$$

    4.Calculate the next time response $\mathbf{Q_{t+1}}$ with plastic-phase.

- If the elastic-plastic index Iflage=[1]
  - Check straining conditions $s = \dot{\lambda}$
    - If s>0

      1.Use return mapping method to calculate .
      2.Calculate the next time response $\mathbf{Q_{t+1}}$ with plastic-phase.
    - If s<0

      1.calculate the next time response $\mathbf{Q_{t+1}}$ with elastic-phase.
      2.Check whether the admissible conditions are satisfied.
      If it is not satisfied
        A.Calculate the switching time response. $\mathbf{Q}_{t_{on}}$ with pull-back.
        B.Use return mapping method to calculate.
        C.Calculate the next time response. $\mathbf{Q_{t+1}}$ with plastic-phase.
      If it is satisfied Iflage=0