Computational Plasticity

Chapter 3 – Numerical integrations in computational plasticity

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Overview

- The Newton method
- Plastic mechanism and on-off switching
- Return mapping integrations

The Newton method

Nonlinear algebraic equations

Newton's method for the solution of $\mathbf{g}(\mathbf{y}) = 0$ is defined by

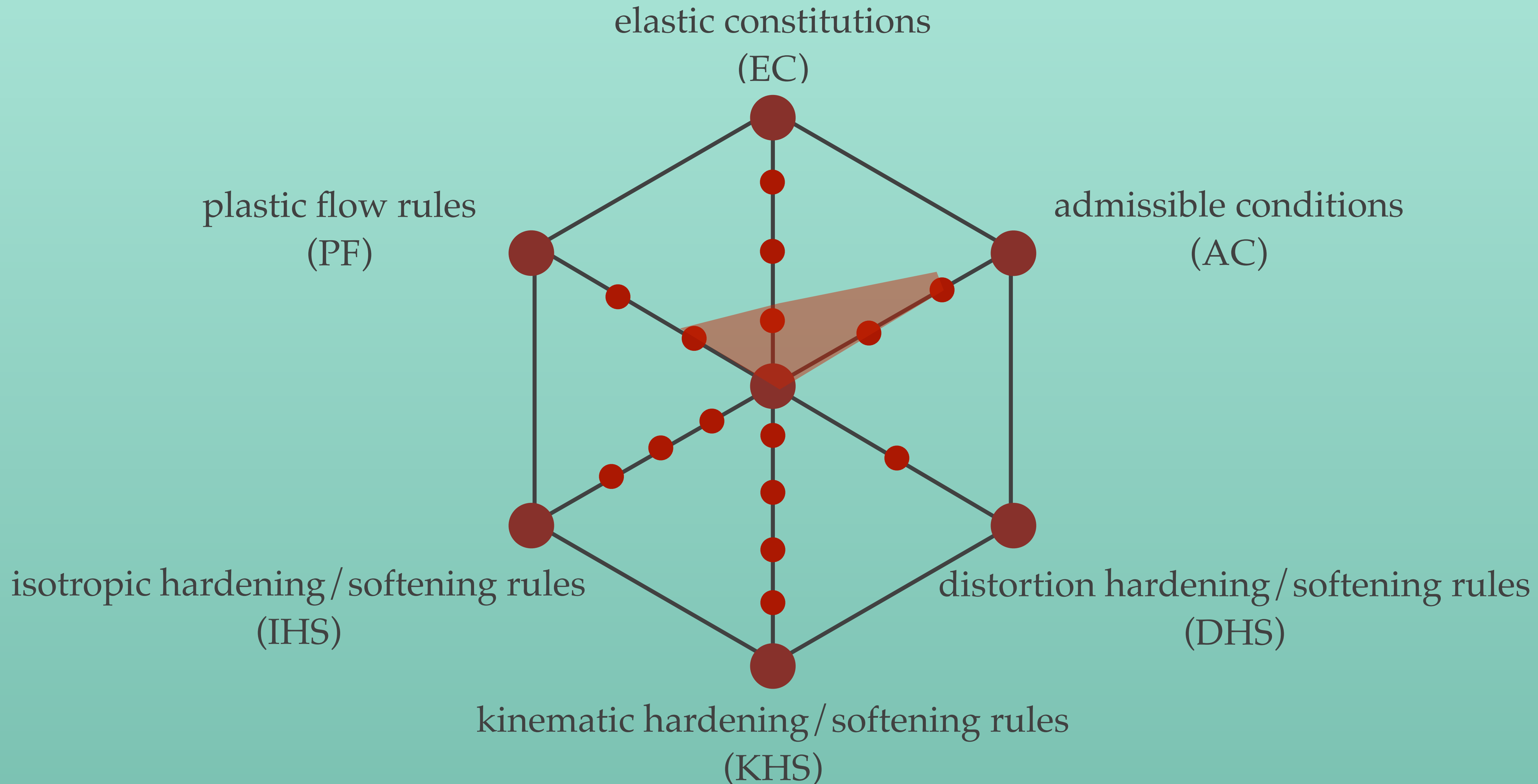
$$\mathbf{y}_{k+1} = \mathbf{y}_k - \left[\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right]^{-1}\Bigg|_{\mathbf{y}=(\mathbf{y}_k)} \mathbf{g}(\mathbf{y}_k), k = 0,1,2,\cdots$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

Hand-in

If $g = x_1^2 + x_2^2 - 1$, please calculate y_1 and y_2 by using the Newton method for the algebraic equation $g = 0$ and $y_0 = 1.5$ where $x_1 = 1 - y$ and $x_2 = 1 + y$.

Plastic mechanism and on-off switching



Models of perfect elastoplasticity

Perfectly elastoplastic models–generalized model 1

Generalized models of perfect elastoplasticity

$$\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},$$

$$\mathbf{Q} = k_e \mathbf{q}^{\mathrm{e}},$$

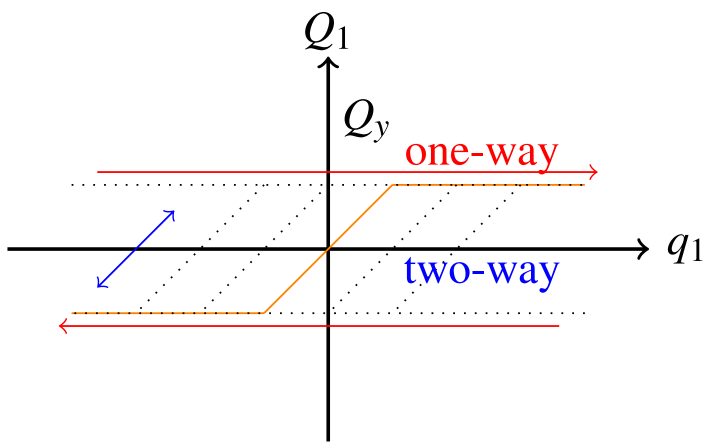
$$\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}} \dot{\lambda},$$

$$f = \|\mathbf{Q}\| - Q_y \le 0,$$

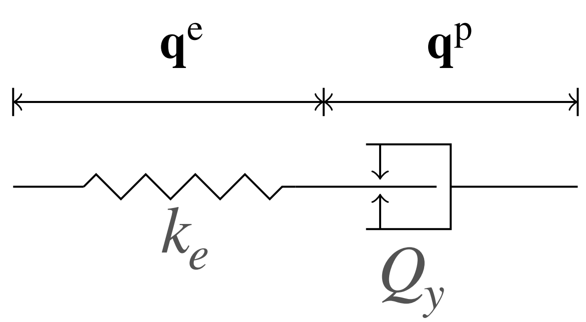
$$\dot{\lambda} \ge 0,$$

$$f \dot{\lambda} = 0.$$

stress-strain curve



mechanical element



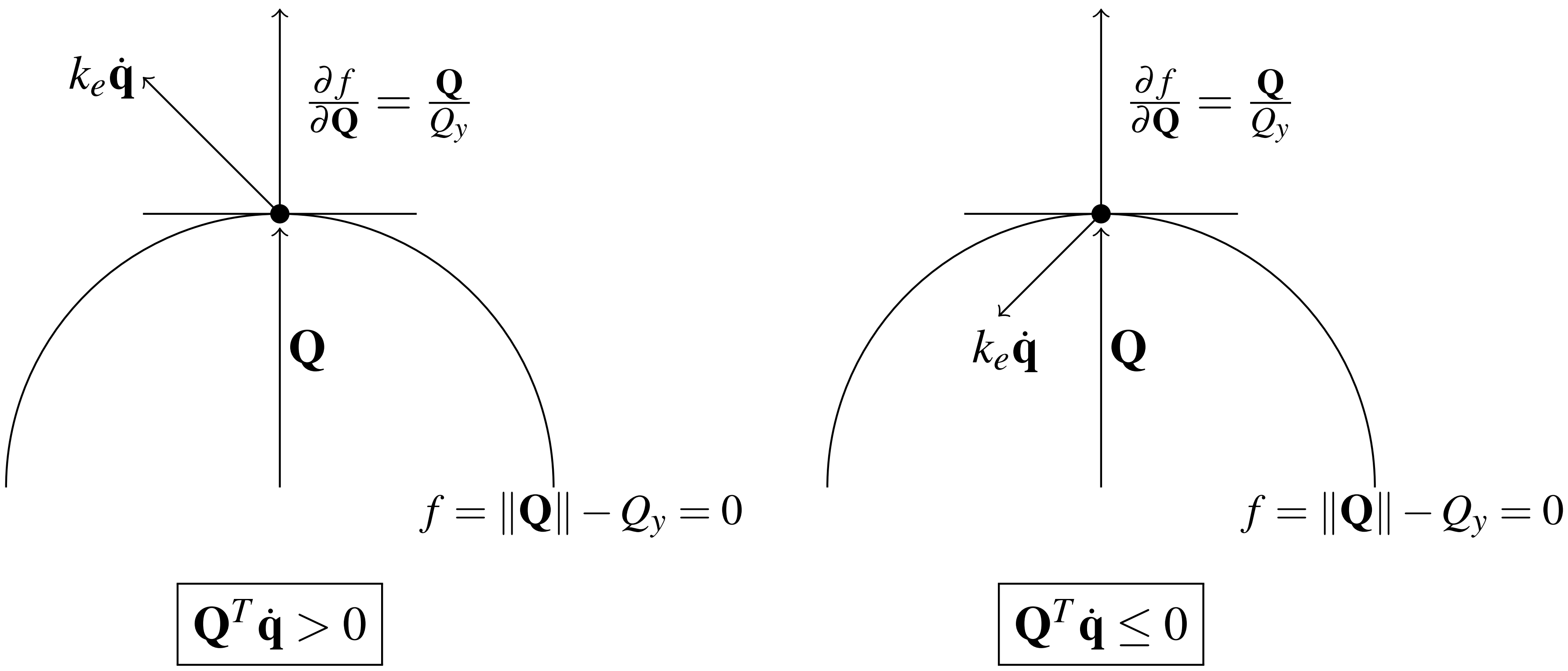
On-off switching

$$\dot{\lambda} = \begin{cases} \dfrac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y} & \text{if} \quad \|\mathbf{Q}\| - Q_y = 0 \quad \text{and} \quad \mathbf{Q}^T \dot{\mathbf{q}} > 0 \quad \text{ON phase} \\ 0 & \text{if} \quad \|\mathbf{Q}\| - Q_y < 0 \quad \text{or} \quad \mathbf{Q}^T \dot{\mathbf{q}} \le 0 \quad \text{OFF phase} \end{cases}$$

Derivation —on-off switching

Generalized models of perfect elastoplasticity

- The straining condition



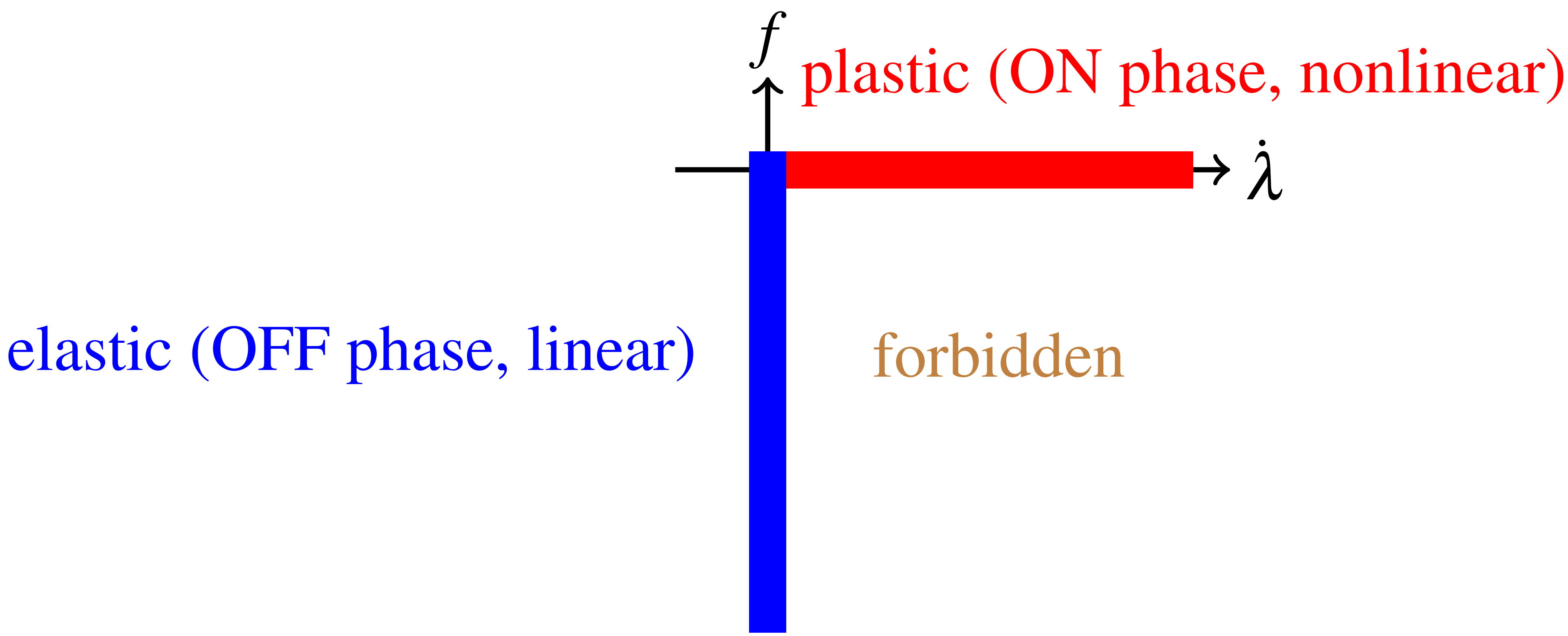
- Sufficient and necessary condition of plastic mechanism

$$\{\,\|\mathbf{Q}\| - Q_y = 0 \text{ and } \mathbf{Q}^T\dot{\mathbf{q}} > 0\} \Longleftrightarrow \{\dot{\lambda} = \frac{\mathbf{Q}^T\dot{\mathbf{q}}}{Q_y} > 0\} \Longleftrightarrow \{\dot{\lambda} > 0\}$$

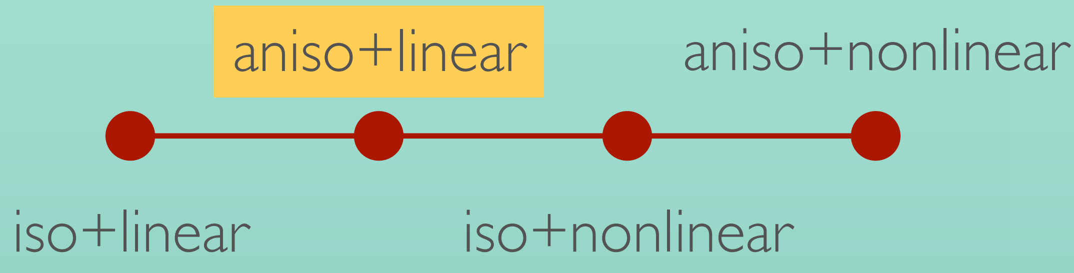
Generalized models of perfect elastoplasticity

- A two-phase dynamical system

$\dot{\mathbf{Q}} = k_e \dot{\mathbf{q}}$	if	$\|\mathbf{Q}\| - Q_y < 0$	or	$\mathbf{Q}^T \dot{\mathbf{q}} \leq 0$	off-phase
$\dot{\mathbf{Q}} = -k_e \dfrac{\mathbf{Q}^T \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}}$	if	$\|\mathbf{Q}\| - Q_y = 0$	and	$\mathbf{Q}^T \dot{\mathbf{q}} > 0$	on-phase



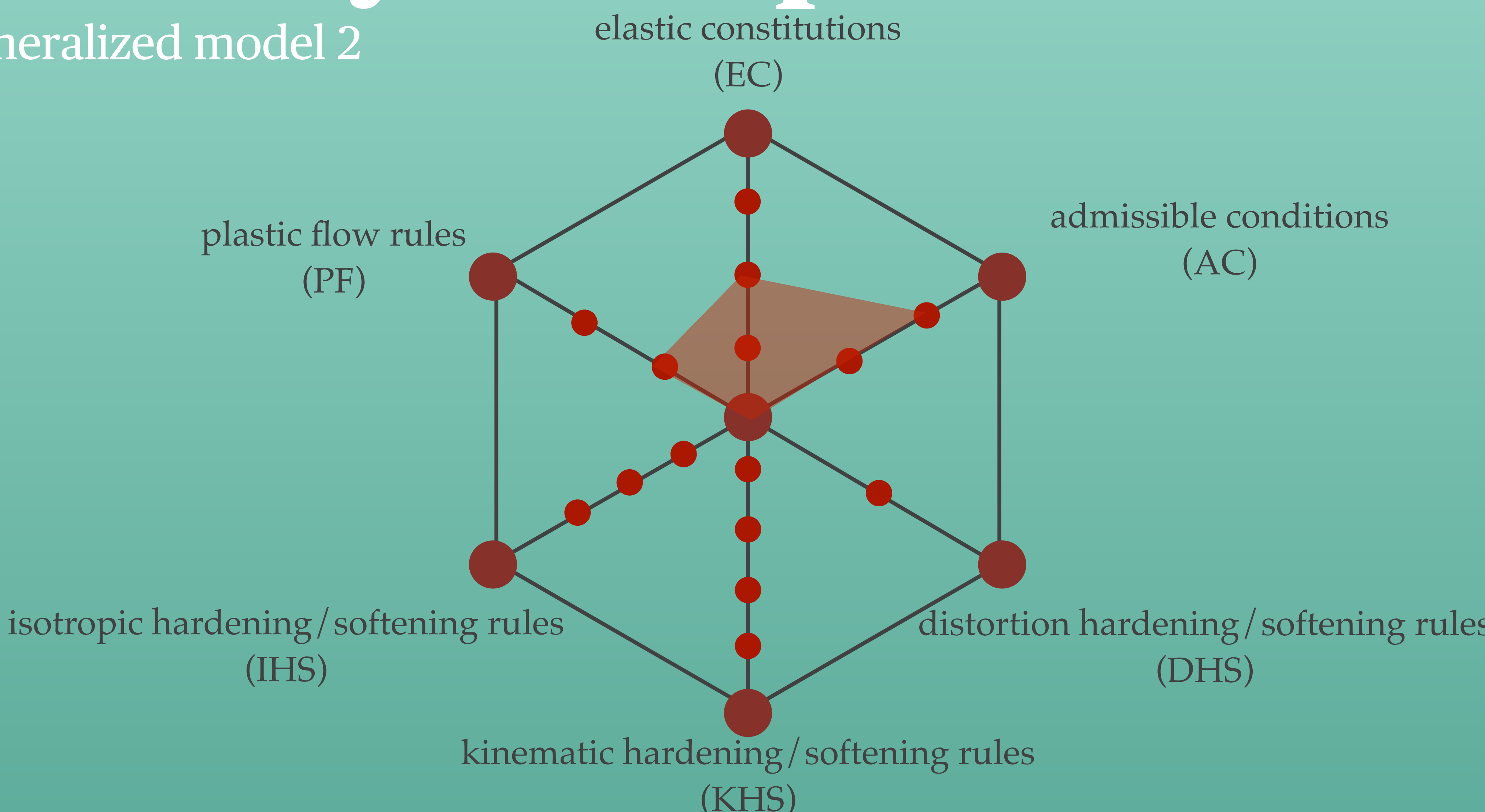
- elastic constitutions



- stress admissible condition

Perfectly elastoplastic models

–Generalized model 2

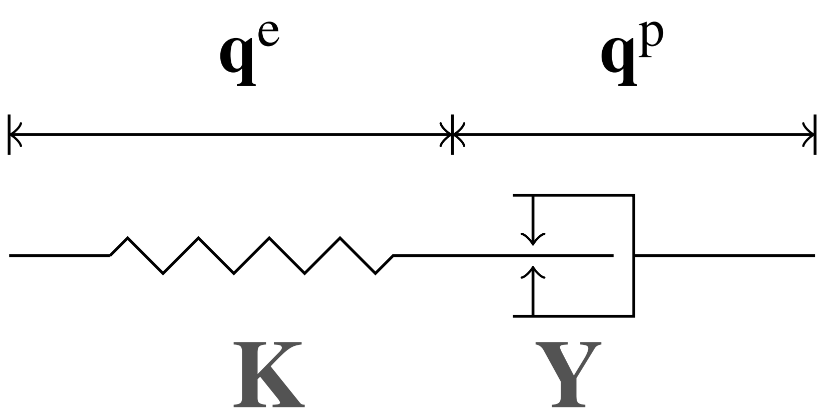


Generalized models of perfect elasticity with quadratic yield surface

- Mathematical formulation

$$
\begin{aligned}
&\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},\\
&\mathbf{Q} = \mathbf{K}\mathbf{q}^{\mathrm{e}},\\
&\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}}\dot{\lambda},\\
&f = \frac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} - 1 \leq 0,\\
&\dot{\lambda} \geq 0,\\
&f\dot{\lambda} = 0.
\end{aligned}
$$

- Mechanical element



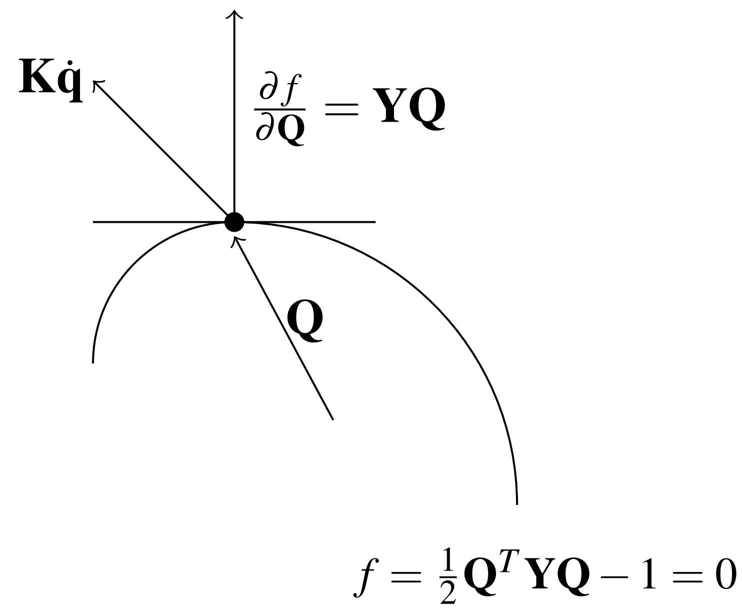
$$\mathbf{K}, \mathbf{Y} \in \mathbb{R}^{N \times N}$$

- On-off switching

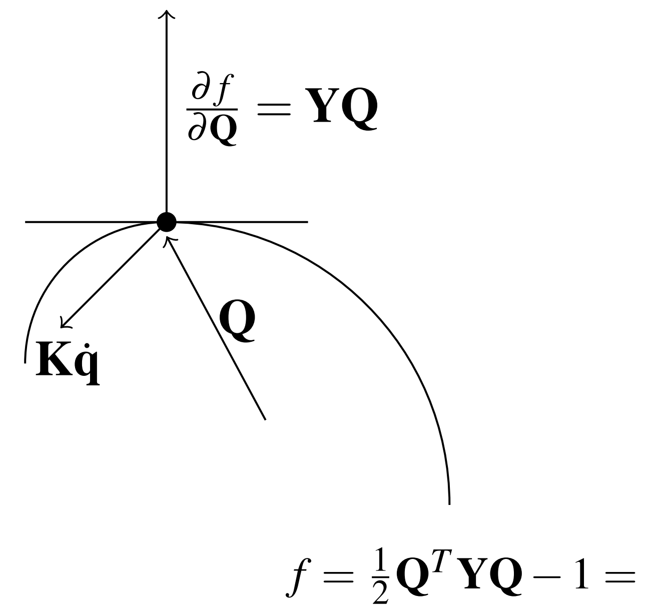
$$
\dot{\lambda} = \begin{cases}
\dfrac{\mathbf{Q}^T\mathbf{Y}\mathbf{K}\dot{\mathbf{q}}}{\mathbf{Q}^T\mathbf{Y}\mathbf{K}\mathbf{Y}\mathbf{Q}} & \text{if} \quad \dfrac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} - 1 = 0 \quad \text{and} \quad \mathbf{Q}^T\mathbf{Y}\mathbf{K}\dot{\mathbf{q}} > 0 \quad \text{ON phase}\\[2ex]
0 & \text{if} \quad \dfrac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} - 1 < 0 \quad \text{or} \quad \mathbf{Q}^T\mathbf{Y}\mathbf{K}\dot{\mathbf{q}} \leq 0 \quad \text{OFF phase}
\end{cases}
$$

Generalized models of perfect elasticity

- The straining condition



$\boxed{\mathbf{Q}^T\mathbf{YK}\dot{\mathbf{q}} > 0}$



$\boxed{\mathbf{Q}^T\mathbf{YK}\dot{\mathbf{q}} \leq 0}$

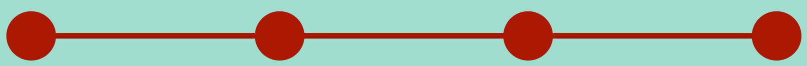
- Sufficient and necessary condition of plastic mechanism

$$\{\frac{1}{2}\mathbf{Q}^T\mathbf{YQ} - 1 = 0 \text{ and } \mathbf{Q}^T\mathbf{YK}\dot{\mathbf{q}} > 0\} \iff \{\dot{\lambda} = \frac{\mathbf{Q}^T\mathbf{YK}\dot{\mathbf{q}}}{\mathbf{Q}^T\mathbf{YKYQ}} > 0\} \iff \{\dot{\lambda} > 0\}$$

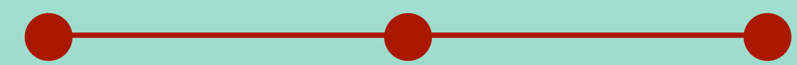
- A two-phase dynamical system

$$\begin{cases} \dot{\mathbf{Q}} = \mathbf{K}\dot{\mathbf{q}} & \text{if } \frac{1}{2}\mathbf{Q}^T\mathbf{YQ} - 1 < 0 \text{ or } \mathbf{Q}^T\mathbf{YK}\dot{\mathbf{q}} \leq 0, \\ \dot{\mathbf{Q}} = -\mathbf{K}\frac{\mathbf{Q}^T\mathbf{YK}\dot{\mathbf{q}}}{\mathbf{Q}^T\mathbf{YKYQ}}\mathbf{YQ} + \mathbf{K}\dot{\mathbf{q}} & \text{if } \frac{1}{2}\mathbf{Q}^T\mathbf{YQ} - 1 = 0 \text{ and } \mathbf{Q}^T\mathbf{YK}\dot{\mathbf{q}} > 0. \end{cases}$$

- elastic constitutions

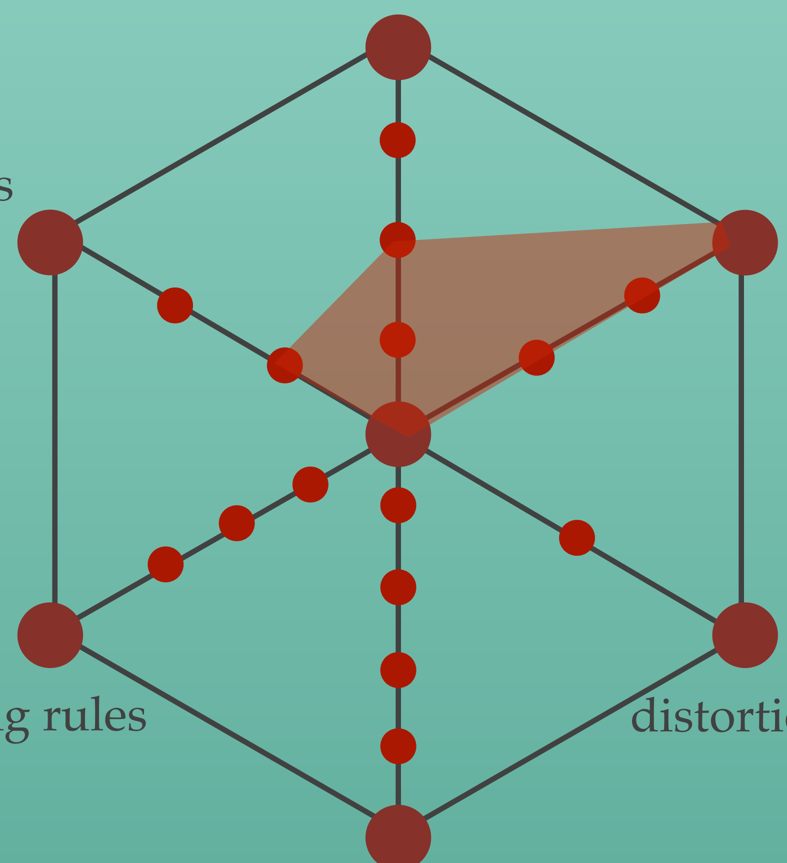


- stress admissible condition



Perfectly elastoplastic models

–Generalized model 3

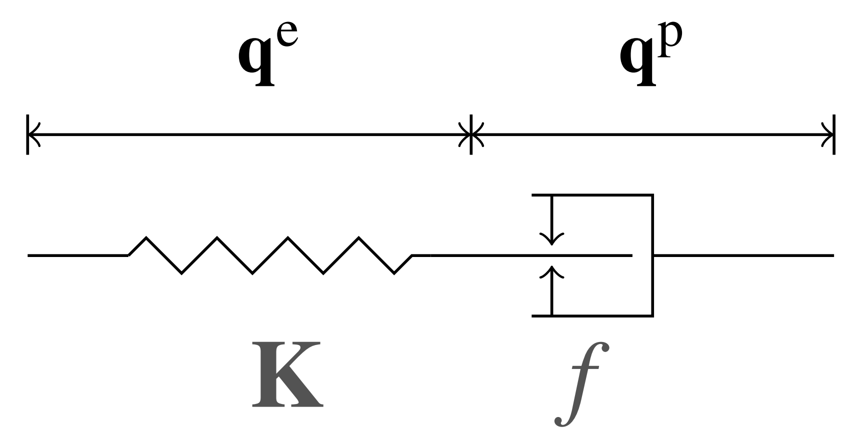


Generalized models of perfect elasticity with arbitrary yield function

- Mathematical formulation

$$
\begin{aligned}
&\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},\\
&\mathbf{Q} = \mathbf{K}\mathbf{q}^{\mathrm{e}},\\
&\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}}\dot{\lambda},\\
&f \le 0,\\
&\dot{\lambda} \ge 0,\\
&f\dot{\lambda} = 0.
\end{aligned}
$$

- Mechanical element



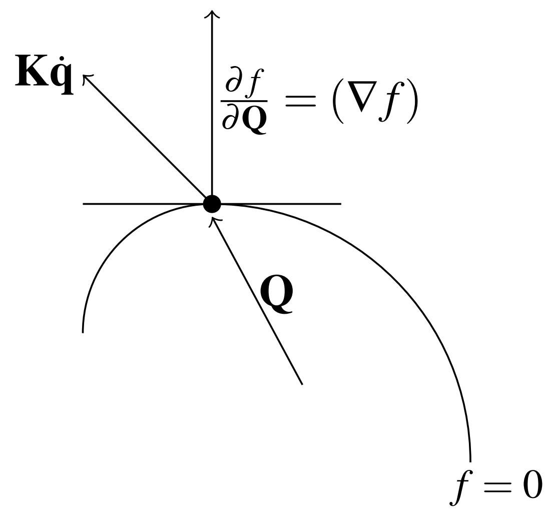
$$\mathbf{K} \in \mathbb{R}^{N\times N}$$

- On-off switching

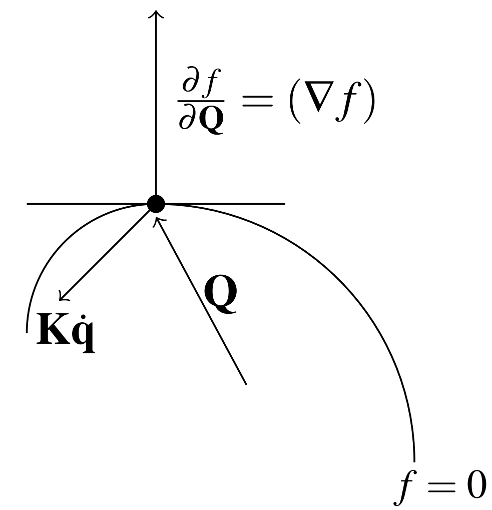
$$
\dot{\lambda} = \begin{cases} \dfrac{(\nabla f)^T \mathbf{K}\dot{\mathbf{q}}}{(\nabla f)^T \mathbf{K}\,\nabla f} & \text{if} \quad f = 0 \quad \text{and} \quad (\nabla f)^T \mathbf{K}\dot{\mathbf{q}} > 0 \quad \text{ON phase} \\ 0 & \text{if} \quad f < 0 \quad \text{or} \quad (\nabla f)^T \mathbf{K}\dot{\mathbf{q}} \le 0 \quad \text{OFF phase} \end{cases}
$$

Generalized models of perfect elasticity with arbitrary yield function

- The straining condition



$\boxed{(\nabla f)^T \mathbf{K}\dot{\mathbf{q}} > 0}$



$\boxed{(\nabla f)^T \mathbf{K}\dot{\mathbf{q}} \leq 0}$

- Sufficient and necessary condition of plastic mechanism

$$\{f = 0 \text{ and } (\nabla f)^T \mathbf{K}\dot{\mathbf{q}} > 0\} \iff \{\dot{\lambda} = \frac{(\nabla f)^T \mathbf{K}\dot{\mathbf{q}}}{(\nabla f)^T \mathbf{K}\,\nabla f} > 0\} \iff \{\dot{\lambda} > 0\}$$

- A two-phase dynamical system

$$\begin{cases} \dot{\mathbf{Q}} = \mathbf{K}\dot{\mathbf{q}} & \text{if } f < 0 \text{ or } (\nabla f)^T \mathbf{K}\dot{\mathbf{q}} \leq 0, \\ \dot{\mathbf{Q}} = -\mathbf{K}\dfrac{(\nabla f)^T \mathbf{K}\dot{\mathbf{q}}}{(\nabla f)^T \mathbf{K}\,\nabla f}\nabla f + \mathbf{K}\dot{\mathbf{q}} & \text{if } f = 0 \text{ and } (\nabla f)^T \mathbf{K}\dot{\mathbf{q}} > 0. \end{cases}$$

Elastoplastic models with kinematic hardening

- elastic constitutions

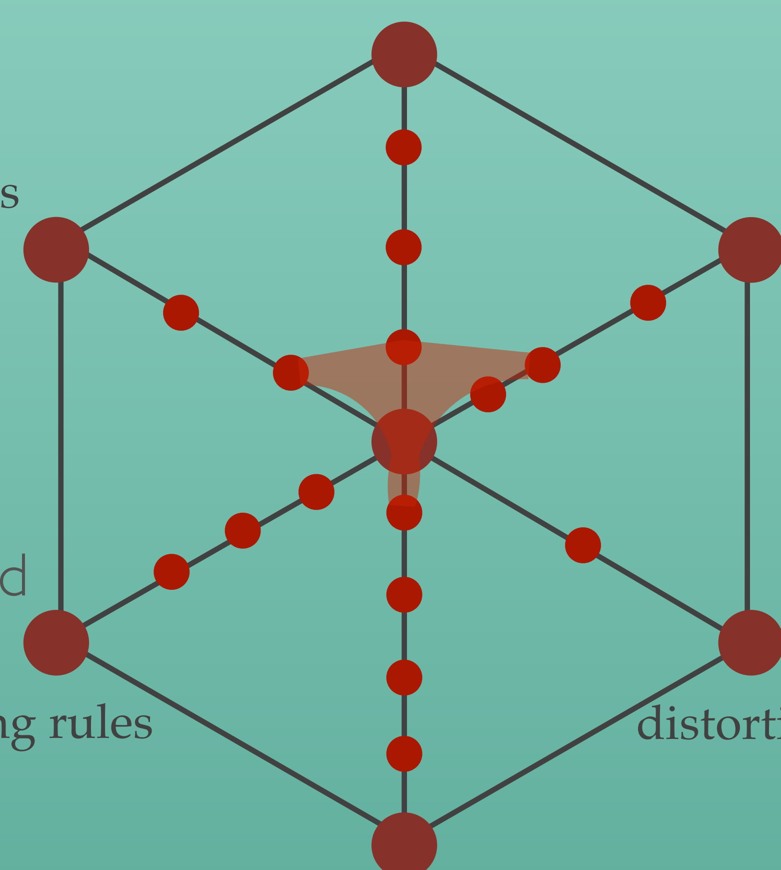
aniso+linear aniso+nonlinear

iso+linear iso+nonlinear

- stress admissible conditions

linear quadratic+linear

quadratic high order (cubic)



- plastic flow rules

generalized associated

associated non-associated

- kinematic hardening

Prager O-W

non A-F A-K

Generalized models of bilinear elastoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p,$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b,$$

$$\mathbf{Q} = k_e \mathbf{q}^e,$$

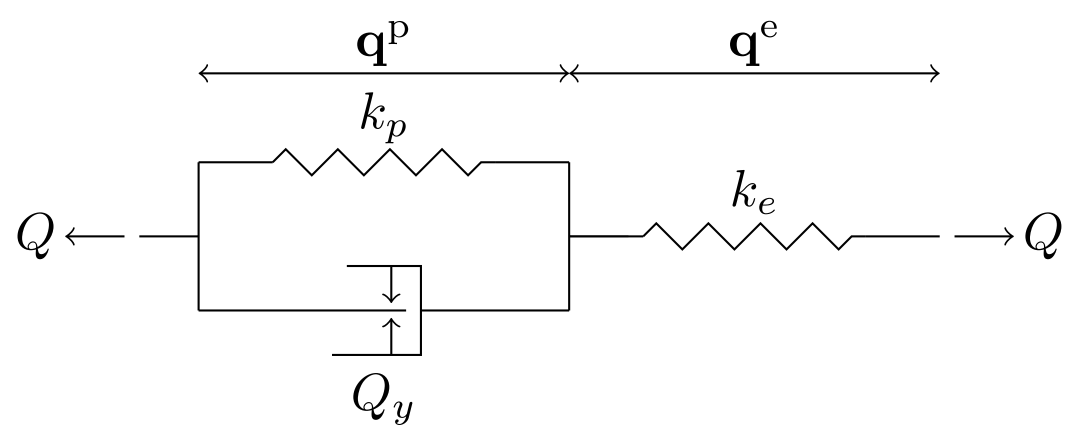
$$\dot{\mathbf{q}}^p = \dot{\lambda}\frac{\partial f}{\partial \mathbf{Q}},$$

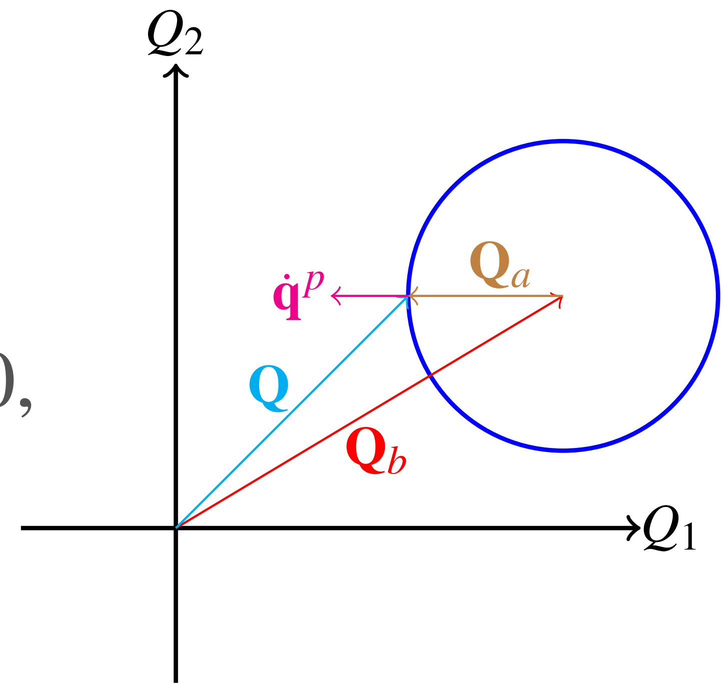
$$\dot{\mathbf{Q}}_b = k_p \dot{\mathbf{q}}^p,$$

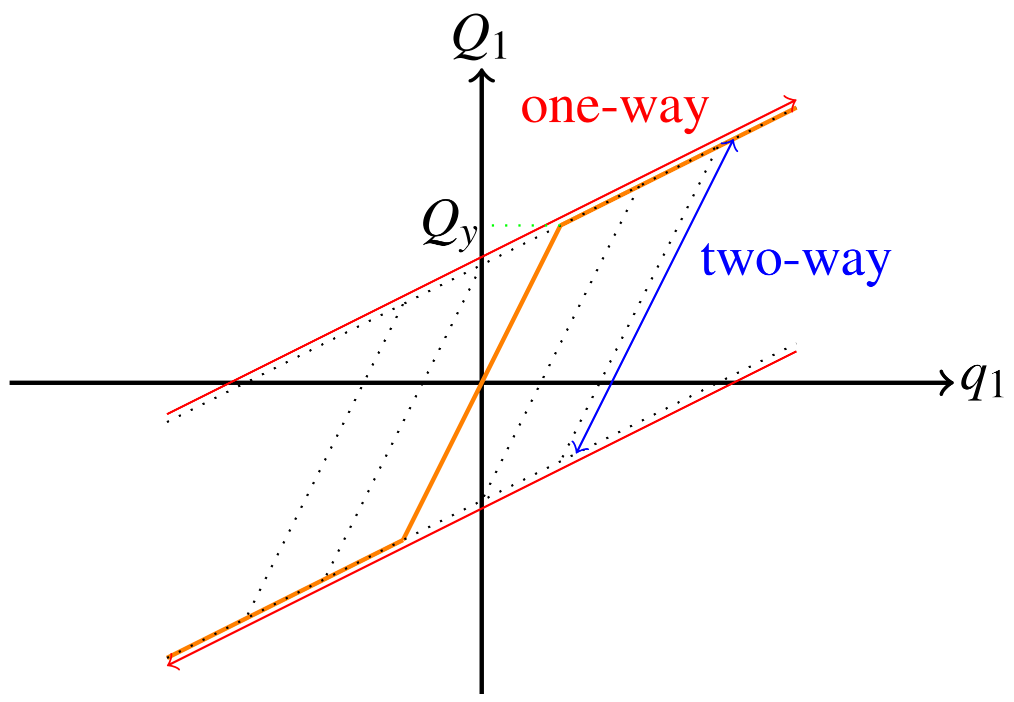
$$f = \|\mathbf{Q}_a\|^2 - Q_y^2 \leq 0,$$

$$\dot{\lambda} \geq 0,$$

$$f\dot{\lambda} = 0.$$







Generalized models of bilinear elastoplasticity

On-off switching

$$\dot{\lambda} = \begin{cases} \dfrac{\beta \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}}}{2Q_y^2} & \text{if} \quad \|\mathbf{Q}\|^2 - Q_y^2 = 0 \quad \text{and} \quad \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}} > 0 \quad \text{ON phase} \\ 0 & \text{if} \quad \|\mathbf{Q}\|^2 - Q_y^2 < 0 \quad \text{or} \quad \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}} \le 0 \quad \text{OFF phase} \end{cases}$$

The straining condition

$$\mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}} \ge 0 \qquad \beta = \frac{k_e}{k_e + k_p}$$

Sufficient and necessary condition of plastic mechanism

$$\{\|\mathbf{Q}_\mathrm{a}\|^2 = Q_y^2 \ \text{ and } \ \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}} > 0\} \Leftrightarrow \{\dot{\lambda} = \frac{\beta \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}}}{2Q_y^2} > 0\} \Leftrightarrow \{\dot{\lambda} > 0\}$$

Generalized models of bilinear elastoplasticity

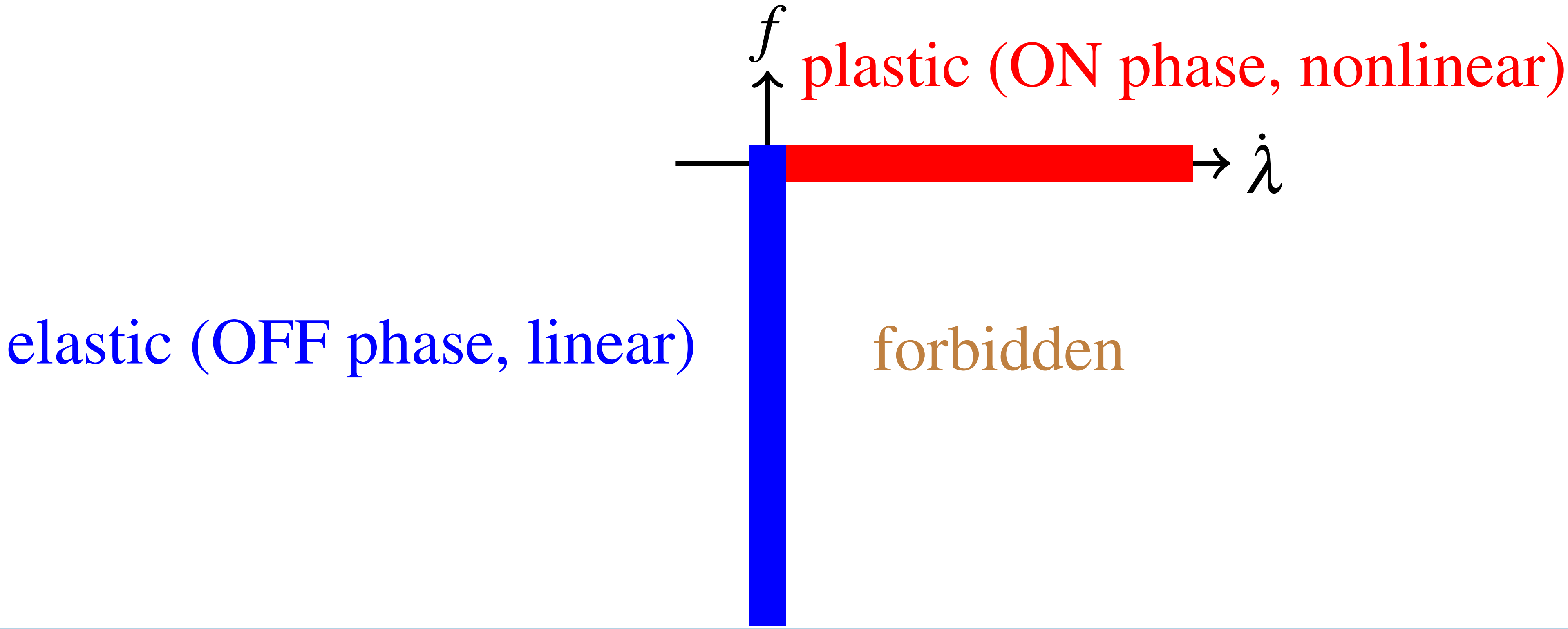
A two-phase dynamical system

$$\dot{\mathbf{Q}} = -k_e \frac{\beta \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}}}{Q_y^2} \mathbf{Q} + k_e \dot{\mathbf{q}} \qquad \text{if} \quad \|\mathbf{Q}_\mathrm{a}\|^2 = Q_y^2 \quad \text{and} \quad \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}} > 0$$

on-phase

$$\dot{\mathbf{Q}} = k_e \dot{\mathbf{q}} \qquad \text{if} \quad \|\mathbf{Q}_\mathrm{a}\|^2 < Q_y^2 \quad \text{or} \quad \mathbf{Q}_\mathrm{a} \cdot \dot{\mathbf{q}} \le 0$$

off-phase



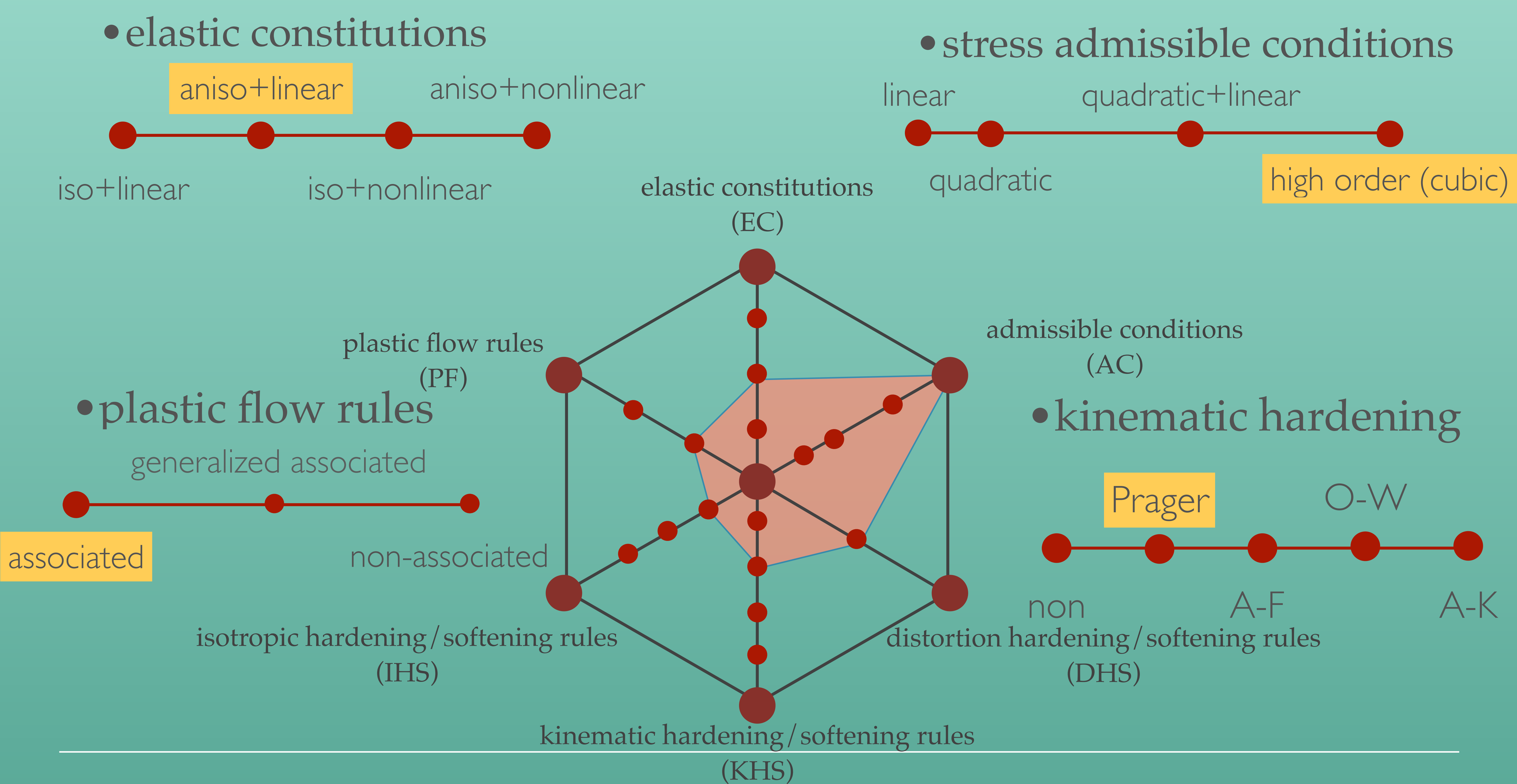
Generalized models of bilinear elastoplasticity

A two-phase dynamical system

$$\begin{cases} \dot{\mathbf{Q}} = -k_e \dfrac{\beta(\mathbf{Q}-\mathbf{Q}_\mathrm{b})\cdot\dot{\mathbf{q}}}{Q_y^2}\mathbf{Q} + k_e\dot{\mathbf{q}} \\ \dot{\mathbf{Q}}_b = k_p \dfrac{\beta(\mathbf{Q}-\mathbf{Q}_\mathrm{b})\cdot\dot{\mathbf{q}}}{Q_y^2}\mathbf{Q} \end{cases} \quad \text{if} \quad \|\mathbf{Q}-\mathbf{Q}_\mathrm{b}\|^2 = Q_y^2 \quad \text{and} \quad \mathbf{Q}_\mathrm{a}\cdot\dot{\mathbf{q}} > 0$$

$$\begin{cases} \dot{\mathbf{Q}} = k_e\dot{\mathbf{q}} \\ \dot{\mathbf{Q}}_b = \mathbf{0} \end{cases} \quad \text{if} \quad \|\mathbf{Q}_\mathrm{a}\|^2 < Q_y^2 \quad \text{or} \quad \mathbf{Q}_\mathrm{a}\cdot\dot{\mathbf{q}} \le 0$$

Models with Directional Distortional hardening



Model of Directional Distortional Hardening

The model of Distortional Hardening elatoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p,$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b,$$

$$\mathbf{Q} = \mathbf{K}_e \mathbf{q}^e,$$

$$\dot{\mathbf{Q}}_b = \dot{\lambda} \left\| \frac{\partial f}{\partial \mathbf{Q}} \right\| a_1(\mathbf{n} - a_2 \mathbf{Q}_b),$$

$$\dot{\mathbf{q}}^p = \dot{\lambda} \frac{\partial f}{\partial \mathbf{Q}},$$

$$f\dot{\lambda} = 0,$$

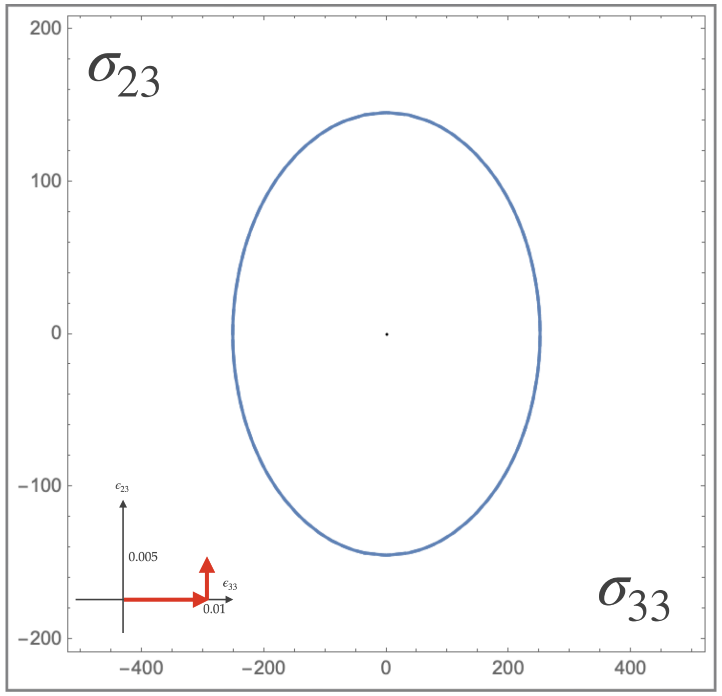
$$f = \frac{3}{2}\left(1 - c\frac{\mathbf{Q}_b^{\mathbf{T}} \mathbf{P}_d(\mathbf{Q} - \mathbf{Q}_b)}{\sqrt{(\mathbf{Q} - \mathbf{Q}_b)^{\mathbf{T}} \mathbf{P}_d(\mathbf{Q} - \mathbf{Q}_b)}}\right)(\mathbf{Q} - \mathbf{Q}_b)^{\mathbf{T}} \mathbf{P}_d(\mathbf{Q} - \mathbf{Q}_b) - \tau_y^2,$$

$$\dot{\lambda} \geq 0,$$

$$\dot{\tau}_y = \frac{1}{2}\lambda k_1(1 - k_2 \tau_y)$$

On-Off switch

$$\dot{\lambda} = \begin{cases} \dfrac{\frac{\partial f}{\partial \mathbf{Q}} : \mathbf{K}\dot{\mathbf{q}}}{\mathbf{K}_p + \frac{\partial f}{\partial \mathbf{Q}} : \mathbf{K} \frac{\partial f}{\partial \mathbf{Q}}} & \text{if } f = 0 \text{ and } \dfrac{\partial f}{\partial \mathbf{Q}} : \mathbf{K}\dot{\mathbf{q}} > 0 \quad \text{on-phase} \\ 0 & \text{if } f < 0 \text{ or } \dfrac{\partial f}{\partial \mathbf{Q}} : \mathbf{K}\dot{\mathbf{q}} \leq 0 \quad \text{off-phase} \end{cases}$$



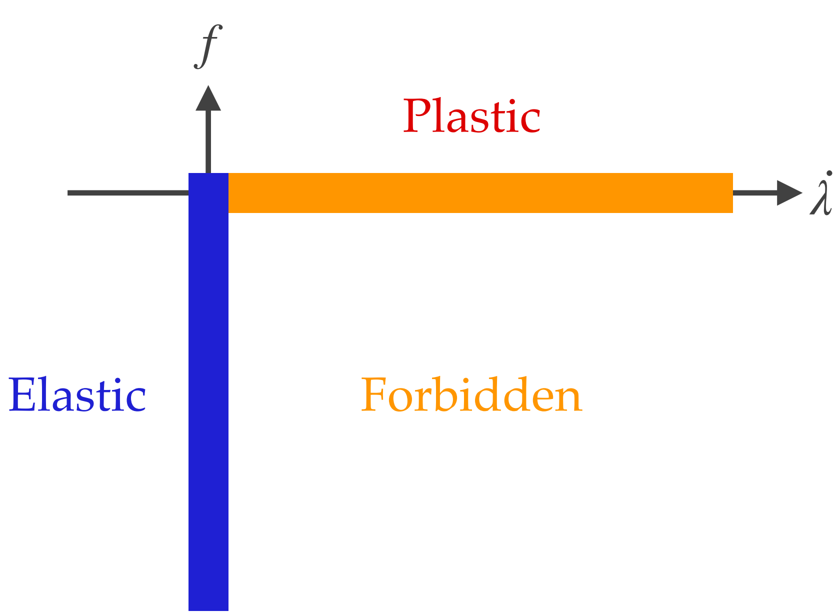
René Marek ; Jiří Plešek2; Zbyněk Hrubý ; Slavomír Parma; Heidi P. Feigenbaum; and Yannis F. Dafalias, M.ASCE Numerical Implementation of A Model With Directional Distortional Hardening, American Society of Civil Engineers,Volume 141, Issue 12, 2015

Model of Directional Distortional Hardening

Two-phase dynamical system

plastic phase: $$\begin{cases} \dot{\mathbf{Q}} = -\dot{\lambda}\mathbf{K}\frac{\partial f}{\partial \mathbf{Q}} + \mathbf{K}\dot{\mathbf{q}} \\ \dot{\mathbf{Q}}_b = \dot{\lambda} \left\| \frac{\partial f}{\partial \mathbf{Q}} \right\| a_1(\mathbf{n} - a_2\mathbf{Q}_b) \\ \dot{\tau}_y = \frac{1}{2}\dot{\lambda}k_1(1 - k_2\tau_y) \end{cases} \quad \text{if } f = 0 \quad \text{and} \quad \mathcal{S} > 0,$$

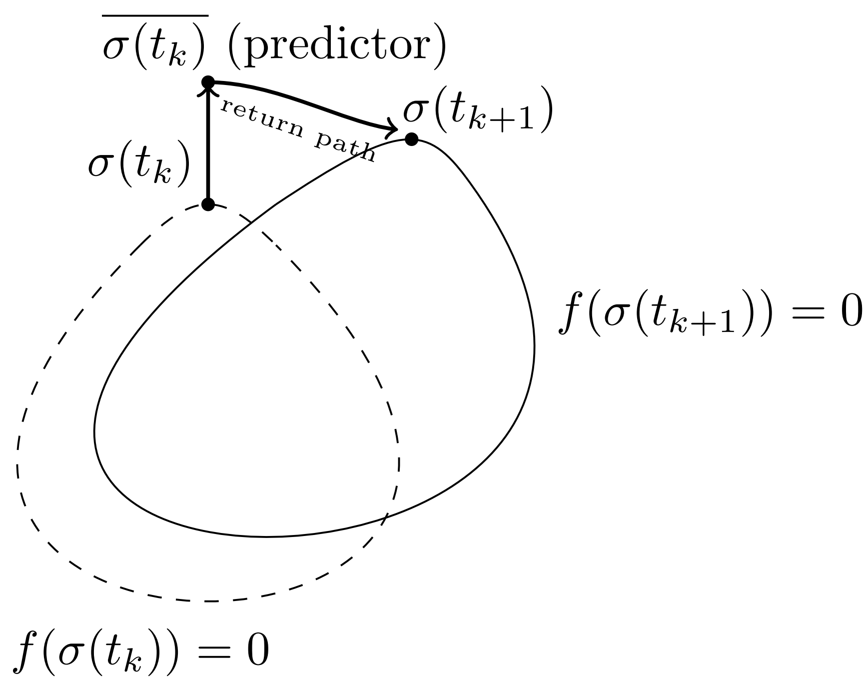
elastic-phase: $$\begin{cases} \dot{\mathbf{Q}} = \mathbf{K}\dot{\mathbf{q}}, \\ \dot{\mathbf{Q}}_b = 0, \\ \dot{\tau} = 0, \end{cases} \quad \text{if } f < 0 \quad \text{and} \quad \mathcal{S} \leq 0,$$



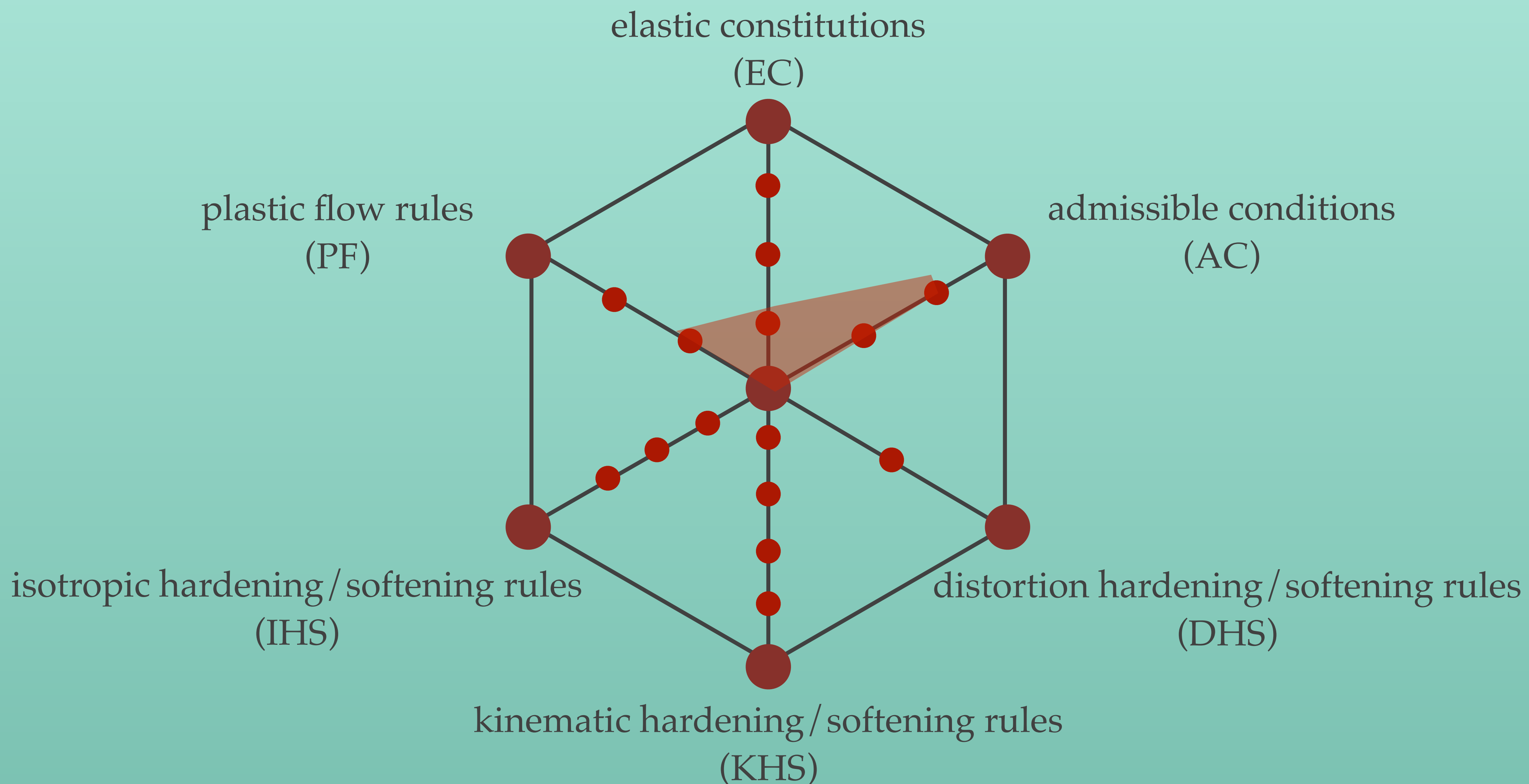
	$\dot{\lambda} = 0$	$\dot{\lambda} > 0$
$f < 0$	Elastic	Forbidden
$f = 0$	Elastic	Plastic

Return mapping integrations

Schematic diagram



- Wilkins (1964)
- Krieg & Krieg (1977)
- Scheryer et al. (1979)
- Ortiz & Pinsky (1983)
- Ortiz & Popov (1985)
- Simo & Taylor (1985)
- Ortiz & Simo (1986)

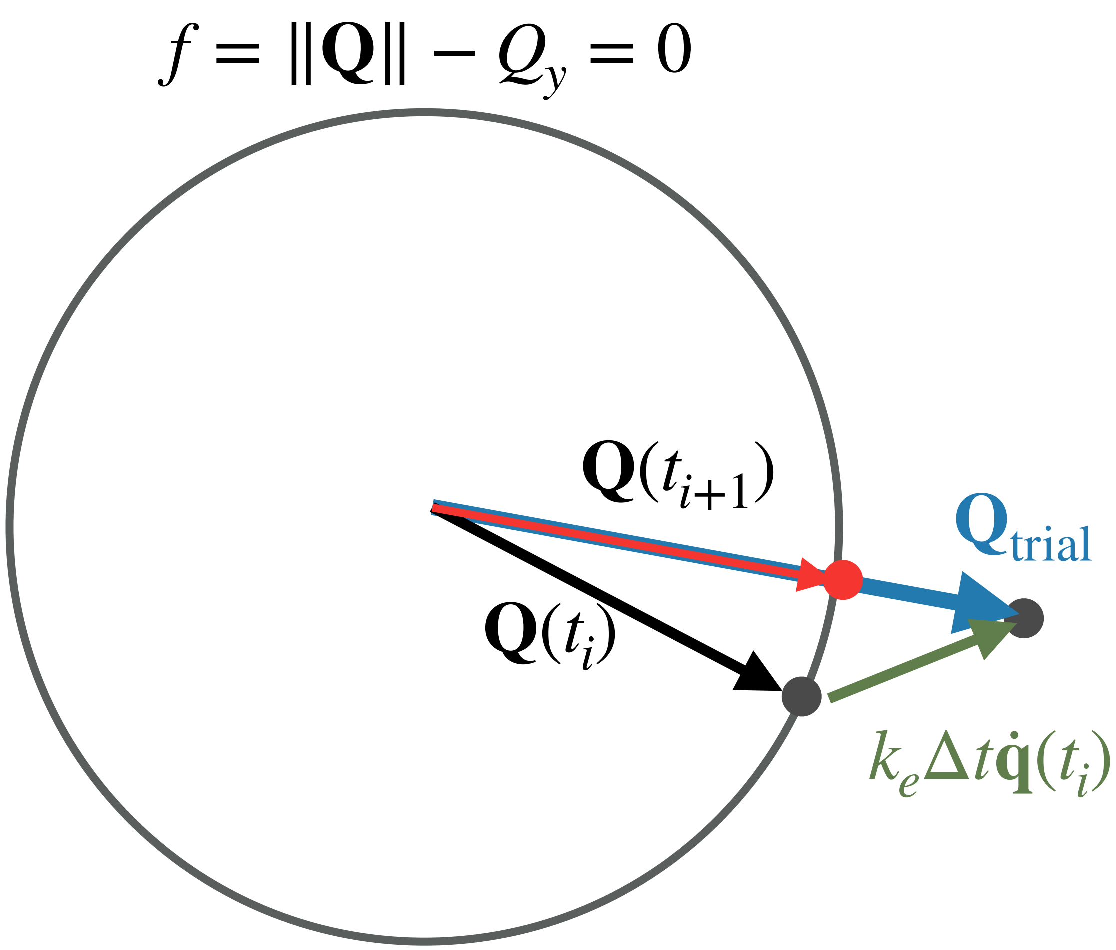


Models of perfect elastoplasticity

Perfectly elastoplastic models–generalized model 1

Return-mapping integrations

The closest-point projection approach



Perfect model–radial return algorithm

PREDICT

Elastic predict

$$\mathbf{Q}_{\text{trial}} = \mathbf{Q}(t_i) + k_e \Delta t \dot{\mathbf{q}}(t_i)$$

Plastic predict

$$\mathbf{Q}_{\text{trial}} = \mathbf{Q}(t_i) + \left[-k_e \frac{\mathbf{Q}^T(t_i)\dot{\mathbf{q}}(t_i)}{Q_y^2} \mathbf{Q}(t_i) + k_e \dot{\mathbf{q}}(t_i) \right] \Delta t$$

CORRECT

$$\mathbf{n}_{\text{trial}} = \frac{\mathbf{Q}_{\text{trial}}}{\|\mathbf{Q}_{\text{trial}}\|}$$

$$\mathbf{Q}(t_{i+1}) = Q_y \mathbf{n}_{\text{trial}}$$

J. C Simo and R. L. Taylor, Consistent tangent operators for rate-independent elastoplasticity, Computer Methods in Applied Mechanics and Engineering, 48, 101-118, 1985

Hand-in

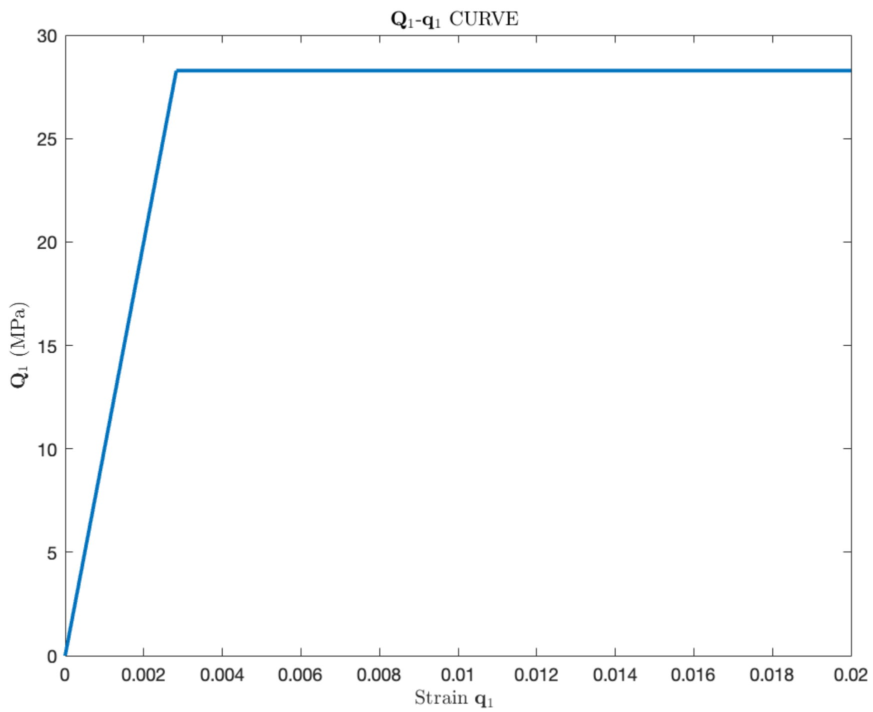
Please calculate $\mathbf{Q}(t_1)$ and $\mathbf{Q}(t_2)$, where $t_{i+1} = t_i + \Delta t, i = 0,1,$ by the closest-point projection approach where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^2$, $k_e = 10$ GPa, $Q_y = 40$ MPa, $\mathbf{Q}(t_0) = \begin{bmatrix} Q_y \\ 0 \end{bmatrix}$, and $\dot{\mathbf{q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Please compare the $\mathbf{Q}(t_1)$ and $\mathbf{Q}(t_2)$ with the result from the substepping integration.

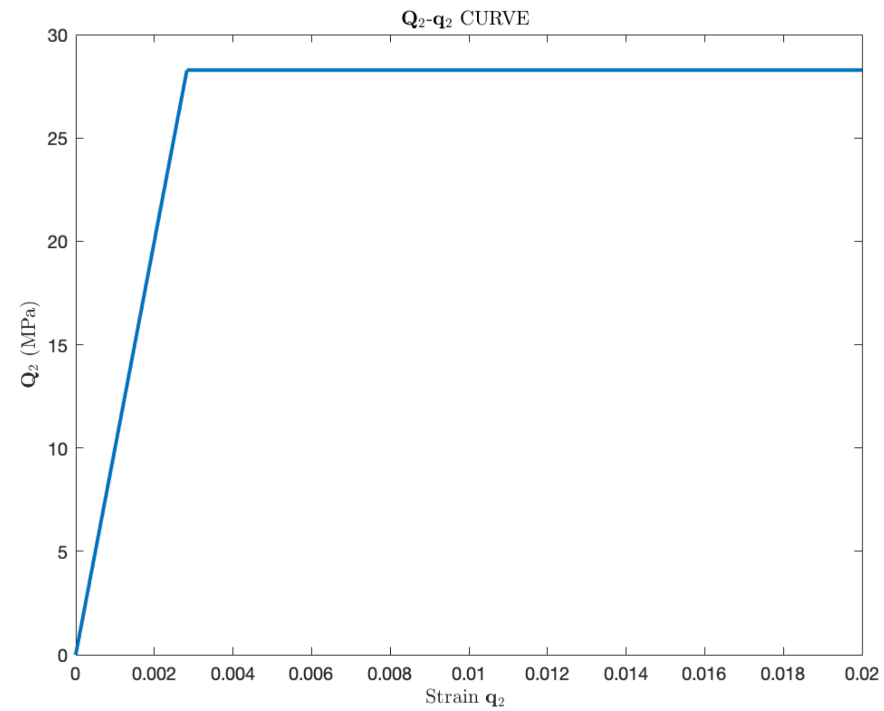
Coding

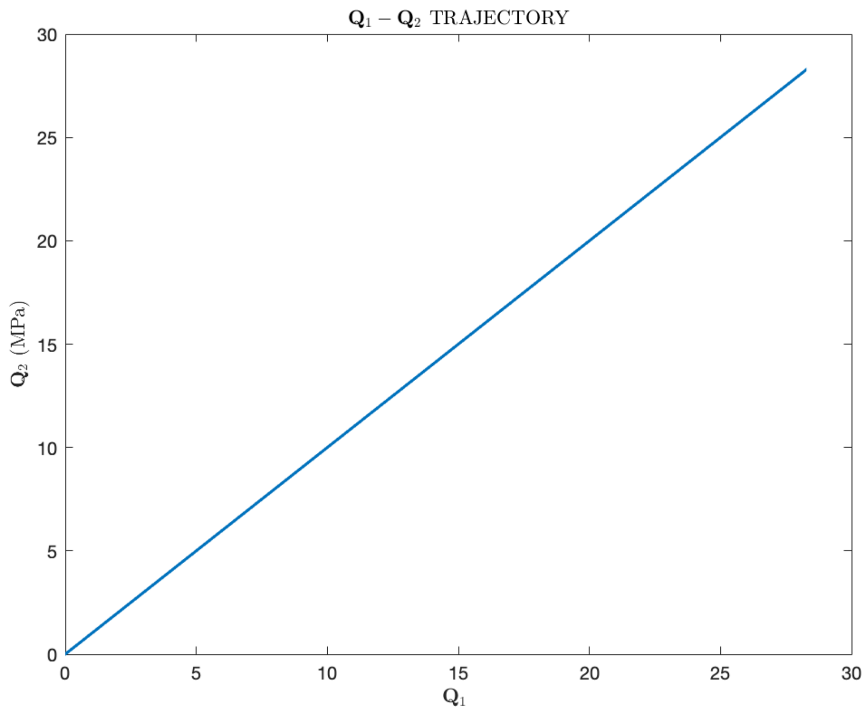
Please code plastic module to compute the $\mathbf{Q}(t_i)$ response by the closest-point projection approach where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^2$, $k_e =$ 10GPa, $Q_y =$ 40MPa, $\mathbf{Q}(t_0) = \begin{bmatrix} Q_y \\ 0 \end{bmatrix}$, and $\dot{\mathbf{q}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

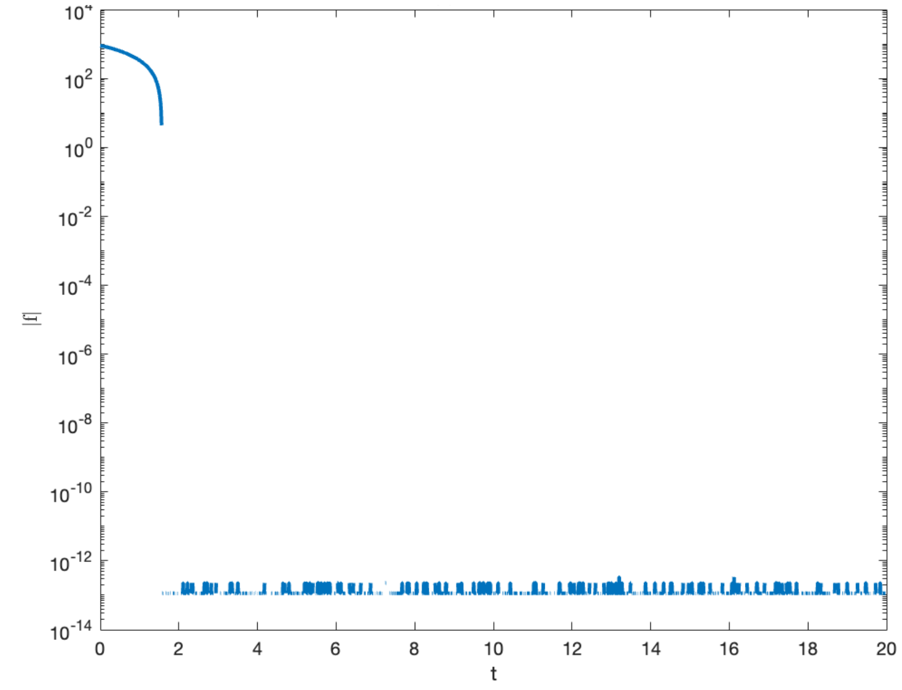
ACMT FISV

Computational results

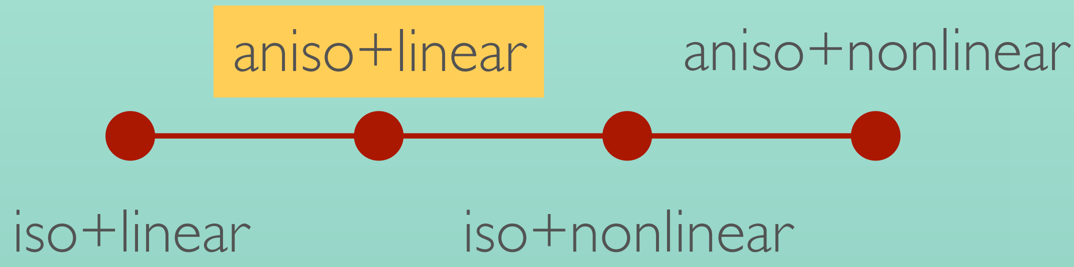




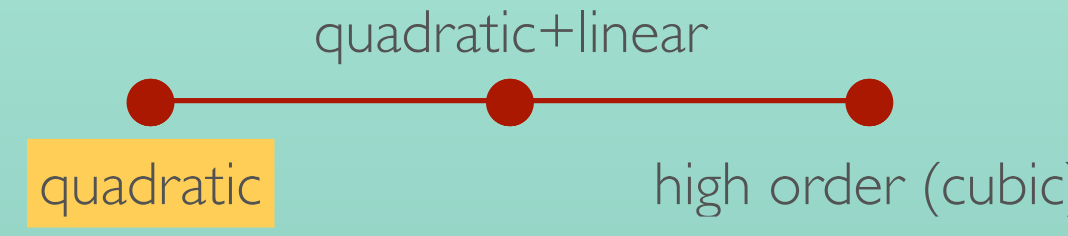




- elastic constitutions

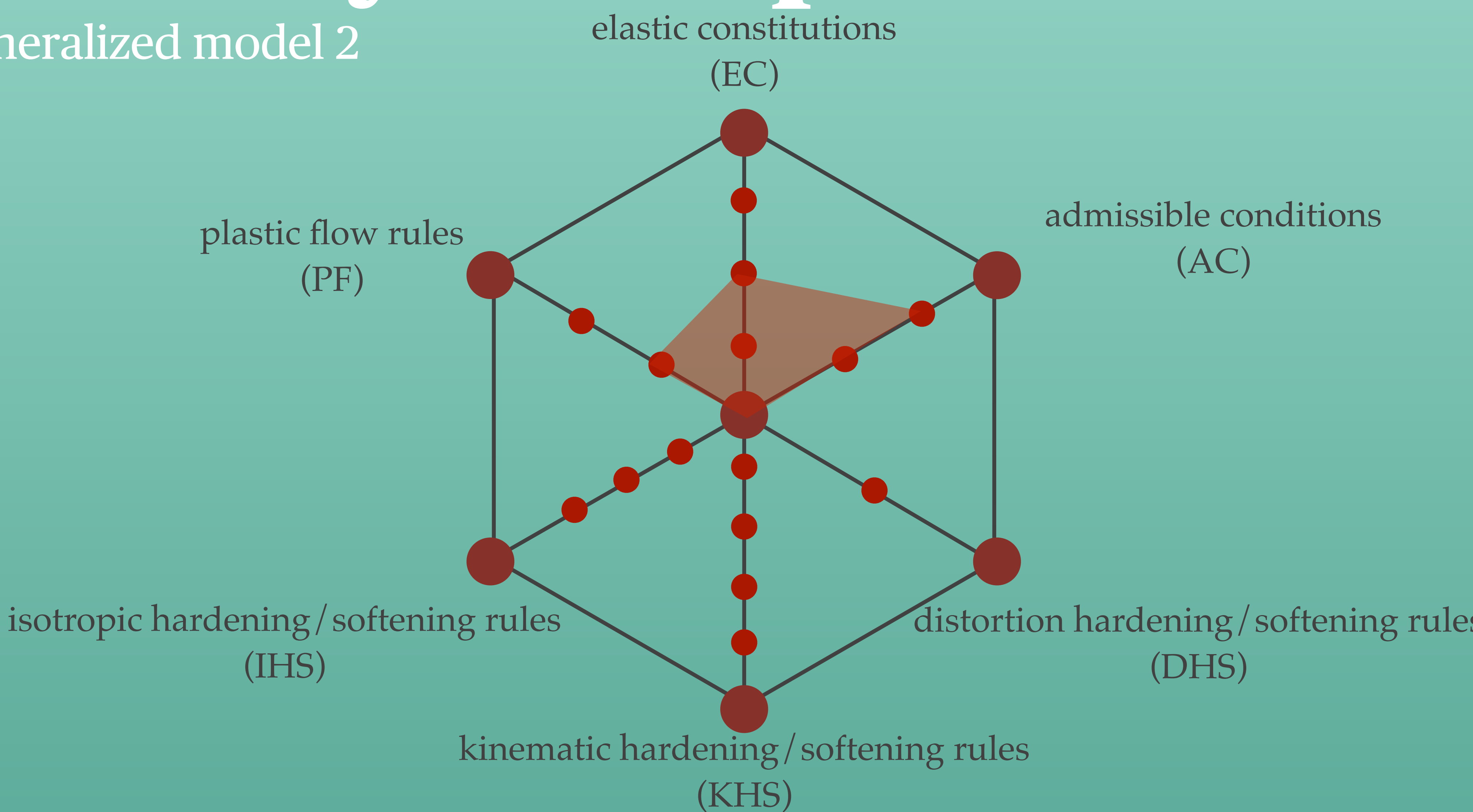


- stress admissible condition



Perfectly elastoplastic models

–Generalized model 2

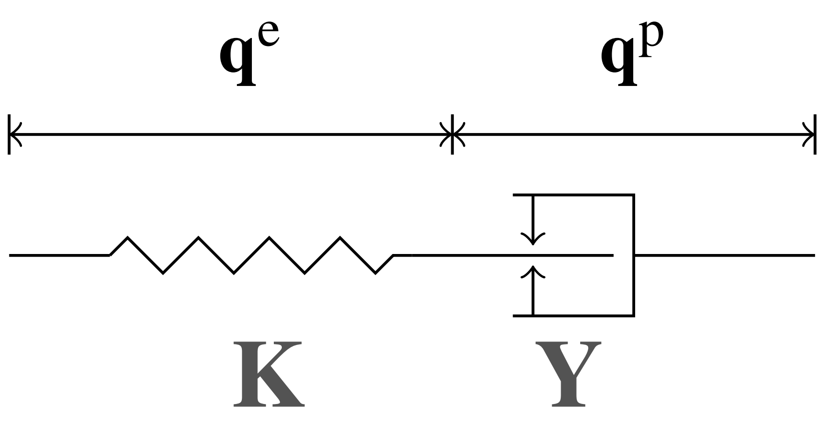


Generalized models of perfect elasticity with quadratic yield surface

- Mathematical formulation

$$\mathbf{q} = \mathbf{q}^{\mathrm{e}} + \mathbf{q}^{\mathrm{p}},$$
$$\mathbf{Q} = \mathbf{K}\mathbf{q}^{\mathrm{e}},$$
$$\dot{\mathbf{q}}^{\mathrm{p}} = \frac{\partial f}{\partial \mathbf{Q}}\dot{\lambda},$$
$$f = \frac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} - 1 \leq 0,$$
$$\dot{\lambda} \geq 0,$$
$$f\dot{\lambda} = 0.$$

- Mechanical element



$$\mathbf{K}, \mathbf{Y} \in \mathbb{R}^{N\times N}$$

- On-off switching

$$\dot{\lambda} = \begin{cases} \dfrac{\mathbf{Q}^T\mathbf{Y}\mathbf{K}\dot{\mathbf{q}}}{\mathbf{Q}^T\mathbf{Y}\mathbf{K}\mathbf{Y}\mathbf{Q}} & \text{if} \quad \frac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} - 1 = 0 \quad \text{and} \quad \mathbf{Q}^T\mathbf{Y}\mathbf{K}\dot{\mathbf{q}} > 0 \quad \text{ON phase} \\ 0 & \text{if} \quad \frac{1}{2}\mathbf{Q}^T\mathbf{Y}\mathbf{Q} - 1 < 0 \quad \text{or} \quad \mathbf{Q}^T\mathbf{Y}\mathbf{K}\dot{\mathbf{q}} \leq 0 \quad \text{OFF phase} \end{cases}$$

Model of anisotropic perfect elastoplasticity

Two phase-dynamic

$$\begin{cases} \text{plastic phase: } \dot{\mathbf{Q}} = -\dot{\lambda}\mathbf{K}\frac{\partial f}{\partial \mathbf{Q}} + \mathbf{K}\dot{\mathbf{q}}, & \text{if } f = 0 \quad \text{and} \quad \mathcal{S} > 0, \\ \text{elastic phase: } \dot{\mathbf{Q}} = \mathbf{K}\dot{\mathbf{q}}, & \text{if } f < 0 \quad \text{or} \quad \mathcal{S} \leq 0. \end{cases}$$

The return mapping method

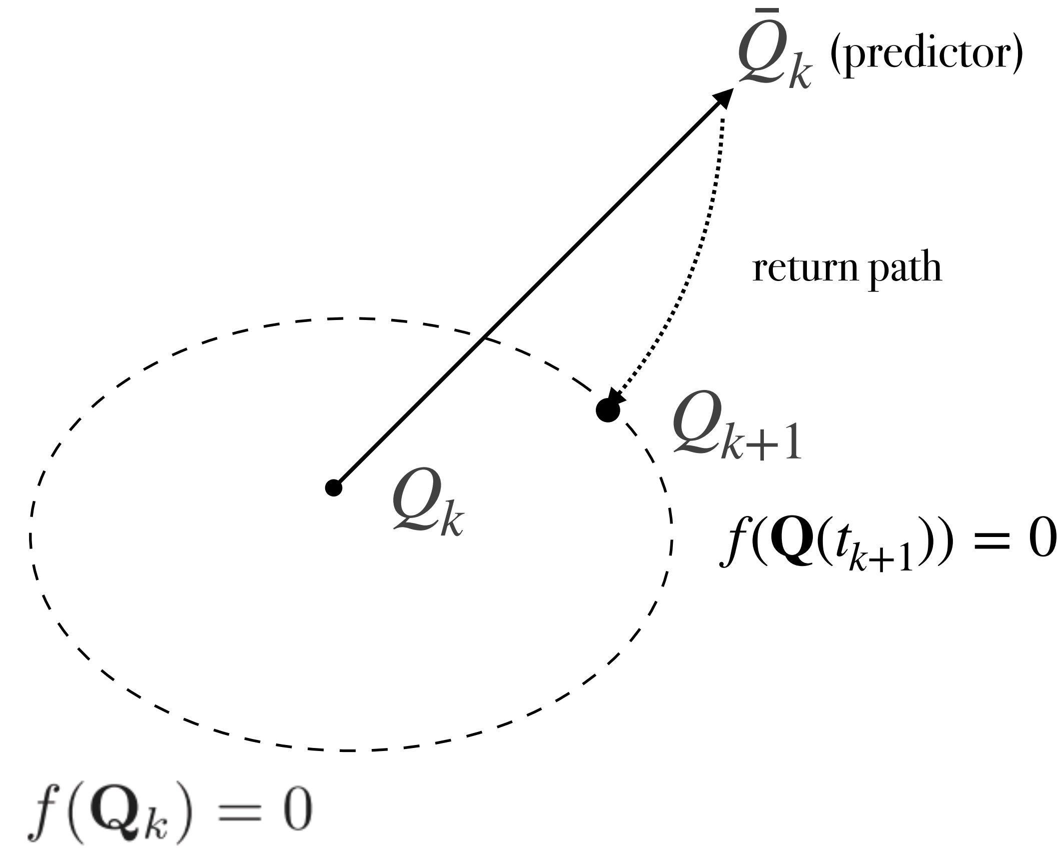
$$\mathbf{Q}(t_{k+1}) = \mathbf{Q}(t_k) + \left(\mathbf{K}^{-1} + \Delta\lambda\mathbf{Y}\right)^{-1}\left(\mathbf{K}^{-1}\mathbf{Q}(t_k) + \Delta\mathbf{q} - \Delta\lambda\mathbf{P}\right)$$

where

$$\Delta\lambda^{i+1} = \Delta\lambda^{i} - \left.\frac{f}{\frac{\partial f}{\partial \Delta\lambda}}\right|_{\Delta\lambda^{i}}$$

where

$$\frac{\partial f}{\partial \Delta\lambda} = -\left(\mathbf{Y}\mathbf{Q}(t_{k+1}) + \mathbf{P}\right)\left(\mathbf{K}^{-1} + \Delta\lambda\mathbf{Y}\right)^{-1}\left(\mathbf{Y}\mathbf{Q}(t_{k+1}) + \mathbf{P}\right)$$



Return mapping integrations

Solve the nonlinear algebraic equation

$$f(\mathbf{Q}(t_{k+1})) = f\left(\mathbf{Q}(t_k) + \left(\mathbf{K}^{-1} + \Delta\lambda\mathbf{Y}\right)^{-1}\left(\mathbf{K}^{-1}\mathbf{Q}(t_k) + \Delta\mathbf{q} - \Delta\lambda\mathbf{P}\right)\right) = 0$$

J.C.J. Schellekens, R. de Borst. The use of the Hoffman yield criterion in finite element analysis of anisotropic composites. Computers & Structures. Volume 37, Issue 6, 1990, Pages 1087-1096

Hand-in

Please calculate $\mathbf{Q}(t_1)$ and $\mathbf{Q}(t_2)$, where $t_{i+1} = t_i + \Delta t, i = 0,1,$ by the return-mapping integration

where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^6$,

$$\mathbf{K} = \begin{bmatrix} 26923 & 11538 & 11538 & 0 & 0 & 0 \\ 11538 & 26923 & 11538 & 0 & 0 & 0 \\ 11538 & 11538 & 26923 & 0 & 0 & 0 \\ 0 & 0 & 0 & 76923 & 0 & 0 \\ 0 & 0 & 0 & 0 & 76923 & 0 \\ 0 & 0 & 0 & 0 & 0 & 76923 \end{bmatrix},$$

$$\mathbf{Y} = \begin{bmatrix} 1.3462 & 0.5769 & 0.5769 & 0 & 0 & 0 \\ 0.5769 & 1.3462 & 0.5769 & 0 & 0 & 0 \\ 0.5769 & 0.5769 & 1.3462 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3846 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3846 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3846 \end{bmatrix}, \mathbf{Q}(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \dot{\mathbf{q}} = \begin{bmatrix} 0.00001 \\ 0.00001 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Coding

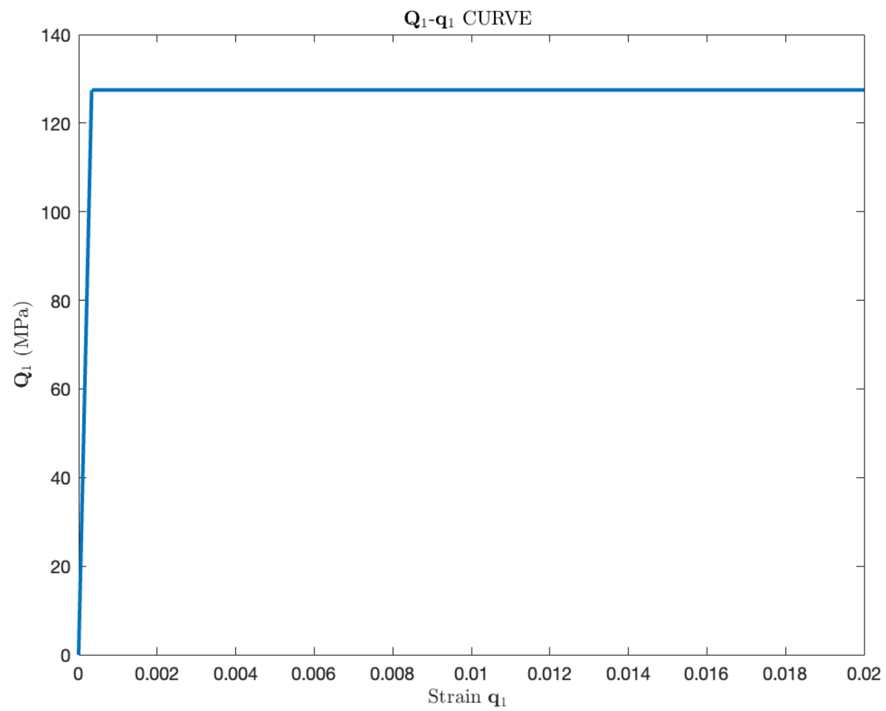
Please code plastic module to compute the $\mathbf{Q}(t_i)$ response by the closest-point projection approach where $\mathbf{Q}, \mathbf{q} \in \mathbb{R}^6$,

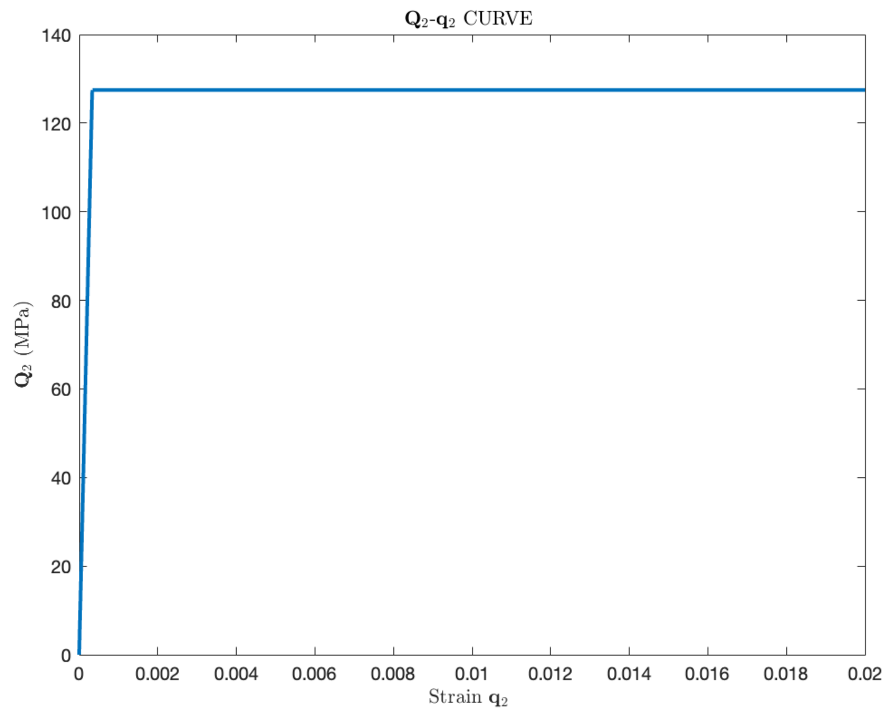
$$\mathbf{K} = \begin{bmatrix} 26923 & 11538 & 11538 & 0 & 0 & 0 \\ 11538 & 26923 & 11538 & 0 & 0 & 0 \\ 11538 & 11538 & 26923 & 0 & 0 & 0 \\ 0 & 0 & 0 & 76923 & 0 & 0 \\ 0 & 0 & 0 & 0 & 76923 & 0 \\ 0 & 0 & 0 & 0 & 0 & 76923 \end{bmatrix},$$

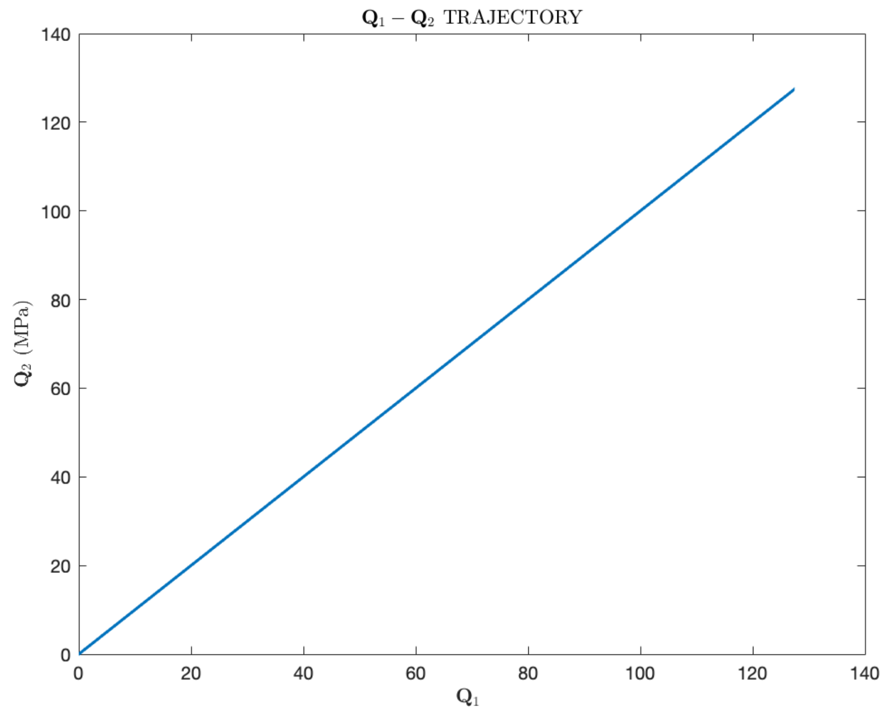
$$\mathbf{Y} = \begin{bmatrix} 1.3462 & 0.5769 & 0.5769 & 0 & 0 & 0 \\ 0.5769 & 1.3462 & 0.5769 & 0 & 0 & 0 \\ 0.5769 & 0.5769 & 1.3462 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3846 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3846 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3846 \end{bmatrix}, \mathbf{Q}(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \dot{\mathbf{q}} = \begin{bmatrix} 0.00001 \\ 0.00001 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

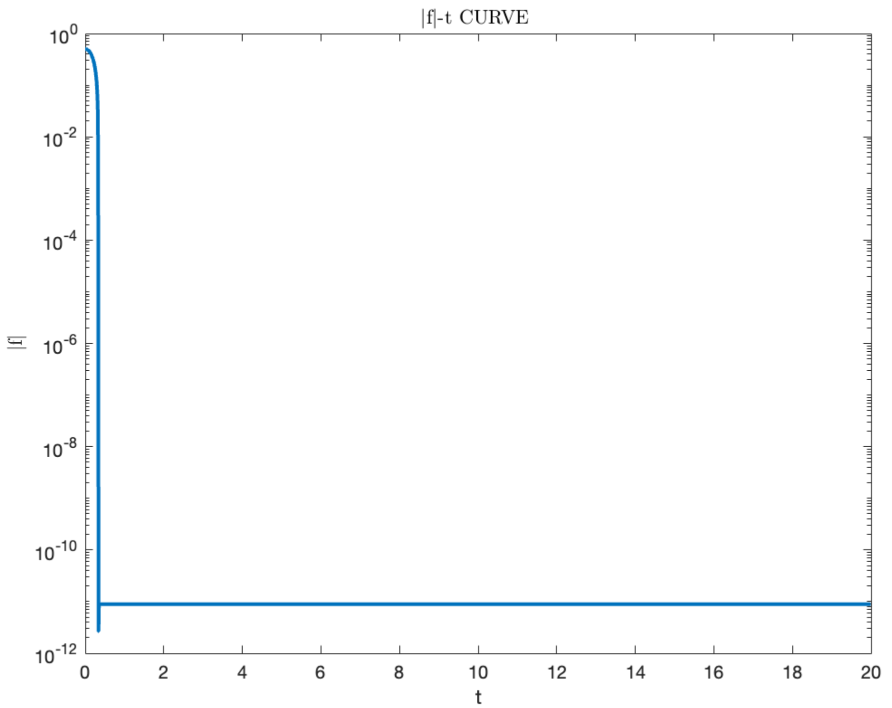
ACMT FISV

Computational results

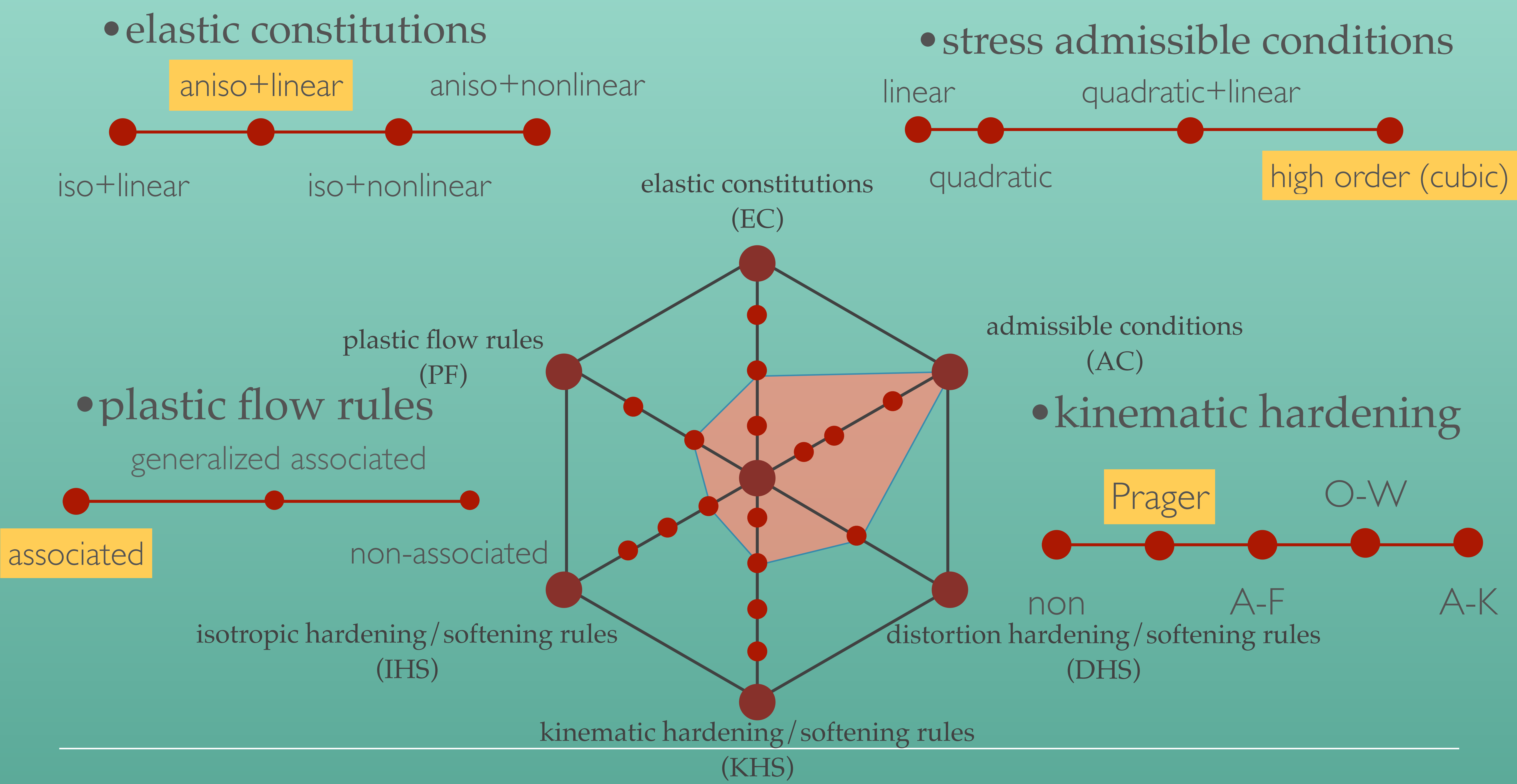








Models with Directional Distortional hardening



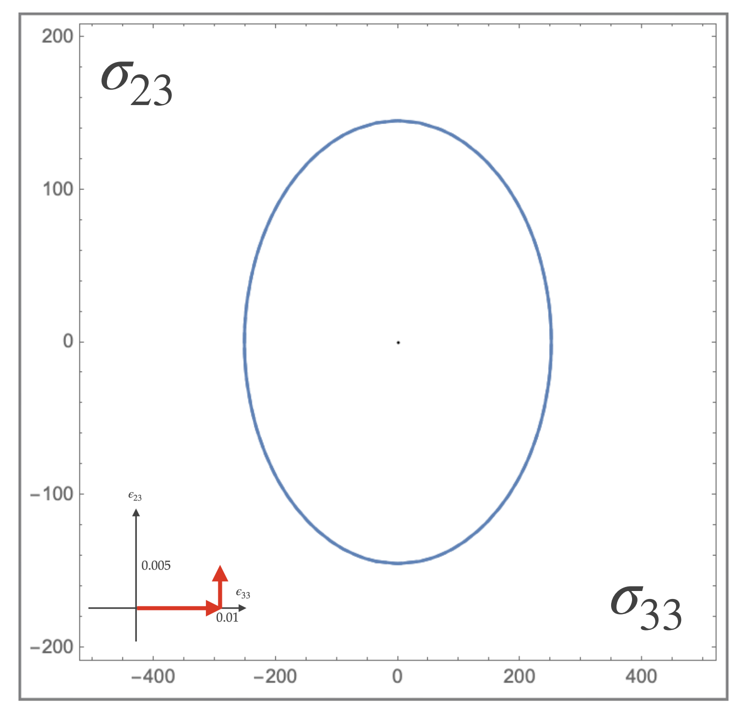
Model of directional distortional hardening

The model of Distortional Hardening elatoplasticity

$$\mathbf{q} = \mathbf{q}^e + \mathbf{q}^p,$$
$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b,$$
$$\mathbf{Q} = \mathbf{K}_e \mathbf{q}^e,$$
$$\dot{\mathbf{Q}}_b = \dot{\lambda} \left\| \frac{\partial f}{\partial \mathbf{Q}} \right\| a_1(\mathbf{n} - a_2 \mathbf{Q}_b),$$
$$\dot{\mathbf{q}}^p = \dot{\lambda} \frac{\partial f}{\partial \mathbf{Q}},$$
$$f\dot{\lambda} = 0,$$
$$f = \frac{3}{2}\left(1 - c\frac{\mathbf{Q}_b^{\mathbf{T}}\mathbf{P}_d(\mathbf{Q}-\mathbf{Q}_b)}{\sqrt{(\mathbf{Q}-\mathbf{Q}_b)^{\mathbf{T}}\mathbf{P}_d(\mathbf{Q}-\mathbf{Q}_b)}}\right)(\mathbf{Q}-\mathbf{Q}_b)^{\mathbf{T}}\mathbf{P}_d(\mathbf{Q}-\mathbf{Q}_b) - \tau_y^2,$$
$$\dot{\lambda} \geq 0,$$
$$\dot{\tau}_y = \frac{1}{2}\lambda k_1(1 - k_2\tau_y)$$

On-Off switch

$$\dot{\lambda} = \begin{cases} \dfrac{\frac{\partial f}{\partial \mathbf{Q}}:\mathbf{K}\dot{\mathbf{q}}}{\mathbf{K}_p + \frac{\partial f}{\partial \mathbf{Q}}:\mathbf{K}\frac{\partial f}{\partial \mathbf{Q}}} & \text{if } f = 0 \text{ and } \dfrac{\partial f}{\partial \mathbf{Q}} : \mathbf{K}\dot{\mathbf{q}} > 0 \quad \text{on-phase} \\ 0 & \text{if } f < 0 \text{ or } \dfrac{\partial f}{\partial \mathbf{Q}} : \mathbf{K}\dot{\mathbf{q}} \leq 0 \quad \text{off-phase} \end{cases}$$



René Marek ; Jiří Plešek2; Zbyněk Hrubý ; Slavomír Parma; Heidi P. Feigenbaum; and Yannis F. Dafalias, M.ASCE Numerical Implementation of A Model With Directional Distortional Hardening, American Society of Civil Engineers,Volume 141, Issue 12, 2015

Model of directional distortional hardening

Return mapping method

Predict

Calculate trial stress

$$\mathbf{Q}^{trial} = \mathbf{K}\Delta\mathbf{q}$$

Update hardening and distortional parameters

$$\mathbf{Q}_b(t+1) = \mathbf{Q}_b(t) + \dot{\lambda} \left\| \frac{\partial f}{\partial Q} \right\| a_1(\mathbf{n} - a_2\mathbf{Q}_b(t))$$

$$\tau_y(t+1) = \tau_y(t) + \frac{1}{2}\lambda k_1(1 - k_2\tau_y(t))$$

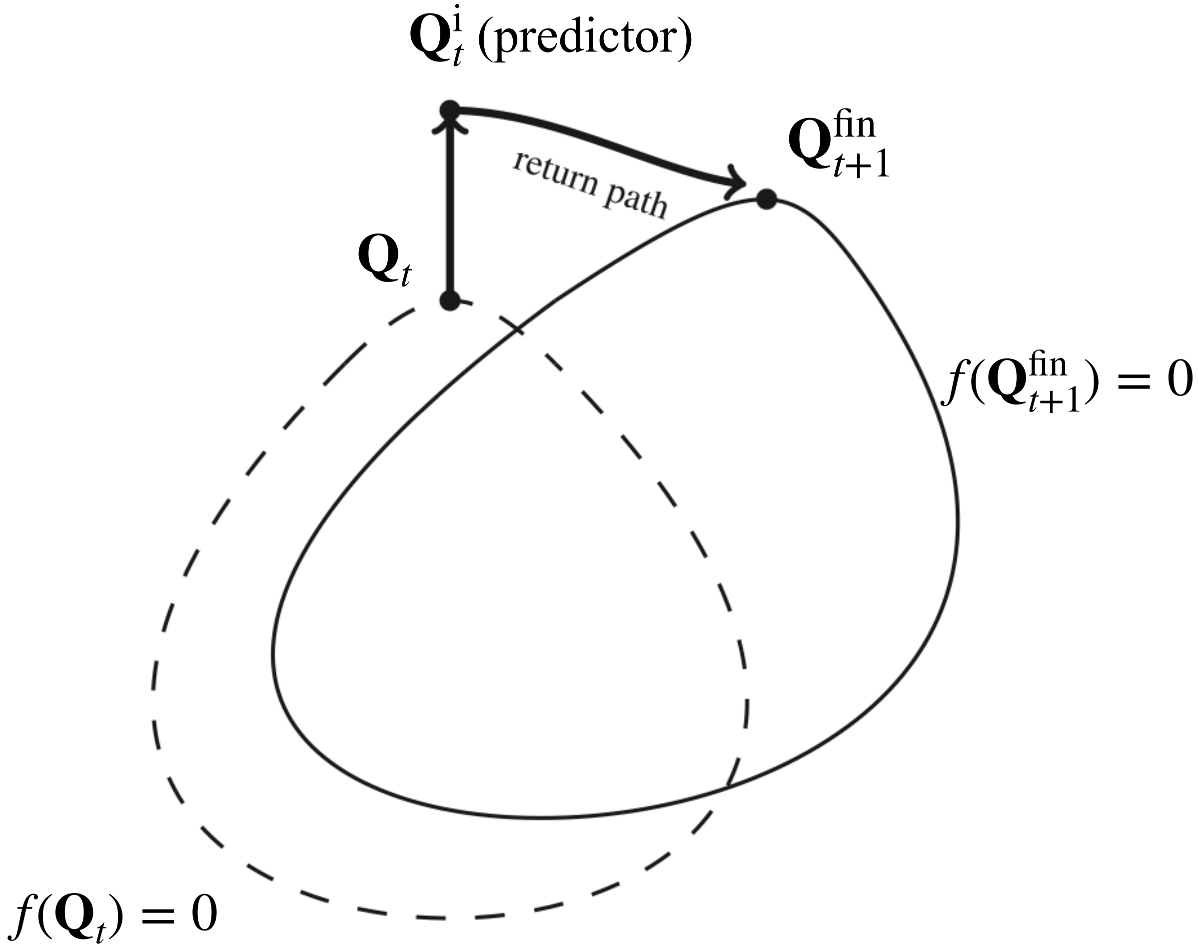
Correct

Use return mapping method to calculate

$$\mathbf{Q}^{\mathrm{fin}}_{t+1} = \mathbf{Q}^{\mathrm{i}}_t - f \cdot \mathbf{n}(\mathbf{n} : \frac{\partial f}{\partial \mathbf{Q}})^{-1}$$

Until

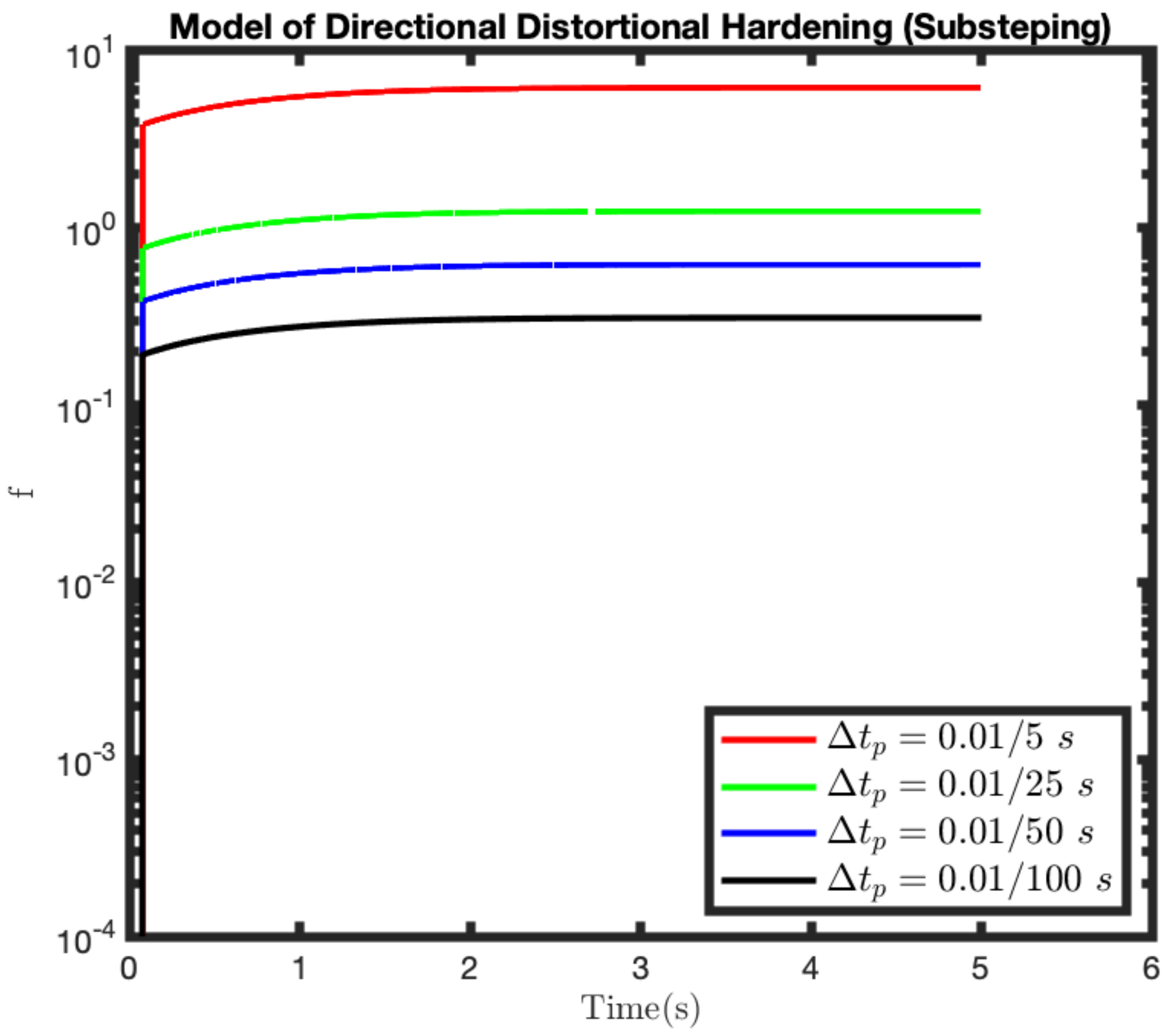
$$f(\mathbf{Q}^{\mathrm{fin}}_{t+1}) = 0$$



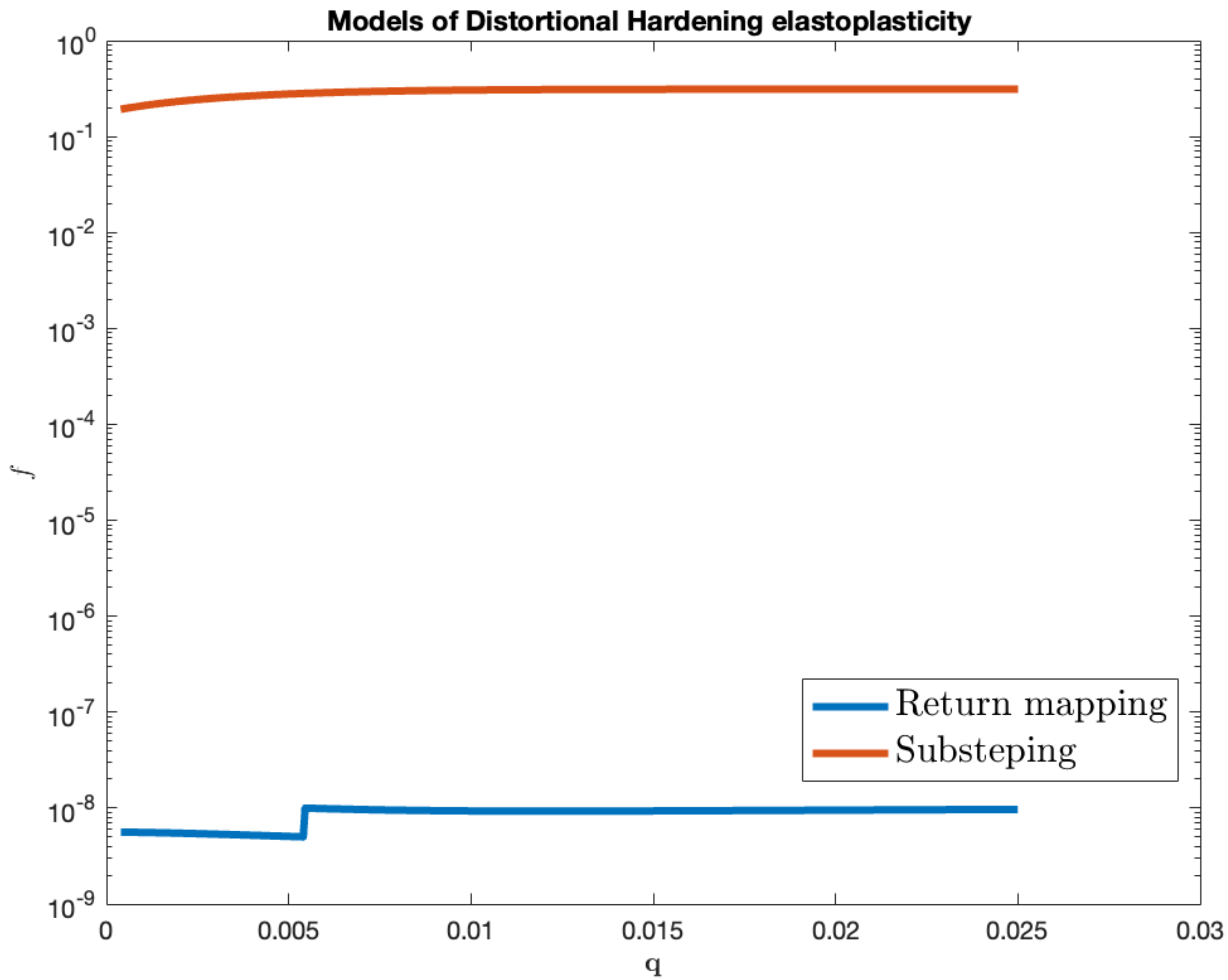
Return mapping integrations

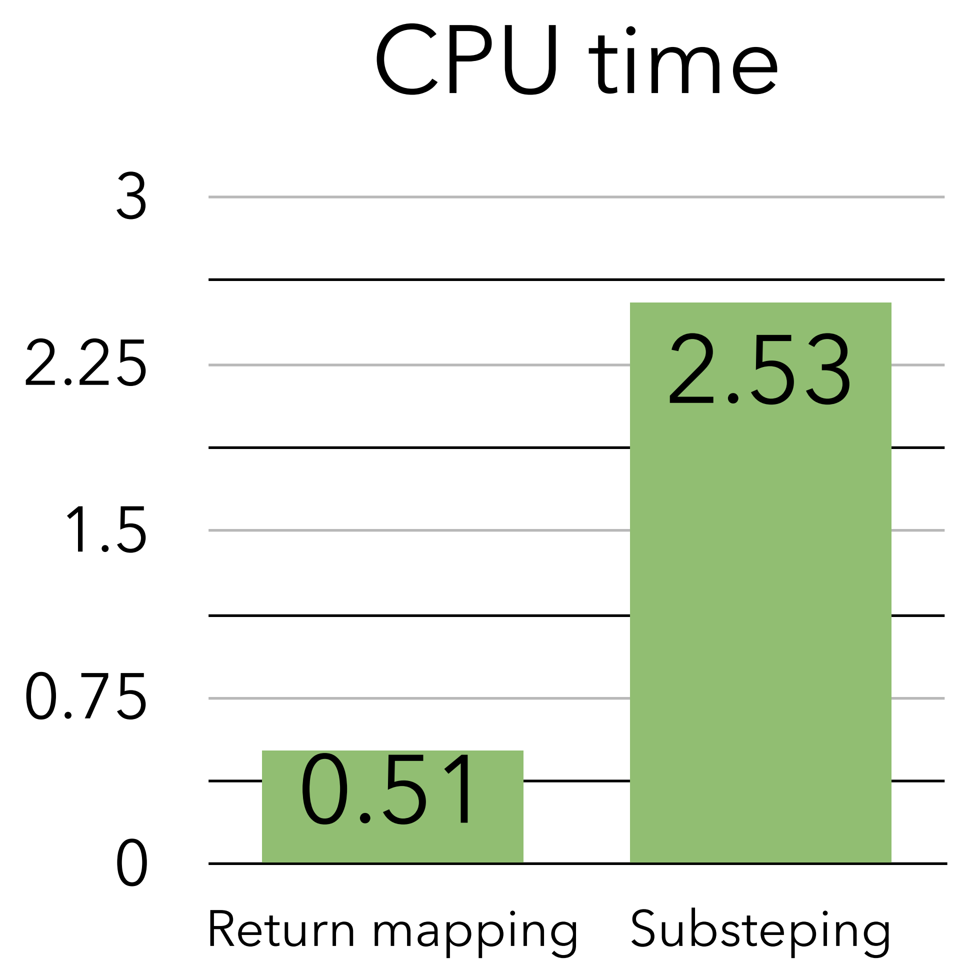
René Marek ; Jiří Plešek2; Zbyněk Hrubý ; Slavomír Parma; Heidi P. Feigenbaum; and Yannis F. Dafalias, M.ASCE Numerical Implementation of A Model With Directional Distortional Hardening, American Society of Civil Engineers,Volume 141, Issue 12, 2015

Model of directional distortional hardening



Model of Directional Distortional Hardening





Numerical integrations for DAEs

Differential-algebraic equations

$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{y}),$$

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{0}\,.$$

- The backward differentiation formulae (BDF)
- The implicit Runge-Kutta methods
- The modified extended backward differentiation formulae (MEBDF)
- The Padé approximation
- The pseudospectral method
- The Adomian decomposition method
- The variational iterative method
- The exponential integration
- The Lie-group differential algebraic equations (LGDAE) method

Thanks for your attention