Chapter 1: The Geometry of Euclidian Space

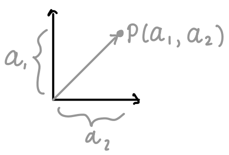
Section 1.1: Points and vectors in 2D and 3D

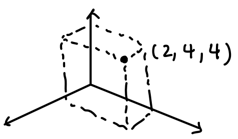
n-dimensional space

For $n \geq 1$, $\mathbb{R}^n$ is the set of all ordered lists of n real numbers (order matters)

e.g. $\binom{-1}{4} \in \mathbb{R}^2$ $\quad \begin{pmatrix} 5 \\ 0 \\ -1 \\ \pi \end{pmatrix} = \mathbb{R}^4$

We can represent points (i.e. locations) in n-dimensional space using elements of $\mathbb{R}^n$.





(a_1, a_2) and $(2,4,4)$ are the cartesian coordinates of P and Q.

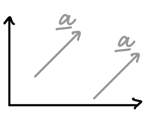
- a_1 and a_2 are called "coordinates" of P.
- Cartisian refers to the fact that we fixed the origin and the mutually perpendicular axis going through the origin. Points are described relative to the origin.

vectors

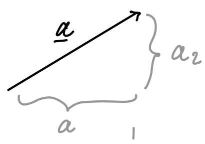
A vector is a physical quantity that has magnitude and direction.

- e.g. displacement (direction, distance), velocity (direction, magnitude), force (direction, strength).
- a vector is "n-dimensional" if it is situated in n-dimensional space.
- write vectors as $\underline{a}$, $\vec{a}$, or $\boldsymbol{a}$ (boldface).

We can use arrows in n-dimensional space to represent n-dimensional vectors.



These represent two of the same vectors.

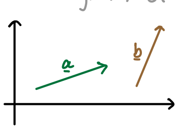


$\underline{a} = \langle a_1, a_2 \rangle$ $\quad a_1$ and a_2 are "components" of the vector $\underline{a}$.

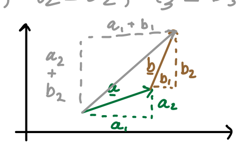
vector addition and subtraction

We add and subtract vectors by adding and subtracting each component.

e.g. in 3D: $\langle a_1, a_2, a_3 \rangle \pm \langle b_1, b_2, b_3 \rangle = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$

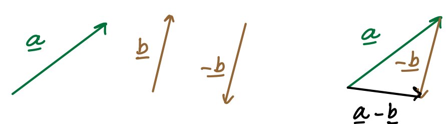


To add visually, put the tail of $\underline{b}$ with the head of $\underline{a}$:



To subtract vectors geometrically, picture $\underline{a} - \underline{b}$ as $\underline{a} + (-\underline{b})$

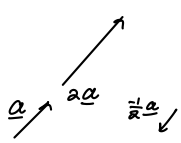
A negative vector = opposite direction



scalar multiplication

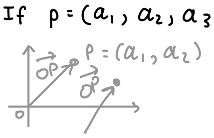
Multiplying a vector by a scalar corresponds to scaling:

If $\alpha \in \mathbb{R}$, $\alpha \langle a_1, a_2, a_3 \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle$



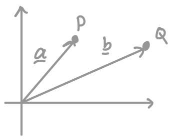
Important examples.

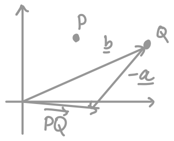
i) If $p = (a_1, a_2, a_3)$ is a point, its position vector is $\langle a_1, a_2, a_3 \rangle$. can be denoted as $\overrightarrow{OP}$



regardless of location, they are the same vector.

ii) If P and Q are two points with position vectors $\underline{a}$ and $\underline{b}$, then $\overrightarrow{PQ} = \underline{b} - \underline{a} = \overrightarrow{OQ} - \overrightarrow{OP}$



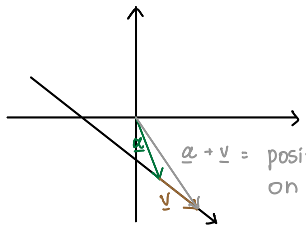


iii) we write
$\underline{i} = \langle 1,0,0 \rangle$
$\underline{j} = \langle 0,1,0 \rangle$
$\underline{k} = \langle 0,0,1 \rangle$
These are the "basis vectors" in $\mathbb{R}^3$

If $\underline{a} = \langle a_1, a_2, a_3 \rangle$, then $\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$

Lines

Q: can we get a formula that generates the position vector of all points on a line given a point which the line passes and a vector parallel to the line.



$\underline{a} + \underline{v}$ = position vector of another point on the line

If you vary the coefficient of $\underline{v}$, you get a different position vector on the line.

A: If the line passes through point P with position vector $\underline{a}$, and if the line is parallel to $\underline{v}$, then as t varies in $\mathbb{R}$, $\underline{a} + t\underline{v}$ generates the position vectors of all points on the line. We write: $\underline{L}(t) = \underline{a} + t\underline{v} \quad (t \in \mathbb{R})$

This is the parametric equation for the line. ← parameter

Example: find the equation of the line through $P = (3,-1,2)$ in direction $\underline{v} = \langle 2,-3,4 \rangle$.

Answer: $\underline{a} = \langle 3,-1,2 \rangle$, hence

$\underline{L}(t) = \langle 3,-1,2 \rangle - t\langle 2,-3,4 \rangle$

$= \langle 3-2t, -1+3t, 2-4t \rangle$

Q: can we do the same if given 2 distinct points on the line?

A: Yes. If P and Q are such points, then vector $\overrightarrow{PQ}$ is parallel to the line, so $\underline{v} = \overrightarrow{PQ}$ in the above. More precisely, if $\underline{a}$ is the position vector of P and $\underline{b}$ is the position vector of Q, then $\underline{v} = \overrightarrow{PQ}$ hence,

$\underline{L}(t) = \underline{a} + t\underline{v} = \underline{a} + t(\underline{b} - \underline{a}) = \underline{a} + t\underline{b} - t\underline{a}$

$= (1-t)\underline{a} + \underline{b}t \quad (t \in \mathbb{R})$

Note: Rather than giving the sign of a line parametrically, we can eliminate the parameter and give the "coordinate" equation.

If $\underline{L}(t) = \underline{a} + t\underline{v}$ then the line is the set of points (x,y,z) such that

$x = a_1 + t v_1 \qquad y = a_2 + t v_2 \qquad z = a_3 + t v_3$

If $v_1, v_2, v_3 \neq \varnothing$, then we can eliminate t as follows:

$x = a_1 + t v_1$, $x - a_1 = t v_1$, $t = \dfrac{x-a_1}{v}$

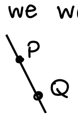
$y = a_2 + t v_2$, $t = \dfrac{y-a_2}{v_2}$

$z = a_3 + t v_3$, $t = \dfrac{z-a_3}{v_3}$

$\therefore \dfrac{x-a_1}{v_1} = \dfrac{y-a_2}{v_2} = \dfrac{z-a_3}{v_3}$

Line segments

If we were to consider t in a finite interval, we'd have a parameterization of a line segment.



Take $t=0$ in $\underline{L}(t) = (1-t)\underline{a} + \underline{b}t \quad (t \in \mathbb{R})$

$\underline{L}(0) = (1-0)\underline{a} + \underline{b}(0) = \underline{a}$

Take $t=1$

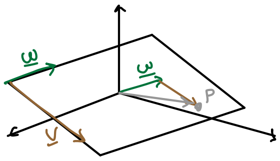
$\underline{L}(1) = (1-1)\underline{a} + \underline{b}(1) = \underline{b}$

So, when we restrict to $t \in [0,1]$, we obtain a parameterization of a line segment between P and Q.

If e.g., $v_1 = 0$, then we can eliminate t as follows: $x = a_1 + t v_1$, $x = a_1 + (0) v_1 = a_1$ $\quad \dfrac{x-a_1}{v_1} = \dfrac{x-x}{v_1} = 0 \quad \therefore \dfrac{y-a_2}{v_2} = \dfrac{z-a_3}{v_3}$

Planes

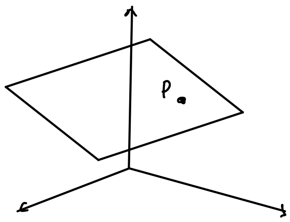
Planes through the origin are determined by two vectors $\underline{v}$, $\underline{w}$ parallel to plane. (Assume $\underline{v}$ is not parallel to $\underline{w}$)



Position vectors of all points in such a plane are of the form $s\underline{v} + s\underline{w}$ $(s, t \in \mathbb{R})$

We say "$\underline{v}$, $\underline{w}$ span the plane", and we write $\underline{\Sigma}(s,t) = s\underline{v} + t\underline{w}$ $(s,t \in \mathbb{R})$

as a parameterization of the plane (spanned by $\underline{v}$, $\underline{w}$).



If the point passes through point P with the position vector $\underline{a}$ and is parallel to $\underline{v}$ and $\underline{w}$, then

$$\underline{\Sigma}(s,t) = \underline{a} + s\underline{v} + t\underline{w} \quad (s,t \in \mathbb{R})$$

Example: find the equation of a plane through points $P(0,0,1)$, $Q(0,1,0)$, and $R(3,2,5)$

vectors $\vec{PQ}$ and $\vec{PR}$ are parallel to the plane.

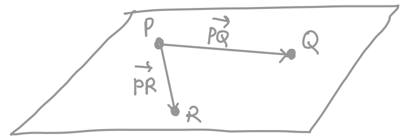
we can take $\underline{v} = \vec{PQ}$ and $\underline{w} = \vec{PR}$

$\underline{v} = \vec{PQ} = \langle 0,1,0\rangle - \langle 0,0,1\rangle = \langle 0,1,-1\rangle$

$\underline{w} = \vec{PR} = \langle 3,2,5\rangle - \langle 0,0,1\rangle = \langle 3,2,4\rangle$

$\underline{a}$ = position vector of $P = \langle 0,0,1\rangle$

we have $\Sigma(s,t) = \langle 0,0,1\rangle + s\langle 0,1,-1\rangle + t\langle 3,2,4\rangle$



Section 1.2: Inner Product, Length, and Distance

Inner Product

If $\underline{a} = \langle a_1, a_2, a_3\rangle$ $(= a_1\underline{i} + a_2\underline{j} + a_3\underline{k})$

$\underline{b} = \langle b_1, b_2, b_3\rangle$ $(= b_1\underline{i} + b_2\underline{j} + b_3\underline{k})$

We define the inner product (also called "dot product") to be the number

$\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$ can also be written as $\langle \underline{a}, \underline{b}\rangle$

e.g.: $\begin{pmatrix}1\\0\\4\end{pmatrix}\cdot\begin{pmatrix}\pi\\6\\5\end{pmatrix} = (1\times\pi) + (0\times 6) + (4\times 5) = \pi \cdot 20$

Algebraic Properties. Show why

i) $\underline{a}\cdot\underline{a} \geq 0$, and $\underline{a}\cdot\underline{a} = 0 \overset{\text{if and only if}}{\iff} \underline{a} = 0$

↳ hen you multiply a vector by itself, it will always be greater than or equal to zero.

ii) $(\alpha\underline{a})\cdot\underline{b} = \underline{a}\cdot(\alpha\underline{b}) = \alpha(\underline{a}\cdot\underline{b})$

iii) $\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{c}$ The inner product is distributive over addition

$(\underline{a}+\underline{b})\cdot\underline{c} = \underline{a}\cdot\underline{c} + \underline{b}\cdot\underline{c}$

iv) $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{c}$

Magnitude of Vectors

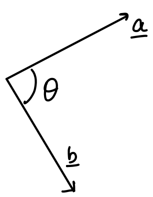
Geometric Properties: recall that the length of a vector $\underline{a} = \langle a_1, a_2, a_3\rangle$ is $\|\underline{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\underline{a}\cdot\underline{a}}$

so, the inner product can be used to find the length of a vector (pythagores)

More generally, if we want to compute the distance from P to Q, with position vectors $\underline{a}$, $\underline{b}$. This is the same as computing the magnitude of $\vec{PQ}$, i.e.

$$\|\vec{PQ}\| = \|\underline{b}-\underline{a}\| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2} = \sqrt{(\underline{b}-\underline{a})\cdot(\underline{b}-\underline{a})}$$

Angles between vectors



Real power of the inner product is computing angles between vectors:

Theorem: If $\underline{a}$, $\underline{b}$ are vectors in $\mathbb{R}^n$ and $0 \leq \theta \leq \pi$ is the angle between $\underline{a}$, $\underline{b}$,

then $\underline{a}\cdot\underline{b} = \|\underline{a}\|\,\|\underline{b}\|\cos\theta$ (*)

Example 1: Angle between $\underline{a} = \underline{b} = \underline{i} = \langle 1,0,0\rangle$, so (*) says that $1 = 1\cos\theta$, i.e., $\cos\theta = 1$ i.e. $\theta = 0$

Example 2: Angle between $\underline{a} = \underline{i}$ and $\underline{b} = -3\underline{i}$, so (*) says that $-3 = 3\cos(\theta)$ i.e. $\cos\theta = -1$ i.e. $\theta = \pi$

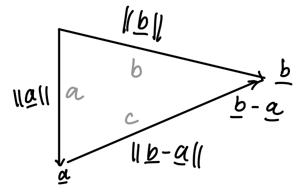
Example 3: Angle between $\underline{a} = \underline{i} + \underline{j} + \underline{k}$ and $\underline{b} = \underline{i} + \underline{j} + \underline{k}$

$\underline{a}\cdot\underline{b} = 1 + 1 - 1 = 1$

$1 = \sqrt{3}\sqrt{3}\cos\theta$ i.e. $\frac{1}{3} = \cos\theta$ i.e. $\theta = \cos^{-1}(\frac{1}{3}) \approx 71°$

$(c^2 = a^2 + b^2 - 2ab\cos\theta)$

Proof:



cosine law: $\|\underline{b}-\underline{a}\|^2 = \|\underline{a}\|^2 + \|\underline{b}\|^2 - 2\|\underline{a}\|\|\underline{b}\|\cos\theta$

$(\underline{b}-\underline{a})\cdot(\underline{b}-\underline{a}) =$

$\underline{a}\cdot\underline{a} - 2\underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{b} =$

$\cancel{\|\underline{a}\|^2} - \cancel{2}\underline{a}\cdot\underline{b} + \cancel{\|\underline{b}\|^2} = \cancel{\|\underline{a}\|^2} + \cancel{\|\underline{b}\|^2} - \cancel{2}\|\underline{a}\|\|\underline{b}\|\cos\theta$

$\underline{a}\cdot\underline{b} = \|\underline{a}\|\|\underline{b}\|\cos\theta$

(*) tells us that if $\underline{a}, \underline{b} \neq 0$, then $\underline{a}\cdot\underline{b} = 0 \iff \cos\theta = 0 \iff \theta = \pi/2$ (because $0 \leq \theta \leq \pi$)

Hence, the inner product provides a test for whether 2 vectors are orthogonal (aka perpendicular).

Consequences of (*)

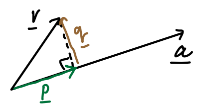
i) Cauchy-Schwarz inequality

$|\underline{a}\cdot\underline{b}| \leq \|\underline{a}\|\|\underline{b}\|$ with equality $\iff \underline{a}, \underline{b}$ parallel or one/both is $\underline{0}$.

ii) Triangle inequality

$\|\underline{a}+\underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|$

Projection



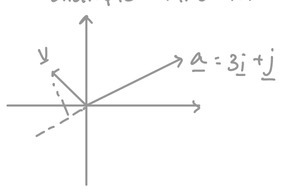
The orthogonal projection of $\underline{v}$ on $\underline{a}$ is "the amount of $\underline{v}$ that is in the direction of $\underline{a}$"

$\underline{v} = \underline{p} + \underline{q} = c\underline{a} + \underline{q} \Rightarrow \underline{v}\cdot\underline{a} = (c\underline{a} + \underline{q})\cdot\underline{a} = c\|\underline{a}\|^2 + \underbrace{\underline{a}\cdot\underline{q}}_{=\emptyset} \Rightarrow c = \frac{\underline{v}\cdot\underline{a}}{\|\underline{a}\|^2}$

Definition: The orthogonal projection of $\underline{v}$ onto $\underline{a}$ is $\underline{p} = \frac{\underline{v}\cdot\underline{a}}{\|\underline{a}\|^2}\underline{a}$

Note: The projection of $\underline{v}$ onto $\underline{a}$ is the same as the projection of $\underline{v}$ onto $\alpha\underline{a}$ for any $\alpha \neq 0$. $\underline{v}$ is not limited by $\underline{a}$.

Example: Find the orthogonal projection $\underline{v} = -\underline{i} + \underline{j}$ onto $\underline{a} = 3\underline{i} + \underline{j}$



$\underline{p} = \frac{\underline{a}\cdot\underline{v}}{\|\underline{a}\|^2}\underline{a} = \frac{-3+1}{10}\underline{a} = \frac{-1}{5}\underline{a} = \langle \frac{-3}{5}, \frac{-1}{5}\rangle$

Definition: The scalar projection of $\underline{v}$ onto $\underline{a}$ is the magnitude (aka length, aka norm) of the orthogonal projection.

i.e. $\|\underline{p}\| = \left\|\frac{\underline{v}\cdot\underline{a}}{\|\underline{a}\|^2}\underline{a}\right\| = \frac{|\underline{v}\cdot\underline{a}|}{\|\underline{a}\|^2}\|\underline{a}\| = \frac{|\underline{v}\cdot\underline{a}|}{\|\underline{a}\|} = \frac{\|\underline{v}\|\cancel{\|\underline{a}\|}|\cos\theta|}{\cancel{\|\underline{a}\|}} = \|\underline{v}\||\cos\theta|$

Section 1.3: Matrices, Determinants, and Cross Products

Matrices

Definition: A $m \times n$ matrix is an m by n array of real numbers.

$$\begin{array}{c} \text{\# row} \\ \downarrow \\ m \end{array} \quad \text{by} \quad \begin{array}{c} n \\ \uparrow \\ \text{\# columns} \end{array}$$

e.g.: a 2x2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ 3x3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

If $m = n$, we call these square matrices.

Note: you should think of an $m \times n$ matrix as being a linear map or function that takes n dimensional vectors and returns m-dimensional vectors.

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} (2\times a) + (1\times b) + (-3\times c) \\ (1\times a) + (5\times b) + (0\times c) \end{pmatrix} = \begin{pmatrix} 2a + b - 3c \\ a + 5b \end{pmatrix}$$

2×3 ← input

output

Multiplying Matrices

If A is a $m \times n$ matrix and B is a $p \times q$, then the product AB is well-defined $\Leftrightarrow n = p$.

To obtain the $[i,j]$ entry of AB, take the i'th row of A and take inner product with the j'th column of B.

e.g.: $\underbrace{\begin{pmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} 0 & 3 \\ -7 & 1 \\ 2 & 2 \end{pmatrix}}_{3 \times 2} = \begin{pmatrix} (2\times 0)+(1\times -7)+(-3\times 2) & (2\times 3)+(1\times 1)+(-3\times 2) \\ (1\times 0)+(5\times -7)+(0\times 2) & (1\times 3)+(5\times 1)+(0\times 2) \end{pmatrix} = \begin{pmatrix} -13 & 1 \\ -35 & 8 \end{pmatrix}$

Note: The previous example is a special case of matrix multiplication if we view a vector as an $n \times 1$ matrix.

Note: if we are viewing matrices as functions taking vectors to vectors then it makes sense to add, subtract, multiply by scalars. This is all done componentwise.

e.g: $\begin{pmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 & -100 \\ 20 & -10 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & -103 \\ 21 & -5 & 0 \end{pmatrix}$

$5\begin{pmatrix} 6 & \pi \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 30 & 5\pi \\ -10 & 5 \end{pmatrix}$

Determinant

We can compute the determinant of any square matrix. The determinant of a 2×2 matrix

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the number $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{23} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{21} & a_{32} \\ a_{31} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{23} \end{vmatrix}$$

E.g: $\begin{vmatrix} 1 & 2 & 0 \\ 5 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 5 & 0 \\ -1 & 1 \end{vmatrix} = 1(-1) - 2(10+1) + 0 = -1 - 22 = -23$

Properties of Determinants:

i) if you interchange 2 rows, the determinant changes signs. $\Leftrightarrow$ if 2 rows are the same, then det $= 0$.

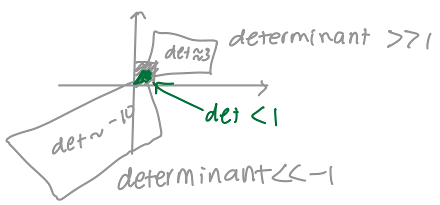
ii) if you multiply a row by a scalar, then the determinant is multiplied by the same scalar.

iii) If you add a multiple of one row to another, then the determinant is unchanged.

Note: you can replace "row" in every statement above with "column".

Intuitively: determinants of an $n\times n$ square matrix measures how much the matrix is considered as a map: $\mathbb{R}^n \to \mathbb{R}^n$, distorts areas/volumes in $\mathbb{R}^n$.

Matrices are a mapping. So if its a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ under this matrix, it might be bigger/smaller, but to what extent?



Cross Product

Defn: The cross product ("vector product") of two 3-D vectors

$\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$

$\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

denoted as $\underline{a}\times\underline{b}$, is the vector defined by

$\underline{a}\times\underline{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\underline{k}$

"=" $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Ex: $(3\underline{i} - 4\underline{k}) \times (\underline{i} - \underline{j} + 2\underline{k})$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & -4 \\ 1 & -1 & 2 \end{vmatrix}$

$= \underline{i}\begin{vmatrix} 0 & -4 \\ -1 & 2 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & -4 \\ 1 & 2 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix}$

$= \underline{i}(0-4) - \underline{j}(6+4) + \underline{k}(-3-1)$

$= -4\underline{i} - 10\underline{j} - 4\underline{k}$

You can check that $\underline{a}\times\underline{b} = -(\underline{b}\times\underline{a})$

This property implies that $\underline{a}\times\underline{a}$ is the zero vector for any vector $\underline{a}$.

The cross product satisfies:

i) $\underline{a}\times(\beta\underline{b}+\gamma\underline{c}) = \beta(\underline{a}\times\underline{b}) + \gamma(\underline{a}\times\underline{c})$

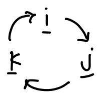
ii) $(\alpha\underline{a}+\beta\underline{b})\times\underline{c} = \alpha(\underline{a}\times\underline{c}) + \beta(\underline{b}\times\underline{c})$

You should also check

$\underline{i}\times\underline{j} = \underline{k}$

$\underline{j}\times\underline{k} = \underline{i}$

$\underline{k}\times\underline{i} = \underline{j}$



Since $\underline{a}\times\underline{b}$ is a vector we can take its inner product with another vector $\underline{c}$ to get $(\underline{a}\times\underline{b})\cdot\underline{c}$.

This is called a "(scalar) triple product."

$$(\underline{a}\times\underline{b})\cdot\underline{c} = \left(\begin{vmatrix} a_2 & a_3 \\ a_2 & b_3\end{vmatrix}\underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix}\underline{j} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}\underline{k}\right)\cdot(c_1\underline{i}+c_2\underline{j}+c_3\underline{k}) = c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3\end{vmatrix} + c_3\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3\end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{vmatrix} \quad (\dagger)$$

The fundamental property of the cross product:

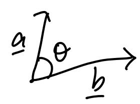
Thus: If $\underline{a}, \underline{b} \neq \underline{0}$ and $\underline{a} \neq \alpha\underline{b}$, then $\underline{a}\times\underline{b}$ is orthogonal to both $\underline{a}, \underline{b}$ (equivalently, it is orthogonal to the plane through the origin spanned by $\underline{a}$ and $\underline{b}$.

Proof: Recall that 2 non-zero vectors are orthogonal $\Leftrightarrow$ their inner product $=0$. Suppose $\underline{c}\neq\underline{0}$ is parallel to the plane through the origin spanned by $\underline{a}$ and $\underline{b}$, i.e., $\underline{c} = \alpha\underline{a}+\beta\underline{b}$ for some α, β not both $=\underline{0}$. Then by $(\dagger)$, $(\underline{a}\times\underline{b})\cdot\underline{c} = 0$ since the 3rd row is a linear combination of the other 2 rows.

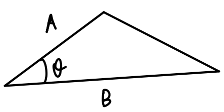
Hence, $\underline{a}\times\underline{b}$ is orthogonal to c, i.e. $\underline{a}\times\underline{b}$ is orthogonal to any vector in the plane through the origin spanned by $\underline{a}, \underline{b}$, and in particular, orthogonal to $\underline{a}, \underline{b}$.

Finally, if $0 \le \theta \le \pi$ is the angle between $\underline{a}\cdot\underline{b}$, then

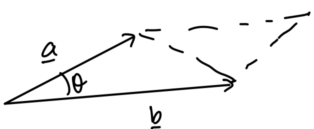
$\|a\times b\| = \|a\|\|b\|\sin\theta$ proof?



Compare formula to $|\underline{a}\cdot\underline{b}| = \|a\|\|b\|\cos\theta$



Area $= \frac{1}{2}AB\sin\theta$



Then area of this triangle is $\frac{1}{2}\|\underline{a}\|\|\underline{b}\|\sin\theta$

$= \frac{1}{2}\|\underline{a}\times\underline{b}\|$

Hence, $\|\underline{a}\times\underline{b}\|$ is the area of the parallelogram spanned by $\underline{a}, \underline{b}$ (when a, b are 3D vectors)

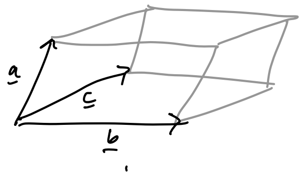
In particular, if $\underline{a}$ and $\underline{b}$ are two vectors in the plane (2D), then, we can consider $\underline{a}, \underline{b}$ as vectors in 3D with zero $\underline{k}$-components. i.e., $\underline{a} = \langle a_1, a_2, 0\rangle$ and $\underline{b} = \langle b_1, b_2, 0\rangle$

Then the area of the parallelogram spanned by $\underline{a}, \underline{b}$

$$= \|\underline{a}\times\underline{b}\| = \left\|\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0\end{vmatrix}\right\| = \left\|\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}\underline{k}\right\| = \left|\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix}\right|$$

From the area of the parallelogram = determinant of its values, you can see what the determinant represents, (area/volume)

Hence, the absolute value of the determinant of a 2x2 matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2\end{pmatrix}$ is equal to the area of the parallelogram spanned by $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle$



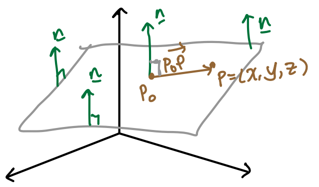
Also: the area of the parallelipiped spanned by vectors $\underline{a}, \underline{b}, \underline{c}$ is the absolute value of the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Plane

Recall that if a plane passes through a point w/ pos. vec. $\underline{a}$ and parallel to the plane spanned by $\underline{v}$, $\underline{w}$, then

$\underline{\Sigma}(s,t) = \underline{a} + s\underline{v} + t\underline{w} \quad (s,t \in \mathbb{R})$

Suppose we're given a point $P_0 = (x_0, y_0, z_0)$ through which the plane passes, and a vector $\underline{n} = A\underline{i} + B\underline{j} + C\underline{k}$ orthogonal to the plane.



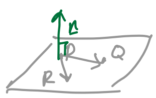
Then a point $P = (x,y,z)$ belongs to the plane $\Leftrightarrow$ $\overrightarrow{P_0P}$ is orthogonal to $\underline{n}$. $\Leftrightarrow$ $\overrightarrow{P_0P} \cdot \underline{n} = 0$

But $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$

Hence $\overrightarrow{P_0P} \cdot \underline{n} = 0 \Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ i.e., $Ax + By + Cz + D = 0$

where $D = -Ax_0 - By_0 - Cz_0$

Example: Find a coordinate of the planes through $P = (1,1,1)$ $Q = (2,0,0)$, $R = (1,1,0)$



well $\overrightarrow{PQ} = \langle 1, -1, -1 \rangle$

$\overrightarrow{PR} = \langle 0, 0, -1 \rangle$

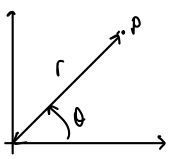
hence, $\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to the plane.

$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = \underline{i} + \underline{j}$ so $A = 1$, $B = 1$, $C = 0$

so the sign of the plane is $1(x-1) + 1(y-1) + 0(z-1) = 0$ i.e., $x + y - 2 = 0$

Note: 2 planes are parallel if their normal vectors are parallel.

Section 1.4: Cylindrical and Spherical Coordinates



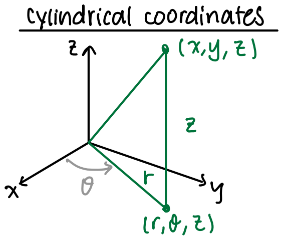
We denote Polar coordinates by (r, θ) where $0 \leq r \leq \infty$ and $0 \leq \theta \leq 2\pi$.

$x = r\cos\theta$

$y = r\sin\theta$

e.g. in 2D, the circle of radius is "$x^2 + y^2 = 1$", but in polar coords, this is "$r = 1$".

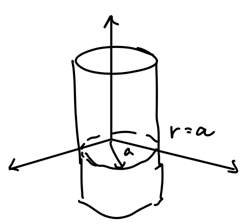
Cylindrical coordinates



To get from cartesian (x,y,z) to cylindrical coordinates, (r, θ, z), we simply convert the first two coordinates x, y into polar coordinates, and we leave the z-variable untouched.

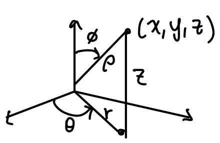
i.e. $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ $\quad r = \sqrt{x^2 + y^2}$

Why "cylindrical"? In 2D polar coords, the curve "$r = a$" is a circle of radius a, but in 3D, "$r = a$" is an infinite cylinder centered on the z-axis.



Spherical Coordinates

(ρ, θ, φ) are derived as follows: $\rho = \sqrt{x^2 + y^2 + z^2}$, θ is as in polar/cylindrical coord $(0 \leq \theta \leq 2\pi)$, and $0 \leq \varphi \leq \pi$ is the angle from the z-axis to the position vector of the point $\underline{v} = x\underline{i} + y\underline{j} + z\underline{k}$



we can see φ is the solution in $0 \leq \varphi \leq \pi$ to $\cos\varphi = \frac{\underline{v} \cdot \underline{k}}{\|\underline{v}\|\|\underline{k}\|}$

From the picture, we see $z = \rho\cos\varphi$, $r = \rho\sin\varphi$

where (r, θ, z) are the cylindrical coords.

But $x = r\cos\theta = \rho\sin\varphi\cos\theta$ and $y = r\sin\theta = \rho\sin\varphi\sin\theta$.

$(0 \leq \rho \leq \infty, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi)$.

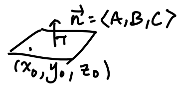
i.e. spherical coords related to cartesian.

Chapter 2: Differentiation

Section 2.1: The geometry of real-valued functions

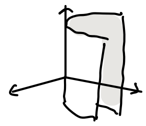
Here, we'll look at surfaces derived by equations that are satisfied by the coordinates (x, y, z) or its points.

i) "The surface $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ $(*)$"
Means "The set of points (x,y,z) satisfying $(*)$



ii) "The surface $x^2+y^2+z^2=1$ $(**)$"
means "the set of points (x,y,z) satisfying $(**)$

iii) The surface $y=x^2$ $(**)$
means the set of points (x,y,z) satisfying $(***)$



is a parabolic cylinder.

It might be hard to picture a surface given just its equation.
However, if our surfaces arises as the graph of a function $\mathbb{R}^2\to\mathbb{R}$, we have the notion of level curves

Multivariable Functions

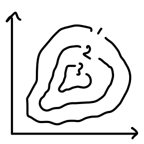
Let $U\subseteq\mathbb{R}^n$ be a domain of f.
Look at functions $f: U\subseteq\mathbb{R}^n\to\mathbb{R}^m$, where $n,m\geq 1$. If $m=1$, we call f scalar-valued, otherwise we call f vector valued. If $n>1$, we call f a function of several variables.

e.g. $f(x)=x^2$, $f:\mathbb{R}\to\mathbb{R}$ (one variable input, one variable output.
$f(x)=(x^2, 5x)$, $f:\mathbb{R}\to\mathbb{R}^2$
$f(x,y)=x^2+y^2$, $f:\mathbb{R}^2\to\mathbb{R}$
$f(x,y)=(x^2,-y^3)$, $f:\mathbb{R}^2\to\mathbb{R}^2$

The graph of function $f: U\subseteq\mathbb{R}^n\to\mathbb{R}$ is obtained by plotting $f(x_1,\cdots,x_n)$ for each $(x_1,\cdots,x_n)\in U$.
e.g. $f:[0,1]\to\mathbb{R}$, $f(x)=x^2$. Then the graph of f is obtained by plotting $f(x)$, i.e. x^2, for each $x\in[0,1]$

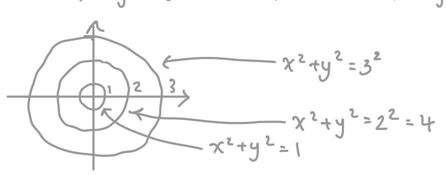
Level Curves

We want to develop an understanding of how to plot graphs of functions $f: U\subseteq\mathbb{R}^2\to\mathbb{R}$.
Keypoint: Build a picture of a graph of $f:\mathbb{R}^2\to\mathbb{R}$ the same way we build a picture of a 3D mountain range using a 2D point with contour lines.

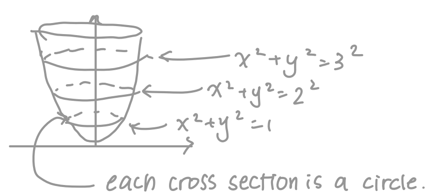


The values correspond with the height of z.

E.g. Consider $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2+y^2$. Curves in the xy plane of constant f correspond to the contour line of the mountain range. (and the mountain range corresponds to the graph of f).



Raise the level curves to the graph



Defn: let $f: U\subseteq\mathbb{R}^2\to\mathbb{R}$ and $c\in\mathbb{R}$. The level curve of value c is the set of points $(x,y)\in U$ s.t. $f(x,y)=c$.

If $f: U\subseteq\mathbb{R}^2\to\mathbb{R}$, then the graph of f is the same as "the surface $z=f(x,y)$" so sometimes we use z in place of $f(x,y)$ and say e.g. "The graph of $z=x^2+y^2$".

e.g, Sketch the graph of $f(x,y) = x^2 - y^2$.

Method: sketch the level curves ("contour lines" corresponding to f = various constants) then assemble the level curves according to the values of these constants (i.e, heights).

What is the level curve of value $\emptyset$? well, $x^2 - y^2 = 0 \Leftrightarrow y = \pm x$, so

Of value 1?: $x^2 - y^2 = 1 \Leftrightarrow y = \pm\sqrt{x^2 - 1}$

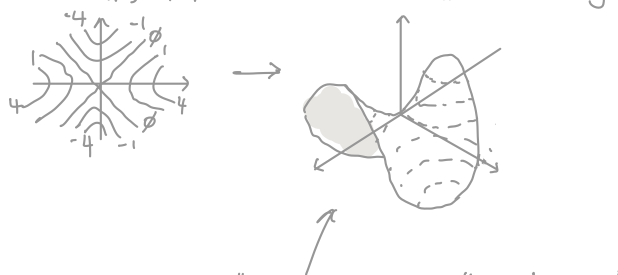
i.e, a hyperbola through the points $(\pm 1, 0)$

of value 4?: $x^2 - y^2 = 4 \Leftrightarrow y = \pm\sqrt{x^2 - 4}$

i.e, a hyperbola through points $(\pm 2, 0)$

of value -1? $x^2 - y^2 = 1 \Leftrightarrow x = \pm\sqrt{y^2 - 1}$

i.e, a hyperbola through $(0, \pm 1)$

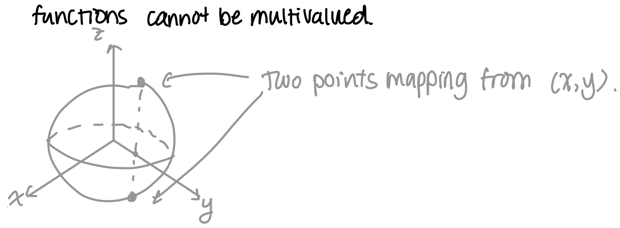


"hyperbolic parabaloid", or "saddle"

Note: we can gain a better understanding of the graph of a function by looking at what happens when we intersect it with constants of x and y (intersecting with planes of constant z are precisely the level curves, after projecting to the xy-plane.

e.g, $f(x,y) = x^2 - y^2$, if we consider the subset of the graph where $x = 0$, i.e, the intersection of the graph with the yz-plane, we get $f(0,y) = -y^2$ (or $z = y^2$)

Note: Not all surfaces arises as the graphs of functions! e.g., the sphere $x^2 + y^2 + z^2 = 1$ cannot be the graph of a function since functions cannot be multivalued.



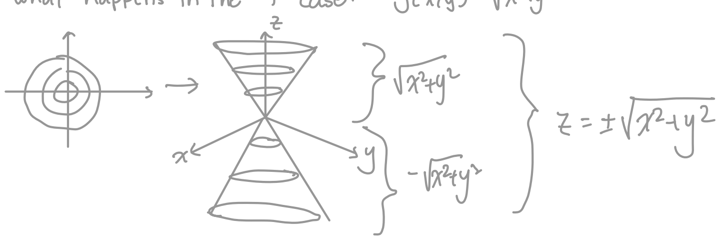
Defn: let $f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$ and $c \in \mathbb{R}$. Then the level surface of value c is the set of points (x,y,z) is the set of points $(x,y,z) \in U$ s.t. $f(x,y,z) = c$.

E.g level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$ of values c^2 are spheres with radii of c.

E.g. Describe the level surfaces of $f(x,y,z) = x^2 + y^2 - z^2$.

$c = 0$: $x^2 + y^2 - z^2 = 0 \Leftrightarrow z = \pm\sqrt{x^2 + y^2}$

what happens in the + case: $g(x,y) = \sqrt{x^2 + y^2}$

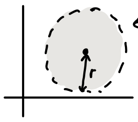


Section 2.2: Limits and Continuity

The continuity and differentiability of multivariable functions. These require the notion of a limit, and limits makes sense in open sets.

Open Set

Defn: The open disc of radius r, centered at $(x_0, y_0) \in \mathbb{R}^2$ is the set of points less than distance r from (x_0, y_0).

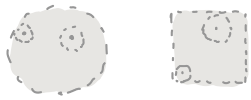


← the dotted line not included in the open disc.

Note: if we include the outer circle, we'd get a "closed disc".

Defn: Let $u \subseteq \mathbb{R}^2$, we call u an "open set" if every point $(x,y) \in u$, there exists an open disc centered at (x,y) which is contained in u.

Ex:



Non-Ex:

← seeps outside the shaded region.

It has to hold for every point in the disc.

Note: the above defns generalize to other dimensions, e.g. in 1D, replace "disc" with "interval", and in 3D, replace "disc" with "ball".

Intuitively: a set is open if it doesn't contain any points on its boundaries.

Boundary

Defn: A "neighborhood" of points $(x,y) \in \mathbb{R}^2$ is any open set containing (x,y).

Ex: any open disc containing (x,y) is a neighborhood of (x,y).

Defn: let $u \subseteq \mathbb{R}^2$. A point $(x,y) \in \mathbb{R}^2$ is a boundary point of u if every neighborhood of (x,y) contains a point in u and a point not inside of u.

Note: a boundary point of u need not belong to u.

Ex: if u = open disc, then all points on its outer circle are boundary points, but none of these points belong to u.

half in circle half out.

If u = closed disc, the outer circle remains as the boundary points, but now these elements are part of the set.

Limits

In what follows, we denote $(x_1, \ldots, x_n) \in \mathbb{R}^n$ by $\underline{x}$ (a vector or point). Assume $n = 1, 2$, or 3.

Let $u \subseteq \mathbb{R}^n$ be open. Let $f: u \subseteq \mathbb{R}^n \to \mathbb{R}$.

Intuitively, "f has the limit $b \in \mathbb{R}$ as $\underline{x}$ tends to $\underline{x}_0$" if the points $f(x)$ gets closer to b as the points $\underline{x}$ get closer to $\underline{x}_0$.

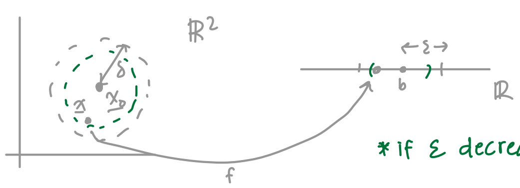
Defn: Suppose that $\underline{x}_0$ is either in u or is a boundary point of u. Let $b \in \mathbb{R}$. We say "f has the limit $b \in \mathbb{R}$ as $\underline{x}$ tends to $\underline{x}_0$" if for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $\underline{x} \in u$ and $0 < \|x - x_0\| < \delta$, then $\|f(x) - b\| < \varepsilon$.

i.e., "f has the limit $b \in \mathbb{R}$ as $\underline{x}$ tends to $\underline{x}_0$" if we can get $f(x)$ as close to b as we like by taking $\underline{x}$ sufficiently close to $\underline{x}_0$."

Note: δ will in general, depend on ε.

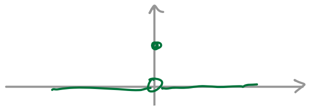
We write $\lim_{x \to x_0} f(x) = b$ or $f(x) \to b$ as $x \to x_0$.

Note: we do not necessarily assume that x_0 belongs to u. This is an important feature, e.g., $g'(x_0) = \lim_{x \to x_0} \underbrace{\frac{g(x) - g(x_0)}{x - x_0}}$ because we must make sure it incorperates certain cases. in undefined at $x = x_0$.



* if ε decreases, then δ decreases.

Ex1: let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



Claim: $\lim_{\underbrace{x \to 0}_{x_0}} f(x) = \underbrace{0}_{b}$

Let $\varepsilon > 0$, we want to show that there exists some $\delta > 0$ s.t. if $0 < |x - \cancelto{0}{x_0}| < \delta$ then $|f(x) - \cancelto{0}{b}| < \varepsilon$.
But if $0 < |x|$, then $x \neq 0$, hence $f(x) = 0$. so $0 < |x| \Rightarrow |f(x)| = 0 < \varepsilon$.
Hence any value of $\delta > 0$ actually works here.

Ex2: let $f: \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}$ be $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Claim: $\lim_{\underbrace{(x,y) \to (0,0)}_{x_0}} f(x,y) = \underbrace{1}_{b}$.

Fix $\varepsilon > 0$, show there exists $\delta > 0$ s.t. if $0 < \|(x,y)\| < \delta$, then $|f(x,y) - 1| < \varepsilon$

By L'Hopital, $\lim_{\alpha \to 0} \dfrac{\sin \alpha}{\alpha} = 1$ (*)

Therefore, there exists $\delta > 0$ s.t. $0 < |\alpha| < \delta \Rightarrow \left| \dfrac{\sin \alpha}{\alpha} - 1 \right| < \varepsilon$

Without loss of generality, $\delta < 1$, thus if $0 < \|(x,y)\| < \delta$, then $0 < \|(x,y)\|^2 = x^2 + y^2 < \delta^2 < \delta$, and hence,
$|f(x,y) - 1| = \left| \dfrac{\sin(x^2+y^2)}{(x^2+y^2)} - 1 \right| < \varepsilon$ ← by taking $\alpha = x^2 + y^2$ in (*)

Ex3: $\lim_{(x,y) \to (0,0)} \dfrac{x^2}{x^2+y^2}$ does not exist.

As we approach $(0,0)$ on the x-axis (where $y=0$), our function takes on the constant value 1.
As we approach $(0,0)$ on the y-axis (where $x=0$), our function takes on the constant value 0.
Hence, there is no limit at $(0,0)$, because limits must necessarily be unique (if they exist)
[If $\lim_{x \to x_0} f(x) = b_1$, and
$\lim_{x \to x_0} f(x) = b_2$, then $b_1 = b_2$]

<u>Properties of Limits</u>

i) $\lim_{\underline{x} \to \underline{x}_0} c f(\underline{x}) = c \lim_{\underline{x} \to \underline{x}_0} f(\underline{x})$

ii) $\lim_{\underline{x} \to \underline{x}_0} (f+g)(\underline{x}) = \lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) + \lim_{\underline{x} \to \underline{x}_0} g(\underline{x})$

iii) If $m = 1$
$\lim_{\underline{x} \to \underline{x}_0} (fg)(\underline{x}) = \left(\lim_{\underline{x} \to \underline{x}_0} f(\underline{x})\right)\left(\lim_{\underline{x} \to \underline{x}_0} g(\underline{x})\right)$

iv) If $m = 1$ and $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) \neq 0$, the
$\lim_{\underline{x} \to \underline{x}_0} \left(\dfrac{1}{f(\underline{x})}\right) = \dfrac{1}{\lim_{\underline{x} \to \underline{x}_0} f(\underline{x})}$

<u>Continuity</u>

Defn: let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\underline{x}_0 \in U$. we say f is continuous at $x_0 \in U$ if $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$
i.e. for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|x - x_0\| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$
we say f is continuous on U if f is continuous at all points $\underline{x}_0 \in U$. We call f discontinuous at $x_0 \in U$ if it is not continuous at $\underline{x}_0$.

<u>Properties</u>

i) f cts (continuous) at $\underline{x}_0 \Rightarrow \alpha f$ cts at x_0 for all $\alpha \in \mathbb{R}$
ii) f, g cts at $\underline{x}_0 \Rightarrow f + g$ cts at $\underline{x}_0$.
iii) f, g cts at $\underline{x}_0 \Rightarrow fg$ cts at $\underline{x}_0$.
iv) f cts at $\underline{x}_0$ and $f(\underline{x}_0) \neq 0 \Rightarrow \dfrac{1}{f}$ cts at $\underline{x}_0$
v) suppose $g: U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $f: W \subseteq \mathbb{R} \to \mathbb{R}$ and $g(U) \subseteq W$ (so $f \cdot g$ is derived on U). If g is cts at $x_0 \subseteq U$ and f is cts at $g(x_0) \subseteq W$, then $f \cdot g: U \subseteq \mathbb{R}^n \to \mathbb{R}$ is cts at x_0.

Ex: let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

then f is cts.

x^2 and y^2 are cts $\overset{i)}{\Rightarrow}$ x^2+y^2 are cts.

sin cts $\Rightarrow$ $\sin(x^2+y^2)$ cts

$(x,y) \neq (0,0) \Rightarrow \frac{1}{x^2+y^2}$. is cts

iii) $\Rightarrow \sin(x^2+y^2)/(x^2+y^2)$ is cts away from $(0,0)$

Finally, because $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} = 1 = f(0,0)$, we are cts at the origin.

Hence, f is cts on the whole of $\mathbb{R}^2$.

Ex2: $f(x,y,z) = \sin(e^{x^2+z}) + (y^2+z)^{19}$

y^2+2 is cts $\overset{iii}{\Rightarrow}$ $(y^2+2)^{19}$ is cts.

x^2+z is cts and e^t is cts $\overset{VI}{\Rightarrow}$ e^{x^2+z} is cts $\overset{v)}{\Rightarrow}$ $\sin(e^{x^2+z})$ is cts

$\overset{i)}{\Rightarrow}$ $\sin(e^{x^2+z}) + (y^2+z)^{19}$ is cts (on every real pt)

Section 2.3: Differentiation

Partial Differentiation

If $f(x,y)$ is a function of 2 variables, then the partial derivative of f with respect to (w.r.t.) x, denoted as $\frac{\partial f}{\partial x}$, is obtained by differentiating f with respect to x while keeping y fixed (treat it as a constant).
The partial derivative w.r.t. y, denoted as $\frac{\partial f}{\partial y}$, is obtained by differentiating f w.r.t. y whilst treating x as fixed.

e.g, $f(x,y) = e^{2x}\sin(y)$

$\frac{\partial f}{\partial x} = 2e^{2x}\sin(y)$ $\quad$ $\frac{\partial f}{\partial y} = e^{2x}\cos(y)$

Defn: let $U \subseteq \mathbb{R}^n$ be open and suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$. Let $(x_1, \dots, x_n)$ denote the variables of our function. Then for each $1 \le i \le n$ the partial derivatives of f wrt x_i (aka the partial derivative of f in the i^{th} direction) is the function

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$ if such a limit exists.

We denote $\frac{\partial f}{\partial x}(x_0, y_0)$ the function of $\frac{\partial f}{\partial x}$ evaluated at the point (x_0, y_0).

e.g. $\frac{\partial f}{\partial x} = 2y\sin(xy)\cos(xy)$

$\frac{\partial x}{\partial x}(1,\pi) = 2\pi\sin(\pi)\cos(\pi)$

Differentiability

It is possible for the partial derivatives of a function to exist at a point, but for the function not to satisfy our intuitive notion of being "differentiable" at that point. (like near its end)

Instead, we define a function $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ to be differentiable at $(x_0, y_0) \in U$ if

i) $\frac{\partial f}{\partial x}, \frac{\partial y}{\partial y}$ exists at (x_0, y_0) and

ii) the tangent plane at (x_0, y_0) provides a good approximation to f near (x_0, y_0), in the sense that

$$\frac{f(x,y) - f(x_0,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}{\|(x-y)-(x_0,y_0)\|} \to 0 \text{ as } (x,y) \to (x_0,y_0) \qquad (*)$$

Notation, we will write $Df(x_0,y_0)$ for the row matrix $\left(\frac{\partial f}{\partial x}(x_0,y_0) \quad \frac{\partial f}{\partial y}(x_0,y_0)\right)$

So, $(*)$ becomes $$\frac{f(x,y) - f(x_0,y_0) - Df(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}{\|(x-y)-(x_0,y_0)\|} \to 0 \quad \text{as} \quad (x,y) \to (x_0,y_0)$$

More generally, if each component function $f_1, \dots, f_m$ of $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has partial derivatives existing at $x_0 \in U$, we derive the $m \times n$ matrix $Df(\underline{x}_0)$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\underline{x}_0) & \frac{\partial f_1}{\partial x_2}(\underline{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\underline{x}_0) \\ \frac{\partial f_2}{\partial x_1}(\underline{x}_0) & \frac{\partial f_2}{\partial x_2}(\underline{x}_0) & \cdots & \frac{\partial f_2}{\partial x_n}(\underline{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_0) & \frac{\partial f_n}{\partial x_2}(x_0) & \cdots & \frac{\partial f_n}{\partial x_n}(x_0) \end{pmatrix}$$

we call this the...

i) derivative of f at $\underline{x}_0$, or
ii) differential of f at $\underline{x}_0$, or
iii) the matrix of partial derivatives of f at $\underline{x}_0$
iv) the Jocodian of f at $\underline{x}_0$.

We then derive $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ to be diffable at $x_0 \in U$ if

i) the partial derivative of each component $f_1, \dots, f_n$ exists at $\underline{x}_0$ and:

ii) $$\lim_{\underline{x} \to \underline{x}_0} \frac{\|f(\underline{x}) - f(\underline{x}_0) - Df(\underline{x}_0)(\underline{x} - \underline{x}_0)\|}{\|\underline{x} - \underline{x}_0\|} = 0.$$ the $n \times 1$ column matrix

Theorem: if $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has partial derivatives $\frac{\partial f_i}{\partial f_j}$, all existing at x_0 and and these partial derivatives are continuous in the neighbornood of $\underline{x}_0$, then f is differentiable at $\underline{x}_0$.

Examples:

1) $f: \mathbb{R}^3 \to \mathbb{R}^2$

$f(x,y,z) = (ye^{\sin x}, xyz^2)$

$f_1(x_1, y_1, z_1) \quad f_2(x_2, y_2, z_2)$

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} y\cos x e^{\sin x} & e^{\sin x} & 0 \\ yz^2 & xz^2 & 2xyz \end{pmatrix}$$

2) $g: \mathbb{R}^2 \to \mathbb{R}^3$ g_1, g_2, g_3

$g(x,y) = (e^{xy}, x-y, y^2)$

$$\underline{D}g = \begin{pmatrix} \frac{\partial g_1}{\partial x}(x,y) & \frac{\partial g_1}{\partial y}(x,y) \\ \frac{\partial g_2}{\partial x}(x,y) & \frac{\partial g_2}{\partial y}(x,y) \\ \frac{\partial g_3}{\partial x}(x,y) & \frac{\partial g_3}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} ye^{xy} & xe^{xy} \\ 1 & -1 \\ 0 & 2y \end{pmatrix}$$

Are these functions differentiable everywhere?

Recall: if partial derivatives exist at a point and they are continuous in the neighborhood of that point, then function is diffble.

3) Going back to $f(x,y) = x^{1/3}y^{1/3}$

$\underline{D}f(x,y) = \left(\frac{1}{3}x^{-2/3}y^{1/3} \quad \frac{1}{3}x^{1/3}y^{-2/3} \right)$ → We can see that these entries are not continuous at the origin.

Thm: Let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be diffble at $x_0 \in U$, then f is cts at $\underline{x}_0$.

Note: existence of partial derivatives at $x_0 \in U$ is not sufficient to continuity at that point.

If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ is scalar valued, so that $\underline{D}f(x_0) = \left(\frac{\partial f}{\partial x}(\underline{x}_0) \cdots \frac{\partial f}{\partial x_1}(\underline{x}_0) \right)$ is a $1 \times n$ row matrix, we derive the gradient of f at x_0 by:

$$\nabla f(\underline{x}_0) = \left\langle \frac{\partial f}{\partial x_1}(\underline{x}_0), \cdots, \frac{\partial f}{\partial x_n}(\underline{x}_0) \right\rangle$$

e.g., $f: \mathbb{R}^3 - \mathbb{R}$ $f(x,y,z) = e^{xy} + z\sin(xy)$

$\frac{\partial f}{\partial x} = ye^{xy} + yz\cos(xy)$

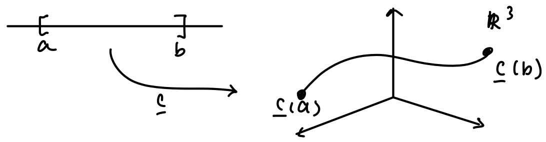
$\frac{\partial f}{\partial y} = xe^{xy} + xz\cos(xy)$

$\frac{\partial f}{\partial z} = \sin xy$

Hence, $\nabla f(x,y,z) = (ye^{xy} + yz\cos(xy))\underline{i} + (xe^{xy} + xz\cos(xy))\underline{j} + (\sin xy)\underline{k}$.

Section 2.4: Paths and curves

Defn: A path in $\mathbb{R}^n$ is a map $\underline{c}: [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$. We call the set of points $\underline{c}(t)$ the "curve parametricized by $\underline{c}$" and $\underline{c}(a)$, $\underline{c}(b)$ are the end points of the curve.



Since a path is a vector valued function, we can write a path $c: [a,b] \to \mathbb{R}^n$ as components $(c_1(t), \dots, c_n(t))$

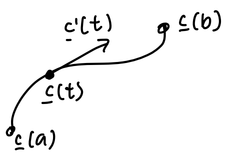
e.g. in 3D $(x(t), y(t), z(t))$.

Think of $\underline{c}(t)$ as being the path traced out by a particle over time with t as the time variable.

It's reasonable to call $c'(t) = \lim_{h \to 0} \frac{\underline{c}(t+h) - \underline{c}(t)}{h} = \left\langle \lim_{h\to 0} \frac{c_1(t+h) - c(t)}{h}, \dots, \lim_{h \to 0} \frac{c_n(t+h) - c(t)}{h} \right\rangle = \langle c_1'(t), \dots, c_n'(t) \rangle$

This is the velocity vector of c at time t (assuming $\underline{c}$ is diff'ble.).

If the vector $\underline{c}'(t)$ is drawn with its tail at $\underline{c}(t)$, then it is tangent to the curve at $\underline{c}(t)$ (assuming that $c'(t) \neq 0$, in which case the particle has momentarily stopped.



In particular, the direction of $\underline{c}'(t)$ gives the instantaneous direction of travel at time t.

Also: we see that the tangent line to the curve at $\underline{c}(t_0)$ is $\ell(t) = \underline{c}(t_0) + c'(t_0)(t - t_0)$

I like branches

– Liwen's dendrophile roomie

$\boxed{\text{Section 2.5: Properties of Derivatives}}$

i) if $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diffble at $\vec{x}_0$ and $c \in \mathbb{R}$, then $h(\vec{x}) = cf(\vec{x})$ is diffble at x_0 with $\vec{D}h(\vec{x}_0) = c\vec{D}_f(\vec{x}_0)$.

ii) if $f, g: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are diffble at $\vec{x}_0$, then $h(x) = f(x) + g(x)$ is diffble at x_0 with $\vec{D}h(\vec{x}_0) = \vec{D}f(x_0) + \vec{D}g(x_0)$.

iii) if $f, g: U \subseteq \mathbb{R}^n \to \mathbb{R}$ are diffble at $\vec{x}_0$, then $h(\vec{x}) = g(x_0)\vec{D}f(x_0) + f(x_0) + \vec{D}g(x_0)$

iv) if $f, g: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are diffble at $\vec{x}_0$, and $g(\vec{x}_0) \neq 0$, then $h(\vec{x}) = \frac{f(\vec{x})}{g(\vec{x})}$ is diffble at $\vec{x}_0$ within $\vec{D}h(\vec{x}_0) = \frac{g(\vec{x}_0)\vec{D}f(\vec{x}_0) - f(\vec{x}_0)\vec{D}g(\vec{x}_0)}{(g(\vec{x}_0))^2}$.

Chain Rule

Recall the 1D chain rule: if $z = f(y)$ and $y = g(x)$, we can view z either as a function of y $(z = f(y))$ or as a function of x $(z = f(g(x)))$. The chain rule relates the derivative of z with respect to x and with respect to y.

$$\frac{dz}{dx}(x) = \frac{dz}{dy}(x)\frac{dy}{dx}(x) = f'(g(x))g'(x)$$

In multivariable:

<u>Theorem</u>: let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets, let $g: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $f: V \subseteq \mathbb{R}^m \to \mathbb{R}^p$.
Suppose $g(U) \subseteq V$ (so the composition mapping of $f \circ g: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is well defined). Suppose g is diffble at $x_0 \subseteq U$ and f is diffble at $g(x_0) \in V$. Then $f \circ g$ is diffble at $x_0 \in U$ with $\underbrace{\vec{D}(f \circ g)(x_0)}_{p \times n \text{ matrix}} = \underbrace{\vec{D}f(g(\vec{x}_0))}_{p \times m \text{ matrix}}\underbrace{\vec{D}g(\vec{x}_0)}_{m \times n \text{ matrix}}$

Example:

suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is diffble and $\underline{c}: \mathbb{R} \to \mathbb{R}^3$ is a diffble path. In components:

$c(t) = (x(t), y(t), z(t))$.

let $h = f \circ \underline{c}: \mathbb{R} \to \mathbb{R}$. Then, the $\underline{D}(f \circ \underline{c})$ is just going to be a number (i.e, a 1×1 matrix).

Then $h(t) = f(\underline{c}(t)) = f(x(t), y(t), z(t))$

satisfies
$$\frac{\partial h}{\partial t}(t_0) = \frac{\partial}{\partial t}(f \circ \underline{c})(t_0)$$
$$= \underline{D}(f \circ \underline{c})(t_0)$$
chain rule
$$= \underline{D}_f(\underline{c}(t_0))\underline{D}g(t_0)$$
$$= \begin{pmatrix} \frac{\partial f}{\partial x}(\underline{c}(t_0)) & \frac{\partial f}{\partial y}(\underline{c}(t_0)) & \frac{\partial f}{\partial z}(\underline{c}(t_0)) \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial t}(t_0) \\ \frac{\partial y}{\partial t}(t_0) \\ \frac{\partial z}{\partial t}(t_0) \end{pmatrix} \quad (*)$$
$$= \frac{\partial f}{\partial x}(\underline{c}(t_0))\frac{\partial x}{\partial t}(t_0) + \frac{\partial f}{\partial y}(\underline{c}(t_0))\frac{\partial y}{\partial t}(t_0) + \frac{\partial f}{\partial z}(\underline{c}(t_0))\frac{\partial z}{\partial t}(t_0)$$

or just $\frac{\partial h}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$

Note: we could also write $(*)$ as $\nabla f(c(t_0)) \cdot c'(t_0)$

vectors

Example 2:

suppose $f = f(u, v, w): \mathbb{R}^3 \to \mathbb{R}$

$g = g(x, y, z) = (u(x,y,z), v(x,y,z), w(x,y,z))$

Derive $h = f \circ g$ so that $h: \mathbb{R}^3 \to \mathbb{R}$ with $h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$,

$$\frac{\partial h}{\partial x}(\underline{x}_0) \; \frac{\partial h}{\partial y}(\underline{x}_0) \; \frac{\partial h}{\partial z}(\underline{x}_0) = \underline{D}h(\underline{x}_0) = \underline{D}(f \circ g)(\underline{x}_0)$$
$$= \underline{D}(f(g(\underline{x}_0)))\underline{D}g(\underline{x}_0)$$

Now,
$$\underline{D}g(\underline{x}_0) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial x} & \frac{\partial w}{\partial z} \end{pmatrix}(\underline{x}_0)$$

$$\underline{D}f(g(\underline{x}_0)) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{pmatrix}(g(\underline{x}_0))$$

Therefore
$$\underline{D}h(\underline{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{pmatrix}(g(\underline{x}_0)) \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial x} & \frac{\partial w}{\partial z} \end{pmatrix}(\underline{x}_0)$$

$$= \begin{array}{l} \begin{pmatrix} \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\ \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \\ \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \text{column 1} \\ \leftarrow \text{column 2} \\ \leftarrow \text{column 3} \end{array}$$

e.g, $\frac{\partial f}{\partial u}(g(\underline{x}_0))\frac{\partial u}{\partial x}(\underline{x}_0) + \frac{\partial f}{\partial v}(g(\underline{x}_0))\frac{\partial v}{\partial y}(\underline{x}_0) + \frac{\partial f}{\partial w}(g(\underline{x}_0))\frac{\partial w}{\partial z}(\underline{x}_0)$

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \qquad \frac{\partial h}{\partial y} = \cdots \qquad \frac{\partial h}{\partial z} = \cdots$$

Ex: $f(u,v,w) = u^3 + wv^2$

$g(x,y,z) = (xyz, e^{-yz}, \sin x)$

Let $h = f \circ g$ i.e., $h(x,y,z) = f(xyz, e^{-yz}, \sin x)$

we're going to compute $\frac{\partial h}{\partial x}$ as a function in terms of x, y, z.

i) using the chain rule

ii) directly.

i) we have $g(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$

$u = xyz$

$v = e^{-yz}$

$w = \sin x$

then: $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$

$\frac{\partial h}{\partial x} = (3u^2)(yz) + (2wv)(0) + v^2(\cos x)$

$= 3(xyz)^2 yz + (e^{-yz})^2 \cos x$

$= 3x^2y^3z^3 + e^{-2yz}\cos x$

ii) $h(x,y,z) = f(\underset{u}{xyz}, \underset{v}{e^{-yz}}, \underset{w}{\sin x})$

$f = u^3 + wv^2$

$f = (xyz)^3 + (\sin x)(e^{-yz})^2$

$= x^3y^3z^3 + \sin x\, e^{-2yz}$

$\frac{\partial h}{\partial x} = 3x^2y^3z^3 + \cos x\, e^{-2yz}$

same results.

Example 3:

$f(u,v,w) = (\underset{f_1(u,v,w)}{u^2 w}, \underset{f_2(u,v,w)}{u+v+w}) : \mathbb{R}^3 \to \mathbb{R}^2$

$g(x,y) = \left(5x^2+1,\ 2y\cos(2x),\ \frac{1}{1+x^2+y^2}\right) : \mathbb{R}^2 \to \mathbb{R}^3$

Then let $h = f \circ g : \mathbb{R}^2 \to \mathbb{R}^3$, so $\underline{D}h(\underline{x}_0) = \underline{D}f(g(\underline{x}_0))\underline{D}g(\underline{x}_0)$

suppose $\underline{x}_0 = (x_0, y_0)$

$$\underline{D}g(\underline{x}_0) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} \end{pmatrix}(\underline{x}_0) = \begin{pmatrix} 10x_0 & 0 \\ -4y_0\sin(2x_0) & 2\cos(2x_0) \\ \frac{-2x_0}{(1+x_0^2+y_0^2)^2} & \frac{-2y_0}{(1+x_0^2+y_0^2)^2} \end{pmatrix}(\underline{x}_0)$$

evaluate each element at $\underline{x}_0$

$$\underline{D}f(g(x_0)) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \end{pmatrix}(g(\underline{x}_0)) = \begin{pmatrix} 2uw & 0 & u^2 \\ 1 & 1 & 1 \end{pmatrix}(g(\underline{x}_0)) \qquad (*)$$

$g(\underline{x}_0) = (\underset{u}{5x_0^2+1},\ \underset{v}{2y_0\cos(2x_0)},\ \underset{w}{\frac{1}{1+x_0^2+y_0^2}})$

so

$$= (*) = \begin{pmatrix} \frac{2(5x_0^2+1)}{1+x_0^2+y_0^2} & 0 & (5x_0^2+1)^2 \\ 1 & 1 & 1 \end{pmatrix}$$

therefore, what is $\underline{D}h(1,1)$ i.e., $\underline{x}_0$'s values.

$$\underline{D}g(1,1) = \begin{pmatrix} 10(1) & 0 \\ -4(1)\sin(2) & 2\cos(2(1)) \\ \frac{-2(1)}{(1+(1)^2+(1)^2)^2} & \frac{-2(1)}{(1+(1)^2+(1)^2)^2} \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ -4\sin 2 & 2\cos 2 \\ \frac{-2}{9} & \frac{-2}{9} \end{pmatrix}$$

$$\underline{D}f(g(1,1)) = \begin{pmatrix} \frac{2(5+1)}{3} & 0 & (5+1)^2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 36 \\ 1 & 1 & 1 \end{pmatrix}$$

multiply matrix

Therefore,

$\underline{D}h(1,1) = \underline{D}(f\circ g)(1,1) = \underline{D}f(g(1,1))\underline{D}g(1,1)$

$$= \begin{pmatrix} 4 & 0 & 36 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 10 & 0 \\ -4\sin 2 & 2\sin 2 \\ \frac{-2}{9} & \frac{-2}{9} \end{pmatrix} = \begin{pmatrix} 36 & -8 \\ \frac{88}{9} - 4\sin 2 & 2\cos 2 - \frac{2}{9} \end{pmatrix} \quad //$$

Section 2.6: Gradients and Directional Derivatives

Defn: let $f: \mathbb{R}^3 \to \mathbb{R}$. The directional derivative of f at the point $\underline{x}$ in the direction of a unit vector $\underline{v}$ is $\lim_{t \to 0} \frac{f(\underline{x}+t\underline{v})-f(\underline{x})}{t}$
or equivalently $\frac{\partial}{\partial t} f(\underline{x}+t\underline{v})\Big|_{t=0}$

Thm: If $\mathbb{R}^3 \to \mathbb{R}$ is diffable, then all directional derivatives exist, and the directional derivative a $\underline{x}$ in the direction of a unit vector $\underline{v} = \langle v_1, v_2, v_3 \rangle$ is $\underbrace{\underline{D} f(\underline{x})}_{1\times 3 \text{ matrix}} \underbrace{(\underline{v})}_{3\times 1 \text{ matrix}}$

i.e, $v_1 \frac{\partial f}{\partial x}(\underline{x}) + v_2 \frac{\partial f}{\partial y}(\underline{y}) + v_3 \frac{\partial f}{\partial z}(\underline{z})$ (*)

Note: if you want the directional unit vector in the direction of a non-unit vector, you first need to normalize the vector before applying any of (*)

Proof:

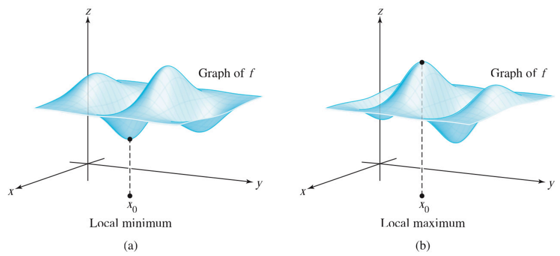
Df: chain rule! Let $\underline{c}(t) = \underline{x} + t\underline{v}$, so $f(\underline{x}+t\underline{v}) = f(c(t))$. Then we know $\frac{\partial}{\partial t} f(\underline{c}(t))\Big|_{t=0} = \nabla f(\underline{c}(t)) \cdot \underline{c}'(t)$.

Evaluate at $t=0$ $c(0)=x$ $c'(0)=\underline{v}$

Hence, $\frac{\partial}{\partial t} f(\underline{c}(t))\Big|_{t=0} = \nabla f(x) \cdot \underline{v}$

Thus: suppose $\nabla f(x) \neq \underline{0}$. Then the vector $\nabla f(\underline{x})$ points in the direction in which f is increasing the fastest at $\underline{x}$.

Chapter 3: Higher Order Derivatives: maxima and minima



Section 3.1: Iterated Partial Derivatives

Suppose $f=f(x,y): U\subseteq\mathbb{R}^2\to\mathbb{R}$ and suppose $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist. Since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are themselves functions, we can take their derivative

$$\frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$ ← definition

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$ ← example of a "mixed partial derivative"

$$\frac{\partial^2 f}{\partial x\partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

If f is "sufficiently smooth", we can repeat this process and obtain "iterated partial derivative".

An iterative partial derivative of function f is any partial derivative of f of 2^{nd} order or higher.

e.g., $\frac{\partial^3 f}{\partial x\partial y\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$

Ex $f(x,y) = 5x^7y + y^4$

$\frac{\partial f}{\partial x} = 35x^6y$

$\frac{\partial^2 f}{\partial x\partial x} = \frac{\partial}{\partial x}(35x^6y) = 210x^5y$

$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}(35x^6y) = 35x^6$

$\frac{\partial f}{\partial y} = 5x^7 + 4y^3$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(5x^7+4y^3) = 12y^2$

$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}(5x^7+4y^2) = 35x^6$

It is no coincidence that

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$$

Theorem: Suppose $f:\mathbb{R}^2\to\mathbb{R}$ has continuous partial derivatives and its second order partial derivatives are also continuous. Then, $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$. (if $f\in C^2$)

Definition: If partial derivatives of f exist and are continuous, we say "f is of class C^1" and write $f\in C^1$.
If $f\in C^1$ and its partial derivatives exist and are continuous, we write $f\in C^2$

Note: Recall $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$

we write: $f_{xy} := (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x}$

$f_{yx} := (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\partial y}$

Section 3.2: Taylor's Theorem

The tangent plane to a graph @ a point gives the best linear approximation to the corresponding function at that point. Taylor theorem gives a higher order extention of approximation. For example, 2^{nd} order taylor approximation of a function at that point will give a quadratic approximation to the function near that point. 3^{rd} order = cubic approximation.

Recal in 1D, if $f: \mathbb{R} \to \mathbb{R}$ is smooth,

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(h^k) + R_k(x_0,h) \qquad (*)$$

where $R_k(x_0,h)$ is an "error term" satisfying $\lim_{h\to 0} \frac{R_k(x_0,h)}{h^k} = 0$ As $h \to 0$, $R_k(x_0,h)$ goes to 0 quicker than $h^k \to 0$.
Call $(*)$ the k'th order Taylor expansion of f at x_0.

The first order Taylor expansion comes from the definition of being differentiable.
If f is differentiable, recall

$$\lim_{\underline{x}\to\underline{x}_0} \frac{f(\underline{x}) - f(\underline{x}_0) - Df(\underline{x}_0)(\underline{x}-\underline{x}_0)}{\|\underline{x}-\underline{x}_0\|} = 0.$$

Equivalently, by taking $\underline{x} = \underline{x}_0 + h$,

$$\lim_{\underline{h}\to 0} \frac{f(\underline{x}_0 - \underline{h}) - f(\underline{x}_0) - Df(\underline{x}_0)h}{\|\underline{h}\|} = 0,$$

Or equivalently,

$$f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + Df(\underline{x}_0)\underline{h} + R_1(x_0+\underline{h})$$

where $\lim_{\underline{h}\to 0} \frac{R_1(x_0,h)}{\|\underline{h}\|} = 0$

Hence we have:
Thm: let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ be diffable at $\underline{x}_0$ then,

$$f(\underline{x}_0 + \underline{h}) = f(\underline{x}_0) + Df(\underline{x}_0)\underline{h} + R_1(\underline{x}_0, h) \qquad (**)$$

where $\lim_{\underline{h}\to 0} \frac{R_1(\underline{x}_0,\underline{h})}{\|\underline{h}\|} = 0.$

We can also write $(**)$ (if $\underline{h} = (h_1, \cdots, h_n)$)

$$f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \underbrace{\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\underline{x}_0)}_{\nabla f(\underline{x}_0)\cdot \underline{h}} + R_1(\underline{x}_0 + \underline{h})$$

comes from the definition of matrix multiplication.

Also have the 2^{nd} order Taylor expansion of a C^2 function.
means f has continuous partial derivatives, and the partial derivatives of the partial derivatives are continuous at $\underline{x}_0$.

Theorem: let $f \in C^2$ at $\underline{x}_0$. Then

$$f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\underline{x}_0) + \underbrace{\frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}_0)}_{} + R_2(\underline{x}_0, \underline{h}) \qquad (***)$$

where $\lim_{\underline{h}\to 0} \frac{R_2(\underline{x}_0,\underline{h})}{\|\underline{h}\|^2} = 0$

sums all possible pairs of (i,j) for $1 \le i \le n$, $1 \le j \le n$.
e.g. $\sum_{i,j=1}^{2} a_{ij} = a_{11} + a_{12} + a_{21} + a_{22}$

In matrix form, if we write

$$\nabla^2 f(\underline{x}_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} (\underline{x}_0)$$

we call $\nabla^2 f(\underline{x}_0)$ the "Hessian" matrix of f at $\underline{x}_0$, and if we also derive the Hessian quadratic form of f at $\underline{x}_0$,
$Hf(\underline{x}_0): \mathbb{R}^n \to \mathbb{R}$ by

$$H_f(\underline{x}_0)(\underline{h}) = \frac{1}{2}(h_1, \cdots, h_n)\nabla^2 f(\underline{x}_0)\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}_0),$$

We can write $(***)$ as

$$f(\underline{x}_0+\underline{h}) = f(\underline{x}_0) + \nabla f(\underline{x}_0)\cdot\underline{h} + H_f(\underline{x}_0)(\underline{h}) + R_2(\underline{x}_0,\underline{h})$$

Example: Find the 2^{nd} order Taylor expansion of $f(x,y) = \sin(x^2+3y)$ at $\underline{x}_0 = (0,0)$

$f(0,0) = \sin(0+0) = 0$

$\frac{\partial f}{\partial x} = 2x\cos(x^2+3y)$ $\quad$ $\frac{\partial f}{\partial x}(0,0) = 0$

$\frac{\partial f}{\partial y} = 3\cos(x^2+3y)$ $\quad$ $\frac{\partial f}{\partial y}(0,0) = 3\cos(0) = 3$

$\frac{\partial^2 f}{\partial x^2} = -2x\sin(x^2+3y)(2x) + 2\cos(x^2+3y)$ $\quad$ $\frac{\partial^2 f}{\partial x^2}(0,0) = 0 + 2 = 2$

$\frac{\partial^2 f}{\partial y^2} = -3\sin(x^2+3y)(3) = -9\sin(x^2+3y)$ $\quad$ $\frac{\partial^2 f}{\partial y^2}(0,0) = 0$

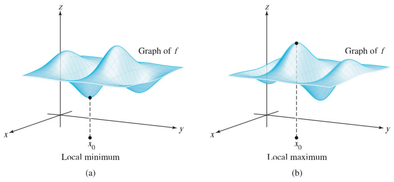
$\frac{\partial^2 f}{\partial x \partial y} = -3\sin(x^2+3y)(2x) = -6x\sin(x^2+3y)$ $\quad$ $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0.$

$$\underset{\substack{\uparrow \\ f((0,0)+(h_1,h_2))}}{f(\underline{x}_0+\underline{h})} = \underset{\substack{\uparrow \\ 0}}{f(\underline{x}_0)} + \underbrace{\sum_{i=1}^{2} h_i \frac{\partial f}{\partial x_i}(\underline{x}_0)}_{(h_1)(0)+(h_2)(3)} + \underbrace{\frac{1}{2}\sum_{i,j=1}^{2} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)}_{\frac{1}{2}\left[h_1h_1(2) + h_1h_2(0) + h_2h_1(0) + h_2h_2(0)\right] = \frac{1}{2}(2h_1^2) = h_1^2} + R_2$$

therefore,

$$f(\underline{h}) = 3h_2 + 2h_1^2 + R_2(\overset{(0,0)}{\underline{x}_0}, \underline{h})$$

Section 3.3: Finding Extrema of real-valued functions



we want to determine the points x_0.

Definition: $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$. we call $x_0 \in U$ a local minimum (resp. local minimum) of f if there exists a neighborhood V of x_0 s.t.:

$f(x) \geq f(x_0)$ for all $x \in V$ (resp. $f(x) \leq f(x_0)$ for all $x \in V$)

↑ or $>$ to find the "strict" local minimum.

If $x_0 \in U$ is either a local minimum or local maximum, we call x_0 a local extrema or relative extrema. This means x_0 is not or max/min globally, just within the points.

we call $x_0 \in U$ a critical point of f if either f is not diffble at x_0 or $Df(x_0)=0$, i.e, $\frac{\partial f}{\partial x_i}(x_0)=0$ for all i, $\nabla f(x_0)=0$.

Note: For us, functions will almost always be diffble, so being asked to find the critical point means finding x_0 s.t. $Df(x_0)=0$.

Theorem: If $U \subseteq \mathbb{R}^n$ (open subset), $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ is diffble. and $x_0 \in U$ a local extremum, then $Df(x_0)=0$ (i.e.: x_0 is a critical point).

Proof: suppose $x_0 \in U$ is a local maximum, since U is open. if $h \in \mathbb{R}^n$ and $t \in \mathbb{R}$ is super small, then $x_0 + th \in U$ too.
Define $g: \mathbb{R} \to \mathbb{R}$ by $g(t) = f(x_0 + th)$ ← add very small vector, $+h$

This has a local maximum at $t=0$ (since f has a local maximum at x_0), hence from 1D calc, we know $g'(0)=0$. But,

$0 = g'(0) = Df(x_0)h$. (*)

Since $h \in \mathbb{R}^n$ was arbitrary, (*) can only hold for all h if $Df(x_0)=0$

If x_0 is a local minimum, proof is analogous.

Note:
This theorem does not say that if $Df(x_0)=0$, then it is an extrema (because of inflection/turning points) or saddle points).
$Df(x_0)=0 \not\Rightarrow x_0$ is an extremum. Could be a saddle points (We define x_0 to be a saddle point of f if $Df(x_0)=0$ but x_0 is not a local extremum).

Sometimes we can determine the type of critical point by examination.

e.g., $f(x,y) = x^2+y^2$

well $f \geq 0$, But also $f(0,0)=0$, hence f must obtain a local (in fact global) minimum at the origin.

Mathematically ...

If $x_0 \in \mathbb{R}^2$ is a local extremum, then $Df(x_0)=0$. But,

$D_n f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \end{pmatrix}$ hence $Df(x_0,y_0)=(0,0) \Leftrightarrow \begin{matrix} 2x_0=0 \\ 2y_0=0 \end{matrix} \Leftrightarrow (x_0,y_0)=(0,0)$.

Summary:

Let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$. Recall:

$x_0 \in U$ a local min if $f(x) \geq f(x_0)$ near x_0

$x_0 \in U$ a local max if $f(x) \leq f(x_0)$ near x_0

$x_0 \in U$ a local extremum if local max or min.

$x_0 \in U$ a critical point (CP) if $Df(x_0)=0$ (or if f is not diffable at x_0)

CP $\not\Rightarrow$ local extremum:

$f(x,y) = x^2 - y^2$

$Df(x,y) = (2x \;\; -2y) = (0,0) \Leftrightarrow x = y = 0$

So the only CP is at $(0,0)$, where $f(0,0) = 0$. But f cannot have a local extrema at $(0,0)$, since $f(x,0) > 0$ for $x \neq 0$, and $f(0,y) < 0$ for $y \neq 0$. So in particular, f takes values $> f(0,0)$ and $< f(0,0)$ arbitrarily close to $(0,0)$, so $(0,0)$ is a saddle point.

Example: Find & classify CP of $f(x,y) = 5x^7 y + y^4$

Ans: $\frac{\partial f}{\partial x} = 35x^6 y$ $\qquad \frac{\partial f}{\partial y} = 5x^7 + 4y^3$

$= 0 \Leftrightarrow$ either $x = 0$ or $y = 0$

If $x = 0$, then $\frac{\partial f}{\partial y} = 0 \Leftrightarrow y = 0$

If $y = 0$, then $\frac{\partial f}{\partial y} = 0 \Leftrightarrow x = 0$

So, $(0,0)$ is the only CP for f and $f(0,0) = 0$.

Note: $f(0,y) > 0$ for $y > 0$

On the other hand let $n \in \mathbb{N}$

$$f\left(\frac{-1}{n^{3/7}}, \frac{1}{n}\right) = 5\left(\frac{-1}{n^{3/7}}\right)^7\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)^4$$
$$= 5\left(\frac{-1}{n^3}\right)\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)^4 = \frac{-5}{n^4} + \frac{1}{n^4} = \frac{-4}{n^4} < 0$$

So f take values $> f(0,0)\ (=0)$ and $< f(0,0)\ (=0)$ arbitrarily close to our CP $(0,0)$. Hence $(0,0)$ saddle.

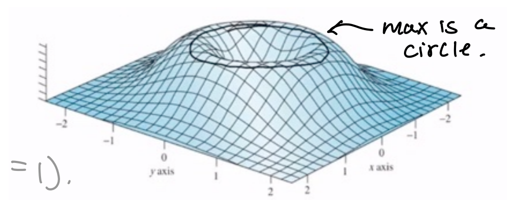
Note: CP need not be "isolated points"

let $f(x,y) = 2(x^2+y^2)e^{-x^2-y^2}$

$\frac{\partial f}{\partial x} = 4x(e^{-x^2-y^2})(1-x^2-y^2)$

$\frac{\partial f}{\partial y} = 4y(e^{-x^2-y^2})(1-x^2-y^2)$

Then $\left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right) = (0,0) \Leftrightarrow$ either $1 - x^2 - y^2 = 0$ (i.e., $x^2 + y^2 = 1$).



max is a circle.

Extrema through Hessian

Recall from 1D:

- $f''(x) > 0 \Rightarrow x_0$ is a local minimum
- $f''(x) < 0 \Rightarrow x_0$ is a local maximum
- $f''(x) = 0 \Rightarrow x_0$ could be a min, max, or inflection point.

Recall that the hessian quadratic form of f at $\underline{x_0}$, $Hf(\underline{x_0}): \mathbb{R}^n \to \mathbb{R}$, is $H_f(\underline{x_0})(\underline{h}) = \frac{1}{2}(h_1 \cdots h_n)\nabla^2 f(\underline{x_0})\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$

$= \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) h_i h_j$

Defn: we say that the hessian of f at $\underline{x}$ is positive definite (resp. negative definite) if $H_f(\underline{x_0})(\underline{h}) > 0$ (resp < 0) for all $\underline{h} \neq \underline{0}$.

2^{nd} Derivative Test for Local Extrema

Thm: let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^2, suppose $\underline{x_0} \in U$ is a CP of f (i.e. $Df(x_0) = \underline{0}$)

- If $\nabla^2 f(\underline{x_0})$ is positive definite, $\underline{x_0}$ is a (strict) local minimum.
- If $\nabla^2 f(\underline{x_0})$ is negative definite, $\underline{x_0}$ is a (strict) local maximum.

How do we determine if $\nabla^2 f(\underline{x_0})$ is positive or negative definite or neither?

Simple example: $f(x) = x^2 + y^2$ $\quad \nabla f = \langle 2x, 2y \rangle$

$$\nabla^2 f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Hence, $H_f(x,y)(\underline{h}) = \frac{1}{2}(h_1\ h_2)\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{1}{2}(h_1\ h_2)\begin{pmatrix} 2h_1 \\ 2h_2 \end{pmatrix} = h_1^2 + h_2^2 > 0$ if $\underline{h} \neq \underline{0}$

Example 2: $f(x,y) = x^2 - y^2$

$\nabla^2 f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad H_f = \frac{1}{2}(h_1\ h_2)\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_1^2 - h_2^2$ can be either positive or negative for $\underline{h} \neq \underline{0}$ so saddle point.

Determine sign of Hessian matrix

Let $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be C^2; U open.

1) If i) $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$

ii) $\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ at (x_0, y_0)

Then, $\nabla^2 f(x_0, y_0)$ is positive definite. In particular, if (x_0, y_0) is a CP (i.e., $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$), and i) and ii) hold, then (x_0, y_0) is a local minimum for f.

2) If iii) $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$

iv) $D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ at (x_0, y_0)

Then, $\nabla^2 f(x_0, y_0)$ is negative definite. In particular, if (x_0, y_0) is a CP and iii) and iv hold, then (x_0, y_0) = local max.

3) If (x_0, y_0) is a CP for f, but $D < 0$, then (x_0, y_0) is a saddle point.

4) If $D = 0$, then test is inconclusive

Example 1: $f(x,y) = x^2 + y^2$ $\quad \nabla f(x) = \langle 2x, 2y \rangle$

$Df(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \frac{\partial^2 f}{\partial x^2} = 2 > 0$

$D = (2 \times 2) - 0 = 4 > 0$

Hence i) and ii) are satisfied at $(0,0)$, so $(0,0)$ is a local minimum.

Example 2: $f(x,y) = x^2 - y^2$

$\nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\quad \frac{\partial^2 f}{\partial x^2} > 0$

$D = -4 - 0 = -4 < 0$ $\quad$ Hence CP $(0,0)$ is a saddle point.

Example 3: $f(x,y) = e^{x^2 - y^2}$ $\quad \nabla f = (2xe^{x^2-y^2}, -2ye^{x^2-y^2})$

$= (0,0) \Leftrightarrow (x,y) = (0,0) \leftarrow$ only CP

$\frac{\partial^2 f}{\partial x^2} = 2e^{x^2-y^2} + 4x^2 e^{x^2-y^2} = 2 @ (0,0)$

$\frac{\partial^2 f}{\partial y^2} = 2e^{x^2-y^2} + 4y^2 e^{x^2-y^2} = -2 @ (0,0)$

$\frac{\partial^2 f}{\partial x \partial y} = -4xye^{x^2-y^2}, = 0 @ (0,0)$

Hence $D = (2)(-2) - (0)^2 = -4 < 0$ $\quad$ Hence, CP $(0,0)$ is a saddle point.

Finding Extrema with Determinants

Theorem: let $f: U \subseteq \mathbb{R}^3 \to \mathbb{R}$ be C^2, U open and:

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

1) If the 3 diagonal submatrices have the determinant at (x_0, y_0, z_0), then $\nabla^2 f(x_0, y_0, z_0)$ is positive definite. (Hence, if (x_0, y_0, z_0) is actually a critical point, then it is a local minimum of f.

2) If the 3 diagonal submatrices have determinants alternately $-, +, -$, then the $\nabla^2 f(x_0, y_0, z_0)$ ("hessian") is negative definite. Hence, if (x_0, y_0, z_0) is a CP, then it is a local max of f.

3) If the determinant of $\nabla^2 f(x_0, y_0, z_0)$ is non-zero, but the hessian is neither positive nor negative definite, then if (x_0, y_0, z_0) is a CP, it is a saddle point

4) If the determinant of $\nabla^2 f(x_0, y_0, z_0)$ is 0, inconclusive.

Example: $f(x,y,z) = x^2 + y^2 + z^2 + 2xyz$

$\nabla f(x,y,z) = \langle 2x + 2yz, 2y + 2xz, 2z + 2xy \rangle$

consider the CPs $(0,0,0)$ and $(-1,1,1)$

$\nabla^2 f(x,y,z) = \begin{pmatrix} 2 & 2z & 2y \\ 2z & 2 & 2x \\ 2y & 2x & 2 \end{pmatrix}$ $\quad$ so $\nabla^2 f(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \nabla^2 f(-1,1,1) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}$

The determinants are all > 0, hence $(0,0,0)$ is a local min (rule 1)

The determinants are $2, 0, -32$. Hence $(-1,1,1)$ is a saddle point.

3.4: constrained extrema and lagrange multipliers

Finding extremas subject to constraints.

we've already seen an example of constrained optimization when looking for global extrema on the closed unit disc of the function $f(x,y) = x^2+y^2-x-y+1$

when finding the extrema of f on the boundary circle, we were finding the critical points of $f(x,y)$ "subject to the constraint $x^2+y^2=1$".

The Language Multiplier Method

In what follows, if $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq U$, we write $f|_S$ to mean the restriction of f to S (i.e. we only consider f as a function of S.

Languange Multiplier Theorem, LMT: suppose $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$, and $g: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ are C^1. Fix $C \in \mathbb{R}$ and let $S \subseteq U$ be the level curve of g of value c (i.e. the set of points $(x,y) \in U$ s.t $g(x,y) = C$)

Then $f|_S$ ("f constricted to s") has a local max or local min at $\underline{x}_0 \in S$, and if $\nabla g(\underline{x}_0) \neq 0$, then there exists $\lambda \in \mathbb{R}$ (possibly 0) s.t. $\nabla f(\underline{x}_0) = \lambda \nabla g(x_0)$ //

Therefore: if we want to minimize or maximize $f(x,y)$ subject to some constraint $g(x,y)=c$, we should look for points $\underline{x}_0$ s.t.

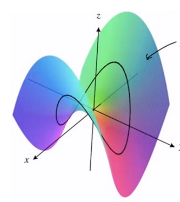
$\nabla f(\underline{x}_0) = \lambda \nabla g(\underline{x}_0)$

for some $\lambda \in \mathbb{R}$ ← if this is a function of 2 variables, then this has 2 equations inside. 3 total with $g(x,y)=c$. and 3 unknowns: x_0, y_0, λ

Note: If both $f, g: U \subseteq \mathbb{R}^3 \to \mathbb{R}$, then the same statement holds when we replace "level curve" with "level surface".

Ex: let $f(x,y) = x^2-y^2$ and let S be the unit circle centered at $(0,0)$. Find the extrema of $f|_S$.

Ans: let $g(x,y) = x^2+y^2$. Then S is the level curve of g of value 1.



$\nabla f(x,y) = (2x, -2y)$

$\nabla g(x,y) = (2x, 2y) \neq \langle 0,0 \rangle$ is $(x,y) \neq \langle 0,0 \rangle$.

Hence LMT ⇒ at an extrenum (x_0,y_0), there exists $\lambda \in \mathbb{R}$ s.t. $\langle 2x_0, -2y_0 \rangle = \lambda \langle 2x_0, 2y_0 \rangle$

i.e. we have 3 equations

$$\begin{cases} 2x_0 = \lambda 2x_0 & ① \\ 2y_0 = \lambda 2y_0 & ② \\ x_0^2 + y_0^2 = 1 & ③ \end{cases}$$

① ⇒ $\lambda = 1$ or $x_0 = 0$. If $\lambda = 1$, then ② ⇒ $y=0$, then ③ $x_0^2 = \pm 1$.

If $x_0 = 0$, then 3 ⇒ $y_0 = \pm 1$.

So the only contenders for constrained extrema are $(\pm 1, 0)$ and $(0, \pm 1)$.

$f(1,0) = 1$, $f(-1,0) = 1$, $f(0,1) = -1$, $f(0,-1) = -1$.

Therefore, $(\pm 1, 0)$ are constrained maxima and $(0, \pm 1)$ are constrained minima.

Ex 2: Find max and min of $f(x,y) = x^2+y^2-x-y=1$ on the closed unit disc.

Ans: the only CP in the open unit disc is $(x,y) = (\frac{1}{2}, \frac{1}{2})$. To find the CP of $f|_{\text{unit circle}}$, set $g(x,y) = x^2+y^2$, $c=1$ and we look for (x_0, y_0) s.t. ($g(x,y) \neq 0$ because $(x,y) \neq \langle 0,0 \rangle$)

$\nabla f(x,y) = \lambda \Delta g(x,y)$ for some $\lambda \in \mathbb{R}$,

i.e. $\langle 2x_0 - 1, 2y_0 - 1 \rangle = \lambda (2x_0, 2y_0)$

i.e. $\begin{cases} 2x_0 - 1 = \lambda 2x_0 & ① \\ 2y_0 - 1 = \lambda 2y_0 & ② \\ x_0^2 + y_0^2 = 1. & ③ \end{cases}$ ① ⇒ $x_0(2-2\lambda) = 1$, ② ⇒ $(2-2\lambda)y_0 = 1$

Clearly from ① we can't have $\lambda = 1$. And by ③ we cannot have both $x_0 = y_0 = 0$.

Then $\frac{②}{①} \Rightarrow \frac{y_0}{x_0} = \frac{1}{1}$ i.e. $x_0 = y_0$ ← any CP satisfying $x_0 \neq 0$ must satisfy this.

Likewise, if $y_0 \neq 0$, then $\frac{①}{②} \Rightarrow \frac{x_0}{y_0} = \frac{1}{1}$ i.e. $x_0 = y_0$

$x_0 = y_0$ ③ ⇒ $2x_0^2 = 1$ i.e. $x_0 = \pm \frac{\sqrt{2}}{2}$

Therefore the contenders to CP of $f|_{\text{unit circle}}$ are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$

↑ $t = \frac{\pi}{4}$ ↑ $\frac{5\pi}{4}$

Functions of 3 variables

$\nabla f(\underset{\sim}{x}_0) = \lambda \nabla g(\underset{\sim}{x}_0)$. $(*)$

Note: just because λ, x_0 satisfies $(*)$ does not mean x_0 is a local extremum for $f|_S$ (could be a saddle point).

In the case that S is closed and bounded, we know that $f|_S$ attains a global min & global max at some points. Therefore, at least one of the solutions to $(*)$ gives a global min of $f|_S$ and at least one solution to $(*)$ gives a global max for $f|_S$, when S is closed and bounded.

Example: maximize and minimize $f(x,y,z) = x^2+y$. subject to $x^2+y^2+z^2=1$

Ans: $g(x,y,z) = x^2+y^2+z^2$, $c=1$.

$\nabla f(x,y,z) = \langle 2x, 1, 0\rangle$

$\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$

and hence we're looking to find (x_0, y_0, z_0) s.t.

$$\begin{cases} 2x = 2\lambda x & ① \\ 1 = 2\lambda y & ② \\ 0 = 2\lambda z & ③ \\ x_0^2 + y_0^2 + z_0^2 = 1 & ④ \end{cases}$$

③ $\Rightarrow \lambda = 0$ or $z_0 = 0$

② $\Rightarrow \lambda \neq 0$

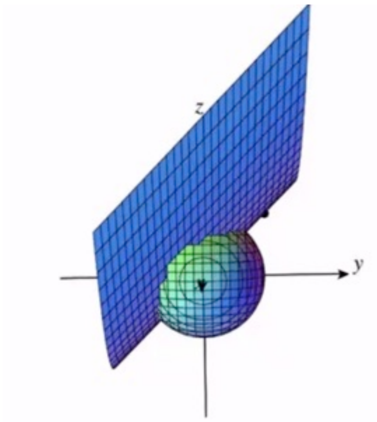
hence ②,③ $\Rightarrow z_0 = 0$.

so ④ $\Rightarrow \boxed{x_0^2 + y_0^2 = 1}$ ⑤

① $\Rightarrow \lambda = 1$ or $x_0 = 0$

If $\lambda = 1$, ② $\Rightarrow y_0 = 1/2$ so ⑤ $\Rightarrow x_0 = \pm\frac{\sqrt{3}}{2}$

If $x_0 = 0$, then ⑤ $\Rightarrow y_0 = \pm 1$



contenders are $\left(\frac{\pm\sqrt{3}}{2}, \frac{1}{2}, 0\right)$ and $(0, \pm 1, 0)$.

$f\left(\frac{\pm\sqrt{3}}{2}, \frac{1}{2}, 0\right) = \frac{5}{4}$ $\quad f(0,1,0) = 1$ $\quad f(0,-1,0) = -1$

So $f|_S$ attains a global max at both $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$ and a global min at $(0,-1,0)$

Not covered in this course

- 2^{nd} derivative test for constrained extrema ("bordered hessian")
- Implicit function theorem (when can we view a surface locally as the graph of a function?)

Chapter 4: Vector Valued Functions

From paths $\underline{c}: [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$ to vector fields $\underline{F}: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$

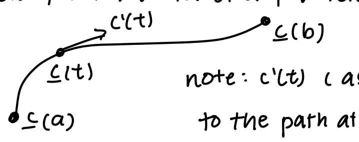
Section 4.1 Acceleration & Newton's 2^{nd} Law

Velocity

Recall: A path in $\mathbb{R}^n$ is a map $\underline{c}: [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$ (or $\underline{c}: \mathbb{R} \to \mathbb{R}^n$).

If $\underline{c}(t) = \langle x(t), \dots, x_n(t) \rangle$ then its derivative is the vector $c'(t) = \langle x'(t), \dots, x_n'(t) \rangle$.

Think of a path as the trajectory of a particle, where t = time, $c(t)$ = position vector of a particle, $c'(t)$ is the velocity vector of the particle.



Sometimes we may write $\underline{v}$ (or $\underline{v}(t)$) instead of $\underline{c}'(t)$.

note: $c'(t)$ (assuming $\neq 0$) is tangent to the path at $\underline{c}(t)$.

We call $s = \|\underline{v}(t)\| = \|c'(t)\|$ the speed of the particle at time t.

We have a dot product rule for paths.

$$\frac{d}{dt}(\underline{b}(t) \cdot c(t)) = b'(t) + c(t) - b(t)c'(t).$$

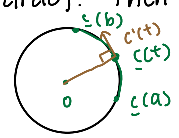
cross product rule for paths in $\mathbb{R}^3$:

$$\frac{d}{dt}(\underline{b}(t) \times \underline{c}(t)) = \underline{b}'(t) \times \underline{c}(t) + \underline{b}(t) \times \underline{c}'(t).$$

Also recall

$$\frac{d}{dt}(\underline{c}(q(t))) = q'(t)\underline{c}'(q(t)) \quad \text{(chain rule)}$$

Prop: suppose $\underline{c}: [a,b] \to \mathbb{R}^n$ s.t. the $\|c(t)\|$ is constant. (always a constant distance from the origin so must be a circle). Then $\underline{c}'(t)$ is always orthogonal to the vector $\underline{c}(t)$. (assuming $c'(t) \neq 0$)



Proof: $\|c(t)\|$ constant $\Rightarrow$ $\underline{c}(t)\underline{c}(t) = \|\underline{c}(t)\|^2$ = a constant.

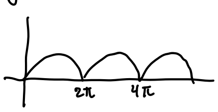
so, $0 = \frac{d}{dt}(\underline{c}(t) \cdot \underline{c}(t)) = \underline{c}'(t)\underline{c}(t) + \underline{c}(t)\underline{c}'(t) = 2\underline{c}(t)\underline{c}'(t)$.

Hence $c(t)$ and $c'(t)$ are orthogonal. //

Smoothness

Unlike graphs of diffable scalar-valued functions, images of diffable paths may not look smooth.

e.g. $\underline{c}(t) = \langle t - \sin t, 1 - \cos t \rangle$



common feature at these corners (when $t = 2n\pi$ for $n = 0, 1, 2 \dots$), then $c'(2n\pi) = \langle 0, 0 \rangle$

since $c'(t) = \langle 1 - \cos t, \sin t \rangle$

Therefore at these points, the tangent line to the path is not well-defined.

With the particle interpretation, the particle has slowed to a rest at these points.

Defn: a diffable path $\underline{c}$ is called "regular" at t_0 if $c'(t_0) \neq 0$.

If $c'(t) \neq 0$ $\forall t$ (for all of t), we call $\underline{c}$ a regular path. "A regular path will always look smooth"

Ex: $\underline{c}(t) = \langle t^3, t^5, \cos t \rangle$

@ which points $t \in \mathbb{R}$ is $\underline{c}$ regular?

Ans: $\underline{c}'(t) = \langle 3t^2, 5t^4, -\sin t \rangle$

$3t^2 = 0 \Leftrightarrow t = 0$, $5t^4 = 0 \Leftrightarrow t = 0$, $-\sin t = 0 \Leftrightarrow t = n\pi$.

The only time every component can be equal to zero is when $t = 0$.

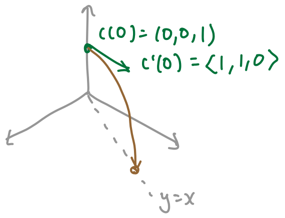
Therefore, $c'(t) = \langle 0, 0, 0 \rangle$ only when $t = 0$.

Hence, the path is regular $\forall t \in \mathbb{R}$ except $t = 0$

Acceleration

Let $c(t) = \langle x_1(t), \dots, x_n(t) \rangle$ be twice differentiable. Then $c'(t) = \langle x_1'(t), \dots, x_n'(t) \rangle$ is the velocity vector, and $a(t) = c''(t) = \langle x_1''(t), \dots, x_n''(t) \rangle$ is the acceleration vector.

Ex: suppose that a particle in $\mathbb{R}^3$ has acceleration $\underline{a}(t)=\langle 0,0,-1\rangle$ ↓ If $\underline{c}(0)=\langle 0,0,1\rangle$ and $c'(0)=\langle 1,1,0\rangle$, then when and where does the particle hit the plane $t=0$? what path is traced out by the particle?



Let $c(t)=\langle x(t), y(t), z(t)\rangle$ be the path of our particle

we have $c''(t)=\langle 0,0,-1\rangle = \langle x''(0), y''(0), z''(0)\rangle$.

Hence $x'(t) = \text{constant } (c_1)$

$y'(t) = \text{constant } (c_2)$

$z'(t) = -t + \text{constant } (c_3)$

But, $x'(0)=1$, $y'(0)=1$, $z'(0)=0$

$\Downarrow c_1=1$, $\Downarrow c_2=1$, $\Downarrow c_3=0$

i.e. $x'(t)=1$, $y'(t)=1$, $z'(t)=$

i.e. $c'(t)=\langle 1,1,-t\rangle$

So, $x(t)=t+d_1$, $y(t)=t+d_2$, $z(t)=\frac{1}{2}t^2+d_3$

But, $x(0)=0 \Rightarrow d_1=0$ $\quad y(0)=0 \Rightarrow d_2=0$ $\quad z(0)=1 \Rightarrow d_3=1$

Hence $\underline{c}(t)=\langle t, t, 1-\frac{1}{2}t^2\rangle$

$1-\frac{1}{2}t^2=0$ with $t=\sqrt{2}$, i.e. the particle hits the plane $z=0$ at time $t=\sqrt{2}$, at $(\sqrt{2},\sqrt{2},0)$.

we have for all $t\geq 0$ that $x(t)=y(t)$. $(=t)$, hence the particle is constrained to the plane $x=y$. Moreover, in this plane, $z(t)=1-\frac{1}{2}t^2=1-\frac{1}{2}x(t)^2$ i.e. $z=1-\frac{1}{2}x^2$ Hence we have our parabolic equation.

Newton's 2^{nd} Law

Suppose a particle is traveling along a path $\underline{c}$ in $\mathbb{R}^3$ under the influence of a force $\underline{F}$. If the mass of the particle is m, then its acceleration vector $(\underline{a}(t))$ satisfies $\underline{F}(\underline{c}(t))=m\,\underline{a}(t)$ i.e. $\underline{F}(\underline{c}(t))=m\,\underline{c}''(t)$ $\forall t$.

Extended example (Kepler's Law).

Suppose the sun is of mass M, centered at the origin in $\mathbb{R}^3$. Then newton's law of gravitation states that a planet of mass m at position vector $\underline{r}$ experiences a force of $\frac{-GMm}{r^3}\underline{r}$ where $r=\|\underline{r}\|$

Kepler's Law: the square of the orbital period of the circular orbit around the sun is proportional to the cube of the radius of the orbit.

Proof: suppose planet of mass m moves at a constant speed s in the xy plane at radius r_0 from the sun.

Then, $\underline{r}(t)=(r_0\cos(\omega t), r_0\sin(\omega t))$ where ω is to be determined.

we have $\underline{r}'(t)=\langle -r_0\omega\sin(\omega t), r_0\omega\cos(\omega t)\rangle$ but b/c $\|r'(t)\|=s$ we know that $s^2=(-r_0\omega\sin(\omega t))^2+(r_0\omega\cos(\omega t))^2$

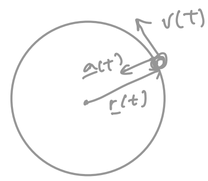
$= r_0^2\omega^2(\sin^2\omega t+\cos^2\omega t) = r_0^2\omega^2$

so $\omega=\frac{s}{r_0}$.

Hence $\underline{r}(t)=\langle r\cos(\frac{st}{r_0}), r_0\sin(\frac{st}{r_0})\rangle$

$= \frac{s}{r_0}$

we call ω the frequency of the orbit, where $\omega=\frac{2\pi}{T}$ if T is the time period of the orbit.



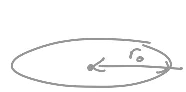
Hence,

$$\underline{a}(t)=\underline{r}''(t)=\left\langle \frac{-s^2}{r_0}\cos\left(\frac{st}{r_0}\right), \frac{-s^2}{r_0}\sin\left(\frac{st}{r_0}\right)\right\rangle = \frac{-s^2}{r_0}\underline{r}(t) \quad (*)$$

Newton $\Rightarrow m\underline{a}(t)=\frac{-GMm}{r_0^3}\underline{r}(t)$ i.e. $\underline{a}(t)=\frac{-GM}{r_0^3}\underline{r}(t)$ $(**)$

Equating $(*)$ and $(**)$ we see

$\frac{-s^2}{r_0^2}\underline{r}(t)=\frac{-GM}{r_0^3}\underline{r}(t)$ i.e. $\underbrace{\frac{s^2}{r_0^2}}_{=\frac{4\pi^2}{T^2}}=\frac{-GM}{r_0^3}$ Hence, $T^2=\frac{4\pi^2 r_0^3}{GM}$



Section 4.2 : Arc Length

For a C^1 path, $\underline{c}(t) = (x(t), y(t))$ for $t \in [a,b]$, you saw that the length of $\underline{c}$ was

$$L(\underline{c}) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{(x'(t))^2 + y'(t)^2}\, dt = \int_a^b \|c'(t)\|\, dt$$

$\hookrightarrow \langle x'(t), y'(t) \rangle$

e.g. $\underline{c}(t) = \langle \cos t, \sin t \rangle$ for $t \in [0, 2\pi)$

Then $c'(t) = \langle -\sin t, \cos t \rangle$

Therefore, $L(\underline{c}) = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} dt = 2\pi.$

In higher dimensions, the length of a C^1 path $\underline{c}(t)$ (for $t \in [a,b]$) is

$L(c) = \int_a^b \|c'(t)\| dt$

i.e. if $\underline{c}(t) = (x(t), \dots, x_n(t))$ then $L(c) = \int_a^b \sqrt{(x_1'(t))^2 + \dots + (x_n'(t))^2}\, dt$

e.g. $\underline{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi$

Then $c'(t) = (-\sin t, \cos t, t)$

Hence, $L(c) = \int_0^{\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\pi$

Sometimes, integrals won't be nice.

e.g. $\underline{c}(t) = (\cos t, \sin t, t^2)$ for $0 \le t \le 2\pi$ Then $c'(t) = \langle -\sin t, \cos t, 2t \rangle$

Hence, $L(c) = \int_0^{\pi} \sqrt{(\sin t)^2 + (\cos t)^2 + (2t)^2}\, dt = \int_0^{\pi} \sqrt{1 + 4t^2}\, dt = 2\int_0^{\pi} \sqrt{t^2 + (\frac{1}{2})^2}\, dt$

we can apply the formula:

$= \frac{1}{2}\left(t\sqrt{t^2+a^2} + a^2 \log(t + \sqrt{t^2+a^2})\right) + C$

Therefore

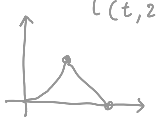
$L(\underline{c}) = 2 \cdot \frac{1}{2}\left[t\sqrt{t^2+(1/2)^2} + (1/2)^2 \log(t + \sqrt{t^2+(1/2)^2})\right]_0^{\pi}$

$\pi = \sqrt{\pi^2 + \frac{1}{4}} + \frac{1}{4}\log(\pi + \sqrt{\pi^2 + 1/4}) - \frac{1}{4}\log(1/2) \approx 10.63$

Sometimes we want to compute lengths of paths that are not C^1 everywhere, e.g.

- As long we can split the path into "C^1 pieces" then we can compute the lengths of these individual pieces and add together the results.
- we refer to these curves as being "piecewise C^1" or as "piecewise smooth"

Ex: $\underline{c}(t) = \begin{cases} (t, t^2) & \text{for } 0 \le t \le 1 \\ (t, 2-t) & \text{for } 1 \le t \le 2 \end{cases}$



Clearly, $\underline{c} \notin C^1$ since we have a corner at $t=1$. But the curve is piecewise C^1 and we can compute its length:

For $0 \le t \le 1$, $\underline{c}'(t) = \langle 1, 2t \rangle$

For $1 \le t \le 2$, $\underline{c}'(t) = \langle 1, -1 \rangle$

So the length of our curve (curve is an image of the path is $\int_0^1 \sqrt{1+4t^2}\, dt + \int_1^2 \sqrt{2}\, dt = \int_0^1 \sqrt{1+4t^2}\, dt + \sqrt{2} = 2\int_0^1 \sqrt{t^2+(1/2)^2}\, dt + \sqrt{2}$

$= 2 \cdot \frac{1}{2}\left[t\sqrt{t^2+(1/2)^2} + (1/2)^2 \log(t + \sqrt{t^2+(1/2)^2})\right]_0^1 + \sqrt{2} \quad \approx 2.89.$

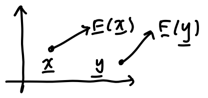
Section 4.3: Vector Field

Definition: A vector field on a subset $A \subseteq \mathbb{R}^n$ is a map

$$\underline{F}: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

(When $n=2$, we refer to a "vector field" in the plane)

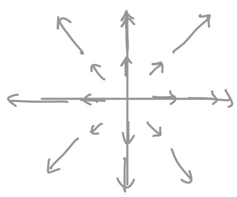
So: a vector field assigns to each point $\underline{x}$ in its domain A a vector $\underline{F}(\underline{x})$. We can picture a vector field by attaching to each point $\underline{x} \in A$ an arrow corresponding to the vector $\underline{F}(\underline{x})$



Note: vector fields are extremely important in physics, engineering etc., where often $\underline{F}(\underline{x})$ represents a physical vector quantity associated with a position $\underline{x}$. e.g.

- the velocity vector field of a fluid
- the gravitational field around a mass.
- the electric field around a charge

Example: let $\underline{V}: \mathbb{R}^2 \to \mathbb{R}^2$ be $\underline{V}(x,y) = \langle x, y \rangle$



$V(x,y) = \langle x, y \rangle$

$\underline{V}(1,0) = \langle 1,0 \rangle$ $\quad V(0,1) = \langle 0,1 \rangle$ $\quad V(1,1) = \langle 1,1 \rangle$

$\underline{V}(-1,0) = \langle -1,0 \rangle$ $\quad V(0,-1) = \langle 0,-1 \rangle$

Gradient vector field

Recall that if $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, then for $\underline{x} \in \mathbb{R}^n$, $\nabla f(\underline{x})$ is the vector $\left\langle \frac{\partial f}{\partial x}(\underline{x}) \dots \frac{\partial f}{\partial x_n}(\underline{x}) \right\rangle$

Hence the map $\nabla f: A: \mathbb{R}^n \to \mathbb{R}^n$ mapping $\underline{x}$ to $\nabla f(\underline{x})$ is a vector field called the "gradient vector field" of f.

Example: let mass M be at the origin in $\mathbb{R}^3$. Recall that the mass m at position vector $\underline{r} = \langle x, y, z \rangle$ experiences a gravitational force of $\underline{F} = \frac{-GMm}{r^3}\underline{r}$ where $r = \|\underline{r}\|$

If we define $V = \frac{-mMG}{r} = \frac{-mMG}{\sqrt{x^2+y^2+z^2}}$

Then $\frac{\partial V}{\partial x} = \frac{\partial}{\partial x} -mMG(x^2+y^2+z^2)^{-1/2} = \frac{mMG}{(x^2+y^2+z^2)^{3/2}} x = \frac{mMG}{r^3} x$

$\frac{\partial V}{\partial y} = \frac{mMG}{r^3} y \qquad \frac{\partial V}{\partial z} = \frac{mMG}{r^3} z$

$-\nabla V = \frac{-mMG}{r^3}\langle x, y, z \rangle = \frac{-mMG}{r^3}\underline{r} = -\underline{F}$

Hence the gravitational field arises as the gradient of V, and we call V the gravitational potential.

<u>Remark</u>: the above is a special case of an important physical principle: a force is conservative (i.e the total work done by the force in moving a particle between 2 points is independent of the path taken) $\Leftrightarrow$ the vector field corresponding to the force arises as the gradient of a function (called the potential)

Not all vector fields arise as the gradient of some smooth function e.g.

Ex: $\underline{F}(x,y) = \langle y, -x \rangle$. There is no function $\mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\nabla f(x,y) = \underline{F}(x,y)$

Proof: suppose for a contradiction that such a function f exists. Then

$\left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle = \nabla f(x,y) = \langle y, -x \rangle$ i.e $\frac{\partial f}{\partial x} = y^{①}$, $\frac{\partial f}{\partial y} = -x^{②}$

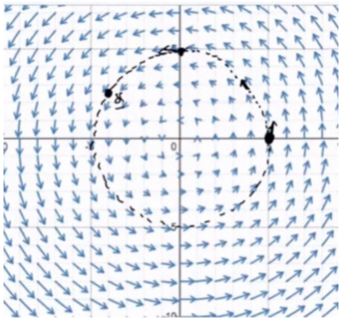
Differentiate ① wrt y, we see $\frac{\partial f}{\partial x \partial y} = 1$ (*)

Differeniate ② wrt x, we see $\frac{\partial f}{\partial y \partial x} = -1$ (**) Remember, if f is a smooth function, its mixed partial derivatives will be equal.

(*) and (**) contradict the fact that mixed 2^{nd} derivatives are equal, hence our initial assertion that such a function f existed must be false.

Flow Line

If you imagine that a given vector field is the velocity vector field for a fluid, then a particle dropped in this fluid would "follow the arrows." The curve traced out by this particle is a "flow line".



Let $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$. We see from the 2D example that at every point $\underline{x}$ in the curve traced out by the particle, the arrow $\underline{F}(\underline{x})$ is tangent to the curve of $\underline{x}$. We take this property to be the defining feature of a flow line (in any dimension):

Definition: If $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, a flow line for $\underline{F}$ (aka streamlines, aka integral curves) is a path $\underline{c}: [a,b] \to \mathbb{R}^n$ s.t.

$$\underline{c}'(t) = \underline{F}(\underline{c}(t)) \quad \forall t \in [a,b]$$

Example: $\underline{F}(x,y) = \langle -y, x \rangle$, show $c(t) = (\cos t, \sin t)$ is a flow line of $\underline{F}$.

Ans: $c'(t) = (-\sin t, \cos t)$

$F(c(t)) = \underline{F}(\cos t, \sin t)$

$= (-\sin t, \cos t)$ //

Example2: $\underline{F}(x,y) = \langle -x, y \rangle$. Check $c(t) = (e^{-t}, e^{t})$ is a flow line for $\underline{F}$.

$y = \frac{1}{x}$

Ans: $c'(t) = \langle -e^{-t}, e^{t} \rangle$

$F(c(t)) = \underline{F}(e^{-t}, e^{t})$

$= (-e^{-t}, e^{t})$ //

Section 4.4: Divergence and Curl

- Divergence takes a vector field and returns a function.
- Curl takes a vector field and returns another vector field

Divergence

Definition: let $\underline{F} = \langle F_1, F_2, F_3 \rangle$ be a vector field.
Then the divergence of $\underline{F}$ is the function div F (or $\nabla \cdot F$) derived by

$$\text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

(more generally, $\text{div } \underline{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$

Example: let $\underline{F}(x,y,z) = \sin x \, \underline{i} + x^2 y \, \underline{j} + z \underline{k}$
Then $\text{div } \underline{F} = \frac{\partial}{\partial x}(\sin x) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(z) = \cos x + x^2 + 1$.

Interpretation: suppose $\underline{F}$ is the velocity vector field of a fluid/gas. Then div $\underline{F}$ at a point $\underline{x}$ measures to what extent the volume of a small region containing $\underline{x}$ would change if moved along the flow lines of $\underline{F}$. i.e.,

div $F > 0 \iff$ expansion

div $F < 0 \iff$ compression.

In other words, div $\underline{F}$ represents the rate of expansion per unit volume under the flow of the gas/fluid.

Example: $\underline{F}(x,y) = \langle x, y \rangle$

As the fluid moves away from the origin, the arrows get larger and the fluid expands.

Indeed,

$\text{div } \underline{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y = 1 + 1 = 2 > 0$

Example 2: $\underline{F}(x,y) = \langle -x, -y \rangle$

$\text{div } \underline{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2 < 0$

As the fluid moves towards the origin, arrows get smaller and the fluid/gas compresses.

Example 3: $\underline{F}(x,y) = \langle -y, x \rangle$

Flow lines are concentric circles, and we should not expect compression or expansion.

$\text{div } \underline{F}(x,y) = -1 + 1 = 0$

Note: it won't always be clear from the picture of a vector field whether is expansion, compression, or neither.

Let $f: \mathbb{R}^n \to \mathbb{R}$. Then $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, so we can take its divergence. Indeed f: if $\mathbb{R}^3 \to \mathbb{R}$, then $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$,
and div f $(= \nabla \cdot \nabla f) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
we call this the laplacian of f (more generally, lapacian of f is given by $\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$).
We will denote the laplacian of f by Δf (note: people often write $\nabla^2 f$ for the lapacian of f, but we already used this notation for the hessian of f.).

Curl

Definition: If $\underline{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field on $\mathbb{R}^3$, then the curl of $\underline{F}$, denoted curl $\underline{F}$ or $\nabla \times F$, is the vector field

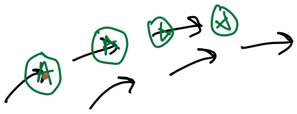
$$\text{curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}$$

Note: If asked to compute the curl of a 2D vector field $\underline{F} = \langle F_1, F_2 \rangle$, we implicitly mean the curl of $\langle F_1, F_2, 0 \rangle$

Example let $\underline{F}(x,y,z) = e^{xy}\underline{i} + \sin t \, \underline{j} + xyz \underline{k}$
then $\text{curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin t & xyz \end{vmatrix} = (xz - \cos t)\underline{i} + (0 - yz)\underline{j} + (0 - xe^{xy})\underline{k} = (xz \cos t)\underline{i} - yz\underline{j} - xe^{xy}\underline{z}$.

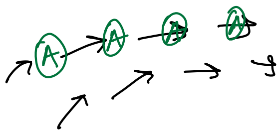
Interpretation: the curl of a velocity vector field $\underline{F}$ at a point is a measure of the local rotation caused by the fluid at that point.

E.g, in a 2D fluid, at $\underline{x}$ place a small solid disc printed with the letter 'A'.



If you released the disc and the disc started to rotate around its axis as it moved with the fluid, then the curl of $\underline{F}$ at $\underline{x}$ would be non-zero.

If the disc did not rotate about its axis, then the curl of $\underline{F}$ at $\underline{x}$ would be $\underline{0}$.



Note the curl of $\underline{F}$ does not care about the "global rotation" of the fluid, as the following example shows.

Ex: $\underline{F}(x,y,z) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

$$\text{curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \underline{k}\left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \right) = 0\underline{k} = \underline{0}$$

Let $f: \mathbb{R}^n \to \mathbb{R}$, then ∇f is a vector field. So what is curl ∇f?

$$\text{curl } \nabla f = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right)\underline{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right)\underline{k} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)\underline{k}.$$

Therefore, if $f \in C^2$, curl $\nabla f = 0$.

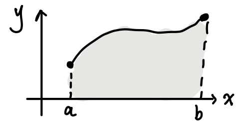
This gives another way of showing that a given vector field is not the gradient of a C^2 function

e.g. $\underline{F}(x,y,z) = (y, -x)$, then curl $\underline{F} = -2\underline{k} \neq 0$, hence by the above, $\underline{F}$ cannot be the gradient vector field of a C^2 func.

$\boxed{\text{Chapter 5: Double and Triple Integrals.}}$

$\boxed{\text{Section 5.1: Introduction}}$

Integrals in 2D

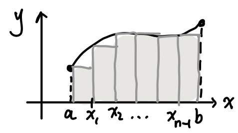


Recap: suppose $f:[a,b] \to \mathbb{R}$, with $f \geq 0$.

To find the area under f, we start by approximating using blocks: for any n,

we can partition $[a,b]$ into n subintervals of equal width,

$[\underset{=a}{x_0}, x_1], [x_1, x_2], \ldots, [x_{n-1}, \underset{=b}{x_n}]$ (the width of each of these subintervals is $\frac{b-a}{n}$).



Then we compute the sum $\sum_{i=0}^{n-1} f(x_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$ $\quad (*)$

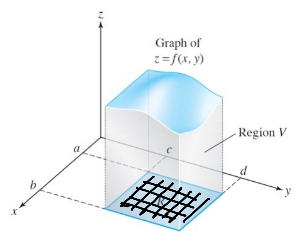
As we take n larger and larger (i.e. "finer partition"), then the sum $(*)$ should give us a closer approximation to the true area under the graph of f.

If the limit exists in $(*)$ as $n \to \infty$, we say that f is integrable and we write

$\int_a^b f(x)\,dx$

o denote this limit,

Double Integrals as Volume

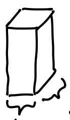


Suppose we have a function $f \geq 0$ of 2 variables, say derived on a rectangle $R \subseteq \mathbb{R}^2$ i.e.

$f: R \subseteq \mathbb{R}^2 \to \mathbb{R}$

We could ask: what is the volume under the graph of f?

One can use a similar approximation argument to estimate the volume above. If $R: [a,b] \times [c,d]$, we start by partitioning R into smaller rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ where x_i are obtained by partitioning $[a,b]$ into n subintervals of equal widths, and the y_j are obtained by partitioning $[c,d]$ into n subintervals of equal length $\frac{d-c}{n}$.



$\frac{b-a}{n} = \Delta x$

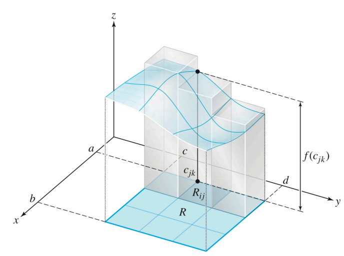
$\frac{d-c}{n} = \Delta y$

$\sum_{i,j=0}^{n-1} f(x_i, y_j)\,\Delta x \Delta y \quad (**)$

As we take n larger, $(**)$ should provide a better approximation to the volume under the graph of f.

If the limit as $n \to \infty$ in $(**)$ exists, we call f integrable and write

$\iint_R f(x,y)\,dxdy$ to denote its limit.

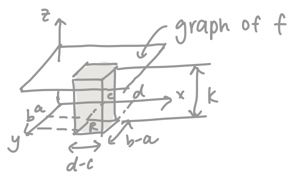


Section 5.2: Double Integral over a Rectangle

Q: How do we calculate $\iint_R f(x,y)\,dx\,dy$?

With the volume interpretation, we don't need to integrate at all...

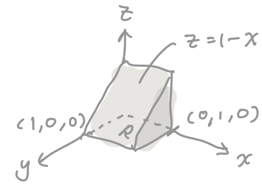
Ex 1: suppose $f(x,y) = k > 0$. Then $\iint_R f(x,y)\,dx\,dy$, where $R = [a,b]\times[c,d]$ is just the volume of the cuboid bounded by the graph of f and the rectangle $R \subseteq \mathbb{R}^2$.



So, $\iint_R f(x,y)\,dx\,dy = (b-a)(d-c)k$.

Ex 2: suppose $f(x,y) = 1-x$, $R = [0,1]\times[1,0]$ what is $\iint_R f(x,y)\,dx\,dy$?

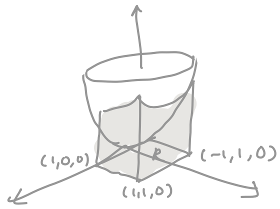
The graph of f is a plane in $\mathbb{R}^3$ intersecting the xy plane along the line $x=1$, hence:



so the answer is $1/2$.

Ex 3: $f(x,y) = x^2+y^2$; $R = [-1,1]\times[0,1]$.

Then, $\iint_R f(x,y)\,dx\,dy = \iint_R (x^2+y^2)\,dx\,dy$ is the volume:



The most common method for these sort of integrals is to write $\iint_R f(x,y)\,dx\,dy$ as an "integrated integral."

Fubini's theorem (version 1)

Suppose $f: R \subseteq \mathbb{R}^2$ is continuous, $R = [a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{\text{treat } y \text{ as a constant to compute this.}} dy$$

$$= \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{\text{treat } x \text{ as a constant to compute this.}} dx$$

} iterated integrals.

Ex 3 continued:

We have $R = [-1,1]\times[0,1]$ and $\iint_R (x^2+y^2)\,dx\,dy$

$= \int_{y=0}^1 \left(\int_{x=-1}^1 (x^2+y^2)\,dx\right) dy$

$= \int_{y=0}^1 \left[\frac{x^3}{3} + y^2x\right]_{-1}^1 dy$

$= \int_{y=0}^1 \left(\frac{1}{3} + y^2 + \frac{1}{3} + y^2\right) dy = \int_{y=0}^1 \left(2y^2 + \frac{2}{3}\right) dy$

$= \frac{2}{3}y^3 + \frac{2}{3}y \Big|_0^1 = \left(\frac{2}{3}+\frac{2}{3}\right) - (0)$

$= \frac{4}{3}$

Could also compute $\iint_R (x^2+y^2)\,dx\,dy$ as

$= \int_{x=-1}^1 \left(\int_{y=0}^1 (x^2+y^2)\,dy\right) dx$

$= \int_{x=-1}^1 \left[x^2y + \frac{1}{3}y^3\right]_0^1 dx = \int_{-1}^1 \left(x^2 + \frac{1}{3}\right) dx$

$= \frac{1}{3}x^3 + \frac{1}{3}x \Big|_{-1}^1 = \left(\frac{1}{3}+\frac{1}{3}\right) - \left(\frac{-1}{3} - \frac{1}{3}\right)$

$= \frac{4}{3}$

Sometimes we leave out parentheses:

$\int_c^d \int_a^b f(x,y)\,dx\,dy$ means $\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$,

$\int_a^b \int_c^d f(x,y)\,dy\,dx$ means $\int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$.

Ex 1: let $f(x,y) = \frac{x}{y} + \frac{y}{x}$

$R = [2,4]\times[1,2]$

compute $\iint_R f(x,y)\,dx\,dy$.

$\int_{y=1}^2 \int_{x=2}^4 \left(\frac{x}{y} + \frac{y}{x}\right) dx\,dy = \int_{y=1}^2 \left[\frac{x^2}{2y} + y\log|x|\right]_2^4 dy$

$= \int_{y=1}^2 \left[\frac{8}{y} + y\log 4 - \frac{2}{y} - y\log 2\right] dy = \int_{y=1}^2 \left(\frac{6}{y} + y\log 2\right) dy$

$= \left[6\log|y| + \frac{y^2}{2}\log 2\right]_1^2 = 6\log 2 + 2\log 2 - \left(6\log 1 + \frac{1}{2}\log 2\right)$

$= \frac{15}{2}\log 2$

Ex 2: $\int_0^{\pi/4}\int_0^{\pi/4} \tan x \sec^2 y\,dx\,dy$ note: $\tan x = \frac{\sin x}{\cos x} = -(\log\cos x)$

$= \int_0^{\pi/4} \left[(-\log|\cos x|)\sec^2 y\right]_0^{\pi/4} dy$

$= \int_0^{\pi/4} \left(-\log\frac{\sqrt{2}}{2}\sec^2 y\right) dy = \int_0^{\pi/4} \log\sqrt{2}\sec^2 y\,dy$

$= \left[(\tan y)\log\sqrt{2}\right]_0^{\pi/4}$

$= \log\sqrt{2}$

Generally,

$\int_c^d\int_a^b g(x)h(y)\,dx\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$

Properties of the double integral over rectangles

suppose f, g are integrable over $R \subseteq \mathbb{R}^2$ and let $c \subseteq \mathbb{R}$ be constant. Then,

i) Linearity: $f+g$ is integrable with $\iint_R (f(x,y)+g(x,y))\,dx\,dy = \iint_R f(x,y)\,dx\,dy + \iint_R g(x,y)\,dx\,dy$

ii) Homogeneity: cf is integrable with $\iint_R cf(x,y)\,dx\,dy = c\iint_R f(x,y)\,dx\,dy$

iii) Monotonicity: if $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then $\iint_R f(x,y)\,dx\,dy \geq \iint_R g(x,y)\,dx\,dy$

iv) Additivity: if $Q \subseteq \mathbb{R}^2$ is a rectangle and $Q = \bigcup_{i=1}^{M} R_i$ where $R_1, \dots, R_m$ are pairwise disjoint rectangles, and if f is integrable over each R_i, then f is integrable over Q with $\iint_Q f(x,y)\,dx\,dy = \sum_{i=1}^{M} \iint_{R_i} f(x,y)\,dx\,dy$

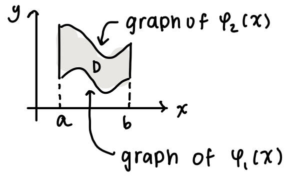
v) $\left|\iint_R f(x,y)\,dx\,dy\right| \leq \iint_R f(x,y)\,dx\,dy$

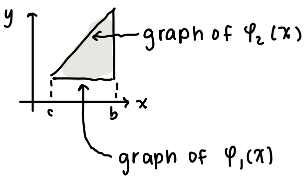
5.3 Double Integral Over General Regions

Q: How do we integrate over regions other than rectangles?

y-simple regions

Let's suppose domain of f looks like:





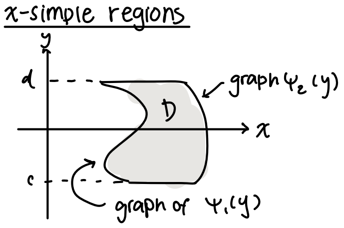
More precisely, we suppose D can be written as the set of all points $(x,y) \in \mathbb{R}^2$ s.t. $a \leq x \leq b$ and $\varphi_1(x) \leq y \leq \varphi_2(x)$, where $\varphi_1, \varphi_2 : [a,b]\ \mathbb{R}$ are two functions with $\varphi_1(x) < \varphi_2(x)$ for all $x \in [a,b]$. We call D a "y-simple region."

Iterated Integrals

We call f integrable if for a rectangle containing D, the function $f^*: R \subseteq \mathbb{R}^2 \to \mathbb{R}$, derived by
$$f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \setminus D \quad \text{(rectangle outside set D.} \end{cases}$$
is integrable on R. (note: this is always going to be the case if f is continuous on D.

One then defines
$$\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA$$ and can show $$\iint_D f(x,y)\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

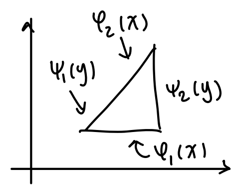
x-simple regions



If a region D can be written as the set of points (x,y) s.t $c \leq y \leq d$ and $\psi_1(y) \leq x \leq \psi_2(y)$ $\forall y [c,d]$, we call D x-simple.

We define the integral $\iint_D f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$ (at least for cts f on D)

Elementary Regions

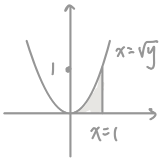


If a region is both x-simple and y-simple, e.g. a triangle, we call this region "simple" or "elementary." In the case of D being a simple region, we can use either the x-simple procedure or the y-simple procedure.

Section 5.4: Changing the Order of Integration

so, $\iint_D f(x,y)\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$

Example 1: $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy \quad (*)$ order swapped.



$\sim \; y = x^2$

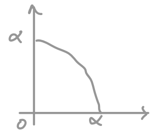
$0 \le x \le 1$

$0 \le y \le x^2$

$\varphi_1(x)$ $\varphi_2(x)$

$(*) = \int_0^1 \int_0^{x^2} e^{x^3}\,dy\,dx$

Example 2: $\int_0^\alpha \int_0^{(\alpha^2-x^2)^{1/2}} (\alpha^2-y^2)^{1/2}\,dy\,dx \quad (**)$



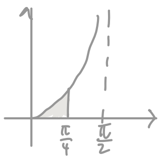
Region is given by

$0 \le y \le \alpha$

$0 \le x \le (\alpha^2-y^2)^{1/2}$

Therefore $(**) = \int_0^\alpha \int_0^{(\alpha^2-y^2)^{1/2}} (\alpha^2-y^2)^{1/2}\,dx\,dy$

Ex 3: $\int_0^1 \int_{\tan^{-1}(y)}^{\pi/4} \sec^5 x\,dx\,dy \quad (***)$



$0 \le x \le \frac{\pi}{4}$

$0 \le y \le \tan(x)$

$(***) = \int_0^{\frac{\pi}{4}} \int_0^{\tan x} \sec^5 x\,dy\,dx$

Estimating Integrals

Recall that if, $f(x,y) \le g(x,y)$ for all $(x,y) \in D$, then $\iint_D f(x,y)\,dA \le \iint_D g(x,y)\,dA$

Therefore, if m is the minimum value attained by f on D, and M is the maximum value,
$\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$, i.e. $m\text{Area}(D) \le \iint_D f(x,y)\,dA \le M\text{Area}(D)$.

Dividing through by Area(D) we see $m \le \underbrace{\frac{1}{\text{Area}(D)} \iint_D f(x,y)\,dA}_{\text{the "mean value" of } D.} \le M$

Section 5.5: The Triple Integral

Definition

Let $B=[a,b]\times[c,d]\times[p,q]\subseteq\mathbb{R}^3$ be a box/cuboid in $\mathbb{R}^3$ and $f: B\subseteq\mathbb{R}^3\to\mathbb{R}$.

We can define integrability of f and the triple integral

$\iiint_B f(x,y,z)\,dx\,dy\,dz$ $(*)$ (also written as $\iiint_B f(x,y,z)\,dV$, or $\iiint_B f\,dV$)

In a similar way to double integral case, using finite sums ("rieman sums") and taking the limit.

We skip the details of the definition of $(*)$ and focus on calculations, assuming e.g. that continuous functions on B are integrable.

Fubini's theorem (version 2)

The 3D version of Fubini's Theorem allows us to compute $(*)$ using iterated integrals.

If f is integrable on B, then any of the 6 possible integrable integrals are equal to $(*)$. i.e,

$$\iiint_B f(x,y,z)\,dV$$
$$=\int_p^q\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$
$$=\int_p^q\int_a^b\int_c^d f(x,y,z)\,dy\,dx\,dz$$
$$=\dots$$

Ex: $B=[0,1]\times[-1,0]\times[0,\tfrac12]$ and $f(x,y,z)=x^2y+xyz$

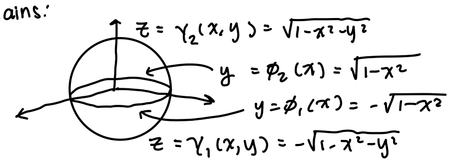
$$\iiint_B(x^2y+xyz)\,dV$$
$$=\int_0^{1/2}\int_{-1}^0\int_0^1(x^2y+xyz)\,dx\,dy\,dz \quad =\int_0^{1/2}\int_{-1}^0\left[\tfrac13x^3y+\tfrac12x^2yz\right]_0^1 dy\,dz$$
$$=\int_0^{1/2}\int_{-1}^0\left(\tfrac13y+\tfrac12yz\right)dy\,dz=\int_0^{1/2}\left[\tfrac16y^2+\tfrac14y^2z\right]_{-1}^0 dz$$
$$=\int_0^{1/2}\left(\tfrac{-1}{6}-\tfrac{z}{4}\right)dz=\left[\tfrac{-z}{6}-\tfrac{z^2}{8}\right]_0^{1/2}=\tfrac{-1}{12}-\tfrac{1}{32}=\tfrac{-11}{96}$$

Instead of the $dx\,dy\,dz$ choice, we could have picked $dy\,dz\,dx$

Integrals over Elementary Regions

A region in $\mathbb{R}^3$ is called "elementary" if the domain of one of the variables (e.g. z) can be described as being between 2 functions of the other 2 variables (e.g. x and y), and the domain D of the other 2 variables is simple (in the sense considered for double integrals.)

Let's describe the following as elementary domains:



i) The closed unit ball in $\mathbb{R}^3$

Ans: (x,y,z) s.t. $x^2+y^2+z^2\le1$

Then $-\sqrt{1-x^2-y^2}\le z\le\sqrt{1-x^2-y^2}$

$\left.\begin{array}{l}-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\\ -1\le x\le1\end{array}\right\}$ so we've described the domain D of x,y as a y-simple region here.

Alternatively, we could describe as

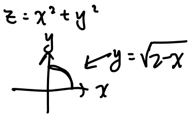
$-\sqrt{1-x^2-z^2}\le y\le\sqrt{1-x^2-z^2}$

$-\sqrt{1-z^2}\le x\le\sqrt{1-z^2}$

$-1\le z\le1$

ii) Let W be the region bounded by the planes $x=0$, $y=0$, $z=2$, and the paraboloid $z=x^2+y^2$

Everything inside the paraboloid is s.t. $z\ge x^2+y^2$, so,



$x^2+y^2\le z\le2$

$\left.\begin{array}{l}0\le y\le\sqrt{2-x^2}\\ 0\le x\le\sqrt2\end{array}\right\}$ y-simple domain.

Alternatively,

$0\le x\le\sqrt{z-y^2}$

$\left.\begin{array}{l}0\le y\le\sqrt{z}\\ 0\le z\le2\end{array}\right\}$ y-simple domain in the yz plane.

Examples

If W is an elementary region in $\mathbb{R}^3$ in the form

$f_1(x,y) \le z \le f_2(x,y)$

$\varphi_1(x) \le y \le \varphi_2(x)$

$a \le y \le b$

and if $f: W \subseteq \mathbb{R}^3 \to \mathbb{R}$ is integrable on W, then

$$\iiint_W f\,dV = \int_a^b \int_{\varphi_1}^{\varphi_2} \int_{f_1}^{f_2} f(x,y,z)\,dz\,dy\,dx$$

An anagolous form holds for elementary domains of other types, e.g. if

$f_1(y,z) \le x \le f_2(y,z)$

$\varphi_1(y) \le z \le \varphi_2(y)$

$c \le y \le d$

then

$$\iiint_W f(x,y)\,dV = \int_c^d \int_{\varphi_1}^{\varphi_2} \int_{f_1}^{f_2} f(x,y,z)\,dx\,dz\,dy$$

Volume

Recall: $\iint_{D \subseteq \mathbb{R}^2} dA = \text{area}(D)$

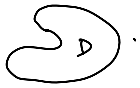
Now: $\iiint_{W \subseteq \mathbb{R}^3} dV = \text{volume}(W)$

L

Chapter 6: Change of variables and integration

Motivation: suppose we want to compute the double integral:

$\iint_D f\,dA$ (*)



where D^* looks like

One way to do this is to transform D^* into a "nicer" region, e.g., a disc via some mapping T.

We will see that (*) can be calculated by computing "suitable" integrals on D.

We first need to understand the geometry of closely diffable maps $T: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$

Section 6.1: The Geometry Map from $\mathbb{R}^2 \to \mathbb{R}^2$

Maps of One Region to Another

For such a map T, we write $T(D^*)$ or D to denote the image set, i.e, the set of points $T(x^*, y^*)$ where (x^*, y^*) are the coords of points in D^*.

Ex: Let $D^* = [0,1] \times [0, 2\pi] \subseteq \mathbb{R}^2$, so that points in D^* are of form (r,θ) for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$

Define $T: D^* \to \mathbb{R}^2$ by

$T(r,\theta) = (r\cos\theta, r\sin\theta)$

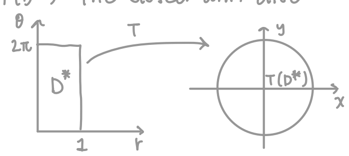
What is $T(D^*)$?

Answer: let $(x,y) = (r\cos\theta, r\sin\theta)$ be an image point. Then,

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$
$$\le 1$$

Therefore, $T(D^*)$ is contained in the closed unit disc.

Conversely every point (x,y) in the closed unit disc can be written as $(x,y) = (r\cos\theta, r\sin\theta)$ for some $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Therefore, $T(D^*)$ = the closed unit disc.

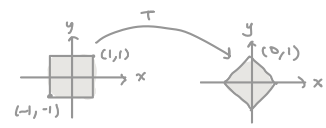


We call T the "polar coordinates change of variables map": it takes the rectangle D^* (in polar coordinates) to the unit disc (in Euclidean coords).

Images of Maps

Theorem: Let A be a 2x2 matrix s.t. det $A \neq 0$, and derive the linear mapping $T: \mathbb{R}^2 \to \mathbb{R}$ by $T\binom{x}{y} = A\binom{x}{y}$

Then T maps parallelograms to parallelograms and vertices to verticies.



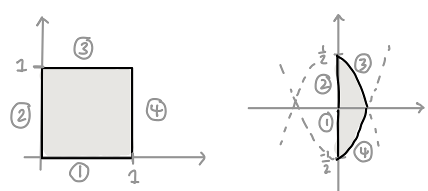
Ex: let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be $T(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ and let $D^* = [-1,1] \times [-1,1]$. What is $T(D^*)$?

Ans: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Hence, D^* will be mapped to some parallelogram with verticies $T\begin{pmatrix} 1 \\ 1 \end{pmatrix}, T\begin{pmatrix} 1 \\ -1 \end{pmatrix}, T\begin{pmatrix} -1 \\ 1 \end{pmatrix}, T\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Ex 2: let $D^* = [0,1] \times [0,1]$ and let $T: D^* \to \mathbb{R}^2$ be $T(u,v) = (uv, \frac{1}{2}(v^2 - u^2))$



We call T the "parabolic coordinate change of variable map"

① Let $\underline{c}_1(t) = (t, 0)$ for $t \in [0,1]$

$T(\underline{c}_1(t)) = \left(t \cdot \emptyset, \frac{1}{2}\left((0)^2 - (t)^2\right)\right) = \left(0, -\frac{1}{2}t^2\right)$

② Let $c_2(t) = (0, t)$ for $t \in [0,1]$

$T(\underline{c}_2(t)) = \left(0, \frac{1}{2}t^2\right)$

③ Let $c_3(t) = (t, 1)$ for $t \in [0,1]$

$T(\underline{c}_3(t)) = \left(t, \frac{1}{2} - \frac{1}{2}t^2\right)$

④ Let $c_4(t) = (1, t)$ $t \in [0,1]$

$T(\underline{c}_4(t)) = \left(t, \frac{1}{2}t^2 - \frac{1}{2}\right)$

One-to-one Map

Drawing the deformations caused by a mapping T does not give us the full picture, and sometimes we'll need to know more about the properties of T to use in the change of variables formula.

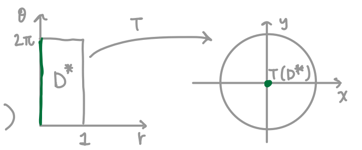
Definition: we call $T: D^* \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ one-to-one (or "injective") if distinct points by T, i.e., if $(u,v), (r,s) \in D^*$ and $T(u,v) = T(r,s)$, then $u=r$, $v=s$.

Ex 1: $T(x,y) = (x^2+y^2, y^4)$ $T: \mathbb{R}^2 \to \mathbb{R}^2$.

this is not one-to-one e.g. $T(1,-1)$ and $T(1,1)$ are both equal to $(2,1)$.

Ex 2: $D^* = [0,1] \times [0, 2\pi]$ and $T: D^* \to \mathbb{R}^2$, $T(r,\theta) = (r\cos\theta, r\sin\theta)$

Then T is not 1-1 because all points of the form $(0,\theta)$ (where $\theta \in [0,2\pi]$) are mapped to the same point (origin). See diagram $\rightarrow$



However, T considered as a map: $(0,1] \times [0, 2\pi)$ is one-to-one because every point in the closed unit disc minus the origin can be uniquely specified by a distance $0 < r \leq 1$ from the origin and an angle $0 \leq \theta < 2\pi$ from the axis

Ex 3: $\underbrace{T: [-1,1] \times [-1,1]}_{D^*} \to \mathbb{R}^2$, $T(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

This is one-to-one: suppose $T(x,y) = T(x',y')$ i.e. $\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = \left(\frac{x'+y'}{2}, \frac{x'-y'}{2}\right)$

i.e. $x+y = x'+y'$

$x-y = x'-y'$

$\Rightarrow 2x = 2x' \Rightarrow x = x' \Rightarrow y = y'$

Onto Maps

We are also interested in a sort of converse question to finding the image of set D^* under a map T given a set of $D \subseteq \mathbb{R}^2$ and a mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$, can we find set $D^* \subseteq \mathbb{R}^*$ s.t. $T(D^*) = D$?

In this case, we call the mapping $T: D^* \to D$ onto (or "subjective"), i.e, for every $(x,y) \in D$, there exists at least one point $(u,v) \in D^*$ s.t. $T(u,v) = (x,y)$.

Fact: for linear mapping T, i.e. those that can be written in the form $T(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix}$ for some matrix A, being one-to-one and onto are equivalent, and occurs if and only if $\det A \neq 0$.

Ex 1. let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be $T(u,v) = (u,0)$.

let $D = [0,1] \times [0,1]$. Is there a subset $D^* \subseteq \mathbb{R}^*$ s.t. $T: D^* \to D$ is onto?

Ans: No: $T(x,v) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$\hookrightarrow$ det A.

Alternatively, we see that T maps the whole of $\mathbb{R}^2$ to one axis, hence cannot be onto the square D for any D^* c

Section 6.2: The Change in Variables Theorem

Central problem: let D, D^* and suppose $T: D^* \to D$ is onto (i.e. $T(D^*) = D$) and diffable. Also let $f: D \to \mathbb{R}$ be integrable. Can we express
$\iint_D f(x,y)\,dA$
in terms of an integral $f \circ T$ over D^*.

$$\begin{tikzcd} D^* \arrow[r, "T"] \arrow[rr, bend right, "f \circ T"'] & D \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Answer: Yes, under certain conditions:

If we write T in components as $T(u,v) = (x(u,v), y(u,v))$

coords on D^* ; coords on D

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v))\,|?|\,du\,dv \quad (*)$$

where $|?|$ is a certain factor which takes into account distortions of area between the two coordinate systems.

Note: T is referred to as a change in variables map or change of coords.

The factor "?" is the "Jacobian determinant."

Jacobian Determinants

Definition: let $T: D^* \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map given by
$T(u,v) = (x(u,v), y(u,v))$.
The Jacobian determinant of T, written $\frac{\partial(x,y)}{\partial(u,v)}$, is the determinant of the derivative matrix $D\underline{T}(u,v)$, i.e.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
← could be neg. or pos.

Example 1: Let T be the polar coordinant change of variables, i.e.

$T(r,\theta) = (r\cos\theta, r\sin\theta)$ ($u = r$, $v = \theta$, $x = r\cos\theta$, $y = r\sin\theta$)

Note: polar coord change in variables may not be 1-1 (e.g. if $D^* = [0,a] \times [0,2\pi]$) but we assume this formula still holds

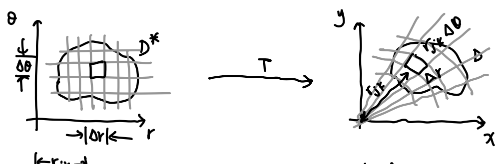
Then,
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial y} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$
"Jacobian Determinant"

note: under $(*)$ we'd have
$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(r,\theta), y(r,\theta))\, r\,dr\,d\theta$$
region in (x,y) plane ; region in (r,θ) plane

Example 2: $T(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = \frac{-1}{2}$$

Example 3: $T(u,v) = (uv, \frac{1}{2}(v^2 - u^2))$
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ -u & v \end{vmatrix} = v^2 + u^2$$

Q: why is $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ the "correct" factor "$|?|$"?



Area $(D) = \iint_D dx\,dy$

Also, Area $(D) \approx \sum_{j,k} r_{jk}\,\Delta r\,\Delta\theta = \iint_{D^*} r\,dr\,d\theta$

Jacobian determinant.

Change in Variables (for Double Integrals)

Let D and D^* be elementary regions in $\mathbb{R}^2$ and let $T: D^* \to D$ be 1 to 1, onto, and C^1. Then for integrable $f: D \to \mathbb{R}$, we have
$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$
absolute value

Note: This is a multivariable generalization of integration by substitution.

E.g. $\int_c^b f(x(u))\frac{dx}{du}\,du = \int_{x(a)}^{x(b)} f(x)\,dx$

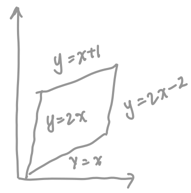
The usual setup: we want to calculate $\iint_D f(x,y)\,dx\,dy$ where f, D are given. We choose a suitable change of variable map $T(u,v) = (x(u,v),\ y(u,v))$ s.t. $\iint_{D^*} f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ is easier to compute.

Example 1: let P be a parallelogram bounded by the lines $y=2x$, $y=x$, $y=2x-2$, $y=x+1$. Find $\iint_P xy\,dx\,dy$

by making the change of variables $x=u-v$, $y=2u-v$ (i.e, $T(u,v) = (u-v,\ 2u-v)$)

T is linear since $T(u,v) = \underbrace{\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}}_{A}\begin{pmatrix} u \\ v \end{pmatrix}$

with $\det A = 1 \neq 0$, hence T is 1-1.



We see $T(0,0) = (0,0)$

$T(1,0) = (1,2)$

$T(0,-2) = (2,2)$

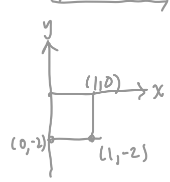
$T(1,-2) = (3,4)$

we have $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$

Therefore

$\iint_P xy\,dx\,dy = \iint_{P^*} (u-v)(2u-v)\,1\,du\,dv = \int_0^{-2}\int_0^1 (2u^3 - 3uv + v^2)\,du\,dv$

$= \cdots = 7$

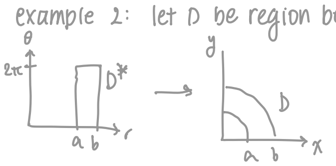


More Examples

example 1: calculate the area of a disc with radius a.

$\iint_D dx\,dy = \int_{D^*} r\,dr\,d\theta = \int_0^{2\pi}\int_0^a r\,dr\,d\theta = 2\pi \int_0^a r\,dr = 2\pi\left[\frac{r^2}{2}\right]_0^a = \pi a^2$

example 2: let D be region between two circles of radius a and b, where $0<a<b$. Find $\iint \log(x^2+y^2)\,dx\,dy$ using polar coord transformation.

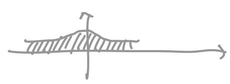


$\iint_D \log(x^2+y^2)\,dx\,dy = \int_a^b\int_0^{\pi/2} \log(r^2)\,r\,d\theta\,dr = \frac{\pi}{2}\int_a^b r\log(r^2)\,dr$

$= \pi\int_a^b r\log r\,dr = \pi\left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]_a^b = \pi\left[\frac{b^2}{2}\log b - \frac{a^2}{2}\log a - \frac{1}{4}(b^2-a^2)\right]$

Note: it is most natural to convert to polar coord if our set D has circular symmetry (i.e. radial symmetry) and/or the function being integrated involves terms like $f(x^2+y^2)$

example 3: show $I := \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ "gaussian integrals"



pf: let D_a be the disc $x^2+y^2 \le a^2$.

consider $\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi(1-e^{-a^2})$ dummy variable

Taking $a\to\infty$ we see that $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \iint_{\mathbb{R}^2} e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{\mathbb{R}} e^{-x^2}dx\right)\left(\int_{\mathbb{R}} e^{-y^2}dy\right) = \underbrace{\left(\int_{\mathbb{R}} e^{-x^2}dx\right)}_{I}\left(\int_{\mathbb{R}} e^{-x^2}dx\right) = I^2$

Triple Integrals

Let $T: W \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be C^1 with $T(u,v,w) = (x(u,v,w),\ y(u,v,w),\ z(u,v,w))$ ($x y z$ are functions of the coords u, v, w via T.

The Jacobian Determinant of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

← measures how the transformation of T distorts volume.

Example: recall spherical coordinates: $x = \rho\sin\varphi\cos\theta$, $y = -\rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ -\sin\varphi\sin\theta & -\rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & \rho\sin\varphi \end{vmatrix}$$

$= \sin\varphi\cos\theta(-\rho^2\sin^2\varphi\cos\theta) + \rho\sin\varphi\sin\theta(-\rho\sin^2\varphi\sin\theta - \rho\cos^2\varphi\sin\theta)$

$+ \rho\cos\varphi\cos\theta(\rho\sin\varphi\cos\varphi\cos\theta)$

$= -\rho^2\sin\varphi\left[\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi\sin^2\theta + \cos^2\varphi\cos^2\theta\right]$

$= -\rho^2\sin\varphi\left[(\sin^2\varphi + \cos^2\varphi)(\cos^2\theta) + (\cos^2\varphi + \sin^2\varphi)(\sin^2\theta)\right]$

$= -\rho^2\sin\varphi.$

Triple Integrals for Change of Variables

Let W and W^* be elemenrary regions in $\mathbb{R}^3$ and suppose $T: W^* \to W$ is C^1, 1-1, and onto, with

$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$

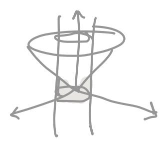
Then for integrable $f: W \to \mathbb{R}$, $\iiint_W f(x,y,z)\,dx\,dy\,dz$

$\iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw.$

Note: the the case of cylindrical coordinates,

$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, t)\, r\,dr\,d\theta\,dz.$

Example: let W be the region where $x^2+y^2 \le 1$, $z \ge 0$ and below cone $z = \sqrt{x^2+y^2}$



Find $\iiint_W z\,dx\,dy\,dz$ using cylindrical change of coords

Our domain: $0 \le r \le 1 \leftarrow x^2+y^2 \le 1$

$0 \le \theta \le 2\pi$

$0 \le z \le r = \sqrt{x^2+y^2}$

So the integral is $\int_0^{2\pi}\int_0^1\int_0^r z\,r\,dz\,dr\,d\theta$

$= 2\pi \int_0^1 r \left[\frac{z^2}{2}\right]_0^r dr = \frac{\pi}{4}$

Spherical Coordinates:

$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2 \sin\varphi\; d\rho\,d\theta\,d\varphi$

Ex: let W = unit ball in $\mathbb{R}^3$. Find $(*) = \iiint_W e^{(x^2+y^2+z^2)^{3/2}} dV$.

Ans: W^* is $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$

$(*) = \int_0^1\int_0^\pi\int_0^{2\pi} e^{\rho^3} \rho^2 \sin\, d\theta\, d\varphi\, d\rho = 2\pi \int_0^1\int_0^\pi \rho^2 e^{\rho^3} \sin\varphi\; d = \frac{4\pi}{3}(e-1)$

Section 6.3: Applications

Let W be a 3D region, interpretted physically as a solid. Suppose the mass density of W at $(x,y,z) \in W$ is $\rho(x,y,z)$. Then the mass W is $\iiint_W \rho(x,y,z)\,dx\,dy\,dz$. and the center of mass of W is the point $(\bar{x},\bar{y},\bar{z})$ where

$$\bar{x} = \frac{\iiint_W x\rho(x,y,z)\,dx\,dy\,dz}{\text{Mass}(W)}, \quad \bar{y} = \frac{\iiint_W y\rho(x,y,z)\,dx\,dy\,dz}{\text{Mass}(W)}, \quad \bar{z} = \frac{\iiint_W z\rho(x,y,z)\,dx\,dy\,dz}{\text{Mass}(W)}$$

Note: generalization of COM of n masses $m_1,\dots,m_n$ at point $x_1,\dots,x_n$ on the x-axis given $\bar{x} = \dfrac{\sum_{x=1}^{n} m_i x_i}{\sum_{x=1}^{n} m_i}$

Note: if we wanted to find the C.O.M of a 2D region, then forget about z and $\bar{z}$.

Example:

Consider the solid hemisphere given by $x^2+y^2+z^2 \le 1$, $z \ge 0$ with uniform density $\rho = 1$, where is C.O.M? $\bar{x} = \bar{y} = 0$

$\text{mass}(W) = \iiint_W 1\,dV = \text{Vol}(W) = \frac{2\pi}{3}$

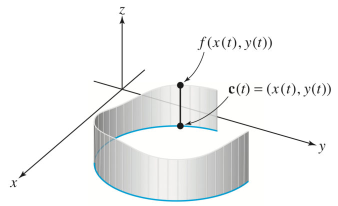
Also $\iiint_W z\,dV = \int_0^1\int_0^{2\pi}\int_0^{\pi/2} \rho\cos\varphi\,\rho^2\sin\varphi\,d\varphi\,d\theta\,d\rho = \frac{\pi}{4}$

So, $\bar{z} = \frac{\pi/4}{M} = \frac{\pi/4}{2\pi/3} = \frac{3}{8}$

Chapter 7: Integrals over paths and surfaces

- Finding integrals and vector fields over general curves and surfaces.

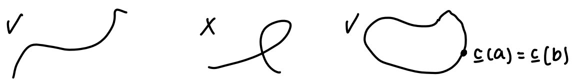
Section 7.1: The Path Integral



Suppose we have a path $\underline{c}: [a,b] \to \mathbb{R}^2$ and $f: \mathbb{R}^2 \to \mathbb{R}$. If $f \geq 0$, we can talk of the area of the "fence" with base $\underline{c}(t) = (x(t), y(t))$ in the xy and the height given by $f(\underline{c}(t)) = f(x(t), y(t))$ (i.e. the area in between $\underline{c}(t)$ and the graph restricted to $\underline{c}$):

The area is given by $\int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$ (can justify this using Rieman's sum type argument).

Definition: a <u>simple curve</u> $C \subseteq \mathbb{R}^n$ is the image of C^1 path $\underline{c}: [a,b] \to \mathbb{R}^n$ with no self intersections except possibly with $c(a) = c(b)$, in which case we refer to C as a "closed simple curve."



Definition: Let $C \subseteq \mathbb{R}^n$ be a simple curve and $\underline{c}: [a,b] \to \mathbb{R}^n$ a C^1 parameterization of C. Suppose $f: \mathbb{R}^n \to \mathbb{R}$ and $f \circ c: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Then the path integral of f along C is $\int_C f\, ds = \int_a^b f(\underline{c}(t)) \|c'(t)\|\, dt$

$f(x_1(t), \dots, x_n(t))$ if $\underline{c}(t) = (x_1(t), \dots x_n(t))$.

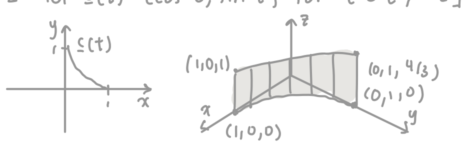
Note: independent of the chosen parameterization of $\underline{c}(t)$.

Note: works with piecewise C^1 or $f(c(t))$ is only piecewise continuous, then we can still define $\int_C f\, ds$ by breaking $[a,b)$ up into pieces over which the hypotheses are satisfied (compute each integral and sum).

What would the path integral represent in higher dimensions?

If e.g. $C \subseteq \mathbb{R}^3$ is a wire and $f(x,y,z) \geq 0$ is the mass density of the wire. Then $\int_C f\, ds$ is the total mass of the wire.

Ex 1: let $\underline{c}(t) = (\cos^3 t, \sin^3 t)$ for $t \in [0, \pi/2]$ and $f(x,y) = 1 + \frac{y}{3}$



$c'(t) = \langle -3\sin t \cos^2 t, 3\sin^2 t \cos t \rangle$

$\|c'(t)\| = \sqrt{9\sin^2 t \cos^4 t + 9\sin^4 t \cos^2 t} = 3\sqrt{\sin^2 t \cos^2 t (\sin^2 t + \cos^2 t)} = 3\sin t \cos t$ Value of the norm.

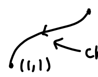
and $f(c(t)) = 1 + \frac{y(t)}{3} = 1 + \frac{\sin^3(t)}{3}$

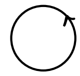
Hence, $\int_C (1 + \frac{y}{3})\, ds = \int_0^{\pi/2} (1 + \frac{\sin^3(t)}{3}) 3\sin t \cos t\, dt = \int_0^{\pi/2} (3\sin t\cos t + \sin^4 t \cos t)\, dt = \left[\frac{3}{2}\sin^2 t + \frac{1}{5}\sin^5 t\right]_0^{\pi/2} = \frac{7}{2}$

Section 7.2: The Line Integral

Integrate a vector field along a path.

A curve $C \subseteq \mathbb{R}^n$ is called "oriented" if it is equipped with a direction:





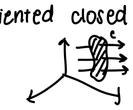
A consistent parameterization would be $(\cos t, \sin t)$ for $t \in [0, 2\pi]$ for this direction.

Definition: let C be an oriented simple curve in $\mathbb{R}^n$ and $\underline{c}: [a,b] \to \mathbb{R}^n$ a C^1 parameterization consistent with the orientation of C.
Let $\underline{F}: \mathbb{R}^n \to \mathbb{R}^n$ be a vector field and suppose $F \circ c: [a,b] \to \mathbb{R}^n$ is cts on $[a,b]$. Define the line integral of $\underline{F}$ along $\underline{c}$ by
$\int_C \underline{F} \cdot ds = \int_a^b \underline{F}(\underline{c}(t)) \cdot c'(t)\, dt$ (Again, if the hypotheses only satisfied piecewise, break $[a,b]$ up accordingly, compute line integral and sum).

Note: the line integral over an oriented curve C is independent of the chosen param. $\underline{c}(t)$ as long as $\underline{c}(t)$ is consistent with orientation. If $\underline{c}(t)$ goes in the opposite direction to the orientation of C, then the integral will change signs. i.e. if $-C$ denotes the same curve as C, but with the opposite orientation, then $\int_C \underline{F} \cdot ds = -\int_{-C} \underline{F} \cdot ds$.

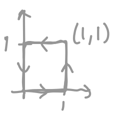
Interpretation: if $\underline{F}$ is a force field in space acting on a test particle moving along an oriented curve C, then the work done by $\underline{F}$ as the particle traverses C (with param. $\underline{c}(t)$ for $t \in [a,b]$) is $\int_C^b \underline{F}(\underline{c}(t))\, \underline{c}'(t)\, dt$
Note: this generalizes the formula work done = force × distance.

Another Example: If $\underline{B}$ is a magnetic field, in $\mathbb{R}^3$ and C is an oriented closed simple curve in $\mathbb{R}^3$. Then Ampere's Law says that the net current through any surface bounded by C is $I = \int_C \underline{B} \cdot ds$



Ex 1: let $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$ and $\underline{c}(t) = \langle t^2, 3t, 2t^3 \rangle$ for $t \in [-1, 2]$.
then $c'(t) = \langle 2t, 3, 6t^2 \rangle$
$\underline{F}(c(t)) = t^2\, \underline{i} + 3t\, \underline{j} + 2t^3\, \underline{k}$
Hence, $\underline{F}(\underline{c}(t)) \cdot c'(t) = 2t^3 + 9t + 12t^5$ so $\int_C \underline{F} \cdot d\underline{s} = \int_{-1}^2 2t^3 + 9t^2 + 12t^5\, dt = 147$.

Ex 2: Let C be the perimeter of the square



oriented counterclockwise. Evaluate $\int_C \underline{F} \cdot d\underline{s}$ where $\underline{F}(x,y) = (x^2, xy)$

Answer: choose any param. of C in CCW...
4 splits:
$\underline{c}_1(t) = (t, 0)$ for $t \in [0,1]$
$\underline{c}_2(t) = (1, t)$ for $t \in [0,1]$
$\underline{c}_3(t) = (-t, -1)$ for $t \in [-1, 0]$
$\underline{c}_4(t) = (0, -t)$ for $t \in [-1, 0]$

then ✓ plugged in $\underline{c}_1(t)$ into $\underline{F} = (x^2, xy)$ $c_1'(t)$

$$(*) = \int_0^1 \begin{pmatrix} t^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} 1 \\ t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \int_{-1}^0 \begin{pmatrix} t^2 \\ t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dt + \int_{-1}^0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt$$
$$= \int_0^1 t^2\, dt + \int_0^1 t\, dt + \int_{-1}^0 -t^2\, dt = 1/2$$

Notation of "differential forms".
✓
Alternative Notation: if $\underline{F} = (F_1, F_2, F_3)$ then you may see $\int_C \underline{F} \cdot d\underline{s}$ written as $\int_C \left[F_1(\underline{c}(t)) \frac{dx}{dt}(t) + F_2(\underline{c}(t)) \frac{dy}{dt}(t) + F_3(\underline{c}(t)) \frac{dz}{dt}(t) \right] dt$
Ex: find $\int_C \cos z\, dx + e^x dy + e^y dz$ where C is parameterized by $\underline{c}(t) = (1, t, e^t)$ for $t \in [0, 2]$.
Ans: $\frac{dx}{dt} = 0$ $\frac{dy}{dt} = 1$ $\frac{dz}{dt} = e^t$
Hence, $= \int_0^2 [(0)\cos(e^t) + (1) e^1 + e^t e^t]\, dt = 2e + \frac{e^4}{2} + \frac{1}{2}$.

Fundamental Theorem of Calculus

Recall the fundamental theorem of calculus (FTC) for 1D functions: if $G: [a,b] \to \mathbb{R}$ is differentiable with $G' = g$, then
$\int_a^b g(x)\, dx = G(b) - G(a)$

Theorem (FTC for line integrals):
Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is C^1 and C is an oriented simple curve (or at least piecewise C^1) and $\underline{c}: [a,b] \to \mathbb{R}^3$ is a param consistent with the orientation on C. then, $\int_C \nabla f \cdot dc = f(\underline{c}(b)) - f(\underline{c}(a))$.

Example, let $\underline{F}(x,y,z) = (y, x, 0)$ and let C be the oriented simple curve param by $\underline{c}(t) = \left(\frac{t^4}{4}, \sin^3\left(\frac{\pi t}{2}\right), 0\right)$ for $t \in [0,1]$

what is $\int_C \underline{F} \cdot d\underline{s}$?

Answer: we see $\underline{F} = \nabla f$ where $f(x,y,z) = xy$

Hence by FTC, $\int_C \underline{F}\, d\underline{s} = f(\underline{c}(1)) - f(\underline{c}(0))$

$\int_C \underline{F}\, d\underline{s} = f(\underline{c}(1)) - f(\underline{c}(0)) = f\left(\frac{1}{4}, 1, 0\right) - f(0,0,0) = \frac{1}{4}$

So even if the curve C is complicated, if we're integrating a gradient vector, then we don't actually need to compute any integrals at all!

Section 7.3: Parameterized surfaces

Two types of surfaces:

i) graphs of function $f: \mathbb{R}^2 \to \mathbb{R}$

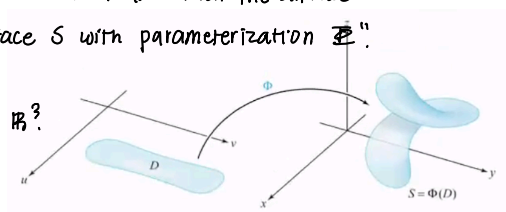
ii) level sets of function $f: \mathbb{R}^3 \to \mathbb{R}$

Note: ii) includes i) as a special case.

Parameterized surfaces as mappings

For integrations over surfaces, it will be useful to have a way of describing surfaces parametrically.

Definition: A parametrisation of a surface is any function $\Phi: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ where D is a domain in $\mathbb{R}^2$. Then the surface corresponding to Φ is the image of D under Φ, i.e, $S = \Phi(D)$. We refer to "the surface S with parameterization Φ".



Idea: Φ takes a flat domain $D \subseteq \mathbb{R}^2$ and it leads/twists it into a surface $\Phi(D)$ is $\mathbb{R}^3$.

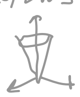
If we write $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, then we are viewing u and v as the parameters.

Note: textbook says, if Φ is diffable/C^1, then we call S a diffable/C^1 surface. But, Φ can be differentiable and S have corners, so we'll just refer to Φ as being diffable/C^1 rather than S.

Example: let $P \subseteq P^3$ be a plane, going through a point with pos. vec $\underline{a}$ and is parallel to 2 vectors, $\underline{b}$ and $\underline{c}$. Then the plane P is the set of points with position vectors $\underline{a} + u\underline{b} + v\underline{c}$ for $(u,v) \in \mathbb{R}^2$.
i.e, $\Phi(u,v) = \underline{a} + u\underline{b} + v\underline{c}$ for $(u,v) \in \mathbb{R}^2$ (so $D = \mathbb{R}^2$ here) This is the parameterization of a plane.

Example 2: unit sphere in $\mathbb{R}^3$ and spherical coords: $(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $0 \le \theta \le 2\pi, 0 \le \varphi \le \pi$.
Thus, $\Phi(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $D = \underbrace{[0, 2\pi)}_{\theta} \times \underbrace{[0, \pi]}_{\varphi} \subseteq \mathbb{R}^2$ is a parameterization of the sphere.

Example 3: cone given $z = \sqrt{x^2+y^2}$ in $\mathbb{R}^3$ we realize as a parametric surface by letting $x = u\cos v$, $y = u\sin v$, and $z = u$ for $v \in [0, 2\pi]$ and $u \in [0, \infty)$ i.e, $D = [0, \infty) \times [0, 2\pi)$.



$\Phi(u,v) = (u\cos v, u\sin v, u)$

Note, Φ is C^1 but S has a corner, so we shouldn't call S a "C^1 surface".

Regular surfaces.

Given the surface $S \subseteq \mathbb{R}^3$ with a diffable parameterization $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$, we want to talk of tangent planes and normal vectors to S. This will make sense if S is a "regular surface."

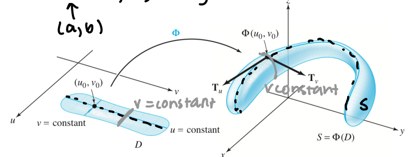
Q: How can we create tangent vectors at $\Phi(u,v) \in S$ using Φ? (If we had to such tangent vectors, we can take their cross product to get a normal vector at $\Phi(u_0, v_0) \in S$).

A: If we hold u constant with value u_0, and we vary v by considering the path $t \mapsto \underset{\substack{\uparrow \\ (a,b)}}{\Phi}(u_0, t) \in S$, then the image of this path is a curve on S with tangent vector at $\Phi(u_0, v_0)$ given by $\frac{\partial \Phi}{\partial v}(u_0, v_0) = \left(\frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\right)$

We denote $\frac{\partial \Phi}{\partial v}$ by T_v

Likewise, $\frac{\partial \Phi}{\partial v}(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\vec{i} + \cdots$

We denote $\frac{\partial \Phi}{\partial u}$ by T_u



The idea is that if the surface is "nice" (or "differentiable") at $\Phi(u_0, v_0)$, then T_u and T_v should be non-zero and non-parallel at (u_0, v_0). Then $(T_u \times T_v)(u_0, v_0)$ is a non-zero vector and normal to the surface S at $\Phi(u_0, v_0)$.

Definition: let S be the surface with diffable parameterization Φ. We call S regular (or "smooth" or "diffable") at $\Phi(u_0, v_0)$ provided that $T_u \times T_v \neq 0$ at (u_0, v_0).

We call S regular if it is regular at all $\Phi(u_0, v_0) \in S$. The vector $T_u \times T_v$ is normal to S at any point.

Example:

Consider the cone $z = x^2 + y^2$. The C^1 parameterization is given by $\Phi(u,v) = (u\cos v, u\sin v, u)$ for $(u,v) \in [0,\infty) \times [0, 2\pi)$

Φ is diffable so we can compute T_u and T_v. But we should expect $(T_u \times T_v)(0,0) = \underline{0}$.

Indeed, $T_u(0,0) = \frac{\partial \Phi}{\partial u}(0,0) = \cos(0)\,\underline{i} + \sin(0)\,\underline{j} + \underline{k} = \underline{i} + \underline{k}$

$T_v(0,0) = \frac{\partial \Phi}{\partial v}(0,0) = -0\sin(0)\,\underline{i} + 0\cos 0\,\underline{j} + 0\,\underline{k} = \underline{0}$

So, $(T_u \times T_v)(0,0) = \underline{0}$. Hence, S is not regular at $(0,0)$

Tangent Plane to Parameterized Surfaces

Definition: Let S be the surface with diffable param $\Phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$,

and suppose S is regular at $\Phi(u_0, v_0)$. Then the tangent plane of S at $\Phi(u_0,v_0)$ is

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \underline{n} = 0$

where $\underline{n} = (\underline{T_u} \times \underline{T_v})(u_0, v_0)$ and $(x_0, y_0, z_0) = \Phi(u_0, v_0)$

Example:

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ where $\Phi(u,v) = (u\cos v, u\sin v, u^2 + v^2)$

$T_u = \langle \cos v, \sin v, 2u \rangle$

$T_v = \langle -u\sin v, u\cos v, 2v \rangle$

$$T_u \times T_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 2v \end{vmatrix} = \begin{pmatrix} 2v\sin v - 2u^2\cos v \\ -2u^2\sin v + 2v\cos v \\ u \end{pmatrix}$$

Clearly, if $T_u \times T_v = \underline{0}$ at some point (u,v), we have $u=0$.

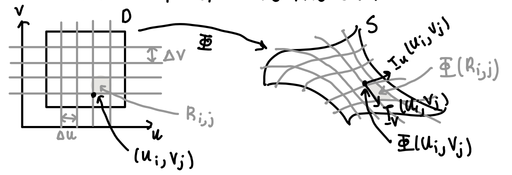
If $u=0$, then $T_u \times T_v = \begin{pmatrix} 2v\sin v \\ 2v\cos v \\ 0 \end{pmatrix}$ so then $T_u \times T_v = 0 \Leftrightarrow (u,v) = (0,0)$

Hence, no tangent plane at $(0,0)$.

But, there is a well-defined tangent plane at all other points on S. e.g. look at $\Phi(1,0) = (1,0,1)$ and $\underline{n} = \langle -2, 0, 1 \rangle$ hence the equation of our tangent plane is $-2(x-1) + (z-1) = 0$ i.e. $z = 2x - 1$.

7.4: Area of Surface

Q: How do we compute/derive the surface area of a surface S with parameterization $\Phi: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$?



We know that $T_u(u_i, v_j)$ and $T_v(u_i, v_j)$ are tangent to S at $\Phi(u_i, v_j)$.
Also, $\Delta u T_u(u_i, v_j)$ and $\Delta v\, T_v(u_i, v_j)$ then these span a parallelogram whose area $\|\Delta u T_u(u_i, v_j) \times \Delta v\, T_v(u_i, v_j)\|$ approximates the area of $\Phi(R_{ij})$, and this approximation gets better as we make the grid finer.

i.e. we can expect that $\sum_{i,j=0}^{n} \| T_u(u_i, v_j) \times T_v(u_i, v_j) \| \Delta v \Delta u$ tends to Area(S) as $n \to \infty$.
$\hookrightarrow \iint_D \| T_u \times T_v \| \, du\, dv$

Definition: suppose S is a regular parameterized surface (possibly except at finitely points) with parameterization $\Phi: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ which is C^1 and 1-1 except possibly at the boundary of D. Then the surface area of S is

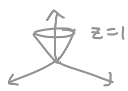
$A(S) = \iint_D \| T_u \times T_v \| \, du\, dv$.

Note: if hypothesis are satisfied only piecewise, compute surface area of each separately and sum.

Note: quantity of $A(S)$ is independent of the chosen parameterization.

Example 1:

Find the area of the cone S
$z = \sqrt{x^2+y^2}$ below the plane $z=1$.



Parameterize S as $x = u\cos v$, $y = u \sin v$, $z = u$ where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

with $\Phi(u,v) = (u\cos v, u \sin v, u)$, we see that Φ is 1-1 away from the boundary of D, since if $\Phi(u,v) = \Phi(u', v')$ for some $0<u, u'<1$ and $0<v, v'<2\pi$

then $\begin{cases} u\cos v = u'\cos v' \\ u \sin v = u' \sin v' \\ u = u' \end{cases} \xrightarrow{\text{sub in } u \text{ for } u'} \begin{matrix} \cos v = \cos v' \\ \sin v = \sin v' \end{matrix}$

so, $v = v' + 2n\pi$ for some $n \in \mathbb{Z}$, but because of bounds,
$v = v'$
$u = u'$

$T_u = \langle \cos v, \sin v, 1 \rangle$
$T_v = \langle -u\sin v, u\cos v, 0 \rangle$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u \cos v & 0 \end{vmatrix} = \begin{pmatrix} -u\cos v \\ -u \sin v \\ u \end{pmatrix}$$

$\| T_u \times T_v \| = \sqrt{u^2\cos^2 v + u^2 \sin^2 v + u^2} = \sqrt{2}\, u$

Note $T_u \times T_v$ only vanishes when $u=0$ and since $\Phi(0,v) = (0,0,0)$, the cone only fails to be regular at one point, so we can compute

$A(S) = \iint_D \sqrt{2}\, u \, du\, dv \quad D = [0,1] \times [0, 2\pi]$
$= \int_0^{2\pi} \int_0^1 \sqrt{2}\, u \, du\, dv = \sqrt{2}\pi$

$\boxed{\text{7.5: Integrals of scalar functions over surfaces}}$

S is a surface parameterized by a C^1 mapping $\Phi: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ which is 1-1 (except possibly at the boundaries of D) and S is regular (except at finitely many points).

Definition: suppose $f: S \to \mathbb{R}$ is continuous. Then the integral of f over S is
$$\iint_S f\, dS = \iint_D f(\Phi(u,v))\, \|T_u \times T_v\|\, du\, dv$$

Note: when $f \equiv 1$, we recover the surface area of S, i.e. $A(S) = \iint_S dS$.

Note: the above is independent of the parameterization Φ (as for the path integral).

Interpretation: if $\rho: S \to \mathbb{R}$ is ≥ 0 and represents the mass density of the surface S, then the total mass of S is $M(S) := \iint_S \rho(x,y,z)\, dS$.

Example 1: let $S \subseteq \mathbb{R}^3$ be a helicoid parameterized by $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ for $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$.
Let $f(x,y,z) = \sqrt{1+x^2+y^2}$. Find $\iint_S f\, dS$

Answer: First compute $\|\vec{T}_r \times \vec{T}_\theta\|$

$\vec{T}_r = \frac{\partial \Phi}{\partial r} = \langle \cos\theta, \sin\theta, 0 \rangle$

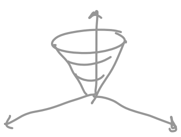
$\vec{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 1 \rangle$

so, $\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ r \end{pmatrix}$

so, $\|\vec{T}_r \times \vec{T}_\theta\| = \sqrt{1+r^2}$

Hence, $\iint_S f\, dS = \int_0^{2\pi} \int_0^1 \underbrace{\sqrt{1+r^2}}_{f(\Phi(r,\theta))} \sqrt{1+r^2}\, dr\, d\theta = \int_0^{2\pi}\int_0^1 1+r^2\, dr\, d\theta = 2\pi \int_0^1 (1+r^2)\, dr = \frac{8\pi}{3}$.

Example 2: let $S \subseteq \mathbb{R}^3$ be the cone $z = \sqrt{x^2+y^2}$ below the plane $z=1$. Suppose S has a mass density $\rho(x,y,z) = e^{x^2+y^2}$. Find $M(S)$.



density ↑ exponentially as we go ↑ the cone.

Answer: we saw S could be parameterized by $\Phi(u,v) = (u\cos v, u\sin v, u)$ for $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.
with $\|\vec{T}_u \times \vec{T}_v\| = \sqrt{2}u$.

Hence $M(S) = \iint_S \rho\, dS = \int_0^{2\pi}\int_0^1 \sqrt{2}u\, e^{u^2}\, du\, dv = \pi\sqrt{2}\left[e^{u^2}\right]_0^1 = \sqrt{2}\pi(e-1)$

$\boxed{\text{7.6: Surface integrals of vector fields}}$ 11/15/2021

Q: why do we integrate a vector field over a surface?

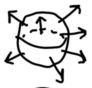
Recall: if $\underline{F}$ is a force field in space and a test particle moves along an oriented curve $C \subseteq \mathbb{R}^3$, then the <u>work done</u> by $\underline{F}$ on the particle is <u>the line integral</u> $\int_C \underline{F} \cdot d\underline{s}$ $\left(= \int_a^b \underline{F}(\underline{c}(t)) \cdot c'(t)\,dt\right)$

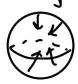
a parameterization: $[a,b] \to \mathbb{R}^3$ of C consistent with the orientation.

Next: $\iint_S \underline{F} \cdot d\underline{S}$ will be the <u>flux</u> of $\underline{F}$ through an oriented surface $\underline{S}$.

<u>Orientation</u>

Definition: An oriented surface S is a two sided surface equipped with a continuous choice of normal vectors across the surface. (A mobius strip is not oriented because $\vec{n}_1 = -\vec{n}_2$)

- "continuous" means the normal vector shouldn't jump to the other side of the surface. i.e, they all lie on one side and hence specify (mathematically) our chosen side.
- <u>Ex</u>:  we might say "the sphere is oriented with the outward normal," or "oriented with normal $x\underline{i} + y\underline{j} + z\underline{k}$."

   "sphere is oriented with inward normal" or "oriented with normal $-x\underline{i} - y\underline{j} - z\underline{k}$."

Every surface of 2 sides has 2 possible orientations. Pick one of the two sides.

Now suppose we have a surface S with a C^1 param $\Phi(u,v)$ and suppose S is regular at (u_0, v_0). Then we know how to get a normal vector at $\Phi(u_0, v_0)$, namely $(T_u \times T_v)(u_0, v_0)$.

Therefore, a parameterization on S <u>induces</u> an orientation. We just take the continuous choice of normal to be $(T_u \times T_v)(u,v)$ as (u,v) ranges over D.

On the other hand, if S is an <u>oriented</u> surface and we have a param $\Phi(u,v)$, then $T_u \times T_v$ will either point in the same direction on S, in which case we call $\Phi(u,v)$ orientation preserving, or it will point in the opposite direction in which case $\Phi(u,v)$ is orientation reversing.

Example: Let S be the unit sphere in $\mathbb{R}^3$ oriented with its outwards normal.

Is $\Phi(u,v) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$ orientation preserving or reversing?

Answer: let $(x,y,z) = \Phi(\theta_0, \varphi_0)$. Then $(T_\theta \times T_\varphi)(\theta_0, \varphi_0) = \langle -\sin^2\varphi_0 \cos\theta_0, -\sin^2\varphi_0 \sin\theta_0, -\sin\varphi_0\cos\varphi_0 \rangle$ $(*)$

$= -\sin\varphi_0 \underbrace{\langle \cos\theta_0 \sin\varphi_0, \sin\theta_0 \sin\varphi_0, \cos\varphi_0 \rangle}_{\text{position vector } (x,y,z)}$ $\to \langle \cos\theta_0 \sin\varphi_0, \sin\theta_0 \sin\varphi_0, \cos\varphi_0 \rangle$

i.e, $(*)$ points inwards, i.e. in the opposite direction to our orientation on S, so $\Phi(\theta,\varphi)$ is orientation reversing.

Note: $\Phi(\varphi,\theta)$ is then orientation preserving.

<u>Integrating Vector Fields over Oriented Surfaces</u>

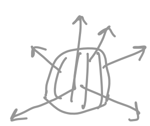
Definition: Let S be an oriented surface and $\Phi = \Phi(u,v): D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ an orientation preserving param with S, Φ, D satisfying the same assumptions as in sect 7.4.

Let $\underline{F}: S \to \mathbb{R}^3$ be a continuous vector field on S. Then the surface integral of $\underline{F}$ over S is $\iint \underline{F} \cdot d\underline{S} = \iint_D F(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v)\,du\,dv$

Note: this is independent of Φ for orientation-preserving. If Φ is orientation reversing,

$\iint_S \underline{F} \cdot d\underline{S} = -\iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\,du\,dv.$

Ex: Let S be the oriented surface consisting of the graph of $z = 1 - x^2 - y^2$ with $z \ge 0$, equipped with outward normal.



Let $\underline{F} = \langle 2x, 2y, z \rangle$. Find $\iint_S \underline{F} \cdot d\underline{S}$

Ans: Try $\Phi = (\underset{x}{r\cos\theta}, \underset{y}{r\sin\theta}, \underset{z}{1-r^2})$

$$T_r \times T_\theta = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} 2r^2\cos\theta \\ 2r^2\sin\theta \\ r \end{pmatrix}$$

always positive. If we swapped the order, it would be $-r$ and therefore be orientation reversing.

Hence, orientation preserving.
$\underline{F}(\Phi(r,\theta)) = (2r\cos\theta, 2r\sin\theta, 1-r^2)$
$\underline{F}(\Phi(r,\theta))\cdot(\underline{T}_r\times\underline{T}_\theta) = 4r^3\cos^2\theta + 4r^3\sin^2\theta + r - r^3$

Hence $\iint_S \underline{F}\cdot d\underline{S} = \int_0^{2\pi}\int_0^1 (3r^3+r)\,dr\,d\theta = 2\pi\left[\frac{3r^4}{4}+\frac{r^2}{2}\right]_0^1 = \frac{5\pi}{2}$.

Simplifying Surface Integral Calculations

11/17/2021

Suppose we have a surface S given by the graph of $z=g(x,y)$ where g is diffable. Then we can param S by $\Phi(u,v)=(u,v,g(u,v))$ (here $(u,v)\in D$, where D is the domain of g).

Then, $\underline{T}_u = \langle 1, 0, \frac{\partial g}{\partial u}\rangle$
$\underline{T}_v = \langle 0, 1, \frac{\partial g}{\partial v}\rangle$

Then $\underline{T}_u\times\underline{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial g}{\partial u} \\ 0 & 1 & \frac{\partial g}{\partial v}\end{vmatrix} = \begin{pmatrix} -\frac{\partial g}{\partial u} \\ -\frac{\partial g}{\partial v} \\ 1 \end{pmatrix}$ ← never the $\underline{0}$ vector

Hence, the param'd surface S is regular with $\|\underline{T}_u\times\underline{T}_v\| = \sqrt{1+(\frac{\partial g}{\partial u})^2+(\frac{\partial g}{\partial v})^2}$

[if $g: D\subseteq\mathbb{R}^2\to\mathbb{R}$ and f is a scalar function, then $\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+(\frac{\partial g}{\partial x})^2+(\frac{\partial g}{\partial y})^2}\,dx\,dy$

By our formula for $\underline{T}_u\times\underline{T}_v$ the 2 possible directions of a normal vector at $(x_0,y_0,g(x_0,y_0))$ are the directions of $\pm\underline{n}$, where $\underline{n} = \langle -\frac{\partial g}{\partial x}(x_0,y_0), -\frac{\partial g}{\partial y}(x_0,y_0), 1\rangle$

unless otherwise specified, we always assume the graphs of $z=g(x,y)$ are oriented in the direction of $\underline{n}$, so always upward pointing.

Then the parameterization $\Phi(u,v)=(u,v,g(u,v))$ is then orientation preserving by construction.

Note that if $\underline{F}=\langle F_1,F_2,F_3\rangle$ is a continuous vector field on the graph S, then $\underline{F}\cdot(\underline{T}_u\times\underline{T}_v) = -\frac{\partial g}{\partial x}F_1 - \frac{\partial g}{\partial y}F_2 + F_3$

Therefore, $\iint_S \underline{F}\cdot d\underline{S} = \iint_D\left(-\frac{\partial g}{\partial x}F_1 - \frac{\partial g}{\partial y}F_2 + F_3\right)dx\,dy$.

Ex1: let S be the graph $z=1-x^2-y^2$ where $z\geq 0$, $\underline{F}=\langle 2x,2y,z\rangle$
then $\frac{\partial g}{\partial x}=-2x$ $\frac{\partial g}{\partial y}=-2y$ so $\iint_S\underline{F}\cdot d\underline{S} = \underbrace{\iint_D}_{\text{unit disk in } xy\text{-plane}}(4x^2+4y^2+(1-x^2-y^2))\,dx\,dy = \iint(1+3x^2+3y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1(1+3r^2)\,r\,dr\,d\theta = 5\pi/2$

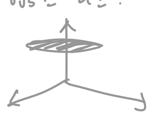
only works when surface S is the graph of a function g.

Also: sometimes it's easy to see what a normal vector to a surface is without an orientation preserving parameterization. $\Phi: D\to\mathbb{R}^3$

Then the normal $\underline{T}_u\times\underline{T}_v$ coincides with the orientation on S, hence

$\iint_S\underline{F}\cdot d\underline{S} = \iint_D\underline{F}\cdot(\underline{T}_u\times\underline{T}_v)\,du\,dv = \iint_D\underline{F}\cdot\underbrace{\left(\frac{\underline{T}_u\times\underline{T}_v}{\|\underline{T}_u\times\underline{T}_v\|}\right)}_{\text{unit normal denoted by } \underline{n}}\|\underline{T}_u\times\underline{T}_v\|\,du\,dv = \iint_D(\underline{F}\cdot\underline{n})\|\underline{T}_u\times\underline{T}_v\|\,du\,dv = \iint_S(\underline{F}\cdot\underline{n})\,dS$

Ex: let S be disk of radius 5 in the plane $z=12$ centered on the z-axis. let $\underline{F}=\langle x,y,z\rangle$
Find $\iint_S\underline{F}\cdot d\underline{S}$.

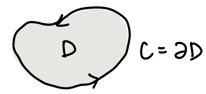
Ans: 
since S is a graph, implicitly oriented upwards.
so, $\underline{n}=\langle 0,0,1\rangle$ hence $\iint_S\underline{F}\cdot d\underline{S} = \iint_S z\,dS = 12\,\text{area}(S) = 300\pi$.

$\boxed{\text{Chapter 8: The Integral Theorems of Vector Analysis}}$

- green's theorem
- stoke's theorem
- gauss' theorem (divergence theorem)

$\boxed{\text{Section 8.1 : Green's Theorem}}$

Green's Theorem gives a formula for the line integral of a 2D vector field along a closed oriented curve $C \subseteq \mathbb{R}^2$ in terms of a double integral over a region enclosed by C



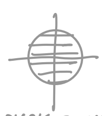
Green's Theorem: let $D \subseteq \mathbb{R}^2$ be on elementary regions and let ∂D be its boundary oriented clockwise.
Let $\underline{F} = \langle F_1, F_2 \rangle$ be a C' vector field on D.
Then, $\int_{\partial D} \underline{F} \cdot d\underline{s} = \iint_D (\text{curl } \underline{F}) \cdot \underline{k} \, dA$. $(*)$

Recall: by curl $\underline{F}$ we mean the curl of vector $\langle F_1, F_2, 0 \rangle$ which is $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}$.
So, $(\text{curl } \underline{F}) \cdot \underline{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$, and if
we write $\underline{F} \cdot d\underline{s} = F_1 \, dx - F_2 \, dy$, then $(*)$ is equivalent to $\int_{\partial D} F_1 \, dx + F_2 \, dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$

Example: let D = unit disc and $\underline{F} = \langle x, xy \rangle$ verify green's theorem.
Ans: ∂D = unit circle, param. by $\underline{c}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ (which is orientation-preserving)

Then, $\int_{\partial D} F_1 dx + F_2 dy = \int_0^{2\pi} F_1(c_1(t), c_2(t)) \frac{dc_1}{dt}(t) + F_2(c_1(t), c_2(t)) \frac{dc_2}{dt}(t) dt$.
$= \int_0^{2\pi} (\cos t \sin t + \cos t \sin t \cos z) dt$
$= \int_0^{2\pi} (\cos^2 t \sin t - \cos t \sin t) dt$
$= \left[\frac{-\cos^3 t}{3} + \frac{\cos^2 t}{2}\right]_0^{2\pi} = 0$

Also, $\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$

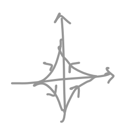


$\iint_D (y - 0) \, dx dy$
we integrate odd function over symmetrical.
$\int_0^{2\pi} \int_0^1 r^2 \sin\theta \, dr d\theta$
$= 0$.

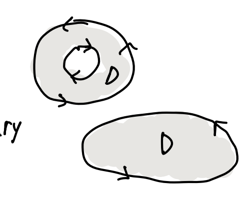
Let $F_1 = -y$, $F_2 = x$. Then green's theorem says

Youtube "planimeter"

$\int_{\partial D} -y dx + x dy = \iint_D 2 \, dx dy$
$\text{Area}(D) = \frac{1}{2} \int_{\partial D} x dy - y dx$



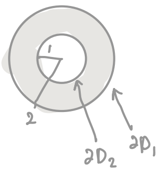
Ex: Let C be hypocycloid in $\mathbb{R}^2$ param'd by $\underline{c}(t) = (\cos^3 t, \sin^3 t)$ for $t \in [0, 2\pi]$
Find enclosed area
$A = \frac{1}{2} \int_C x dy - y dx$
$A = \frac{1}{2} \int_0^{2\pi} \cos^3 t (3 \cos t \sin^2 t) - \sin^3(t)(-2\cos^2 t \sin t)] dt$
$A = \frac{3}{2} \int_0^{2\pi} (\sin^2 t + \cos^4 t + \cos^2 t + \sin^4 t) dt$
$A = \frac{3}{2} \int_0^{2\pi} \sin^2 t \cos^2 t \, dt$
$A = \frac{3}{8} \int_0^{2\pi} \sin^2(2t) \, dt$
$A = \frac{3}{8} \int_0^{2\pi} \frac{1}{2}(1 - \cos(4t)) \, dt$
$A = 3\pi/8$



For a more general region $D \subseteq \mathbb{R}^2$ (not necessarily elementary) green's theory still holds true if we orient each boundary curve correctly.
If you walk along a boundary curve C with correct orientation for green's theorem, then region D should be on your left.

i.e. $\int_{\partial D} F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right) dxdy$ where $\int_{\partial D}$ = sum of integrals over each boundary curve (with the correct orientation).

Ex: Let D = annulus with outer radius 2 and inner radius 1.



Confirm green's theorem for $\underline{F} = \langle 2x^3, 2y^3 \rangle$

we have

$$\int_{\partial D} 2x^3 dx + y^3 dy = \int_{\partial D_1} 2x^3 dx + y^3 dy + \int_{\partial D_2} 2x^3 dx + y^3 dy \quad \Big\} (*)$$

Param ∂D_1 by $(2\cos t, 2\sin t)$

Param ∂D_2 by $(\cos t, -\sin t)$

$$(*) = \int_0^{2\pi} \left[16\cos^3 t(-2\sin t) + 8\sin^3 t(2\cos t)\right] dt + \int_0^{2\pi} \left[(2\cos^3 t)(-\sin t) + (-\sin^3 t)(-\cos t)\right] dt$$

$$= \int_0^{2\pi} (-34\cos^3 t \sin t + 17\sin^3 t \cos t)\, dt$$

$$= \left[\frac{34}{4}\cos^4 t + \frac{17}{4}\sin^4 t\right]_0^{2\pi} = 0$$

And $\iint_D \left(\underbrace{\frac{\partial F_2}{\partial x}}_{0} - \underbrace{\frac{\partial F_1}{\partial y}}_{0}\right) dxdy$

$= 0$

Section 8.2: Stoke's Theorem

Recall Green's Theorem ($\int_{\partial D} \underline{F} \cdot d\underline{s} = \iint_D (\text{curl}\, \underline{F}) \cdot \underline{k}\, dA$ (*))

Also: since $\text{curl}\, \underline{F} = \text{curl} \langle F_1, F_2, 0 \rangle = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \underline{k}$ and $\underline{F} \cdot d\underline{s} = F_1\, dx + F_2\, dy$, we wrote (*) as
$\int_{\partial D} F_1\, dx + F_2\, dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$.

Stoke's Theorem will generalize (*) to oriented surfaces $S \subseteq \mathbb{R}^3$ with boundary ∂S under certain assumptions we'll have
$\int_{\partial S} \underline{F} \cdot d\underline{s} = \iint_S (\text{curl}\, \underline{F}) \cdot d\underline{S}$.

Stoke's Theorem for Parameterized surfaces

Theorem (Stoke's theorem): Let $S \subseteq \mathbb{R}^3$ be an oriented surface derived by a C^1 parameterization $\Phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ s.t.

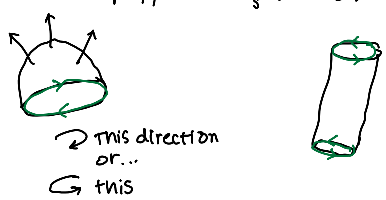
i) Φ is 1-1 (including on the boundary ∂D of D)

ii) Φ is regular (except possibly at finitely many points).

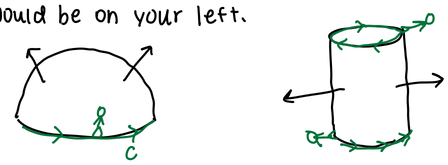
iii) D has smooth boundary curves (can be piecewise smooth)

Let ∂S denote the boundary curve(s) of S, each oriented "positively" (w.r.t. the orientation on S). Let $\underline{F} = \langle F_1, F_2, F_3 \rangle$ be a C^1 vector field on S. Then,
$\iint_S (\text{curl}\, \underline{F}) \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{s}$

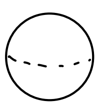
If ∂S is empty, then $\iint_S (\text{curl}\, \underline{F}) \cdot d\underline{S} = 0$



Remarks: A) let C be one of the boundary curves in ∂S. Then the "positive" orientation on C is s.t. if you were to walk around C in the direction of "positive" orientation with the orientation of S as your upright direction, then the surface S would be on your left.



B) The reason we impose Φ is 1-1 (including on ∂D) is so that $\Phi(\partial D) = \partial S$. If e.g. Φ were the standard param. of the sphere in $\mathbb{R}^3$, i.e. $\Phi = \Phi(\theta, \varphi) : \underbrace{[0, 2\pi) \times (0, \pi]}_{D} \to \mathbb{R}^3$



which is not 1-1, the boundary would be a curve in the sphere, but the sphere has no boundaries.

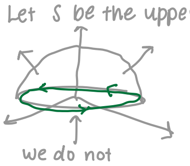
C) If $\underline{c}(t) = \langle c_1(t), c_2(t) \rangle$ is a param of ∂D oriented positively. (in the same considered for green's thm), then $\Phi \circ \underline{c}$ is a param. of ∂S oriented positively (in the sense of remark A).

In Practice

You don't always need to find Φ explicitly if it's obvious such a parameterization exists, e.g. for hemispheres.

Ex 1:

Let S be the upper unit hemisphere in $\mathbb{R}^3$ oriented outwards



Let $\vec{F} = \langle y, -x, e^{xz} \rangle$

Find $\iint (\text{curl}\, \vec{F}) \cdot d\vec{S}$ $(*)$

If we were to attempt this directly, then $\text{curl}\, \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial k} \\ y & -x & e^{xz} \end{vmatrix} = \begin{pmatrix} 0 \\ -ze^{xz} \\ -2 \end{pmatrix}$

$\underline{n} = \langle x, y, z \rangle$ so $\text{curl}\, \underline{F} \cdot \underline{n} = -yze^{xz}$

$(*) = \iint_S -yze^{xz}\, dA$. $\leftarrow$ Too hard to compute.

By Stoke's Theorem,

$(*) = \int_{\partial S} \underline{F} \cdot d\underline{s}$ with ∂S oriented positively w.r.t. the orientation on S.

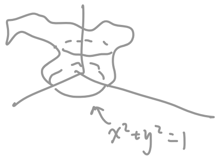
So the positive orientation of ∂S is counter clockwise i.e. we can take $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 0$ for $t \in [0, 2\pi)$.

so $\int_{\partial S} \underline{F} \cdot d\underline{s} = \int_{\partial S} F_1\, dx + F_2\, dy + F_3\, dz = \int_0^{2\pi} [\underbrace{F_1(x(t))}_{\sin t} \underbrace{\frac{dx}{dt}}_{-\sin t} + \underbrace{F_2(y(t))}_{-\cos t} \underbrace{\frac{dy}{dt}}_{\cos t} + F_3(z(t)) \underbrace{\frac{dz}{dt}}_{=0}]\, dt$

$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = \int_0^{2\pi} (-1)\, dt$

$= -2\pi$. //

Note: nothing special about the hemisphere only the fact its boundary is the unit sphere in the xy-plane and the surface S is nice.

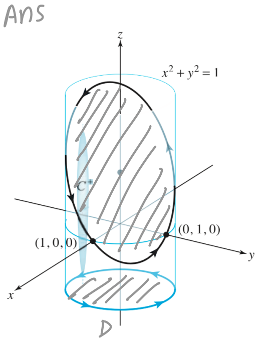


Then again, $\iint_S (\text{curl}\, \underline{F}) \cdot d\underline{S} = -2\pi$.
(independent of surface).

Example 2:

Let C be the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$, oriented s.t. the projection down onto the xy plane is counterclockwise. Find $\int_C -y^3 dx + x^3 dy - z^3 dz$. $(\dagger)$

Ans



We have $\underline{F} = \langle -y^3, x^3, -z^3 \rangle$

C bounds the portion of the graph $z = 1-x-y$ where $x^2+y^2 \leq 1$.

$\text{curl}\, \vec{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3x^2+3y^2 \end{pmatrix}$

But recall the surface integral of $G = \langle G_1, G_2, G_3 \rangle$ over the graph of a function $z = f(x,y)$ can be computed as.

$\iint_S \underline{G} \cdot d\underline{S} = \iint_D \left(-\frac{\partial f}{\partial x} G_1 - \frac{\partial f}{\partial y} G_2 + G_3\right) dx\,dy$

$\uparrow$ the domain of f.

Taking $\underline{G} = \text{curl}\, \underline{F}$, i.e. $G_1 = G_2 = 0$ $G_3 = 3x^2+3y^2$.

we get

$\iint_S \text{curl}\, \underline{F} \cdot d\underline{S} = \iint_{\{x^2+y^2 \leq 1\}} 3(x^2+y^2)\, dx\,dy = 3\int_{r=0}^{1} \int_{\theta=0}^{2\pi} r^3\, d\theta\, dr = 6\pi \int_0^1 r^3\, dr = \frac{3\pi}{2}$ //

By Stoke's theorem $(\dagger) = 3\pi$.

Note: check Stoke's theorem holds in this example by computing $(\dagger)$ directly using the param of C given by $\underline{c}(t) = \langle \cos t, \sin t, 1-\cos t - \sin t \rangle$.

Applications

Recall if $\underline{n}$ is the unit normal of an oriented surface S, then $\iint_S \underline{G} \cdot d\underline{S} = \iint_S (\underline{G} \cdot \underline{n})\, dS$ (*)

and similarly, if $\underline{T}$ is the unit tangent vector to a curve C, then $\iint_S \underline{G} \cdot d\underline{s} = \int_C (\underline{G} \cdot \underline{T})\, ds$ (**)

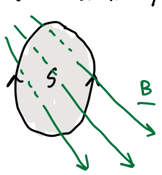
Taking $\underline{G} = \text{curl}\, \underline{F}$ in (*)

and $\underline{G} = \underline{F}$ in (**)

then Stoke's tells us that $\underbrace{\iint_S (\text{curl}\, \underline{F} \cdot \underline{n})\, dS}_{\text{surface integral of normal component of curl } \underline{F}.} = \underbrace{\iint_{\partial S} (\underline{F} \cdot \underline{T})\, ds}_{\text{path integral of the tangential component of } F \text{ along } \partial S.}$

Stoke's theorem allows us to convert between "pointwise" (or "differential") formulations and "integral" formulations of physical laws

For example, let $\underline{B}$ be a line-dependent magnetic field in $\mathbb{R}^3$, inducing an electric field $\underline{E}$. Let S be an oriented surface in $\mathbb{R}^3$ with boundary curve C.



one of maxwell's equations is $\text{curl}\, \underline{E} = \frac{-\partial B}{\partial t}$ (*)

Integrating (*) over S gives: $\iint_S \text{curl}\, \underline{E} \cdot d\underline{S} = \iint_S \frac{-\partial B}{\partial t} \cdot d\underline{S}$

But stoke's theorem says: $\iint_S \text{curl}\, \underline{E} \cdot d\underline{S} = \int_C \underline{E} \cdot d\underline{s}$

and also: $\iint_S \frac{-\partial B}{\partial t} \cdot d\underline{S} = \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{S}$

Hence: $\boxed{\int_C \underline{E} \cdot d\underline{s} = -\frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{S}}$

which is Faraday's law: the voltage $(\int_C \underline{E} \cdot d\underline{s}$ around C is minus the rate of change of the magnetic flux through S.

Section 8.3: conservative vector fields

Recall:

i) If $f:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}$ is C^2 and $n=2$ or 3, then $\text{curl}(\nabla f)=\underline{0}$
(with the convention that when $n=2$, $\nabla f=\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, 0\rangle$)

ii) if $f\in C^1$ and $\underline{c}:[a,b]\to\mathbb{R}^n$ is a piecewise C^1 param of $C\subseteq\mathbb{R}^n$, then $\int_C \nabla f\cdot d\underline{s} = f(\underline{c}(b))-f(\underline{c}(a))$

Conservative Fields

Q: When is a given vector field a gradient field?

Note: The $\text{curl}\,\underline{F}=\underline{0}$ is not sufficient to conclude $\underline{F}=\nabla f$ for some f.

Ex: $\underline{F}=\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$, which is C^1 on $\mathbb{R}^2\setminus\{0\}$ and $\text{curl}\,\underline{F}=\langle 0,0,\frac{\partial F_2}{\partial x}-\frac{\partial F}{\partial y}\rangle=\langle 0,0,0\rangle$ on $\mathbb{R}^2\setminus\{0\}$

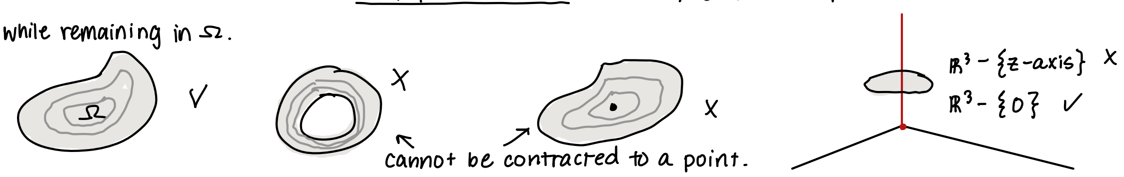
Also, $\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right)=\frac{-y}{x^2+y^2} \leftarrow F_1$

$\frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right)=\frac{x}{x^2+y^2} \leftarrow F_2$

But $\tan^{-1}\left(\frac{y}{x}\right)$ is not defined on all of $\mathbb{R}^2\setminus\{0\}$ (e.g. not defined wherever $x=0$), hence there is no $f\in C^2(\mathbb{R}^2\setminus\{0\})$ s.t. $\nabla f=\underline{F}$ in $\mathbb{R}^2\setminus\{0\}$ //

However, this equality "$\text{curl}\,\underline{F}=0$" is sufficient to obtain $\underline{F}=\nabla f$ as long as your domain is "simply connected"

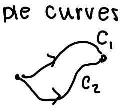
Definition: A domain $\Omega=\mathbb{R}^n$ simply connected if every closed (startpoint = endpoint) simple (no self intersections) curve can be continuously contracted to a point while remaining in Ω.



Theorem: let $n=2$ or 3, let $\underline{F}:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^n$ be a C^1 vector field, derived on a simply connected domain Ω. Then the following are equivalent:

i) For any oriented simple closed curve $\underline{c}\subseteq\Omega$, $\int_C \underline{F}\cdot d\underline{s}=0$.

ii) For any 2 oriented simple curves c_1 and c_2 (contained in Ω) with the same start and end points,
$\int_{c_1}\underline{F}\cdot d\underline{s}=\int_{c_2}\underline{F}\cdot d\underline{s}$



iii) $\underline{F}=\nabla f$ for some $f\in C^2(\Omega)$

iv) $\text{curl}\,\underline{F}=\underline{0}$ (where $\text{curl}\,\underline{F}$ means $\text{curl}\langle F_1,F_2,0\rangle$ if $n=2$.

A vector field satsisfying any one (and hence all) of the above is called conservative.

Sketch proof in 3D: i) → ii) exercise

ii) → iii) Suppose for simplicity $(0,0,0)\in\Omega$

let $f(x,y,z)=\int_C \underline{F}\cdot d\underline{s}$ where C is any oriented simple curve joining $(0,0,0)$ to (x,y,z). Then show $\nabla f=\underline{F}$

iii) → iv) seen before

iv) → i) Let $C\subseteq\Omega$ be any simple closed curve and let S be any surface whose boundary is C. Then, $\int \underline{F}\cdot d\underline{s} \overset{\text{Stoke's}}{=} \iint_S \text{curl}\,\underline{F}\cdot d\underline{S}=0$.

We call a vector field $\underline{F}$ satisfying $\text{curl}\,\underline{F}=\underline{0}$ irrotational. Hence, a vector field is irrotational ⟺ it is a gradient field and is on a simply connected domain.

If $\underline{F}=\nabla f$, we call f a potential for $\underline{F}$.

Ex 1: Does ∃ (exists) f s.t. $\nabla f=\underline{F}$ where $\underline{F}=\langle 2xyz+\sin x, x^2z, x^2y\rangle$ on $\mathbb{R}^3$? If so, find f.

Answer: $\mathbb{R}^3$ is simply connected and $\underline{F}\in C^1$ with $\text{curl}\,\underline{F}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz+\sin x & x^2z & x^2y\end{vmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ so such a function f exists by our theorem.

To find f,

$\frac{\partial f}{\partial x} = 2xy + \sin x$ ①   ① $\Rightarrow f = x^2yz + \cos x + g(y,z)$ Hence, $-\cos(x) + g(y,y,z) = h(x,y) + p(x,y)$

$\frac{\partial f}{\partial y} = x^2 z$ ②   ② $\Rightarrow f = x^2zy + h(x,z)$ so

$\frac{\partial f}{\partial z} = x^2 y$ ③   ③ $\Rightarrow f = x^2yz + p(x,y)$ $f(x,y,z) = x^2yz + \cos x + C$. //

Ex 2: let $\underline{F} = \langle e^x \sin y, e^x \cos y, z^2 \rangle$ and let C be param'd by $c(t) = \langle \sqrt{t}, t^3, e^{\sqrt{t}} \rangle$ for $0 \le t \le 1$. Find $\int_c \underline{F} \cdot d\underline{s}$.

Ans: curl $\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x \cos y & z^2 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Hence by our theorem, $\underline{F} = \nabla f$ for some f and $\int_c \underline{F} \cdot d\underline{s} = f(\underline{c}(1)) - f(\underline{c}(0)) = f(1,1,e) - f(0,0,1)$.

To find f:

$\frac{\partial f}{\partial x} = e^x \sin y \Rightarrow f = e^x \sin y + g(y,z)$ Take $p(x,y) = e^x \sin y$

$\frac{\partial f}{\partial y} = e^x \cos y \Rightarrow f = e^x \sin y + h(x,z)$ $h(x,z) = p(y,z) = \frac{1}{3}z^3$, so

$\frac{\partial f}{\partial y} = z^2 \Rightarrow f = \frac{1}{3}z^3 + p(x,y)$ $f(x,y,z) = e^x \sin y + \frac{1}{3}z^3$

so $f(1,1,e) - f(0,0,1) = e\sin(1) + \frac{1}{3}e^3 - \frac{1}{3}$. //

Q: Does $\text{div}\,\underline{F} = 0 \Rightarrow \underline{F} = \text{curl}\,\underline{G}$ for some vector field $\underline{G}$?

Answer: yes, but we have to impose more on our domain.

Theorem: If $\underline{F}$ is a C^1 vector field on $\mathbb{R}^3$ with div $\underline{F} = 0$ on $\mathbb{R}^3$, then there exists a C^2 vector fields $\underline{G}$ s.t. curl $\underline{G} = \underline{F}$.

Idea: we can find a vector field $\underline{G}$ which works s.t $G = \langle G_1, G_2, 0 \rangle$

Then curl $\underline{G} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ G_1 & G_2 & 0 \end{vmatrix} = \begin{pmatrix} -\partial_z G_2 \\ \partial_z G_1 \\ \partial_x G_2 - \partial_y G_1 \end{pmatrix}$ want this to be equal to $\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$

$-\partial_z G_2 = F_1 \Rightarrow G_2(x,y,z) = -\int_0^z F_1(x,y,t)\,dt + g(x,y)$

$\partial_z G_1 = F_2 \Rightarrow G_1 = \int_0^z F_2(x,y,t)\,dt$

where $g(x,y)$ is to be determined. //

Ex: let $\underline{F} = \langle x^2+1, z-2xy, y \rangle$ in $\mathbb{R}^3$. Does there exist $\underline{G}$ s.t. curl $\underline{G} = \underline{F}$?

If so, find one

Ans: div $\underline{F} = 2x - 2x + 0 = 0$, so $\underline{G}$ exists.

$G_1(x,y,z) = \int_0^z (t - 2xy)\,dt = \left[\frac{1}{2}t^2 - 2xyt\right]_0^z = \frac{1}{2}z^2 - 2xyz$

$G_2(x,y,z) = -\int_0^z (x^2+1)\,dt + g(x,y) = (x^2+1)t\Big|_0^z + g(x,y) = -z(x^2+1) + g(x,y)$

curl $\underline{G} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{z^2}{z} - 2xyz & \begin{matrix} -z(x^2+1) \\ +g(x,y) \end{matrix} & 0 \end{vmatrix} = \begin{pmatrix} x^2+1 \\ 2z - 2xy \\ 2xz + \partial_x g(x,y) + 2xz \end{pmatrix} = \begin{pmatrix} x^2+1 \\ z - 2xy \\ \partial_x g(x,y) \end{pmatrix}$

want this $= \underline{F} = \begin{pmatrix} x^2+1 \\ z-2xy \\ y \end{pmatrix}$ so we want $\partial_x g(x,y) = y$ so we can take $g(x,y) = xy$.

Ex 2: $\underline{F} = \langle y^2, z^2, x^2 \rangle$ then div $\underline{F} = 0$, so there exists $\underline{G}$ s.t. curl $\underline{G} = \underline{F}$

$G_1 = \int_0^z t^2\,dt = \frac{1}{3}z^3$

$G_2 = -\int_0^z y^2\,dt + g(x,y) = -y^2 z + g(x,y)$

$G_3 = 0$

curl $\underline{G} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{1}{3}z^3 & -y^2z + g(x,y) & 0 \end{vmatrix} = \begin{pmatrix} \partial_z y^2 z \\ -\partial_z \frac{1}{3}z^3 \\ -\partial_x y^2 z + \partial_x g(x,y) \end{pmatrix}$ want $= \begin{pmatrix} y^2 \\ z^2 \\ x^2 \end{pmatrix}$ i.e $\partial_x g(x,y) = x^2$

i.e $g(x,y) = \frac{1}{3}x^3$

So, $\underline{G} = \left\langle \frac{1}{3}z^3, -y^2 z + \frac{1}{3}x^3, 0 \right\rangle$

Section 8.4: Gauss' Divergence Theorem

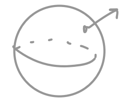
Recall: Green's + Stokes Related integrals over a surface to integrals over the surface's boundary curve.

Divergence theorem will relate an integral over a domain in $\mathbb{R}^3$ to an intregral over the boundary surface(s) of the domain.

Divergence Theorem: let $W \subseteq \mathbb{R}^3$ be a region with piecewise smooth boundary ∂W, orient each boundary surface in ∂W with the normal pointing <u>out</u> of the region W. Suppose $\underline{F}$ is a C^1 vector field on W. Then $\iiint_W \operatorname{div} \underline{F} \, dV = \iint_{\partial W} \underline{F} \cdot d\underline{S}$.

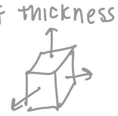
Example of regions:

i) If W is the unit ball (points (x,y,z) s.t. $x^2+y^2+z^2 \leq 1$) then ∂W is the unit sphere (points (x,y,z) s.t. $x^2+y^2+z^2=1$)



ii) Let B_1 be the ball of radius 1 and $B_{1/2}$ is the ball of radius 1/2. Let $W = B_1 / B_{1/2}$ This is a spherical shell. of thickness 1/2. Outer sphere must point out and inner point in, so that its away from shell.

iii)



Recall: Interpretation of divergence: If $\underline{F}$ is velocity vector field of a fluid, then $\operatorname{div} \underline{F}$ is the rate of expansion per unit volume under the flow of the fluid.

$\operatorname{div} \underline{F} > 0 \rightsquigarrow$ expansion

$\operatorname{div} \underline{F} < 0 \leftrightsquigarrow$ compression.

So $\iiint_W \operatorname{div} \underline{F} \, dV$ measures the net rate of expansion of the fluid within W.

On the other hand, $\iint_{\partial W} \underline{F} \cdot d\underline{S}$ (with ∂W oriented out from W) is the total flux out of the region W, i.e, the net rate of which the fluid flows out of W.

So the div theorem says: the net rate of expansion of fluid is within a volume W is equal to the net rate of flow out of the volume (no suprise that the div theorem comes up in conservation laws in physics.

Note: when a fluid in incompressible, i.e. when $\operatorname{div} \underline{F} = 0$, then div theorem says $\iint_{\partial W} \underline{F} \cdot d\underline{S} = 0$

This is to be expected: if a fluid is incompressible, then the amount of fluid in W must be constant along the flow. Any fluid flowing out of region W must be balanced by the same amount of fluid flowing into the region W. i.e, the total flux is zero.

More generally, for a physical vector field $\underline{F}$, one should view points where $\operatorname{div} \underline{F} > 0$ as being sources for $\underline{F}$, and points where $\operatorname{div} \underline{F} < 0$ as being sinks for $\underline{F}$. So div theorem says that the net contribution from sources and sinks must match exactly the net flow out of W.

Ex 1: $\underline{F} = \langle 2x, y^2, z^2 \rangle$ and let S be the unit sphere oriented outwards. Find $\iint_S \underline{F} \cdot d\underline{S}$.

Ans: If W = unit ball in $\mathbb{R}^3$, then $\partial W = S$, so by div theorem, $\iint_S \underline{F} \cdot d\underline{S} = \iiint_W \operatorname{div} \underline{F} \, dV$

$= \iiint_W (2 + 2y + 2z) \, dV \quad (*)$

could compute using spherical, but don't need to.

$\iiint_W y \, dV = \iiint_W z \, dV = 0$

by symmetry and the fact that y, z are odd functions, so

$(*) = \iiint_W 2 \, dV = 2 \operatorname{vol}(W) = \frac{8\pi}{3}$.

Ex 2: $\iint_{\partial W} (x^2 + y + z) \, dS \quad (\dagger)$ where W = unit ball.

Ans: Apply div theorem: find $\underline{F} \langle F_1, F_2, F_3 \rangle$ where $\underline{F} \cdot \underline{n} = x^2 + y^2 + z$

Where $\underline{n}$ = outwards pointing unit normal on ∂W. i.e, $\underline{n} = \langle x, y, z \rangle$, so $\underline{F} \cdot \underline{n} = F_1 x + F_2 y + F_3 z = x^2 + y + z$

i.e. $\underline{F} = \langle x, 1, 1 \rangle$ Then $\operatorname{div} \underline{F} = 1$ Hence, $(\dagger) = \iiint_W 1 \, dV = \operatorname{vol}(W) = \frac{4\pi}{3}$.

Ex 3: $\underline{F} = \langle y, x, xz \rangle$