# Problem Set I

1. The problem is to prove four properties of convolution. I assume the space is continuous.

   (a) Associativity: $(f * g) * h = f * (g * h)$
   Let $f(t) * g(t) = a(t)$, and $g(t) * h(t) = b(t)$,
   $a(t) * h(t) = \int_{-\infty}^{\infty} a(\tau)h(t-\tau)d\tau$
   $a(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)g(\tau)h(t-\tau)d\tau$
   $a(t) * h(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\lambda)g(\tau-\lambda)d\lambda h(t-\tau)d\tau$
   Let $\tau - \lambda = \gamma$, $\frac{d\gamma}{d\tau} = 1$
   $a(t) * h(t) = \int_{-\infty}^{\infty} f(\lambda)\int_{-\infty}^{\infty} g(\gamma)h(t-\gamma-\lambda)d\gamma d\lambda$
   $a(t) * h(t) = \int_{-\infty}^{\infty} f(\lambda)b(t-\lambda)d\lambda = f(t) * b(t)$
   Therefore, $(f * g) * h = f * (g * h)$

   (b) Distributivity: $f * (g + h) = f * g + f * h$
   $f * (g + h) = \int_{-\infty}^{\infty} f(\tau)(g(t-\tau) + h(t-\tau))d\tau$
   $f * (g + h) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau) + f(\tau)h(t-\tau)d\tau$
   $f * (g + h) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau + \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$
   $f * (g + h) = f * g + f * h$

   (c) Differentiation rule: $(f * g)' = f' * g = f * g'$
   $(f * g)'(t) = \frac{d(\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau)}{dt}$
   $(f * g)'(t) = \int_{-\infty}^{\infty} f(\tau)\frac{d(g(t-\tau))}{dt}d\tau$
   $(f * g)'(t) = \int_{-\infty}^{\infty} f(\tau)\frac{d(g(t-\tau))}{dt}d\tau = f(t) * g'(t)$
   based on commutativity:
   $(f * g)'(t) = \int_{-\infty}^{\infty} g(\tau)\frac{d(f(t-\tau))}{dt}d\tau = g(t) * f'(t)$

   (d) Convolution theorem: $F(g * h) = F(g)F(h)$
   Suppose $Fg(s)$ and $Fh(s)$ are Fourier transform of $g(t)$ and $h(t)$.
   $F(g * h) = \int_{-\infty}^{\infty}(g * h)e^{-2\pi ist}dt$
   $F(g * h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau e^{-2\pi ist}dt$
   Let $t - \tau = a$, $\frac{da}{dt} = 1$
   $F(g * h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau)h(a)e^{-2\pi is(a+\tau)}d\tau da$
   $F(g * h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau)e^{-2\pi is(\tau)}h(a)e^{-2\pi is(a)}d\tau da$
   $F(g * h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau)e^{-2\pi is(\tau)}d\tau h(a)e^{-2\pi is(a)}da$
   $F(g * h) = \int_{-\infty}^{\infty} Fg(s)h(a)e^{-2\pi is(a)}da$
   $F(g * h) = Fg(s)Fh(s)$
   $F(g * h) = F(g)F(h)$

2. The problem requires denoising using Fourier transform. First, I convert the entire image to frequency domain by Fourier transform. Then I aggregate low frequency to the middle of the domain based on periodic properties of Fourier transform to make it easier to drop high frequency pixels. Then I zero out high frequency points while keeping low frequency points with window size 10 (Figure: 1), 20 (Figure: 2), 40 (Figure: 3), 100 (Figure: 4) and up to the

full dimension. At last, I convert the frequency domain back to corresponding image spatial domain to get smoothed images. Low frequency points keep the main object of original image. So, if I gradually extend non-zero area, I will get the original image on Figure: 5.

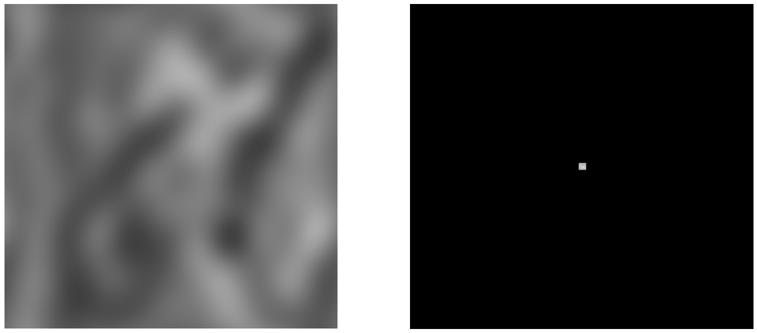

Figure 1: Number of low frequencies: $10^2$

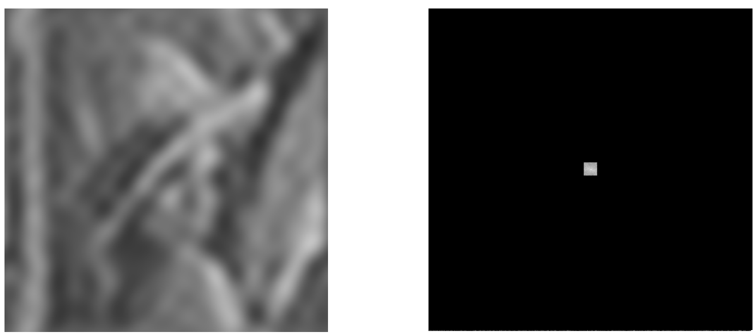

Figure 2: Number of low frequencies: $20^2$

Figure 3: Number of low frequencies: $40^2$

Figure 4: Number of low frequencies: $100^2$

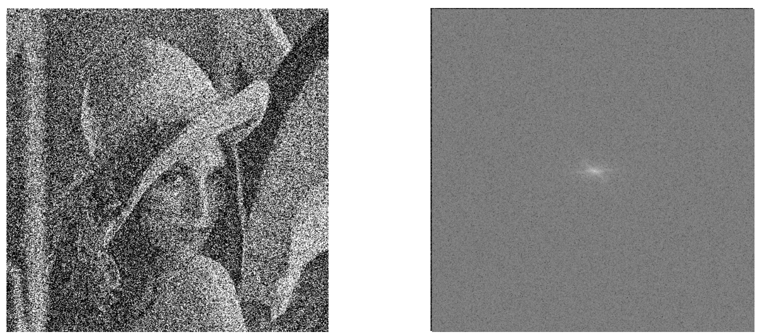

Figure 5: Original Image

3. The problem is to implement a ROF model for total variation denoising. First, energy function and gradient function of it are derived. The original noisy image is shown on Figure 6. In order to perform iterations, parameters including epsilon,learning rate (lr), lambda and initial values of u are initialized. Then I iteratively perform gradient descent algorithm on the estimated image u. The energy loss keeps decreasing and converges after about 80 iterations (Figure: 8). Central difference is applied to calculate gradient of the image. After it converges, u is optimized shown on Figure: 7.

Figure 6: Original Noisy Image

Figure 7: Denoised Image

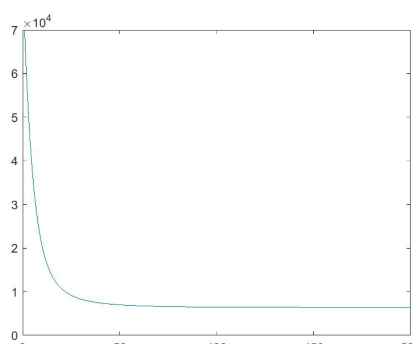


Figure 8: Graph of Decreasing Loss