Project

Due on August 7th, 2022

Consider the following ordinary differential equation for u:

$$-u''(x) + \pi^2 \cos^2(\pi x) u(x) = f(x) \quad x \in [0,1] \tag{1}$$

with boundary conditions:

$$\begin{aligned} u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{2}$$

1. Consider $f(x) = \pi^2 \sin(\pi x) \cosh(\sin(\pi x))$, and check that the function $u(x) = \sinh(\sin(\pi x))$ is the solution to the boundary value problem (BVP) (1)+(2).

solution. Consider $u(x) = \sinh(sin(\pi x))$,we have:

$$\begin{aligned} u(0) = \sinh(0) = 0, \\ u(1) = \sinh(0) = 0. \end{aligned} \tag{3}$$

and

$$u^{'}(x) = \cosh(\sin(\pi x))\pi \cos(\pi x) \tag{4}$$

$$u^{''}(x) = \pi^2 \sinh(\sin(\pi x))\pi \cos^2(\pi x) - \pi^2 \cosh(\sinh(\pi x)) \sin(\pi x) \tag{5}$$

so

$$\begin{aligned} &-u'' + \pi^2 \cos^2(\pi x) \\ =&\pi^2 \sin(\pi x) \cosh(\sinh(\pi x)) - \pi^2 \sinh(\sin(\pi x))\pi \cos^2(\pi x) \\ &+ \pi^2 \cos^2(\pi x) \sinh(\sin(\pi x)) \\ =&\pi^2 \sin(\pi x) \cosh(\sin(\pi x)) = f(x) \end{aligned} \tag{6}$$

2. We want to solve this BVP numerically. We begin by discretizing the interval $[0,1]$. For this, consider the gridpoints:

$$x_i = ih, \quad i = 0, 1, \ldots, n+1, \quad h = \frac{1}{n+1}. \tag{7}$$

Note that $h_i = x_{i+1} - x_i = h$ for all i. Now we approximate the second derivative. Show that if g has four continuous derivatives, then

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + O(h^2), \tag{8}$$

where $g_i = g(x_i)$.

solution. According to the Taylor expansion,we have the following equations:

$$g_{i+1} - g_i = g_i^{'}h + \frac{1}{2!}g_i^{''}h^2 + \frac{1}{3!}g_i^{'''}h^3 + O(h^4) \tag{9}$$

$$g_{i-1} - g_i = -g_i^{'}h + \frac{1}{2!}g_i^{''}h^2 - \frac{1}{3!}g_i^{'''}h^3 + O(h^4) \tag{10}$$

(9)+(10)

$$g_{i+1} - 2g_i + g_{i-1} = h^2 g_i^{''} + O(h^4) \tag{11}$$

that is:

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i^{''} + O(h^2) \tag{12}$$

3. Consider now the linear system of equations

$$-\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + \pi^2 \cos^2(\pi x_i)\, g_i = f(x_i) \quad i = 1, 2, \ldots, n. \tag{13}$$

Show that this can be rewritten in matrix form as

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f},$$

where $\boldsymbol{g} = (g_1, \ldots, g_n)^T$, $\boldsymbol{f} = (f_1, \ldots, f_n)^T$, and the matrix $\boldsymbol{A}$ is tridiagonal, with entries:

$$a_{i,j} = \begin{cases} -\frac{1}{h^2} & |i-j| = 1, \\ \frac{2}{h^2} + \pi^2 \cos^2(\pi x_i) & i = j, \\ 0 & Otherwise. \end{cases} \tag{14}$$

solution.

$$A = \begin{pmatrix} \frac{2}{h^2} + \pi^2 \cos^2(\pi x_1) & -\frac{1}{h^2} & & \\ -\frac{1}{h^2} & \frac{2}{h^2} + \pi^2 \cos^2(\pi x_2) & -\frac{1}{h^2} & \\ & & \ddots & \\ & & -\frac{1}{h^2} & \frac{2}{h^2} + \pi^2 \cos^2(\pi x_n) \end{pmatrix} \tag{15}$$

so according to the

$$\begin{aligned} & -\frac{1}{h^2}g_{i+1} + (\frac{2}{h^2} + \pi^2 \cos^2(\pi x_2)) \times g_i - \frac{1}{h^2}g_{i-1} \\ & = -\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + \pi^2 \cos^2(\pi x_i)\, g_i = f(x_i) \quad i = 1, 2, \ldots, n. \end{aligned} \tag{16}$$

and

$$g_0 = g_n = 0 \tag{17}$$

4. Show that Scheme (13) is second-order accurate.

solution. If $g(x)$ is the exact solution to the problem(1)+(2), that is:

$$-g''(x) + \pi^2 \cos^2(\pi x) g(x) = f(x) \tag{18}$$

according to the Taylor expansion

$$-\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + \pi^2 \cos^2(\pi x_i) + O(h^2) = f(x_i) \tag{19}$$

compaired with (13),the local truncation error of (13) is $O(h^2)$

5. Solve the system of equations (13). Use the following values: $n = 10$, 20, 40, 80, 160, 320. For each $h = 1/(n+1)$, compute the error

$$e(h) = \sup_{1 \le i \le n} |g_i - u(x_i)| \tag{20}$$

and do a log-log plot of $e(h)$, that is, plot $\log(e(h))$ as a function of $\log(h)$. Show, using this plot, that $e(h) = O(h^2)$, consistent with **4**.

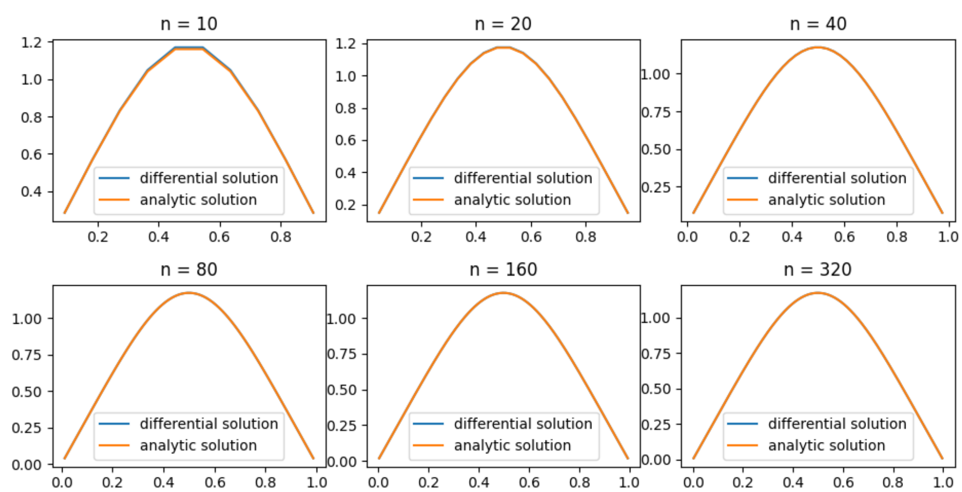

Figure 1: solution

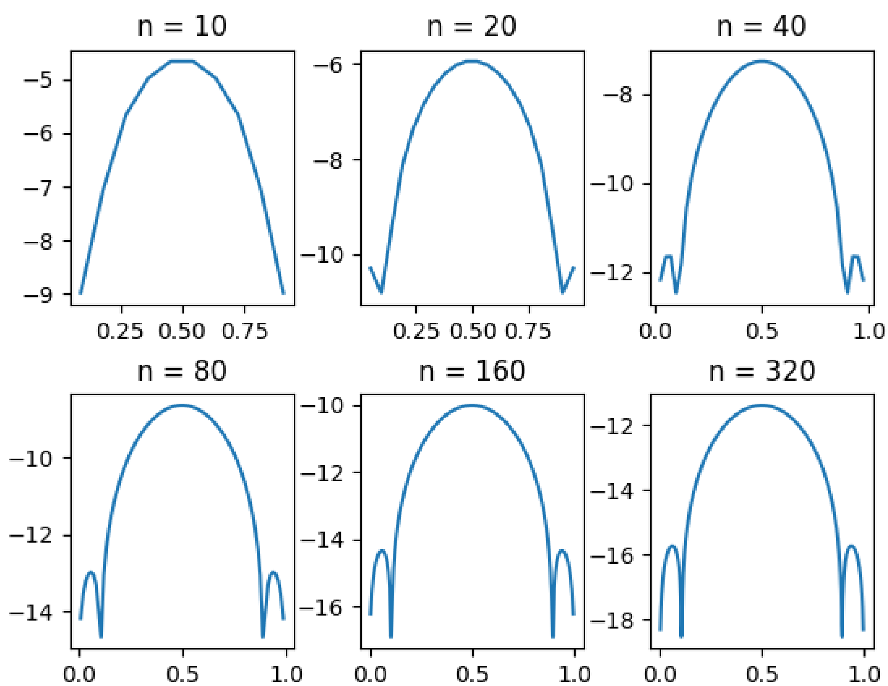

Figure 2: error

solution.

6. Consider a neural network solution in the following form

$$u_{\mathrm{NN}}(x) = x(1-x)\left(\sum_{i=1}^{n-1} u_i \sin(w_i x + b_i)\right), \tag{21}$$

which satisfies the boundary condition. The loss function in the least-squares sense is

$$\sum_{i=1}^{n-1} \left(-u''_{\mathrm{NN}}(x_i) + \pi^2 \cos^2(\pi x_i) u_{\mathrm{NN}}(x_i) - f(x_i)\right)^2. \tag{22}$$

Use Adam or stochastic gradient descent method to find the optimal set of parameters $\{u_i, w_i, b_i\}_{i=1}^{n-1}$. Use the following values: $n = 10$, 20, 40, 80, 160, 320. For each n, compute the error

$$e(n) = \sup_{1 \le i \le n} |u_{\mathrm{NN}}(x_i) - u(x_i)|. \tag{23}$$

What is the conclusion we can draw for the neural network solution? Compare this result with that of the second-order difference scheme.

Send your project with your name, your student ID to jingrunchen@ustc.edu.cn.