

# The Solution of Model Predictive Control: Theory, Computation, and Design [1]

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# Chapter 1

## Getting Started with Model Predictive Control

### 1.1 Brief Review

In this section, we just consider state space linear time invariant system with zero steady state.

**Lemma 1.3** (LQR convergence). For  $(A, B)$  controllable, the infinite LQR gives a convergent closed-loop system.

*Proof.* Because  $(A, B)$  is controllable, there exists a sequence of  $n$  inputs that transfers the state to zero. When  $k > n$ , we let  $u = 0$ , then the objective function  $V(x, u) = \sum_{k=0}^{\infty} x_k^T Q x_k + u^T R u$  is finite, which implies the optimization problem is feasible. On the other hand, the solution is unique since  $R > 0$  and the objective function is strict convex with  $u$ .

So the solution of the LQR problem exists and is unique. This implies to that the objective function is non-increasing with time, and we have  $x \rightarrow 0, u \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

**Remark.** The optimal solution can be calculate from Riccati equation, which is from backward dynamic programming similar to Kalman filter.

$$\begin{aligned} K &= -(B^T P B + R)^{-1} B^T P A \\ P &= Q + A^T P A - A^T P B (B^T P B + R)^{-1} B^T P A \end{aligned}$$

### 1.2 The Solution of Exercises

**Exercise 1.1.** State space form for chemical reaction model.

Consider the following chemical reaction kinetics for a two-step series reaction



We wish to follow the reaction in a constant volume, well-mixed, batch reactor. As taught in the undergraduate chemical engineering curriculum, we proceed by writing material balances for the three species giving

$$\frac{dc_A}{dt} = -r_1 \quad \frac{dc_B}{dt} = r_1 - r_2 \quad \frac{dc_C}{dt} = r_2 \quad (1.2)$$

in which  $c_j$  is the concentration of species  $j$ , and  $r_1$  and  $r_2$  are the rates (mol/(time·vol)) at which the two reactions occur. We then assume some rate law for the reaction kinetics, such as

$$r_1 = k_1 c_A \quad r_2 = k_2 c_B \quad (1.3)$$

We substitute the rate laws into the material balances and specify the starting concentrations to produce three differential equations for the three species concentrations.

- (a) write the linear state space model for the deterministic series chemical reaction model. Assume we can measure the component A concentration. What are  $x$ ,  $y$ ,  $A$ ,  $B$ ,  $C$ , and  $D$  for this model?
- (b) Simulate this model with initial conditions and parameters given by

$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \quad k_1 = 2 \quad k_2 = 1$$

**Answer 1.** (a) the linear state space model is

$$\frac{dx}{dt} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x = Ax \quad (1.4)$$

where  $x = [c_A, c_B, c_C]^\top$ .  $B$  does not exist because there is no system input variables.  $C = [1, 0, 0]^\top$ ,  $D = 0$ ,  $y = Cx$ .

- (b) the simulation result is shown as Fig.1.1. The code used in all of the exercise can be found in github <https://github.com/lixc21/MPC-Solution>.

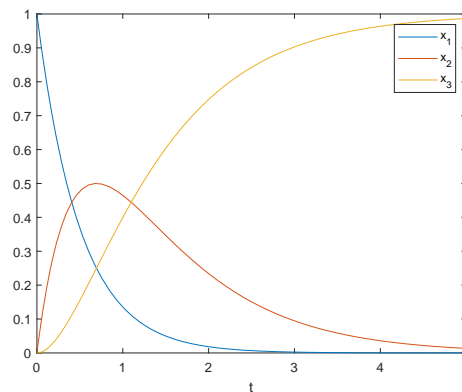


Figure 1.1: system simulation

**Exercise 1.2.** Distributed systems and time delay.

We assume familiarity with the transfer function of a time delay from an undergraduate systems course

$$\bar{y}(s) = e^{-\theta s} \bar{u}(s) \quad (1.5)$$

Let's see the connection between the delay and the distributed systems, which give rise to it. A simple physical example of a time delay caused by transport in a flowing system. Consider plug flow in a tube depicted in Fig.1.2.

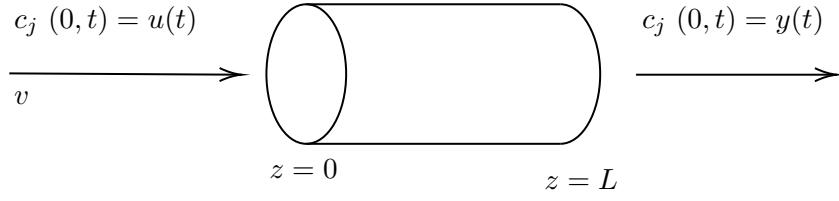


Figure 1.2: Plug-flow reactor

- (a) Write down the equation of change for moles of component  $j$  for an arbitrary volume element and show that

$$\frac{\partial c_j}{\partial t} = -\nabla \cdot (c_j v_j) + R_j \quad (1.6)$$

in which  $c_j$  is the molar concentration of component  $j$ ,  $v_j$  is the velocity of component  $j$ , and  $R_j$  is the production rate of component  $j$  due to chemical reaction. Plug flow means the fluid velocity of all components is purely in the  $z$  direction, and is independent of  $r$  and  $\theta$  and, we assume here,  $z$

$$v_j = v \delta_z \quad (1.7)$$

- (b) Assuming plug flow and neglecting chemical reaction in the tube, show that the equation of change reduces to

$$\frac{\partial c_j}{\partial t} = -v \frac{\partial c_j}{\partial z} \quad (1.8)$$

This equation is known as a hyperbolic, first-order partial differential equation.

$$c_j(z, t) = u(t) \quad 0 = z \quad t \geq 0 \quad (1.9)$$

$$c_j(z, t) = c_{j0}(t) \quad 0 \leq z \leq L \quad t = 0 \quad (1.10)$$

In other words, we are using the feed concentration as the manipulated variable,  $u(t)$ , and the tube starts out with some initial concentration profile of component  $j$ ,  $c_{j0}(z)$ .

- (c) Show that the solution to (1.8) with these boundary conditions is

$$c_j(z, t) = \begin{cases} u(t - z/v) & vt > z \\ c_{j0}(z - vt) & vt < z \end{cases} \quad (1.11)$$

- (d) If the reactor starts out empty of component  $j$ , show that the transfer function between the outlet concentration,  $y = c_j(L, t)$ , and the inlet concentration,  $c_j(0, t) = u(t)$ , is a time delay. What is the value of  $\theta$ ?

**Answer 2.** (a) let  $f$  be the moles of one of the component, then from 3D Leibniz formula, we get

$$\frac{\partial c_j}{\partial t} = \frac{d}{dt} \int_V f(\vec{x}, t) dV = \int_V \frac{\partial f}{\partial t} dV - \int_A f \vec{v} \cdot \vec{n} dV \quad (1.12)$$

where  $V$  is a small unit volume,  $A$  is the responding surface,  $v$  represent the velocity of the point on the surface,  $n$  is the outward unit normal vector related

to  $u$ .

By using Gauss divergence theorem, we know that

$$\int_V \frac{\partial f}{\partial t} dV - \int_A f \vec{v} \cdot \vec{n} dV = \int_V \frac{\partial f}{\partial t} - \nabla \cdot f \vec{v} dV = -\nabla \cdot (c_j v_j) + R_j \quad (1.13)$$

where the last equation comes from  $v_j = v \delta_z$ .

- (b) Neglecting chemical reaction in the tube, we get  $R_j = 0$ . Then we know that

$$\frac{\partial c_j}{\partial t} = -\nabla \cdot (c_j v_j) = -\left(\frac{\partial}{\partial x} \delta_x + \frac{\partial}{\partial y} \delta_y + \frac{\partial}{\partial z} \delta_z\right) \cdot v \delta_z = -v \frac{\partial c_j}{\partial z} \quad (1.14)$$

- (c) Assuming that  $u(t - z/v) = c_{j0}(z - vt)$  when  $vt < z$ , we just need to prove the solution is  $c_j(z, t) = u(t - z/v)$ . The variables of original partial differential equation has already been separated, so we get  $c_j(z, t) = u(t - z/v)$  easily from the method of characteristics.

- (d) We know that  $y = u(t - L/v)$ , which is a time delay. The value of  $\theta$  could be  $L/v$ .

**Exercise 1.3.** Pendulum in the state space.

Consider the pendulum suspended at the end of a rigid link depicted in Figure 1.3. Let  $r$  and  $\theta$  denote the polar coordinates of the center of the pendulum, and let  $p = r \delta_r$  be the position vector of the pendulum, in which  $\delta_r$  and  $\delta_\theta$  are the unit vectors in polar coordinates. We wish to determine a state space description of the system. We are able to apply a torque  $T$  to the pendulum as our manipulated variable. The pendulum has mass  $m$ , the only other external force acting on the pendulum is gravity, and we neglect friction. The link provides force  $-t \delta_r$  necessary to maintain the pendulum at distance  $r = R$  from the axis of rotation, and we measure the force  $t$ .

- (a) Provide expressions for the four partial derivatives for changes in the unit vectors with  $r$  and  $\theta$

$$\frac{\partial \delta_r}{\partial r} \quad \frac{\partial \delta_r}{\partial \theta} \quad \frac{\partial \delta_\theta}{\partial r} \quad \frac{\partial \delta_\theta}{\partial \theta} \quad (1.15)$$

- (b) Use the chain rule to find the velocity of the pendulum in terms of the time derivatives of  $r$  and  $\theta$ . Do not simplify yet by assuming  $r$  is constant. We want the general result.

- (c) Differentiate again to show that the acceleration of the pendulum is

$$\ddot{p} = (\ddot{r} - r\dot{\theta}^2)\delta_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\delta_\theta \quad (1.16)$$

- (d) Use a momentum balance on the pendulum mass (you may assume it is a point mass) to determine both the force exerted by the link

$$t = mR\dot{\theta}^2 + mg \cos \theta \quad (1.17)$$

and an equation for the pendulum due to gravity and the applied torque

$$mR\ddot{\theta} - T/R + mg \sin \theta = 0 \quad (1.18)$$

- (e) Define a state vector and give a state space description of your system. What is the physical significance of your state. Assume you measure the force exerted by the link.

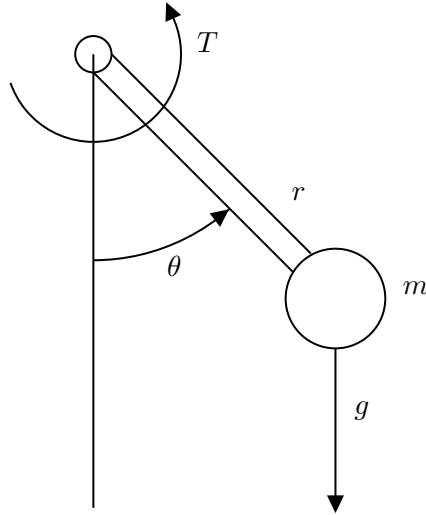


Figure 1.3: Pendulum with applied torque

One answer is

$$\frac{dx_1}{dt} = x_2 \quad (1.19)$$

$$\frac{dx_2}{dt} = -(g/R) \sin x_1 + u \quad (1.20)$$

$$y = mRx_2^2 + mg \cos x_1 \quad (1.21)$$

in which  $u = T/(mR)$

**Answer 3.** 1.

## Chapter 2

# Appendix A. Mathematical Background

Since the mathematical background is basic, we jump to the exercises section.

### 2.1 The solution of Exercises

**Exercise 2.1.** Norms in  $\mathbb{R}^n$

Show that the following three functions are all norms in  $\mathbb{R}^n$

$$\|x\|_2$$

# Bibliography

- [1] James Blake Rawlings, David Q Mayne, and Moritz Diehl. *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI, 2017.