# The Solution of Model Predictive Control: Theory, Computation, and Design [1]

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# Chapter 1

# Getting Started with Model Predictive Control

## 1.1 Brief Review

In this section, we just consider state space linear time invariant system with zero steady state.

**Lemma 1.3** (LQR convergence). For $(A, B)$ controllable, the infinite LQR gives a convergent closed-loop system.

*Proof.* Because $(A, B)$ is controllable, there exists a sequence of $n$ inputs that transfers the state to zero. When $k > n$, we let $u = 0$, then the objective function $V(x, u) = \sum_{k=0}^{\infty} x_k^\top Q x_k + u^\top R u$ is finite, which implies the optimization problem is feasible. On the other hand, the solution is unique since $R > 0$ and the objective function is strict convex with $u$.

So the solution of the LQR problem exists and is unique. This implies to that the objective function is non-increasing with time, and we have $x \to 0$, $u \to 0$ as $k \to 0$. $\square$

**Remark.** *The optimal solution can be calculate from Riccati equation, which is from backward dynamic programming similar to Kalman filter.*

$$
\begin{aligned}
K &= -(B^\top P B + R)^{-1} B^\top P A \\
P &= Q + A^\top P A - A^\top P B (B^\top P B + R)^{-1} B^\top P A
\end{aligned}
$$

## 1.2 The Solution of Exercises

**Exercise 1.1.** State space form for chemical reaction model.
Consider the following chemical reaction kinetics for a two-step series reaction

$$
A \xrightarrow{k_1} B \qquad B \xrightarrow{k_2} C \tag{1.1}
$$

We wish to follow the reaction in a constant volume, well-mixed, batch reactor. As taught in the undergraduate chemical engineering curriculum, we proceed by writing material balances for the three species giving

$$
\frac{\mathrm{d}c_A}{\mathrm{d}t} = -r_1 \qquad \frac{\mathrm{d}c_B}{\mathrm{d}t} = r_1 - r_2 \qquad \frac{\mathrm{d}c_C}{\mathrm{d}t} = r_2 \tag{1.2}
$$

in which $c_j$ is the concentration of species $j$, and $r_1$ and $r_2$ are the rates (mol/(time·vol)) at which the two reactions occur. We then assume some rate law for the reaction kinetics, such as

$$r_1 = k_1 c_A \qquad r_2 = k_2 c_B \tag{1.3}$$

We substitute the rate laws into the material balances and specify the starting concentrations to produce three differentia equations for the three species concentrations.

(a) write the linear state space model for the deterministic series chemical reaction model. Assume we can measure the component A concentration. What are $x$, $y$, $A$, $B$, $C$, and $D$ for this model?

(b) Simulate this model with initial conditions and parameters given by

$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \quad k_1 = 2 \quad k_2 = 1$$

**Answer 1.** (a) the linear state space model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x = Ax \tag{1.4}$$

where $x = [c_A,\ c_B,\ c_C]^\top$. $B$ does not exist because there is no system input variables. $C = [1,\ 0,\ 0]^\top$, $D = 0$, $y = Cx$.

(b) the simulation result is shown as Fig.1.1. The code used in all of the exercise can be found in github `https://github.com/lixc21/MPC-Solution`.

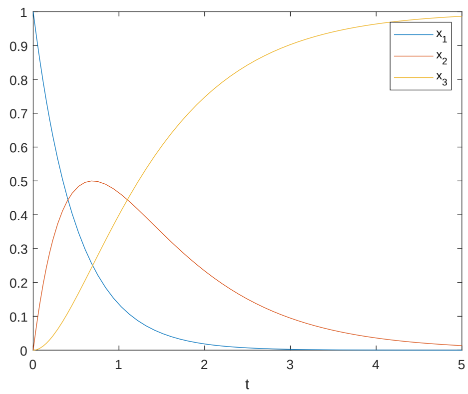


Figure 1.1: system simulation

**Exercise 1.2.** Distributed systems and time delay.
We assume familiarity with the transfer function of a time delay from an undergraduate systems course

$$\bar{y}(s) = e^{-\theta s}\bar{u}(s) \tag{1.5}$$

Let's see the connection between the delay and the distributed systems, which give rise to it. A simple physical example of a time delay caused by transport in a flowing system. Consider plug flow in a tube depicted in Fig.1.2.

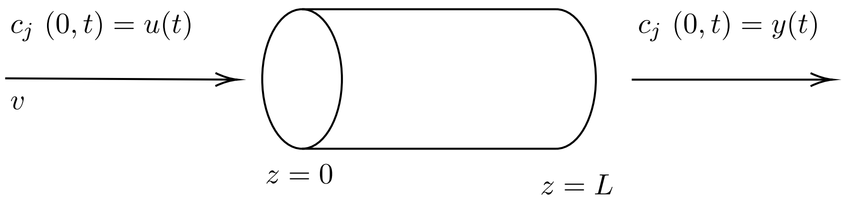


Figure 1.2: Plug-flow reactor

(a) Write down the equation of change for moles of component $j$ for an arbitrary volume element and show that

$$\frac{\partial c_j}{\partial t} = -\nabla \cdot (c_j v_j) + R_j \tag{1.6}$$

in which $c_j$ is the molar concentration of component $j$, $v_j$ is the velocity of component $j$, and $R_j$ is the production rate of component $j$ due to chemical reaction.

Plug flow means the fluid velocity of all components os purely in the $z$ direction, and os independent of $r$ and $\theta$ and, we assume here, z

$$v_j = v\delta_z \tag{1.7}$$

(b) Assuming plug flow and neglecting chemical reaction in the tube, show that the equation of change reduces to

$$\frac{\partial c_j}{\partial t} = -v\frac{\partial c_j}{\partial z} \tag{1.8}$$

This equation is known as a hyperbolic, first-order partial differential equation.

$$c_j(z,t) = u(t) \qquad 0 = z \qquad t \geq 0 \tag{1.9}$$

$$c_j(z,t) = c_{j0}(t) \qquad 0 \leq z \leq L \qquad t = 0 \tag{1.10}$$

In other words, we are using the feed concentration as the manipulated variable, $u(t)$, and the tube starts out with some initial concentration profile of component $j$, $c_{j0}(z)$.

(c) Show that the solution to (1.8) with these boundary conditions is

$$c_j(z,t) = \begin{cases} u(t - z/v) & vt > z \\ c_{j0}(z - vt) & vt < z \end{cases} \tag{1.11}$$

(d) If the reactor start out empty of component $j$, show that the transfer function between the outlet concentration, $y = c_j(L,t)$, and the inlet concentration, $c_j(0,t) = u(t)$, is a time delay. What is the value of $\theta$?

**Answer 2.** (a) let $f$ be the moles of one of the component, then from 3D Leibniz formula, we get

$$\frac{\partial c_j}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V f(\vec{x},t)\ \mathrm{d}V = \int_V \frac{\partial f}{\partial t}\ \mathrm{d}V - \int_A f\vec{v}\cdot\vec{n}\ dV \tag{1.12}$$

where $V$ is a small unit volume, $A$ is the responding surface, $v$ represent the velocity of the point on the surface, $n$ is the outward unit normal vector related

to $u$.
By using Gauss divergence theorem, we know that

$$\int_V \frac{\partial f}{\partial t}\,\mathrm{d}V - \int_A f\vec{v}\cdot\vec{n}\,\mathrm{d}V = \int_V \frac{\partial f}{\partial t} - \nabla\cdot f\vec{v}\,\mathrm{d}V = -\nabla\cdot(c_j v_j) + R_j \tag{1.13}$$

where the last equation comes from $v_j = v\delta_z$.

(b) Neglecting chemical reaction in the tube, we get $R_j = 0$. Then we know that

$$\frac{\partial c_j}{\partial t} = -\nabla\cdot(c_j v_j) = -\left(\frac{\partial}{\partial x}\delta_x + \frac{\partial}{\partial y}\delta_y + \frac{\partial}{\partial z}\delta_z\right)\cdot v\delta_z = -v\frac{\partial c_j}{\partial z} \tag{1.14}$$

(c) Assuming that $u(t - z/v) = c_{j0}(z - vt)$ when $vt < z$, we just need to prove the solution is $c_j(z,t) = u(t - z/v)$. The variables of original partial differential equation has already been separated, so we get $c_j(z,t) = u(t - z/v)$ easily from the method of characteristics.

(d) We know that $y = u(t - L/v)$, which is a time delay. The value of $\theta$ could be $L/v$.

**Exercise 1.3.** Pendulum in the state space.
Consider the pendulum suspended at the end of a rigid link depicted in Figure 1.3. Let $r$ and $\theta$ denote the polar coordinates of the center of the pendulum, and let $p = r\delta_r$ be the position vector of the pendulum, in which $\delta_r$ and $\delta_\theta$ are the unit vectors in polar coordinates. We wish to determine a state space description of the system. We are able to apply a torque $T$ to the pendulum as our manipulated variable. The pendulum has mass $m$, the only other external force acting on the pendulum is gravity, and we neglect friction. The link provides force $-t\delta_r$ necessary to maintain the pendulum at distance $r = R$ from the axis of rotation, and we measure the force $t$.

(a) Provide expressions for the four partial derivatives for changes in the unit vectors with $r$ and $\theta$

$$\frac{\partial \delta_r}{\partial r} \quad \frac{\partial \delta_r}{\partial \theta} \quad \frac{\partial \delta_\theta}{\partial r} \quad \frac{\partial \delta_\theta}{\partial \theta} \tag{1.15}$$

(b) Use the chain rule to find the velocity of the pendulum in terms of the time derivatives of $r$ and $\theta$. Do not simplify yet by assuming $r$ is constant. We want the general result.

(c) Differentiate again to show that the acceleration of the pendulum is

$$\ddot{p} = (\ddot{r} - r\dot{\theta}^2)\delta_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\delta_\theta \tag{1.16}$$

(d) Use a momentum balance on the pendulum mass (you may assume it is a point mass) to determine both the force exerted by the link

$$t = mR\dot{\theta}^2 + mg\cos\theta \tag{1.17}$$

and an eqaution for the pendulum due to gravity and the applied torque

$$mR\ddot{\theta} - T/R + mg\sin\theta = 0 \tag{1.18}$$

(e) Define a state vector and give a state space description of your system. What is the physical significance of your state. Assume you measure the force exerted by the link.

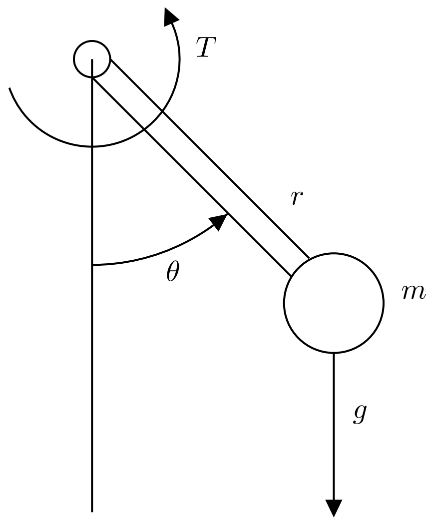


Figure 1.3: Pendulum with applied torque

One answer is

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \tag{1.19}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -(g/R)\sin x_1 + u \tag{1.20}$$

$$y = mRx_2^2 + mg\cos x_1 \tag{1.21}$$

in which $u = T/(mR)$

# Chapter 2

# Appendix A. Mathematical Background

Since the mathematical background is basic, we jump to the exercises section.

## 2.1 The solution of Exercises

**Exercise 2.1.** Norms in $\mathbb{R}^n$
Show that the following three functions are all norms in $\mathbb{R}^n$

$$\|x\|_2$$

# Bibliography

[1] James Blake Rawlings, David Q Mayne, and Moritz Diehl. *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI, 2017.