The Solution of Model Predictive Control: Theory, Computation, and Design

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Chapter 1

Getting Started with Model Predictive Control

1.1 Brief Review

In this section, we just consider state space linear time invariant system with zero steady state.

Lemma 1.3 (LQR convergence). For (A, B) controllable, the infinite LQR gives a convergent closed-loop system.

Proof. Because (A, B) is controllable, there exists a sequence of n inputs that transfers the state to zero. When $k > n$, we let $u = 0$, then the objective function $V(x, u) = \sum_{k=0}^{\infty} x_k^\top Q x_k + u^\top R u$ is finite, which implies the optimization problem is feasible. On the other hand, the solution is unique since $R > 0$ and the objective function is strict convex with u.

So the solution of the LQR problem exists and is unique. This implies to that the objective function is non-increasing with time, and we have $x \to 0$, $u \to 0$ as $k \to 0$. $\square$

Remark. *The optimal solution can be calculate from Riccati equation, which is from backward dynamic programming similar to Kalman filter.*

$$
\begin{aligned}
K &= -(B^\top P B + R)^{-1} B^\top P A \\
P &= Q + A^\top P A - A^\top P B (B^\top P B + R)^{-1} B^\top P A
\end{aligned}
$$

1.2 The Solution of Exercises

Exercise 1.1. State space form for chemical reaction model.
Consider the following chemical reaction kinetics for a two-step series reaction

$$A \xrightarrow{k_1} B \qquad B \xrightarrow{k_2} C \tag{1.1}$$

We wish to follow the reaction in a constant volume, well-mixed, batch reactor. As taught in the undergraduate chemical engineering curriculum, we proceed by writing material balances for the three species giving

$$\frac{\mathrm{d}c_A}{\mathrm{d}t} = -r_1 \qquad \frac{\mathrm{d}c_B}{\mathrm{d}t} = r_1 - r_2 \qquad \frac{\mathrm{d}c_C}{\mathrm{d}t} = r_2 \tag{1.2}$$

in which c_j is the concentration of species j, and r_1 and r_2 are the rates (mol/(time·vol)) at which the two reactions occur. We then assume some rate law for the reaction kinetics, such as

$$r_1 = k_1 c_A \qquad r_2 = k_2 c_B \tag{1.3}$$

We substitute the rate laws into the material balances and specify the starting concentrations to produce three differentia equations for the three species concentrations.

(a) write the linear state space model for the deterministic series chemical reaction model. Assume we can measure the component A concentration. What are x, y, A, B, C, and D for this model?

(b) Simulate this model with initial conditions and parameters given by

$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \quad k_1 = 2 \quad k_2 = 1$$

Answer 1. (a) the linear state space model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x = Ax \tag{1.4}$$

where $x = [c_A,\ c_B,\ c_C]^\top$. B does not exist because there is no system input variables. $C = [1,\ 0,\ 0]^\top$, $D = 0$, $y = Cx$.

(b) the simulation result and code is shown bellow.

Listing 1.1: ./code_ch1/exer1.m

```
%% init
x=[1 0 0]';
k1=2;
k2=1;
A=[-k1 0 0; k1 -k2 0; 0 k2 0];
for
ode45
```

Chapter 2

Appendix A. Mathematical Background

Since the mathematical background is basic, we jump to the exercises section.

2.1 The solution of Exercises

Exercise 2.1. Norms in $\mathbb{R}^n$
Show that the following three functions are all norms in $\mathbb{R}^n$

$$\|x\|_2$$