The Solution of Model Predictive Control: Theory, Computation, and Design [1]

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Chapter 1

Getting Started with Model Predictive Control

1.1 Brief Review

In this section, we just consider state space linear time invariant system with zero steady state.

Lemma 1.3 (LQR convergence). For (A, B) controllable, the infinite LQR gives a convergent closed-loop system.

Proof. Because (A, B) is controllable, there exists a sequence of n inputs that transfers the state to zero. When $k > n$, we let $u = 0$, then the objective function $V(x, u) = \sum_{k=0}^{\infty} x_k^\top Q x_k + u^\top R u$ is finite, which implies the optimization problem is feasible. On the other hand, the solution is unique since $R > 0$ and the objective function is strict convex with u.

So the solution of the LQR problem exists and is unique. This implies to that the objective function is non-increasing with time, and we have $x \to 0$, $u \to 0$ as $k \to 0$. $\square$

Remark. *The optimal solution can be calculate from Riccati equation, which is from backward dynamic programming similar to Kalman filter.*

$$
\begin{aligned}
K &= -(B^\top P B + R)^{-1} B^\top P A \\
P &= Q + A^\top P A - A^\top P B (B^\top P B + R)^{-1} B^\top P A
\end{aligned}
$$

1.2 The Solution of Exercises

Exercise 1.1. State space form for chemical reaction model.
Consider the following chemical reaction kinetics for a two-step series reaction

$$A \xrightarrow{k_1} B \qquad B \xrightarrow{k_2} C \tag{1.1}$$

We wish to follow the reaction in a constant volume, well-mixed, batch reactor. As taught in the undergraduate chemical engineering curriculum, we proceed by writing material balances for the three species giving

$$\frac{\mathrm{d}c_A}{\mathrm{d}t} = -r_1 \qquad \frac{\mathrm{d}c_B}{\mathrm{d}t} = r_1 - r_2 \qquad \frac{\mathrm{d}c_C}{\mathrm{d}t} = r_2 \tag{1.2}$$

in which c_j is the concentration of species j, and r_1 and r_2 are the rates (mol/(time·vol)) at which the two reactions occur. We then assume some rate law for the reaction kinetics, such as

$$r_1 = k_1 c_A \qquad r_2 = k_2 c_B \tag{1.3}$$

We substitute the rate laws into the material balances and specify the starting concentrations to produce three differentia equations for the three species concentrations.

(a) write the linear state space model for the deterministic series chemical reaction model. Assume we can measure the component A concentration. What are x, y, A, B, C, and D for this model?

(b) Simulate this model with initial conditions and parameters given by

$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \quad k_1 = 2 \quad k_2 = 1$$

Answer 1. (a) the linear state space model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x = Ax \tag{1.4}$$

where $x = [c_A,\ c_B,\ c_C]^\top$. B does not exist because there is no system input variables. $C = [1,\ 0,\ 0]^\top$, $D = 0$, $y = Cx$.

(b) the simulation result is shown as Fig.1.1. The code used in all of the exercise can be found in github `https://github.com/lixc21/MPC-Solution`.

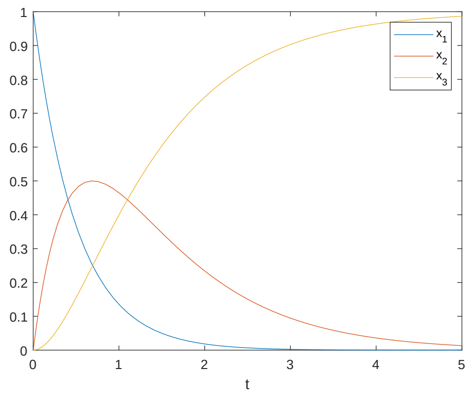

Figure 1.1: system simulation

Exercise 1.2. Distributed systems and time delay.
We assume familiarity with the transfer function of a time delay from an undergraduate systems course

$$\bar{y}(s) = e^{-\theta s}\bar{u}(s) \tag{1.5}$$

Let's see the connection between the delay and the distributed systems, which give rise to it. A simple physical example of a time delay caused by transport in a flowing system. Consider plug flow in a tube depicted in Fig.1.2.

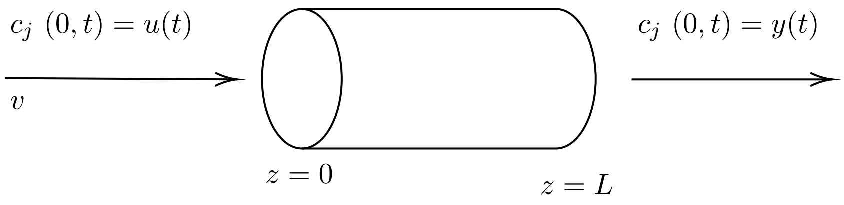

Figure 1.2: Plug-flow reactor

(a) Write down the equation of change for moles of component j for an arbitrary volume element and show that

$$\frac{\partial c_j}{\partial t} = -\nabla \cdot (c_j v_j) + R_j \tag{1.6}$$

in which c_j is the molar concentration of component j, v_j is the velocity of component j, and R_j is the production rate of component j due to chemical reaction.

Plug flow means the fluid velocity of all components os purely in the z direction, and os independent of r and θ and, we assume here, z

$$v_j = v\delta_z \tag{1.7}$$

(b) Assuming plug flow and neglecting chemical reaction in the tube, show that the equation of change reduces to

$$\frac{\partial c_j}{\partial t} = -v\frac{\partial c_j}{\partial z} \tag{1.8}$$

This equation is known as a hyperbolic, first-order partial differential equation.

$$c_j(z,t) = u(t) \qquad 0 = z \qquad t \geq 0 \tag{1.9}$$
$$c_j(z,t) = c_{j0}(t) \qquad 0 \leq z \leq L \qquad t = 0 \tag{1.10}$$

In other words, we are using the feed concentration as the manipulated variable, $u(t)$, and the tube starts out with some initial concentration profile of component j, $c_{j0}(z)$.

(c) Show that the solution to (1.8) with these boundary conditions is

$$c_j(z,t) = \begin{cases} u(t - z/v) & vt > z \\ c_{j0}(z - vt) & vt < z \end{cases} \tag{1.11}$$

(d) If the reactor start out empty of component j, show that the transfer function between the outlet concentration, $y = c_j(L,t)$, and the inlet concentration, $c_j(0,t) = u(t)$, is a time delay. What is the value of θ?

Answer 2. (a) let f be the moles of one of the component, then from 3D Leibniz formula, we get

$$\frac{\partial c_j}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V f(\vec{x},t)\ \mathrm{d}V = \int_V \frac{\partial f}{\partial t}\ \mathrm{d}V - \int_A f\vec{v}\cdot\vec{n}\ dV \tag{1.12}$$

where V is a small unit volume, A is the responding surface, v represent the velocity of the point on the surface, n is the outward unit normal vector related to u.

By using Gauss divergence theorem, we know that

$$\int_V \frac{\partial f}{\partial t}\ \mathrm{d}V - \int_A f\vec{v}\cdot\vec{n}\ \mathrm{d}V = \int_V \frac{\partial f}{\partial t} - \nabla\cdot f\vec{v}\ \mathrm{d}V = -\nabla\cdot(c_j v_j) + R_j \tag{1.13}$$

(b)

Chapter 2

Appendix A. Mathematical Background

Since the mathematical background is basic, we jump to the exercises section.

2.1 The solution of Exercises

Exercise 2.1. Norms in $\mathbb{R}^n$
Show that the following three functions are all norms in $\mathbb{R}^n$

$$\|x\|_2$$

Bibliography

[1] James Blake Rawlings, David Q Mayne, and Moritz Diehl. *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI, 2017.