The Solution of Model Predictive Control: Theory, Computation, and Design [1]

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Chapter 1

Getting Started with Model Predictive Control

1.1 Brief Review

In this section, we just consider state space linear time invariant system with zero steady state.

Lemma 1.3 (LQR convergence). For (A, B) controllable, the infinite LQR gives a convergent closed-loop system.

Proof. Because (A, B) is controllable, there exists a sequence of n inputs that transfers the state to zero. When $k > n$, we let $u = 0$, then the objective function $V(x, u) = \sum_{k=0}^{\infty} x_k^\top Q x_k + u^\top R u$ is finite, which implies the optimization problem is feasible. On the other hand, the solution is unique since $R > 0$ and the objective function is strict convex with u.

So the solution of the LQR problem exists and is unique. This implies to that the objective function is non-increasing with time, and we have $x \to 0$, $u \to 0$ as $k \to 0$. $\square$

Remark. *The optimal solution can be calculate from Riccati equation, which is from backward dynamic programming similar to Kalman filter.*

$$
\begin{aligned}
K &= -(B^\top P B + R)^{-1} B^\top P A \\
P &= Q + A^\top P A - A^\top P B (B^\top P B + R)^{-1} B^\top P A
\end{aligned}
$$

1.2 The Solution of Exercises

Exercise 1.1. State space form for chemical reaction model.
Consider the following chemical reaction kinetics for a two-step series reaction

$$A \xrightarrow{k_1} B \qquad B \xrightarrow{k_2} C \tag{1.1}$$

We wish to follow the reaction in a constant volume, well-mixed, batch reactor. As taught in the undergraduate chemical engineering curriculum, we proceed by writing material balances for the three species giving

$$\frac{\mathrm{d}c_A}{\mathrm{d}t} = -r_1 \qquad \frac{\mathrm{d}c_B}{\mathrm{d}t} = r_1 - r_2 \qquad \frac{\mathrm{d}c_C}{\mathrm{d}t} = r_2 \tag{1.2}$$

in which c_j is the concentration of species j, and r_1 and r_2 are the rates (mol/(time·vol)) at which the two reactions occur. We then assume some rate law for the reaction kinetics, such as

$$r_1 = k_1 c_A \qquad r_2 = k_2 c_B \tag{1.3}$$

We substitute the rate laws into the material balances and specify the starting concentrations to produce three differentia equations for the three species concentrations.

(a) write the linear state space model for the deterministic series chemical reaction model. Assume we can measure the component A concentration. What are x, y, A, B, C, and D for this model?

(b) Simulate this model with initial conditions and parameters given by

$$c_{A0} = 1 \quad c_{B0} = c_{C0} = 0 \quad k_1 = 2 \quad k_2 = 1$$

Answer 1. (a) the linear state space model is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} x = Ax \tag{1.4}$$

where $x = [c_A,\ c_B,\ c_C]^\top$. B does not exist because there is no system input variables. $C = [1,\ 0,\ 0]^\top$, $D = 0$, $y = Cx$.

(b) the simulation result is shown as Fig.1.1. The code used in all of the exercise can be found in github `https://github.com/lixc21/MPC-Solution`.

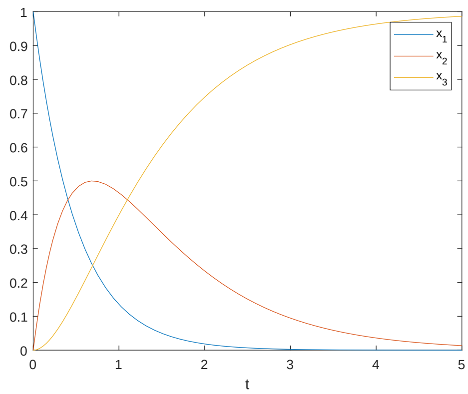

Figure 1.1: system simulation

Exercise 1.2. Distributed systems and time delay.
We assume familiarity with the transfer function of a time delay from an undergraduate systems course

$$\bar{y}(s) = e^{-\theta s}\bar{u}(s) \tag{1.5}$$

Let's see the connection between the delay and the distributed systems, which give rise to it. A simple physical example of a time delay caused by transport in a flowing system. Consider plug flow in a tube depicted in Fig.1.2.

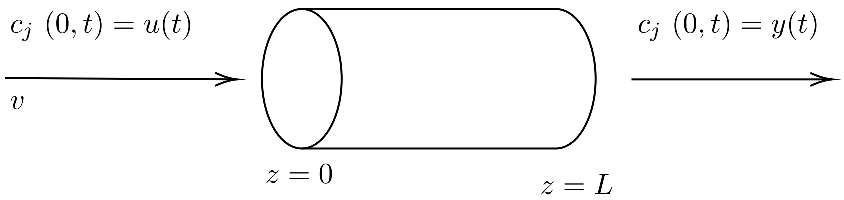

Figure 1.2: Plug-flow reactor

(a) Write down the equation of change for moles of component j for an arbitrary volume element and show that

$$\frac{\partial c_j}{\partial t} = -\nabla \cdot (c_j v_j) + R_j \tag{1.6}$$

in which c_j is the molar concentration of component j, v_j is the velocity of component j, and R_j is the production rate of component j due to chemical reaction.

Plug flow means the fluid velocity of all components os purely in the z direction, and os independent of r and θ and, we assume here, z

$$v_j = v\delta_z \tag{1.7}$$

(b) Assuming plug flow and neglecting chemical reaction in the tube, show that the equation of change reduces to

$$\frac{\partial c_j}{\partial t} = -v\frac{\partial c_j}{\partial z} \tag{1.8}$$

This equation is known as a hyperbolic, first-order partial differential equation.

$$c_j(z,t) = u(t) \qquad 0 = z \qquad t \geq 0 \tag{1.9}$$
$$c_j(z,t) = c_{j0}(t) \qquad 0 \leq z \leq L \quad t = 0 \tag{1.10}$$

In other words, we are using the feed concentration as the manipulated variable, $u(t)$, and the tube starts out with some initial concentration profile of component j, $c_{j0}(z)$.

(c) Show that the solution to (1.8) with these boundary conditions is

$$c_j(z,t) = \begin{cases} u(t - z/v) & vt > z \\ c_{j0}(z - vt) & vt < z \end{cases} \tag{1.11}$$

(d) If the reactor start out empty of component j, show that the transfer function between the outlet concentration, $y = c_j(L,t)$, and the inlet concentration, $c_j(0,t) = u(t)$, is a time delay. What is the value of θ?

Answer 2. (a) let f be the moles of one of the component, then from 3D Leibniz formula, we get

$$\frac{\partial c_j}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V f(\vec{x},t)\ \mathrm{d}V = \int_V \frac{\partial f}{\partial t}\ \mathrm{d}V - \int_A f\vec{v}\cdot\vec{n}\ dV \tag{1.12}$$

where V is a small unit volume, A is the responding surface, v represent the velocity of the point on the surface, n is the outward unit normal vector related

to u.
By using Gauss divergence theorem, we know that

$$\int_V \frac{\partial f}{\partial t}\,\mathrm{d}V - \int_A f\vec{v}\cdot\vec{n}\,\mathrm{d}V = \int_V \frac{\partial f}{\partial t} - \nabla\cdot f\vec{v}\,\mathrm{d}V = -\nabla\cdot(c_j v_j) + R_j \tag{1.13}$$

where the last equation comes from $v_j = v\delta_z$.

(b) Neglecting chemical reaction in the tube, we get $R_j = 0$. Then we know that

$$\frac{\partial c_j}{\partial t} = -\nabla\cdot(c_j v_j) = -\left(\frac{\partial}{\partial x}\delta_x + \frac{\partial}{\partial y}\delta_y + \frac{\partial}{\partial z}\delta_z\right)\cdot v\delta_z = -v\frac{\partial c_j}{\partial z} \tag{1.14}$$

(c) Assuming that $u(t - z/v) = c_{j0}(z - vt)$ when $vt < z$, we just need to prove the solution is $c_j(z,t) = u(t - z/v)$. The variables of original partial differential equation has already been separated, so we get $c_j(z,t) = u(t - z/v)$ easily from the method of characteristics.

(d) We know that $y = u(t - L/v)$, which is a time delay. The value of θ could be L/v.

Exercise 1.3. Pendulum in the state space.
Consider the pendulum suspended at the end of a rigid link depicted in Figure 1.3. Let r and θ denote the polar coordinates of the center of the pendulum, and let $p = r\delta_r$ be the position vector of the pendulum, in which δ_r and δ_θ are the unit vectors in polar coordinates. We wish to determine a state space description of the system. We are able to apply a torque T to the pendulum as our manipulated variable. The pendulum has mass m, the only other external force acting on the pendulum is gravity, and we neglect friction. The link provides force $-t\delta_r$ necessary to maintain the pendulum at distance $r = R$ from the axis of rotation, and we measure the force t.

(a) Provide expressions for the four partial derivatives for changes in the unit vectors with r and θ

$$\frac{\partial \delta_r}{\partial r} \quad \frac{\partial \delta_r}{\partial \theta} \quad \frac{\partial \delta_\theta}{\partial r} \quad \frac{\partial \delta_\theta}{\partial \theta} \tag{1.15}$$

(b) Use the chain rule to find the velocity of the pendulum in terms of the time derivatives of r and θ. Do not simplify yet by assuming r is constant. We want the general result.

(c) Differentiate again to show that the acceleration of the pendulum is

$$\ddot{p} = (\ddot{r} - r\dot{\theta}^2)\delta_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\delta_\theta \tag{1.16}$$

(d) Use a momentum balance on the pendulum mass (you may assume it is a point mass) to determine both the force exerted by the link

$$t = mR\dot{\theta}^2 + mg\cos\theta \tag{1.17}$$

and an equation for the pendulum due to gravity and the applied torque

$$mR\ddot{\theta} - T/R + mg\sin\theta = 0 \tag{1.18}$$

(e) Define a state vector and give a state space description of your system. What is the physical significance of your state. Assume you measure the force exerted by the link.

One answer is

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \tag{1.19}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -(g/R)\sin x_1 + u \tag{1.20}$$

$$y = mRx_2^2 + mg\cos x_1 \tag{1.21}$$

in which $u = T/(mR)$

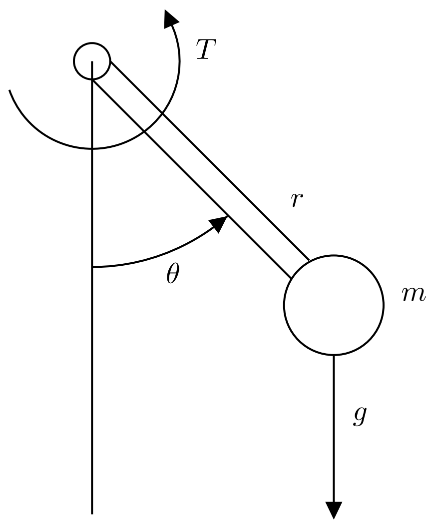

Figure 1.3: Pendulum with applied torque

Answer 3. (a) we assume that δ_θ is rotated from δ by anticlockwise.

$$\frac{\partial \delta_r}{\partial r} = 0 \quad \frac{\partial \delta_r}{\partial \theta} = \delta_\theta \quad \frac{\partial \delta_\theta}{\partial r} = 0 \quad \frac{\partial \delta_\theta}{\partial \theta} = -\delta_r \tag{1.22}$$

(b) Since $p = r\delta_r$

$$\dot{p} = \dot{r}\delta_r + r\frac{\partial \delta_r}{\partial t} = \dot{r}\delta_r + r\frac{\partial \delta_r}{\partial \theta}\dot{\theta} = \dot{r}\delta_r + r\dot{\theta}\delta_\theta \tag{1.23}$$

(c) Differentiate again

$$\ddot{p} = \ddot{r}\delta_r + \dot{r}\dot{\theta}\delta_\theta + r\ddot{\theta}\delta_\theta + r\dot{\theta}\left(-\delta_r\dot{\theta}\right) + \dot{r}\dot{\theta}\delta_\theta \tag{1.24}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\delta_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\delta_\theta \tag{1.25}$$

(d) By the Newton's second law of motion, we get

$$F = -t\delta_r + T/R\delta_\theta + mg\sin\theta\delta_r + mg\cos\theta\delta_\theta = -mR\dot{\theta}^2\delta_r \tag{1.26}$$

Simplify it by two direction

$$t = mR\dot{\theta}^2 + mg\cos\theta \tag{1.27}$$

$$mR\ddot{\theta} - T/R + mg\sin\theta = 0 \tag{1.28}$$

(e) State vector could be $x = [\theta, \dot{\theta}]^\top$, and the system

$$\dot{x} = \begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ -g\sin\theta/R + T/\left(mR^2\right)\end{bmatrix} \tag{1.29}$$

$$y = \theta \tag{1.30}$$

Exercise 1.4. Time to Laplace domain.
Take the Laplace transform of the following set of differential equations and find the transfer function, $G(s)$, connecting $\bar{u}(s)$ and $\bar{y}(s)$, $\bar{y} = G\bar{u}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu \tag{1.31}$$

$$y = Cx + Du \tag{1.32}$$

For $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$, what is the dimension of the G matrix? What happens to the initial condition, $x(0) = x_0$?

Answer 4. The Laplace transform of the differential equation is

$$sx = Ax + Bu \tag{1.33}$$

and the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{R}^{p\times m} \tag{1.34}$$

the initial condition does not appear in the Laplace transform, because the Laplace transform explains the dynamic from u to y, and when we need to determine the accurate trajectory of the system, the initial condition is needed by inverse Laplace transform.

Exercise 1.5. Converting between continuous and discrete time models.
Given a prescribed $u(t)$, derive and check the solution to (1.31). Given a prescribed $u(k)$ sequence, what is the solution to the discrete time model

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k) \tag{1.35}$$

$$y(k) = \tilde{C}x(k) + \tilde{D}u(k) \tag{1.36}$$

(a) Compute $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ so that the two solutions agree at the sample times for a zero-order hold input, i.e., $y(k) = y(t_k)$ for $u(t) = u(k), t \in (t_k, t_{k+1})$ in which $t_k = k\Delta$ for sample time Δ.

(b) Is your result valid for A singular? If not, how can you find $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ for this case?

Answer 5. (a) the solution to (1.31) is

$$x(\Delta) = e^{A\Delta}x_0 + \int_0^\Delta e^{A(\Delta-\tau)}Bu(\tau)\mathrm{d}x \tag{1.37}$$

so the accurate discrete time model is

$$\tilde{A} = e^{A\Delta} \quad \tilde{B} = \int_0^\Delta e^{A(\Delta-\tau)}B\mathrm{d}x \quad \tilde{C} = C \quad \tilde{D} = D \tag{1.38}$$

(b) Yes, this result can be calculate normally even if A is singular.

Exercise 1.6. Continuous to discrete time conversion for nonlinear models
Consider the autonomous nonlinear differential equation model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, u) \tag{1.39}$$

$$x(0) = x_0 \tag{1.40}$$

Given a zero-order hold on the input, let $s(t, u, x_0), 0 \le t \le \Delta$, be the solution to (1.39) given initial condition x_0 at the time $t = 0$, and constant input u is applied for t in the interval $0 \le t \le \Delta$. Consider also the nonlinear discrete time model

$$x(k+1) = F(x(k), u(k)) \tag{1.41}$$

(a) What is the relationship between F and s so that the solution of the discrete time model agrees at the sample times with the continuous time model with a zero-order hold?

(b) Assume f is linear and apply this result to check the result of Exercise 1.5.

Answer 6. (a) The relationship is

$$F(x(k), u(k)) = s(\Delta, u(k-1), x(k-1)) \tag{1.42}$$

(b) It is obvious.

Exercise 1.7. Commuting functions of a matrix.
Although matrix multiplication does not commute in general

$$AB \neq BA \tag{1.43}$$

multiplication of functions of the same matrix do commute. You may have used the following fact in Exercise 1.5

$$A^{-1} \exp\{At\} = \exp\{At\} A^{-1} \tag{1.44}$$

(a) Prove that (1.44) is true assuming A has distinct eigenvalues and can therefore be represented as

$$A = Q\Lambda Q^{-1} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{1.45}$$

in which Λ is a diagonal matrix containing the eigenvalues of A, and Q is the matrix of eigenvectors such that

$$Aq_i = \lambda_i q_i, \quad i = 1, \cdots, n \tag{1.46}$$

in which q_i is the ith column of the matrix Q.

(b) Prove the more general relationship

$$f(A)g(A) = g(A)f(A) \tag{1.47}$$

in which f and g are any functions definable by Taylor series.

(c) Prove that (1.47) is true without assuming the eigenvalues are distinct.
Hint: use the Taylor series defining the functions and apply the Cayley-Hamilton theorem [1].

Answer 7. By using Cayley-Hamilton theorem, we know the matrix functions definable by Taylor series can be written in the sum of finite power terms of matrix A [2]. So all the question is answered obviously by matrix commutability of finite power terms.

Exercise 1.8. Finite difference formula and approximating the exponential.
Instead of computing the exact conversion of a continuous time to a discrete time system as in Exercise 1.5, assume instead one simply approximates the time derivative with a first-order finite difference formula

$$\frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{x(t_{k-1}) - x(t_k)}{\Delta} \tag{1.48}$$

with step size equal to the sample time, Δ. For this approximation of the continuous time system, compute $\tilde{A}$ and $\tilde{B}$ so that the discrete time system agrees with the approximate continuous time system at the sample times. Comparing these answers to the exact solution, what approximation of $e^{A\Delta}$ results from the finite difference approximation? When is this a good approximation of $e^{A\Delta}$?

Answer 8. In this case, the system function can be written as

$$\frac{x(t_{k+1}) - x(t_k)}{\Delta} = Ax(t_k) + Bu(t_k) \tag{1.49}$$

which could be written as

$$x(t_{k+1}) = (I + \Delta A)x(t_k) + \Delta Bu(t_k) \tag{1.50}$$

So $\tilde{A} = I + \Delta A$, $\tilde{B} = \Delta B$, and $e^{A\Delta} \approx I + \Delta A$. When Δ is small, this is a good approximation.

Exercise 1.9. Mapping eigenvalues of continuous time systems to discrete time systems. Consider the continuous time differential equation and discrete equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax \tag{1.51}$$

$$x^+ = \tilde{A}x \tag{1.52}$$

and the transformation

$$\tilde{A} = e^{A\Delta} \tag{1.53}$$

Consider the scalar A case.

(a) What A represents an integrator in continuous time? What is the corresponding A value for the integrator for discrete time?

(b) What A give purely oscillatory solutions? What are the corresponding $\tilde{A}$?

(c) For what A is the solution of the ODE stable? Unstable? What are the corresponding $\tilde{A}$?

(d) Sketch and label these A and $\tilde{A}$ regions in two complex-plane diagrams.

Answer 9. (a) In continuous time, $A = 0$, and the corresponding $\tilde{A} = I$.

(b) A with all eigenvalues on the image axis give purely oscillatory solutions, and the corresponding $\tilde{A}$ has all its eigenvalues on the unit circle.

(c) A with all eigenvalues on the left-half plane give the solution of the ODE stable (right-half plane give the solution of the ODE unstable), and the corresponding $\tilde{A}$ has its all eigenvalues in the unit circle (out of the unit circle).

(d) See Fig.1.4. The orange area denote the stable A, and the blue area denote the unstable A.

Exercise 1.10. State space realization
Define a state vector and realize the following models as the state models by hand. One should do a few by hand to understand what the Octave or MATLAB calls are doing. Answer the following questions. What is the connection betweenthe poles of G and the state space description? For what kinds of $G(s)$ does one obtain a nonzero D matrix? What is the order and gain of these systems? Is there a connection between order and the numbers of inputs and outputs?

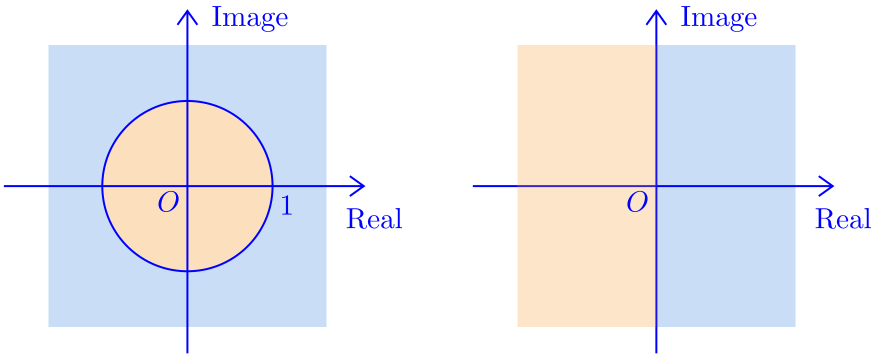

Figure 1.4: Stable and Unstable A and $\tilde{A}$ in two complex plane

(a)
$$G(s) = \frac{1}{2s+1}$$

(b)
$$G(s) = \frac{1}{(2s+1)(3s+1)}$$

(c)
$$G(s) = \frac{2s+1}{3s+1}$$

(d)
$$y(k+1) = y(k) + 2u(k)$$

(e)
$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_1 u(k) + b_2 u(k-1)$$

Answer 10. 1.

Bibliography

[1] James Blake Rawlings, David Q Mayne, and Moritz Diehl. *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI, 2017.

[2] The cayley-hamilton theorem and the matrix exponential. `http://web.archive.org/web/20041123013742/http://web.mit.edu/2.151/www/Handouts/CayleyHamilton.pdf`. Accessed: 2004-11-23.