

CS270: LAB #14

FOL Natural Deduction

You may work in teams of one or two people (three is acceptable in the event of an unscheduled absence). Unless stated otherwise, the lab is due to be submitted into Gradescope at the end of the day.

In order to receive credit, follow these instructions:

[a] Every team member should be discussing simultaneously the same problem – do NOT try to divvy up the labor and assign different problems to different students since the material is cumulative.

[b] Directly edit this lab PDF with your answers (extra pages can be added in the rare event you need more than the allotted space)

[c] Each lab, rotate which member has the responsibility of being the Scribe. This is the person that is typing the answers and uploading the final PDF – note that only a single copy of the filled in PDF is turned into Gradescope. Only one lab needs to be submitted for the entire team, and all members receive the same score. Make sure to use a font that your PDF editor is compatible with (otherwise you might find your answers appear as weird shapes/sizes or simply disappear entirely!)

[d] The Gradescope submission must have each answer properly tagged with the appropriate question. Moreover, every member of the team must be listed as a submitter. Although it is the Scribe which executes these actions, it is still the responsibility of the entire team to make certain this is done properly (thus it is highly recommended that the Scribe share their screen so the entire team can witness it). Answers which are improperly tagged cannot be seen by the grader and thus cannot be scored.

REMOTE ONLY:

[e] Each lab, rotate which member has the responsibility of being the Recorder. This is the person who hits the Zoom Record button (once the technical permission is granted by the TA/RCF/Professor) and ensures that everyone has their camera/microphone on. They are also the member that is responsible to make sure the DrexelStream video is marked as viewable and entered into the <https://tinyurl.com/VidLinkForm> webform before 11:59pm (they should also email the rest of their team as confirmation.) Note that the video file doesn't get created/processed until after the Recorder has quit Zoom.

[f] Each lab, rotate which member has the responsibility of being the Manager. This is the person that ensures that everyone is participating equally and honestly, keeps the group on task, ensures that all team members understand a solution before going on to the next question, and presses the “hand up” button in Zoom to summon a TA or the professor (but they only do so after surveying the group to make sure everyone has the same question).

Team Name (CS Pioneer): Marvin Minsky

Scribe name: Lixiao Yang

Recorder name: Terie Ha

Manager name: Jerry Li

Other team member (if any): Evelyn Thai

Question 1: 26 points

In previous weeks, we have looked at **Predicate Logic**. We will now see how it works with Deduction. The Greatest Common Denominator was defined as follows.

$$\text{divides}(g, a) \wedge \text{divides}(g, b) \wedge \forall e (\text{divides}(e, a) \wedge \text{divides}(e, b) \implies \text{divides}(e, g))$$

The number of parenthesis can quickly become confusing in these expressions. We use use a shorter notation in this lab. Functions will be capital letters and variables will be lowercase letters.

$$Dga \wedge Dgb \wedge \forall e (Dea \wedge Deb \implies Deg)$$

(a) (6 points) Convert the following expression to the new format.

$$\forall x (\text{even}(x)) \implies \exists y (ge(y, x) \wedge \text{even}(y))$$

$$\forall x (Ex) \Rightarrow \exists y (Gyx \wedge Ey)$$

(b) (6 points) A **Bound Variable** is a variable that has a quantifier (either \forall or \exists).

What is the bound variables in the following expression?

$$Dga \wedge Dgb \wedge \forall e (Dea \wedge Deb \implies Deg)$$

e

(c) (6 points) A **Free Variable** is a variable that does not have a quantifier.

What are the free variables in the following expression?

$$Dga \wedge Dgb \wedge \forall e (Dea \wedge Deb \implies Deg)$$

gab

(d) (8 points) For each variable in the below expression, state if it is **bound** or **free**.

$$Na \implies \forall x (Ma \implies Mx), Ma, \neg Mb \therefore \neg Na$$

a: free
x: bound

Question 2: 14 points

In an **Interpretation** we assign meanings to the functions and values in Predicate Logic expressions.

The following predicate logic expression can be interpreted in many ways.

$$Pxy \implies \exists z(Pxz \wedge Pzy)$$

If we interpret $P(x, y) = x < y$ then we are saying

$$(x < y) \implies \exists z((x < z) \wedge (z < y))$$

This is true if x and y are real numbers. It is false if x and y are integers and $y = x + 1$.

Come up with two interpretations of the following expression. Hint: try $<, >, \neq, =, \leq, \geq$.

$$(Pxy \wedge Pyz) \implies Pxz$$

(a) (7 points) Come up with a function for predicate P that will make the statement True for integers

Which of the following operators ($<, >, \neq, =, \leq, \geq$) can P be defined as: $P(a, b) = (a ? b)$ so that the statement $\forall(x, y, z) \in \mathbb{Z} ((Pxy \wedge Pyz) \implies Pxz)$ is True.

$<, >, =, \leq, \geq$

(b) (7 points) Come up with a function for predicate P that will make the statement False for integers

Which of the following operators ($<, >, \neq, =, \leq, \geq$) can P be defined as: $P(a, b) = (a ? b)$ so that the statement $\forall(x, y, z) \in \mathbb{Z} ((Pxy \wedge Pyz) \implies Pxz)$ is False.

\neq

Question 3: 1 point

Quantifiers can be used in our proof checker. <https://proof-tool.herokuapp.com/>

There are two rules for working with **For All** statements.

We will prove $\forall x \in V (Px \rightarrow Qx) ; \forall x \in V Px \therefore \forall x \in V Qx$

| Line # | Expression | Rule |
|--------|---------------------------------------|----------------------|
| 1 | $\forall x \in V (Px \rightarrow Qx)$ | Premise |
| 2 | $\forall x \in V Px$ | Premise |
| 3 | $Pa \rightarrow Qa$ | $\forall E$ 1 |
| 4 | Pa | $\forall E$ 2 |
| 5 | Qa | $\rightarrow E$ 3, 4 |
| 6 | $\forall x \in V Qx$ | $\forall I$ 5 |

On Line 3, we state that since the statement is true for all variables, we can create a variable c that the statement holds for. This is called a **For All Elimination**. We complete the same process on Line 4.

Line 5 uses **Conditional Elimination** from deduction.

Line 6 says that since we have shown Qc to be true for an arbitrary c under our premise, we can conclude that it is true for all variables. This uses **For All Introduction**.

(a) Copy the above proof into the checker.

Note: To enter these into the Proof Checker, select FOL, or choose the "Lab 14" assignment in the proof tool. Put a screenshot of your solution below.

| Line # | Expression | Rule | | | | | | | |
|--------|---------------------------------------|----------------------|---|---|---|---|---|---|---|
| 1 | $\forall x \in V (Px \rightarrow Qx)$ | Premise | + | ← | → | ▲ | ▼ | ✖ | 📄 |
| 2 | $\forall x \in V Px$ | Premise | + | ← | → | ▲ | ▼ | ✖ | 📄 |
| 3 | $Pa \rightarrow Qa$ | $\forall E$ 1 | + | ← | → | ▲ | ▼ | ✖ | 📄 |
| 4 | Pa | $\forall E$ 2 | + | ← | → | ▲ | ▼ | ✖ | 📄 |
| 5 | Qa | $\rightarrow E$ 3, 4 | + | ← | → | ▲ | ▼ | ✖ | 📄 |
| 6 | $\forall x \in V Qx$ | $\forall I$ 5 | + | ← | → | ▲ | ▼ | ✖ | 📄 |

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Question 4 : 12 points

Using the same proof (duplicated below), answer the following questions. Use the proof tool to test each question.

| Line # | Expression | Rule |
|--------|---------------------------------------|----------------------|
| 1 | $\forall x \in V (Px \rightarrow Qx)$ | Premise |
| 2 | $\forall x \in V Px$ | Premise |
| 3 | $Pa \rightarrow Qa$ | $\forall E$ 1 |
| 4 | Pa | $\forall E$ 2 |
| 5 | Qa | $\rightarrow E$ 3, 4 |
| 6 | $\forall x \in V Qx$ | $\forall I$ 5 |

(a) (4 points) Change the a to an x in lines 3-5. Does the proof still pass the checker? Explain why or why not.

It would not pass since x should always be bounded.

(b) (4 points) In lines 3 and 4, change Pa to Pc. In lines 3 and 5, change Qa to Qb.

Does the proof still pass the checker? Why or why not?

Note that as of May 2023, Proof Buddy has a bug and thinks the proof is valid with this change, even though it is not (that bug is currently being fixed). Can you explain why in the real world it should NOT logically work.

No, because according to the premise, $Px \Rightarrow Qx$ is only satisfied when the variables are the same, so Pc would only get to Qc instead of Qb.

(c) (4 points) In lines 3 and 4, change Pa to Pb. In lines 3 and 5, change Qa to Qb.

Does the proof still pass the checker? Why or Why not?

It will pass since $Pb \Rightarrow Qb$ is satisfiable according to the premise and the variables are the same in P and Q.

Question 5: 1 point

There are also two rules related to the **Exists** law.

They are both shown in the example below. $\forall x \in V (Qx \rightarrow Rx) ; \exists x \in V (Px \wedge Qx) \therefore \exists x \in V (Px \wedge Rx)$

| Line # | Expression | Rule | | | | | |
|--------|---------------------------------------|--------------------------|---|---|---|---|---|
| 1 | $\forall x \in V (Qx \rightarrow Rx)$ | Premise | + | ← | → | ↕ | × |
| 2 | $\exists x \in V (Px \wedge Qx)$ | Premise | + | ← | → | ↕ | × |
| 3.1 | $Pc \wedge Qc$ | Assumption | + | ← | → | ↕ | × |
| 3.2 | $Qc \rightarrow Rc$ | $\forall E$ 1 | + | ← | → | ↕ | × |
| 3.3 | Qc | $\wedge E$ 3.1 | + | ← | → | ↕ | × |
| 3.4 | Rc | $\rightarrow E$ 3.2, 3.3 | + | ← | → | ↕ | × |
| 3.5 | Pc | $\wedge E$ 3.1 | + | ← | → | ↕ | × |
| 3.6 | $Pc \wedge Rc$ | $\wedge I$ 3.5, 3.4 | + | ← | → | ↕ | × |
| 3.7 | $\exists x \in V (Px \wedge Rx)$ | $\exists I$ 3.6 | + | ← | → | ↕ | × |
| 4 | $\exists x \in V (Px \wedge Rx)$ | $\exists E$ 2, 3 | + | ← | → | ↕ | × |

In this problem, we need to make an assumption. On line 3, we assume that c is the value that exists to make the statement true.

On line 4, we state that since this premise is true for all, it must also be true for our c .

The basic deduction rules are now used. On line 9, we says that since $Pc \wedge Rc$ is true there must exists a value.

We can now leave the subproof. On line 10 we draw a conclusion. We state that a value exists (line 2). That value causes another statement to be true. We conclude this means $\exists x(Px \wedge Rx)$ true.

(a) Copy the above proof into the checker.

Put a screenshot of your solution below.

| Line # | Expression | Rule | | | | | | | |
|--------|---------------------------------------|--------------------------|---|---|---|---|---|---|--|
| 1 | $\forall x \in V (Qx \rightarrow Rx)$ | Premise | + | ← | → | ↕ | × | □ | |
| 2 | $\exists x \in V (Px \wedge Qx)$ | Premise | + | ← | → | ↕ | × | □ | |
| 3.1 | $Pc \wedge Qc$ | Assumption | + | ← | → | ↕ | × | □ | |
| 3.2 | $Qc \rightarrow Rc$ | $\forall E$ 1 | + | ← | → | ↕ | × | □ | |
| 3.3 | Qc | $\wedge E$ 3.1 | + | ← | → | ↕ | × | □ | |
| 3.4 | Rc | $\rightarrow E$ 3.2, 3.3 | + | ← | → | ↕ | × | □ | |
| 3.5 | Pc | $\wedge E$ 3.1 | + | ← | → | ↕ | × | □ | |
| 3.6 | $Pc \wedge Rc$ | $\wedge I$ 3.5, 3.4 | + | ← | → | ↕ | × | □ | |
| 3.7 | $\exists x \in V (Px \wedge Rx)$ | $\exists I$ 3.6 | + | ← | → | ↕ | × | □ | |
| 4 | $\exists x \in V (Px \wedge Rx)$ | $\exists E$ 2, 3 | + | ← | → | ↕ | × | □ | |

Check Proof

Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Result: The proof is valid and complete!

Question 6 : 6 points

Using the same proof as the previous page answer the following questions.

| Line # | Expression | Rule | | | | | |
|--------|---------------------------------------|--------------------------|---|---|---------------|---------------|---|
| 1 | $\forall x \in V (Qx \rightarrow Rx)$ | Premise | + | - | \rightarrow | \forall | x |
| 2 | $\exists x \in V (Px \wedge Qx)$ | Premise | + | - | \rightarrow | \exists | x |
| 3.1 | $Pc \wedge Qc$ | Assumption | + | - | \rightarrow | \wedge | c |
| 3.2 | $Qc \rightarrow Rc$ | $\forall E$ 1 | + | - | \rightarrow | \forall | c |
| 3.3 | Qc | $\wedge E$ 3.1 | + | - | \rightarrow | \wedge | c |
| 3.4 | Rc | $\rightarrow E$ 3.2, 3.3 | + | - | \rightarrow | \rightarrow | c |
| 3.5 | Pc | $\wedge E$ 3.1 | + | - | \rightarrow | \wedge | c |
| 3.6 | $Pc \wedge Rc$ | $\wedge I$ 3.5, 3.4 | + | - | \rightarrow | \wedge | c |
| 3.7 | $\exists x \in V (Px \wedge Rx)$ | $\exists I$ 3.6 | + | - | \rightarrow | \exists | x |
| 4 | $\exists x \in V (Px \wedge Rx)$ | $\exists E$ 2, 3 | + | - | \rightarrow | \exists | x |

(a) (3 points) In plain English, explain why line 3.7 is true.

Because in step 3.6 it is already shown that for variable c, there exist $Pc \wedge Rc$ is true.

(b) (3 points) In plain English, explain why line 4 is true. What is the key difference between it and line 3.7?

Subproof has shown that for a specific free variable c, we can conclude that $\exists x \in V (Pc \wedge Rc)$ is satisfiable, therefore, line 4 shows that there exists a free variable that can satisfy $\exists x \in V (Px \wedge Rx)$.

Question 7 : 10 points

Prove the following. Put a screenshot of your proof below.

$$\forall x \in V \forall y \in V Gxy \therefore \exists x \in V Gxx$$

| Line # | Expression | Rule | | | | | | | |
|--------|---------------------------------------|---------------|---|---|---|---|---|---|---|
| 1 | $\forall x \in V \forall y \in V Gxy$ | Premise | + | ← | → | ▲ | ▼ | × | 📄 |
| 2 | $\forall y \in V Gay$ | $\forall E$ 1 | + | ← | → | ▲ | ▼ | × | 📄 |
| 3 | Gaa | $\forall E$ 2 | + | ← | → | ▲ | ▼ | × | 📄 |
| 4 | $\exists x \in V Gxx$ | $\exists I$ 3 | + | ← | → | ▲ | ▼ | × | 📄 |

Check Proof

Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Result: The proof is valid and complete!

Question 8 : 10 points

Prove the following. Put a screenshot of your proof below.

$$\forall x \in V (Px \rightarrow Bx) ; \exists x \in V Px \therefore \exists x \in V Bx$$

Consider what this is saying in English. For instance, suppose V is the set of all arctic animals, P is the predicate for Puffins, and B is the predicate for birds. As example $P(\text{Bob the polar bear})$ is false, and $B(\text{Tom the tern})$ is true. Then our first premise technically states "for all arctic animals, if it's a puffin, then it's a bird", however in plain English this would be stated simply as "all puffins are birds".

The second premise is "there is a puffin in the arctic". How would you state the conclusion in plain English?

| Line # | Expression | Rule | | | | | | | |
|--------|---------------------------------------|--------------------------|---|---|---|---|---|---|---|
| 1 | $\forall x \in V (Px \rightarrow Bx)$ | Premise | + | ← | → | ▲ | ▼ | × | 📄 |
| 2 | $\exists x \in V Px$ | Premise | + | ← | → | ▲ | ▼ | × | 📄 |
| 3.1 | Pa | Assumption | + | ← | → | ▲ | ▼ | × | 📄 |
| 3.2 | $Pa \rightarrow Ba$ | $\forall E$ 1 | + | ← | → | ▲ | ▼ | × | 📄 |
| 3.3 | Ba | $\rightarrow E$ 3.2, 3.1 | + | ← | → | ▲ | ▼ | × | 📄 |
| 3.4 | $\exists x \in V Bx$ | $\exists I$ 3.3 | + | ← | → | ▲ | ▼ | × | 📄 |
| 4 | $\exists x \in V Bx$ | $\exists E$ 2, 3 | + | ← | → | ▲ | ▼ | × | 📄 |

Check Proof

Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Result: The proof is valid and complete!

There is a bird in the arctic.

Question 9 : 10 points

Prove the following. Put a screenshot of your proof below. $(\neg Ma) \rightarrow (\forall x \in V (Ca \rightarrow Cx)) ; Ca ; \neg Cb \therefore Ma$

Suppose V is the set of my relatives. M is the predicate for wearing a mask, and C is the predicate for having COVID.

a is the constant for my aunt alice, and b is the constant for my brother barry. Translate the above argument into plain English.

| Line # | Expression | Rule |
|--------|---|------------------------|
| 1 | $(\neg Ma) \rightarrow (\forall x \in V (Ca \rightarrow Cx))$ | Premise |
| 2 | Ca | Premise |
| 3 | $\neg Cb$ | Premise |
| 4.1 | $\neg Ma$ | Assumption |
| 4.2 | $\forall x \in V (Ca \rightarrow Cx)$ | $\rightarrow E$ 1, 4.1 |
| 4.3 | $Ca \rightarrow Cb$ | $\forall E$ 4.2 |
| 4.4 | Cb | $\rightarrow E$ 4.3, 2 |
| 4.5 | \perp | $\neg E$ 3, 4.4 |
| 5 | Ma | IP 4 |

Check Proof Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Result: The proof is valid and complete!

Question 10: 10pts. Prove or disprove following sequent: $\exists x \in V Px \therefore \forall x \in V Px$

If you believe it to be true, then screenshot a proof from the proof checker. If you don't think the conclusion logically follows from the premises, then you need to provide a counter-model. This means coming up with your own set V and predicate P where all the premises are true in your model, but the conclusion is false.

a: Mike
V: Every human being in the world
P: believes that $1+1=3$

If there exists a person, Mike, who believes $1+1=3$ does not means everyone in the world believes $1+1=3$.