Homework 3

Solutions

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Question 1

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Part 1

Find $|\mathcal{P}(S)|$ - i.e., find the total number of subsets of S. Don't forget $\{\}$ is a subset of every set.

$$\boxed{|\mathcal{P}(S)| = 2^8}$$

Part 2

How many subsets of S have 6 elements?

There are $\binom{8}{6}$ subsets of S with 6 elements.

Part 3

How many of the subsets of cardinality 6 contain $\{2, 4, 6, 7\}$ as a subset?

We need two more elements to make $\{2, 4, 6, 7\}$ a set of 6 elements. We can choose from $\{1, 3, 5, 8\}$, so there are $\binom{4}{2} = 6$ such subsets.

Question 2

Consider the set of all 12-bit strings.

Part 1

How many 12-bit strings start with the substring 1101?

With the first four digits chosen for us, we're left with eight digits. Each of those eight can be a 0 or a 1, so there are 2^8 such strings.

Part 2

How many 12-bit strings have weight 6 and begin with 1101?

Again, we have eight digits to set to either 0 or 1. We want to choose 2 of those eight to make 1s (and the rest will be set to 0). This gives us $\binom{8}{2}$ such strings.

Question 3

Consider lattice paths that start at $(1, 2)$.

Part 1

How many lattice paths start at $(1, 2)$ and end at $(9, 13)$.

We have to move right 8 times and up 11 times. This gives us a total of 19 moves. We can pick which 8 will be `right` and set the rest to `up` - this gives us a total of $\boxed{\binom{19}{8}}$. Or, alternatively, we could choose the 11 `up` moves, which gives us $\boxed{\binom{19}{11}}$.

Part 2

How many lattice paths start at $(1, 2)$, end at $(9, 13)$, and pass through $(5, 6)$.

To get from $(1, 2)$ to $(5, 6)$, we need to make a total of 8 moves (4 `right` and 4 `up`). This gives us $\boxed{\binom{8}{4}}$ lattice paths.

To get from $(5, 6)$ to $(9, 13)$, we need to make a total of 11 moves (4 `right` and 7 `up`). This gives a total of $\boxed{\binom{11}{4}}$ or $\boxed{\binom{11}{7}}$.

So, to get from $(1, 2)$ to $(9, 13)$, passing through $(5, 6)$, there are $\boxed{\binom{8}{4}\binom{11}{7}}$ paths.

Part 3

How many lattice paths start at $(1, 2)$, end at $(9, 13)$, and avoid $(5, 6)$.

We can subtract the number of paths that *do* pass through $(5, 6)$ from the total number of lattice paths:

$$\boxed{\binom{19}{11} - \binom{8}{4}\binom{11}{7}}$$

Question 4

What is the coefficient of x^{14} in $(x + 3)^{19}$?

Since $14 = 19 - 5$, the x^{14}-term will be

$$\binom{19}{5}x^{14} \cdot 3^5 = \boxed{2825604\, x^{14}}$$

Question 5

In how many ways may 8 people form a circle for a folk dance?

$$\boxed{(8 - 1)! = 7!}$$

Question 6

How many anagrams are there of each of the following words?

1. train
2. falafel
3. expediently

Train has no repeated letters, so there are $5!$ anagrams.

Falafel has 2 f's, 2 a's, and 2 l's, so there are $\frac{7!}{2!2!2!}$ anagrams.

Expediently has 3 e's, so there are $\frac{11!}{3!}$ anagrams.

Question 7

Mr. Jones owns 4 pairs of pants, 7 shirts, and 3 sweaters. In how many ways may he choose 2 of the pairs of pants, 3 of the shirts, and 1 of the sweaters to pack for a trip?

In total, there are

$$\binom{4}{2}\binom{7}{3}\binom{3}{1}$$

Question 8

Consider sets A and B where $|A| = 8$ and $|B| = 15$.

Part 1

How many functions $f : A \to B$ are there?

There are 15^8 functions because each of the 8 elements in A can be mapped to one of 15 elements in B.

Part 2

How many functions $f : A \to B$ are injective?

For an injective function, we don't want repeats, so that gives us

$$\boxed{\frac{15!}{(15-8)!} \text{ or } \frac{15!}{7!}}$$