Math 180: Homework 1

Due: Wednesday, January 18th

Question 1

Let $A = \{1, 3, 5, 7, 11\}$ and $B = \{4, 5, 6, 7, 8\}$.

Find each of the following sets (list their elements).

1. $A \cap B = \{5, 7\}$
2. $A \cup B = \{1, 3, 4, 5, 6, 7, 8, 11\}$
3. $A \setminus B = \{1, 3, 11\}$
4. $B \setminus A = \{4, 6, 8\}$

Question 2

Find the cardinality of each set.

Part 1

$A = \{x \in \mathbb{N} : 5 \leq x \leq 31\}$

$$A = \{5, 6, 7, 8, \ldots, 31\} \implies |A| = 27$$

Part 2

$B = \{x \in \mathbb{Z} : x^2 = 36\}$

$$B = \{-6, 6\} \implies |B| = 2$$

Part 3

$C = \{x \in \mathbb{N} : 10 \leq x \leq 60 \text{ and } x \text{ is prime}\}$

$$C = \{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59\} \implies |C| = 13$$

There are many possible solutions.

Question 3

Find a set with the largest possible size (cardinality) that is a subset of both $\{2, 4, 6, 8, 10, 12\}$ and $\{1, 2, 3, 4, 5, 6, 7\}$

$$\{2, 4, 6\}$$

Question 4

Find a set with the smallest possible size (cardinality) that has both $\{2, 4, 6, 8, 10, 12\}$ and $\{1, 2, 3, 4, 5, 6, 7\}$ as subsets.

$$\{1, 2, 3, 4, 5, 6, 7, 8, 10, 12\}$$

Question 5

Find an example of sets A and B such that $|A| = 3$, $|B| = 4$, and $|A \cup B| = 5$.

$$A = \{1, 2, 3\}, \quad B = \{2, 3, 4, 5\}$$

Question 6

Are there sets A and B such that $|A| = |B|$, $|A \cup B| = 10$, and $|A \cap B| = 5$?

Explain.

Hint: $|A \cup B| = |A| + |B| - |A \cap B|$.

Since $|A| = |B|$, we have

$$|A \cup B| = |A| + |A| - |A \cap B|$$
$$|A \cup B| = 2|A| - |A \cap B|$$
$$10 = 2|A| - 5$$
$$5 = 2|A|$$

No such sets exist because $|A|$ must be a natural number.

Question 7

Let $A = \{2, 4, 6, 7\}$. Suppose B is a set such that $|B| = 5$.

Part 1

What are the smallest and largest possible values of $|A \cup B|$?

If $A \subset B$, then $|A \cup B| = |B| = 5$. This is the smallest possible size.

If $A \cap B = \{\}$, then $|A \cup B| = 9$. This is the largest possible size.

Part 2

What are the smallest and largest possible values of $|A \cap B|$?

If $A \subset B$, then $|A \cap B| = |B| = 5$. This is the largest possible size.

If $A \cap B = \{\}$, then $|A \cap B| = 0$. This is the smallest possible size.

Part 3

What are the smallest and largest possible values of $|A \times B|$?

$|A \times B| = |A||B| = 20$: This is both the largest and smallest size.

Question 8

Let $X = \{n \in \mathbb{N} : 10 \leq n \leq 20\}$.
Find examples of sets with the properties below and explain why your examples work.

Part 1

A set $A \subseteq \mathbb{N}$ with $|A| = 10$ such that $X \setminus A = \{10, 12, 14\}$

$$A = \{11, 13, 15, 16, 17, 18, 19, 20, 21, 22\}$$

Part 2

A set $B \in \mathcal{P}(X)$ with $|B| = 5$ (here, $B \in \mathcal{P}(X)$ is just another way of writing that B is a subset of X)

$$B = \{10, 11, 12, 13, 14, 15\}$$

Part 3

A set $E \subseteq X$ such that $|E| \in E$

We can just let $E = X$ since $|X| = 11$ and $11 \in X$.