Homework 2

Due: Wednesday, January 25th

Question 1

Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ given by

$$f(n) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$$

Part a

Find $f(3)$

$$f(3) + 3$$

Part b

Find an element n in the domain for which $f(n) = n$

$$f(3) = 3 \implies \boxed{n = 3}$$

Part c

Is this function a bijection? Explain.

Yes. The image of the domain (range) is equal to the co-domain, so it is surjective. There are no repeated elements in the range, so it is injective.

Question 2

Each of the following functions has $\{1, 2, 3, 4, 5, 6\}$ as its domain and co-domain.

For each, determine whether it is (only) injective, (only) surjective, bijective, or none of the above.

Part a

$$f(n) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 1 & 4 & 1 \end{pmatrix}$$

Neither. There's no n for which $f(n) = 6$, so it's not a surjection, and $f(4) = f(6)$, so it's not injective.

Part b

$$f(n) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 1 & 4 \end{pmatrix}$$

It is both.

Part c

$$f(x) = |5 - x|$$

There's a typo here: $f(5) = 0$ which is not in the co-domain, so it's neither (not well-defined).

Question 3

Write out all of the functions $f : \{1, 2\} \to \{a, b, c\}$ (using two-line notation). - How many functions are there? - How many are injective? - How many are surjective? - How many are bijective?

For each of the two elements in the domain, we have three choices in the co-domain. This gives us $3 \cdot 3 = 9$ total functions.

If a function is injective, we can't have repeated elements. This means we have 3 choices for $f(1)$ and 2 remaining choices for $f(2)$, which gives us $3 \cdot 2 = 6$ injective functions.

There are no surjective functions because the image of the domain $f(\{1, 2\})$ can have at most two elements. Since the co-domain has three elements, we can't have a surjection.

Since there are no surjections, there are no bijections.

Question 4

Consider the function $f : \mathbb{N} \to \mathbb{N}$ given recursively by $f(0) = 2$ and $f(n+1) = 3 \cdot f(n)$. Find $f(6)$.

This is easiest to do by listing them.

n:	0	1	2	3	4	5	6
$f(n)$:	2	6	18	54	162	486	1458

So, $f(6) = 1458$.

Question 5

Suppose $f : X \to Y$ is a function. Which of the following are possible? Explain.

Part a

f is injective but not surjective.

This is possible if $|X| < |Y|$. For example, we might have $f : \{1, 2\} \to \{a, b, c\}$ defined by

$$\begin{pmatrix} 1 & 2 \\ a & c \end{pmatrix}$$

This function is an injection ($f(1) \neq f(2)$) but is not a surjection because there is no n for which $f(n) = b$.

Part b

f is surjective but not injective.

This is possible if $|X| > |Y|$. For example, we might have $f : \{1, 2, 3\} \to \{a, b\}$ defined by

$$\begin{pmatrix} 1 & 2 & 3 \\ a & b & b \end{pmatrix}$$

This function is surjective since $f(\{1, 2, 3\}) = \{a, b\}$, but it is not injective since $f(2) = f(3)$.

Part c

$|X| = |Y|$ and f is injective but nor surjective.

Note: X and Y could be infinite sets.

This is possible. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = 2n$$

This is injective because if $a \neq b$ then $2a \neq 2b$, so $f(a) \neq f(b)$.

However, it is not injective since the output of f is always even; therefore, there is no $n \in \mathbb{N}$ for which $f(n) = 5$ (or any other odd for that matter!).

Part d

$|X| = |Y|$ and f is surjective but nor injective.

Note: X and Y could be infinite sets.

This is possible, too! Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = \lfloor n/2 \rfloor$$

This is the floor function, which always rounds down. E.g., $\lfloor 3.6 \rfloor = 3$. Consider a few values of f:

n	0	1	2	3	4	5	6
$f(n)$	0	0	1	1	2	2	3

This function is surjective, but not injective.

Part e

$|X| = |Y|$, X and Y are finite, and f is injective but not surjective.

No, this isn't possible. Since f is injective, we know that $|f(X)| = |X|$. But $|X| = |Y|$, so $|f(X)| = |Y|$, which makes f surjective as well. ### Part f

$|X| = |Y|$, X and Y are finite, and f is surjective but not injective.

No, this isn't possible. Since f is surjective, we know that $|f(X)| = |Y|$. But $|X| = |Y|$, so $|f(X)| = |X|$, which makes f injective as well.

Question 6

Your wardrobe consists of 7 shirts, 5 pairs of pants, and 12 hats. How many different outfits can you make?

$$7 \cdot 5 \cdot 12 = 420$$

Question 7

A group of college students were asked about their TV watching habits. Of those surveyed, 30 students watch The The Office, 23 watch Parks and Rec, and 18 watch Superstore. Additionally, 14 watch The Office and Parks and Rec, 12 watch Parks and Rec and Superstore, and 11 watch The Office and Superstore. There are 7 students who watch all three shows. How many students surveyed watched at least one of the shows?

This is another inclusion-exclusion problem.

Let's define our sets:

- O: Office watchers
- P: Parks and Rec watchers
- S: Superstor watchers
- $|O| = 30$
- $|P| = 23$
- $|S| = 18$
- $|O \cap P| = 14$
- $|P \cap S| = 12$
- $|O \cap S| = 11$
- $|O \cap P \cap S| = 7$

Putting it all together:

$$|O \cup P \cup S| = 30 + 23 + 18 - 14 - 12 - 11 + 7 = 41$$

So, 41 students watch at least one.

Question 8

For how many $n \in \{1, 2, 3, \ldots, 800\}$ is n a multiple of 2, 3, or 5? Explain your answer using the Principle of Inclusion/Exclusion.

Let's define our sets:

- $X = \{1, 2, 3, \ldots, 800\}$
- A: numbers in X divisible by 2
- B: numbers in X divisible by 3
- C: numbers in X divisible by 5

We'll do this by inclusion-exclusion.

- $|A| = \lfloor 800/2 \rfloor = 400$
- $|B| = \lfloor 800/3 \rfloor = 266$
- $|C| = \lfloor 800/5 \rfloor = 160$
- $|A \cap B| = \lfloor 800/6 \rfloor = 133$
- $|A \cap C| = \lfloor 800/10 \rfloor = 80$

- $|B \cap C| = \lfloor 800/15 \rfloor = 53$
- $|A \cap B \cap C| = \lfloor 800/30 \rfloor = 26$

$$|A \cup B \cup C| = 400 + 266 + 160 - 133 - 80 - 53 + 26 = 586$$

Question 9

Consider all 6 letter "words" made from the letters a through j. (In this context words are just strings of letters, not necessarily actual English words.)

Part a

How many of these words are there total?

There are 10^6.

Part b

How many of these words contain no repeated letters?

There are $10!/4! = 151200$

Part c

How many of these words start with the sub-word "kid"?

There are 10^3.

Part d

How many of these words either start with "aha" or end with "bah" or both?

There are 10^3 that begin with "aha" and 10^3 that end with "bah," but these sets have one in common. The total is $10^3 + 10^3 - 1 = 1999$.

Part e

How many of the words containing no repeats also do not contain the sub-word "hid"?

There are four ways to put "hid" in a six-letter word:

1. `H I D _ _ _`
2. `_ H I D _ _`
3. `_ _ H I D _`
4. `_ _ _ H I D`

In each case, we've got $7 \cdot 6 \cdot 5 = 210$ choices for the remaining three slots. In total, we have $4 \cdot 210 = 840$ words.

Question 10

For how many two-digit numbers (10 - 99) is the *sum of the digits* even? Explain your solution.

If the sum of two numbers is even, then either both are even or both are odd.

- If both digits are even, we have four choices for the first digit (2,4,6,8) and five choices for the second (0,2,4,6,8). This gives us $4 \cdot 5 = 20$ numbers.
- If both digits are odd, we have five choices for each giving us $5 \cdot 5 = 25$ numbers.

In total, we have $20 + 25 = 45$ numbers.