Math 180

Homework 5

Section 2.3

Question 1

Use polynomial fitting to find the formula for the n^{th} term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$4, 5, 8, 13, 20, 29, 40, \ldots$$

Show all your work.

The first differences are: $1, 3, 5, 7, 9, \ldots$

The second differences are: $2, 2, 2, 2, 2, \ldots$

So, the closed form will be quadratic:

$$a_n = An^2 + Bn + C$$

Let's consider a_0, a_1, a_2.

$$\begin{aligned} a_0 &= 4 \\ A(0)^2 + B(0) + C &= 4 \\ C &= 4 \end{aligned}$$

We can update a_n:

$$a_n = An^2 + Bn + 4$$

$$\begin{aligned}
a_1 &= 5 \\
A(1)^2 + B(1) + 4 &= 5 \\
A + B &= 1
\end{aligned}$$

$$\begin{aligned}
a_2 &= 8 \\
A(2)^2 + B(2) + 4 &= 8 \\
4A + 2B &= 4 \\
2A + B &= 2
\end{aligned}$$

This gives us a system of equations to solve:

$$\begin{aligned}
A + B &= 1 \\
2A + B &= 2 \\
A &= 1 \\
B &= 0
\end{aligned}$$

Finally, we have $\boxed{a_n = n^2 + 4.}$

Question 2

Use polynomial fitting to find the formula for the n^{th} term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$2, 5, 9, 14, 20, 27, 35, \ldots$$

Show all your work.

The first differences are $3, 4, 5, 6, 7, \ldots$

The second differences are $1, 1, 1, 1, 1, \ldots$

Once again, we have a quadratic closed form: $a_n = An^2 + Bn + C$.

$$
\begin{aligned}
a_0 &= 2 \\
A(0)^2 + B(0) + C &= 2 \\
C &= 2
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= 5 \\
A(1)^2 + B(1) + 2 &= 5 \\
A + B &= 3
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= 9 \\
A(2)^2 + B(2) + 2 &= 9 \\
4A + 2B &= 7
\end{aligned}
$$

Solve the system of equations:

$$
\begin{cases} A + B & = 3 \\ 4A + 2B & = 7 \end{cases} \Longrightarrow \begin{cases} 2A + 2B & = 6 \\ 4A + 2B & = 7 \end{cases} \Longrightarrow 2A = 1
$$

So, $A = 1/2$ and $B = 5/2$, giving us

$$
\boxed{a_n = \frac{1}{2}n^2 + \frac{5}{2}n + 2 \text{ or } \frac{1}{2}\left(n^2 + 5n + 4\right)}
$$

Question 3

Use polynomial fitting to find the formula for the n^{th} term of the sequence $(a_n)_{n\geq 0}$ which starts,

$$
0, 2, 5, 12, 26, 50, 87, \ldots
$$

Show all your work.

First differences: $2, 3, 7, 14, 24, 37, \ldots$

Second differences: $1, 4, 7, 10, 13, \ldots$

Third differences: $3, 3, 3, 3, \ldots$

This time, we have a cubic closed form: $a_n = An^3 + Bn^2 + Cn + D$.

$$
\begin{aligned}
a_0 &= 0 \\
A(0)^3 + B(0)^2 + C(0) + D &= 0 \\
D &= 0
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= 2 \\
A(1)^3 + B(1)^2 + C(1) &= 2 \\
A + B + C &= 2
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= 5 \\
A(2)^3 + B(2)^2 + C(2) &= 5 \\
8A + 4B + 2C &= 5
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= 12 \\
A(3)^3 + B(3)^2 + C(3) &= 12 \\
27A + 9B + 3C &= 12 \\
9A + 3B + C &= 4
\end{aligned}
$$

This gives us a system of three equations:

$$
\begin{cases}
A + B + C = 2 \\
8A + 4B + 2C = 5 \\
9A + 3B + C = 4
\end{cases}
$$

We can solve by substitution:

$$
A + B + C = 2 \implies \boxed{C = 2 - A - B}
$$

$$
\begin{cases}
8A + 4B + 2(2 - A - B) = 5 \\
9A + 3B + 2 - A - B = 4
\end{cases}
$$

$$
\begin{cases}
8A + 4B + 4 - 2A - 2B = 5 \\
9A + 3B + 2 - A - B = 4
\end{cases}
$$

$$
\begin{cases}
6A + 2B = 1 \\
8A + 2B = 2
\end{cases}
\implies 2A = 1 \implies \boxed{A = 1/2}
$$

Solve for B:

$$6(1/2) + 2B = 1 \implies 2B = -2 \implies \boxed{B = -1}$$

Solve for C:

$$\begin{aligned} C &= 2 - (1/2) - (-1) \\ &= 5/2 \end{aligned}$$

Finally,

$$\boxed{a_n = \frac{1}{2}n^3 - n^2 + \frac{5}{2}n}$$

Question 4

Complete Exercise 12 from the textbook.

Part (a)

P(1)	P(2)	P(3)	P(4)	P(5)
1	4	10	20	35

Part (b)

Let's look at the differences:

- First: 3, 6, 10, 15
- Second: 3, 4, 5, ...
- Third: 1, 1, 1,...

$$\begin{aligned} a_n &= an^3 + bn^2 + cn + d \\ a_1 &= a + b + c + d = 1 \\ a_2 &= 8a + 4b + 2c + d = 4 \\ a_3 &= 27a + 9b + 3c + d = 10 \\ a_4 &= 64a = 16b + 4c + d = 30 \\ a_n &= \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n \end{aligned}$$

We can also write the solution as

$$a_n = \frac{n(n+1)(n+2)}{6}$$

Notice that this is also $\binom{n+2}{3}$. All of the "Choose 3" binomials lie along a diagonal in Pascal's triangle.

Section 2.4

Question 5

Solve the recurrence relation $a_n = a_{n-1} + 2^n$ with $a_0 = 3$

Let's start by writing out some terms in the sequence:

a_0	a_1	a_2	a_3	a_4	a_5
3	5	9	17	33	65

Notice that each term is 1 greater than a power of 2:

a_0	a_1	a_2	a_3	a_4	a_5
$3 = 1 + 2^1$	$5 = 1 + 2^2$	$9 = 1 + 2^3$	$17 = 1 + 2^4$	$33 = 1 + 2^5$	$65 = 1 + 2^6$

Therefore, $a_n = 2^{n+1} + 1$.

Question 6

Find a solution to the recurrence relation $a_n = 5a_{n-1} + 6a_{n-2}$ with $a_0 = 1$ and $a_1 = 13$.

$$\begin{aligned} r^2 - 5r - 6 &= 0 \\ (r-6)(r+1) &= 0 \\ r &= -1, 6 \\ a_n &= A(-1)^n + B6^n \\ a_0 = A + B &= 1 \\ a_1 = -A + 6B &= 13 \\ a_0 + a_1 = 7B &= 14 \\ B &= 2 \\ A &= -1 \end{aligned}$$

So, $a_n = (-1)(-1)^n + 2 \cdot 6^n$ or $\boxed{a_n = (-1)^{n+1} + 2 \cdot 6^n}$

Question 7

Find a solution to the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_0 = 4$ and $a_1 = 11$.

$$\begin{aligned} r^2 - 5r + 6 &= 0 \\ (r-3)(r-2) &= 0 \\ r &= 3, 2 \\ a_n &= A3^n + B2^n \\ a_0 = A + B &= 4 \\ a_1 = 3A + 2B &= 11 \\ a_1 - 2a_0 = A &= 3 \\ \longrightarrow B &= 1 \end{aligned}$$

$$\boxed{a_n = 3^{n+1} + 2^n}$$

Question 8

Find a solution to the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ with $a_0 = 3$ and $a_1 = 10$.

$$\begin{aligned} r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \\ r &= 4 \\ a_n &= A4^n + Bn4^n \\ a_0 = A &= 3 \\ a_1 = 4(3) + 4B &= 10 \\ \longrightarrow B &= -1/2 \end{aligned}$$

$$\boxed{a_n = 3 \cdot 4^n - \frac{1}{2}n \cdot 4^n}$$

Question 9

Complete Exercise 9 from the textbook.

(a) Find both a recursive and closed formula for how many Skittles the *n*th customer gets.

1	2	3	4	5
1	4	16	64	256

Recursively, we can define this sequence as $a_n = 4a_{n-1}$ with $a_1 = 1$. In closed form, we can write $a_n = 4^{n-1}$.

(b) Check your solution for the closed formula by solving the recurrence relation using the Characteristic Root technique.

$$\begin{aligned} r^2 - 4r &= 0 \\ r &= 0, 4 \\ a_n &= A4^n \\ a_1 = 4A &= 1 \\ A &= 1/4 \\ a_n = \frac{1}{4}4^n &= 4^{n-1} \end{aligned}$$