Math 180

Homework 4 Solutions

Question 1

For each of the following sequences, find a closed formula for a_n, the n^{th} term of the sequence. Assume the first term is a_0. Briefly show how you got your answer.

Many of these can be completed by considering some of sequences we encounter frequently:

- $n^2 : 0, 1, 4, 9, 16, 25, 36, \ldots$
- $2^n : 1, 2, 4, 8, 16, 32, 64, \ldots$
- $\frac{n(n+1)}{2} : 1, 3, 6, 10, 15, 21, 28, 36, \ldots$

Part a

$$4, 5, 7, 11, 19, 35, \ldots$$

We can compare this sequence to $b_n = 2^n$. Writing out the first few terms of b, we have

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

Since $a_n = b_n + 3$, we conclude that $\boxed{a_n = 2^n + 3}$.

Part b

$$0, 3, 8, 15, 24, 35, \ldots$$

We can compare this one with $b_n = n^2$. The first several terms of b are

$$0, 1, 4, 9, 16, 25, 36, \ldots$$

If we subtract 1 from each term, we have $n^2 - 1$

$$-1, 0, 3, 8, 15, 24, 35, \ldots$$

We want to start at 0, rather than -1, so we should shift the index to make it $(n+1)^2 - 1$. This way, starting at $n = 0$, we have

$$0, 3, 8, 15, 24, 35, \ldots$$

So, $\boxed{a_n = (n+1)^2 - 1}$.

Part c

$$6, 12, 20, 30, 42, \ldots$$

We can compare this one with $\frac{n(n+1)}{2}$:

$$1, 3, 6, 10, 15, 21, 28, 36, \ldots$$

Doubling the sequence gives us $n(n+1)$:

$$2, 6, 12, 20, 30, 42, 72, \ldots$$

Once again, we want to shift up by one term: $(n+1)(n+2)$. This yields exactly what we want:

$$6, 12, 20, 30, 42, 72, \ldots$$

So, $\boxed{a_n = (n+1)(n+2)}$ or $\boxed{n^2 + 3n + 2}$.

Part d

$$0, 2, 7, 15, 26, 40, 57, \ldots$$

$$\boxed{a_n = \frac{n(3n+1)}{2}}$$

Hint: These are sometimes called *house numbers*.

Question 2

Write out the first five terms, starting with a_0, for the sequence below. Then, provide a closed formula.

$$a_n = 3a_{n-1} - 2a_{n-2}, \text{ with } a_0 = 0,\ a_1 = 1$$

We can start by writing out the first several terms:

$$0, 1, 3, 7, 15, 31, ...$$

Compare this with 2^n:

$$1, 2, 4, 8, 16, 32, \ldots$$

Each term of a_n is one less than the sequence 2^n:

$$\boxed{a_n = 2^n - 1}$$

Question 3

Write out the first five terms, starting with a_0, for the sequence below. Then, provide a recursive formula.

$$a_n = n^2 + n$$

Hint: Consider 2.1.3a from the book.

Question 4

Show that $a_n = 2^n - 5^n$ is a solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

To confirm that this closed form satisfies the recurrence relation, we want to plug it into the recurrence relation. First, we'll simplify a_{n-1} and a_{n-2}:

- $a_n = 2^n - 5^n$
- $a_{n-1} = 2^{n-1} - 5^{n-1} = 2^n 2^{-1} - 5^n 5^{-1} = \frac{1}{2} 2^n - \frac{1}{5} 5^n$
- $a_{n-2} = 2^{n-2} - 5^{n-2} = 2^n 2^{-2} - 5^n 5^{-2} = \frac{1}{4} 2^n - \frac{1}{25} 5^n$

$$
\begin{aligned}
7a_{n-1} - 10a_{n-2} &= 7\left(\frac{1}{2}2^n - \frac{1}{5}5^n\right) - 10\left(\frac{1}{4}2^n - \frac{1}{25}5^n\right) \\
&= \frac{7}{2}2^n - \frac{7}{5}5^n - \frac{5}{2}2^n + \frac{2}{5}5^n \\
&= \frac{2}{2}2^n - \frac{5}{5}5^n \\
&= 2^n - 5^n \\
&= a_n \checkmark
\end{aligned}
$$

The initial conditions should be a_0 and a_1:

- $a_0 = 2^0 - 5^0 = 0$
- $a_1 = 2^1 - 5^1 = -3$

Hint: Consider 3.1.9

Question 5

Compute the sum $2 + 5 + 8 + \cdots + 59$.

$$
\begin{aligned}
S &= 2 + 5 + 8 + \cdots + 56 + 59 \\
S &= 59 + 56 + \cdots + 8 + 5 + 2 \\
2S &= 61 + 61 + \cdots + 61 + 61 + 61 \\
2S &= 20(61) \\
S &= \boxed{610}
\end{aligned}
$$

Question 6

Compute the sum $-3 + 1 + \cdots + 77$.

$$
\begin{aligned}
S &= -3 + 1 + \cdots + 73 + 77 \\
S &= 77 + 73 + \cdots + 1 - 3 \\
2S &= 74 + 74 + \cdots + 74 + 74 \\
2S &= 21(74) \\
S &= \boxed{777}
\end{aligned}
$$

Question 7

Find a closed form for the sum of the sequence $a_n = 1 + 4n$ for $n \geq 0$.

$$
\begin{aligned}
S &= 1 + 5 + \cdots + (1 + 4(n-1)) + (1 + 4n) \\
S &= (1 + 4n) + (1 + 4(n-1)) + \cdots + 5 + 1 \\
2S &= (2 + 4n) + (2 + 4n) + \cdots + (2 + 4n) + (2 + 4n) \\
2S &= (n)(2 + 4n) \\
S &= \boxed{2n^2 + n}
\end{aligned}
$$

Question 8

Compute the sum of the sequence $3 \cdot 2^n$ for $n \geq 0$.

$$
\begin{aligned}
S &= 3 + 6 + 12 + \cdots + 3 \cdot 2^{n-1} + 3 \cdot 2^n \\
-2S &= -6 - 12 - \cdots - 3 \cdot 2^n - 3 \cdot 2^{n+1} \\
-S &= 3 - 3 \cdot 2^{n+1} \\
S &= \boxed{3 \cdot 2^{n+1} - 3}
\end{aligned}
$$

Question 9

Express the repeating decimal $0.37373737373737\ldots$ as a fraction (without decimals, of course).

$$
\begin{aligned}
N &= 0.37373737373737\ldots \\
-0.01N &= -0.0037373737373737\ldots \\
0.99N &= 0.37 \\
N &= \boxed{\frac{37}{99}}
\end{aligned}
$$

Question 10

Express the repeating decimal $0.213213213213\ldots$ as a fraction (without decimals, of course).

$$
\begin{aligned}
N &= 0.213213213213\ldots \\
-0.001N &= -0.000213213213213213\ldots \\
0.999N &= 0.213 \\
N &= \boxed{\frac{213}{999}}
\end{aligned}
$$