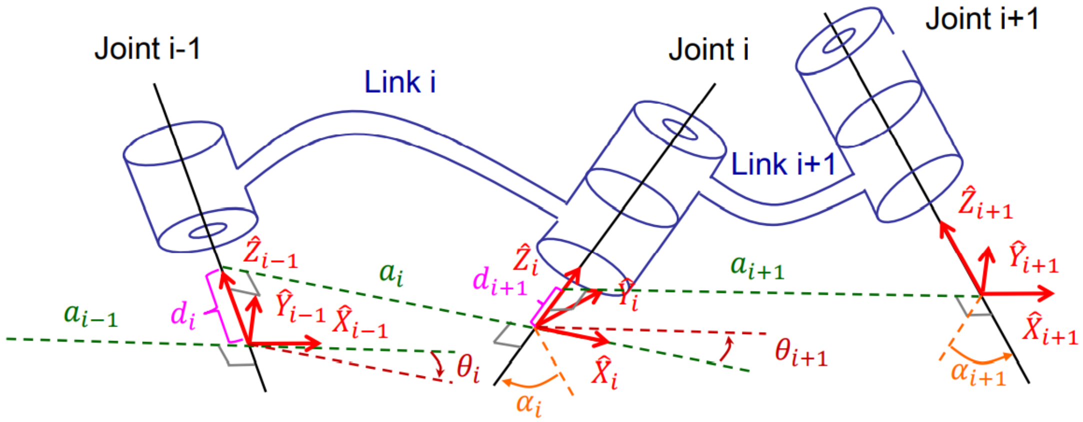
一、 机械臂的运动学正解和逆解

SDH 参数表

Link	连杆偏移量 d_i/m	连杆长度 a_i/m	连杆扭角 $\alpha_i/°$	关节转角 $\theta_i/°$	Offset /°
1	D1	0	-90	θ_1	0
2	0	L2	0	θ_2	-90
3	0	L3	-90	θ_3	0
4	D4	0	90	θ_4	0
5	0	0	-90	θ_5	0
6	D6	0	0	θ_6	±180

关于 offset 项是叠加在θ上面的偏置量，可以理解为机器人处于零位时，驱动轴和转动轴依然存在夹角。

其中 D1=0.3991，L2=0.448，L3=0.042，D4=0.451，D6=0.082，单位（m）。



通过连杆偏移量 d_i、连杆长度 a_i、连杆扭角α_i、关节转角θ_i，就能相邻连杆之间的DH坐标系变换矩阵如下：

$$ {}^{i-1}_{i}T = Rot(Z_{i-1},\theta_i)Trans(0,0,d_i)Trans(a_i,0,0)Rot(X_i,\alpha_i)$$

$$=\begin{bmatrix}\cos\theta_i & -\sin\theta_i & 0 & 0\\ \sin\theta_i & \cos\theta_i & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d_i\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & a_i\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha_i & -\sin\alpha_i & 0\\ 0 & \sin\alpha_i & \cos\alpha_i & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$
{}^{i-1}_{i}T=\begin{bmatrix}\cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i\\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i\\ 0 & \sin\alpha_i & \cos\alpha_i & d_i\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
=\begin{bmatrix}c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i\\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i\\ 0 & s\alpha_i & c\alpha_i & d_i\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

其中，$s\theta_i$ 表示 $\sin\theta_i$，$c\theta_i$ 表示 $\cos\theta_i$，将 DH 参数代入变换矩阵：

$$
{}^{0}_{1}T=\begin{bmatrix}\cos\theta_1 & 0 & -\sin\theta_1 & 0\\ \sin\theta_1 & 0 & \cos\theta_1 & 0\\ 0 & -1 & 0 & D_1\\ 0 & 0 & 0 & 1\end{bmatrix},\quad {}^{1}_{2}T=\begin{bmatrix}\cos\theta_2 & -\sin\theta_2 & 0 & L_2\cos\theta_2\\ \sin\theta_2 & \cos\theta_2 & 0 & L_2\sin\theta_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{2}_{3}T=\begin{bmatrix}\cos\theta_3 & 0 & -\sin\theta_3 & L_3\cos\theta_3\\ \sin\theta_3 & 0 & \cos\theta_3 & L_3\sin\theta_3\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad {}^{3}_{4}T=\begin{bmatrix}\cos\theta_4 & 0 & \sin\theta_4 & 0\\ \sin\theta_4 & 0 & -\cos\theta_4 & 0\\ 0 & 1 & 0 & D_4\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{4}_{5}T=\begin{bmatrix}\cos\theta_5 & 0 & -\sin\theta_5 & 0\\ \sin\theta_5 & 0 & \cos\theta_5 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad {}^{5}_{6}T=\begin{bmatrix}\cos\theta_6 & -\sin\theta_6 & 0 & 0\\ \sin\theta_6 & \cos\theta_6 & 0 & 0\\ 0 & 0 & 1 & D_6\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{0}_{6}T={}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T
$$

$$
{}^{0}_{6}T=\begin{bmatrix}n_x & o_x & a_x & p_x\\ n_y & o_y & a_y & p_y\\ n_z & o_z & a_z & p_z\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
n_x=S_6(C_4S_1-S_4C_1C_{23})+C_6\left[C_5(S_1S_4+C_4C_1C_{23})-S_5C_1S_{23}\right]
$$

$$
n_y=-S_6(C_1C_4+S_4S_1C_{23})-C_6\left[C_5(C_1S_4-C_4S_1C_{23})+S_5S_1S_{23}\right]
$$

$$
n_z=S_{23}S_4S_6-C_6(C_{23}S_5+S_{23}C_4C_5)
$$

$$
o_x=C_6(C_4S_1-S_4C_1C_{23})-S_6\left[C_5(S_1S_4+C_4C_1C_{23})-S_5C_1S_{23}\right]
$$

$$
o_y=S_6\left[C_5(C_1S_4-C_4S_1C_{23})+S_5S_1S_{23}\right]-C_6(C_1C_4+S_4S_1C_{23})
$$

$$o_z = S_6(C_{23}S_5 + S_{23}C_4C_5) + S_{23}C_6C_4$$

$$a_x = -S_5(S_1S_4 + C_4C_1C_{23}) - C_5C_1S_{23}$$

$$a_y = S_5(C_1S_4 - C_4S_1C_{23}) - C_5S_1S_{23}$$

$$a_z = S_{23}C_4C_5 - C_{23}C_5$$

$$p_x = L_2C_1C_2 - D_4S_{23}C_1 - D_6S_{23}C_1C_5 + L_3C_1C_{23} - D_6S_1S_4S_5 - D_6C_1C_2C_3C_4C_5 + D_6C_1C_4S_2S_3S_5$$

$$p_y = L_2C_2S_1 - D_4S_{23}S_1 - D_6S_{23}C_5S_1 + L_3S_1C_{23} + D_6C_1S_4S_5 + D_6C_4S_1S_2S_3S_5 - D_6C_2C_3C_4S_1S_5$$

$$p_z = D_1 - D_4C_{23} - L_2S_2 - D_6\left[C_5C_{23} - C_4C_5S_{23}\right] - L_3S_{23}$$

这里使用了和差化积公式：

$$C_{23} = C_2C_3 - S_2S_3$$

$$S_{23} = C_2S_3 + S_2C_3$$

二、 机械臂逆运动学求解

解析法优点：具有计算可靠、能够获得所有解的优势。

缺点：需要处理的方程组为复杂的非线性方程组，需要花费大量时间用于求解方程组，且机器人是否存在解析解的要求严格。必须严格遵守 Pieper 准则，例如常用的六自由度机械臂通常将后三轴设计交于一点，相当于给复杂的方程增加了限制，降低了求解的复杂度。一般工业机器人在设计和制造时会考虑逆运动学求解的问题。

数值法：以末端执行器位姿和目标位姿之间误差最小为优化指标，建立目标函数，将非线性方程求解问题转化为最优化问题。解法较为灵活、通用，但数值法需要的运算量更大，不适合实时性要求较高的场合，且数值解通循环迭代寻找最优解，只能输出一个解．因此，在工业制造场合（如用多自由度机器人进行 3D 打印），人们更倾向于使用解析法对逆运动学进行求解。

商用的机械臂一般都会采用解析解，因为求解速度快且准确，而不会采用迭代的数值解法。如下图采用数值法对 3D 打印规划的路径求取逆解。

STL 模型	三角面片数量	切片层厚	轮廓点	逆解时间
恐龙	311898	2mm	41707	2732s(45min)

机器人逆运动学分析(解析法)：

已知机械臂各关节和连杆参数以及执行机构与固定参考坐标系相对位姿的情况下，推导出机械臂连杆之间的夹角θ_i，这就是机器人逆运动学分析。

$$ {}_6^0T = {}_1^0T\,{}_2^1T\,{}_3^2T\,{}_4^3T\,{}_5^4T\,{}_6^5T = T_{tool} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

求解运动学逆解，对等式${}_6^0T = {}_1^0T\,{}_2^1T\,{}_3^2T\,{}_4^3T\,{}_5^4T\,{}_6^5T$两边同时乘以${}_1^0T^{-1}$，为方便计算，设$D_6 = 0$即

$$ {}_1^0T^{-1}\,{}_6^0T = {}_2^1T\,{}_3^2T\,{}_4^3T\,{}_5^4T\,{}_6^5T $$

等式左边

$$ {}_1^0T^{-1}\,{}_6^0T = \begin{bmatrix} n_xC_1 + n_yS_1 & o_xC_1 + o_yS_1 & a_xC_1 + a_yS_1 & p_xC_1 + p_yS_1 \\ -n_z & -o_z & -a_z & D_1 - p_z \\ n_yC_1 - n_xS_1 & o_yC_1 - o_xS_1 & a_yC_1 - a_xS_1 & p_yC_1 - p_xS_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}_2^1T\,{}_3^2T\,{}_4^3T\,{}_5^4T\,{}_6^5T = \begin{bmatrix} -C_6(S_{23}S_5 - C_{23}C_4C_5) - C_{23}S_4S_6 & S_6(S_{23}S_5 - C_{23}C_4C_5) - C_{23}C_6S_4 & -S_{23}C_5 - C_{23}C_4C_5 & L_3C_{23} - D_4S_{23} + L_2C_2 \\ C_6(C_{23}S_5 + S_{23}C_4C_5) - S_{23}S_4S_6 & -S_6(C_{23}S_5 + S_{23}C_4C_5) - S_{23}C_6S_4 & C_{23}C_5 - S_{23}C_4C_5 & D_4C_{23} + L_3S_{23} + L_2S_2 \\ -C_4S_6 - C_5C_6S_4 & C_5S_4S_6 - C_4C_6 & S_4S_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

1. 求解θ_1，根据等式两边对应元素相等，由第 3 行 4 列的元素可得

py*cos(theta1) - px*sin(theta1) == 0

$$ p_yC_1 - p_xS_1 = 0 $$

由上式得

$$ \theta_1 = \arctan(p_y, p_x) $$

2. 求解θ_3，根据等式 1 行 4 列和 2 行 4 列元素相等，得

px*cos(theta1) + py*sin(theta1) == L3*cos(theta2 + theta3) - D4*sin(theta2 + theta3) + L2*cos(theta2)
D1 - pz == D4*cos(theta2 + theta3) + L3*sin(theta2 + theta3) + L2*sin(theta2)

$p_xC_1 + p_yS_1 = L_3C_{23} - D_4S_{23} + L_2C_2$， $D_1 - p_z = D_4C_{23} + L_3S_{23} + L_2S_2$

分别将两边平方后相加，得

$$\left(p_xC_1+p_yS_1\right)^2+\left(D_1-p_z\right)^2=\left(L_3C_{23}-D_4S_{23}+L_2C_2\right)^2+\left(D_4C_{23}+L_3S_{23}+L_2S_2\right)^2$$
$$=L_2{}^2+L_3{}^2+D_4{}^2+2L_2L_3(C_2C_{23}+S_2S_{23})+2L_2D_4(S_2C_{23}-C_2S_{23})$$
$$=L_2{}^2+L_3{}^2+D_4{}^2+2L_2L_3C_3-2L_2D_4S_3$$

$$L_3C_3-D_4S_3=\frac{\left(p_xC_1+p_yS_1\right)^2+\left(D_1-p_z\right)^2-L_2{}^2-L_3{}^2-D_4{}^2}{2L_2}$$

假设有$-\sin(\theta)p_x+\cos(\theta)p_y=d$，求$\theta$。

首先进行三角恒等变换，令$p_x=\rho\cos(\phi),p_y=\rho\sin(\phi)$

其中：$\rho=\sqrt{p_x^2+p_y^2}$，$\phi=\operatorname{atan2}(p_x,p_y)$，然后带入原方程：

注：atan2(x,y)返回的是原点至点(x,y)的方位角。返回值的单位为弧度，取值范围为$(-\pi,\pi]$

$\cos(\theta)\sin(\phi)-\sin(\theta)\cos(\phi)=d/\rho$

$\sin(\phi-\theta)=d/\rho$，则$\cos(\phi-\theta)=\pm\sqrt{1-\frac{d^2}{\rho^2}}$

则 $$\phi-\theta=\operatorname{atan2}\left(\frac{d}{\rho},\pm\sqrt{1-\frac{d^2}{\rho^2}}\right)$$

$\theta=\operatorname{atan2}(p_y,p_x)-\operatorname{atan2}(-d,\pm\sqrt{p_x{}^2+p_y{}^2-d^2})$,其中$p_x{}^2+p_y{}^2-d^2\geq 0$

设 $$d_3=\frac{\left(p_xC_1+p_yS_1\right)^2+\left(D_1-p_z\right)^2-L_2{}^2-L_3{}^2-D_4{}^2}{2L_2}$$

$\theta_3=\arctan\left(L_3,D_4\right)-\arctan\left(d_3,\pm\sqrt{L_3{}^2+D_4{}^2-d_3{}^2}\right)$，其中 $L_3{}^2+D_4{}^2-d_3\geq 0$

3. 求解θ_2。同理，上式左右两边依次左乘${}_1^0T^{-1}$的逆、${}_2^1T^{-1}$的逆，令左右矩阵中1行4列和2行4列元素相等，可得：

D1*sin(theta2) - L2 - pz*sin(theta2) + px*cos(theta1)*cos(theta2) + py*cos(theta2)*sin(theta1) == L3*cos(theta3) - D4*sin(theta3)

D1*cos(theta2) - pz*cos(theta2) - px*cos(theta1)*sin(theta2) - py*sin(theta1)*sin(theta2) == D4*cos(theta3) + L3*sin(theta3)

$$D_1S_2-L_2-p_zS_2+p_xC_1C_2+p_yC_2S_1=L_3C_3-D_4S_3$$

$$D_1C_2-p_zC_2-p_xC_1S_2-p_yS_1S_2=D_4C_3+L_3S_3$$

$$C_2\left(D_1-p_z+p_xC_1+p_yS_1\right)-S_2\left(p_xC_1+p_yS_1+p_z-D_1\right)=D_4C_3+L_3S_3+L_3C_3-D_4S_3+L_2$$

设 $m_2=D_1-p_z+p_xC_1+p_yS_1$，$n_2=p_xC_1+p_yS_1+p_z-D_1$

$$d_2=D_4C_3+L_3S_3+L_3C_3-D_4S_3+L_2$$

$$\theta_2 = \arctan(m_2, n_2) - \arctan(d_2, \pm\sqrt{m_2^{\ 2} + n_2^{\ 2} - d_2^{\ 2}})$$

4. 求解θ_4，在等式两边同时依次左乘${}^0_1T^{-1}\,{}^1_2T^{-1}\,{}^2_3T^{-1}\,{}^3_4T^{-1}$，根据两边矩阵 3 行 3 列元素相等，可得

ay*cos(theta1)*cos(theta4) - ax*cos(theta4)*sin(theta1) - az*cos(theta2)*sin(theta3)*sin(theta4) - az*cos(theta3)*sin(theta2)*sin(theta4) + ax*cos(theta1)*cos(theta2)*cos(theta3)*sin(theta4) + ay*cos(theta2)*cos(theta3)*sin(theta1)*sin(theta4) - ax*cos(theta1)*sin(theta2)*sin(theta3)*sin(theta4) - ay*sin(theta1)*sin(theta2)*sin(theta3)*sin(theta4) == 0

$$m_4 = a_y C_1 - a_x S_1$$

$$n_4 = a_z C_2 S_3 + a_z C_3 S_2 - a_x C_1 C_2 C_3 - a_y C_2 C_3 S_1 + a_x C_1 S_2 S_3 + a_y S_1 S_2 S_3$$

$$\theta_4 = \arctan(m_4, n_4), \theta_4 = \theta_4 + 180$$

5. 求解θ_5，在等式式两边同时依次左乘${}^0_1T^{-1}\,{}^1_2T^{-1}\,{}^2_3T^{-1}\,{}^3_4T^{-1}$，得到的等式两边 1 行 3 列、 2 行 3 列元素对应相等，可得

ax*sin(theta1)*sin(theta4) - ay*cos(theta1)*sin(theta4) - az*cos(theta2)*cos(theta4)*sin(theta3) - az*cos(theta3)*cos(theta4)*sin(theta2) + ax*cos(theta1)*cos(theta2)*cos(theta3)*cos(theta4) + ay*cos(theta2)*cos(theta3)*cos(theta4)*sin(theta1) - ax*cos(theta1)*cos(theta4)*sin(theta2)*sin(theta3) - ay*cos(theta4)*sin(theta1)*sin(theta2)*sin(theta3) == -sin(theta5)

az*sin(theta2)*sin(theta3) - az*cos(theta2)*cos(theta3) - ax*cos(theta1)*cos(theta2)*sin(theta3) - ax*cos(theta1)*cos(theta3)*sin(theta2) - ay*cos(theta2)*sin(theta1)*sin(theta3) - ay*cos(theta3)*sin(theta1)*sin(theta2) == cos(theta5)

$$-S_5 = a_x S_1 S_4 - a_y C_1 S_4 - a_z C_2 C_4 S_3 - a_z C_3 C_4 S_2 + a_x C_1 C_2 C_3 C_4 + a_y C_2 C_3 C_4 S_1 - a_x C_1 C_4 S_2 S_3 - a_y C_4 S_1 S_2 S_3$$

$$C_5 = a_z S_2 S_3 - a_z C_2 C_3 - a_x C_1 C_2 S_3 - a_x C_1 C_3 S_2 - a_y C_2 S_1 S_3 - a_y C_3 S_1 S_2$$

$$\theta_5 = -\arctan\frac{a_x S_1 S_4 - a_y C_1 S_4 - a_z C_2 C_4 S_3 - a_z C_3 C_4 S_2 + a_x C_1 C_2 C_3 C_4 + a_y C_2 C_3 C_4 S_1 - a_x C_1 C_4 S_2 S_3 - a_y C_4 S_1 S_2 S_3}{a_z S_2 S_3 - a_z C_2 C_3 - a_x C_1 C_2 S_3 - a_x C_1 C_3 S_2 - a_y C_2 S_1 S_3 - a_y C_3 S_1 S_2}$$

6. 求解θ_6，在等式两边同时依次左乘${}^0_1T^{-1}\,{}^1_2T^{-1}\,{}^2_3T^{-1}\,{}^3_4T^{-1}\,{}^4_5T^{-1}$，根据等式两边矩阵 2 行 1 列和 2 行 2 列元素，可以推得

nx*cos(theta4)*sin(theta1) - ny*cos(theta1)*cos(theta4) + nz*cos(theta2)*sin(theta3)*sin(theta4) + nz*cos(theta3)*sin(theta2)*sin(theta4) - nx*cos(theta1)*cos(theta2)*cos(theta3)*sin(theta4) - ny*cos(theta2)*cos(theta3)*sin(theta1)*sin(theta4) + nx*cos(theta1)*sin(theta2)*sin(theta3)*sin(theta4) + ny*sin(theta1)*sin(theta2)*sin(theta3)*sin(theta4) == sin(theta6)

ox*cos(theta4)*sin(theta1) - oy*cos(theta1)*cos(theta4) + oz*cos(theta2)*sin(theta3)*sin(theta4) + oz*cos(theta3)*sin(theta2)*sin(theta4) - ox*cos(theta1)*cos(theta2)*cos(theta3)*sin(theta4) - oy*cos(theta2)*cos(theta3)*sin(theta1)*sin(theta4) + ox*cos(theta1)*sin(theta2)*sin(theta3)*sin(theta4)

+ oy*sin(theta1)*sin(theta2)*sin(theta3)*sin(theta4) == cos(theta6)

$$S_6 = n_x C_4 S_1 - n_y C_1 C_4 + n_z C_2 S_3 S_4 + n_z C_3 S_2 S_4 - n_x C_1 C_2 C_3 S_4 - n_y C_2 C_3 S_1 S_4 + n_x C_1 S_2 S_3 S_4 + n_y S_1 S_2 S_3 S_4$$

$$C_6 = o_x C_4 S_1 - o_y C_1 C_4 + o_z C_2 S_3 S_4 + o_z C_3 S_2 S_4 - o_x C_1 C_2 C_3 S_4 - o_y C_2 C_3 S_1 S_4 + o_x C_1 S_2 S_3 S_4 + o_y S_1 S_2 S_3 S_4$$

$$\theta_6 = \arctan \frac{n_x C_4 S_1 - n_y C_1 C_4 + n_z C_2 S_3 S_4 + n_z C_3 S_2 S_4 - n_x C_1 C_2 C_3 S_4 - n_y C_2 C_3 S_1 S_4 + n_x C_1 S_2 S_3 S_4 + n_y S_1 S_2 S_3 S_4}{o_x C_4 S_1 - o_y C_1 C_4 + o_z C_2 S_3 S_4 + o_z C_3 S_2 S_4 - o_x C_1 C_2 C_3 S_4 - o_y C_2 C_3 S_1 S_4 + o_x C_1 S_2 S_3 S_4 + o_y S_1 S_2 S_3 S_4}$$