An equivalent form of Kusner's conjecture and proof of its weaker version

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Abstract

In this article, I proposed an equivalent form of Kusner's conjecture on l_1^n and proved its weaker version. The findings maybe had been included in other published paper.

1 Introduction

An equilateral set in a metric space is a set A that the distance between any pair of distinct members of A is a constant. l_1^d denotes the space $\mathbb{R}^d$ with the l_1-distance $||a-b|| = \Sigma_{i=1}^d |a_i - b_i|$, where $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d)$. Kusner conjectured that the maximum possible cardinality of an equilateral set in l_1^d is $2d$ [1].

On the following part, I will propose an equivalent form of Kusner's conjecture on section 2 and prove its weaker version on section 3.

2 An equivalent form of Kusner's conjecture on $\mathbb{Z}^d$

Proposition 1. For any given integer n greater than 1, we define that an equilateral set in $\{0, 1, 2, \ldots, n\}^d$ is a set A that the l_1-distance between any pair of distinct members of A is a constant. Then the maximum possible cardinality of an equilateral set in $\{0, 1, 2, \ldots, n\}^d$ is $2d$.

Proof of the equivalence between Kusner's conjecture and Proposition 1.
We only need to show that Kusner's conjecture is true under the proposition. For any equilateral set A in $\mathbb{R}^d$. Consider the vectors $\{(\{t * a_{11}\}, \{t * a_{12}\}, \ldots, \{t * a_{md}\}), t \in \mathbb{Z}\}$, where a_{ij} is the j-th coordinate of a_i in A and $\{x\}$ denotes the decimal part of x. These vectors are in the hypercube $[0, 1]^{m*d}$. Therefore, there exists different $t, s \in \mathbb{Z}$ such that the l_1-distance between $(\{t * a_{11}\}, \{t * a_{12}\}, \ldots, \{t * a_{md}\})$ and $(\{s * a_{11}\}, \{s * a_{12}\}, \ldots, \{s * a_{md}\})$ is smaller than $1/2$.

Next, we show that $B = \{([(s-t) * a_{i1}], [(s-t) * a_{i2}], \ldots, [(s-t) * a_{id}]), i \in \{1, 2 \ldots, m\}\}$ is an equilateral set, where $[x]$ denotes the integer that has the smallest distance with x. We assume the distance between any pair of members in A is L, then for any given $i \neq j$,

$$
\begin{aligned}
&|\Sigma_{k=1}^d |[(s-t) * a_{ik}] - [(s-t) * a_{jk}]| - (s-t)L| \\
=&|\Sigma_{k=1}^d |[(s-t) * a_{ik}] - [(s-t) * a_{jk}]| - \Sigma_{k=1}^d |(s-t) * a_{ik} - (s-t) * a_{jk}|| \\
\leq&\Sigma_{k=1}^d |[(s-t) * a_{ik}] - (s-t) * a_{ik}| + \Sigma_{k=1}^d |[(s-t) * a_{jk}] - (s-t) * a_{jk}| \\
\leq&\Sigma_{k=1}^d |\{s * a_{ik}\} - \{t * a_{ik}\}| + \Sigma_{k=1}^d |\{s * a_{jk}\} - \{t * a_{jk}\}| \\
<&1/2.
\end{aligned}
$$

Therefore, the distance between any pair of distinct numbers in B is an integer in $((s-t)L-1/2,(s-t)L+1/2)$, which is a constant. Thus, B is an equilateral set. By adding a big integer on each coordinate of all members in B, there is an integer n such that $B \subset \{0,1,2,\ldots,n\}^d$. Therefore, m is not greater than $2d$ according to the proposition, which confirms Kusner's conjecture.

3 A weaker version of the equivalent form

Theorem 1. For any given positive integer n and d, $n \geq 2$, we define that an equilateral set in $\{0,1,2,\ldots,n\}^d$ is a set A that the l_1-distance between any pair of distinct members of A is a constant. Then the maximum possible cardinality of an equilateral set in $\{0,1,2,\ldots,n\}^d$ is not greater than $n*d$.

Proof. We construct a mapping $f : \{0,1,2,\ldots,n\}^d \to \{-1,1\}^{n*d}$.

$$f((a_1,a_2,\ldots,a_d)) = (p_1,p_2,\ldots,p_{n*d}),$$

where p is defined as

$$p_{n*i+j} = \text{sign}\{a_{i+1} < j\} \quad 0 \leq i \leq d-1;\ 1 \leq j \leq n,$$

where sign is a function that equals 1 when $a_{i+1} < j$ and equals -1 otherwise. Therefore,

$$\begin{aligned}
&f((a_1,a_2,\ldots,a_d)) * f((b_1,b_2,\ldots,b_d)) \\
&= \Sigma_{i=0,1,\ldots,d-1;j=1,2,\ldots,n}\ \text{sign}\{a_{i+1} < j\}\text{sign}\{b_{i+1} < j\} \\
&= \Sigma_{0\leq i\leq d-1}\ (n-2|a_{i+1}-b_{i+1}|) \\
&= n*d - 2*\Sigma_{0\leq i\leq d-1}\ |a_{i+1}-b_{i+1}|.
\end{aligned} \tag{1}$$

For the problem, we assumed that there exists an equilateral set A in $\{0,1,2,\ldots,n\}^d$ which has $n*d+1$ members. The l_1-distance between any pair of distinct members of A is k ($k > 0$). We define B as $f(A)$. The dot product of coordinate vectors in B is $n*d-2k$ according to equation (1). Then we consider the coordinate matrix constructed by $n*d+1$ points in B, denoted by $(n*d+1)*(n*d)$ dimensional matrix O. We have

$$O*O^T = \begin{pmatrix} n*d & n*d-2k & \cdots & n*d-2k \\ n*d-2k & n*d & \cdots & n*d-2k \\ \vdots & \vdots & \ddots & \vdots \\ n*d-2k & n*d-2k & \cdots & n*d \end{pmatrix}.$$

As the rank of O is not greater than $d*n$, there exists nonzero vector $\vec{x}$ such that $\vec{x}*O = \vec{0}$, then

$$\vec{0} = \vec{x}*O*O^T = \vec{x}*\begin{pmatrix} n*d & n*d-2k & \cdots & n*d-2k \\ n*d-2k & n*d & \cdots & n*d-2k \\ \vdots & \vdots & \ddots & \vdots \\ n*d-2k & n*d-2k & \cdots & n*d \end{pmatrix}.$$

Therefore $\vec{x}$ is constant on each coordinate, which means that the summation of coordinate vectors in B is $\vec{0}$. By using the definition of f, we can get that each coordinate of members in A is 0 or n. Then problem is converted to the case $n = 1$ by dividing all vectors in A by n.

Therefore, there exists an equilateral set in $\{0,1\}^d$ with $n*d+1 \geq d+2$ members. By using the above method, we can get

$$\vec{0} = \vec{x} * O * O^T = \vec{x} * \begin{pmatrix} d & d-2k & \cdots & d-2k \\ d-2k & d & \cdots & d-2k \\ \vdots & \vdots & \ddots & \vdots \\ d-2k & d-2k & \cdots & d \end{pmatrix}, \tag{2}$$

where O is a $(d+2)*(d+2)$ dimensional matrix. Thus $d + (d-2k)*(d+1) = 0$, which is not true. Therefore, the assumption that A has $n*d+1$ members is not true and the problem is proved.

4 Discussion

I got these findings by some efforts after I knew Kusner's conjecture. After that, I found it interesting that the method adopted in [2] is partly similar to my thought. They both split the coordinate and discuss the rank of a matrix product. However, my perspective on Kusner's conjecture is different with that paper. I converted the conjecture from $\mathbb{R}^d$ into $\{0,1,\ldots,n\}^d$. Although the weaker version I proved here is far from the bound $2d$, there raises a route that tackling the conjecture on integer space. In [2], authors thought it seems unlikely that the methods described there will suffice to prove the precise statement of the conjecture. Therefore, the methods maybe need to be combined with number theory to achieve the precise upper bound. In addition, there maybe exists an effective computational method basing on number theory to find a counter-example if Kusner's conjecture is false.

I am not going to submit this paper to any journal, as I am not an expert on this field. If you find it worthy to publish or know some published paper including this paper's finding, please feel free to email me.

References

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[2] N. Alon and P. Pudl'ak. Equilateral sets in l_p^n. *Geom. Funct. Anal.*, 78, 13(3):467-482, 2003.