Relations on Some Cohomology Theories

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1 Some Tools in Sheaf Theory

To prove some relation of cohomology theories, we need some tools from sheaf theory. First, we present a general construction, Godement canonical resolution.

Let $\mathcal{F}$ be a sheaf of Abelian groups on a topology space X, we define a sheaf $\mathcal{C}^0\mathcal{F}$, $\mathcal{C}^0\mathcal{F}(U) := \prod_{p\in U}\mathcal{F}_p$ and the restriction map is clearly induced from $\mathcal{F}$. $\mathcal{C}^0\mathcal{F}$ can be interpreted as sheaf of all (not necessarily continuous) sections, clearly there is an monomorphism from $\mathcal{F}$ to $\mathcal{C}^0\mathcal{F}$ and an exact sequence.

$$0 \to \mathcal{F} \to \mathcal{C}^0\mathcal{F} \to \mathcal{Q}^1 \to 0$$

where $\mathcal{Q}^1$ is the quotient $\mathcal{C}^0\mathcal{F}/\mathcal{F}$. This construction can be repeated, i.e.

$$0 \to \mathcal{Q}^1 \to \mathcal{C}^0\mathcal{Q}^1 \to \mathcal{Q}^2 \to 0$$
$$0 \to \mathcal{Q}^2 \to \mathcal{C}^0\mathcal{Q}^2 \to \mathcal{Q}^3 \to 0$$
$$\cdots$$

These short exact sequences can be spliced together to a long exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0\mathcal{F} \to \mathcal{C}^1\mathcal{F} \to \mathcal{C}^2\mathcal{F} \to \cdots$$

where $\mathcal{C}^k\mathcal{F} := \mathcal{C}^0\mathcal{Q}^k$. This is called Godement resolution.

This construction is actually functorial. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$, $\mathcal{C}^0\phi : \mathcal{C}^0\mathcal{F} \to \mathcal{C}^0\mathcal{G}$ is clearly defined by a collection of morphism on stalks. And the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "\phi"'] & \mathcal{C}^0\mathcal{F} \arrow[d, "\mathcal{C}^0\phi"'] \\
\mathcal{G} \arrow[r] & \mathcal{C}^0\mathcal{G}
\end{tikzcd}$$

yields a morphism between quotients $\mathcal{Q}^1_{\mathcal{F}} \to \mathcal{Q}^1_{\mathcal{G}}$. Repeating this process, $\mathcal{C}^k\phi : \mathcal{C}^k\mathcal{F} \to \mathcal{C}^k\mathcal{G}$ is defined.

Proposition: The Godement sheaf functor $\mathcal{F}$ is exact.

Proof. For any short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

It's exact on each stalk. Notice that $\mathcal{C}^0\mathcal{F}(U)$ is the product of all stalks of $\mathcal{F}$ in U. So the following sequence is exact.

$$0 \to \mathcal{C}^0\mathcal{F}(U) \to \mathcal{C}^0\mathcal{G}(U) \to \mathcal{C}^0\mathcal{H}(U) \to 0$$

Taking direct limit, we get exact sequence

$$0 \to (\mathcal{C}^0\mathcal{F})_p \to (\mathcal{C}^0\mathcal{G})_p \to (\mathcal{C}^0\mathcal{H})_p \to 0$$

So, $\mathcal{C}^0$ is exact. Then

$$0 \to \mathcal{Q}_F \to \mathcal{Q}_G \to \mathcal{Q}_H \to 0$$

is exact by classical discussion in homological algebra(zig-zag lemma or nine lemma). These two steps suffice to prove all $\mathcal{C}^k$ are exact by definition. □

We define the Godement cohomology for a sheaf $\mathcal{F}$ on a topological space X

$$H^k_{\mathcal{C}}(X, \mathcal{F}) := h^k(\mathcal{C}^\bullet\mathcal{F}(X))$$

where h^k takes k-th cohomology of a cochain complex. (Later, we will prove it coincides with the usual sheaf cohomology)

Proposition: $H^0_{\mathcal{C}}(X, \mathcal{F}) = \mathcal{F}(X)$

Proof.

$$H^0_{\mathcal{C}}(X, \mathcal{F}) = \ker(\mathcal{C}^0\mathcal{F}(X) \to \mathcal{C}^1\mathcal{F}(X))$$

From the definition, $\mathcal{C}^0\mathcal{F} \to \mathcal{C}^1\mathcal{F}$ is the composition of

$$\mathcal{C}^0\mathcal{F} \to \mathcal{Q}^1 \to \mathcal{C}^1\mathcal{F}$$

where the second map is monomorphism. So $\mathcal{C}^0\mathcal{F}(X) \to \mathcal{C}^1\mathcal{F}(X)$ is the composition of

$$\mathcal{C}^0\mathcal{F}(X) \to \mathcal{Q}^1(X) \to \mathcal{C}^1\mathcal{F}(X)$$

where the second map is still monomorphism by the left exactness of $\Gamma(X, -)$.

So

$$\ker(\mathcal{C}^0\mathcal{F}(X) \to \mathcal{C}^1\mathcal{F}(X)) = \ker(\mathcal{C}^0\mathcal{F}(X) \to \mathcal{Q}^1(X)) = \mathcal{F}(X)$$

□

Definition: A sheaf of Abelian groups on a topological space X is called flasque, if for every open subset U, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

For any sheaf $\mathcal{F}$, the Godement sheaf $\mathcal{C}^0\mathcal{F}$ is flasque, since $\mathcal{C}^0\mathcal{F}(X) = \prod_{p\in X} \mathcal{F}_p \to \mathcal{C}^0\mathcal{F}(U) = \prod_{p\in U} \mathcal{F}_p$ is clearly surjective. And $\mathcal{C}^k\mathcal{F} = \mathcal{C}^0\mathcal{Q}^k$, so all $\mathcal{C}^k\mathcal{F}$ are flasque.

Proposition:

1. For a short exact sequence of sheaves over topological space

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

if $\mathcal{F}$ is flasque, then for any open subset U,

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

is exact

2. If $\mathcal{F}$ and $\mathcal{G}$ above are flasque, so is $\mathcal{H}$.

3. If

$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots$$

is an exact sequence of flasque sheaves on X, then for any open set $U \subset X$ the sequence of abelian groups

$$0 \to \mathcal{F}(U) \to \mathcal{L}^0(U) \to \mathcal{L}^1(U) \to \mathcal{L}^2(U) \to \cdots$$

is exact.

Proof. 1. Let i and j be the map for $\mathcal{F}(U) \to \mathcal{G}(U)$ and $\mathcal{G}(U) \to \mathcal{H}(U)$ respectively. By abuse of notation, we may use i and j to represent the maps restricted to a smaller open set. It suffices to prove the surjectivity of $j : \mathcal{G}(U) \to \mathcal{H}(U)$. Let $g \in \mathcal{H}(U)$. Since $\mathcal{G} \to \mathcal{H}$ is surjective as a sheaf map, all stalk maps $\mathcal{G}_p \to \mathcal{H}_p$ are surjective. Hence, every point $p \in U$ has a neighborhood $U_\alpha \subset U$ on which there exists a section $f_\alpha \in \mathcal{G}(U_\alpha)$ such that $j(f_\alpha) = g|_{U_\alpha}$.

Let V be the largest union $\bigcup_\alpha U_\alpha$ on which there is a section $f_V \in \mathcal{G}(V)$ such that $j(f_V) = g|_V$. We claim that $V = U$. If not, then there is a set U_α not contained in V and $f_\alpha \in \mathcal{G}(U_\alpha)$ such that $j(f_\alpha) = g|_{U_\alpha}$. If $V \cap U_\alpha = \emptyset$, the gluing axiom gives a compatible section $f_{V\cup U_\alpha} \in \mathcal{G}(V \cup U_\alpha)$ which contradicts with the maximality of V. So we may assume $V \cap U_\alpha$ is not empty. On $V \cap U_\alpha$, still writing j for $j_{V\cap U_\alpha}$, we have

$$j(f_V - f_\alpha) = 0.$$

$$f_V - f_\alpha = i(e_{V,\alpha}) \text{ for some } e_{V,\alpha} \in \mathcal{F}(V \cap U_\alpha).$$

Since $\mathcal{F}$ is flasque, one can find a section $e_U \in \mathcal{F}(U)$ such that $e_U|_{V\cap U_\alpha} = e_{V,\alpha}$.

On $V \cap U_\alpha$,

$$f_V = i(e_{V,\alpha}) + f_\alpha.$$

If we modify f_α to

$$\bar{f}_\alpha = i(e_U) + f_\alpha \quad \text{on } U_\alpha,$$

then $f_V = \bar{f}_\alpha$ on $V \cap U_\alpha$, and $j(\bar{f}_\alpha) = g|_{U_\alpha}$. By the gluing axiom for the sheaf $\mathcal{F}$, the elements f_V and $\bar{f}_\alpha$ piece together to give an element $f \in \mathcal{F}(V \cup U_\alpha)$ such that $j(f) = g|_{V\cup U_\alpha}$. This contradicts the maximality of V.

2. Basic diagram chasing can show $\mathcal{H}(X) \to \mathcal{H}(U)$ is surjective.

3. The long exact sequence is equivalent to a list of short exact sequences

$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{Q}^0 \to 0$$

$$0 \to \mathcal{Q}^0 \to \mathcal{L}^1 \to \mathcal{Q}^1 \to 0$$

$$\cdots$$

The proposition above shows that each $\mathcal{Q}^k$ is flasque. So the following sequences are exact.

$$0 \to \mathcal{F}(U) \to \mathcal{L}^0(U) \to \mathcal{Q}^0(U) \to 0$$

$$0 \to \mathcal{Q}^0(U) \to \mathcal{L}^1(U) \to \mathcal{Q}^1(U) \to 0$$

$$\cdots$$

They splice together to get the long exact sequence we want.

□

Corollary: Flasque sheaf has no nontrivial Godement cohomology.

Corollary: The functor from sheaves to Abelian groups $\mathcal{F} \to \mathcal{C}^k\mathcal{F}(X)$ is exact.

We say a sheaf $\mathcal{F}$ on X is $\mathcal{C}$-acyclic if $H^k_{\mathcal{C}}(X, \mathcal{F}) = 0$ for all $k > 0$.

Now it's time to relate Godement cohomology with the usual sheaf cohomology we defined using injective resolution.

First we have the following fact.

Proposition: Injective sheaf is flasque, thus $\mathcal{C}$-acyclic.

Proof. Let $\mathbb{Z}_X$ be the locally constant sheaf of $\mathbb{Z}$ on X, we have a natural isomorphism

$$\mathcal{F}(X) \cong \mathrm{Hom}(\mathbb{Z}_X, \mathcal{F})$$

We define a sheaf $\mathbb{Z}_U$ on X as a subsheaf of $\mathbb{Z}_X$

$$\mathbb{Z}_U(V) := \{s \in \mathbb{Z}_X(U \cap V) : \mathrm{supp} s \text{ is closed in } V\}$$

where the support of a section f is defined to be $\mathrm{supp} f := \{x \in X : f_x \neq 0\}$, which is an inborn closed set unlike the continuous function case.

In fact, we have a clear interpretation for locally constant sheaf. We know $\mathbb{Z}_X(V) \cong \mathbb{Z}\{V_i : i \in I\}$ where each V_i is a connected component of V. From the definition, it's also easy to see $\mathbb{Z}_U(V) \cong \mathbb{Z}\{V_i : V_i \subset U\}$. Then following isomorphism is valid.

$$\mathcal{F}(U) \cong \mathrm{Hom}(\mathbb{Z}_U, \mathcal{F})$$

So any section $s \in \mathcal{F}(U)$ can be identified as a morphism $\mathbb{Z}_U \to \mathcal{F}$, if $\mathcal{F}$ is injective the morphism can be extended to be a morphism $\mathbb{Z}_X \to \mathcal{F}$ which corresponds to a section in $\mathcal{F}(X)$. So $\mathcal{F}$ is flasque. □

Proposition: The Godement cohomology coincides with usual sheaf cohomology.

Proof. For a sheaf $\mathcal{F}$, pick any injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$. Then take Godement functor for each injective sheaf and apply $\Gamma(X, -)$, we get a double complex.

$$\begin{tikzcd}
(\mathcal{C}^0\mathcal{I}^2)(X) \arrow[r] & (\mathcal{C}^1\mathcal{I}^2)(X) \arrow[r] & (\mathcal{C}^2\mathcal{I}^2)(X) \\
(\mathcal{C}^0\mathcal{I}^1)(X) \arrow[r] \arrow[u] & (\mathcal{C}^1\mathcal{I}^1)(X) \arrow[r] \arrow[u] & (\mathcal{C}^2\mathcal{I}^1)(X) \arrow[u] \\
(\mathcal{C}^0\mathcal{I}^0)(X) \arrow[r] \arrow[u] & (\mathcal{C}^1\mathcal{I}^0)(X) \arrow[r] \arrow[u] & (\mathcal{C}^2\mathcal{I}^0)(X) \arrow[u]
\end{tikzcd}$$

$$E_0^{p,q} := (\mathcal{C}^p\mathcal{I}^q)(X)$$

First, take horizontal cohomology, since each $\mathcal{I}^q$ is $\mathcal{C}$-acyclic, we get

$$E_1^{p,q} = \begin{cases} \mathcal{I}^q(X) & p = 0 \\ 0 & p > 0 \end{cases}$$

Then taking vertical cohomology, we get

$$E_2^{p,q} = \begin{cases} H^q(X, \mathcal{F}) & p = 0 \\ 0 & p > 0 \end{cases}$$

On the other side, taking vertical cohomology first, since $(\mathcal{C}^p -)(X)$ is an exact functor, we get

$$E_1'^{p,q} = \begin{cases} (\mathcal{C}^p\mathcal{F})(X) & q = 0 \\ 0 & q > 0 \end{cases}$$

Then taking horizontal cohomology yields

$$E_2'^{p,q} = \begin{cases} H^p_{\mathcal{C}}(X, \mathcal{F}) & q = 0 \\ 0 & q > 0 \end{cases}$$

So we have

$$H^n(X, \mathcal{F}) \cong H^n_{\mathcal{C}}(X, \mathcal{F})$$

which is true from spectral sequence(or just diagram chasing since it's a degenerate case) □

Remark: The same argument can show any $\mathcal{C}$-acyclic resolution computes Godement cohomology, thus sheaf cohomology. In particular, flasque resolution does.

Now we introduce another type of sheaf which is useful in de-Rham cohomology.

Definition: Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X and $\{U_\alpha\}$ a locally finite open cover of X. A partition of unity of a sheaf $\mathcal{F}$ subordinate to $\{U_\alpha\}$ is a collection $\{\eta_\alpha \colon \mathcal{F} \to \mathcal{F}\}$ of sheaf maps such that

1. $\mathrm{supp}\eta_\alpha \subset U_\alpha$;
2. for each point $x \in X$, $\sum_\alpha \eta_{\alpha,x} = \mathrm{id}_{\mathcal{F}_x}$.

Definition: A sheaf $\mathcal{F}$ on a topological space X is said to be fine if for every locally finite open cover $\{U_\alpha\}$ of X, the sheaf $\mathcal{F}$ admits a partition of unity subordinate to $\{U_\alpha\}$.

Definition: A topological space is X called paracompact if any open cover admits a locally finite refinement.

It's called hereditarily paracompact if any open subset is paracompact.

On nice space, fine sheaf has the following property similar to flasque sheaf.

Proposition:

1. For a short exact sequence of sheaves over a paracompact space X,
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$
if $\mathcal{F}$ is fine, then
$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$$
is exact

   Furthermore, if X is hereditarily paracompact, we have

2. If $\mathcal{F}$ is fine and $\mathcal{G}$ is flasque, $\mathcal{H}$ is flasque.
3. If
$$0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \cdots$$
is an exact sequence of sheaves on X in which $\mathcal{F}$ is fine and $\mathcal{L}^i$ is flasque, then for any open set $U \subset X$ the sequence of abelian groups
$$0 \to \mathcal{F}(U) \to \mathcal{L}^0(U) \to \mathcal{L}^1(U) \to \mathcal{L}^2(U) \to \cdots$$
is exact.

Proof. Let i and j be the map for $\mathcal{F}(X) \to \mathcal{G}(X)$ and $\mathcal{G}(X) \to \mathcal{H}(X)$ respectively. By abuse of notation, we may use i and j to represent the maps restricted to a smaller open set. It suffices to show $j \colon \mathcal{G}(X) \to \mathcal{H}(X)$ is surjective.

Let $g \in \mathcal{H}(X)$. Since $\mathcal{G}_p \to \mathcal{H}_p$ is surjective for all $p \in X$, there exist an open cover $\{U_\alpha\}$ of X and elements $f_\alpha \in \mathcal{G}(U_\alpha)$ such that $j(f_\alpha) = g|_{U_\alpha}$. By the paracompactness of X, we may assume that the open cover $\{U_\alpha\}$ is locally finite. On $U_{\alpha\beta} := U_\alpha \cap U_\beta$,

$$j(f_\alpha|_{U_{\alpha\beta}} - f_\beta|_{U_{\alpha\beta}}) = j(f_\alpha)|_{U_{\alpha\beta}} - j(f_\beta)|_{U_{\alpha\beta}} = g|_{U_{\alpha\beta}} - g|_{U_{\alpha\beta}} = 0.$$

By the exactness of the sequence

$$0 \to \mathcal{F}(U_{\alpha\beta}) \xrightarrow{i} \mathcal{G}(U_{\alpha\beta}) \xrightarrow{j} \mathcal{H}(U_{\alpha\beta}),$$

there is an element $e_{\alpha\beta} \in \mathcal{F}(U_{\alpha\beta})$ such that on $U_{\alpha\beta}$, s.t.

$$f_\alpha - f_\beta = i(e_{\alpha\beta}).$$

Note that on the triple intersection $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$i(e_{\alpha\beta} + e_{\beta\gamma}) = f_\alpha - f_\beta + f_\beta - f_\gamma = i(e_{\alpha\gamma}).$$

Since $\mathcal{F}$ is a fine sheaf, it admits a partition of unity $\{\eta_\alpha\}$ subordinate to $\{U_\alpha\}$. Then the section $\eta_\gamma(e_{\alpha\gamma}) \in \mathcal{F}(U_{\alpha\gamma})$ can be extended by zero to a section over U_α. To simplify the notation, we still write the extension $\eta_\gamma(e_{\alpha\gamma}) \in \mathcal{F}(U_\alpha)$ Let e_α be the locally finite sum

$$e_\alpha = \sum_\gamma \eta_\gamma e_{\alpha\gamma} \in \mathcal{F}(U_\alpha).$$

On the intersection $U_{\alpha\beta}$,

$$\begin{aligned} i(e_\alpha - e_\beta) &= i\Big(\sum_\gamma \eta_\gamma e_{\alpha\gamma} - \sum_\gamma \eta_\gamma e_{\beta\gamma}\Big) = i\Big(\sum_\gamma \eta_\gamma (e_{\alpha\gamma} - e_{\beta\gamma})\Big) \\ &= i\Big(\sum_\gamma \eta_\gamma e_{\alpha\beta}\Big) = i(e_{\alpha\beta}) = f_\alpha - f_\beta. \end{aligned}$$

Hence, on $U_{\alpha\beta}$,

$$f_\alpha - i(e_\alpha) = f_\beta - i(e_\beta).$$

By the gluing sheaf axiom for the sheaf $\mathcal{G}$, there is an element $f \in \mathcal{G}(X)$ such that $f|_{U_\alpha} = f_\alpha - i(e_\alpha)$. Then

$$j(f)|_{U_\alpha} = j(f_\alpha) = g|_{U_\alpha} \quad \text{for all } \alpha.$$

By the uniqueness sheaf axiom for the sheaf $\mathcal{H}$, we have $j(f) = g \in \mathcal{H}(X)$.

The proof for the other two properties is essentially the same as the flasque case. □

Corollary: Fine sheaves over hereditarily paracompact space are $\mathcal{C}$-acyclic.

2 Relation with Some Particular Cohomology Theories

2.1 Singular Cohomology

Theorem: Let X be a locally contractible and hereditarily paracompact topological space and $H^n_{sing}(X,\mathbb{R})$ be it's singular cohomology group with value in $\mathbb{R}$, then we have $H^n_{sing}(X,\mathbb{R}) \cong H^n(X,\mathbb{R}_X)$ where $\mathbb{R}_X$ is the locally constant sheaf with value in $\mathbb{R}$ on X.

To prove this theorem, we give a resolution of $\mathbb{R}_X$ which is related to singular cohomology. Namely, we claim such resolution exists for locally contractible space X.

$$0 \to \mathbb{R}_X \to \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \mathcal{C}^3 \to \cdots$$

$\mathcal{C}^p$ is the sheafification of the singular cochain presheaf $\mathcal{C}^p_{sing}$, where $\mathcal{C}^p_{sing}(U) := C^p(U,\mathbb{R}) = Hom_{\mathbb{Z}}(C_p(U),\mathbb{R})$.

It's indeed a resolution because the exactness can be check at each stalk which coincides with the stalk of presheaf where the exactness comes from the singular cohomology of a contractible space is trivial.

Actually, we have a good expression of $\mathcal{C}^p$, i.e.

$$\mathcal{C}^p(U) \cong \mathcal{C}^p_{sing}(U)/\mathcal{C}^p_{sing}(U)_0$$

where $\mathcal{C}^p_{sing}(U)_0 = \{\phi \in \mathcal{C}^p_{sing}(U) : \exists$ open cover of U, $\mathcal{U} = (U_i)_{i\in I}$ s.t. $\phi|_{C_p(U_i)} = 0\}$. The isomorphism comes from the sheafification map $\mathcal{C}^p_{sing}(U) \to \mathcal{C}^p(U)$. Definition of sheafification tells $\mathcal{C}^p_{sing}(U)_0$ is exactly the kernel. To prove such map is surjective, we need a technical lemma.

Lemma: Let X be hereditarily paracompact and let $\mathcal{F}$ be a presheaf on X. Let $\hat{\mathcal{F}}$ denote its sheafification. Suppose that the presheaf $\mathcal{F}$ already satisfies gluability in the sense that, for any covering $\mathcal{U} = (U_i)$ of an open subset U and choice of sections $\alpha_i \in \mathcal{F}(U_i)$ which agree on pairwise intersections, there is a section $\alpha \in \mathcal{F}(U)$ which restricts to α_i over U_i. Then for any open set U , the natural map $\mathcal{F}(U) \to \hat{\mathcal{F}}(U)$ is a surjection.

Proof. See Prop 1.14 in [1] □

The presheaf $\mathcal{C}^p_{sing}$ clearly satisfies the gluing condition. So the isomorphism is valid. Also, one can easily see $\mathcal{C}^p$ is flasque since the surjectivity condition can be verified for $\mathcal{C}^p_{sing}$ and preserved after the quotient.

Then we claim the two cohomology computed from $\mathcal{C}^\bullet_{sing}$ and $\mathcal{C}^\bullet$ coincide. i.e. $\mathcal{C}^\bullet_{sing}(X)$ and $\mathcal{C}^\bullet(X)$ are quasi-isomorphic. For any open cover $\mathcal{U} = (U_i)$, we have a homotopy equivalence $C^{\mathcal{U}}_p(X) \to C_p(X)$. Taking dual, $\pi^p_{\mathcal{U}} : C^p(X,\mathbb{R}) \to C^p_{\mathcal{U}}(X,\mathbb{R})$ is a homotopy equivalence. It's surjective since we can define the value for simplex not in $C^{\mathcal{U}}_p(X)$ arbitrarily. We have an exact sequence

$$0 \to Ker\pi^p_{\mathcal{U}} \to C^p(X,\mathbb{R}) \to C^p_{\mathcal{U}}(X,\mathbb{R}) \to 0$$

From the induced long exact sequence, one can see that $Ker\pi_{\mathcal{U}}^{\bullet}$ is acyclic. Then taking direct limit for opencover and refinement, we have $\lim\limits_{\rightarrow} Ker\pi_{\mathcal{U}}^{\bullet} = \mathcal{C}_{sing}^{\bullet}(X)_0$ by definition and it's still acyclic since taking direct limit is exact. Then the right term in the short exact sequence becomes $\mathcal{C}^p(X)$ from the isomorphism $\mathcal{C}^p(X) \cong \mathcal{C}_{sing}^p(X)/\mathcal{C}_{sing}^p(X)_0$.

So we complete the argument since flasque resolution computes the sheaf cohomology.

Remark:

1. The discussion is essentially the same if we replace $\mathbb{R}$ with any other Abelian group.
2. The theorem is true for more general space, i.e. the hereditary paracompactness is not needed. See [2] for details.

2.2 de Rham Cohomology

For a smooth manifold X, we have a natural resolution

$$0 \to \mathbb{R}_X \to \mathcal{A}^1 \to \mathcal{A}^2 \to \mathcal{A}^3 \to \cdots$$

where $\mathcal{A}^k$ is the sheaf of smooth k-forms.

It's indeed a resolution due to Poincare Lemma, i.e. locally every closed form is exact. We have

Proposition: $\mathcal{A}^k$ is a fine sheaf.

Proof. Let (U_α) be a locally finite open cover. Then there is a smooth partition of unity on the manifold (ρ_α) subordinate to (U_α). Then we define $\eta_{\alpha,U} : \mathcal{A}^k(U) \to \mathcal{A}^k(U)$ by $\omega \mapsto \rho_\alpha\omega$. It is the desired partition of unity. □

Since smooth manifold is hereditarily paracompact, on which fine sheaves are $\mathcal{C}$- acyclic. We have the following isomorphism.

Theorem: Let X be a smooth manifold, $H^n(X, \mathbb{R}_X) \cong H^n_{dR}(X, \mathbb{R})$.

2.3 Čech Cohomology

Let $\mathcal{U} = (U_\alpha)_{\alpha\in\Lambda}$ be a open cover of a space X, where the index set is linearly ordered. Let $\mathcal{F}$ be a sheaf on X, for any open subset $U \subset X$, we can define the restricted sheaf $\mathcal{F}_U$, $\mathcal{F}_U(V) := \mathcal{F}(U \cap V)$. We use the notation $U_{\alpha_0\alpha_1\cdots\alpha_p} := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$. Then we have a resolution

$$0 \to \mathcal{F} \to \prod_{\alpha_0} \mathcal{F}_{U_{\alpha_0}} \to \prod_{\alpha_0<\alpha_1} \mathcal{F}_{U_{\alpha_0\alpha_1}} \to \prod_{\alpha_0<\alpha_1<\alpha_2} \mathcal{F}_{U_{\alpha_0\alpha_1\alpha_2}} \to \cdots$$

Let $\mathcal{F}^p := \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{F}_{U_{\alpha_0\cdots\alpha_p}}$, the Čech coboundary operator

$$\delta_V : \mathcal{F}^p(V) \to \mathcal{F}^{p+1}(V)$$

is defined by

$$(\delta_V \omega)_{\alpha_0 \cdots \alpha_{p+1}} := \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_{p+1}}$$

for $\omega = (\omega_{\alpha_0 \cdots \alpha_p})_{\alpha_0 \cdots \alpha_p} \in \mathcal{F}^p(V)$. (the restriction map is omitted).

$\delta^2 = 0$ is easily checked.

We need to prove what we give is indeed a resolution.

For $s \in \mathcal{F}^{n+1}(V)$, if $\delta s = 0$, define $t_\beta = (s_{\beta\alpha_0 \cdots \alpha_n})_{\alpha_0 \cdots \alpha_n} \in \mathcal{F}^n(V \cap U_\beta)$, we can verify that $(\delta t_\beta)_{\alpha_0 \cdots \alpha_{n+1}} = s_{\alpha_0 \cdots \alpha_{n+1}} - (\delta s)_{\beta\alpha_0 \cdots \alpha_{n+1}} = s_{\alpha_0 \cdots \alpha_{n+1}} \in \mathcal{F}^{n+1}(V \cap U_\beta)$, and if $V \cap U_{\beta_1} \cap U_{\beta_2} \cap U_{\alpha_1 \cdots \alpha_n}$ is not empty, $\delta s = 0$ implies $t_{\beta_1} = t_{\beta_2}$ on their intersection. So all t_β glue together to give a $t \in \mathcal{F}^n(V)$ s.t. $\delta t = s$.

This resolution defines Čech cohomology with respect to $\mathcal{U}$,

$$\check{H}^q_{\mathcal{U}}(X, \mathcal{F}) := h^q(\mathcal{F}^\bullet(X))$$

Definition: A sheaf $\mathcal{F}$ is acyclic with respect to open cover $\mathcal{U} = (U_\alpha)_{\alpha \in \Lambda}$ if the sheaf cohomology $H^k(U_{\alpha_0 \cdots \alpha_p}, \mathcal{F}) = 0$ for $k > 0$ and all finite intersection in $\mathcal{U}$.

Proposition: If a sheaf $\mathcal{F}$ is acyclic with respect to an open cover $\mathcal{U}$, then

$$\check{H}^q_{\mathcal{U}}(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$$

Proof. Take any flasque resolution(Godement resolution will do) for $\mathcal{F}$, $0 \to \mathcal{F} \to \mathcal{L}^\bullet$. And applying Čech functor for each flasque sheaf, we get a double complex.

$$\begin{tikzcd}
\mathcal{L}^1 \arrow[r] & \mathcal{L}^{1,0} \arrow[r] & \mathcal{L}^{1,1} \\
\mathcal{L}^0 \arrow[r] \arrow[u] & \mathcal{L}^{0,0} \arrow[r] \arrow[u] & \mathcal{L}^{0,1} \arrow[u] \\
\mathcal{F} \arrow[r] \arrow[u] & \mathcal{F}^0 \arrow[r] \arrow[u] & \mathcal{F}^1 \arrow[u]
\end{tikzcd}$$

There are zeros in the bottom and left which we omit. Consider the first quadrant double complex, $\mathcal{L}^{p,q}$, and its associated total complex, $\mathcal{K}^n = \oplus_{p+q=n} \mathcal{L}^{p,q}$. Claim $\mathcal{K}^n$ is flasque. It's true because being flasque is preserved under restriction, product and sum. Notice that all rows are exact (for the whole double complex not the first quadrant part), so the cohomology of $\mathcal{K}^\bullet$ can be compute by the first column which is exactly the resolution of $\mathcal{F}$ with vanishing cohomology. So $\mathcal{K}^\bullet$ is a flasque resolution of $\mathcal{F}$. We have

$$H^q(X, \mathcal{F}) \cong h^q(\mathcal{K}^\bullet(X))$$

On the other side, apply $\Gamma(X, -)$ functor to this first quadrant double complex. It's vertical cohomology vanish except the 0-th row, since we have $\mathcal{F}$ is acyclic with respect to $\mathcal{U}$. Then taking horizontal cohomology gives the cohomology of the total complex.

$$h^q(\mathcal{K}^\bullet(X)) \cong \check{H}^q_{\mathcal{U}}(X, \mathcal{F})$$

Then we see the desired isomorphism. $\square$

For example, take $\mathcal{F} = \mathbb{R}_X$ and let $\mathcal{U}$ be an open cover in which all finite intersection is contractible. Then we have

$$H^q(X, \mathbb{R}_X) \cong \check{H}^q_{\mathcal{U}}(X, \mathbb{R}_X)$$

where the right hand side carries only combinational information.

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