Milnor's Exotic Sphere

Lixiong Wu

February 22, 2024

Abstract

This report explores characteristic classes and their application in constructing Milnor's exotic spheres. The first several chapters are based on material from Milnor's classic textbook [1], while the construction of the exotic sphere is presented in a more concise manner, drawing from another report [3].

Contents

1 Vector Bundles

Definition: A real vector bundle is $\xi = (E, p, B)$ satisfying the following:

1. E and B are topological space and $p : E \to B$ is continuous.
2. $\forall b \in B$, $p^{-1}(b)$ is a real vector space.
3. $\forall b \in B$, there exists a neighborhood of U of b, a natural number n and a homeomorphism $h : U \times \mathbb{R}^n \to p^{-1}(U)$ such that $\forall u \in U$, the map $x \mapsto h(u, x)$ is an isomorphism between vector space $\mathbb{R}^n$ and $p^{-1}(u)$

such (U, h) is called a local chart of ξ of b. When U can be taken the whole B, the vector bundle is said to be trivial.

E is called the total space of ξ, and B the base space.

Let F_b denote $p^{-1}(b)$ being a vector space, usually called the fiber of b. The dimension of F_b is a locally constant function of b, thus constant when B is connected, which is exactly the case we are interested. In this case, we say such a vector bundle has rank n, or call it an $\mathbb{R}^n$ bundle.

Remark: We get the definition of complex vector bundle if we replace $\mathbb{R}$ with $\mathbb{C}$ in the definition above. We get the definition of smooth vector bundle if we require the topological spaces to be smooth manifolds and all homeomorphism to be diffeomorphism.

Definition: A section of a vector bundle $\xi = (E, p, B)$ is a continuous map $s : B \to E$ such that $p \circ s = id$. The zero section embed B into E, let E_0 denote E minus the image of B.

Here are some examples.

1. Trivial bundle $(B \times \mathbb{R}^n, p, B)$, where p is the projection to the first component. It's usually denoted by ε^n when the base space is understood.
2. When M is a differential manifold of dimension n, the tangent bundle consisting of all tangent vectors of M is a differential manifold of rank n.
3. Canonical line bundle: let $\mathbb{RP}^n$ denote the n dimensional projective space. We define the total space of γ_n^1 to be $E(\gamma_n^1) := \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | v$ belongs to the line represented by $x\}$, and $p : E(\gamma_n^1) \to \mathbb{RP}^n$ the projection to the first component.
4. Universal bundle. Let Gr_n be the space of n dimensional subspace in $\mathbb{R}^\infty$. Then bundle γ^n has total space $\{(V, x) \in Gr_n \times \mathbb{R}^\infty | x \in V\}$. With obvious projection map, it's a real vector bundle with rank n. The complex version is defined similarly.

Definition : A bundle map from η to ξ is a pair of continuous map $(g, \bar{g})$ making the following diagram commute and g mapping the fiber of b isomorphically to the fiber of $\bar{g}(b)$.

$$\begin{tikzcd}
E(\eta) \arrow[r, "g"] \arrow[d, "p_1"'] & E(\xi) \arrow[d, "p_2"] \\
B(\eta) \arrow[r, "\bar{g}"] & B(\xi)
\end{tikzcd}$$

Notice that $\bar{g}$ is determined by g, so we can consider g to be the bundle map. After defining bundle maps, we can see there is a vector bundles category.

Theorem: Let $f : \xi \to \eta$ be a bundle map between bundles over the same base space B covering the identity map

$$\begin{tikzcd}
E(\eta) \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E(\xi) \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

then f has an inverse bundle map, in which case ξ and η are called isomorphic.

The proof is based on local triviality of vector bundles and taking inverse to non singular matrix is a continuous transform.

The operations of vector spaces can be induced to the category of vector bundles. Let $Vect_{\mathbb{R}}$ be the category of finite dimensional vector spaces. We give the following lemma without proof.

Lemma : For each continuous functor $T : Vect_{\mathbb{R}} \times \cdots \times Vect_{\mathbb{R}} \to Vect_{\mathbb{R}}$, and $\xi_1, \cdots, \xi_n$ vector bundles over B, there exists a vector bundle $T(\xi_1, \cdots, \xi_n)$ over B such that its fiber over $b \in B$ is exactly $T(F_b(\xi_1), \cdots, F_b(\xi_n))$. It's unique up to isomorphism.

Notice that there is a natural topology on space of linear maps between vector spaces, we say T is continuous if it's continuous in that sense.

Now we can define $\xi \oplus \eta$, $\xi \otimes \eta$ and $\mathrm{Hom}(\xi, \eta)$. In particular, $\mathrm{Hom}(\xi, \varepsilon)$ is called the dual bundle of ξ.

Here are some other operations:

1. Restriction: For $\xi = (E, p, B)$, $B' \subset B$, $\xi|_{B'}$ is defined. The base space is just B', the total space E' is $p^{-1}(B)$ and p' is the restriction of p to E'.

2. Induced bundles(pullback): let $\xi = (E, p, B)$ be a bundle and B_1 a topological space and $f : B_1 \to B$ a continuous map. Then we can define the induced bundle $f^*\xi$ over B_1. The total space $E_1 :=\{(b, e) \in B \times E | f(b) = p(e)\}$. $p_1 : E_1 \to B_1$ takes (b, e) to b. Define $\hat{f} : E_1 \to E$ taking (b, e) to e. We have the commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\hat{f}"] \arrow[d, "p_1"'] & E \arrow[d, "p"] \\ B_1 \arrow[r, "f"] & B \end{tikzcd}$$

It's easy to see $\hat{f}$ is a bundle map.

3. Cartesian products: Given $\xi_i = (E_i, p_i, B_i)$, for $i = 1, 2$. $\xi_1 \times \xi_2$ is obviously defined by $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$. It's not hard to verify that $\xi_1 \times \xi_2 \cong \pi_1^*(\xi_1) \oplus \pi_2^*(\xi_2)$, where π_1 and π_2 are projections form $B_1 \times B_2$ to its two components.

Theorem: (Classifying Theorem) For a paracompact space B, we have the following bijection.
$\{$Isomorphic classes of n dimensional vector bundles over $B\} \leftrightarrow \{$Homotopy classes $f : B \to Gr_n\}$

In other words, given an $\mathbb{R}^n$ bundle ξ over B, there exist a continuous map $f : B \to Gr_n$ such that

$$f^*(\gamma^n) = \xi$$

such f is unique up to homotopy equivalent. The complex version is similar.

Definition: ξ and η are two vector bundles defined on the same base space B. We say ξ is a subbundle of η if $\forall b \in B$, $F_b(\xi)$ is a subspace of $F_b(\eta)$ (denoted by $\xi \subset \eta$). Then we have the concept of quotient bundle η/ξ with fiber on b $F_b(\eta)/F_b(\xi)$.

Definition: For smooth manifolds $N \subset M$. The normal bundle of N in M is defined to be $(\tau_M)|_N/\tau_N$, which is the quotient bundle of the restriction of tangent bundle of M on N and the tangent bundle of N.

Definition: An Euclidean metric on a real vector bundle ξ is to assign an inner product on each fiber.

$$< \cdot, \cdot >_b : F_b \times F_b \to \mathbb{R}$$

due to local triviality, such inner product can be identified to a positive definite matrix with variable b, we require it to be continuous.

Remark: Due to partition of unity, any vector bundle over a manifold admits an Euclidean metric.

Remark: In the complex case, we have the concept of Hermitian metric, it's an inner product for complex linear space on each fiber.

Definition: Let $\xi \subset \eta$, where η admits an Euclidean metric. there exists $\xi^\perp$ with fiber $F_b(\xi^\perp)$ being the orthogonal complement of $F_b(\xi)$ in $F_b(\eta)$. $\xi^\perp$ is called the orthogonal complement of ξ in η.

Lemma: η is isomorphic to $\xi \oplus \xi^\perp$.

Theorem: Let $\xi \subset \eta$ be subbundle, where η admits an Euclidean metric, then η/ξ is isomorphic to $\xi^\perp$.

Theorem: When ξ admits an Euclidean metric, it's isomorphic to its dual bundle $\mathrm{Hom}(\xi, \varepsilon)$.

2 Thom Isomorphism and Euler Class

For an n dimensional vector space V, $V_0 := V - \{0\}$, $H^n(V, V_0; \mathbb{Z}) = \mathbb{Z}$ has two generators. A choice of generator gives an orientation on V. Notice that a choice of such orientation is equivalent to a choice of an ordered basis for V.

Definition: An orientation of a vector bundle ξ is to choose a generator for each fiber F_b. $u_b \in H^n(F_b, F_{b0}; \mathbb{Z})$ such that for each point in B, there exists a neighborhood N and $u \in H^n(p^{-1}(U), p^{-1}(U)_0; \mathbb{Z})$ with $u|_{(F_b,F_{b0})} = u_b$ for each $b \in U$.

Remark: The orientation concept for a manifold coincides with the orientation of its tangent bundle. For $x \in M$, consider the normal bundle of the zero dimensional manifold x in M, which is exactly T_xM. Combine tubular neighborhood theorem and excision theorem. We have isomorphism $H^n(M, M-x) \cong H^n(T_xM, (T_xM)_0)$.

Theorem:(Thom isomorphism) Let $\xi = (E, p, B)$ be an n dimensional vector bundle with preferred orientation. Then there exists an unique $u \in H^n(E, E_0; \mathbb{Z})$ such that $u|_{(F_b,F_{b0})} = u_b$ for each $b \in B$. And the map $x \mapsto x \smile u$ gives an isomorphism from $H^j(E; \mathbb{Z})$ to $H^{j+n}(E, E_0; \mathbb{Z})$ for any integer j. Such u is called Thom class.

the inclusion $(E, \emptyset) \to (E, E_0)$ induces $H^n(E, E_0; \mathbb{Z}) \to H^n(E; \mathbb{Z})$, the image of u under this map is denoted by $u|_E$

Definition: The Euler class of $\xi = (E, p, B)$ is an element $e(\xi)$ in $H^n(B; \mathbb{Z})$, where n is the rank of the vector bundle satisfying

$$p^*e(\xi) = u|_E$$

where $u \in H^n(E, E_0; \mathbb{Z})$ is the Thom class.

If $f : B \to B'$ is covered by an orientation preserving bundle map $\xi \to \xi'$, then f^* carries the Thom class of ξ' to the Thom class of ξ. It follows that $f^*e(\xi') = e(\xi)$

Notice that when the orientation is reversed, the Euler class change sign. And when ξ has odd dimension, $(b, v) \mapsto (b, -v)$ covering the identity map reverses orientation, in which case $e(\xi) = -e(\xi)$.

Lemma: If ξ admits a nowhere zero section, it has Euler class 0.

Proof. Let $s : B \to E_0$ be such section, the composition

$$B \xrightarrow{\quad s \quad} E_0 \xrightarrow{\qquad} E \xrightarrow{\quad p \quad} B$$

is the identity on B. The corresponding composition

$$H^n(B) \xrightarrow{p^*} H^n(E) \longrightarrow H^n(E_0) \xrightarrow{s^*} H^n(B)$$

is the identity on $H^n(B)$. So $e(\xi) = s^*(p^*(e(\xi))|_{E_0}) = s^*((u|_E)|_{E_0})$, but $(u|_E)|_{E_0}$ is 0 due to the exact sequence

$$H^n(E, E_0) \to H^n(E) \to H^n(E_0)$$

$\square$

Theorem: $e(\xi \times \xi') = e(\xi) \times e(\xi')$

Proof. For given orientations u_b and u'_b on F_b and F'_b. $u_b \times u'_b \in H^{m+n}(F_b \times F'_b, (F_b \times F'_b)_0)$ determines an orientation on $F_b \times F'_b$. We have the following commutative diagram

$$\begin{array}{ccc} H^m(E, E_0) \times H^n(E', E'_0) & \xrightarrow{\times} & H^{m+n}(E \times E', (E \times E')_0) \\ \downarrow & & \downarrow \\ H^m(F_b, F_{b0}) \times H^n(F'_b, F'_{b0}) & \xrightarrow{\times} & H^{m+n}(F_b \times F'_b, (F_b \times F'_b)_0) \end{array}$$

where the vertical maps are restrictions. So if u and u' are Thom classes of ξ and ξ', $u \times u'$ is Thom class of $\xi \times \xi'$. Then it's easy to verify the assertion holds. $\square$

It follows immediately that $e(\xi \oplus \xi') = e(\xi) \smile e(\xi')$
These are results from intersection theory.

1. Let $E \to M$ be a smooth oriented rank n real vector bundle over a closed oriented manifold M. Let ψ be a section which intersects the zero section transversely and let $Z = \psi(M) \cap M$ where M is identified with the zero section of E. Then its Euler class is the Poincaré Dual of the fundamental class of Z.

2. Let M be a compact oriented n-manifold. Then its Euler characteristic is $\chi(M) = \langle e(M), [M] \rangle$, where $e(M)$ denotes the Euler class of tangent bundle of M.

For proofs, see [4]

Theorem:(Gysin sequence) $\xi = (E, p, B)$ is a real oriented vector bundle of rank n, then there exists a long exact sequence

$$\cdots \longrightarrow H^i(B) \xrightarrow{\smile e} H^{i+n}(B) \xrightarrow{p_0^*} H^{i+n}(E_0) \to H^{i+1}(B) \xrightarrow{\smile e} \cdots$$

where p_0 is the restriction of $p : E \to B$ to E_0 and e is the Euler class.

Proof. Consider the long exact sequence with respect to (E, E_0) and the isomorphism $H^j(B) \to H^{j+n}(E, E_0), x \mapsto p^*(x) \smile u$, where u is the Thom class. Then the theorem is easily verified. $\square$

3 Chern and Pontryagin Classes

For an n dimensional complex vector space, we have a canonical orientation for the $2n$ dimensional underlying real vector space $V_{\mathbb{R}}$ determined by $\{a_1, ia_1, \cdots, a_n, ia_n\}$, where $\{a_1, \cdots, a_n\}$ is a basis for V. Such orientation does not depend on the choice of basis. So, for a complex vector bundle ω, the underlying real vector bundle $\omega_{\mathbb{R}}$ has a canonical bundle.

For a complex vector bundle $\omega = (E, p, B)$ of rank n with Hermitian metric, we can define a vector bundle ω_0 over E_0 of rank $n-1$. For $v \in E_0$, the fiber of ω_0 at v is the orthogonal complement of $\mathbb{C}v$ in $p^{-1}(v)$. (or equivalently $p^{-1}(v)/\mathbb{C}v$)

Recall the Gysin sequence for a $2n$ dimensional real vector bundle.

$$\cdots \longrightarrow H^{i-2n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{p_0^*} H^i(E_0) \to H^{i-2n+1}(B) \xrightarrow{\smile e} \cdots$$

Notice that when $i < 2n-1$, p_0^* is an isomorphism.

Definition: The n-th Chern class of an n dimensional complex vector bundle ω is defined to be $e(\omega_{\mathbb{R}}) \in H^{2n}(B; \mathbb{Z})$. For $i < n$, define $c_i(\omega) := {p_0^*}^{-1} c_i(\omega_0)$ inductively. For $i > n$ Chern class is defined to be 0. $c(\omega) := 1 + c_1(\omega) + \cdots + c_n(\omega)$ is called the Chern class.

The naturality of Chern class follows from that of Euler class and induction.

Lemma: $c(\omega \oplus \varepsilon^k) = c(\omega)$

Proof. We only need to prove when $k = 1$. Let $\eta = \omega \oplus \varepsilon$ be a bundle of rank $n+1$, which obviously processes a nowhere zero section s. So $c_{n+1}(\eta) = e(\eta_{\mathbb{R}}) = 0 = c_{n+1}(\omega)$. Let $\eta_0 = (E(\eta_0), p', E_0(\eta))$, we have the bundle map

$$\begin{tikzcd}
E(\omega) \arrow[r, "\hat{s}"] \arrow[d, "p"'] & E(\eta_0) \arrow[d, "p'"] \\
B \arrow[r, "s"] & E_0(\eta)
\end{tikzcd}$$

for $v \in p^{-1}(b)$, $\hat{s}$ sends v to $[(v, 0)] \in {p'}^{-1}(s(b)) \cong (p^{-1}(b) \oplus \mathbb{C})/\mathbb{C}s(b)$ being isomorphism on each fiber. So $s^* c_i(\eta_0) = c_i(\omega)$, and by definition $p_0^* c_i(\eta) = c_i(\eta_0)$. $s^* \circ p_0^* = id$, the conclusion follows. $\square$

Theorem: The cohomology ring $H^*(\mathbb{CP}^n; \mathbb{Z})$ is $\mathbb{Z}[c_1]/\langle c_1^{n+1} \rangle$, where $c_1 = c_1(\gamma_n^1)$ is the first Chern class of the canonical line bundle.

Proof. Consider the Gysin sequence of γ_n^1, notice that c_1 is the Euler class in this case.

$$\cdots \longrightarrow H^{i+1}(E_0) \to H^i(\mathbb{CP}^n) \xrightarrow{\smile c_1} H^{i+2}(\mathbb{CP}^n) \xrightarrow{p_0^*} H^{i+2}(E_0) \longrightarrow \cdots$$

But $E_0 = E_0(\gamma_n^1)$ is just all the nonzero point in $\mathbb{C}^{n+1}$ with homotopy type S^{2n+1}. Then when $0 \le i \le 2n-2$, we have

$$0 \to H^i(\mathbb{CP}^n) \to H^{i+2}(\mathbb{CP}^n) \to 0$$

an isomorphism by multiplying c_1.It's follows that $H^{2k}(\mathbb{CP}^n)$ is an infinite cyclic group generated by c_1^k and $H^{2k-1}(\mathbb{CP}^n) = 0$ $\square$

Theorem: The cohomology ring $H^*(Gr_n(\mathbb{C}^\infty); \mathbb{Z})$ is the polynomial ring $\mathbb{Z}[c_1(\gamma^n), \cdots, c_n(\gamma^n)]$ with no polynomial relation between these generators.

Proof. Take $n \to \infty$ in the last theorem, we know this conclusion is true for $n = 1$. Now when $n \le 2$, we assume it holds for $n-1$. Consider the Gysin sequence

$$\cdots \longrightarrow H^i(Gr_n) \xrightarrow{\smile c_n} H^{i+2n}(Gr_n) \xrightarrow{p_0^*} H^{i+2n}(E_0) \longrightarrow H^{i+1}(Gr_n) \longrightarrow \dots$$

associated with the bundle γ^n.

We will first show that the cohomology ring $H^*(E_0)$ can be identified with $H^*(Gr_{n-1})$. Define $f : E_0 \to Gr_{n-1}$, $(X, v) \mapsto X \cap v^\perp$ being the orthogonal complement of v in X.

Consider the sub-bundle $\gamma^n(\mathbb{C}^N) \subset \gamma^n$, consisting of complex n-planes in $\mathbb{R}^N$ where N is large natural number. Let $f_N : E_0(\gamma^n(\mathbb{C}^N)) \to Gr_{n-1}(\mathbb{C}^N)$ be the corresponding restriction of f. For any Y in $Gr_{n-1}(\mathbb{C}^N)$, $f_N^{-1}(Y) \subset E_0(\gamma^n(\mathbb{C}^N))$ consists of all pairs (X, v) where $v \in \mathbb{C}^N$ is a non-zero vector perpendicular to Y, and $X = Y + \mathbb{C}$. Thus f_N can be identified with the projection map

$$E_0(\omega^{N-n+1}) \longrightarrow Gr_{n-1}(\mathbb{C}^N)$$

where ω^{N-n+1} is the complex vector bundle whose fiber, over $Y \in Gr_{n-1}(\mathbb{C}^N)$, is the orthogonal complement of Y in $\mathbb{C}^N$.

Using the Gysin sequence of this new vector bundle, it follows that f_N induces cohomology isomorphisms in dimensions $\le 2(N-n)$. Therefore, taking the direct limit as N tends to infinity, f induces cohomology isomorphisms in all dimensions.

Thus we can replace E_0 with Gr_{n-1} in the Gysin sequence

$$\cdots \longrightarrow H^i(Gr_n) \longrightarrow H^{i+2n}(Gr_n) \xrightarrow{\lambda} H^{i+2n}(Gr_{n-1}) \longrightarrow H^{i+1}(Gr_n) \longrightarrow \dots$$

with $\lambda = f^{*-1}p_0^*$.

We will show that $\lambda = f^{*-1}p_0^*$ maps $c_i(\gamma^n)$ to $c_i(\gamma^{n-1})$. It is clear for $i = n$, so we may assume that $i < n$. By the definition of Chern classes,$p_0^* c_i(\gamma^n) = c_i(\gamma_0^n)$. But $f : E_0 \longrightarrow Gr_{n-1}$ is covered by a bundle map $\gamma_0^n \to \gamma^{n-1}$. Therefore $f^* c_i(\gamma^{n-1}) = c_i(\gamma_0^n)$ and it follows that

$$\lambda c_i(\gamma^n) = f^{*-1} p_0^* c_i(\gamma^n) = c_i(\gamma^{n-1})$$

By induction $H^*(Gr_{n-1})$ is generated by $c_1(\gamma^{n-1}), \ldots, c_{n-1}(\gamma^{n-1})$, so λ is surjective, the sequence reduces to

$$0 \longrightarrow H^i(Gr_n) \xrightarrow{\smile c_n} H^{i+2n}(Gr_n) \xrightarrow{\lambda} H^{i+2n}(Gr_{n-1}) \longrightarrow 0.$$

For the fixed n, suppose our assertion holds for some i. For an element$x \in H^{i+2n}(Gr_n)$, $\lambda(x)$ can be expressed uniquely as a polynomial $p(c_1(\gamma^{n-1}), \ldots, c_{n-1}(\gamma^{n-1}))$. Then $x - p(c_1(\gamma^n), \ldots, c_{n-1}(\gamma^n))$ belongs to the kernel of λ, and hence can be expressed as a product $yc_n(\gamma^n)$ for some uniquely determined $y \in H^i(Gr_n)$. Now y can be expressed uniquely as a polynomial $q(c_1(\gamma^n), \ldots, c_n(\gamma^n))$ by our hypothesis, hence

$$x = p(c_1(\gamma^n), \ldots, c_{n-1}(\gamma^n)) + c_n(\gamma^n)q(c_1(\gamma^n), \ldots, c_n(\gamma^n)).$$

If x were also equal to

$$p'(c_1(\gamma^n), \ldots, c_{n-1}(\gamma^n)) + c_n(\gamma^n)q'(c_1(\gamma^n), \ldots, c_n(\gamma^n))$$

apply λ on both side, we get $p = p'$, and from the exact sequence $\smile c_n$ is an injection, so $q = q'$. That is if the assertion holds for i, it holds for $i + 2n$, but we know it holds for all $i < 0$, and hence for all i. $\square$

Lemma: For natural number m, n, there exists an unique polynomial $p_{m,n}$ such that

$$c(\omega \oplus \phi) = p_{m,n}(c_1(\omega), \cdots, c_m(\omega), c_1(\phi), \cdots, c_n(\phi))$$

holds for all complex bundles ω and ϕ with rank m and n over a paracompact space B.

Proof. We first show it holds for the universal situation. γ^m and γ^n are the universal bundles over Gr_m and Gr_n. Let π_1, π_2 be the projections from $Gr_m \times Gr_n$ to its two components. We have bundles $\gamma_1^m := \pi_1^*\gamma^m$ and $\gamma_2^n := \pi_2^*\gamma^n$ over $Gr_m \times Gr_n$. Künneth formula asserts

$$H^*(Gr_m) \otimes H^*(Gr_n) \longrightarrow H^*(Gr_m \times Gr_n)$$

taking $a \otimes b$ to $\pi_1^*(a) \smile \pi_2^*(b)$ is an isomorphism.

So, $H^*(Gr_m \times Gr_n)$ is the polynomial ring generated by these algebraically independent elements.

$$\pi_1^* c_i(\gamma^m) = c_i(\pi_1^*\gamma^m) = c_i(\gamma_1^m), 1 \leq i \leq m$$

$$\pi_2^* c_j(\gamma^n) = c_j(\pi_2^*\gamma^n) = c_j(\gamma_2^n), 1 \leq j \leq n$$

For the bundle $\gamma_1^m \oplus \gamma_2^n$ over $Gr_m \times Gr_n$, there exists an unique polynomial $p_{m,n}$ such that

$$c(\gamma_1^m \oplus \gamma_2^n) = p_{m,n}(c_1(\gamma_1^m), \cdots, c_m(\gamma_1^m), c_1(\gamma_2^n), \cdots, c_n(\gamma_2^n))$$

If ω and ϕ are complex bundles over a paracompact space B with rank m and n, we have $f : B \to Gr_m$ and $g : B \to Gr_n$ such that $f^*\gamma^m \cong \omega, g^*\gamma^n \cong \phi$. There is an $h : B \to Gr_m \times Gr_n$ making the diagram commutative.

$$\begin{tikzcd}
 & B \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
Gr_m & Gr_m \times Gr_n \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Gr_n
\end{tikzcd}$$

Applying this diagram to vector bundles and Chern classes, it's easy to verify the conclusion.

Actually we have

$$p_{m,n}(c_1, \cdots, c_m, c'_1, \cdots, c'_n) = (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots, +c'_n)$$

Clearly, it holds for $m = n = 0$, we apply induction on $m + n$, suppose we have

$$c(\gamma_1^{m-1} \oplus \gamma_2^n) = (1 + c_1(\gamma_1^{m-1}) + \cdots + c_{m-1}(\gamma_1^{m-1}))(1 + c_1(\gamma_2^n) + \cdots, +c_n(\gamma_2^n))$$

Then

$$c(\gamma_1^{m-1} \oplus \gamma_2^n) = c(\gamma_1^{m-1} \oplus \varepsilon \oplus \gamma_2^n) = p_{m,n}(c_1(\gamma_1^{m-1}), \cdots, c_{m-1}(\gamma_1^{m-1}), 0, c_1(\gamma_2^n), \cdots, c_n(\gamma_2^n))$$

We get

$$p_{m,n}(c_1, \cdots, c_{m-1}, 0, c'_1, \cdots, c'_n) = (1 + c_1 + \cdots + c_{m-1})(1 + c'_1 + \cdots, +c'_n)$$

This yields

$$p_{m,n}(c_1, \cdots, c_m, c'_1, \cdots, c'_n) = (1+c_1+\cdots+c_m)(1+c'_1+\cdots, +c'_n) \pmod{c_m}$$

Similarly, the equality holds for $\mod c'_n$. So there is a polynomial u such that

$$p_{m,n}(c_1, \cdots, c_m, c'_1, \cdots, c'_n) = (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots, +c'_n) + uc_mc'_n$$

due to the degree limitation, u must be zero dimensional. But the product formula for Euler class asserts $c_{m+n}(\omega \oplus \phi) = c_m(\omega)c_n(\phi)$, so u must be 0. □

Definition: For a complex vector bundle ω, the conjugate bundle $\bar{\omega}$ has the same total space with ω but with the conjugate structure. In other words, the multiplication of λ and v is $\bar{\lambda}v$, where the latter multiplication is understood in the original bundle.

Theorem: $c_k(\bar{\omega}) = (-1)^k c_k(\omega)$

Proof. The canonical orientation of ω and $\bar{\omega}$ are determined by $\{a_1, ia_1, \cdots, a_n, ia_n\}$ and $\{a_1, -ia_1, \cdots, a_n, -ia_n\}$ respectively. They differ by $(-1)^n$, so $c_n(\bar{\omega}) = (-1)^n c_n(\omega)$. For $k < n$ it's also easy to prove because of the inductive definition of Chern classes. □

Lemma: If ω admits an Hermitian metric, there is a canonical isomorphism between $\bar{\omega}$ and $\mathrm{Hom}(\omega, \varepsilon)$

Let τ^n denote the tangent bundle of $\mathbb{CP}^n$.

Lemma: $\tau^n \cong \mathrm{Hom}(\gamma^1, \omega^n)$, where ω^n is the orthogonal bundle of γ^1 in ε^{n+1}

Proof. The total space of the three bundles are

$$E(\tau^n) = \{\{(\lambda x, \lambda v) | \lambda \in S^1\} | x \in S^{2n+1}, v \in \mathbb{C}^{n+1}, v \perp x\}$$

$$E(\gamma^1) = \{\{(\lambda x, v) | \lambda \in S^1\} | x \in S^{2n+1}, v \in \mathbb{C}^{n+1}, v \parallel x\}$$

$$E(\omega^n) = \{\{(\lambda x, v) | \lambda \in S^1\} | x \in S^{2n+1}, v \in \mathbb{C}^{n+1}, v \perp x\}$$

elements in these space are represented by equivalent classes.

Given $[(x, v)] \in E(\tau^n)$, it corresponds to a linear map sending $[(x, w)]$ ($w = \mu x$ for some μ) to $[(x, \mu v)]$. (these [ ] actually represent different equivalent relations). It's easy to verify it's a well-defined linear isomorphism on each fiber. □

Theorem: $c(\tau^n) = (1 + a)^{n+1}$, where $a = -c_1(\gamma^1)$

Proof. $\tau^n \oplus \varepsilon \cong \mathrm{Hom}(\gamma^1, \omega^n) \oplus \mathrm{Hom}(\gamma^1, \gamma^1) \cong \mathrm{Hom}(\gamma^1, \varepsilon^{n+1}) \cong (n+1)\overline{\gamma^1}$ □

Then we have $e(\tau^n) = c_n(\tau^n) = (n + 1)a^n$, and we've already known the Euler characteristic of $\mathbb{CP}^n$ from it's cohomology structure. $n + 1 = \chi(\mathbb{CP}^n) = \langle (n + 1)a^n, [\mathbb{CP}^n] \rangle$ gives us $\langle a^n, [\mathbb{CP}^n] \rangle = 1$

For a real bundle ξ, tensor each fiber with $\mathbb{C}$ gives us a complex bundle $\xi \otimes \mathbb{C}$, which is called the complexification of ξ.

Lemma: $\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}}$

It follows that the all odd Chern classes of $\xi \otimes \mathbb{C}$ has order 2.

Definition: The i-th Pontrjagin class of a real bundle ξ is defined by

$$p_i(\xi) := (-1)^i c_{2i}(\xi \otimes \mathbb{C})$$

notice that for an n bundle $p_i(\xi) = 0$ if $i > \frac{n}{2}$.

$$p(\xi) := 1 + p_1(\xi) + \cdots + p_{[\frac{n}{2}]}(\xi)$$

From the property of Chern classes, the followings are easily verified.

1. $p(\xi \oplus \varepsilon^k) = p(\xi)$
2. $p(\xi \oplus \eta) \equiv p(\xi)p(\eta) \mod 2$

Lemma: ω is a complex bundle, there is a canonical isomorphism between $\omega_{\mathbb{R}} \otimes \mathbb{C}$ and $\omega \oplus \bar{\omega}$.

Theorem: For a complex bundle ω of rank n, $c_i(\omega)$ and $p_k(\omega|_{\mathbb{R}})$ satisfy

$$1 - p_1 + p_2 - \cdots + (-1)^n p_n = (1 - c_1 + c_2 - \cdots + (-1)^n c_n)(1 + c_1 + c_2 + \cdots c_n)$$

Direct calculation shows that $p(\tau^n|_{\mathbb{R}}) = (1+a^2)^{n+1}$, where τ^n stills represents the tangent bundle of $\mathbb{CP}^n$ and $a = -c_1(\gamma^1)$

Let $\widetilde{G}r_n = \widetilde{G}r_n(\mathbb{R}^\infty)$ denote the space of oriented real n–planes in $\mathbb{R}^\infty$. It's the classifying space for oriented bundles. Since the product formula for Pontrjagin classes only holds mod 2, we will study the cohomology of $\widetilde{G}r_n$ with coefficients in an integral domain Λ containing $\frac{1}{2}$.

Theorem: If Λ is an integral domain containing $\frac{1}{2}$, then the cohomology ring $H^*(\widetilde{G}r_{2m+1};\Lambda)$ is a polynomial ring over Λ generated by the Pontrjagin classes

$$p_1(\widetilde{\gamma}^{2m+1}), \ldots, p_m(\widetilde{\gamma}^{2m+1}).$$

Similarly $H^*(\widetilde{G}r_{2m};\Lambda)$ is a polynomial ring over Λ generated by the Pontrjagin classes $p_1(\gamma^{2m}), \ldots, p_{m-1}(\gamma^{2m})$ and the Euler class $e(\widetilde{\gamma}^{2m})$.

In other words for every value of n, even or odd, the ring $H^*(\widetilde{G}r_n;\Lambda)$ is generated by the characteristic classes $p_1, \ldots, p_{\lfloor n/2 \rfloor}$ and e. These generators are subject only to the relations:

$e = 0$ for n odd,

$e^2 = p_{n/2}$ for n even.

The proof is only slightly difficult than that of complex version. For a complete proof see 15.6 of [1].

4 Chern numbers and Pontryagin numbers

Let K^n be a compact complex manifold of complex dimension n. Then for each partition $I = i_1, \ldots, i_r$ of n, the Chern number

$$c_I[K^n] = c_{i_1} \cdots c_{i_r}[K^n]$$

is defined to be the integer

$$\langle c_{i_1}(\tau^n) \ldots c_{i_r}(\tau^n), [K^n]] \rangle$$

Here τ^n denotes the tangent bundle of K^{2n} as real manifold. We adopt the convention that $c_I[K^n]$ is zero if I is a partition of some integer other than n.

As an example, for the complex projective space $\mathbb{CP}^n$, since $c_i(\tau^n) = \binom{n+1}{i} a^i$, we have the formula

$$c_{i_1} \cdots c_{i_r}[\mathbb{CP}^n] = \binom{n+1}{i_1} \cdots \binom{n+1}{i_r}$$

for any partition $i_1, \ldots, i_r$ of n.

Now consider a smooth, compact, oriented manifold M^{4n}. For each partition $I = i_1, \ldots, i_r$ of n, the I–th Pontrjagin number $p_I[M^{4n}] = p_{i_1} \cdots p_{i_r}[M^{4n}]$ is defined to be the integer

$$\langle p_{i_1}(\tau^{4n}) \cdots p_{i_r}(\tau^{4n}), [M^{4n}] \rangle$$

Here τ^{4n} denotes the tangent bundle. Similarly, we have

$$p_{i_1}\cdots p_{i_r}[\mathbb{CP}^n] = \binom{2n+1}{i_1}\cdots\binom{2n+1}{i_r}$$

Theorem: If M^{4n} is the boundary of a compact smooth oriented $4n+1$ manifold W, all it's Pontrjagin number are 0.

Proof. Take the fundamental class $[W] \in H_{4n+1}(W, M)$, $\partial[W] = [M] \in H_{4n}(M)$ is the fundamental class of M.

Form collar neighborhood theorem, we have $\tau_W|_M \cong \tau_M \oplus \varepsilon$. $p_I(\tau_M) = i^* p_I(\tau_W)$ mod 2, where $i : M \to W$ is the embedding.

$$\langle p_I(\tau_M), [M]\rangle = \langle i^* p_I(\tau_W), \partial[W]\rangle = \langle \delta i^* p_I(\tau_W), [W]\rangle$$

where $\delta i^* = 0$ due to the exact sequence. □

For indeterminates $t_1, \cdots, t_n$, we have symmetric polynomials $\sigma_1, \cdots, \sigma_n$ satisfying

$$1 + \sigma_1 + \cdots + \sigma_n = (1 + t_1)\cdots(1 + t_n)$$

By summing all monomials with the same form of the given $t_1^{i_1}\cdots t_r^{i_r}$, we get its symmetrization $\Sigma t_1^{i_1}\cdots t_r^{i_r}$. For example $\sigma_k = \Sigma t_1 \cdots t_k$.

For $n \geq k$, $\sigma_1, \cdots, \sigma_k$ are algebraically independent and generate all symmetric polynomials of degree k. Let $I = \{i_1, \cdots, i_r\}$ be a partition of k, there is a unique polynomial s_I satisfying

$$s_I(\sigma_1, \cdots, \sigma_k) = \Sigma t_1^{i_1} t_2^{i_2}\cdots t_r^{i_r}$$

Such s_I independent of n.

Let ω be a complex n–plane bundle with base space B and with total Chern class $c = 1 + c_1 + \ldots + c_n$. For any $k \geq 0$ and any partition I of k the cohomology class

$$s_I(c_1, \ldots, c_k) \in H^{2k}(B; \mathbb{Z})$$

will be denoted briefly by the symbol $s_I(c)$ or $s_I(c(\omega))$.

Theorem: $s_I(c(\omega \oplus \omega')) = \sum_{JK=I} s_J(c(\omega)) s_K(c(\omega'))$

Remark: For $J = \{j_1 \cdots, j_r\}$ and $K = \{k_1 \ldots k_t\}$, the juxtaposition $JK = \{j_1 \cdots, j_r, k_1 \ldots k_t\}$. The operation is clearly associative and commutative because we're considering unordered partition.

Proof. By the product formula of Chern classes, it suffices to prove.

$$s_I(c''_1, \cdots, c''_k) = \sum_{JK=I} s_J(c_1, \cdots, c_k) s_K(c'_1, \cdots, c'_k)$$

where $c''_i = \sum_{j=0}^{i} c_j c'_{i-j}$.

To make use of the definition of s_I, we construct as follows. Let σ_k and σ'_k be the k-th symmetric polynomial of $t_1, \cdots, t_n$ and $t_{n+1}, \cdots, t_{2n}$ respectively and $\sigma''_i = \sum_{j=0}^{i} \sigma_j \sigma'_{i-j}$. It's easy to see σ''_i is exactly the i-th symmetric polynomial of $t_1, \cdots, t_{2n}$ so $s_I(\sigma''_1, \cdots, \sigma''_k) = \Sigma t_1^{i_1}\cdots t_k^{i_r}$ by definition. Then by counting, we ge the formula. □

Now consider a compact complex manifold K^n of complex dimension n. For each partition I of n the notation $s_I(c)[K^n]$, or briefly $s_I[K^n]$, will stand for the characteristic number

$$\langle s_I(c(\tau^n)), [K^n]\rangle \in \mathbb{Z}.$$

The next theorem is easily verified.

Theorem: The characteristic number $s_I[K^m \times L^n]$ of a product of complex manifolds is equal to

$$\sum_{I_1 I_2 = I} s_{I_1}[K^m] s_{I_2}[L^n],$$

Completely analogous formulas are true for Pontrjagin classes and Pontrjagin numbers. If ξ is a real vector bundle over B, then for any partition I of n the characteristic classes

$$s_I(p_1(\xi), \ldots, p_n(\xi)) \in H^{4n}(B; \mathbb{Z})$$

is denoted briefly by $s_I(p(\xi))$. The congruence

$$s_I(p(\xi \oplus \xi')) = \sum_{JK=I} s_J(p(\xi)) s_K(p(\xi'))$$

modulo 2 clearly follows. Hence there is a corresponding equality

$$s_I(p)[M \times N] = \sum_{JK=I} s_J(p)[M] s_K(p)[N]$$

for characterisitc numbers because there is no integer with order 2. In particular, these characteristic numbers of $M \times N$ are zero unless the dimensions of M and N are divisible by 4.

5 Oriented Cobordism Ring

For smooth close manifolds M and N with dimension n, we say they are in the same cobordism class if $M \sqcup -N$ is diffeomorphic(orientation preserving) to the boundary of some $n+1$ dimensional smooth compact oriented manifold, with induced orientation. It's easy to check it's an equivalence relation.

Let Ω_n denote the set consisting all cobordism classes of n-manifold. It's an abelian group with disjoint union being the addition. And $[M_1^m] \times [M_2^n] \mapsto [M_1^m \times M_2^n]$ is a well-defined map from $\Omega_m \times \Omega_n$ to Ω_{m+n}. It's associative and compatible with the disjoint union. So $\Omega_* := \bigoplus_{k=0}^{\infty} \Omega_k$ is a graded ring. A point (zero dimensional manifold) with orientation $+1$ is the unity. And $[M_1^m] \times [M_2^n] = (-1)^{mn} [M_2^n] \times [M_1^m]$ makes it commutative in the graded sense.

We list the processes to derive information of the oriented cobordism ring. For details see chapter 18 of [1].

Definition: Let ξ be a k–plane bundle with a Euclidean metric, and let $A \subset E(\xi)$ be the subset of the total space consisting of all vectors v with $|v| \geq 1$.

Then $E(\xi)/A$ in which A is pinched to a point will be called the $Th(\xi)$. Thus $Th(\xi)$ has a natural base point, denoted by t_0, and the complement $Th(\xi) - t_0$ consists of all vectors $v \in E(\xi)$ with $|v| < 1$.

The next lemma can be understood intuitively.

Lemma: If the base space B is a CW–complex, then the Thom space $Th(\xi)$ is a $(k-1)$–connected CW–complex, having (in addition to the base point t_0) one $(n+k)$–cell corresponding to each n–cell of B.

Considering exact sequence of (Th, Th_0, t_0),together with excision and Thom isomorphism, we get the following.

Lemma: If ξ is an oriented k–plane bundle over B, then each integral homology group $H_{k+i}(Th(\xi), t_0)$ is canonically isomorphic to $H_i(B)$.

Definition: A homomorphism $h : A \longrightarrow B$ between abelian groups is called a $Ab_{<\infty}$–isomorphism if both the kernel $h^{-1}(0)$ and the cokernel $B/h(A)$ are finite abelian groups.

Theorem: Let X be a finite complex which is $(k-1)$–connected, $k \geq 2$. Then the Hurewicz homomorphism

$$\pi_r(X) \longrightarrow H_r(X; \mathbb{Z})$$

is a $Ab_{<\infty}$–isomorphism for $r < 2k - 1$.

Combining the theorem above, we get

Theorem: If Th is the Thom space of an oriented k–plane bundle over the finite complex B, then there is a $Ab_{<\infty}$–isomorphism

$$\pi_{n+k}(Th) \longrightarrow H_n(B; \mathbb{Z})$$

for all dimensions $n < k - 1$.

For a smooth map $f : M \to N$ between smooth manifolds and a smooth submanifold $Y \subset N$, we know that if f is transversal to Y, then $f^{-1}(Y)$ is a submanifold in M. For $x \in f^{-1}(Y)$, $[v] \mapsto [f_*(v)]$ gives a well-defined map from $(N_{f^{-1}(Y)}M)_x := T_xM/T_xf^{-1}(Y)$ to $(f^*(N_YN))_x = (N_YN))_{f(x)} := T_{f(x)}M/T_{f(x)}Y$. It's easy to see it's injective and by transversality it's surjective. It follows that the normal bundle of $f^{-1}(Y)$ in M can be identified with the induced bundle of normal bundle of Y in N. Notice that the bundle induced from oriented bundle admits a natural orientation. In particular, if the normal bundle of Y in N is oriented and M is oriented, we have the induce bundle over $f^{-1}(Y)$ is oriented.

Given $\xi = (E, p, B)$, the normal bundle of B in $Th(\xi)$ is isomorphic to ξ, so if ξ is oriented, so is the normal bundle.

With knowledge for smooth approximation and transversal approximation but no further difficulty, we have the following theorem.

Theorem: Let ξ be a smooth oriented k-bundle. Every continuous map $f : S^m \longrightarrow Th(\xi)$ is homotopic to a map g which is smooth throughout $g^{-1}(Th - t_0)$, and is transverse to the zero cross–section B. The oriented cobordism class of the resulting smooth $(m-k)$–dimensional manifold $g^{-1}(B)$ depends only on the homotopy class of g. Hence the correspondence

$$g \mapsto g^{-1}(B)$$

gives rise to a homomorphism from the homotopy group $\pi_m(Th, t_0)$ to the oriented cobordism group Ω_{m-k}.

Let $\widetilde{\gamma}_p^k = \widetilde{\gamma}^k(\mathbb{R}^{k+p})$ be the bundle of oriented k–planes in $(k+p)$–space.

Theorem: If $k \geq n$ and $p \geq n$, then the homomorphism

$$\pi_{n+k}(Th(\widetilde{Gr}_p^k)) \longrightarrow \Omega_n$$

from the last theorem is surjective.

Proof. Let M^n be an arbitrary smooth, compact, oriented n–dimensional manifold. Then, by Whitney embedding theorem, M^n can be embedded in the Euclidean space $\mathbb{R}^{n+k}$. By tubular neighborhood theorem, we can choose a neighborhood U of M^n in $\mathbb{R}^{n+k}$ which is diffeomorphic to the total space $E(\nu^k)$ of the normal bundle. Using the Gauss map(sending a normal vector v to (the oriented k-plane v lies, v)), we have

$$U \cong E(\nu^k) \longrightarrow E(\widetilde{Gr}_n^k) \subset E(\widetilde{Gr}_p^k),$$

and composing with the canonical map $E(\widetilde{Gr}_p^k) \longrightarrow Th(\widetilde{Gr}_p^k)$, we obtain a map $g : U \longrightarrow Th(\widetilde{Gr}_p^k)$ which is transverse to the zero cross–section B, and satisfies $g^{-1}(B) = M^n$.

Now extend g to the one–point compactification $\mathbb{R}^{n+k} \cup \{\infty\} \cong S^{n+k}$ by mapping $S^{n+k} - U$ to the base point t_0. The resulting map $\hat{g} : S^{n+k} \longrightarrow Th(\widetilde{Gr}_p^k)$ is an element in $\pi_{n+k}(Th(\widetilde{Gr}_p^k)$ giving M by the construction from last theorem. $\square$

Theorem: The oriented cobordism group Ω_n is finite for $n \not\equiv 0 \pmod 4$, and is a finitely generated group with rank equal to $p(r)$, the number of partitions of r, when $n = 4r$.

Proof. The group Ω_n is a homomorphic image of $\pi_{n+k}(Th(\widetilde{Gr}_p^k))$ for k and p large, and this latter group is $Ab_{<\infty}$ isomorphic to $H_n(\widetilde{Gr}_k(\mathbb{R}^{k+p}); \mathbb{Z})$, being finite generated because $\widetilde{Gr}_k(\mathbb{R}^{k+p})$ admits a finite cell structure. $\widetilde{Gr}_k(\mathbb{R}^{k+p})$ can be identified with $\widetilde{Gr}_p(\mathbb{R}^{k+p})$. We know the structure of $H^*(\widetilde{Gr}_p; \mathbb{Q})$. For fixed n, $H_n(\widetilde{Gr}_p(\mathbb{R}^{k+p})); \mathbb{Q}) \cong H_n(\widetilde{Gr}_p; \mathbb{Q})$ when k is sufficiently large and the right hand side is trivial for $n \not\equiv 0 \pmod 4$ when p is sufficiently large. So we must have $H_n(\widetilde{Gr}_k(\mathbb{R}^{k+p}); \mathbb{Z})$ is finite when $n \not\equiv 0 \pmod 4$ and finitely generated with rank $p(r)$ when $n = 4r$. The surjectivity of above theorem implies that Ω_n is finite for $n \not\equiv 0 \pmod 4$, and is a finitely generated group with rank not greater than $p(r)$

$[M^{4r}] \mapsto s_I(p)[M^{4r}]$ defines a group homomorphism from $\Omega_{4r} \otimes \mathbb{Q}$ to $\mathbb{Q}$. Let $I = \{i_1, \cdots, i_l\}, J = \{j_1, \cdots, j_k\}$ ba partitions of r, notice that the $p(r) \times p(r)$ matrix $(s_I(p)(\mathbb{CP}^{2j_1} \times \cdots \mathbb{CP}^{2j_k}))$ with entries in $\mathbb{Q}$ has nonzero determinant, because $s_I(p)(\mathbb{CP}^{2j_1} \times \cdots \mathbb{CP}^{2j_k}) = 0$ unless I is a refinement of J. It follows that the $p(r)$ elements $\{\mathbb{CP}^{2j_1} \times \cdots \mathbb{CP}^{2j_k}\}$ in $\Omega_{4r} \otimes \mathbb{Q}$(or in Ω_{4r}) are linearly independent. $\square$

It follows that $\Omega_* \otimes \mathbb{Q} = \bigoplus_{k \geq 0} \Omega_{4k} \otimes \mathbb{Q}$ and is an algebra over $\mathbb{Q}$ with basis $\mathbb{CP}^2, \mathbb{CP}^4, \cdots$

6 Signature of Manifolds

Let M be a oriented closed manifold of dimension $4k$ with fundamental class $[M]$, we have the bilinear form

$$H^{2k}(M, \mathbb{Q}) \times H^{2k}(M, \mathbb{Q}) \to \mathbb{Q}$$

sending (α, β) to $\langle \alpha \smile \beta, [M] \rangle$

This bilinear form is non-degenerate due to Poincaré Duality. Since they have even degree, the bilinear map is symmetric, hence admits a signature which is defined to be the number of positive eigenvalues of the representation matrix minus that of negative ones.

We define $\sigma(M)$ the signature of M to be the signature of the bilinear map. Notice that it change sign when reversing orientation. Notice that $[M \sqcup N] = [M] + [N]$, it's easy to verify that $\sigma(M \sqcup N) = \sigma(M) + \sigma(N)$

Lemma: Let $\phi : V \times V \to \mathbb{Q}$ be a non-degenerate bilinear form on the $2n$ dimensional $\mathbb{Q}$-vector space V. If there exists a subspace W with dimension n such that $\phi : W \times W \to \mathbb{Q}$ is identically zero, then ϕ has signature 0.

Proof. Let r be the number of positive eigenvalues, by symmetry it suffices to prove $r \geq n$. Let $\{x_1, \cdots, x_n\, z_1, \cdots, z_n\}$ be a basis of V, where $\{x_1, \cdots, x_n\}$ is a basis of W. Define $\theta : V \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ by

$$\theta(x) = (\phi(x, x_1), \phi(x, x_2), \cdots, \phi(x, x_n)) \quad \psi(x) = (\phi(x, z_1), \phi(x, z_2), \cdots, \phi(x, z_n))$$

Since ϕ is non-degenerate, $\ker \theta \cap \ker \phi = 0$. But they are linear maps from $2n$ dimensional space to n dimensional space, so their kernels must have dimension greater than or equal to n. Thus they are both surjective.

Take y_1 such that $\theta(y_1) = (1, 0, \cdots, 0)$. We have $\phi(x_1, x_1) = 0$, $\phi(x_1, y_1) = 1$. Take $a = (1 - \phi(y_1, y_1))/2$, we get $\phi(ax_1 + y_1, ax_1 + y_1) = 1$ which gives a eigenvector corresponding to positive eigenvalue completing the case $n = 1$. When $n > 1$, define $\omega(x) = (\phi(x, x_1), \phi(x, y_1))$. $\omega(x_1) = (0, 1)$ and $\omega(y_1) = (1, \phi(y_1, y_1))$, ω is a surjective. Let $V' := \ker \omega$. $\phi(V', span(x_1, y_1)) = 0$. ϕ can be decomposed onto these two subspaces. Let $W' := span(x_2, \cdots, x_n)$, the conclusion follows by induction. $\square$

Theorem: If M is the boundary of a $4k + 1$ dimensional compact oriented manifold W, then $\sigma(M) = 0$.

Proof. Let $z \in H_{4k+1}(W, M)$ be the fundamental class, ∂z is the fundamental class of M, where ∂ is the connecting map. Let $i : M \to W$

We have $\langle i^*\alpha \smile i^*\beta, \partial z \rangle = \langle \alpha \smile \beta, i_* \partial z \rangle = 0$, because $i_* \partial = 0$ from the exact sequence. It suffices to show the image of $i^* : H^{2k}(W) \to H^{2k}(M)$ has half the dimension of $H^{2k}(M)$.

Poincaré Duality gives the following commutative diagram with exact rows.

$$\begin{array}{ccccc} H^{2k}(W) & \xrightarrow{i^*} & H^{2k}(M) & \xrightarrow{\delta} & H^{2k+1}(W,M) \\ \downarrow D & & \downarrow D & & \downarrow D \\ H_{2k+1}(W,M) & \xrightarrow{\partial} & H_{2k}(M) & \xrightarrow{i_*} & H_{2k}(W) \end{array}$$

We have $H^{2k}(M) \cong \ker \delta \oplus im\delta \cong imi^* \oplus im\delta \cong imi^* \oplus imi_*$
and i^* is the dual map of i_*, their image have the same dimension. □

Theorem: $\sigma(M \times N) = \sigma(M)\sigma(N)$

Proof. The Künneth formula gives isomorphism

$$H^*(M \times N; \mathbb{Q}) \cong H^*(M; \mathbb{Q}) \otimes H^*(N; \mathbb{Q})$$

In particular

$$H^{2(m+n)}(M \times N; \mathbb{Q}) \cong \bigoplus_{k+l=2(m+n)} H^k(M; \mathbb{Q}) \otimes H^l(N; \mathbb{Q})$$

The bilinear form on the the space above can be decomposed into $H^{2m}(M;\mathbb{Q}) \otimes H^{2n}(N;\mathbb{Q})$ and $\bigoplus_{k+l=2(m+n), k\neq 2m} H^k(M;\mathbb{Q}) \otimes H^l(N;\mathbb{Q})$ because the cup products between elements from the first space and those from the second space are automatically 0 due to degree limitation.

The fundamental class of $M \times N$ is $[M \times N] = [M] \times [N]$, where $\times$ represents homology cross product. Comparing with cohomology cross product appears in the Künneth isomorphism, it's easy to verify that the first space contribute to the signature by $\sigma(M)\sigma(N)$.

As for the second space

$$\bigoplus_{\substack{k+l=2(m+n) \\ k\neq 2m}} H^k(M;\mathbb{Q}) \otimes H^l(N;\mathbb{Q}) =$$

$$\bigoplus_{\substack{k+l=2(m+n) \\ k>2m}} H^k(M;\mathbb{Q}) \otimes H^l(N;\mathbb{Q}) \oplus \bigoplus_{\substack{k+l=2(m+n) \\ l>2n}} H^k(M;\mathbb{Q}) \otimes H^l(N;\mathbb{Q})$$

The restriction of the bilinear form on each summand is 0 due to degree limitation. At least one of the summand has dimension greater than of equal to that of the whole space, so from the lemma, the contribution of this space is 0. □

Combining the theorems above we get $[M] \mapsto \sigma(M)$ is a $\mathbb{Q}$-algebra homomorphism from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$.(though the signature only take integer values)

7 The Hirzebruch Signature Theorem

Let A^* be a graded Λ-algebra. A^{Π} consists of all formal sums $a_0 + a_1 + a_2 + \cdots$ with $a_i \in A^i$. We focus on elements with leading term 1, i.e. $1 + a_1 + a_2 + \cdots$ with $a_i \in A^i$, all of which form a multiplicative group.

Consider a sequence of polynomials $K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \cdots$ with coefficients in Λ. When x_i is considered to have degree i, we require K_n to be a homogeneous polynomial of degree n. For $a = 1 + a_1 + a_2 + \cdots$, define

$$K(a) := 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots$$

We say K_n is a multiplicative sequence if $K(ab) = K(a)K(b)$ for all a, b with leading term 1.

Let A^* be the graded polynomial ring $\Lambda[t]$ where t is an indeterminate of degree 1. Then an element of A^Π with leading term 1 can be thought of as a formal power series

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3 + \cdots$$

with coefficients in Λ. In particular $1 + t$ is such an element.

Theorem: Given a formal power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$ with coefficients in Λ, where t has degree 1, there is an unique multiplicative sequence with coefficients in Λ satisfying $K(1 + t) = f(t)$.

Remark: If the theorem holds, then for any A^* and any $a_1 \in A^1$ we have $K(1 + a_1) = f(a_1)$.

Proof. Uniqueness

Let $t_1, \cdots, t_n$ be algebraically independent elements with degree 1. $\sigma := (1 + a_1) \cdots (1 + a_n)$. Then

$$K(\sigma) = K(1 + a_1) \cdots K(1 + a_n)$$

focus on the summand of degree n, we notice that $K_n(\sigma_1, \cdots, \sigma_n)$ is determined by f. But $\sigma_1, \cdots, \sigma_n$ have no polynomial relation, which proves the uniqueness.

Existence

For partition $I = \{i_1, \cdots i_r\}$, $\lambda_I := \lambda_{i_1} \cdots \lambda_{i_r}$ Just verify

$$K_n(x_1, \cdots, x_n) := \sum_I \lambda_I s_I(x_1, \cdots, x_n)$$

where the summation runs over all partition of n, is the desired sequence.

□

Now, consider the case $\Lambda = \mathbb{Q}$. Let M^m be a smooth compact oriented manifold.

Definition: The K-genus $K[M^m]$ is zero if the dimension m is not divisible by 4, and is equal to the rational number

$$K_n[M^{4n}] = \langle K_n(p_1, \cdots, p_n), [M^{4n}] \rangle$$

if $m = 4n$, where p_i denotes the i-th Pontrjagin class of the tangent bundle.

$K[M^m]$ is a certain rational linear combination of the Pontrjagin numbers of M^m, so it's 0 when M is a boundary. From the property of multiplicative sequence, it's easy to verify that $K[M \times M'] = K[M]K[M']$.

So $M \mapsto K[M]$ defines a ring homomorphism form Ω_* to $\mathbb{Q}$. It lifts to a $\mathbb{Q}$-algebra homomorphism $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$.

Theorem: Let $\{L_k(p_1, \cdots, p_k)\}$ be the multiplicative sequence of polynomials associated with the power series

$$\frac{\sqrt{t}}{\tanh \sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + \frac{(-1)^{k-1}2^{2k}B_k t^k}{(2k)!} \cdots$$

Then the signature $\sigma(M^{4k})$ of any smooth compact oriented manifold M^{4k} is equal to the L-genus $L[M^{4k}]$.

Proof. Since the correspondences $M \mapsto \sigma(M)$ and $M \mapsto L[M]$ both give rise to $\mathbb{Q}$-algebra homomorphisms from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$, it suffices to check this theorem on a set of generators for the algebra $\Omega_* \otimes \mathbb{Q}$, namely $\mathbb{CP}^{2k}$.

Earlier, we have proved that $H^{2k}(\mathbb{CP}^{2k}; \mathbb{Q})$ is a one dimensional space generated by a^k and $\langle a^k \smile a^k, [\mathbb{CP}^{2k}]\rangle = 1$. It follows that $\sigma(\mathbb{CP}^{2k}) = 1$.

We know $p(\tau) = (1 + a^2)^{2k+1}$. Notice that a^2 has cohomology degree 4, but for the multiplicative sequence $\{L_k(p_1, \cdots, p_k)\}$, a^2 has degree 1. So $L(1+a^2) = \frac{\sqrt{a^2}}{\tanh \sqrt{a^2}}$. Hence, $L(p) = (\frac{a}{\tanh a})^{2k+1}$, whose $2k$-th coefficient is $L[\mathbb{CP}^{2k}]$. Let $\tanh z = u$, then $dz = \frac{du}{1-u^2} = (1 + u^2 + u^4 + \cdots)du$.

$$L[\mathbb{CP}^{2k}] = \frac{1}{2\pi i z^{2k+1}} \oint (\frac{z}{\tanh z})^{2k+1} dz = \frac{1}{2\pi i} \oint \frac{1 + u^2 + u^4 + \cdots}{u^{2k+1}} du = 1$$

□

It's easy to verify that $L_2 = \frac{7p_2 - p_1^2}{45}$, then we ge the crucial result. For a smooth oriented compact manifold of dimension 8, we have

$$\sigma(M) = \langle \frac{7p_2(\tau_M) - p_1(\tau_M)^2}{45}, [M]\rangle$$

8 Construction of Exotic Sphere

We will construct a space as a S^3 bundle over S^4. First, we give the definition of fiber bundles which is a generalization of vector bundles.

Definition: A fiber bundle over B with fiber F and structural group G is:

1. A topological space E called the total space.
2. A (continuous) map $p : E \to B$ called the projection map.
3. For each $b \in B$, the fiber $p^{-1}(b)$ is homeomorphic to F.
4. An action of the (topological) group G on F.
5. For each $b \in B$, there exists a neighborhood $b \in U \subset B$ and a homeomorphism $h : U \times F \to p^{-1}(U)$.

6. The map $h_i^{-1} \circ h_j : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ has the form $(b, x) \mapsto (b, g_{ij}(b)x)$ for some continuous $g_{ij} : U_i \cap U_j \to G$.

Notice that a real vector bundle of rank n is just a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL_n(\mathbb{R})$. The concept of smooth fiber bundle are defined similarly.

$\mathbb{H}$ denotes the quaternion numbers and $\mathbb{HP}^1$ denotes the corresponding projective line consisting of two open subsets: $U_1 = \{[z : w] \in \mathbb{HP}^1 | w \neq 0\}$, $U_2 = \{[z : w] \in \mathbb{HP}^1 | z \neq 0\}$. We have the coordinate maps.

$$\begin{aligned} \phi_1 \colon U_1 &\to \mathbb{H} \\ [z : w] &\mapsto w^{-1} z \end{aligned}$$

$$\begin{aligned} \phi_2 \colon U_2 &\to \mathbb{H} \\ [z : w] &\mapsto z^{-1} w \end{aligned}$$

Identifying $\mathbb{H}$ with $\mathbb{R}^4$, we can easily see $\mathbb{HP}^1 \cong S^4$. S^3 can be seen as elements with norm 1 in $\mathbb{H}$.

For integers h, l, we define a transition map

$$\begin{aligned} f_{h,l} \colon \phi_1(U_1 \cap U_2) \times S^3 &\to \phi_2(U_1 \cap U_2) \times S^3 \\ (z, y) &\mapsto (z^{-1}, \frac{z^h y z^l}{\|z\|^{h+l}}) \end{aligned}$$

Then we get a S^3 bundle over S^4. Specifically $M_{h,l} := (\phi_1(U_1) \times S^3 \sqcup \phi_2(U_2) \times S^3)/\sim$, where $(z_1, w_1) \sim (z_2, w_2)$ iff $(z_2, w_2) = f_{h,l}(z_1, w_1)$ is the total space. Projection map p maps $[z, w]$ to $\phi_1^{-1}(z)$ if $(z, w) \in \phi_1(U_1) \times S^3$ and to $\phi_2^{-1}(z)$ if it comes from $\phi_2(U_2) \times S^3$). This bundle is denoted by $\sigma_{h,l}$. Notice that it consists of all vector with norm 1 in a $\mathbb{R}^4$ bundle $\xi_{h,l}$ with the same transition function.

To prove some $M_{h,l}$ is homeomorphic to S^7, we use Reeb Theorem from differential topology.

Theorem: If M is a compact manifold with a Morse function F such that F has exactly two critical points, then M is homeomorphic to the sphere in the corresponding dimension.

Now we define a Morse function on $M_{h,l}$. For piece one, define

$$\begin{aligned} F_1 \colon \phi_1(U_1) \times S^3 &\to \mathbb{R} \\ (z, v) &\mapsto \frac{\Re(v)}{\sqrt{1 + \|z\|^2}} \end{aligned}$$

To compute its critical points, notice that for fixed z, the function attains its extreme value on critical point, in which case $v = \pm 1$. And direct computation shows that $z \mapsto \frac{1}{\sqrt{1+|z\|^2}}$ admits critical points at $z = 0$. So F_1 has exactly 2 critical points, namely $(0, \pm 1)$.

For piece two, define

$$\begin{aligned} F_2\colon \phi_2(U_2) \times S^3 &\to \mathbb{R} \\ (w,u) &\mapsto \frac{\Re(wu^{-1})}{\sqrt{1+\|wu^{-1}\|^2}} = \frac{\Re(wu^{-1})}{\sqrt{1+\|wu^{-1}\|^2}} \end{aligned}$$

for fixed u, direct computation shows that the gradient with respect to w is never 0. Hence F_2 has no critical point.

We need to piece this two functions together. It suffices to prove $\frac{\Re(v)}{\sqrt{1+\|z\|^2}} = \frac{\Re(wu^{-1})}{\sqrt{1+\|w\|^2}}$, where $w = z^{-1}$ and $u = \frac{z^h v z^l}{\|z\|^{h+l}}$.

We have

$$u^{-1} = \bar{u} = \frac{\bar{z}^l \bar{v} \bar{z}^h}{\|z\|^{h+l}}$$

$$wu^{-1} = \frac{\bar{z}^{l+1} \bar{v} \bar{z}^h}{\|z\|^{h+l+2}}$$

Notice that $\Re(xyx^{-1}) = \Re(y)$, so when $h+l = -1$,

$$\Re(\frac{\bar{z}^{l+1} \bar{v} \bar{z}^h}{\|z\|^{h+l+2}}) = \Re(\frac{\bar{z}^{l+1} \bar{v} \bar{z}^{-1-l}}{\|z\|}) = \frac{\Re(v)}{\|z\|}$$

and

$$\|wu^{-1}\|^2 = \frac{1}{\|z\|^2}$$

then the equality is verified.

So, we've proved that when $h+l = -1$, $M_{h,l}$ is homeomorphic to S^7. Later discussion will show it remains true when $h+l = 1$.

Lemma: The map $\mathbb{Z} \to \pi_3(S^3, 1)$ given by $a \mapsto (x \mapsto x^a)$ is a group isomorphism.(the multiplication and '1' are understood as subset of quaternion)

Proof. Let $\cdot$ denote the multiplication on S^3, and $+$ the usual addition on $\pi_3(S^3, 1)$. By verifying the definition of $+$, we know

$$f(x) \cdot g(x) + f'(x) \cdot g'(x) = (f + f')(x) \cdot (g + g')(x)$$

Then Eckman-Hilton argument shows that they are actually the same operation, the lemma follows. $\square$

Define $P : S^3 \times S^3 \to SO(4)$ $(u, v) \mapsto f_{u,v}$ where $f_{u,v}(x) := uxv^{-1}$. It's easy to verify that it gives to a homeomorphism between $(S^3 \times S^3)/\sim$ and $SO(4)$ where $(u, v) \sim (-u, -v)$. Combining with the lemma above, there exists a group isomorphism $\pi_3(SO(4)) \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ where $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ corresponds to $f_{a,b}(x)(v) := x^a v y^b$.

We give a non rigorous classification argument for oriented bundles over spheres. Notice that for any oriented $\mathbb{R}^4$ bundle over S^4 (with structure group

$SO(4)$), it's trivial when restricted on contractible space U_1 or U_2, the homotopy class of transition function from $U_1 \cap U_2 \sim S^3$ to $SO(4)$ determines the bundle. And on the other side, we have classification theorem. There is a bijection between F bundle over paracompact space X with structure group G and $[X, BG]$, any bundle over F is given by pullback of the universal bundle η over BG. In our case, $\mathbb{R}^4$ bundle over S^4 with structure group $SO(4)$ can be classified by $\pi_4(BSO(4))$. The arguments above hold in a natural sense. i.e. the bundle determined by $f \in \pi_3(SO(4))$ is isomorphic to $g^*\eta$ for $g \in \pi_4(BSO(4))$. The corresponding of f and g gives a group isomorphism.

Choose any element $u \in H^4(BSO(4))$, we have a group homomorphism

$$\begin{aligned}\Phi\colon \pi_4(BSO(4)) &\to H^4(S^4)\\ [f] &\mapsto f^*(u)\end{aligned}$$

Let $u = p$ denote the first Pontryagin class of the universal $\mathbb{R}^4$ bundle over $BSO(4)$. We have $p_1(\xi_{h,l}) = g_{h,l}^*(p) = \Phi(g_{h,l}) = (mh + kl)\alpha$, where α is a generator of $H^4(S^4)$ and the coefficients m, k are determined later.

Consider vector bundle $\xi_{h,l}$ and $\xi_{-l,-h}$, we can construct a bundle isomorphism using conjugate of quaternion.(so $\xi_{h,l}$ is a sphere when $h + l = 1$) Specifically,

$$\begin{aligned}\Phi_{h,l}\colon (\phi_1(U_1) \times \mathbb{H} \sqcup \phi_2(U_2) \times \mathbb{H})/\sim_{h,l} &\to (\phi_1(U_1) \times \mathbb{H} \sqcup \phi_2(U_2) \times \mathbb{H})/\sim_{-l,-h}\\ [z, w] &\mapsto [z, \bar{w}]\end{aligned}$$

It's easy to verify that when $(z_1, w_1) \sim_{h,l} (z_2, w_2)$, we have $(z_1, \bar{w_1}) \sim_{-l,-h} (z_2, \bar{w_2})$.

So we have $mh + kl = m(-l) + k(-h)$ for any h, l, then we know $m = -k$.

Let $\gamma = (E, p, \mathbb{HP}^1)$ is the canonical line bundle, we claim $\gamma \cong \xi_{0,1}$.

Proof.

$$\begin{aligned}p^{-1}(U_1) &= \{([z : 1], (x, y)) \in \mathbb{HP}^1 \times \mathbb{H}^2 | yz = x\}\\ p^{-1}(U_2) &= \{([1 : w], (x, y)) \in \mathbb{HP}^1 \times \mathbb{H}^2 | xw = y\}\end{aligned}$$

local trivialization

$$\begin{aligned}&\psi_1\colon p^{-1}(U_1) \to \phi_1(U_1) \times \mathbb{H}\\ &[z : 1], (x, y) \mapsto (z, y)\end{aligned}$$

$$\begin{aligned}&\psi_2\colon p^{-1}(U_2) \to \phi_2(U_2) \times \mathbb{H}\\ &[1 : w], (x, y) \mapsto (w, x)\end{aligned}$$

then the transition function is $\psi_2 \circ \psi_1^{-1}(z, y) = (z^{-1}, yz)$, which is homotopic to $f_{0,1}$. $\square$

Just like the case $\mathbb{CP}^n$, applying Gysin sequence to canonical line bundle over $\mathbb{HP}^n$ gives us the ring structure of $H^*(\mathbb{HP}^n)$. We have $H^*(\mathbb{HP}^n) = \mathbb{Z}[e]/\langle e^{n+1}\rangle$, where e is the Euler class of γ.

The fiber of γ is $\mathbb{H}$ admitting a complex structure. Hence γ can be seen as a complex bundle of rank 2. $c_1(\gamma) \in H^2(\mathbb{HP}^1)$ must equal to 0 and $c_2(\gamma) = e$.

Then computation shows $p(\gamma) = 1 - 2e + e^2$, $p_1(\gamma) = -2e = m(h-l)\alpha$.

We get $p_1(\xi_{h,l}) = \pm 2(h-l)\alpha$.

Let $N_{h,l}$ denotes the space consisting all vectors in $\xi_{h,l}$ with norm less than or equal to 1, then we have $\partial N_{h,l} = M_{h,l}$. If $M_{h,l}$ is diffeomorphic to S^7, we can attach D^8 smoothly along the boundary and get a 8-manifold $K_{h,l}$.

Let $\pi : N_{h,l} \to S^4$ be the projection and $E_{h,l}$ denote the total space of $\xi_{h,l}$. We have the following exact sequence of bundles.

$$0 \to \pi^*\xi_{h,l} \to \tau_{N_{h,l}} \to \pi^*\tau_{S^4} \to 0$$

where the first map sends (x, y), where y is a vector in the fiber over x, to (x, y), where y is seen as the vector in $T_x N_{h,l}$ (tangent space of a vector space can be identified with the vector space itself) and the second map sends (x, v) to $(x, \pi_*(v))$. The existence of Euclidean metric makes it split. So we have

$$\tau_{N_{h,l}} \cong \pi^*\xi_{h,l} \oplus \pi^*\tau_{S^4}$$

$\tau_{N_{h,l}} \oplus \varepsilon \cong \pi^*\xi_{h,l} \oplus \pi^*\tau_{S^4} \oplus \varepsilon \cong \pi^*\xi_{h,l} \oplus \pi^*(\tau_{S^4} \oplus \varepsilon) \cong \pi^*\xi_{h,l} \oplus \pi^*\varepsilon^5 \cong \pi^*\xi_{h,l} \oplus \varepsilon^5$

$p_1(\tau_{N_{h,l}}) = p_1(\pi^*\xi_{h,l} \oplus \varepsilon^5) = p_1(\pi^*\xi_{h,l}) = \pi^*(p_1(\xi_{h,l})) = \pm 2(h-l)\pi^*(\alpha)$

Since $K_{h,l}$ is obtained by attaching an 8-cell on $N_{h,l}$, the injection $i : N_{h,l} \to K_{h,l}$ induce cohomology isomorphism $i^* : H^4(K_{h,l}) \to H^4(N_{h,l})$ and by definition of tangent bundles we have $i^*(\tau_{K_{h,l}}) = \tau_{N_{h,l}}$. Consequently, we get

$$p_1(\tau_{K_{h,l}}) = 2(h-l)\beta$$

where β is a generator of $H^4(K_{h,l})$.

Since $H^4(K_{h,l}; \mathbb{Q})$ is one-dimensional space generated by β, $\sigma(K_{h,l}) = \pm 1$. Choose an orientation such that $\sigma(M) = 1$, in which case $\langle \beta^2, [K_{h,l}]\rangle = 1$. Then

$$45 = 7\langle p_2(\tau_{K_{h,l}}), [K_{h,l}]\rangle - 4(h-l)^2$$

$$3 \equiv -4(h-l)^2 \equiv 3(h-l)^2 \mod 7$$

$$(h-l)^2 \equiv 1 \mod 7$$

But this does not always hold which gives a contradiction.

References

[1] [J. W. Milnor and J. D. Stashef] *Characteristic Classes*

[2] [J. P. May] *A Concise Course in Algebraic Topology*

[3] [Julio Sampietro and Carlos Segovia] *The Exotic World of Milnors Spheres*

[4] [Matthias Gorg] *Euler Class and Intersection Theory*