Spectral Sequences and Applications on Homotopy groups of spheres

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Abstract

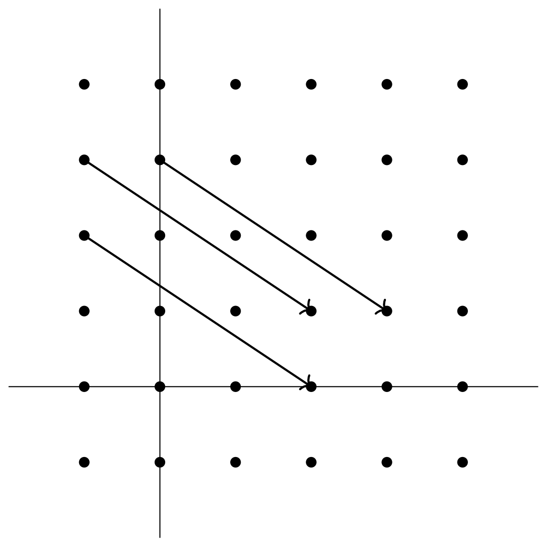
This report mainly introduces the Serre spectral sequence and it's applications in computing homotopy groups of spheres. The first section gives spectral sequences as algebraic preparation. The following two sections introduces Serre spectral sequences (also called Leray-Serre spectral sequences) and Cartan-Leray spectral sequences(without proof) which reveal the connection of (co)homology among fiber, base and total spaces. The fourth section gives some topology constructions or facts. Section 5 gives Serre's result about cohomology of Eilenberg-Maclane space. Section 6 gives the key result on homotopy groups of spheres using former construction.

Contents

1 Spectral Sequence

Definition 1. *A differential bigraded module over a ring R, is a collection of R-modules, $\{E^{p,q}\}$, where p and q are integers, together with an R-linear mapping $d : E^{*,*} \to E^{*,*}$,the differential of bidegree $(s, 1-s)$ or $(-s; s-1)$, for some integer s, and satisfying $d \circ d = 0$.*

We can visualize such data in a plain where each lattice is a module $E^{p,q}$ and each differential is represented by an arrow. The differentials in the following example have bidegree (3,-2).



We can take the homology of a given differential bigraded module

$$H^{p,q}(E^{*,*}, d) := \ker d : E^{p,q} \to E^{p+s,q-s+1} / \operatorname{im} d : E^{p-s,q+s-1} \to E^{p,q}$$

Notice that it does not give a new differential bigraded module since the new differential is not defined.

Definition 2. *A spectral sequence is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$, where $r = 1, 2, \ldots$; the differentials are either all of bidegree $(-r, r-1)$ (for a spectral sequence of homological type) or all of bidegree $(r, 1-r)$ (for a spectral sequence of cohomological type) and for all p, q, r, $E_{r+1}^{p,q}$ is isomorphic to $H^{p,q}(E_r^{*,*}, d_r)$.*

Note that d_{r+1} is not completely determined by E_r page and most of the time this leads key difficulty in computing spectral sequence.

If, for some reason, we have $E_r^{p,q} \cong E_{r+1}^{p,q} \cong E_{r+2}^{p,q} \cong \cdots$, we can define the limit term $E_\infty^{p,q}$ to be the stabilized module. If all the differentials $d_r = 0$ for $r \geq N$, the we say the spectral sequence collapses at E_N-page.

Spectral sequences are used to compute complicated stuff from something computable. For example, something we desire is linked with the E_∞ page in a given spectral sequence with known E_2 page. To make this precise, we give the following definition.

Definition 3. *A filtration F^* on an R-module A is a family of submodules $\{F^pA\}$ for p in $\mathbb{Z}$ such that*

$$\cdots \subseteq F^{p+1}A \subseteq F^pA \subseteq F^{p-1}A \subseteq \cdots \subseteq A \quad \textit{(decreasing filtration)}$$

or

$$\cdots \subseteq F^{p-1}A \subseteq F^pA \subseteq F^{p+1}A \subseteq \cdots \subseteq A \quad \textit{(increasing filtration)}.$$

Its associated graded module, $E_0^*(A)$ is given by

$$E_0^p(A) = \begin{cases} F^pA/F^{p+1}A, & \textit{when } F \textit{ is decreasing,} \\ F^pA/F^{p-1}A, & \textit{when } F \textit{ is increasing.} \end{cases}$$

We say the filtration is bounded if $F^r = A$ and $F^s = 0$ for some r,s. So there are only finite nontrivial $E_0^p(A)$.

Moreover, when the module is graded H^*, we have the degree-wise filtration given by $F^pH^n := F^pH^* \cap H^n$. And the associated graded module is bigraded.

$$E_0^{p,q}(H^*, F) = \begin{cases} F^pH^{p+q}/F^{p+1}H^{p+q}, & \text{if } F^* \text{ is decreasing,} \\ F^pH^{p+q}/F^{p-1}H^{p+q}, & \text{if } F^* \text{ is increasing.} \end{cases}$$

Notice that the associated module can not fully determine the original module. It's unsolved by an extension problem. But when R is a field or a lot of $E_0^p(A)$ are 0, we can do better.

Definition 4. *A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to converge to H^*, a graded R-module, if there is a filtration F on H^* such that*

$$E_\infty^{p,q} \cong E_0^{p,q}(H^*, F),$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence.

Generally, the differentials in a spectral sequence are hard to determine, but with some initial condition, we can derive some information. For example when the spectral sequence is first quadrant i.e. $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$, each $E_*^{p,q}$ eventually stabilizes.

Theorem 1 (Gysin sequence)**.** *Suppose $E_2^{p,q} = \{0\}$ unless $q = 0$ or $q = n$, for some $n \geq 2$ and the filtration on H^* is bounded. Then there is a long exact sequence*

$$\cdots \longrightarrow H^{p+n} \longrightarrow E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \longrightarrow H^{p+n+1} \longrightarrow E_2^{p,n+2} \xrightarrow{d_{n+1}} E_2^{p+n+2,0} \longrightarrow \cdots$$

Proof. Notice that the only possible nonzero differential is $d_{n+1} : E_2^{p,n} \to E_2^{p+n+1,0}$. Therefore we have

$$E_2 \cong \cdots \cong E_{n+1} \quad \text{and} \quad E_{n+2} = H(E_{n+1}, d_{n+1}) \cong E_\infty.$$

This yields the short exact sequence for each p,

$$0 \to E_\infty^{p,n} \to E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \to E_\infty^{p} \to 0.$$

Since $F^k H^{p+n}/F^{k+1}H^{p+n} \cong E_\infty^{k,p+n-k}$ where the only nontrivial E_∞ terms are $E_\infty^{0,p+n}$ and $E_\infty^{p,n}$, combining with the fact that the filtration is bounded, we have $H^{p+n} = F^n H^{p+n}$, $F^{n+1}H^{p+n} = F^{p+n}H^{p+n}$ and $F^{p+n+1}H^{p+n} = 0$.

Then we get a short exact sequence

$$0 \to E_\infty^{p+n,0} \to H^{p+n} \to E_\infty^{p,n} \to 0.$$

Splicing these short exact sequences together, we get desired long exact sequence.

$$\begin{tikzcd}
 & H^{p+n-1} \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E_\infty^{p-1,n} \arrow[r] \arrow[d] & E_2^{p-1,n} \arrow[r] & E_2^{p+n,0} \arrow[r] \arrow[dr, dashed] & E_\infty^{p+n,0} \arrow[r] \arrow[d] & 0 \\
 & 0 & & & H^{p+n} \arrow[d] \arrow[dr, dashed] & \\
 & & & 0 & E_\infty^{p,n} \arrow[l] \arrow[r] \arrow[d] & E_2^{p,n} \\
 & & & & 0 &
\end{tikzcd}$$

□

Sometimes spectral sequences enjoys more structure other than modules.

Definition 5. *A differential graded algebra, (A^*, d), is a graded algebra with a degree 1 linear mapping, $d : A^* \to A^*$, such that d is a derivation, that is, satisfies the Leibniz rule*

$$d(a \cdot a') = d(a) \cdot a' + (-1)^{\deg a} a \cdot d(a').$$

A differential bigraded algebra, $(E^{*,*}, d)$, is a bigraded algebra with a total degree one mapping

$$d : \bigoplus_{p+q=n} E^{p,q} \to E^{r,s}$$

where $r + s = n + 1$ that satisfies the Leibniz rule

$$d(e \cdot e') = d(e) \cdot e' + (-1)^{p+q} e \cdot d(e'),$$

when e is in $E^{p,q}$ and e' is in $E^{r,s}$.

$\{E^{*,*}, d_r\}$ is a spectral sequence of algebras if for each r, $(E^{*,*}, d_r)$ is a differential bigraded algebra and furthermore, the product on E^{*}_{r+1} is induced by the product of $E^{*,*}$ on homology.

Definition 6. *Suppose F^* is a filtration of H^*, a graded algebra. The filtration is said to be stable with respect to the product if*

$$F^r H^* \cdot F^s H^* \subseteq F^{r+s} H^*.$$

A filtration F^* on H^* that is stable with respect to a product on H^* induces a bigraded algebra structure on the associated bigraded module $E^{*,*}(H^*)$.

We say a spectral sequence of algebras converge to H^* as a graded algebra if there is a spectral sequence $\{E^{*,*}_r, d_r\}$ of algebras and a stable filtration on H^* with the E_∞-term of the spectral sequence isomorphic as a bigraded algebra to the associated bigraded algebra, $E^{*,*}_0(H^*, F^*)$.

Proposition 2. *Suppose $E^{*,*}_2 = V^* \otimes W^*$ converges to $H^* \cong \mathbb{Q}$ (that is, H^* is the graded algebra with $H^0 \cong \mathbb{Q}$ and $H^i = \{0\}$ for $i \geq 1$). If $V^* \cong \mathbb{Q}[x_{2n}]$, then $W^* \cong \Lambda(y_{2n-1})$. If $V^* \cong \Lambda(x_{2n+1})$, then $W^* \cong \mathbb{Q}[y_{2n}]$. ($\Lambda(x)$ denotes the exterior algebra generated by x)*

Proof. In the first case, $V^* \cong \mathbb{Q}[x_{2n}]$. Since x_{2n} does not survive to E_∞, there must be a y_{2n-1} in W^* such that

$$d_{2n}(1 \otimes y_{2n-1}) = x_{2n} \otimes 1.$$

y_{2n-1} is of minimal degree in W^* since any elements with lower degree will survive to E_∞ page.

Now, with y_{2n-1} in W^*, we have generated new elements in E^*_2, namely $(x_{2n})^m \otimes y_{2n-1}$. By the derivation property of differentials,

$$d_{2n-1}((x_{2n})^m \otimes y_{2n-1}) = (x_{2n})^{m+1} \otimes 1.$$

If W^* contains any other elements, let w be such element with lowest degree, by dimension argument, w must have degree $4n - 2$, $d_{2n}(w) = kx_{2n} \otimes y_{2n-1}$. Then

$$0 = d_{2n}(d_{2n}(w)) = kx^2_{2n}$$

we have $k = 0$. But then w will survive to E_∞ which gives contraction. Thus $W^* \cong \Lambda(y_{2n-1})$

For the second case, $V^* \cong \Lambda(x_{2n+1})$, similarly, there is $y_{2n} \in W^{2n}$ such that $d_{2n+1}(1 \otimes y_{2n}) = x_{2n+1} \otimes 1$. Then $x_{2n+1} \otimes y_{2n}$ must be killed by $z \in W^{4n}$. i.e. $d_{2n+1} : 1 \otimes z \mapsto x_{2n+1} \otimes y_{2n}$. This map should be isomorphism to prevent all elements from surviving to E_∞ by dimension reason. $d_{2n+1}(1 \otimes y^2) = 2x_{2n+1} \otimes y_{2n}$, so $z = \frac{1}{2}y_{2n}^2$. Continuing this argument, we find that W^* has $\{1, y_{2n}, y_{2n}^2, y_{2n}^3, \cdots\}$ as a vector space basis. i.e. $W^* \cong \mathbb{Q}[y_{2n}]$ □

2 The Serre Spectral Sequence

Definition 7. *Let R be a commutative ring with unit. Suppose $F \to E \to B$ is a fibration, where B is path-connected and F is connected. Then there is a first quadrant spectral sequence of algebras, $\{E_r^{*,*}, d_r\}$, converging to $H^*(E; R)$ as an algebra, with*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F; R))$$

the cohomology of the space B with local coefficients in the cohomology of the fibre . This spectral sequence is natural with respect to fibre-preserving maps of fibrations. Furthermore, the cup product $\smile$ on cohomology with local coefficients and the product $\cdot_2$ on $E_2^{*,*}$ are related by

$$u \cdot_2 v = (-1)^{pq} u \smile v$$

when $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$.

In many applications, the base space B is simply-connected or local coefficients on B is simple and we can reduce $H^p(B; \mathcal{H}^q(F; R))$ to $H^p(B; H^q(F; R))$ where the latter can be derived from universal coefficient theorem.

For homology situation, we have similar result except the product structure.

Example 3. *Suppose $F \to E \to B$ is a fibration with B path-connected,F connected and the system of local coefficients on B induced by the fibre is simple. If R is Noetherian and two of the spaces F, E, or B have cohomology finitely generated R-module in each dimension, then the other space also has cohomology finitely generated in each dimension.*

The Noetherian condition guarantees that submodule of a finitely generated R-module is finitely generated. Then by induction, it's easy to complete the theorem in all 3 cases. Let's prove for E, B is given to be finite type(That is, all (co)homology is finitely generated).

Proof. Suppose i is minimal s.t. $H^i(F)$ is not finitely generated, then $E_r^{p,q}$ is finitely generated for $p \leq i-1$ because it's true for $r = 2$ and subquotient of finitely generated module is f.g.(Noethrian condition). Every differential from $E_r^{i,0}$ has f.g. image. So $E_\infty^{i,0}$ is not f.g. if $E_2^{i,0}$ is not contradicting with the fact that E is of finite type. □

Theorem 4. *Suppose $F \to E \to B$ is a fibration with B path-connected,F connected and the system of local coefficients on B induced by the fibre is simple.*

Suppose further that $H^i(B;R) = \{0\}$ for $0 < i < p$ and $H^j(F;R) = \{0\}$ for $0 < j < q$. Then there is an exact sequence called Serre exact sequence

$$\begin{aligned} 0 &\to H^1(B;R) \to H^1(E;R) \to H^1(F;R) \to H^2(B;R) \to \cdots \\ &\to H^{p+q-2}(F;R) \to H^{p+q-1}(B;R) \to H^{p+q-1}(E;R) \to H^{p+q-1}(F;R) \end{aligned}$$

Proof. Notice that when the total degree is less than $p+q$, the only nontrivial differentials are of the form $d_j : H^{j-1}(F;R) = E_2^{0,j-1} \to E_2^{j,0} = H^j(B;R)$. Similar argument as the Gysin sequence gives the desired long exact sequence. □

Theorem 5 (Wang sequence)**.** *For a fibration $F \to E \to B$ with B, a simply-connected homology n-sphere and F path-connected, we have the following exact sequence*

$$\cdots \to H^k(E;R) \to H^k(F;R) \xrightarrow{\theta} H^{k-n+1}(F;R) \to H^{k+1}(E;R) \to \cdots$$

In this case, if n is even, the mapping θ can be shown to be a (graded) derivation ($\theta(x \cdot y) = \theta(x) \cdot y + (-1)^{deg\ x} x \cdot \theta(y)$) and, if n is odd, a ordinary derivation ($\theta(x \cdot y) = \theta(x) \cdot y + x \cdot \theta(y)$).

Proof. Consider the Serre spectral sequence of the fibration, its term E_2 is isomorphic to

$$H^*(B) \otimes H^*(F) = H^*(S^k) \otimes H^*(F).$$

(R is omitted)

It follows that for a given total degree, the term E_2 has at most two non-zero elements. Similar argument as Gysin exact sequence gives the exact sequence

$$\cdots \to H^i(E) \to H^i(F) \xrightarrow{d_k} H^k(B) \otimes H^{i-k+1}(F) \to H^{i+1}(E) \to \cdots.$$

Choose an isomorphism

$$g : H^{i-k+1}(F) \to H^k(B) \otimes H^{i-k+1}(F)$$

and to take

$$\theta = g^{-1} \circ d_k.$$

Let s be a generator of $H^k(B)$. $g(x) := s \otimes x$. Then, by definition, we have $d_k(x) = s \otimes \theta(x)$ for $x \in H^*(F)$.

Now, we calculate $d_k(xy)$: On the one hand, we have

$$d_k(xy) = s \otimes (xy),$$

on the other hand:

$$d_k(xy) = d_k(x) \cdot y + (-1)^{\deg x} \cdot d_k(y) \cdot y.$$

Now, we calculate:

$$d_k(xy) = s\otimes\theta(x)\cdot y+(-1)^{\deg x}\cdot(s\otimes\theta(y))\cdot y = s\otimes(\theta(x)\cdot y+(-1)^{(k+1)}\deg x\cdot\theta(y)).$$

due to the definition of multiplication in the tensor of two algebra.

So, we obtain

$$\theta(xy) = \theta(x)\cdot y + (-1)^{(k+1)\deg x} x\cdot\theta(y),$$

$\square$

3 Group Homology and Cartan-Leray Spectral Sequence

Here we give a brief introduction for group homology. For more details, see chapter 6 of [5]

For a group π, we say π acts on an abelian group M, if there is a group homomorphism $\pi \to \mathrm{Aut}(M)$. Then M is endowed with the structure of $\mathbb{Z}\pi$-module, where $\mathbb{Z}\pi$ is the group-ring of π.

We present a functor

$$(\ )_\pi : M \mapsto M_\pi$$

where M_π is given by $M/\langle m - am : m \in M, a \in \pi\rangle$. i.e. the largest quotient group on which π acts trivially. The functor is right exact once we notice the following

$$M_\pi \cong \mathbb{Z}\otimes_{\mathbb{Z}\pi} M$$

where π acts on $\mathbb{Z}$ trivially.

Definition 8. *The homology of a group π with coefficients in a (left) $\mathbb{Z}\pi$-module M is defined by*

$$H_i(\pi, M) = \mathrm{Tor}_i^{\mathbb{Z}\pi}(\mathbb{Z}, M).$$

i.e. the derived functor of $(\ )_\pi$. We write $H_i(\pi)$ for $H_i(\pi,\mathbb{Z})$ when $\mathbb{Z}$ is the trivial left π-module.

For some simple cases, it's easy to compute.

$$H_0(\pi) = \mathbb{Z}$$

$$H_i(\mathbb{Z}_m) = \begin{cases} \mathbb{Z}, & \text{if } i = 0 \\ \mathbb{Z}_m, & \text{if } i = 1,3,5,\cdots \\ 0 & \text{if } i = 2,4,6,\cdots \end{cases}$$

$$H_i(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0,1 \\ 0 & \text{if } i = 2,3,4,\cdots. \end{cases}$$

We also have the Kunneth formula(only works for trivial action)

Proposition 6. *For every G and H there is a split exact sequence:*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(G)\otimes H_q(H) \longrightarrow H_n(G\times H) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(G), H_q(H)) \longrightarrow 0.$$

It follows that for a finitely generated abelian group π, all its homology are finitely generated and $H_1(\pi) = \pi$.

Moreover, we have

Proposition 7. *If π is a finite group of order $|\pi|$ and M is a π-module, then every element of $H_i(\pi, M)$ for $i > 0$ has order a divisor of $|\pi|$.*

The proof requires a detailed analysis on bar-construction which can be found in [5].

Given such tools, we can handle fibration with discrete fibers.

Theorem 8 (Cartan-Leray spectral sequence)**.** *If X is a connected space on which the group π acts freely and properly, then there is a spectral sequence of first quadrant, homological type, with*

$$E_2^{p,q} \cong H_p(\pi, H_q(X; R))$$

and converging to $H_*(X/\pi; R)$

In most of our application, the action of π on $H_*(X)$ is trivial (when X is an H-space).

4 Some Topology

4.1 Path-loop Fibration

Now we introduce the famous path-loop fibration. Let X be a topological space. We associate two important spaces to X as follows: Suppose $*$ is a basepoint in X. Let

$$PX = \{\lambda : [0, 1] \to X \mid \lambda \text{ is continuous and } \lambda(0) = *\}$$

denote the space of paths in X based at $*$. Let

$$\Omega X = \{\lambda : [0, 1] \to X \mid \lambda \text{ is continuous and } \lambda(0) = \lambda(1) = *\},$$

the space of based loops in X at $*$ with the compact-open topology. The evaluation mapping

$$\mathrm{ev}_1 : PX \to X$$

given by $\mathrm{ev}_1(\lambda) = \lambda(1)$ is a fibration (it can be shown from definition, see [4])and has fibre ΩX. It's easy to see PX is a contractible space.

Path-loop fibration and theorem 4 gives

Proposition 9. *If X is q-connected, $H_i(\Omega X; R) \cong H_{i+1}(X; R)$ for $i < 2q - 2$.*

We say a space X is of finite type if all of it's singular homology groups are finitely generated.

Example 3 gives

Theorem 10. *If X is of finite type, so is ΩX.*

4.2 Fundamental Class and Eilenberg-MacLane Space

Suppose space X is $(n-1)$ -connected, by UCT, we have the isomorphism, $H^n(X;\pi) \cong \mathrm{Hom}(H_n(X), \pi)$. Recall that Hurewicz homomorphism $h : \pi_n(X) \to H_n(X)$ is an isomorphism when X is $(n-1)$-connected.

Definition 9. *Let X be $(n-1)$-connected. The fundamental class of X is the cohomology class $\iota \in H^n(X; \pi_n(X))$ which corresponds to h^{-1} under the above isomorphism.*

Let's introduce the Eilenberg-Maclane space $K(\pi, n)$ with only one nontrivial homotopy group G at degree n. G is required to be abelian when $n \geq 2$.

Theorem 11. *There is a one-to-one correspondence*

$$[X, K(\pi, n)] \leftrightarrow H^n(X; \pi),$$

given by

$$[f] \leftrightarrow f^*(\iota_n).$$

The theorem can be proved by obstruction theory.

This gives the following corresponding

$$[K(\pi, n), K(\pi', n)] \leftrightarrow H^n(K(\pi, n); \pi') \cong \mathrm{Hom}(H_n(K(\pi, n)), \pi') = \mathrm{Hom}(\pi, \pi').$$

So the weak homotopy type of $K(\pi, n)$ is determined.

Applying loop-path fibration, we get

$$\Omega K(\pi, n) \to PK(\pi, n) \to K(\pi, n)$$

where $PK(\pi, n)$ is contractible and $\Omega K(\pi, n) \simeq K(\pi, n-1)$ is a weak homotopy equivalence.

S^1 is a model for $K(\mathbb{Z}, 1)$, then by induction and proposition 2 , we can get

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \Lambda(x_n), & \text{if } n \text{ is odd} \\ \mathbb{Q}[x_n], & \text{if } n \text{ is even} \end{cases}$$

4.3 Postnikov Tower and Whitehead Tower

The importance of Eilenberg-Maclane spaces can be shown in the following constructions.

Theorem 12 (Postnikov tower)**.** *For any connected space X, there is a 'tower' of fibrations*

$$P_1(X) \xleftarrow{\psi_1} P_2(X) \xleftarrow{\psi_2} P_3(X) \leftarrow \cdots$$

and compatible maps $f_i : X \to P_i(X)$ (compatible in the sense that $\psi_n \circ f_{n+1} = f_n : X \to P_n(X)$), with the following properties:

1. $\pi_k(P_n(X)) = 0$ *for* $k > n$.

2. $\pi_k(X) \to \pi_k(P_n(X))$ *is an isomorphism for* $k \leq n$

3. *The fiber of* ψ_{n-1} *is* $K(\pi_n(X), n)$.

Proof. Let $P_n(X)$ be the space obtained form attaching cells of dimension larger than $n+1$ to kill π_k for $k \geq n+1$. The map $P_n(X) \to P_{n-1}(X)$ is an extension of $X \to P_{n-1}(X)$ which is possible due to the vanishing higher homotopy groups of $P_{n-1}(X)$. Finally, replacing the map by fibrations one by one and we can find that the fiber is indeed $K(\pi_n(X), n)$. □

Theorem 13. *Let* X *be a CW complex. There is a sequence of fibrations:*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

where the fiber of $X_n \to X_{n-1}$ *is* $K(\pi_n(X), n-1)$*, in particular* X_1 *is the universal cover of* X*. They satisfy* $\pi_i(X_n) = 0$ *for all* $i \leq n$*, and the map* $X_n \to X$ *induces isomorphisms on* π_i *for* $i > n$.

Proof. Firstly, just take X_1 to be the universal cover. Suppose we have X_{n-1} with $\pi_i(X_{n-1}) = 0$ for $i \leq n-1$, let's attach cells of dimension $\geq n+2$ to kill $\pi_i(X_{n-1})$ for $i \geq n+1$. The resulting space is $K(\pi_n(X), n) \supset X_{n-1}$. Let X_n be the space of paths in $K(\pi_n(X), n)$ that start from a basepoint and end in X_{n-1}. It is a fibration over X_{n-1}, with fiber $\Omega K(\pi_n(X), n) \simeq K(\pi_n(X), n-1)$. So we get a fibration:

$$K(\pi_n(X), n-1) \to X_n \to X_{n-1}$$

It follows from the long exact sequence that $\pi_i(X_n) = \pi_i(X_{n-1})$ for all $i \geq n+1$, and $\pi_i(X_n) = \pi_i(X_{n-1}) = 0$ for all $i \leq n-2$. The rest of the long exact sequence looks like:

$$0 \to \pi_n(X_n) \to \pi_n(X_{n-1}) \xrightarrow{\partial} \pi_n(X) \to \pi_{n-1}(X_n) \to 0$$

From the following lemma, the map $\partial : \pi_n(X_n) \to \pi_n(K(\pi_n(X), n))$ is an isomorphism, so $\pi_n(X_n) = \pi_{n-1}(X_n) = 0$. This shows that the X_n has constructed the desired properties.

□

Lemma 14. *Let* A *be a subspace of* X*, evaluating at 1 in the path space* $P(X, A) := \{\gamma \in \Omega X : \gamma(1) \in A\}$ *gives a fibration over* A *with fiber* ΩX*. The boundary map* $\partial : \pi_n(A) \to \pi_{n-1}(\Omega X) \cong \pi_n(X)$ *coincides with* $\pi_n(A) \to \pi_n(X)$ *induced by inclusion.*

Proof. What we gave is exactly the pullback of the path-loop fibration of X along inclusion $A \to X$. The naturality of boundary map gives the property. □

4.4 H-space

ΩX has an important property which we will prove for general H-space.

Definition 10. *Let G be a topological space with the composition law, denoted by $\vee$. The pair $(G, \vee)$ is called an H-space, if the following conditions are satisfied:*

(I) A map $(x, y) \mapsto x \vee y$ is a continuous map of $G \times G$ to G.

(II) There is an element $e \in G$, such that $e \vee e = e$, and the maps $x \to x \vee e$, $x \to e \vee x$ are homotopic to the identical map of G relative to $\{e\}$.

Notice that topological groups are H-spaces. And it's easy to verify the fact that

Proposition 15. *Let X be a topological space and $* \in X$. The loop space ΩX at the point $*$ endowed with the compact open topology, and with the path composition law is an H-space.*

For an H-space G, we shall prove it's simplicity, i.e. the deck transformation on the universal covering is homotopic to the identity. More specifically,

Proposition 16. *Let G be a path connected, locally path connected, and locally simply connected H-space. Then automorphisms of the universal covering T of G defined by $\pi_1(G)$ are homotopic to the identity.*

Proof. Recall that we have a standard construction for an universal covering of G, the path homotopy class of path starting at e. i.e.

$$T := \{[\gamma] : \gamma \text{ is a path in } G, \gamma(0) = e\}$$

where the equivalence is taken to be homotopy relative to $\{0, 1\}$.

For $[u] \in \pi_1(G)$ the map $[\gamma] \mapsto [u * \gamma]$ is exactly the automorphism on T defined by $\pi_1(G)$.

Let PG consist of all path in G starting at e, there is a natural quotient map $PG \to T$. It suffices to show the map $PG \to PG$ $\gamma \mapsto u * \gamma$ is homotopic to the identity map and the homotopy H_t is required to send path homotopic paths to path homotopic paths.

The space PG can be endowed with a composition law, denoted also by symbol $\vee$, and defined by $(f \vee g)(t) = f(t) \vee g(t)$, $t \in I$, $f, g \in PG$. This operation is well defined, since $e \vee e = e$.

The property of H-space gives that the map $\gamma \mapsto e \vee \gamma$ is homotopic to identity. $F_\theta(t) := u(\theta t) \vee \gamma(t)$ gives homotopy between $\gamma \mapsto e \vee \gamma$ and $\gamma \mapsto u \vee \gamma$.

$$G_\theta(t) = \begin{cases} u(2t) \vee e, & \text{if } t \leq \frac{\theta}{2}, \\ u(\theta) \vee \gamma(2t - \theta), & \text{if } \frac{\theta}{2} < t < \theta, \\ u(t) \vee \gamma(t), & \text{if } t > \theta, \end{cases}$$

gives homotopy between $\gamma \mapsto u \vee \gamma$ and $\gamma \mapsto V(\gamma)$.

where

$$V(\gamma)(t) = \begin{cases} u(2t) \vee e, & \text{if } t \leq \frac{1}{2}, \\ e \vee \gamma(2t-1), & \text{if } t > \frac{1}{2}. \end{cases}$$

Again, the property of H-space gives homotopy between $\gamma \mapsto V(\gamma)$ and $\gamma \mapsto u * \gamma$.

A number of technical details are needed to be checked, but the main picture is shown. $\square$

Corollary 17. *If G is an H-space, the group $\pi_1(G)$ acts trivially on the homology groups, on the cohomology groups, and on the homotopy groups of the universal covering space T.*

5 Steenrod Algebra and Cohomology of Eilenberg–MacLane Spaces

It turns that $H^*(X;\mathbb{Z}_2)$ enjoys additional structure than a ring.

Proposition 18. *There exists for each pair of integers $i, n \geq 0$ a natural linear map*

$$Sq^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$$

called the ith Steenrod square, with the following properties:

- *Sq^0 is the identity map.*
- *Sq^1 is the "Bockstein homomorphism," the connecting homomorphism in the long exact sequence that arises from*

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0.$$

- *$Sq^n(x) = x^2$. (Note that n is the degree of x.)*
- *$Sq^i(x) = 0$ when $i > n$.*
- *Sq^i commutes with the connecting homomorphism in the long exact sequence on cohomology. In particular, it commutes with the suspension isomorphism*

$$H^n(X) \cong H^{n+1}(\Sigma X).$$

- *The Cartan formula:*

$$Sq^n(x \smile y) = \sum_{i+j=n} Sq^i(x) \smile Sq^j(y).$$

- *The Adem relations hold: when $a < 2b$,*

$$Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c,$$

where $Sq^a Sq^b$ denotes the composition of the Steenrod squares. The binomial coefficient in the formula is taken mod 2.

The Steenrod Algebra $\mathcal{A}$ is the free $\mathbb{Z}_2$-algebra generated by the symbols $\{Sq^i : i > 0\}$, modulo the Adem relations. We can make A into a graded algebra by declaring that Sq^i has degree i. Then for any space X, the graded abelian group

$$H^*(X; \mathbb{Z}_2) = \bigoplus_{n=0}^{\infty} H^n(X; \mathbb{Z}_2)$$

is a graded module over $\mathcal{A}$.

For example, taking $X = \mathbb{RP}^\infty$, we have $H^*(X; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$, where α has degree 1. Then $Sq^1 \alpha = \alpha^2$. Further, Cartan formula shows $Sq^i \alpha^n = \binom{n}{i} \alpha^{n+i}$.

For convenience, let $I = (i_1, i_2, \ldots, i_r, 0, 0, \ldots)$ be any sequence of nonnegative integers which is 0 except for finitely many terms. Then define

$$Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_r}.$$

Therefore the Sq^I generate $\mathcal{A}$. However, they do not form a basis. We still have the Adem relations.

By an inductive argument, we can use the Adem relations to take any monomial Sq^I and express it in terms of monomials Sq^J for which

$$J = (j_1, j_2, \ldots, j_r, 0, \ldots)$$

satisfies

$$j_1 \geq 2j_2, \quad j_2 \geq 2j_3, \quad \ldots$$

Call such a sequence J admissible. So $\{Sq^I : I \text{ admissible}\}$ generate $\mathcal{A}$. The linearly independence come from the universal example $K(\mathbb{Z}_2, 1) \cong \mathbb{RP}^\infty$.

If I is admissible, define the excess of I to be

$$e(I) = \sum_k (i_k - 2i_{k+1}) = 2i_1 - \sum_k i_k.$$

Definition 11. *A graded ring R over $\mathbb{Z}_2$ is said to have the ordered set $x_1, x_2, \ldots$ as a simple system of generators if the monomials*

$$\{x_{i_1}, x_{i_2}, \ldots, x_{i_r} : i_1 < i_2 < \cdots < i_r\}$$

form a $\mathbb{Z}_2$-basis for R.

Examples of rings with a simple system of generators include exterior algebras and the locally finite graded polynomial ring $\mathbb{Z}_2[x_1, x_2, \ldots]$; in the latter case, the $\{x_i^{2^k}\}$ form a simple system of generators.

In the cohomology Serre spectral sequence of a fibration $F \to X \to B$, the differential

$$d_r : E_r^{0,r-1} \to E_r^{r,0}$$

from the left edge to the bottom edge is called the transgression τ. This has domain a subgroup of $H^{r-1}(F)$, the elements on which the previous differentials $d_2, \ldots, d_{r-1}$ are zero. Such elements are called transgressive. The target group of τ is the quotient of $H^r(B)$ obtained by factoring out the images of $d_2, \ldots, d_{r-1}$.

τ coincides with an operator defined without using spectral sequence. (See 6.2 of [3]). The following is due to naturality.

Proposition 19. *If $x \in H^*(F; \mathbb{Z}_2)$ is transgressive then so is $Sq^i(x)$, and $\tau(Sq^i i(x)) = Sq^i(\tau(x))$.*

Theorem 20 (Borel)**.** *Let $(E, p, B; F)$ be a fibre space with E acyclic, and suppose $H^*(F; \mathbb{Z}_2)$ has a simple system $\{\alpha_k\}$ of transgressive generators. Then $H^*(B; \mathbb{Z}_2)$ is the polynomial ring in $\{\tau(\alpha_k)\}$.*

Corollary 21. *$H^*(K(\mathbb{Z}_2, q); \mathbb{Z}_2)$ is the polynomial ring over $\mathbb{Z}_2$ with generators $\{Sq^i(\iota_q)\}$ where i runs through all admissible sequences of excess less than q.*

Proof. The case $q = 1$ is valid. With path-loop fibration, we get $K(\mathbb{Z}_2, q) \to * \to K(\mathbb{Z}_2, q+1)$. By induction, $K(\mathbb{Z}_2, q)$ has a simple system of generators $\{(Sq^I \iota_q)^{2^k} : e(I) < q\} = \{(Sq^I \iota_q) : e(I) \leq q\}$ (by checking Adem relation, the equality is valid). Since $\tau(\iota_q) = \iota_{q+1}$, it follows immediately from the above that $\{(Sq^I \iota_{q+1}) : e(I) < q+1\}$ is the polynomial generator of $K(\mathbb{Z}_2, q+1)$. $\square$

Essentially the same argument shows that

Proposition 22. *For $q \geq 2$, $H^*(K(\mathbb{Z}, q); \mathbb{Z}_2)$ is the polynomial ring over $\mathbb{Z}_2$ with generators $\{Sq^i(\iota_q)\}$ where I runs through all admissible sequences of excess less than q in which the last term i_r is not 1.*

once noticing that the initial case is $K(\mathbb{Z}, 2) \cong \mathbb{CP}^\infty$ with cohomology ring $\mathbb{Z}_2[\iota_2]$ and $Sq^1 \iota_2 = 0$.

Sometimes, we are interested in integral coefficient and Bockstein homomorphism gives us insight.

The Bockstein homomorphism β can be factor as $\rho \circ \tilde{\beta}$ where ρ is the map on cohomology induced by $\mathbb{Z} \to \mathbb{Z}_2$ and $\tilde{\beta}$ is the connecting homomorphism associated with the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. We have $\tilde{\beta} \circ \rho = 0$, thus $\beta \circ \beta = 0$. So $H^*(X; \mathbb{Z}_2)$ is a chain complex and the Bockstein cohomology groups $BH^n(X; \mathbb{Z}_2) := \ker \beta / \mathrm{im} \beta$ is defined. See [6] for the following proposition.

Proposition 23. *If $H_n(X; \mathbb{Z})$ is finitely generated for all n, then the Bockstein cohomology groups $BH^n(X; \mathbb{Z}_p)$ are determined by the following rules:*

1. *Each $\mathbb{Z}$ summand of $H^n(X;\mathbb{Z})$ contributes a $\mathbb{Z}_p$ summand to $BH^n(X;\mathbb{Z}_p)$.*
2. *Each $\mathbb{Z}_{p^k}$ summand of $H^n(X;\mathbb{Z})$ with $k > 1$ contributes $\mathbb{Z}_p$ summands to both $BH^{n-1}(X;\mathbb{Z}_p)$ and $BH^n(X;\mathbb{Z}_p)$.*
3. *A $\mathbb{Z}_p$ summand of $H^n(X;\mathbb{Z})$ gives $\mathbb{Z}_p$ summands of $H^{n-1}(X;\mathbb{Z}_p)$ and $H^n(X;\mathbb{Z}_p)$ with β an isomorphism between these two summands, hence there is no contribution to $BH^*(X;\mathbb{Z}_p)$.*

Proposition 24. *In the situation of the preceding proposition, $H^*(X;\mathbb{Z})$ contains no elements of order p^2 if and only if the dimension of $BH^n(X;\mathbb{Z}_p)$ as a vector space over $\mathbb{Z}_p$ equals the rank of $H^n(X;\mathbb{Z})$ for all n. In this case, $\rho : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}_p)$ is injective on the p-torsion, and the image of this p-torsion under ρ is equal to $Im\,\beta$.*

Then we can recover some information for the mod p cohomology. For example, $H^*(K(\mathbb{Z},7);\mathbb{Z}_2)$ is spanned by $\{1, \iota_7, Sq^2\iota_7, Sq^3\iota_7, Sq^4\iota_7, Sq^5\iota_7, Sq^6\iota_7, Sq^{4,2}\iota_7\}$ in degree no large 13. $\beta = Sq^1$ sends $Sq^2\iota_7$ to $Sq^3\iota_7$, $Sq^4\iota_7$ to $Sq^5\iota_7$, $Sq^6\iota_7$ to $Sq^7\iota_7 = \iota_7^2$ and $Sq^{4,2}\iota_7$ to $Sq^{5,2}\iota_7 \neq 0$. The Bockstein cohomology vanish form degree 8 to 13. From the proposition above, we can get $H^*(K(\mathbb{Z},7);\mathbb{Z})$ has a copy of $\mathbb{Z}$ at degree 7, and $\mathbb{Z}_2$ at degree 10, 12 each. (mod odd torsion)

6 Homotopy group of spheres

Recall the Freudenthal suspension theorem

Theorem 25. *Let X be an n-connected pointed space, the map*

$$\pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k \leq 2n$ and epimorphism if $k = 2n+1$.

Corollary 26. *$\pi_{n+k}(S^n)$ stabilizes when $n \geq k+2$ denoted by π_k^s.*

The suspension theorem gives $\pi_0^s \cong \mathbb{Z}$.

6.1 Finiteness of Higher Homotopy Groups of Spheres

First we give a general construction to study higher homotopy groups of a space.

Let X be a path-connected space, we define a sequence by recursion. $X_0 = X, T_1 = \tilde{X_0}$, the universal cover of X_0 and $X_1 = \Omega T_1$. Let $T_2 = \tilde{X_1}$ and $X_2 = \Omega T_2$. By recursion, we define $T_n = \tilde{X}_{n-1}$ and $X_n = \Omega T_n$.

To make such construction valid, we need a technical requirement on point set topology.

Definition 12. *A space Y is ULC (Uniformly Locally Contractible) if there is a neighborhood U of the diagonal in $Y \times Y$ and a homotopy $F : U \times I \to Y$ such that $F(x,x,t) = x$ for all $x \in Y$ and $t \in I$; and $F(x,y,0) = x$, $F(x,y,1) = y$ for all $(x,y) \in U$.*

It's not a strong restriction. All CW-complexes are ULC. If Y is ULC, then $\tilde{Y}$ exists, $\tilde{Y}$ is ULC and ΩY is ULC.

For all $i > 0$, we have

$$\pi_i(X_n) = \pi_i(\Omega T_n) \cong \pi_{i+1}(T_n) = \pi_{i+1}(\tilde{X}_{n-1}) \cong \pi_{i+1}(X_{n-1})$$

Then

$$\pi_i(X_n) \cong \pi_{i+1}(X_{n-1}) \cong \cdots \cong \pi_{i+n}(X_0) = \pi_{i+n}(X)$$

Theorem 27. *If X is ULC, of finite type, connected and simply-connected, then $\pi_i(X)$ is finitely-generated for all i.*

Proof. We have Cartan-Leray spectral sequence, converging to $H_*(X; A)$ for A, an abelian group, and for which

$$E^2_{p,q} \cong H_p(\pi_1(X), H_q(\tilde{X}; A)),$$

where we are using the homology of the group $\pi_1(X)$ with coefficients in the $\pi_1(X)$-module $H_*(\tilde{X}; A)$.

We proceed by induction. For $X_0 = X$, since X is simply-connected, $\tilde{X} = X = T_1$ and so $X_1 = \Omega T_1$ is of finite type by theorem 10. By induction we suppose that X_{n-1} is of finite type and consider $T_n = \tilde{X}_{n-1}$. The abelian group $\pi_1(X_{n-1})$ acts trivially on $H_*(T_n)$ because X_{n-1} is an H-space. The E_2-term of the Cartan-Leray spectral sequence for the covering $T_n \to X_{n-1}$ simplifies for a trivial action:

$$\begin{aligned} E^2_{p,q} &\cong H_p(\pi_1(X_{n-1}), H_q(T_n)) \\ &\cong H_p(\pi_1(X_{n-1})) \otimes H_q(T_n) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_p(\pi_1(X_{n-1})), H_q(T_n)) \end{aligned}$$

By induction, $\pi_1(X_{n-1}) \cong H_1(X_{n-1})$ is finitely generated, from which it follows that the homology groups of the group $\pi_1(X_{n-1})$ with coefficients in the trivial module $\mathbb{Z}$, $H_i(\pi_1(X_{n-1}))$, are finitely generated (structure theorem for finitely generated Abelian group and Kunneth formula). Similar argument as Example 3 showed that $E^2_{0,q} \cong H_q(T_n)$ is finitely generated for all q. And so is $X_n = \Omega T_n$.

We have proved that all X_n are of finite type, then $\pi_n(X) \cong \pi_1(X_{n-1}) \cong H_1(X_{n-1})$ is finitely generated. □

Theorem 28. *Let k be a field, and X be a ULC space such that $\pi_0(X) = \pi_1(X) = 0$, and the groups $H_i(X, k)$ be finitely generated for all i. If $H_i(X, k) = 0$ for $0 < i < n$, then $\pi_i(X) \otimes k = 0$ for $0 < i < n$, and $\pi_n(X) \otimes k = H_n(X, k)$.*

Proof. We establish the following fact for the sequence of spaces $\{X_i\}$.
Fact: With X as in the theorem above, $H_i(X_j; k) = \{0\}$ if $i > 0$ and $i + j < n$ and $H_i(X_j; k) = H_n(X; k)$, if $i + j = n$.

proof of fact. If $j = 0$, the fact follows from the identification $X_0 = X$. Suppose it is true for $0 \leq m \leq j - 1$. For $j \geq 2$, X_{j-1} is a loop space and the abelian group $\pi_1(X_{j-1})$ acts trivially on $H_*(T_j;k)$. The Cartan-Leray spectral sequence takes the form

$$E^2_{p,q} \cong H_p(\pi_1(X_{j-1})) \otimes H_q(T_j;k) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(\pi_1(X_{j-1})), H_q(T_j;k)).$$

The exact sequence associated to the lower left hand corner of the E^2-term gives

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \to E^2_{0,1} \to H_1(X_{j-1};k) \to E^2_{1,0} \to 0.$$

By induction, $H_1(X_{j-1};k) = \{0\}$ and this implies that

$$E^2_{0,1} \cong H_1(\pi_1(X_{j-1})) \otimes H_0(T_j;k) = \{0\}.$$

Since $\pi_1(X_{j-1})$ is finitely generated and abelian, it follows that $H_1(\pi_1(X_{j-1})) \cong \pi_1(X_{j-1})$ is finitely generated. Finally, $H_0(T_j;k) \cong k$ implies that $\pi_1(X_{j-1})$ is finite whose order is relatively prime to the characteristic of k. So $H_p(\pi_1(X_{j-1}))$ is finite for all $p > 0$ and can be annihilated by $|\pi_1(X_{j-1})|$, so when tensored with a k-vector space, gives $E^2_{p,q} = \{0\}$ for $p > 0$. The Cartan-Leray spectral sequence collapses to the leftmost column and, by convergence, we have that $H_i(T_j;k) \cong H_i(X_{j-1};k)$ for all $i > 0$.

for $i < 2(n-j+1)-2$, we have $H_i(X_j;k) = H_i(\Omega T_j;k) = H_{i+1}(T_j;k) \cong H_{i+1}(X_{j-1};k)$. By the induction hypothesis, we have that $H_i(X_j;k) = \{0\}$ for $i > 0$ and $i + j < n$ and $H_{n-j}(X_j;k) \cong H_n(X;k)$. $\square$

From the fact we prove the theorem. Consider $H_1(X_j) \cong \pi_1(X_j) \cong \pi_{j+1}(X)$. For $j < n-1, j+1 < n$ and so $H_1(X_j;k) = \{0\}$. By the Universal Coefficient theorem, it follows that $\{0\} = H_1(X_j) \otimes k \cong \pi_{j+1}(X) \otimes k$. For $j = n-1$, $\pi_n(X) \otimes k \cong \pi_1(X_{n-1}) \otimes k \cong H_1(X_{n-1};k) \cong H_n(X;k)$. $\square$

Theorem 29. *$\pi_i(S^n)$ is finite for odd n and $i > n$.*

Proof. We may assume $n > 1$, which will make all base spaces in the proof simply connected, so that Serre spectral sequences apply.

Consider the Whitehead Tower of S^n, we take X_n alone. Recall that X_n is n-connected and $\pi_i(X_n) \cong \pi_i(S^n)$ for $i > n$. $X_n \to S^n$ is a fibration and the fiber is $K(\mathbb{Z}, n-1)$ (by homotopy exact sequence). Then apply Serre spectral sequence, $E_2^{*,*} \cong \Lambda(x) \otimes \mathbb{Q}(a)$, where deg $x = (n,0)$, deg $a = (0, n-1)$.

$$\begin{array}{|cccccc}
\mathbb{Q}a^2 & 0 & 0 & 0 & \mathbb{Q}x \cdot a^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Q}a & 0 & 0 & 0 & \mathbb{Q}x \cdot a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Q} & 0 & 0 & 0 & \mathbb{Q}x & 0 \\
\hline
\end{array}$$

The first possible nontrivial differential $d_n : \mathbb{Q}a \to \mathbb{Q}x$ must be an isomorphism, otherwise it would be zero and the term $\mathbb{Q}a$ would survive to E_∞, contradicting the fact that X is $(n-1)$-connected. The differentials $\mathbb{Q}a^i \to \mathbb{Q}a^{i-1}x$

must then be isomorphisms as well, so we conclude that

$$H^i(X; \mathbb{Q}) = 0, i > 0$$

The same is therefore true for homology, and thus $\pi_i(X)$ is finite for all i, hence also $\pi_i(S^n)$ for $i > n$. $\square$

Moreover, we have

Theorem 30. *Let $p \geq 3$ be a prime, and $\mathbb{F}_p$ the finite field contain p elements. If $m \geq 3$ is odd, then*

$$\begin{cases} \pi_i(S^m) \otimes \mathbb{F}_p = 0 & \textit{for } m < i < m + 2p - 3, \\ \pi_i(S^m) \otimes \mathbb{F}_p = \mathbb{F}_p & \textit{for } i = m + 2p - 3. \end{cases}$$

The essential idea is still the generalized Hurewicz theorem applying to the space sequence $\{X_j\}, \{T_j\}$ constructed from X. First, we present the following lemma which is useful in determining the homology of such spaces.

In the following, coefficient ring is considered to be a field with characteristic p without mentioning.

Lemma 31. *Let $q \geq 3$ be odd, and $H^i(X) = H^i(S^q)$ for $i \leq p(q-1)+1$, then the subspace of $H^*(\Omega X)$ composed by elements of degree $\leq p(q-1)$ has a homogeneous basis composed by elements*

$$\{1, y, y^2, \ldots, y^{p-1}, z\},$$

where $\deg y = q - 1$, $\deg z = p(q-1)$, $y^p = 0$.

Proof. In E_2 page of the spectral sequence of path loop fibration, all only non-trivial elements with base degree no larger than $p(q-1)+1$ appear as elements in $H^*(\Omega)$ or $x \otimes H^*(\Omega X)$ where x is a generator of $H^q(X)$. By dimension reason, all differentials sourcing from $H^*(\Omega X)$ with degree no larger than $p(q-1)$, except possibly d_q, are trivial. Since the term E_∞ should be zero in all strictly positive dimensions, for $0 < i \leq p(q-1)$, the differential d_q defines an isomorphism as in the Wang exact sequence

$$\theta : H^i(\Omega) \to H^{i-q-1}(\Omega).$$

then $H^i(\Omega) = 0$ for $i \neq 0 \mod (q-1)$ and $i \leq p(q-1)$ and $H^i(\Omega) = k$ for $i = 0 \mod (q-1)$. Since q is odd, this isomorphism is an ordinary derivation. Let $y \in H^q(\Omega)$, $z \in H^{p(q-1)}(\Omega)$ be nonzero elements.

$$\theta(y^j) = j \cdot y^{j-1} \cdot \theta(y), \text{ and hence, } y^j \neq 0 \text{ for } j < p, \text{ and } y^p = 0.$$

$\square$

Lemma 32. *Let $m \geq 2$ be even. If the subspace of $H^*(X)$ composed by elements of degree $\leq mp$ has a basis composed by homogeneous elements*

$$\{1, y, y^2, \ldots, y^{p-1}, z\},$$

where $\deg y = m$, $\deg z = pm$, and $y^p = 0$, then the subspace of $H^*(\Omega X)$ composed by elements of degree $\leq mp-2$ has a basis composed by homogeneous elements $\{1, v, t\}$, where $\deg v = m-1$, and $\deg t = mp-2$.

Proof. Similarly, by dimension reason, $H^i(\Omega X) = 0$ for $0 < i < m-1$, and $H^{m-1}(\Omega X)$ is generated by one element v such that $d_m v = y$.

Hence, the elements of $E_m \cong E_2$, which have the fiber degree $\leq m-1$, and the total degree $\leq mp-1$ are spanned by the following homogeneous basis:

$$\{1, y, y^2, \ldots, y^{p-1}, v, y \otimes v, y^2 \otimes v, \ldots, y^{p-1} \otimes v\},$$

The action of this differential is described by the formula:

$$d_m y^k = 0 \quad d_m(y^k \otimes v) = y^{k+1}.$$

It follows that in E_{m+1} page, elements of fiber degree $\leq m-1$ and total degree $\leq mp-1$ are generated by 1 and $y^{p-1} \otimes v$.

Suppose $t \in H^*(\Omega X)$ is of the minimal degree except v. We must have $\deg t \geq 2m-2$ otherwise it will survive to E_∞. If $p = 2$, such t must exist and generate $H^{2(m-2)}(\Omega X)$ to kill $y \otimes v$. If $p > 2$, and if $\deg t < mp-2$, the only possible nontrivial differential souring from t is d_m. So we must have $\deg t = 2m-2$. Suppose $d_m t = ky \otimes v$, $0 = d_m^2 t = ky^2$, $y^2 \neq 0$, so $k = 0$. But then, t will survive to E_∞. So $\deg t \geq mp-2$. And indeed, we do need an generator in $H^{mp-2}(\Omega X)$ to kill $y^{p-1} \otimes v$. $\square$

Lemma 33. *Let $X = S^{2n+1}$, $n \geq 1$, X_i the associated space. The cohomology algebras $H^*(X_{2i-1})$ and $H^*(X_{2i})$ in dimensions $1 \leq i \leq n$ have the following homogeneous bases:*

$$H^*(X_{2i-1}) : \textit{basis } \{1, x, x^2, \ldots, x^{p-1}, y\} \textit{ in dimensions } \leq p(2n-2i+2),$$

where $\deg x = 2n-2i+2$, $\deg y = p(2n-2i+2)$, $x^p = 0$.

$$H^*(X_{2i}) : \textit{basis } \{1, v, t\} \textit{ (in dimensions } \leq p(2n-2i+2)-2),$$

where $\deg v = 2n-2i+1$, and $\deg t = p(2n-2i+2)-2$.

Proof. For $i = 1$, the lemma follows immediately from the following lemma 31 32 and the 2-connectivity of X, so we argue by induction on i for $2 \leq i \leq n$.

First, we have to determine the algebra $H^*(X_{2i-1})$ in dimensions $\leq p(2n-2i+2)$. Since $\pi_1(X_{2i-2}) = \pi_{2i-1}(S^{2n+1}) = 0$, $X_{2i-1} = \Omega T_{2i-1} = \Omega X_{2i-2}$ by definition. Here, we can apply Lemma 31 in the case that base space is X_{2i-2} and $q = 2n-2i+3$.

According to the inductive hypothesis, we have $H^*(X_{2i-2}) = H^*(S^q)$ for $\leq p(2n-2i+4)-3$, and indeed $p(2n-2i+4)-3 \geq p(q-1)+1$.

From $H^*(X_{2i-1})$ to determine $H^*(X_{2i})$ is much more straightforward. Just take $m = 2n-2i+2$. □

Now we can give a proof for theorem 30.

proof of theorem 30. Let $m = 2n+1$, according to the conventions of this section. As it was shown in Lemma 6, the cohomology groups of the space X_{2n}, with coefficients in $\mathbb{F}_p$, have the form:

$$H^0(X_{2n}) = H^1(X_{2n}) = H^{2p-2}(X_{2n}) = \mathbb{F}_p,$$
$$H^i(X_{2n}) = 0 \text{ for } 1 < i < 2p-2.$$

The fundamental group of X_{2n} is $\pi_{2n+1}(S_{n+1}) = \mathbb{Z}$ acting trivially on its universal covering T_{2n+1}. Cartan-Leray spectral sequence gives

$$H^0(T_{2n+1}) = H^{2p-2}(T_{2n+1}) = \mathbb{F}_p, \quad H^i(T_{2n+1}) = 0 \text{ for } 0 < i < 2p-2,$$

And we get from the general Hurewicz theorem that

$$\pi_i(T_{2n+1}) \otimes \mathbb{F}_p = 0 \text{ for } 0 < i < 2p-2, \quad \text{and } \pi_{2p-2}(T_{2n+1}) \otimes \mathbb{F}_p = \mathbb{F}_p.$$

Since $\pi_i(T_{2n+1}) = \pi_i(X_{2n}) = \pi_{i+2n}(S^{2n+1})$ for $i \geq 2$, the theorem is proved. □

For the even dimensional cases, we need an auxiliary space. The space W_{2m-1} is defined to be the sphere bundle of the tangent bundle of S^m (i.e. the unit vectors tangent to S^m) with fiber S^{m-1}. Then the cohomology of W_{2m-1} can be derived from the Gysin sequence from the view of Euler class view.

$$0 \to H^{m-1}(W_{2m-1}) \to H^0(S^m) \xrightarrow{\smile e} H^m(S^m) \to H^m(W_{2m-1}) \to 0$$

where e is 2 times of generator when m is even. This gives $H^m(W_{2m-1}) \cong \mathbb{Z}_2$, $H^{m-1}(W_{2m-1}) = 0$. Other dimension is easily derived from the spectral sequence. $H^0(W_{2m-1}) \cong \mathbb{Z}$, $H^{2m-1}(W_{2m-1}) \cong \mathbb{Z}$ and all others are 0.

Theorem 34. *If $m \geq 2$ is even, then the homotopy groups $\pi_i(W_{2m-1})$ are finite for all i, except $\pi_{2m-1}(W_{2m-1})$, which is isomorphic to the direct sum of $\mathbb{Z}$ and a finite group, whose order is a power of 2. Moreover, for any odd prime p:*

$$\begin{cases} \pi_i(W_{2m-1}) \otimes \mathbb{F}_p = 0 & \text{for } 0 \leq i < 2m-1 \text{ and } 2m-1 < i < 2m+2p-4, \\ \pi_i(W_{2m-1}) \otimes \mathbb{F}_p = \mathbb{F}_p & \text{for } i = 2m+2p-4 \end{cases}$$

Proof. When $m = 2$, notice that W_3 consists of all pairs ordered orthonormal vectors in $\mathbb{R}^3$ which can be identified with $SO(3)$, admitting S^3 as universal cover and sharing the same higher homotopy group. And this case is proved in earlier ($\pi_1(W_3) \cong \mathbb{Z}_2$ which has trivial p-component indeed).

When $m \geq 4$ is even. W_{2m-1} is simply connected. The general Hurewicz theorem gives the result for $0 \leq i \leq 2m-1$. Notice that for a field k whose characteristic is not 2, W_{2m-1} has the same cohomology k-algebra as S^{2m-1}. So the exactly same argument applied to odd dimension sphere works and the full result follows. $\square$

In particular, the order of $\pi_i(W_{2m-1})$ is a power of 2 for $i < 2m-1, i = 2m, i = 2m+1$.

Considering the homotopy exact sequence of fibration $S^{m-1} \to W_{2m-1} \to S^m$, we get the following.

Corollary 35. *For even m, $\pi_i(S^m)$ is finite for $i > m$ except for $i = 2m-1$. $\pi_{2m-1}(S^m)$ is isomorphic to $\mathbb{Z}$ plus a finite group.*

6.2 Low Dimension Computation

Theorem 36. $\pi_4(S^3) \cong \mathbb{Z}_2$. $\pi_5(S^3) \cong \mathbb{Z}_2$

Proof. Take the Whitehead tower for S^3:

$$\begin{array}{ccc} K(\pi_4(S^3),3) & \to & X_4 \\ & & \downarrow \\ K(\mathbb{Z},2) & \to & X_3 \\ & & \downarrow \\ & & S^3 \end{array}$$

Serre spectral sequence of the lower fibration in cohomological type can be easily computed. We get $H^{2k+1}(X_3) = \mathbb{Z}_k$ for $k \geq 2$ and trivial elsewhere. X_3 is 3-connected, $\pi_4(S^3) \cong \pi_4(X_3) \cong H_4(X_3) \cong \mathbb{Z}_2$.

To compute $H^*(K(\mathbb{Z}_2,3);\mathbb{Z})$, we have $H^*(K(\mathbb{Z}_2,3);\mathbb{Z}_2)$ has

$$1, \iota_3, Sq^1\iota_3, Sq^2\iota_3, Sq^{2,1}\iota_3, \iota_3^2$$

as basis under degree 6. Bockstein argument shows that $H^i(K(\mathbb{Z}_2,3)) = 0$ for $i = 1,2,3,5$ and $\mathbb{Z}_2$ for $i = 4,6$. We don't need mod odd torsion here because we know $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}))$ can not have odd torsion. (It's true for $n = 1$, and it follows from induction using path-loop fibration)

Then we compute the cohomology of X_4 from the upper fibration. The first nontrivial differential is $d_5 : H^4(K(\mathbb{Z}_2,3)) \cong \mathbb{Z}_2 \to H^5(X_3) \cong \mathbb{Z}_2$ must be isomorphism since X_4 is 4-connected. The next possible nontrivial differential $d_7 : H^6(K(\mathbb{Z}_2,3)) \cong \mathbb{Z}_2 \to H^7(X_3) \cong \mathbb{Z}_3$ is actually 0. So $H^6(X_4) \cong \mathbb{Z}_2$ and all lower degree is trivial. Then $\pi_5(S^3) \cong \pi_5(X_4) \cong H_5(X_4) \cong \mathbb{Z}_2$. $\square$

The following lemma is directly from the detailed computation of Serre spectral sequence for $\mathbb{CP}^\infty \cong K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$

Lemma 37. *For $i \leq 9$,*

$$H^i(K(\mathbb{Z},3);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=3 \\ \mathbb{Z}_2 & i=6,9 \\ \mathbb{Z}_3 & i=8 \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 38. $\pi_6(S^4) \cong \mathbb{Z}_2$

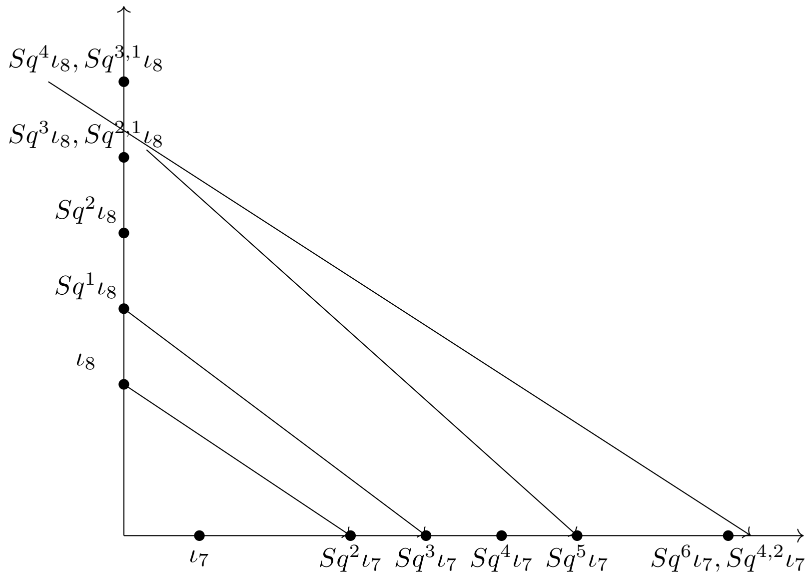
Proof. Similarly, taking the Whitehead tower for S^4.

$$\begin{array}{ccc} K(\mathbb{Z}_2,4) & \to & X_5 \\ & & \downarrow \\ K(\mathbb{Z},3) & \to & X_4 \\ & & \downarrow \\ & & S^4 \end{array}$$

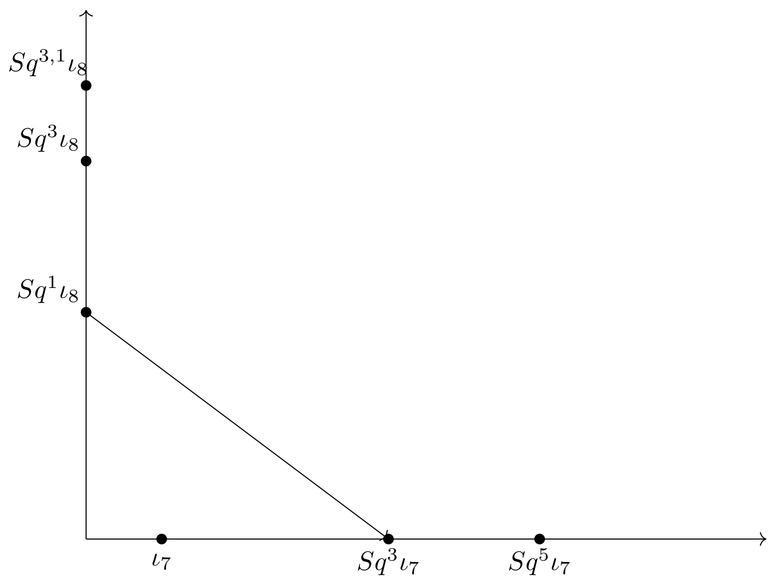
From the lemma, it's easy to compute the first several cohomology of X_4. It's $\mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}_3$ on degree 6,7 and 8 with trivial lower cohomology. $H^*(K(\mathbb{Z}_2,4))$ has a copy of $\mathbb{Z}_2$ on degree 5 and 7 each. (can be sen directly from the path-loop fibration) Then we can compute the first nontrivial cohomology appears at degree 7 which is $\mathbb{Z} \oplus \mathbb{Z}_2$. $\pi_6(S^4) \cong \pi_6(X_5) \cong H_6(X_5) \cong \mathbb{Z}_2$. $\square$

Theorem 39. $\pi_3^s \cong \mathbb{Z}_8$ *(mod odd torsion)*

Proof. Let's compute $\pi_{10}(S^7)$ using Postnikov tower of S^7. The virtue of handling higher dimension is the E_2 page won't produce cross term in lower dimension. The first fibration is $K(\mathbb{Z}_2,8) \to X_8 \to K(\mathbb{Z},7)$. Notice that X_8 can be realized by attaching cells of dimension 10 or higher to S^7. (Cells of each dimension can be taken to be finite due to finiteness of higher homotopy group of S^7) Then we have $H^8(X_8;\mathbb{Z}) = H^9(X_8;\mathbb{Z}) = 0$. So the map $\tau : \mathbb{Z}_2\iota_8 \to \mathbb{Z}_2 Sq^2\iota_7$ must be isomorphism. The commutativity of transgression τ and Sq^i gives the following solid arrows implying an isomorphism. The following shows the E_2 page in the case with coefficient $\mathbb{Z}_2$.



Argument using Bockstein cohomology gives the integral coefficient(mod odd torsion).

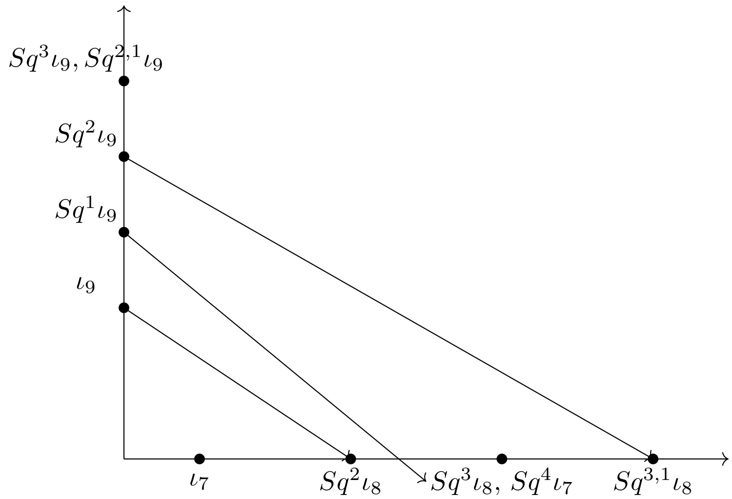


So we have $H^{12}(X_8;\mathbb{Z})$ has order 4 and $H^{12}(X_8;\mathbb{Z}_2) \cong \mathbb{Z}_2$, thus $H^{12}(X_8;\mathbb{Z}) \cong \mathbb{Z}_4$. The fact that E_∞ corresponds to a filtration of $H^{12}(X_8;\mathbb{Z})$ gives a short exact sequence.

$$0 \to H^{12}(K(\mathbb{Z}_2, 8);\mathbb{Z}) \to H^{12}(X_8;\mathbb{Z}) \to H^{12}(K(\mathbb{Z}, 7);\mathbb{Z}) \to 0$$

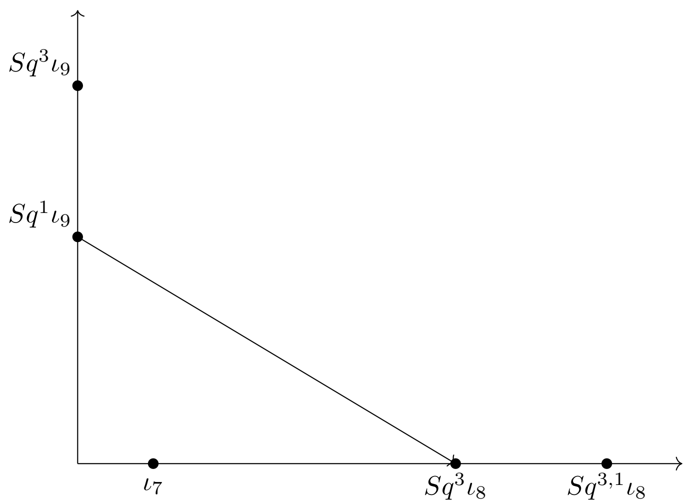
We can see that $Sq^{3,1}\iota_8$ corresponds to a generator of $\mathbb{Z}_4$.

The fibration $K(\mathbb{Z}_2, 9) \to X_9 \to X_8$ gives



The reason for the leftdown arrow being solid is the same as above. And the rest arrows follow.

With integral coefficient,



It follows that $H^{12}(X_9; \mathbb{Z})$ has order 8 and $H^{12}(X_9; \mathbb{Z}_2) \cong \mathbb{Z}_2$. So $H^{12}(X_9; \mathbb{Z}) \cong \mathbb{Z}_8$.

Notice that for finite abelian group π, $K(\pi, n)$ has its first nontrivial cohomology group at degree $n+1$ with group π. Considering $K(\pi_{10}(S^7), 10) \to X_{10} \to X_9$, X_{10} can be realized by attaching 12 or higher cells on S^7, so they share the same homology in degree no large than 11. $H^{12}(X_{10}; \mathbb{Z}) \cong H^{12}(X_{10}, S^7; \mathbb{Z})$ which is a direct sum of some $\mathbb{Z}$. This forces $H^{11}(K(\pi_{10}(S^7)); \mathbb{Z}) \cong H^{12}(X_9; \mathbb{Z}) \cong \mathbb{Z}_8$. (The map is injective due to $H^{11}(X_{10}; \mathbb{Z}) = 0$ and surjective since there is no torsion in $H^{12}(X_{10}; \mathbb{Z})$)

□

References

[1] [J.P. Serre] *Singular Homology of Fibered Spaces.*

[2] [J.P. Serre] *Cohomology Mod 2 of Eilenberg–MacLane Complexes.*

[3] [J. McCleary] *A User's Guide to Spectral Sequences*

[4] [R, Mosher and M, Tangora] *Cohomological Operation and Applications in Homotopy Theory*

[5] [C. Weibel] *An Introduction to Homological Algebra*

[6] [A. Hatcher] *Algebraic Topology*