Strategies in Lowest Unique Bid Auctions

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Abstract

Lowest Unique Bid Auctions are a kind of games in which the agent with the lowest unique bid wins. In this research, I studied the behaviors of agents in the LUBA game. First, I mathematically defined the game and then proposed several models using different decision rules. I presented the models' simulations with comparison to recorded behaviors from real-life games. I analyzed the discrepancy between my models and the data, and proposed potential solutions to this problem. Lastly, I discussed the potential applications of my research in other games with similar structures and in the research at industrial organization.

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Contents

1 Introduction

1.1 Motivation

As the internet continues to expand, online auctions have begun to draw public attention. With the soaring investments in advertising by those online auction sites,[1] online auctions have been more popular than ever. There are many kinds of online auctions that people enjoy, and the Lowest Unique Bid Auctions (LUBA) games are a type of online auctions that attract many participants.

In this kind of game, the agent who has the lowest unique bid wins the auction. Each agent is given the value of the auction item, the bidding fee of placing one bid in the game, the discrete range within which the agent can bid and the expiration time of the auction. When the game begins, every agent can place one bid at a time, and will receive different feedbacks on their bids according to the setting and rules they are playing with. For instance, in the UBH-LUBA games, hosted by an online auction site called Unique Bid Home, agents only know whether bids they submit are currently winning or not; in the VIP-LUBA games, hosted by another online auction site called VIP Auction, agents know whether their bids are currently winning, whether their bids are too high or whether their bids are not unique. When the auction ends, the agent with the uniquely lowest bid wins the game. If such agent does not exist, then none of them wins.

Figure 1 on the following page provides an example of a game with four possible bids. If three agents bid one, no agent bids two, one agent bids three and two agents bid four, the agent who bids three wins the game and can purchase the item at price of 3.

Unlike normal auctions, in which agents have their own reservation values on the auction items and the agent with the highest reservation value is likely to win, in LUBA, reservation values may not exist. Furthermore, in almost every LUBA game, the highest potential bid is significantly lower than the value of the auction item, so as long as the agent wins the auction as bidding uniquely at the highest amount, and all lower bids are not unique, the agent will still have a positive gain. Thus, it is possible that agents will still bid at high level. Agents, however, do have an incentive to bid at lower level because only the lowest unique bid wins.

The unique features of LUBA games raise several questions: if agents do not aim for their reser-

1. "Auction Site A Losing Bet for Consumers According to Ad Watchdog TINA.org" [in eng], *NASDAQ OMX's News Release Distribution Channel* (New York), 2017, `http://search.proquest.com/docview/1906072091/`.

Figure 1: 3 is the winning bid in this auction.

vation values, how do they play in LUBA? Will they have specific strategies? By the time they submit their bids, the public information is limited, so how do they make their decisions? Moreover, how will the anti-coordination property of LUBA affect agents' strategic behaviors? These questions are the motivation for this research.

1.2 Goals

In this research, I will, first, theoretically define the game. Then, I will propose models to simulate agents' behaviors in LUBA games. I will study how agents make decisions under the assumptions of my proposed models. I will also acquire real-life data of the LUBA and compare the data to my models as well as my simulation results to determine whether agents follow those strategies in reality.

1.3 Methodology

In this research, I will define LUBA games mathematically, and then propose quantitative models. Moreover, by using simulations with parameters from data, I will be able to compare those models with data from UBH-LUBA games collected by Radicchi (2012).[2]

Empirical analysis on agents' behavior from the data will not help because the size and other pa-

2. Filippo Radicchi, Andrea Baronchelli, and Luís A. N Amaral, "Rationality, Irrationality and Escalating Behavior in Lowest Unique Bid Auctions (Levy Flights in Online Auctions)" [in eng], *PLoS ONE* (San Francisco, USA) 7, no. 1 (2012), ISSN: 1932-6203.

rameters of each auction are different and there is no direct way to compare agents' behaviors in different games.

In each model I propose, I set up the formal description of the game within the model and try to find the optimal strategies for every agent. Each model will provide a mathematical description of agents' behavior. Then I can use MATLAB to create simulation data with given parameters.

After gathering the mathematical descriptions as well as the simulation data from models, I will propose some general predictions with arbitrary parameters. Then, I compare the simulation data with actual data.

This method I adapt has been widely used in the literature on this subject and can provide a deeper understanding of agents' behavior. In the data, there are 189 sets of auctions recorded with 163 sets of auctions that can be used. The other 26 sets of auctions are useless since they miss crucial details like the maximum bid allowed. In the models, that parameter is essential in determining the agents' strategy within the game.

Potential limitation of this method is that it is impossible for me to encounter every factor in this game. This is the classic dilemma between including all factors to create mathematical chaos and omitting factors that lead to model unfitness. Further details will be discussed in each model.

1.4 Results

I proposed three models in this research. The first model, the single-bid Nash equilibrium model, was inspired by current research undertaken by other scholars. The other two models, the multiple-bid Nash equilibrium model and the multiple-bid probability matching model were proposed, derived and simulated by myself with technical assistance from Professor Jonathan Touboul. I also proposed two-bid Nash equilibrium model and two-bid probability matching model as two special illustrative cases.

In the single-bid Nash equilibrium model, I followed the previous researchers' approach by restricting agents to placing only one bid. In this model, the equilibrium strategy is unique. The optimal strategy can be analytically represented as a recursive relation. By using MATLAB, I was able to numerically calculate agents' strategy profiles under the model's assumptions. Based on the strategy profiles, I was also able to display some features of this model, such as the monotonoicity of bidding probability and winning probability.

In the multiple-bid Nash equilibrium model, I allowed agents to place more than one bid in the game. This created a relatively more realistic environment. In this model, the equilibrium strategies are not unique. The total bidding probability at each bid, however, is unique. This is not surprising because at equilibrium, agents only care about the total bidding number at each bid. The multiple-bid Nash equilibrium model is a direct extension of the single-bid Nash equilibrium model, and by using MATLAB, I was able to calculate agents' bidding behaviors under the model's assumptions.

In the multiple-bid probability matching model, I used a new decision rule other than the Nash equilibrium. Probability matching is a psychological phenomenon that has a significant impact in decision making. I stumbled upon this decision rule when I was writing the simulation codes for the multiple-bid Nash equilibrium model, and using probability matching was a breakthrough in this research. In this setup, agents' bidding probability is proportional to the winning probability. Given the complexity of this model, I failed to present the solution analytically. However, after proving that the probability matching strategy is unique, I numerically calculated the bidding strategy with given parameters. This model offered a potential explanation on why agents submit large bids in real LUBA games. One of the largest drawbacks of this model, however, was that the simulations may be unsuccessful in a relatively large bid support due to the complexity of the model. Nevertheless, I used this model to show several statements that are helpful for further research on this topic.

1.5 Potential Applications

My research results can be directly applied to other games with similar structures of LUBA. The most related game structure is the Lowest Unique Positive Integer (LUPI) game, in which agents will choose a positive integer in a pre-specified range. The agent choosing the lowest unique integer wins the game. LUPI is usually regarded as the cousin of LUBA, since it uses the same structure, with provisions of only allowing agents to bid once, setting the fee to 0 and setting the auction value to infinity.

Another similar game is the Minority Game (El Farol bar problem),[3] in which 100 agents have 2 bars that they can go to. The capacity of either bar is 49. All agents are required to enter one of the bars.

3. D. Challet, M. Marsili, and Y.C. Zhang, *Minority Games*, Oxford finance (Oxford University Press, 2005), ISBN: 9780198566403.

None of them knows how many agents are currently in the bar and they have no prior communication with other agents. The ultimate goal is that if one bar has less than or equal to 49 agents inside, all agents in that bar will be satisfied; otherwise no one in that bar is happy. In other words, the Minority Game can be transformed as 100 agents placing bid on a bid support of 2, and agents receive a positive payoff if they end up in a bar that does not reach the capacity.

Furthermore, my research methods and results will have a further influence on the research on the innovations by firms. In my thesis defense, Professor Jean-Paul L'Huillier brought up the idea that my research may have applications in this. Each firm in a certain industry want to improve their productivities through innovations. It is possible to order their possible attempts of innovation by the costs in the R&D processes. Firms have incentive to be "unique", since they will have the property rights of such innovations. They also want to minimize the cost to maximize their profit. So, the innovation process by firms can be viewed as a LUBA game, and my models can be applied directly.

2 Literature Review

LUBA games with associated strategies have been a topic of important research activities for over a decade and they drew interest from many well-regarded researchers because LUBA is a perfect platform for researchers to study behaviors in anti-coordination games. Researchers from different disciplines, especially from Economics and Physics, have made promising progress.

First, Schmeidler (1973)[4] and Mas-Colell (1984)[5] recognized and developed equilibrium theory for anti-coordination games in a general framework independent of the LUBA game. They both derived conditions for the existence of equilibrium strategies by two different approaches. Schmeidler's approach[6] was based on measurable functions giving the mean of individual strategies. Mas-Colell's approach[7] was based on the distribution of different strategies among individuals composing the non-cooperative game. Both of their works should be considered as the cornerstones for future research.

4. David Schmeidler, "Equilibrium points of nonatomic games" [in eng], *Journal of Statistical Physics* (New York) 7, no. 4 (1973): 295–300, ISSN: 0022-4715.
5. Andreu Mas-Colell, "On a theorem of Schmeidler" [in eng], *Journal of Mathematical Economics* 13, no. 3 (1984): 201–206, ISSN: 0304-4068.
6. Schmeidler, "Equilibrium points of nonatomic games."
7. Mas-Colell, "On a theorem of Schmeidler."

For LUBA related research, Rapoport et al. (2009)[8] and Houba et al. (2011)[9] both reported general properties of the Nash equilibrium within LUBA. In pure strategies, LUBA possesses an asymmetric equilibrium: if one agent bids the minimum bid, other agents will deliberately avoid the minimum bid. Inversely, in mixed strategies, LUBA possesses a symmetric equilibrium: agents bid across the bidding support with decreasing bidding probabilities as the bid value increases. Both authors examined different structures for LUBA. Rapoport and collaborators[10] considered a LUBA structure where agents either bid once or stay out on each auction, whereas in Houba's work,[11] agents can submit multiple bids. They used different techniques: Rapoport's team developed an algorithm based on non-stationary Markov chain, whereas Houba's team used direct numerical methods to solve for equilibrium solutions. What was common in their work is the assumption that agents are anonymous and that the population size is known. My first model was influenced by those two, and these assumptions follow in my other models as well.

In LUBA, agents have limited information about the state of the game. This condition conveys uncertainties to the decision-making process. For instance, each agent makes an estimate without certain information such as the population size or the actions of other agents before making a bidding decision. Östling et al. (2011)[12] and Mohlin (2015)[13] focused on population uncertainty in LUBA by applying the Poisson Game theory reported in Myerson (1998).[14] In Poisson games, agents are uncertain about how many agents are participating in the game and assume that the population is Poisson distributed. In the above cited papers, the authors modeled the LUBA game as a large Poisson game and they reported the existence of a unique equilibrium solution with decreasing bidding probabilities. They also reported that in very large populations, the bidding distribution converges to a uniform distribution. The uniqueness of the work of Östling and collaborators[15] was that they tested their model with real data collected from the Swedish Lottery Game, which is a field version of LUBA. Östling and collaborators[16] reported a mismatch between predicted equilibrium and observed distributions, arguing that the real LUBA game do not converge to the equilibrium. In addition, Crawford et al. (2013)[17] reported that in games such as LUBA, the intrinsic

8. Amnon Rapoport et al., "Unique Bid Auction Games" [in eng], *IDEAS Working Paper Series from RePEc* (St. Louis), 2009, `http://search.proquest.com/docview/1698649736/`.

9. Harold Houba, Dinard Laan, and Dirk Veldhuizen, "Endogenous entry in lowest-unique sealed-bid auctions" [in eng], *Theory and Decision* (Boston) 71, no. 2 (2011): 269–295, ISSN: 0040-5833.

10. Rapoport et al., "Unique Bid Auction Games."

11. Houba, Laan, and Veldhuizen, "Endogenous entry in lowest-unique sealed-bid auctions."

12. Robert Östling et al., "Testing Game Theory in the Field: Swedish LUPI Lottery Games" [in eng], *American Economic Journal: Microeconomics* 3, no. 3 (2011): 1–33, ISSN: 1945-7669.

13. Erik Mohlin, Robert Östling, and Joseph Tao-Yi Wang, "Lowest unique bid auctions with population uncertainty" [in eng], *Economics Letters* 134, no. C (2015): 53–57, ISSN: 0165-1765.

14. Roger B. Myerson, "Population uncertainty and Poisson games" [in eng], *International Journal of Game Theory* (Berlin/Heidelberg) 27, no. 3 (1998): 375–392, ISSN: 0020-7276.

15. Östling et al., "Testing Game Theory in the Field: Swedish LUPI Lottery Games."

16. Ibid.

17. Vincent P Crawford, Miguel A Costa-Gomes, and Nagore Iriberri, "Structural Models of Nonequilibrium Strategic Think-

structure leads agents to adopt strategic decisions that deviate systematically from the equilibrium. From their work, Crawford and collaborators[18] concluded that Nash Equilibrium can be theoretically defined but empirically agents do not behave in equilibrium. This is one of the fundamental reasons that I observed inconsistency between my models' predictions and real data in my research.

In the meantime, other researchers have developed different mathematical models to understand the strategic thinking in LUBA. Zeng (2007),[19] Flitney (2008),[20] Baek & Bernhardsson (2010),[21] Pigolotti et al. (2012)[22] and Costa-Gomes & Shimoji (2014)[23] reported different polynomial or probabilistic models which allow them to conclude that agents optimize bidding strategies not according to Nash Equilibrium. Chen et al (2015),[24] Zhou et al (2015)[25] and Hu et al. (2017)[26] reported different multi-agent models based on microscopic non-linear interactions. For instance, Hu et al.[27] developed a model for UBH-LUBA in which each agent performs bidding optimizations linked to three psychological forces: one attractive force towards the minimum bid, one repulsive force towards low bid values and one random force reproducing fluctuations of behaviors. Other factors are not specified or modeled. They reported results in agreement with observed behaviors in LUBA. More precisely, they reported that real LUBA distributions follow an inverted-J curve with an exponential decay. Three comments on this literature are in order: first, these models are closer to real UBH-LUBA as each agent draws his own expectations about the state of the game and the population size; second, contrary to real UBH-LUBA, they assumed that the lowest unique bid is commonly known at all time allowing agents to behave having more information that in real-life; third, they assumed that agents have equal decision rules. This was unrealistic because in real UBH-LUBA, agents may have different expectations which lead to different behaviors. In other words, to advance in the modeling of the LUBA game by behavioral modes, population shall be assumed as being heterogeneous. In the Discussion section, I will discuss about how the heterogenity assumption will affect and may potentially help me further study

ing: Theory, Evidence, and Applications" [in eng], *Journal of Economic Literature* 51, no. 1 (2013): 5–62, ISSN: 0022-0515.

18. Crawford, Costa-Gomes, and Iriberri, "Structural Models of Nonequilibrium Strategic Thinking: Theory, Evidence, and Applications."

19. Q. Zeng, B.R. Davis, and D. Abbott, "Reverse auction: The lowest unique positive integer game," *Fluctuation and Noise Letters* 7, no. 4 (2007): L439–L447, ISSN: 02194775.

20. Adrian P. Flitney, "Comments on "Reverse auction: the lowest positive integer game"," 2008,

21. S.K. Baek and S. Bernhardsson, "Equilibrium solution to the lowest unique positive integer game," *Fluctuation and Noise Letters* 9, no. 1 (2010): 61–68, ISSN: 02194775.

22. Simone Pigolotti et al., "Equilibrium strategy and population-size effects in lowest unique bid auctions" [in eng], *Physical Review Letters* 108 (2012), ISSN: 0031-9007.

23. Miguel A. Costa-Gomes and Makoto Shimoji, "Theoretical approaches to lowest unique bid auctions" [in eng], *Journal of Mathematical Economics* 52 (2014): 16–24, ISSN: 0304-4068.

24. Qinghua Chen et al., "Multiagent model and mean field theory of complex auction dynamics" [in eng], *New Journal of Physics* 17, no. 9 (2015), ISSN: 1367-2630.

25. Cancan Zhou et al., "Smarter than Others? Conjectures in Lowest Unique Bid Auctions.(Research Article)" [in English], *PLoS ONE* 10, no. 4 (2015), ISSN: 1932-6203.

26. Rui Hu et al., "The Psychological Force Model for Lowest Unique Bid Auction" [in eng], *Computational Economics* (New York) 50, no. 4 (2017): 655–667, ISSN: 0927-7099.

27. Zhou et al., "Smarter than Others? Conjectures in Lowest Unique Bid Auctions.(Research Article)."

this topic.

In this research, I adopted usual approaches in game theory research, described in those standard textbooks that are widely use in undergraduate game theory courses: Fudenberg (1991),[28] Osborne (1994),[29] Dutta (1999),[30] Barron (2013),[31] Maschler (2013)[32] and Tadelis (2013).[33]

As for my own research contribution to this topic, I first began to consider the Nash equilibrium strategies of agents when they are restricted to placing only single bid, like other researchers had done. This strategy is unique. Several properties of this strategy from the literature were proved and verified in my studies. Then, I observed how the changes in parameters would affect the strategy. I moved forward by allowing agents to place more bids and proposed two other models using different decision rules. In the Nash equilibrium setup, I noticed that agents will always bid the highest amount of bids they are allowed. An interesting finding is that when I conditioned the bidding probability, I would recover the single-bid NE strategy. Next, I adopted the concept of probability matching from psychology. First, I focused on allowing agents to bid no more than two bids. In this setup, agents will have a unique strategy. Hence, I could numerically calculate the exact strategy with given parameters. The simulation method I used was different from all other models and had never been used in any literature. Lastly, I relaxed the restriction on the number of bids allowed for agents. I derived the generalized form of probability matching conditions and modified the simulation codes. I noticed that the previous simulation method was not efficient. Hence, I used a new method to bypass the efficiency problem. With the increase of the value of the maximum bid, unfortunately, the complexity of the system will increase exponentially. The algorithm failed at real parameters. Nevertheless, the new algorithm was able to help me simulate the desired result, which I present in this paper.

3 Building Intuition Using Simple Examples

In this part, I use examples to illustrate LUBA game and show how I can analyze this game theoretically.

28. Drew Fudenberg, *Game theory* [in eng] (Cambridge, Mass.: MIT Press, 1991), ISBN: 0262061414.
29. Martin J Osborne, *A course in game theory* [in eng] (Cambridge, Mass.: MIT Press, 1994), ISBN: 0262150417.
30. Prajit K Dutta, *Strategies and games : theory and practice* [in eng] (Cambridge, Mass.: MIT Press, 1999), ISBN: 0262041693.
31. E. N. (Emmanual N.) Barron, *Game Theory: An Introduction* [in eng], 2nd ed., Wiley series in operations research and management science (Hoboken, N.J.: John Wiley & Sons, Inc., 2013), ISBN: 9781118216934.
32. Michael Maschler, Eilon Solan, and Shmuel Zamir, *Game Theory* (Cambridge: Cambridge University Press, 2013), ISBN: 9780511794216.
33. S. Tadelis, *Game theory: An introduction* (Princeton University Press, 2013), ISBN: 9780691129082.

In the first example, there are only 2 agents participating in this game, and the possible bids are 1 or 2. Each agent is restricted to place one and only one bid. Suppose both agents are asked to submit their bids at the same time. Denote the value of the auction item is V. The agent who submits the lowest unique bid k_{unique} will win this game and gain a payoff of $V - k_{unique}$, whereas the other agent will receive a payoff of 0. If there is no lowest unique bid, then both agents receive a payoff of 0. In order to give agents an incentive to play this game, agents need a positive payoff if they win. In terms of the parameters, I have: $V - 2 > 0$.

Then, this game can be represented as the payoff matrix in Table 1.

(A,B)	B 1	B 2
A 1	(0,0)	(V-1,0)
A 2	(0,V-1)	(0,0)

Table 1: Payoff Matrix of a LUBA game with 2 agents and 2 possible bids.

Proposition 3.1. *If $V - 2 > 0$, the LUBA game with 2 agents and 2 possible bids has a pure-strategy Nash Equilibrium at (1,1).*

Proof. Since $V - 2 > 0$, I have $V - 1 > 0$ as well. Then for agent A, the strategy of bidding one weakly dominates strategy of bidding two. By symmetry, B has the same weakly dominant strategy. Hence strategy $(1, 1)$ is a pure Nash Equilibrium for this game. □

This proposition shows that as long as agents have incentives to play in this game, they will only choose the lowest bid. And, ironically, if agents are not cooperative, neither of them will win this game. I can extend this to the next example, in which there are 2 agents in the game and the bids are positive integers from 1 to M. To provide agents with incentive to bid any bids, I restrict $V - M > 0$.

Proposition 3.2. *If $V - M > 0$, the LUBA game with 2 agents and M possible bids has a pure strategy Nash Equilibrium at (1,1).*

Proof. If $V - M > 0$, bidding one weakly dominates all other strategies for both agents. □

From the examples, I conclude that in LUBA of 2 agents, both agents bidding at the minimum bid is a pure strategy Nash equilibrium. However, will the multiple-agent LUBA have the same properties as the 2-agent LUBA does? The answer is no. As in the 2-agent LUBA, agents do not have incentive to bid at bids other than 1. In multiple-agents LUBA, if there are at least 2 agents bidding at 1, the other agents will benefit from bidding at 2. Hence, although these two examples are illustrative, the construction and analysis of the following models will go on a different path.

4 Single-Bid Nash Equilibrium Model

To further study LUBA games, I want to deal with more realistic parameters such as sufficiently large amount of agents and sufficiently large amount of possible bids. From the literature, several research groups like Rapoport et al. (2009),[34] Houba et al. (2011)[35] and Raviv & Virag (2008)[36] proposed their Single-Bid Nash equilibrium models and general properties of Nash Equilibrium in their respective setups. In my single-bid Nash equilibrium model, I extend their analysis in a rigorous approach.

The single-bid LUBA game is defined in the following way: suppose the value of auction item is V, the range of possible bids is M, and number of agents is N. Agents can submit one and only one bid at the same time. Hence, the action set for each agent contains all the one-element subsets of B. The agent who submits the lowest unique bid k_{unique} wins this auction, and receive a positive payoff of $V - k_{unique}$. Otherwise, agents receive a payoff of 0. Assume that all agents share the same strategy and information regarding the game. Also assume that $N > 20$.

Now, define the probability of agent bidding at $k \in B = \{1, 2, \ldots, M\}$ by P_k, the probability of the bid k being the winning bid by Q_k and the number of agents placing bid k by Z_k. Then the strategy profile for each agent is $\Delta = \{P_k : k \in \{1, 2, \ldots, M\}, \sum_1^M P_k = 1\}$. Define the expected payoff of each bid k by U_k. By substituting, I can get $U_k = Q_k \cdot (V - k)$.

Proposition 4.1. *$Q_k = \prod_1^{k-1} P(Z_i \neq 1) \cdot P(Z_k = 0)$, where Z_k follows a binomial distribution $B(N, P_k)$. Furthermore, Z_k follows a Poisson distribution $Poisson(N \cdot P_k)$.*

Proof. First, Z_k follows $Poisson(N \cdot P_k)$ from Poisson Limit Theorem since $N > 20$ and $P_k < 0.05$. As for Q_k, Myerson (1998)[37] introduced the notion of Environmental Equivalence. In Poisson Games, $P(Z_i = i)$ is equal to the probability that the bid k is chosen by i agents, which is also equal to the probability that i competitors chose the bid k. Thus $Q_k = \prod_1^{k-1} P(Z_i \neq 1) \cdot P(Z_k = 0)$. □

At equilibrium, agents should have no incentive to deviate from their strategy. Hence, the expected payoff should be equalized throughout the whole bid support. So, for all $k, k+1 \in \{1, 2, ..., M\}$, $U_k = U_{k+1}$.

34. Rapoport et al., "Unique Bid Auction Games."
35. Zeng, Davis, and Abbott, "Reverse auction: The lowest unique positive integer game."
36. Yaron Raviv and Gabor Virag, "Gambling by auctions" [in eng], *International Journal of Industrial Organization* 27, no. 3 (2009): 369–378, ISSN: 0167-7187.
37. Myerson, "Population uncertainty and Poisson games."

By algebraic manipulation, which is included in the Appendix, I can get a recursive relation for P_k:

$$P_{k+1} = \frac{1}{N}[\ln(e^{N \cdot P_k} - N \cdot P_k) + \ln(1 - \frac{1}{V-k-1})] \tag{1}$$

Also notice that $\sum_1^M P_k = 1$. With the recursive equation (1), I can numerically compute the bidding probability at bid k P_k for every given V, M, N, F. And this following proposition shows some predictions on arbitrary parameters.

Proposition 4.2. *P_k is monotonically decreasing in k; the winning probability of bidding k is monotonically increasing; As $N \to \infty$, Δ converges to a random strategy, i.e. $P_k = P_{k+1}$.*

Proof. This can be proved by a number of algebraic manipulations, which is included in the Appendix. □

This proposition shows that, first, the winning probability is monotonically decreasing with respect to the bidding value. This means that in this game, agents have higher incentive to bid at the lower chance. Secondly, the winning probability is increasing with respect to the bidding value. This implies that since agents have lower incentive to bid higher, in equilibrium, the average number of agents bidding at higher bids is small. Last, when more agents enter this game, the agents eventually bid uniformly, given that smaller bids are too crowded.

I put the recursive relation and the probability constraint with parameters V, M, N in MATLAB to numerically calculate the strategy, and I gather four sets of simulations with different parameters in Figure 2 on page 34 to Figure 5 on page 36 in the in the Simulation Results.

Now, I manipulate parameters in this model and see how those changes will affect agents' bidding strategy.

The first parameter I manipulate is the number of agents. Intuitively, as more and more agents enter this game, the lower bids will get too crowded. Hence, agents will start to bid at higher bids. Figure 6 on page 36 to Figure 11 on page 39 in the in the Simulation Results are 6 sets of simulation results with same values and bid supports, but different numbers of agents. As N increases, agents do start to bid at higher bids. When there are 7000 agents bidding for the item, the bidding strategy is nearly uniform. This verifies Proposition 4.2.

Next, I control the number of agents participating and the bid support. Then, I change the value of the auction item. From Figure 12 on page 39 to Figure 15 on page 41 in the in the Simulation Results, bidding probability will not significantly change. In my model, the value of auction items enters at the log terms. As long as $V > k$, the bidding probability should not change largely. This corrsponds with the model as well.

Lastly, I fix the number of agents participating as well as the value of the auction item. Then, I alter the bid support. From Figure 16 on page 41 to Figure 19 on page 43 in the in the Simulation Results, the bidding probability will not change, given that the recursive relations set the ratio of bidding probability with given numbers of agents. Even when M increases, the ratio between two adjacent bids will not change. Hence, this also corresponds with the model.

After manipulating different parameters, I compare this model's predictions with real data. From Figure 20 on page 43 to Figure 25 on page 46 in the in the Simulation Results, I present 6 sets of simulation results with real data parameters and compare them with real data. This model does not fit with real life data. I will discuss this in the discussion section more.

5 Multiple-Bid Models

One of the limitations from the previous model is that agents are only allowed to place one bid in a game. In reality, however, agents are allowed to submit several bids in a game and I take this feature into my study on LUBA games.

The multiple-bid LUBA game is defined in the following way: suppose the value of auction item is V, the range of possible bids is M, and number of agents is N. Agents can submit up to ω bids in a single game, and all bids are submitted at the same time. Hence, the action set for each agent contains all the subsets of B with less than or equal to ω elements. The agent who submits the lowest unique bid k_{unique} wins this auction, and receive a positive payoff of $V - k_{unique}$. Otherwise, agents receive a payoff of 0. Assume that all agents share the same strategy and information regarding the game.

A immediate question rises: in this setup, I define this game in a static game setting, but is this reasonable? The answer is yes. In the following proposition, I show that it is reasonable to transform a multiple-bid LUBA game to a static game setting.

Proposition 5.1. *For all possible bidding choice* $\{k_1, k_2, ..., k_m\} \subset B = \{1, 2, \ldots, M\}$ *and two possible permutations of set* $\{1, 2, ..., m\}$ $\{t_1, t_2, ..., t_m\}$ *and* $\{t'_1, t'_2, ..., t'_m\}$*, there is no difference between bidding in* *order of* $k_{t_1}, k_{t_2}, ..., k_{t_m}$ *and bidding in order of* $k_{t'_1}, k_{t'_2}, ..., k_{t'_m}$. *More importantly, there is no difference between submitting* $k_{t_1}, k_{t_2}, ..., k_{t_m}$ *one at a time or submitting all* $k_{t_1}, k_{t_2}, ..., k_{t_m}$ *at the same time.*

Proof. Since agents are do not get any feedback after placing each bid, the strategy of bidding $k_{t_1}, k_{t_2}, ..., k_{t_m}$ is the same as bidding $k_{t'_1}, k_{t'_2}, ..., k_{t'_m}$. Moreover, since there will be no useful updates after bidding, bidding $k_{t_1}, k_{t_2}, ..., k_{t_m}$ one at a time is the same as submitting $k_{t_1}, k_{t_2}, ..., k_{t_m}$ at the same time. □

With this proposition, I confirm that I can use the static multiple-bid LUBA setup. Agents will decide all the bids they want to place and place those bids at the same time. And by doing so, the structure of action sets for each agent is simplified.

In this section, I will introduce two models of multiple-bid LUBA: the multiple-bid Nash equilibrium model and the multiple-bid probability matching model. Each model will be divided into two seperate parts: a special case of 2-bid and a generalized case of ω-bid.

5.1 Nash Equilibrium Models

5.1.1 Two-Bid Nash Equilibrium Model

In this model, I allow agents to bid either one bid or two bids.Here, I will introduce the specific notations for the setup. The two-bid LUBA game is defined in the following way: suppose the value of auction item is V, the range of possible bids is M, and number of agents is N. Agents can submit one bid or two bids in a single game, and all bids are submitted at the same time. Hence, the action set for each agent contains all one-element and two-element subsets of B. The agent who submits the lowest unique bid k_{unique} wins this auction, and receive a positive payoff of $V - k_{unique}$. Otherwise, agents receive a payoff of 0. Assume that all agents share the same strategy and information regarding the game.

One of the difficulties of studying multiple-bid LUBA games is that I need to arrange the order of agents' action and then define the bidding probability afterwards. In this model, I denote the bidding probability as an $M \times M$ matrix P, where $P_{i,j}$ represents the probability of placing bids at i, j, and $P_{i,i}$ represents the probability of placing only one bid at i. Here, Proposition 5.1 shows that $P_{i,j} = P_{j,i}$. Hence, for simplicity, I will refer $P_{i,j}$ with an order of $i \leq j$. Then, the strategy profile for each agent is $\Delta = \{P_{i,j} : i, j \subset \{1, 2, ..., M\}, i \leq j\}$.

Next, denote the probability of bid k being the winning bid by Q_k, the number of agents placing bids at bid k by Z_k and the expected payoff to agents when bidding $\{i,j\}, i \le j$ by $\Phi_{i,j}$. Also denote $V_k = V - k$ as the winning payoff at k. Then I have the following equations:

$$Q_k = N\dot{(}1 - \sum_{i=1}^{M} P_{k,i})^{N-1} \cdot (\sum_{i=1}^{M} P_{k,i})$$

$$\begin{aligned} for\ 1 \le i < j \le M,\ \Phi_{i,j} &= [\prod_{\alpha=1}^{i-1}(1-Q_\alpha)] \cdot Q_i \cdot V_i + [\prod_{\alpha=1}^{j-1}(1-Q_\alpha)] \cdot Q_j \cdot V_j \\ for\ 1 \le i \le M,\ \Phi_{i,i} &= [\prod_{\alpha=1}^{i-1}(1-Q_\alpha)] \cdot Q_i \cdot V_i \end{aligned}$$

At equilibrium, agents should have no incentive to deviate from their strategy.

Proposition 5.2. *If agents are allowed to bid once or twice, then they will always bid twice.*

Proof. Fix $i \in B$, $\forall j \in B$, $\Phi_{i,j} \ge \Phi_{i,i}$. Hence, at equilibrium, agents will always be better off if they place two bids. By contradiction, if $P_{i,i} > 0$, then agents will always have incentive to deviate. Hence $P_{i,i} = 0$. □

With this proposition, the problem will be simplified. What I need to find is that $\forall i, j, i^{'}, j^{'} \in B, i \ne j, i^{'} \ne j^{'}$, $\Phi_{i,j} = \Phi_{i^{'},j^{'}}$.

Now, fix $i \in B$. For $j, j+1 \in B$, I have:

$$\begin{aligned} \Phi_{i,j} &= \Phi_{i,j+1} \\ [\prod_{\alpha=1}^{i-1}(1-Q_\alpha)] \cdot Q_i \cdot V_i + [\prod_{\alpha=1}^{j-1}(1-Q_\alpha)] \cdot Q_j \cdot V_j &= [\prod_{\alpha=1}^{i-1}(1-Q_\alpha)] \cdot Q_i \cdot V_i + [\prod_{\alpha=1}^{j}(1-Q_\alpha)] \cdot Q_{j+1} \cdot V_{j+1} \\ [\prod_{\alpha=1}^{j-1}(1-Q_\alpha)] \cdot Q_j \cdot V_j &= [\prod_{\alpha=1}^{j}(1-Q_\alpha)] \cdot Q_{j+1} \cdot V_j + 1 \\ Q_j &= (1-Q_j) \cdot Q_{j+1} \cdot \frac{V-j-1}{V-j} \end{aligned}$$

Notice that this looks similar as the equilibrium condition I get in the single-bid Nash equilibrium model. As $N > 20$ and $\sum_{j=1}^{M} P_{i,j} \le 0.05$, I can apply Poisson distribution to simplify this and get:

$$\sum_{j=1}^{M} P_{i+1,j} = \frac{1}{N}[\ln(e^{N \cdot \sum_{j=1}^{M} P_{i,j}} - N \cdot \sum_{j=1}^{M} P_{i,j}) + \ln(1 - \frac{1}{V-j-1})] \quad (2)$$

However, this also shows that the uniqueness of the mixed strategy Nash equilibrium does not exist.

Proposition 5.3. *In a two-bid LUBA game that has more than 3 possible bids, the Nash equilibrium strategies are not unique.*

Proof. Suppose P is a Nash equilibrium mixed strategy in a two-bid LUBA game. Denote V_k as the winning payoff at bid k. Then, for $i, j, i^{'}, j^{'} \in B$ and $i \neq j$, $i^{'} \neq j^{'}$, I have $\Phi_{i,j} = \Phi_{i^{'},j^{'}}$. By $\Phi_{i,j} = \Phi_{i^{'},j^{'}}$, I get:

$$[\prod_{\alpha=1}^{i-1}(1-Q_\alpha)] \cdot Q_i \cdot V_i + [\prod_{\alpha=1}^{j-1}(1-Q_\alpha)] \cdot Q_j \cdot V_j = [\prod_{\alpha=1}^{i^{'}-1}(1-Q_\alpha)] \cdot Q_{i^{'}} \cdot V_{i^{'}} + [\prod_{\alpha=1}^{j^{'}-1}(1-Q_\alpha)] \cdot Q_{j^{'}} \cdot V_{j^{'}}$$

Then, denote $P^{'}$ as another mixed strategy such that for all $i \in B$, $\sum_{j=1}^{M} P_{i,j} = \sum_{j=1}^{M} P^{'}_{i,j}$ and $P \neq P^{'}$. Here, I use the assumption that $M > 3$; otherwise such $P^{'}$ does not exist. Denote $Q^{'}$ as the winning probability at each bid when playing $P^{'}$. Now, denote $\Phi^{'}$ as the winning probability of each choice of bids when playing $P^{'}$.

It is clear that $Q_i = Q^{'}_i$, $Q_j = Q^{'}_j$, $Q_{i^{'}} = Q^{'}_{i^{'}}$ and $Q_{j^{'}} = Q^{'}_{j^{'}}$. Then we can see that $\Phi_{i,j} = \Phi_{i^{'},j^{'}} = \Phi^{'}_{i,j} = \Phi^{'}_{i^{'},j^{'}}$. Hence by definition, $P^{'}$ is also a mixed NE strategy. □

Nevertheless, I get that $\sum_{j=1}^{M} P_{i,j}$ is unique.

Proposition 5.4. *Suppose that in a two-bid LUBA game, P and $P^{'}$ are two Nash equilibrium mixed strategies. Then for all $i \in B$,*

$$\sum_{j=1}^{M} P_{i,j} = \sum_{j=1}^{M} P^{'}_{i,j}.$$

Proof. This is shown in Theorem 6.1. □

Then, I can use the recursive relation of $\sum_{j=1}^{M} P_{i,j}$ to calculate $\sum_{j=1}^{M} P_{i,j}$. Given that:

$$\sum_{1\leq i\leq j\leq M} P_{i,j} = 1, \quad P_{i,i} = 0 \; \forall i \in B$$

We have:

$$\sum_{i=1}^{M}(\sum_{j=1}^{M} P_{i,j}) = 2$$

To numerically calculate $\sum_{j=1}^{M} P_{i,j}$, I use the "nested" strategy, which is one of the NE mixed strategies. Denote the single bid Nash Equilibrium strategy as $\tilde{\text{P}}$. Fix $i \in B$, for $j, j+1 \in B \setminus \{i\}$, we have $\frac{P_{i,j}}{P_{i,j+1}} = \frac{\tilde{P}_j}{\tilde{P}_{j+1}}$. The reason I call this the "nested" strategy is that the conditional bidding probability at each bid is proportional to the bidding probability of the one-bid model.

I put the recursive relation and the probability constraint with parameters V, M, N in MATLAB to numerically calculate the strategy, and I gathered four sets of simulations with different parameters in Figure 26 on page 47 to Figure 29 on page 49 in the in the Simulation Results.

Now, I manipulate the parameters in this model and to see how those changes will affect agents' bidding strategies.

The first parameter I change is the number of agents. Intuitively, as more and more agents enter this game, the lower bids will get too crowded. Hence, agents will start to bid at higher bids. Figure 30 on page 49 to Figure 33 on page 51 in the in the Simulation Results are 4 sets of simulations with same values and bid supports, but different numbers of agents. As N increases, agents do start to bid at higher bids.

Next, I control the number of agents participating and the bid support. Then, I change the value of the auction item. From Figure 34 on page 51 to Figure 37 on page 53 in the in the Simulation Results, bidding probability will not significantly change. In my model, the value of auction items enters at the log terms. As long as $V > k$, the bidding probability should not change largely.

Lastly, I fix the number of agents participating as well as the value of the auction item. Then, I alter the bid support. From Figure 38 on page 53 to Figure 41 on page 55 in the in the Simulation Results, the bidding probability will not change, given that the recursive relations set the ratio of bidding probability with given numbers of agents. Even when M increases, the ratio between two adjacent bids will not change.

Not surprisingly, what I get for this model is similar to the single-bid Nash Equilibrium model. I will explain more about this in the discussion session.

5.1.2 Multiple-Bid Nash Equilibrium Model

From data, I observe that on average, agents place nearly 6 bids per game, and I need a model to allow such behaviors. So in this model, I allow agents to submit up to ω bids in one game.

The multiple-bid LUBA game is defined in the following way: suppose the value of auction item is V, the range of possible bids is M, and number of agents is N. Agents can submit up to ω bids in a single game, and all bids are submitted at the same time. Hence, the action set for each agent contains all the subsets of B with less than or equal to ω elements. The agent who submits the lowest unique bid

k_{unique} wins this auction, and receive a positive payoff of $V - k_{unique}$. Otherwise, agents receive a payoff of 0. Assume that all agents share the same strategy and information regarding the game.

Then, denote $P_{i_1,i_2,...,i_\omega}$ as the probability that agents bid at $i_1, i_2, ..., i_\omega$, Q_k as the total probability that only one agent bids at k, p_i as the probability that agents bid i. and $\Phi_{i_1,i_2,...,i_\omega}$ as the expected payoff to agents when bidding at $i_1, i_2, ..., i_\omega$. Also denote $V_k = V - k$ as the winning payoff at k. Then the following equations are analogous to the 2-bid model:

$$Q_k = N \cdot (1 - \sum_{k \in K} P_K)^{N-1} \cdot (\sum_{k \in K} P_K) = N \cdot (1 - p_k)^{N-1} \cdot p_k, \quad K \subset B$$

$$\Phi_{i_1,i_2,...,i_\omega} = \sum_{j=1}^{\omega} \{[\prod_{\alpha=1}^{i_j - 1} (1 - Q_\alpha)] \cdot Q_{i_j} \cdot V_{i_j}\}, \quad i_1 < i_2 < ... < i_R$$

From the two-bid Nash equilibrium model, I observe that agents will always place two bids instead of one bid. Also, the total bidding probability at each bid can be defined recursively. So, by analogy, I have the following proposition and theorem:

Proposition 5.5. *For any natural number ω such that $1 < \omega \leq M$, in an ω-bid game, agents will always place ω bids.*

Proof. By placing one more bid, agents will increase their expected payoff. Suppose, at equilibrium, that agents have positive probability to bid less than ω. This contradicts with the equilibrium condition since the agents do have incentives to bid more to increase their winning. Hence in an ω-bid game, agents will always place ω bids. □

Theorem 5.1. *Denote $\tilde{P}_i$ as the one-bid NE strategy. Then for $i, i^{'} \in B$,*

$$\frac{p_i}{p_{i'}} = \frac{\tilde{P}_i}{\tilde{P}_{i'}} \tag{3}$$

Proof. Suppose $i_1, i_2, ..., i_R$ are R distinct integers from $[1, M]$. At equilibrium, $\Psi_{i_1,i_2,...,i_R} = \Psi_{i_1,i_2,...,i_{R-1},i_R+1}$. This can be simplified as:

$$[\prod_{\alpha=1}^{i_R - 1} (1 - Q_\alpha)] \cdot Q_{i_R} \cdot V_{i_R} = [\prod_{\alpha=1}^{i_R} (1 - Q_\alpha)] \cdot Q_{i_R+1} \cdot V_{i_R+1}$$

$$Q_{i_R} = (1 - Q_{i_R}) \cdot Q_{i_R+1} \cdot \frac{V - i_R - 1}{V - i_R}$$

$$p_{i_R} = (1 - p_{i_R}) \cdot p_{i_R+1} \cdot \frac{V - i_R - 1}{V - i_R}$$

By using Poisson approximation, we get:

$$p_{i+1} = \frac{1}{N}[\ln(e^{N\cdot p_i} - N\cdot p_i) + \ln(1 - \frac{1}{V-i-1})] \tag{4}$$

Then by Equation 4, we have:

$$\frac{p_i}{p_{i'}} = \frac{\tilde{P}_i}{\tilde{P}_{i'}}$$

□

By the same analogy, the multiple-bid Nash equilibrium strategies are not unique. Suppose the equalibrium strategy is unique and denote it as P. Then by contradiction, any strategy $\tilde{P}$ such that $\tilde{p}_i = p_i$ will also equalize the winning probability at all states.

Nevertheless, I get that p_i is unique.

Proposition 5.6. *Suppose that in a multiple-bid LUBA game, P and $P^{'}$ are two Nash equilibrium mixed strategies. Then for all $i \in B$,*

$$p_i = p_i^{'}$$

Proof. This is shown in Theorem 6.1. □

Then, I use the recursive relation of p_i to calculate p_i. Given that:

$$\sum_{1\leq i_1<i_2<\ldots<i_\omega\leq M} P_{i_1,i_2,\ldots,i_\omega} = 1$$

and agents only place ω bids, I have:

$$\sum_{i=1}^{M} p_i = \omega$$

To numerically calculate p_i, I use the "nested" strategy, which is one of the Nash equilibrium mixed strategies. Denote the single bid Nash equilibrium strategy as $\tilde{P}$. Fix $i_1, i_2, ..., i_{\omega-1} \in B$, for $i_\omega, i_{\omega+1} \in B \setminus \{i_1, i_2, ..., i_{\omega-1}\}$, we have:

$$\frac{P_{i_1,i_2,\ldots,i_{\omega-1},i_\omega}}{P_{i_1,i_2,\ldots,i_{\omega-1},i_\omega+1}} = \frac{\tilde{P}_{i_\omega}}{\tilde{P}_{i_\omega+1}}.$$

The reason I call this the "nested" strategy is that the conditional bidding probability at any combination of $\omega - 1$ bids is proportional to the bidding probability of the one-bid model.

I do not run any actual simulation for this model. By the similarity of Nash Equilibrium models in

LUBA, I do not expect any significant difference with the previous single-bid and 2-bid model. I will discuss the similarity of Nash Equilibrium models in the dicsussion section.

5.2 Probability Matching Models

From previous Nash equilibrium models, Nash equilibrium strategies exhibit cutoff bids: agents will not place bids higher than a certain amount. In real LUBA games, however, some agents have actually placed large bids that was close to M. Is there a way to explain this phenomenon?

Here, I need to introduce a new decision rule: probability matching. Probability matching is a decision rule in which the bidding probability is proportional to the winning probability. For example, assume that a probability matching agent faces a forked cross and he knows that one and only one of the branch leads to his destination. He is not sure which branch is the right one, but he knows that there is 40% probability that the left branch is correct and 60% probability that the right branch is correct. Then, by probability matching, his decision will be a mixed strategy with 40% probability going left and 60% probability going right. Well-known researchers have been studied probability matching over the past several decades. Studies of human behaviors from psychology, especially from the work of Shanks et al. (2002),[38] Otto et al. (2011),[39] Rivas (2012)[40] and Newell et al. (2013),[41] showed the evidence of the existence of probability matching, and I apply this decision rule into this model.

5.2.1 Two-Bid Probability Matching Model

In this model, I allow agents to bid either one bid or two bids. Here, I will introduce the specific notations for the setup. The two-bid LUBA game is defined in the following way: suppose the value of auction item is V, the range of possible bids is M, and number of agents is N. Agents can submit one bid or two bids in a single game, and all bids are submitted at the same time. Hence, the action set for each agent contains all one-element and two-element subsets of B. The agent who submits the lowest unique bid k_{unique} wins this auction, and receive a positive payoff of $V - k_{unique}$. Otherwise, agents receive a payoff of 0. Assume that all agents share the same strategy and information regarding the game.

38. David R. Shanks, Richard J. Tunney, and John D. Mccarthy, "A re-examination of probability matching and rational choice" [in eng], *Journal of Behavioral Decision Making* (Chichester, UK) 15, no. 3 (2002): 233–250, ISSN: 0894-3257.
39. A. Ross Otto, Eric G. Taylor, and Arthur B. Markman, "There are at least two kinds of probability matching: Evidence from a secondary task" [in eng], *Cognition* 118, no. 2 (2011): 274–279, ISSN: 0010-0277.
40. Javier Rivas, "Probability matching and reinforcement learning" [in eng], *Journal of Mathematical Economics* 49, no. 1 (2012), ISSN: 0304-4068.
41. Ben Newell et al., "Probability matching in risky choice: The interplay of feedback and strategy availability" [in eng], *Memory & Cognition* (New York) 41, no. 3 (2013): 329–338, ISSN: 0090-502X.

One of the difficulties of studying multiple-bid LUBA games is that I need to arrange the order of agents' action and then define the bidding probability afterwards. In this model, I denote the bidding probability as an $M \times M$ matrix P, where $P_{i,j}$ represents the probability of placing bids at i, j, and $P_{i,i}$ represents the probability of placing only one bid at i. Here, Proposition 5.1 shows that $P_{i,j} = P_{j,i}$. Hence, for simplicity, I will refer $P_{i,j}$ with an order of $i \leq j$. Then, the strategy profile for each agent is $\Delta = \{P_{i,j} : i, j \subset \{1, 2, ..., M\}, i \leq j\}$.

Next, denote the probability of bid k being the winning bid by Q_k, the number of agents placing bids at bid k by Z_k and the winning probability at $\{i, j\}, i \leq j$ by $\Psi_{i,j}$. Then I have the following equations:

$$Q_k = N\dot{(}1 - \sum_{i=1}^{M} P_{k,i})^{N-1} \cdot (\sum_{i=1}^{M} P_{k,i})$$

$$for\ 1 \leq i < j \leq M,\ \Psi_{i,j} = [\prod_{\alpha=1}^{i-1}(1 - Q_\alpha)] \cdot Q_i + [\prod_{\alpha=1}^{j-1}(1 - Q_\alpha)] \cdot Q_j$$
$$for\ 1 \leq i \leq M,\ \Psi_{i,i} = [\prod_{\alpha=1}^{i-1}(1 - Q_\alpha)] \cdot Q_i$$

Then, for a probability matching agent, the bidding probability is proportional to the winning probability. This means that for $i, j, i^{'}, j^{'} \in B$,

$$\frac{\Psi_{i,j}}{\Psi_{i^{'},j^{'}}} = \frac{P_{i,j}}{P_{i^{'},j^{'}}}.$$

Fortunately, this can be turned into a fixed point problem. The complexity of this model is exponential, so I cannot derive any analytic solution with the constraints. I can, however, numerically calculate the bidding probability with given parameters.

I start to run simulations and compare behaviors of agents in each of those games. Figure 42 on page 56 to Figure 49 on page 60 in the in the Simulation Results show four sets of simulations with their image of bidding probability with scaled colors and overall bidding probability.

In this setup, agents have higher incentive to place two bids compared to placing only one bid. Define $P_{single} = \sum_{i=1}^{M} P_{i,i}$ as the total probability of agents placing one bid, and $P_{2bids} = \sum_{1 \leq i < j \leq M} P_{i,j}$ as the total probability of agents. Also define $P_{only\ bid\ k} = P_{k,i}$ as the probability of only placing bid at k, and

$P_{k\, and\, other} = (\sum_{i=1}^{M} P_{k,i}) - P_{i,i}$ as the probability of placing two bids including bid k. At last, denote the total probability of placing a bid at k by p_k.

Proposition 5.7. *$P_{single} < P_{2bids}$. Moreover, for all $k \in B$, $P_{only\, bid\, k} < P_{k\, and\, other}$.*

Proposition 5.8. *$P_{only\, bid\, k}$ is decreasing in k. $P_{k\, and\, other}$ is decreasing in k. Moreover, p_k is decreasing in k.*

Proposition 5.9. *As j, $j+1$ getting closer to M, $p_j \approx p_{j+1}$. In other words, we should expect a nin the Simulation Resultsearly uniform distribution of bidding probability at higher bids.*

Proof. All three propositions are proved by algebraic manipulations shown in the Appendix. □

Then, I check how changing parameters affects the bidding probability. First, I fix the number of agents in the game and change the maximum possible bid. From Figure 50 on page 60 to Figure 57 on page 64 in the in the Simulation Results, I acquire 4 sets of simulations with the same N but different M. From those simulation results, the flat bidding probability in each simulation starts at around 20 to 25 and the bidding probability before flat part seems identical.

Next, I fix the maximum possible bid and alter the number of agents. In Figure 58 on page 64 to Figure 65 on page 68 in the in the Simulation Results, I acquire 4 sets of simulations with the same M but different N. As N increases, the bidding probabilities across the bid support become more and more flat, nearly converging to a uniform distribution. This means that as more and more agents enter the game, the winning probability becomes uniform for all two-bid strategies and one-bid strategies.

After manipulating different parameters, I compare this model's predictions with real data. From Figure 66 on page 68 to Figure 69 on page 70 in the in the Simulation Results, I present 4 sets of simulations with real data parameters. Comparing to the simulation results from the single-bid Nash equilibrium model, I notice that the two-bid probability matching model successfully predicts the possible bidding at large bids. This offers a possible explanation for the large bids submitted in each auction. However, the goodness of fit is poor. I will discuss about the goodness of fit in the Discussion section.

5.2.2 Multiple-Bid Probability Matching Model

From data, I observe that on average, agents place nearly 6 bids per game, and I need a model to allow such behaviors. So in this model, I allow agents to submit up to ω bids in one game.

The multiple-bid LUBA game is defined in the following way: suppose the value of auction item is V, the range of possible bids is M, and number of agents is N. Agents can submit up to ω bids in a single game, and all bids are submitted at the same time. Hence, the action set for each agent contains all the subsets of B with less than or equal to ω elements. The agent who submits the lowest unique bid k_{unique} wins this auction, and receive a positive payoff of $V - k_{unique}$. Otherwise, agents receive a payoff of 0. Assume that all agents share the same strategy and information regarding the game.

As I mentioned before, the difficulties of allowing multiple bids is representing the bidding probability. Originally, I want to repeat what I did for the two-bid probability matching model to represent agents' bidding strategy as a tuple of size $M \times M \times ... \times M$. As the maximum allowed bid increases, the complexity of this system increases exponentially. Indeed, I attempted to create a three-bid probability matching model at first, and the simulations are unsuccessful. The problem with this representation is that the symmetry causes redundancy, and it is unnecessary to recur the method used in two-bid model.

My first attempt to bypass the redundancy problem is to arrange the possible bidding plays by the binary representation method. Notice that if the maximum possible bid is M, agents will have $2^M - 1$ possible ways to bid. Then the one-to-one correspondence between 1 to $2^M - 1$ and those $2^M - 1$ possible bidding plays is clear: let 1 correspond with bidding 1; let 2 corresponds with bidding 2; let 3 correspond with bidding both 1 and 2; let 4 correspond with bidding 3;... let $2^{b_1-1} + 2^{b_2-1} + ... + 2^{b_n-1}$ correspond with bidding $b_1, b_2, ..., b_n$. By the uniqueness of binary representation of positive integers, the one-to-one correspondence is easily shown. Then I can define the bidding probability as an $1 \times (2^M - 1)$ vector.

The problem with this method is whether I still want to put an upper bound for the number of bids allowed. This representation will not work if such upper bound is needed.

My second attempt is to use the order from a MATLAB command "nchoosek", which can be used to list all subsets of size n from set $\{1, 2, \ldots, M\}$. Then, I can define an ordered set A such that it consists all one-bid actions, two-bid actions, and all the way to those m-bid actions. In this way, I would have the same one-to-one correspondence and I am able to control the maximum bid again.

Now, denote the action sets of the agents as A, where $A = \{K \subset B : \ |K| \leq \omega\}$. For each element $K \in A$, denote the probability of agent placing all bids $k \in K$ by P_K. Denote Q_k as the probability that bid $k \in B$ is a unique bid, and denote Ψ_K as the winning probability of placing all bids $k \in K$.

Then, I can get the following relations:

$$\Psi_K = \sum_{k \in K} [\prod_{n=1}^{k-1} (1 - Q_n) \cdot Q_k]$$
$$Q_k = N \cdot (1 - \sum_{k \in K} P_K)^{N-1} \cdot (\sum_{k \in K} P_K)$$

Then for a probability matching agent, the bidding probability is proportional to the winning probability. This means that for any two subsets $K, K^{'} \subset B$,

$$\frac{\Psi_K}{\Psi_{K'}} = \frac{P_K}{P_{K'}}.$$

This problem can be transformed to a fixed point problem. Like the previous two-bid probability matching model, I cannot derive any analytic solution with the constraints due to the complexity. I can, however, try to numerically calculate the bidding probability with given parameters. Unfortunately, as I increase ω, the simulations are unsuccessful due to the size of potential action set.

I run several multiple-bid-allowed simulations with relatively small parameters. From Figure 70 on page 71 to Figure 72 on page 72 in the in the Simulation Results, three sets of simulation results are shown. In all three setups, the total bidding probabilities are decreasing in the bids.

This model has flaws in both analytic and numerical tractabilities due to the exponentially growing complexity. However, this is the most generalized model I have.

6 Discussion

6.1 Validity

Given that I use numerical methods to calculate the strategy, I need to show that such strategy exists and is unique in each model in order to prove the validity.

Theorem 6.1. *In the single-bid Nash equilibrium model, the equilibrium strategy is unique. In the multiple-bid Nash equilibrium model, p_i is unique.*

Theorem 6.2. *In the multiple-bid probability matching model, the equilibrium strategy is unique.*

Corollary 6.2.1. *In the 2-bid probability matching model, the equilibrium strategy is unique.*

Proof. For both theorems, it is enough to check that each equilibrium condition requires a contraction map, then by using Banach-Caccioppoli fixed-point theorem (contraction mapping theorem), the uniqueness is proved.

For both single-bid and multiple-bid Nash equilibrium model, given that the equilibrium conditions can be written as a recursive relation, the contraction mappings are clear in both cases.

For the multiple-bid probability matching model, the contraction will be shown as the map $T : P \to P$ is by a square matrix in which the absolute value of the sum of each row is less 1. Then by Perron–Frobenius theorem, the eigenvalue of T is less than 1. Hence T is indeed a contraction map.

Therefore, the corollary is clear since the 2-bid model can be considered as a special case from multiple-bid model. □

With these three statements, the validity of those models is verified.

6.2 Similarities within Nash Equilibrium Models

The similarities within single-bid Nash equilibrium model and multiple-bid Nash equilibrium model are one of the interesting findings in this research. I show that the recursive relation in each model is nearly the same. However, in the single-bid model, it is guaranteed to have a unique strategy, whereas in the multiple-bid model, the Nash equilibrium strategies are not unique.

This is because the Nash equilibrium conditions in LUBA only have bearing on the total probability of bidding at each bid. In the single-bid Nash equilibrium model, since agents are only allowed to submit one bid, the total probability of bidding is the actual bidding probability. Hence, the recursive relation provides a unique strategy. In the multiple-bid Nash equilibrium model, the recursive relation cannot provide uniqueness.

6.3 Discrepancy between Models and Data

Although most of my models can provide predictions when parameters are realistic, from Figure 20 on page 43 to Figure 25 on page 46 and from Figure 66 on page 68 to Figure 69 on page 70 in the in the Simulation Results, there are significant differences between the predictions and data: both single-bid Nash equilibrium model and two-bid probability matching model predict that the lowest bid 1 will receive the highest amount of bids, and the amount of bids should be decreasing. In the data, however, the monotonicity is violated and the peak is not always reached at bid 1. Moreover, the single-bid model predicts a cut-off point, after which agents should not place any bids; in the data, however, there are agents placing large bids after the predicted cut-off point.

There are several potential explanations for this disparity between the model predictions and the data. First, in the setup of those models, I assume that each agent knows the exact number of participants. In reality, agents do not possess this information. A reasonable solution for this is to replace the deterministic number of agents in a game with a stochastic process. Instead of including N as a deterministic parameter, I can define N as a Poisson random variable. Then the bids placed after the cut-off point could be the consequence of this uncertainty.

Second, I rely on the assumption that there is homogeneity in each agent's decision making. Moreover, whether agents are fully rational when playing LUBA is unclear. As an online game, I have no demographic information regarding participating agents. I assume that each agent shares the same understanding of the game, but this is not possible in reality. A potential solution is to define a cognitive hierarchy within agents. Crawford et al. (2013)[42] and Radicchi et al. (2012)[43] suggested that different levels of understanding among agents can explain the late peak in data. For example, 0-level agents may bid randomly. 1-level agents are aware of the existence of 0-level agents, and they decide their actions based on 0-level agents playing randomly, etc. k+1-level agents are aware of agents from 0-level to k-level agents, and they decide their actions taking all prior levels into consideration.

Lastly, creating models that fit data is not the main goal of this research. I want to understand what the equilibrium strategy in LUBA games is, how agents adopt this strategy and what properties the equilbrium strategy has. A consistency between model predictions and data is preferable, but not required.

6.4 Future Research Direction

While these findings are useful, they all reveal a need for further research in this topic. As I have mentioned before, heterogeneity can potentially be the key to understand behaviors of agents from real data.

Another key feature of LUBA is the bidding fee, which is completely ignored in all of my models. The single-bid model can easily implement the bidding fee by including it into the payoff. When deriving the equilibrium condition, I can cancel the fee from both sides of the equation, and get the same result. However,

42. Crawford, Costa-Gomes, and Iriberri, "Structural Models of Nonequilibrium Strategic Thinking: Theory, Evidence, and Applications."

43. Radicchi, Baronchelli, and Amaral, "Rationality, Irrationality and Escalating Behavior in Lowest Unique Bid Auctions (Levy Flights in Online Auctions)."

this will result in a negative expected payoff, meaning that agents will be better off if they do not participate in this game. A potential solution is to rescale the positive payoff, but this is debateful since additional assumptions will be made, and those assumptions may violate risk preference. The implementation of the bidding fee in the multiple-bid model will require even more finesse.

Lastly, I hope to make progress on improving the tractability of the multiple-bid model. I look forward to further studying LUBA games with a more comprehensive consideration.

7 Conclusion

In this research, I proposed three models to understand how agents behave in LUBA games: the single-bid Nash equilibrium model, the multiple-bid Nash equilibrium model and the multiple-bid probability matching model. The first two models tried to find agents' optimal strategy by using Equilibrium concepts, and the last model tried to find agents' optimal strategy by using probability matching as the decision rule.

In the single-bid Nash equilibrium model, I restricted agents to placing only one bid. I successully derived the equilibrium conditions and gave an analytic solution to the problem. I was able to prove the uniqueness of the equilibrium, monotonicity in bidding probabilities and winning probabilities, and showed that as more and more agents enter the game, agents' strategy profile will converges into a uniform mixed strategy. This model was a verification of previous researches done by other scholars.

In the multiple-bid Nash equilibrium model, I restricted agents to placing multiple bids. The analysis was similar to the analysis from the first model, and surprisingly the conditional bidding probability is proportional to the single-bid NE strategy. And I found that the equilibrium strategies are not unique. It is because that the Nash equilibrium condition only has restriction on the total number of bidding at each bid. And given the similarity between multiple-bid Nash Equilibrium model and single-bid Nash Equilibrium model, some of the analogous statements were obvious. I viewed this model as an extension from previous researches on LUBA.

Unlike the first two models, in the multiple-bid probability matching model, I used a different decision rule. I successully derived the conditions, but the I could not solve the problem analytically. Instead, I was able to calculate the bidding probability numerically. Moreover, I was able to prove the uniqueness of the equilibrium, monotonicity in bidding probabilities and winning probabilities, and showed that as more and more agents enter the game, agents' strategy profile will converges into a uniform mixed strategy. Also, this model provided explanations on why agents submit large bids. Unfortunately, the multiple-bid probability model lacked of both numerical and analytic tracabilities. Nevertheless, the multiple-bid probability matching model is the most innovative part in my research.

When I tried to compare my models' predictions with data, I found that the discrepancy between my predictions and the data was significant. I proposed several potential solutions to this setback, and I hope to continue studying LUBA games in my future research.

In my future research on LUBA, I will try to extend my current models by including stochastic parameters, heterogeneity among agents and the bidding fee. Moreover, I will apply my research results in the settings of the innovations by firm, suggested by my advisor Professor L'Huillier. The possible implementations are described in the Potential Application section, and it will be an audacious and feasible attempt at research on industrial organization.

My current research results contribute by helping fellow researchers to develop a deeper understanding on multiple-bid strategies in LUBA games, and my methods can inspire fellow researchers in their own studies of LUBA and other complex games.

A Appendix

Algebraic Derivation of the Recursive Relation in Single-Bid Model

By Proposition 5.1, $P(Z_k = n) = \frac{(N \cdot P_k)^n \cdot e^{-N \cdot P_k}}{n!}$, and then I can get:

$$Q_k = [\prod_{i=1}^{k-1}(1 - N \cdot P_i \cdot e^{-N \cdot P_i})] \cdot e^{-N \cdot P_k}$$

At equilibrium, the payoff should be equal for all bids, i.e. $U_{k+1} = U_k$ for every bid k. Then by writting out $U_{k+1} = U_k$, I should get:

$$\begin{aligned}
U_k &= U_{k+1} \\
(V-k) \cdot [\prod_{i=1}^{k-1}(1 - N \cdot P_i \cdot e^{-N \cdot P_i})] \cdot e^{-N \cdot P_k} &= (V-k-1) \cdot [\prod_{i=1}^{k}(1 - N \cdot P_i \cdot e^{-N \cdot P_i})] \cdot e^{-N \cdot P_{k+1}} \\
\frac{V-k}{V-k-1}[\prod_{i=1}^{k-1}(1 - N \cdot P_i \cdot e^{-N \cdot P_i})] \cdot e^{-N \cdot P_k} &= (V-k-1) \cdot [\prod_{i=1}^{k}(1 - N \cdot P_i \cdot e^{-N \cdot P_i})] \cdot e^{-N \cdot P_{k+1}} \\
\frac{V-k}{V-k-1} &= \frac{1 - N \cdot P_k \cdot e^{-N \cdot P_k}}{e^{-N \cdot P_k}} \cdot e^{-N \cdot P_{k+1}} \\
e^{N \cdot P_{k+1}} &= e^{N \cdot P_k} \cdot (1 - N \cdot P_k \cdot e^{-N \cdot P_k}) \cdot \frac{V-k-1}{V-k} \\
e^{N \cdot P_{k+1}} &= (e^{N \cdot P_k} - N \cdot P_k) \cdot \frac{V-k-1}{V-k}
\end{aligned}$$

Then by taking log of each side, I get the following:

$$\begin{aligned}
N \cdot P_{k+1} &= \ln(e^{N \cdot P_k} - N \cdot P_k) + \ln(1 - \frac{1}{V-k-1}) \\
P_{k+1} &= \frac{1}{N}[\ln(e^{N \cdot P_k} - N \cdot P_k) + \ln(1 - \frac{1}{V-k-1})]
\end{aligned}$$

This is how I get for the recursive relation in the single-bid model.

Detailed Proofs

Proposition. *(4.2)P_k is monotonically decreasing in k; the winning probability of bidding k is monotonically increasing; As $N \to \infty$, Δ converges to a random strategy, i.e. $P_k = P_{k+1}$.*

Proof. As $V > M$, $\ln(1 - \frac{1}{V-k-1}) \approx 0$. Then it is enough to check that:

$$P_{k+1} \approx \frac{1}{N} \ln(e^{N \cdot P_k} - N \cdot P_k)$$

Then take the exponential on both sides I get:

$$e^{N\cdot P_{k+1}} \approx e^{N\cdot P_k} - N\cdot P_k$$

Then it remains to check that $P_k \geq 0$ for all k. Notice that the function $f(x) = \frac{1}{N}\cdot \ln(e^x - N\cdot x)$ is positive when $x > 0$ and f(0)=0. This shows that I will always have $P_k \geq 0$ for all k.

Now denote the winning probability at k as ρ_k. It is enough to show that:

$$\rho_{k+1} - \rho_k = [\prod_{i=1}^{k}(1 - N\cdot P_i \cdot e^{-N\cdot P_i})]\cdot e^{-N\cdot P_{k+1}} - [\prod_{i=1}^{k}(1 - N\cdot P_i\cdot e^{-N\cdot P_i})]\cdot e^{-N\cdot P_k} \geq 0$$

Indeed, by rearranging terms, I get:

$$\rho_{k+1} - \rho_k = [\prod_{i=1}^{k}(1 - N\cdot P_i\cdot e^{-N\cdot P_i})]\cdot (e^{-N\cdot P_{k+1}} - e^{-N\cdot P_k} - N\cdot P_k\cdot e^{-N\cdot(P_{k+1}+P_k)})$$

Notice that $e^{N\cdot P_k+1} < e^{N\cdot P_k}$ by the first part of this proposition, I have:

$$e^{-N\cdot P_{k+1}} - e^{-N\cdot P_k} - N\cdot P_k\cdot e^{-N\cdot(P_{k+1}+P_k)} = e^{-N\cdot(P_{k+1}+P_k)}\cdot(e^{N\cdot P_k} - e^{N\cdot P_{k+1}} - N\cdot P_k) \geq 0$$

Hence, I have $\rho_{k+1} - \rho_k > 0$.

Lastly, when $N \to \infty$, by L'Hôpital's rule:

$$\lim_{N\to\infty} P_{k+1} = \lim_{N\to\infty}\frac{\ln(e^{N\cdot P_k} - N\cdot P_k)}{N} = P_k$$

Thus, when $\lim_{N\to\infty}$, $P_k = P_{k'}$ $\forall k, k' \in B$. ☐

Proposition. *(5.5)$P_{single} < P_{2bids}$. Moreover, for all $k \in B$, $P_{only\, bid\, k} < P_{k\, and\, other}$.*

Proof. Since $P_{single} = \sum_{k=1}^{M}$, it is enough to check that $P_{1,1} + P_{2,2} \leq P_{1,2}$ and $P_{k,k} < P_{1,k} \forall k > 2$. Then I have $P_{single} < \sum_{k=2}^{M} P_{1,k}$. Indeed, $\Psi_{1,2} = \Psi 1, 1 + \Psi_{2,2}\times(1 - \Psi_{1,1})$. Then I have $P_{1,1} + P_{2,2} \leq P_{1,2}$. $\forall k > 2$, $\Psi_{1,k} = \Psi 1, 1 + \Psi_{k,k}\times(1 - \Psi_{1,1})$, so I have $P_{k,k} < P_{1,k}$. ☐

Proposition. *(5.6)$P_{only\, bid\, k}$ is decreasing in k. $P_{k\, and\, other}$ is decreasing in k. Moreover, p_k is decreasing in k.*

Proof. First, notice that p_k is decreasing in k is obvious if $P_{only\, bid\, k}$ and $P_{k\, and\, other}$ are both decreasing since $p_k = P_{k\, and\, other} + P_{only\, bid\, k}$.

For $P_{only\, bid\, k}$, notice that $\Psi_{k,k}\cdot(V - k) = \Psi_{k',k'}\cdot(V - k')$. Then by applying Poisson approximation for

Q_k, I have $P_{k,k} = P_{only\ bid\ k}$ is decreasing in k.

As for $P_{k\ and\ other}$, by using symmetry in P, and noticing that $\Psi_{k,k_1} = \Psi_{k,k} + \Psi_{k_1,k_1} \cdot (1 - \Psi_{k,k})$ and $\Psi_{k,k_2} = \Psi_{k,k} + \Psi_{k_2,k_2} \cdot (1 - \Psi_{k,k})$, if $k_1 < k_2$, I have $P_{k,k_1} < P_{k,k_2}$. Hence I have $P_{k\ and\ other}$ is decreasing in k. $\square$

Proposition. *(5.7)As j, $j+1$ getting closer to M, $p_j \approx p_{j+1}$. In other words, we should expect a nearly uniform distribution of bidding probability at higher bids.*

Proof. It is enough to show that for i closer or equal to 1, $\Psi_{i,j} \approx \Psi_{i,j+1}$. This is obvious since as j closer to M, by previous two propositions, $\prod_{\alpha=1}^{j-1}(1 - Q_\alpha) \cdot Q_j \approx \prod_{\alpha=1}^{j}(1 - Q_\alpha) \cdot Q_{j+1}$. Then as j, $j+1$ getting closer to M, $p_j \approx p_{j+1}$. $\square$

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Simulation Results

Here are the simulation results from previous model sections.

Single-Bid Nash Equilibrium Model

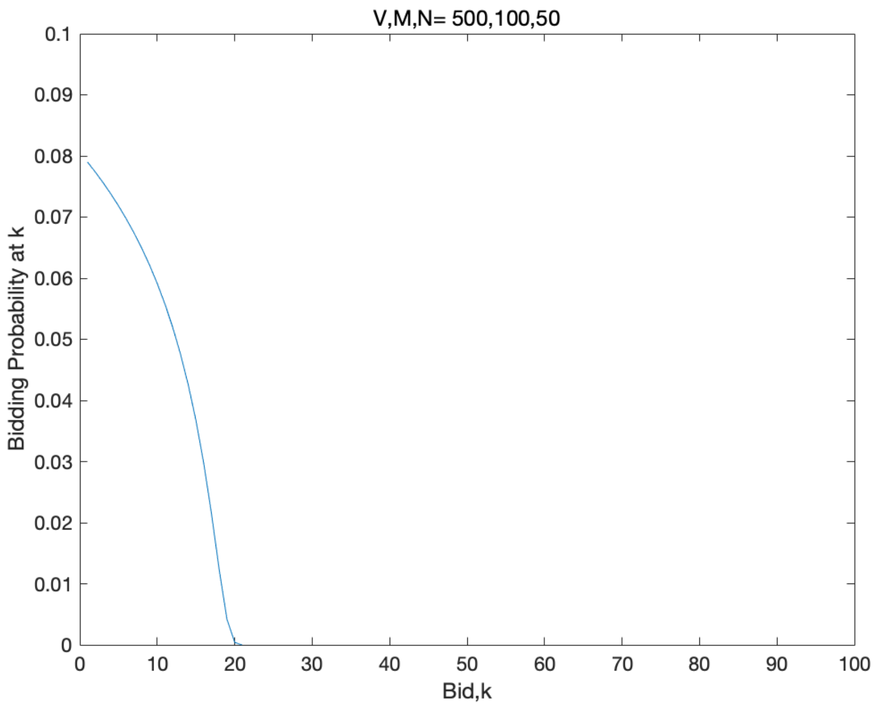

Figure 2: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

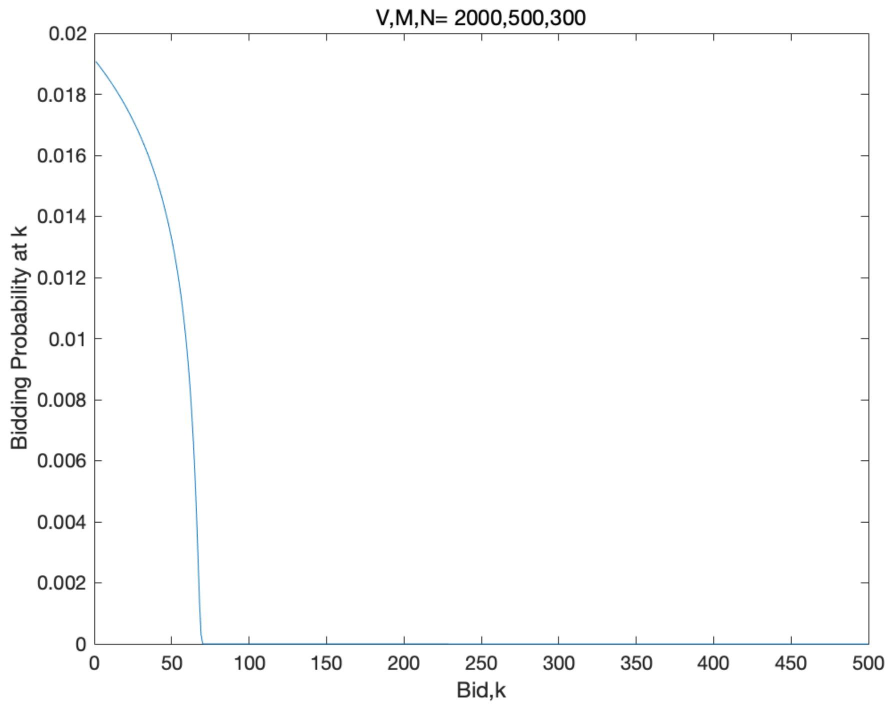

Figure 3: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

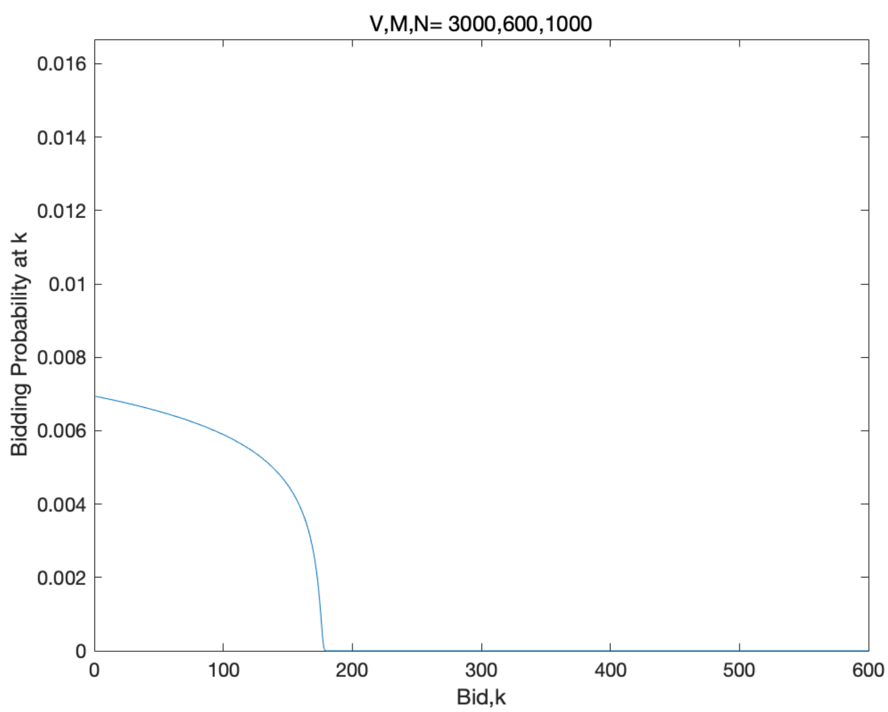

Figure 4: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

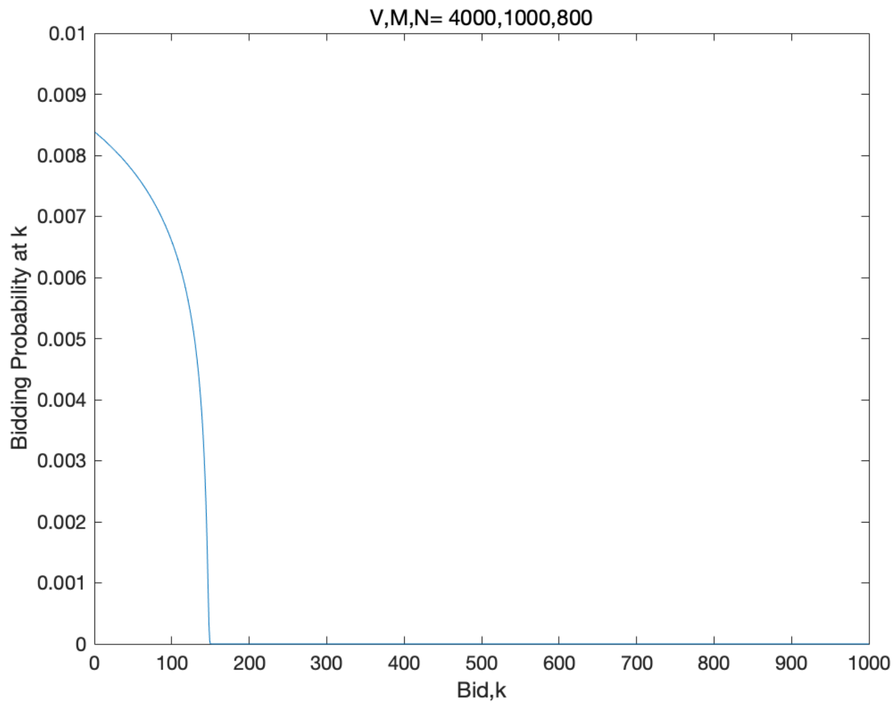

Figure 5: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

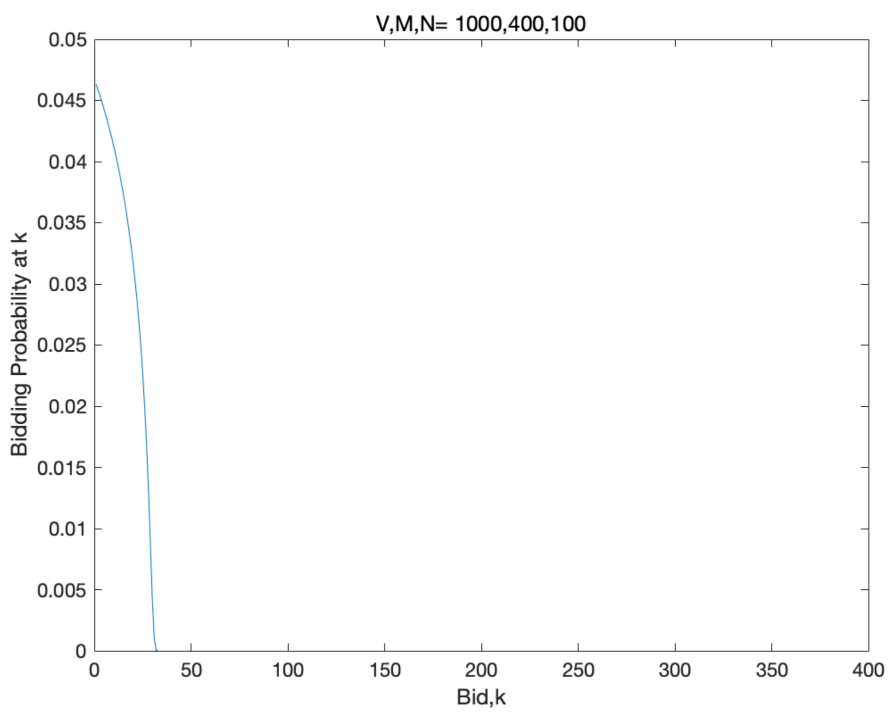

Figure 6: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

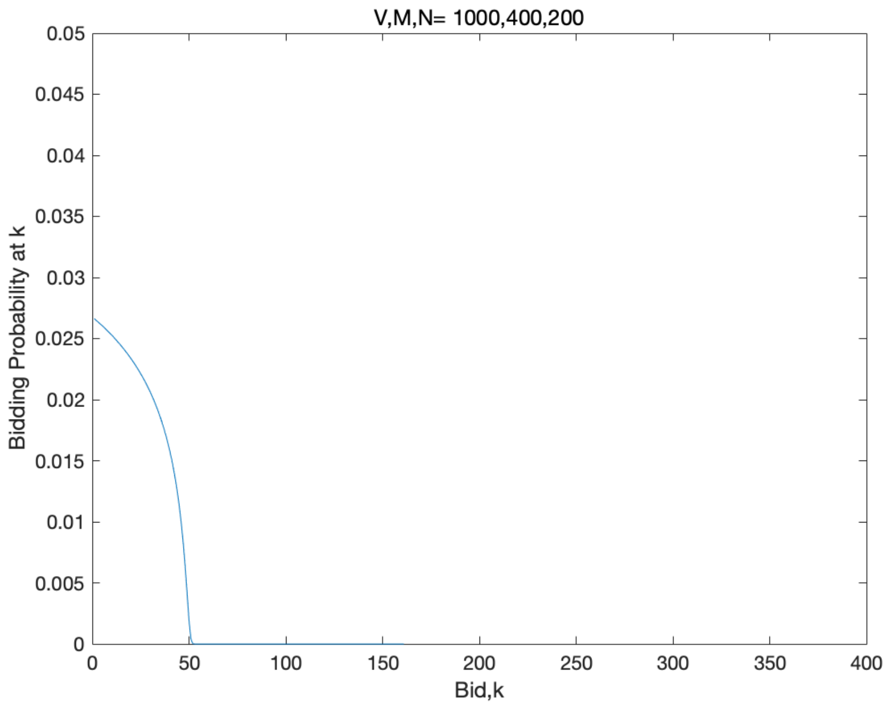

Figure 7: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

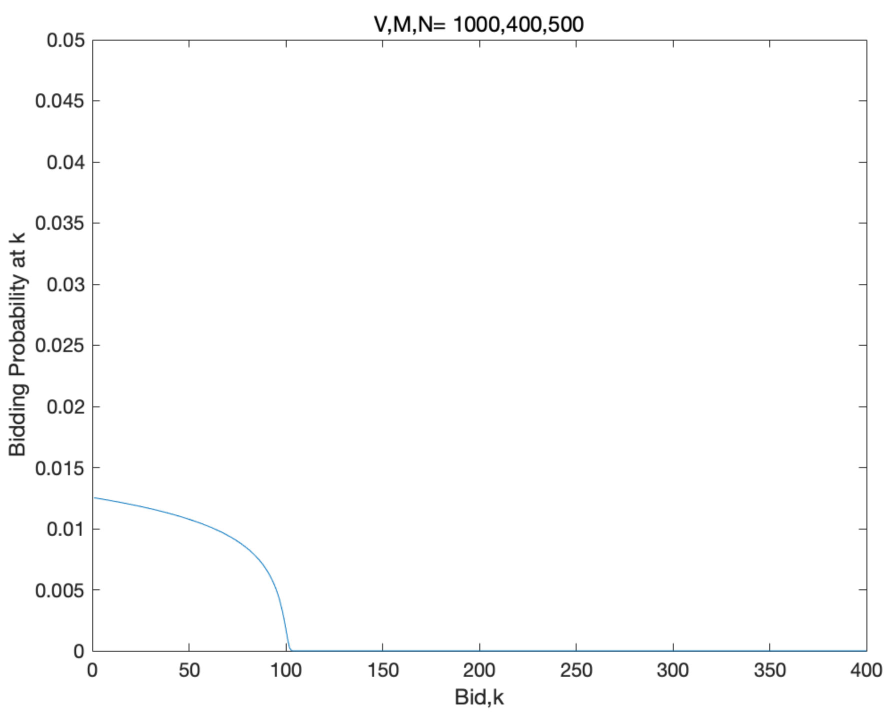

Figure 8: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

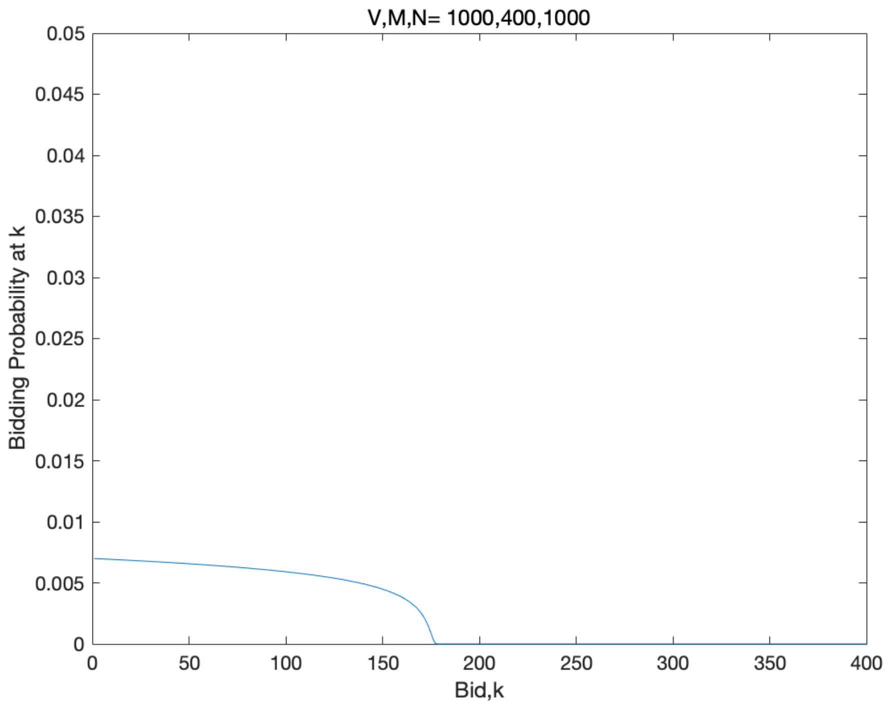

Figure 9: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

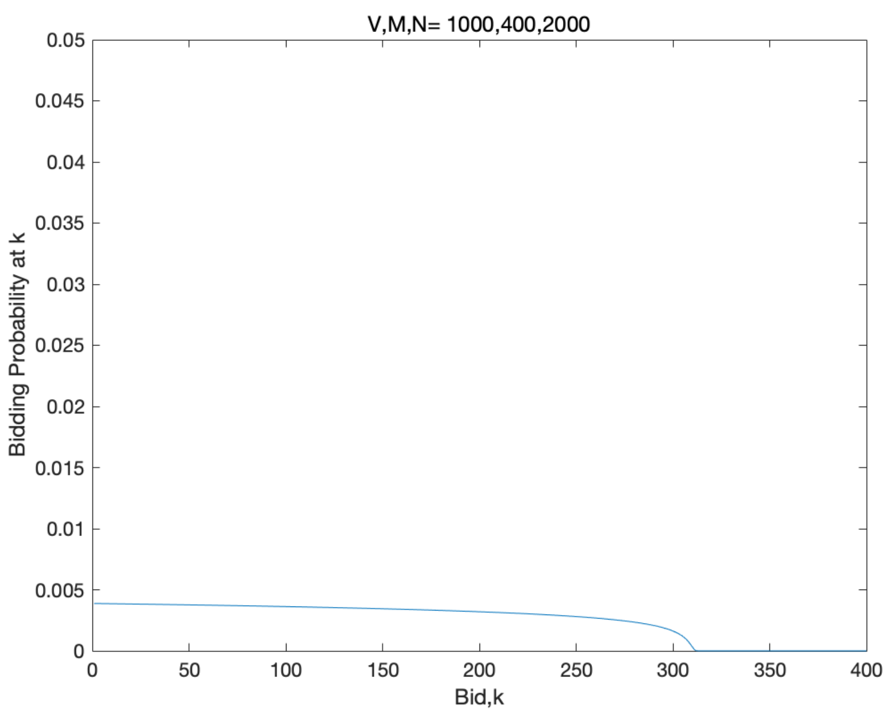

Figure 10: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

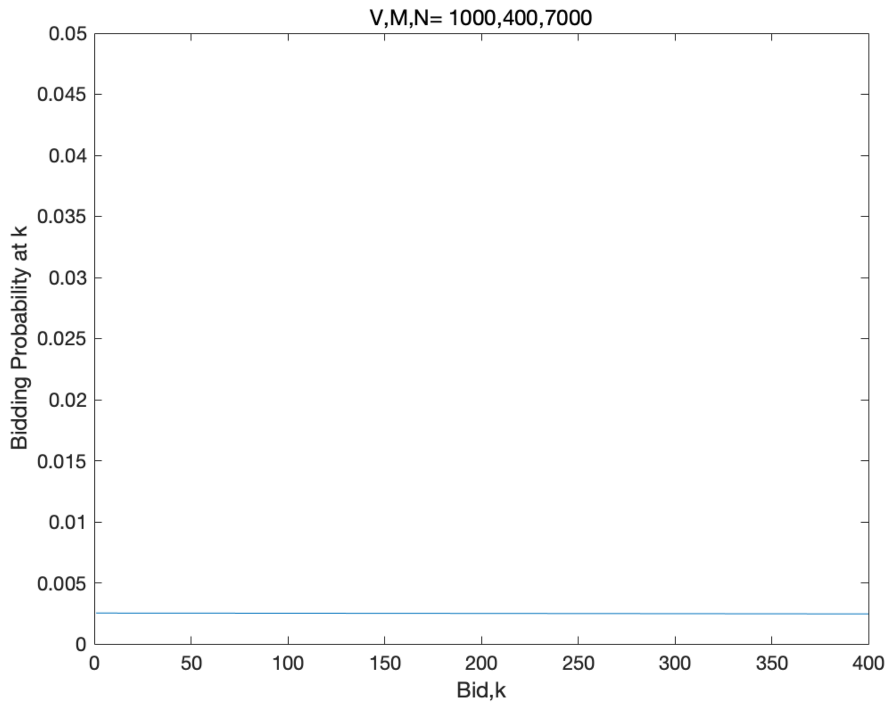

Figure 11: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

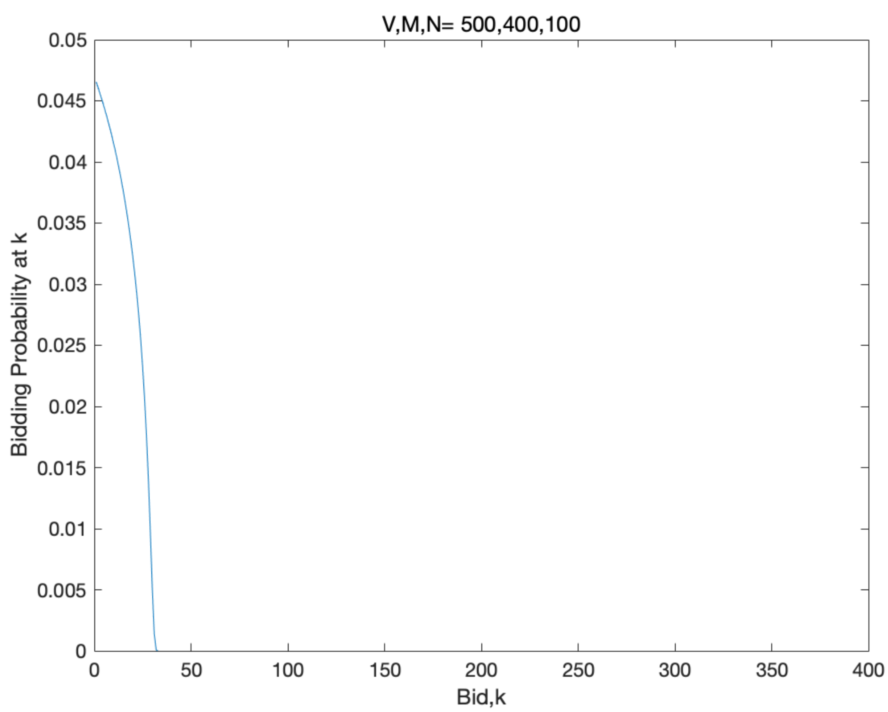

Figure 12: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

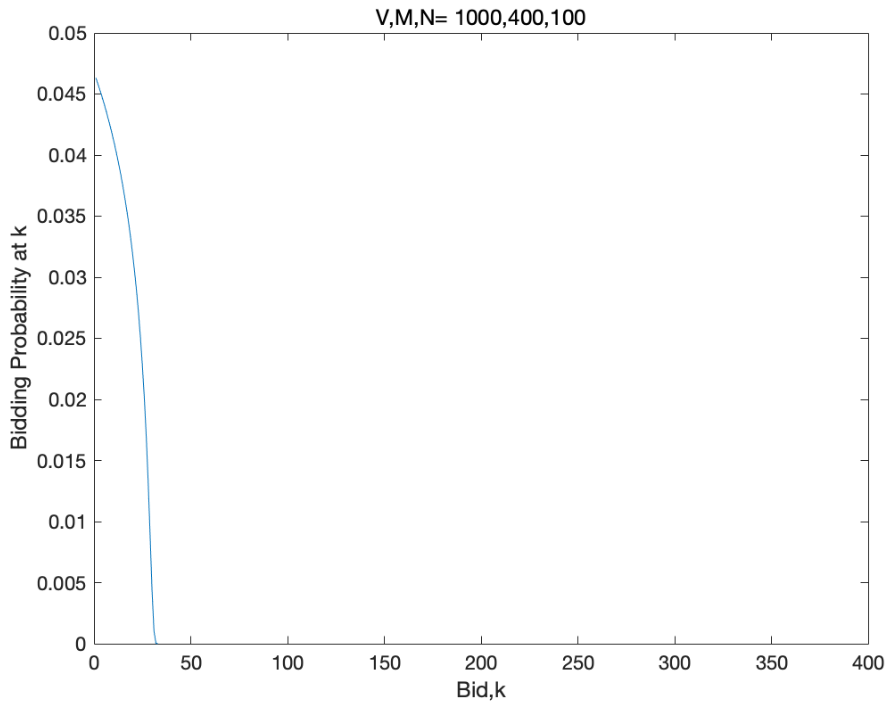

Figure 13: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

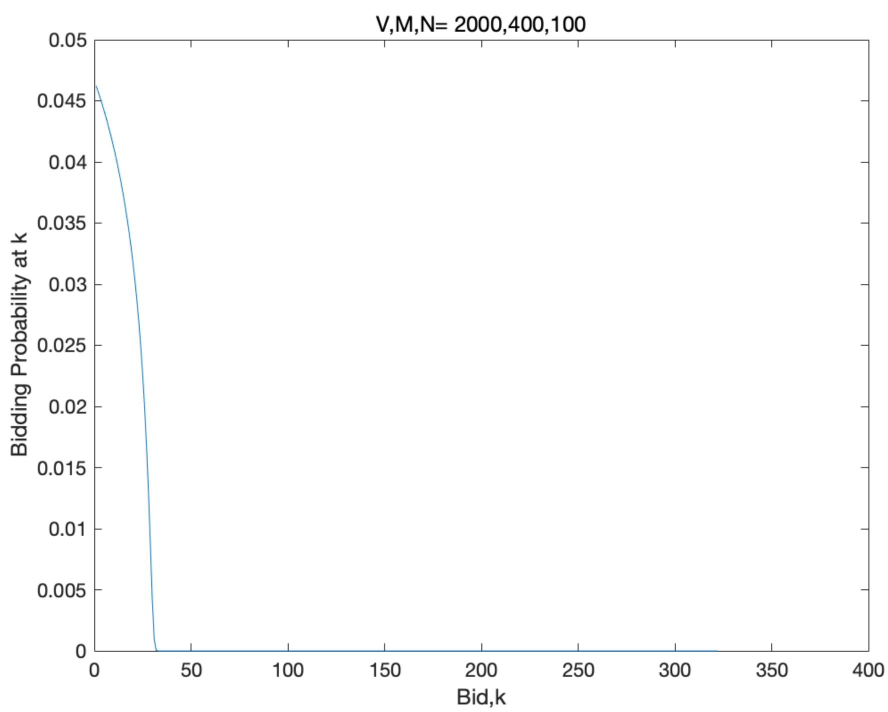

Figure 14: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

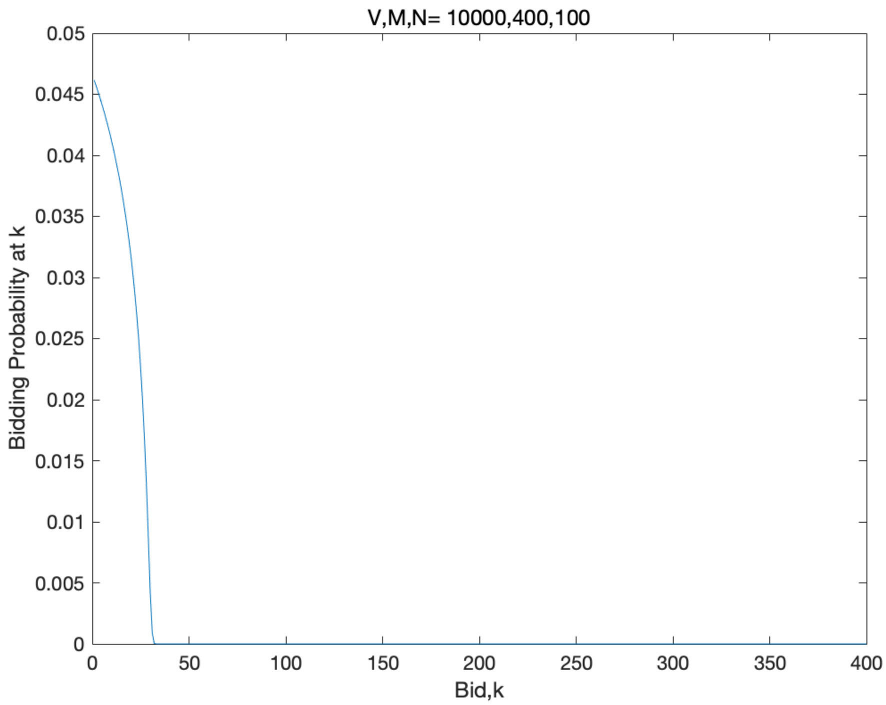

Figure 15: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

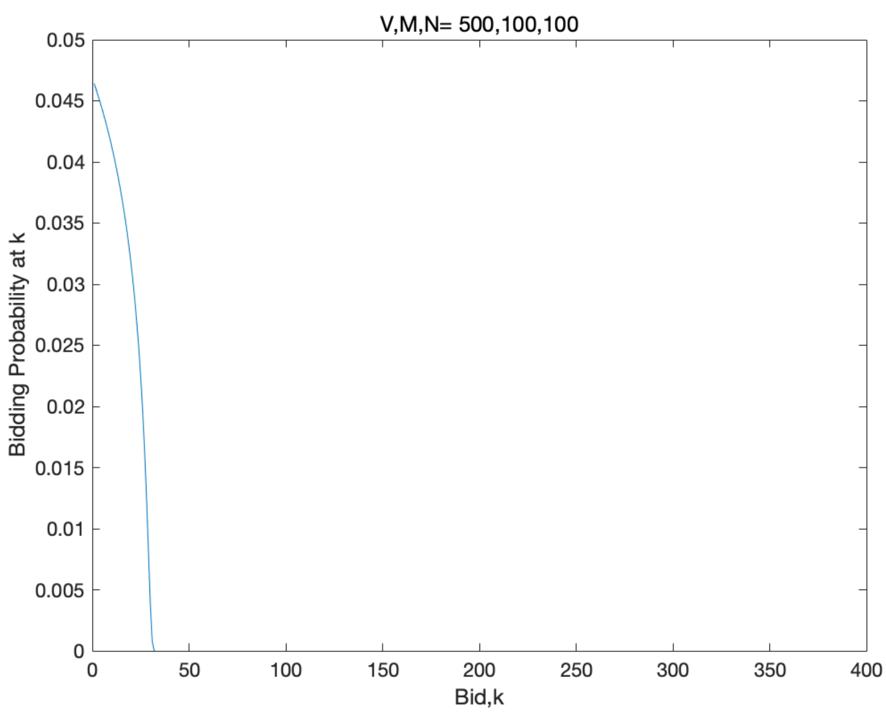

Figure 16: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

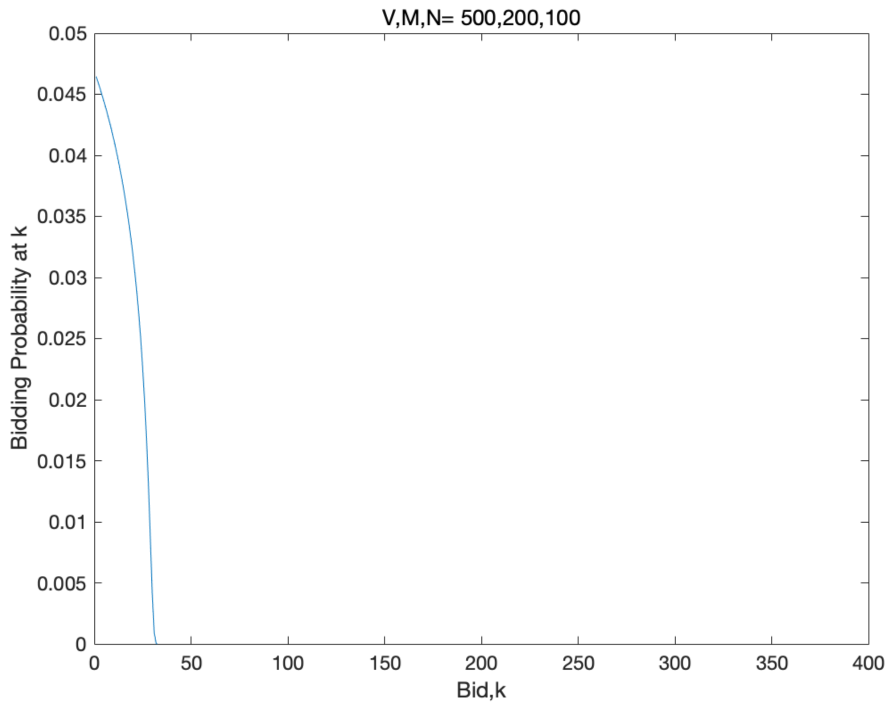

Figure 17: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

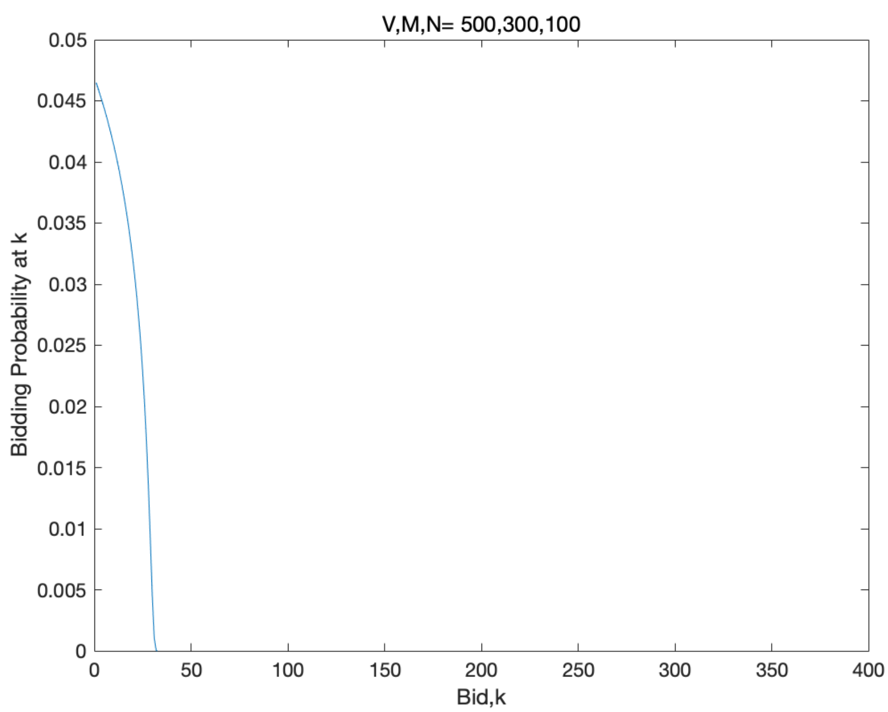

Figure 18: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

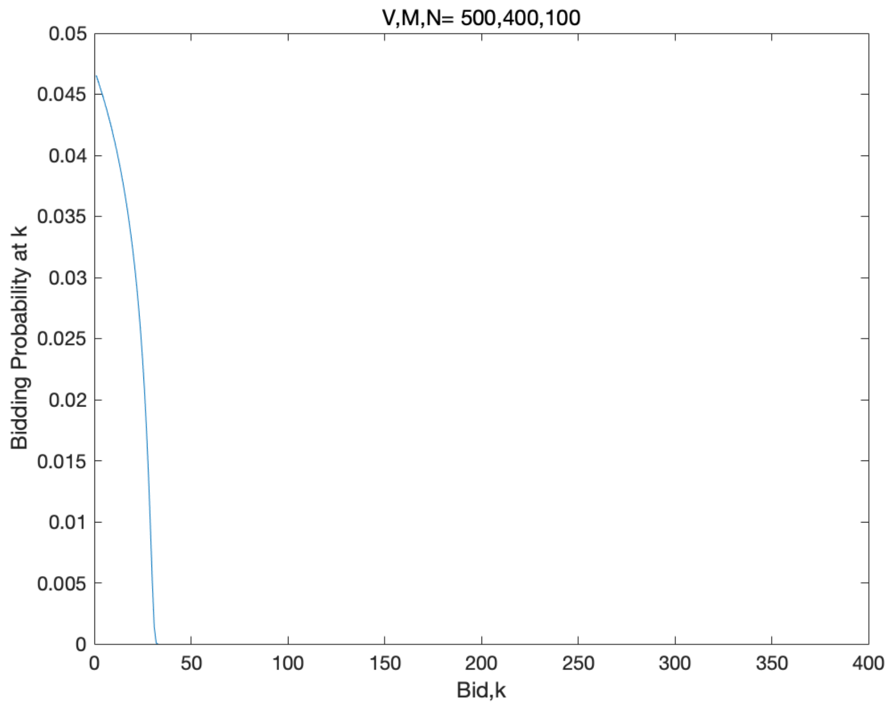

Figure 19: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

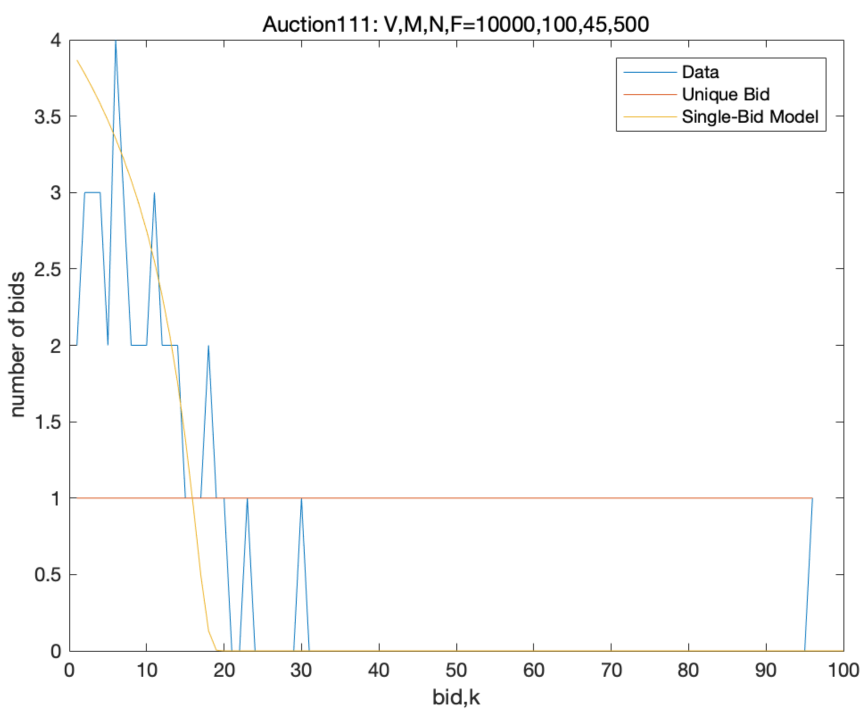

Figure 20: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

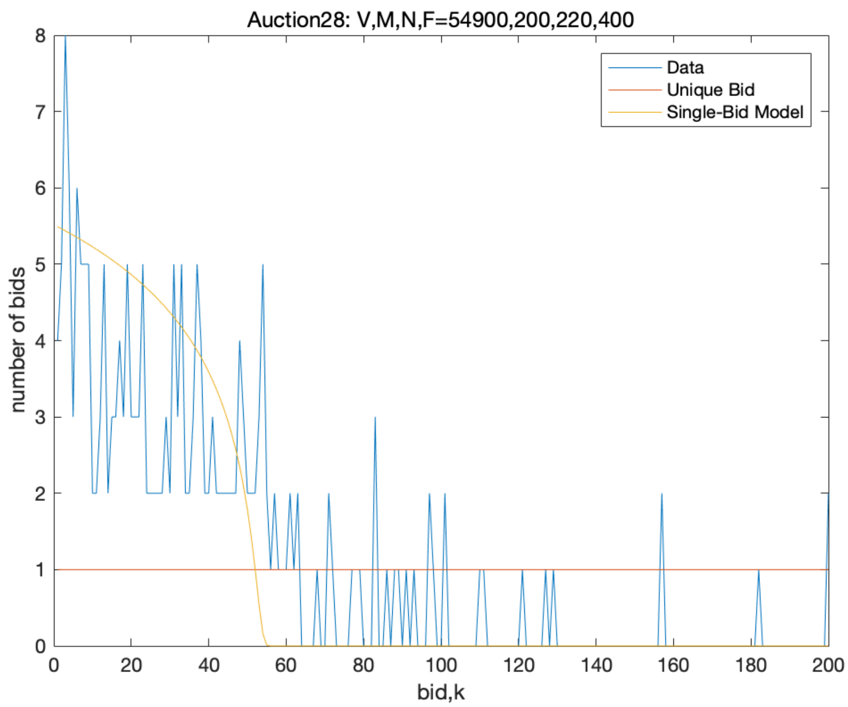

Figure 21: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

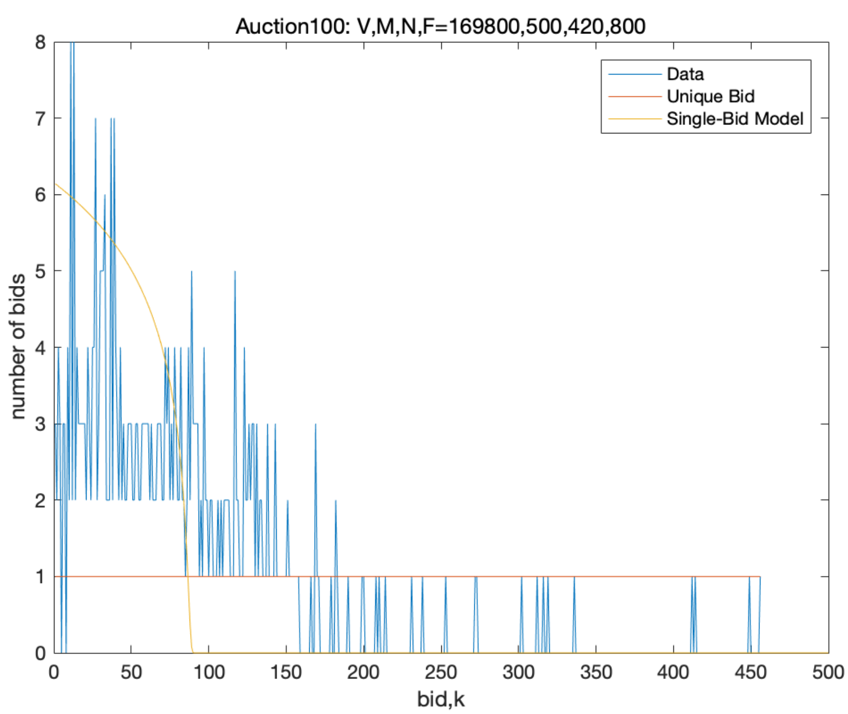

Figure 22: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

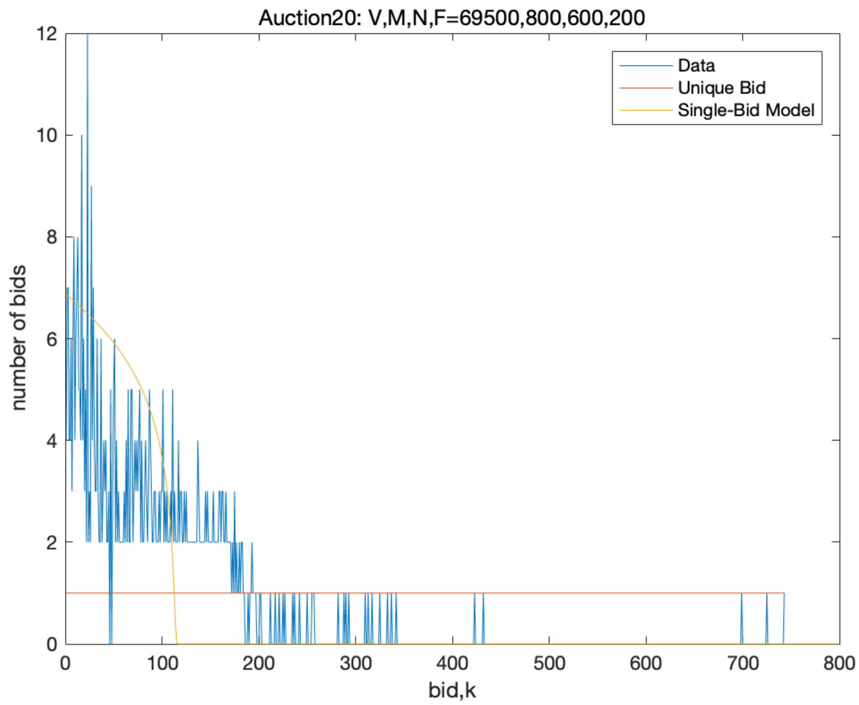

Figure 23: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

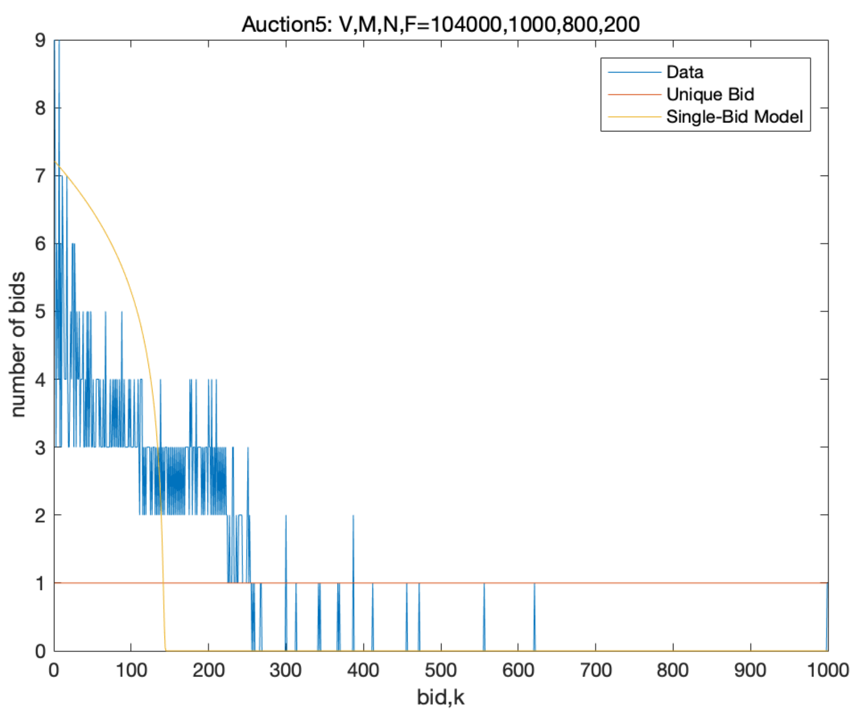

Figure 24: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

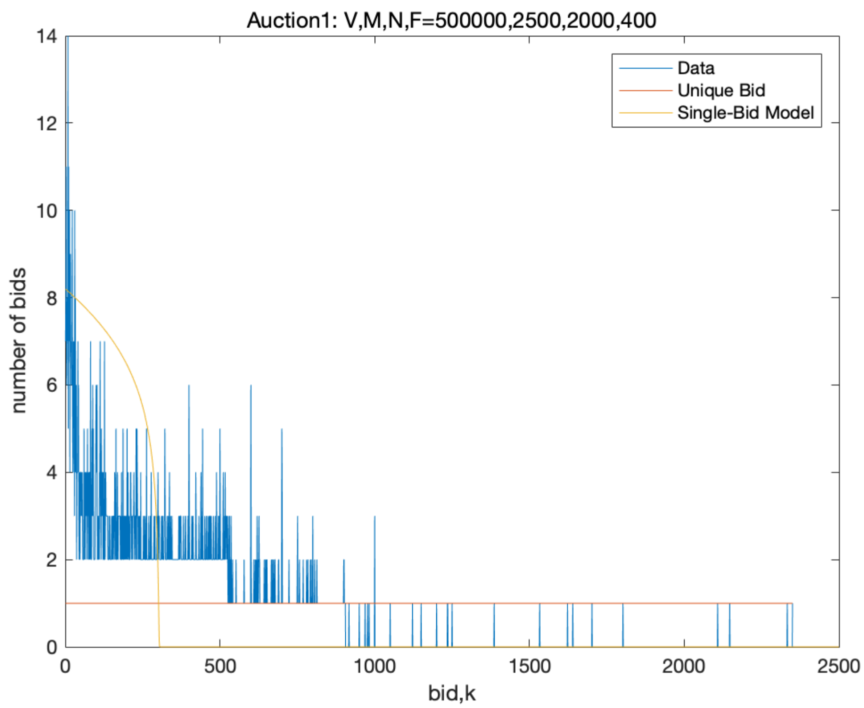

Figure 25: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

Two-Bid Nash Equilibrium Model

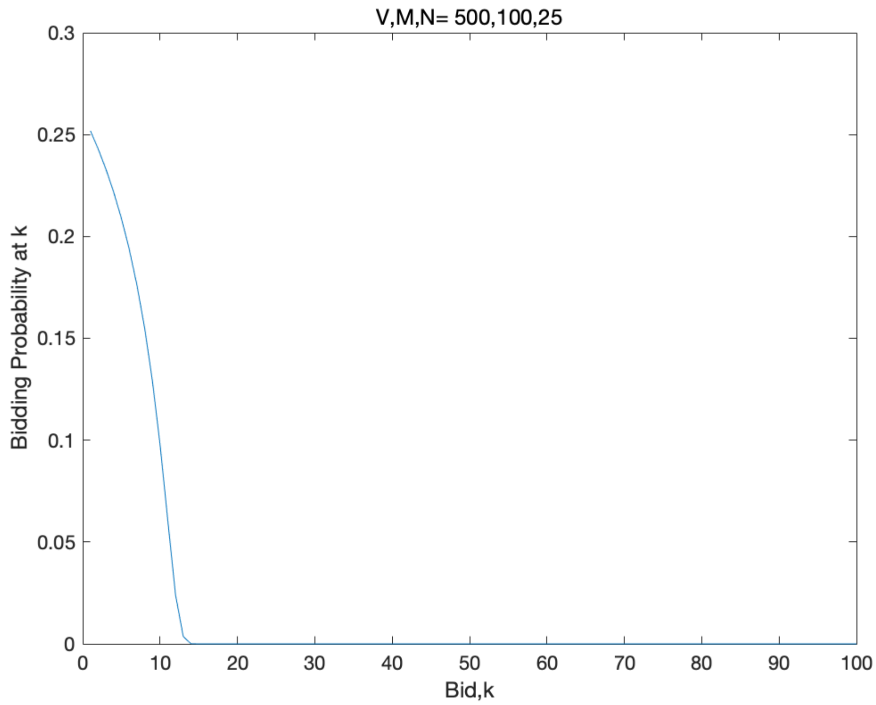

Figure 26: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

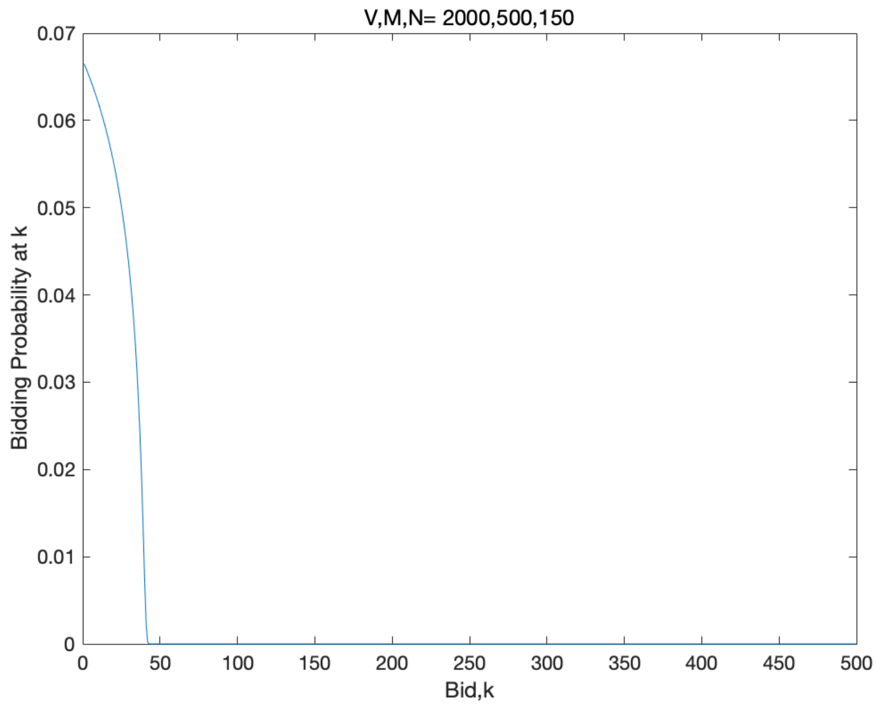

Figure 27: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

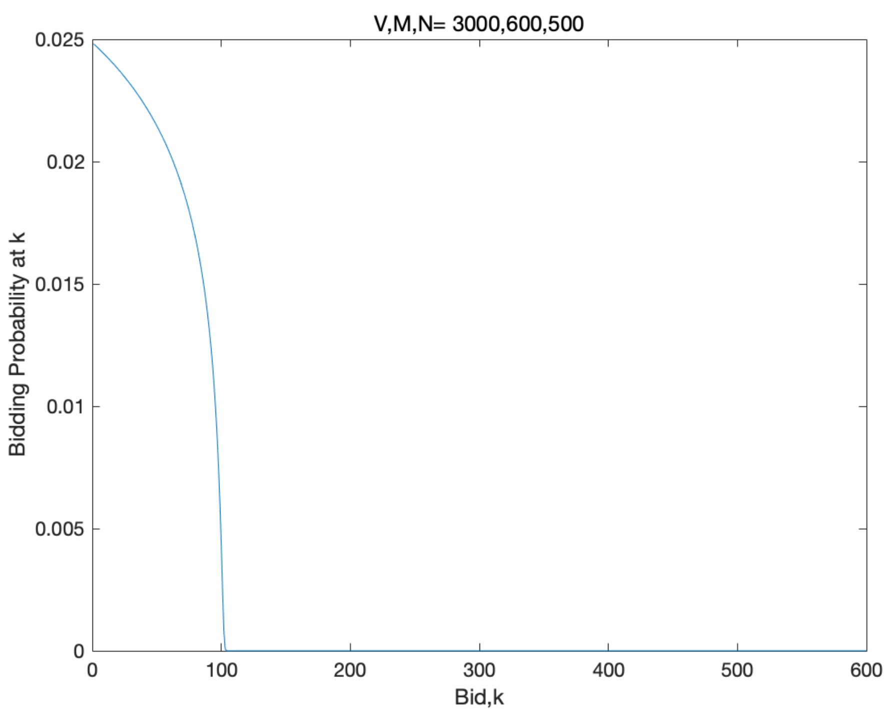

Figure 28: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

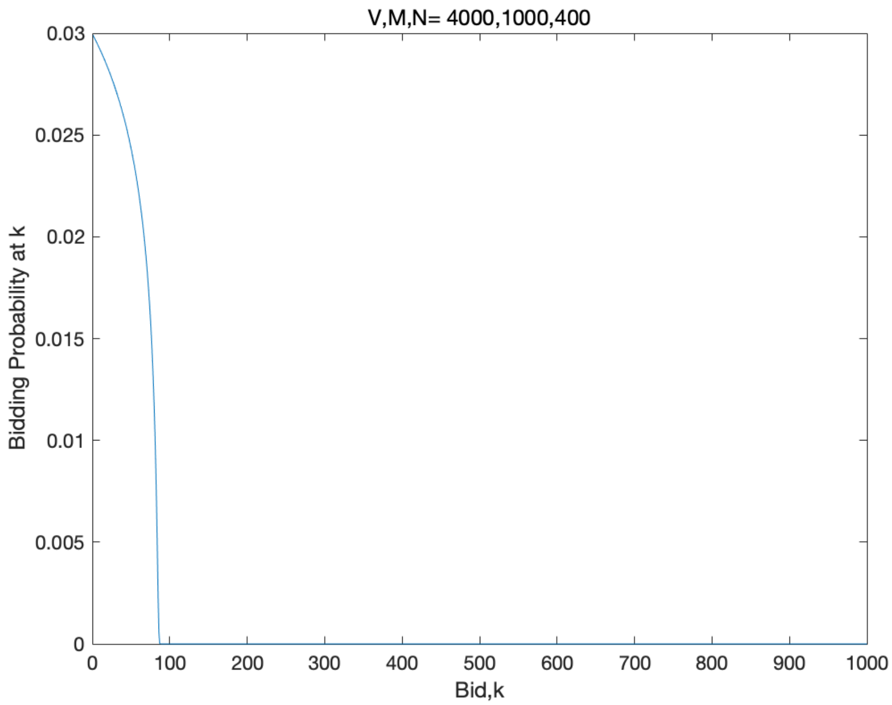

Figure 29: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

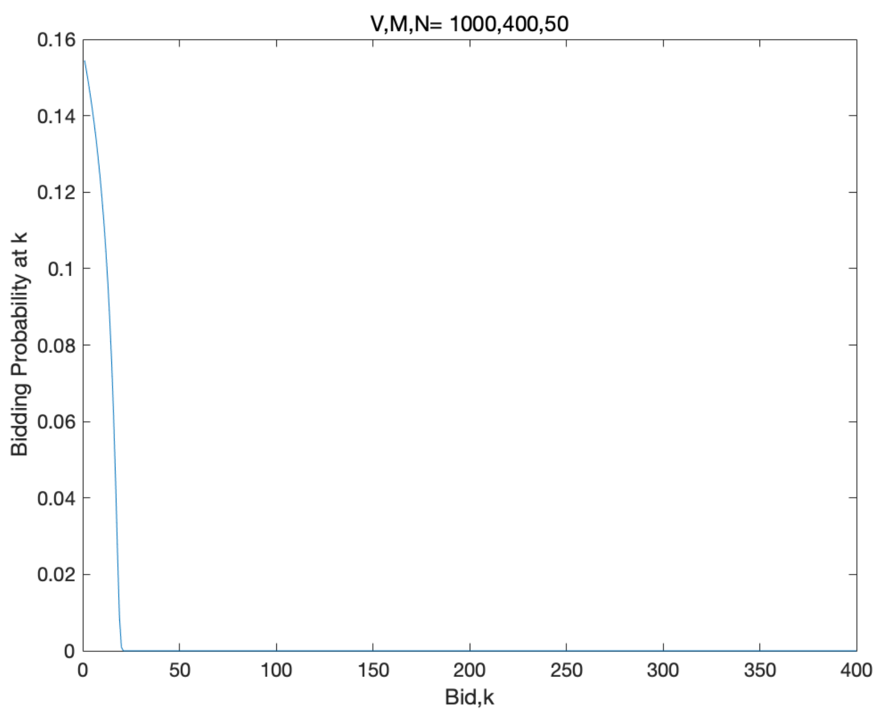

Figure 30: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

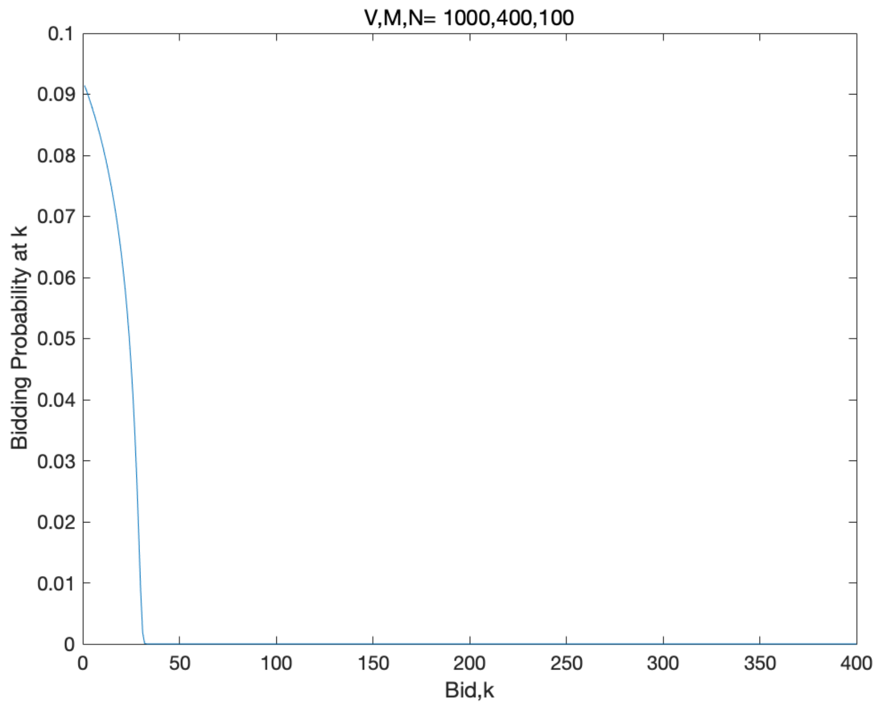

Figure 31: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

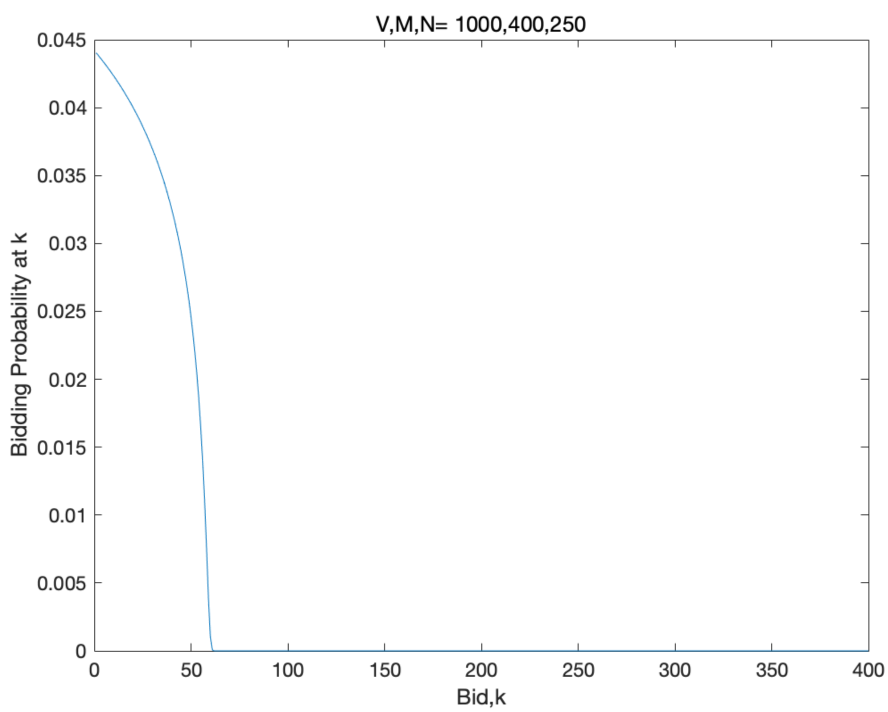

Figure 32: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

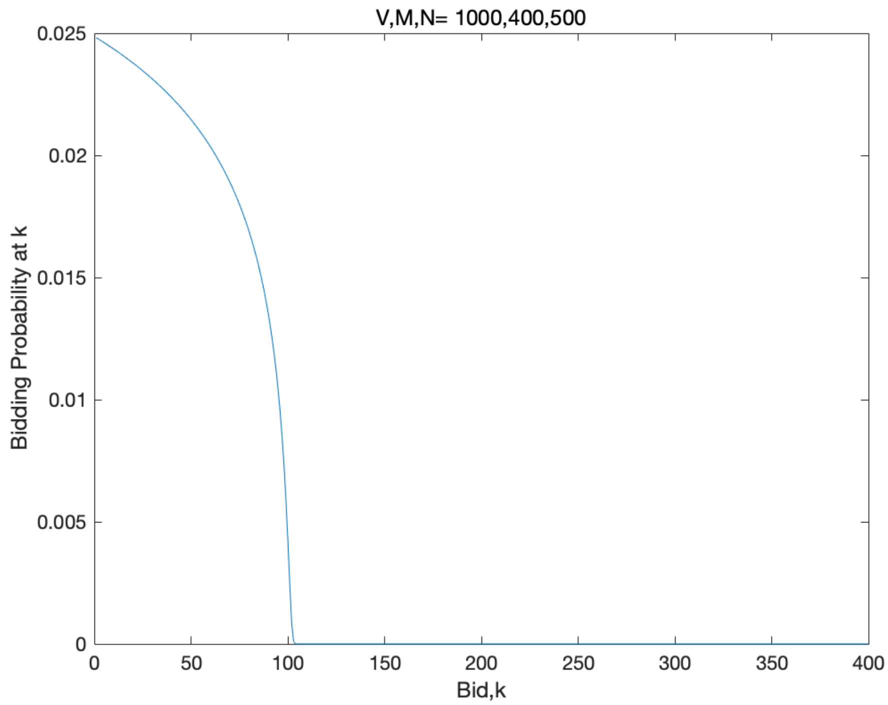

Figure 33: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

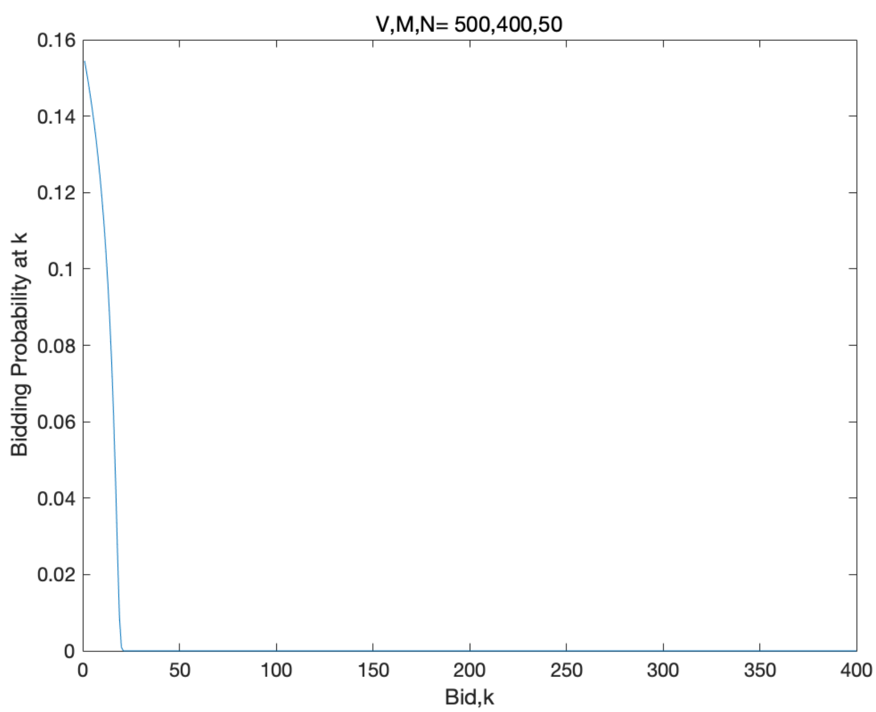

Figure 34: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

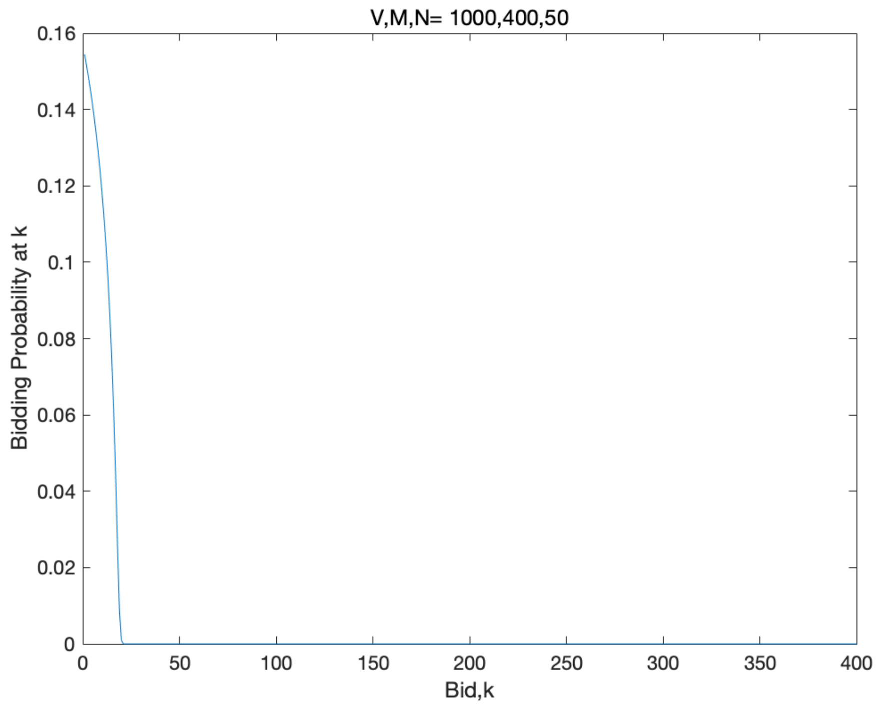

Figure 35: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

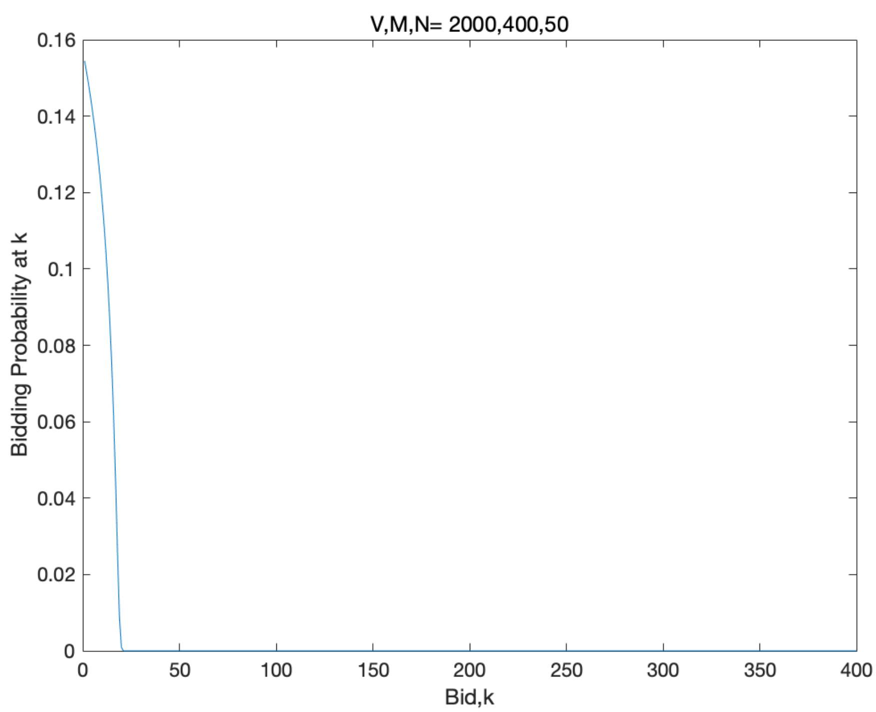

Figure 36: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

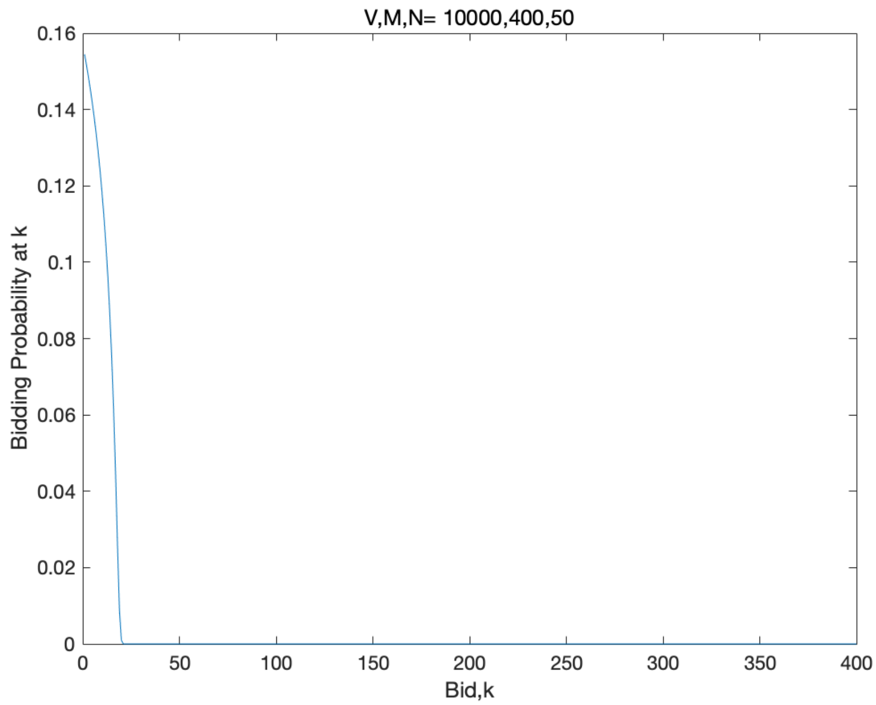

Figure 37: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

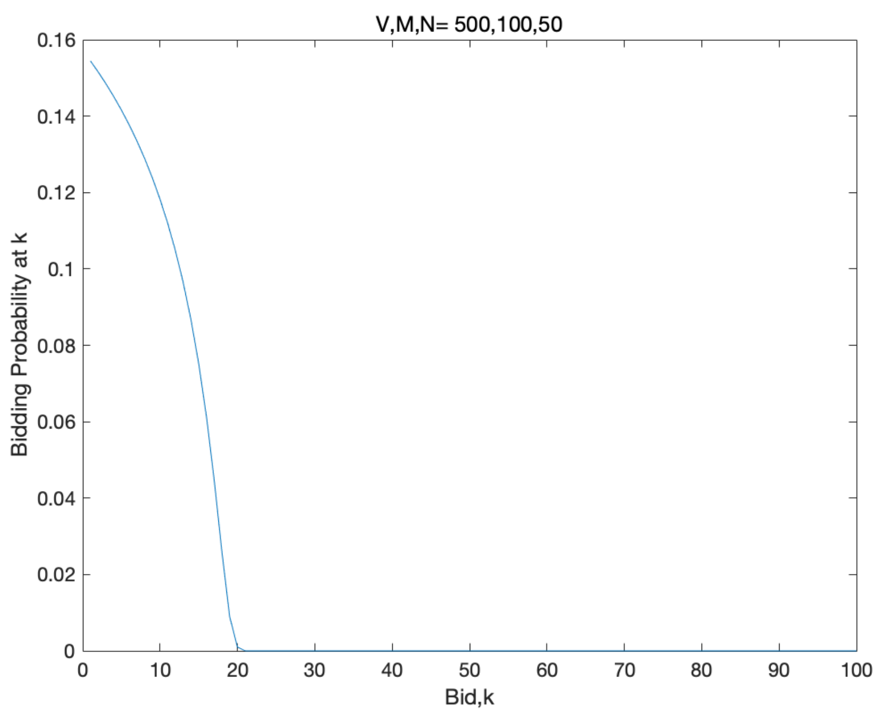

Figure 38: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

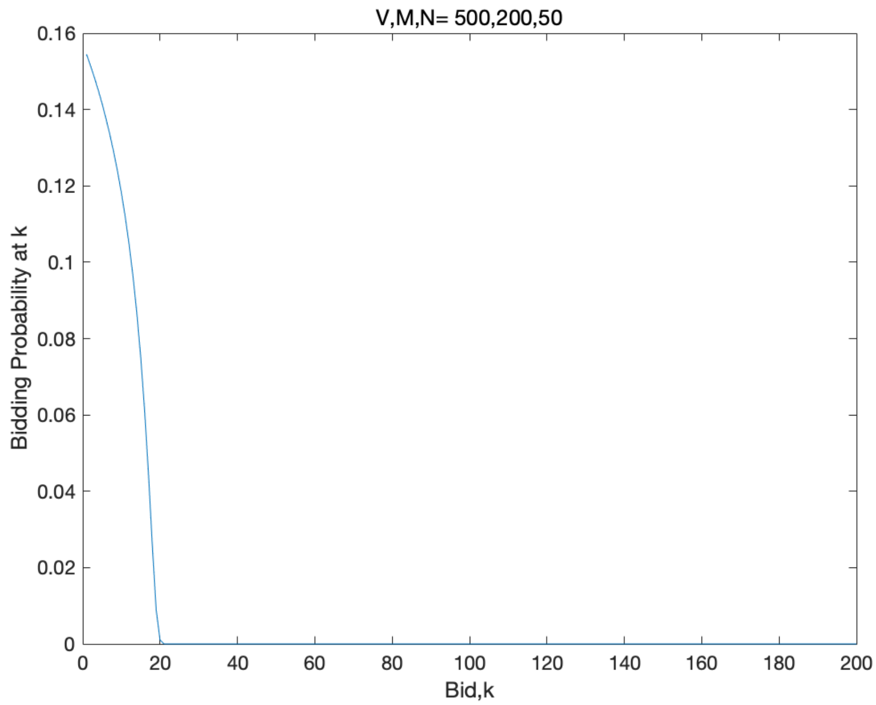

Figure 39: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

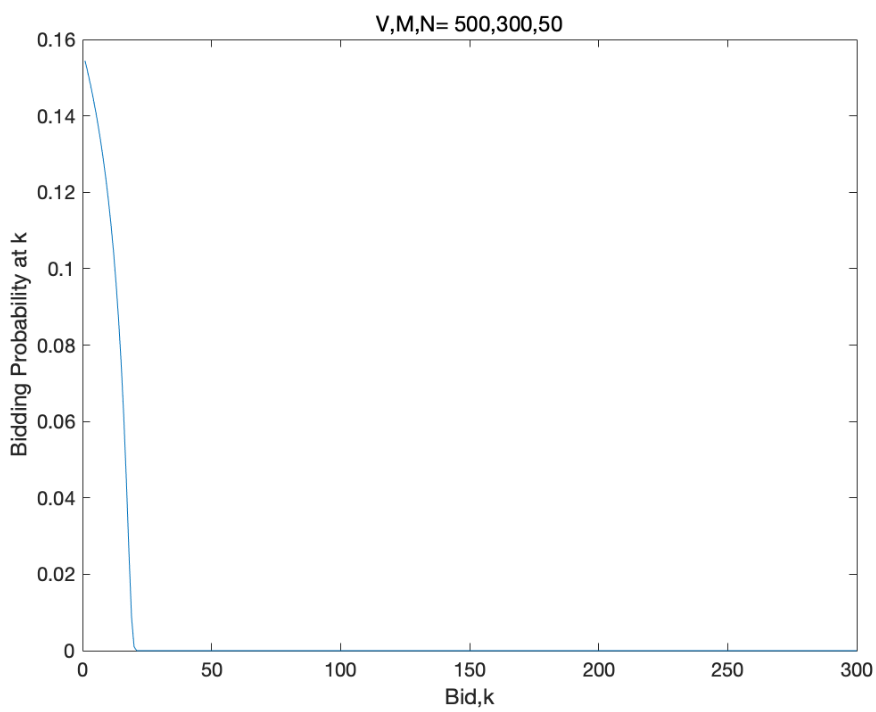

Figure 40: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

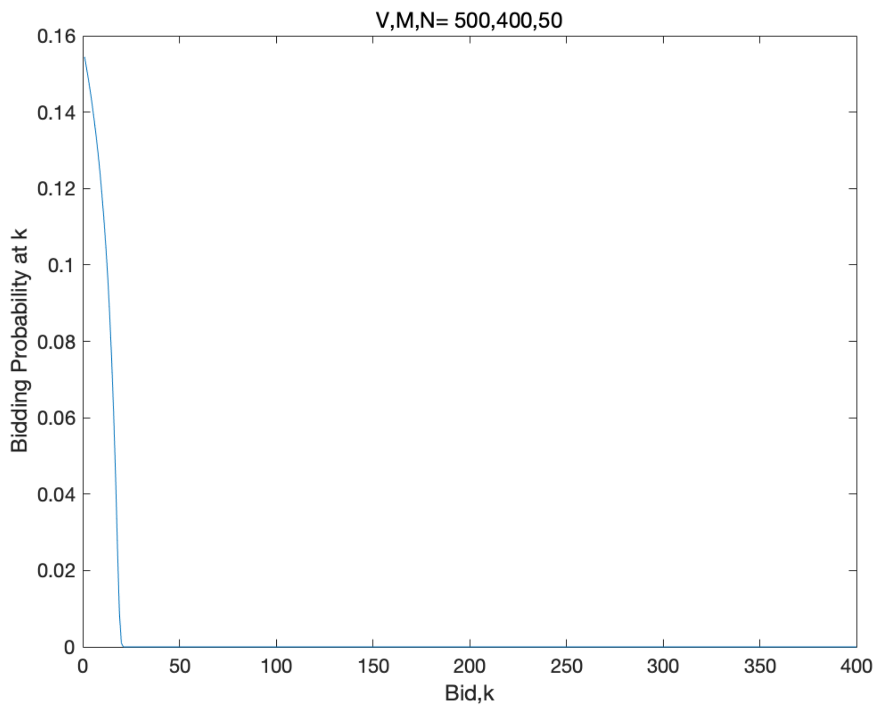

Figure 41: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

Two-Bid Probability Matching Model

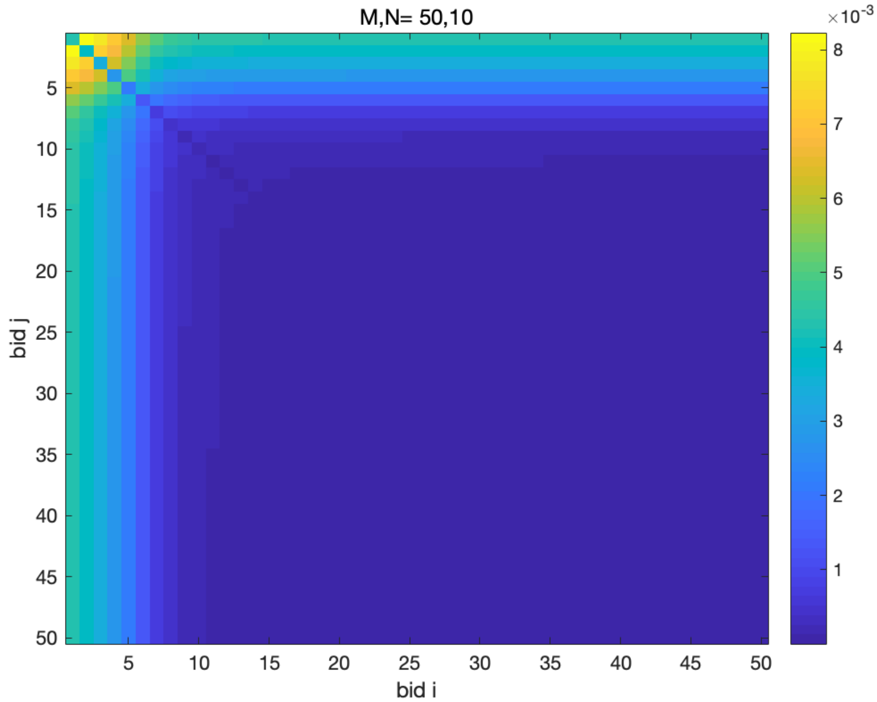

Figure 42: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

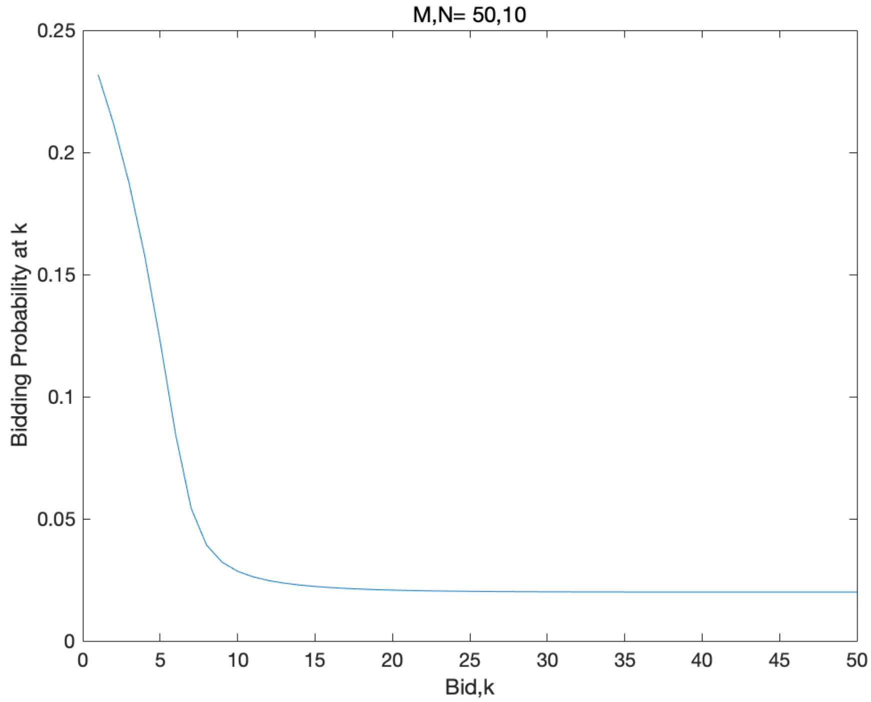

Figure 43: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

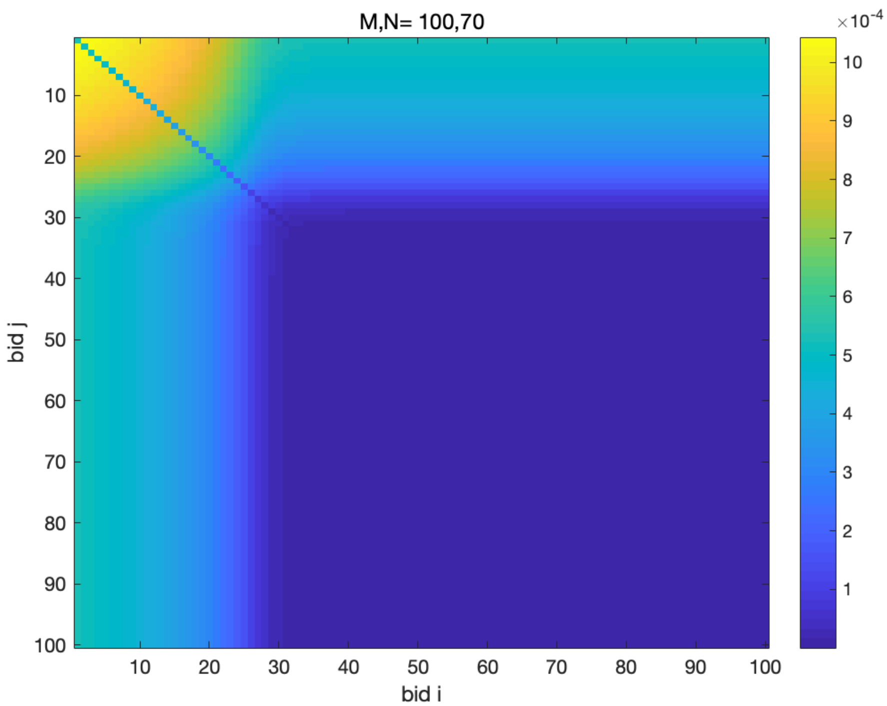

Figure 44: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

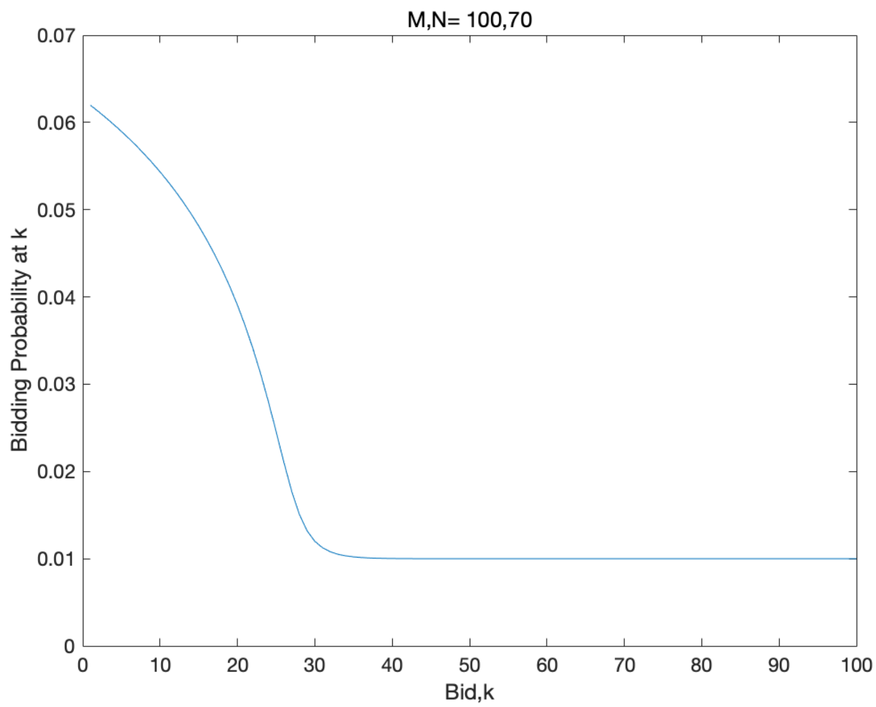

Figure 45: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

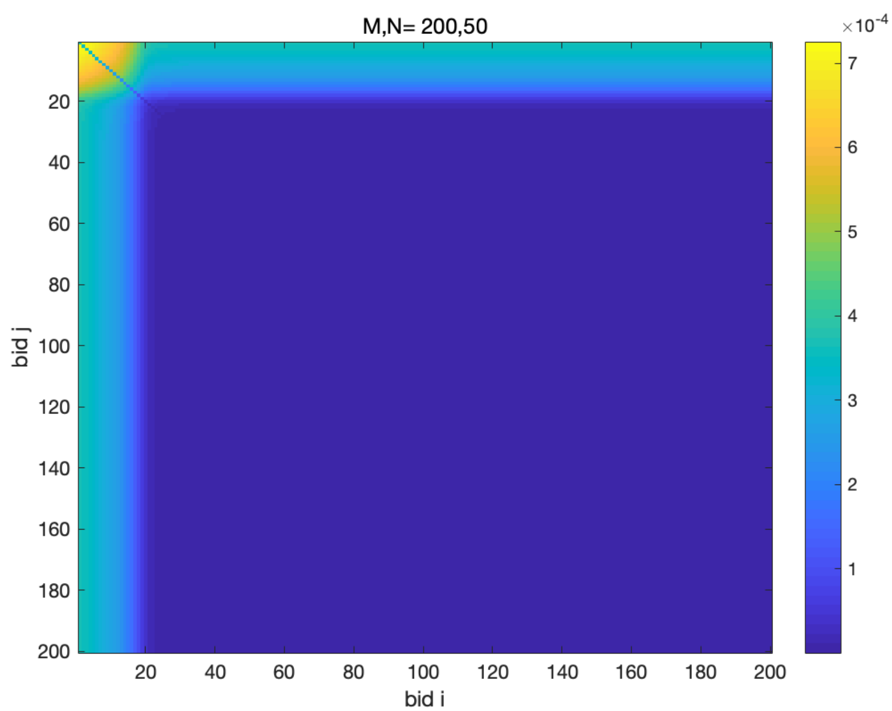

Figure 46: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

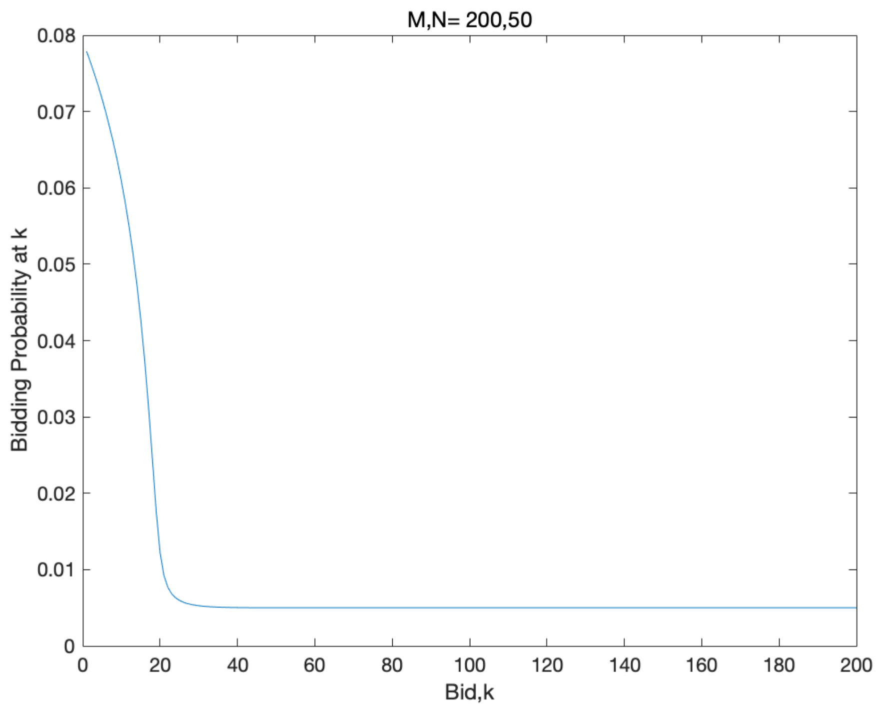

Figure 47: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

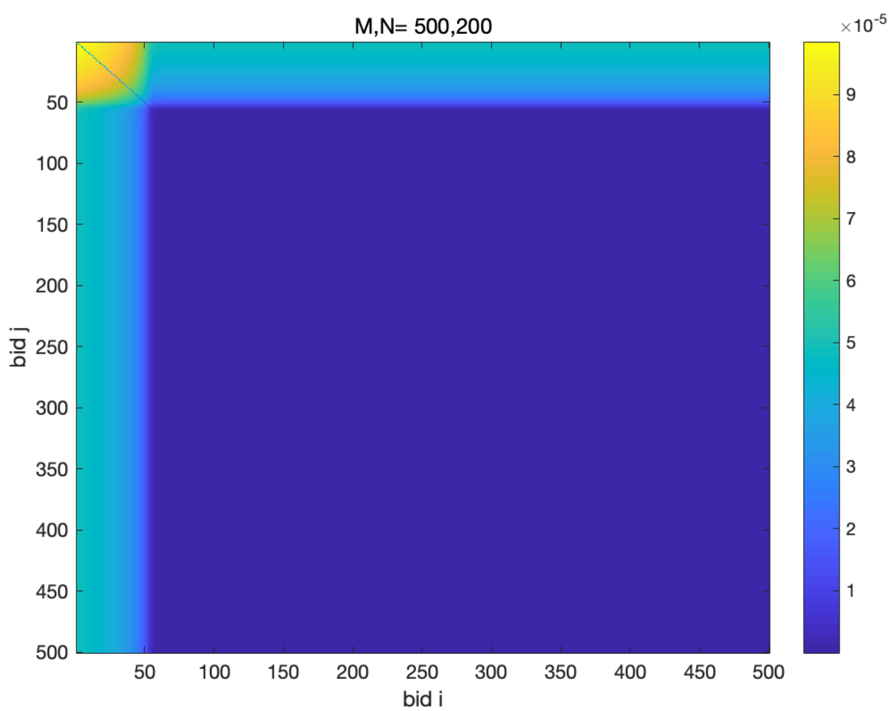

Figure 48: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

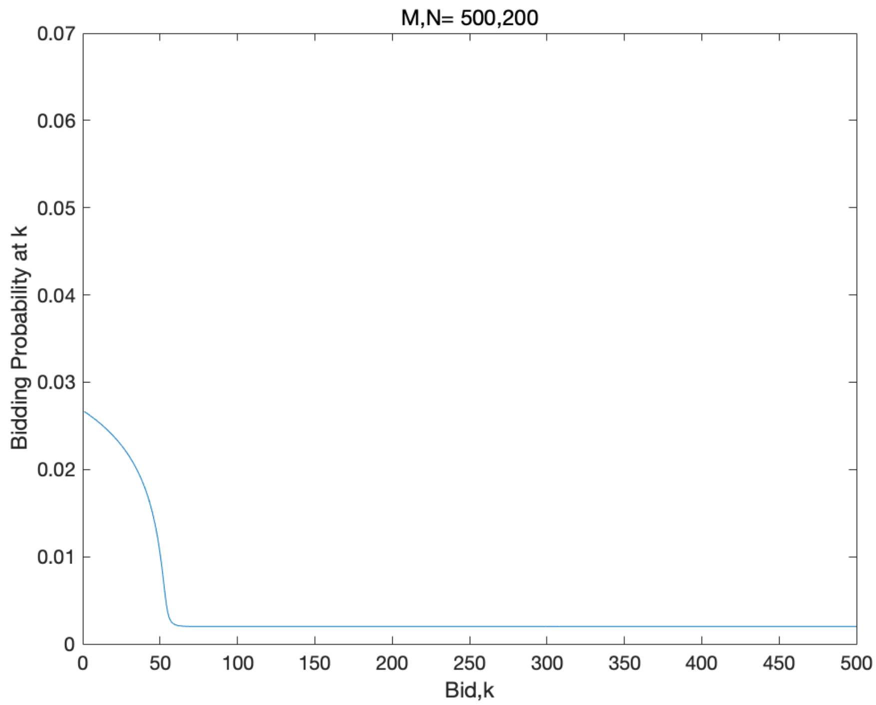

Figure 49: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

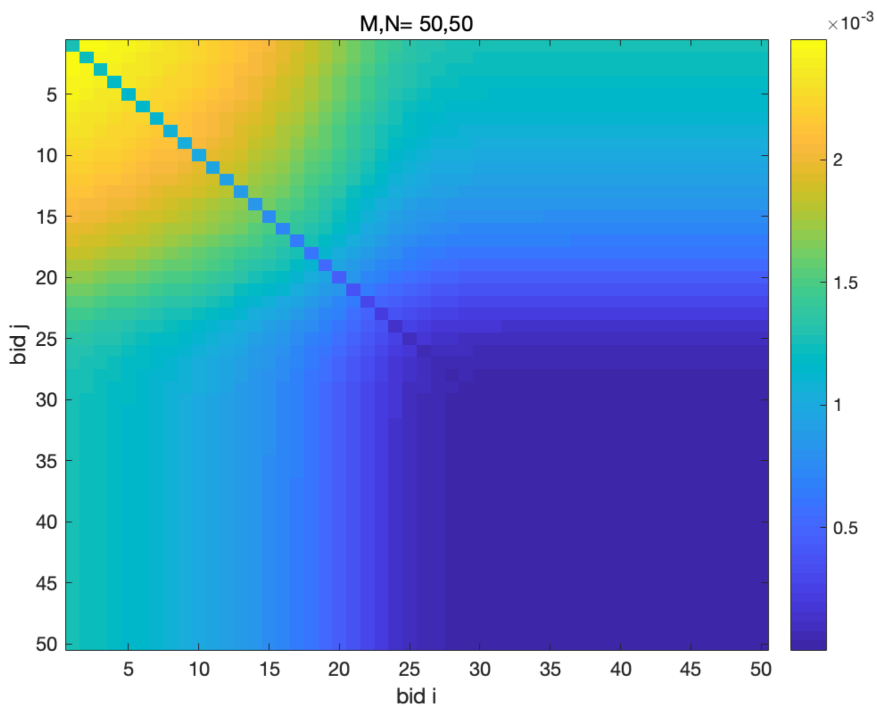

Figure 50: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

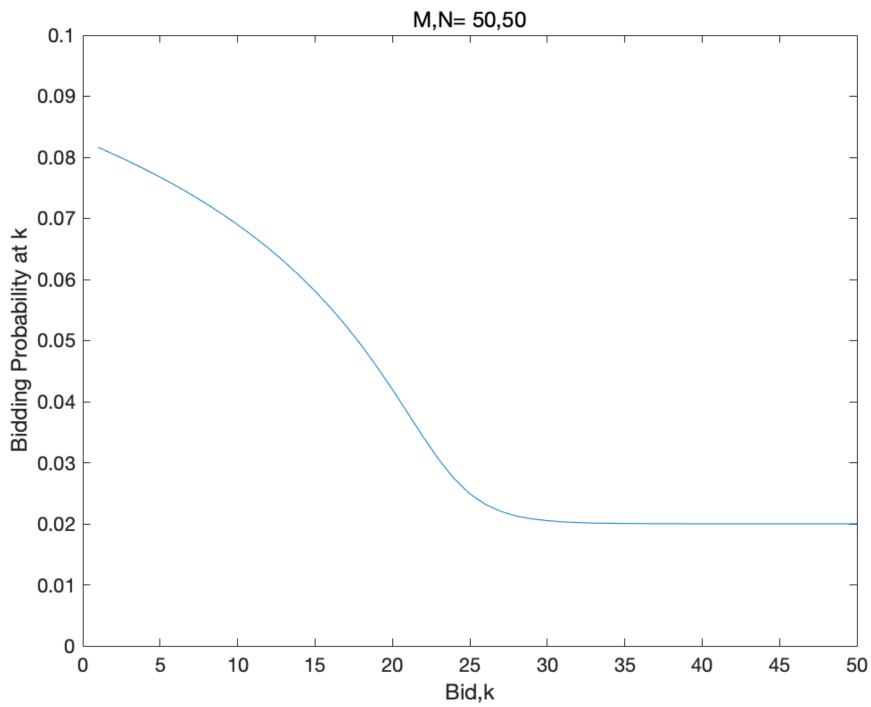

Figure 51: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

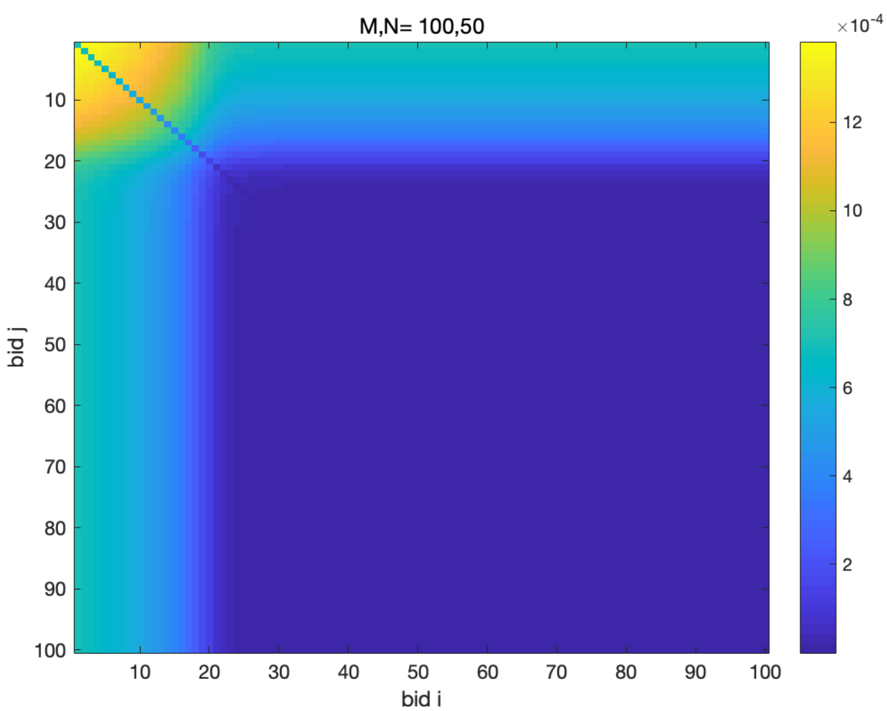

Figure 52: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

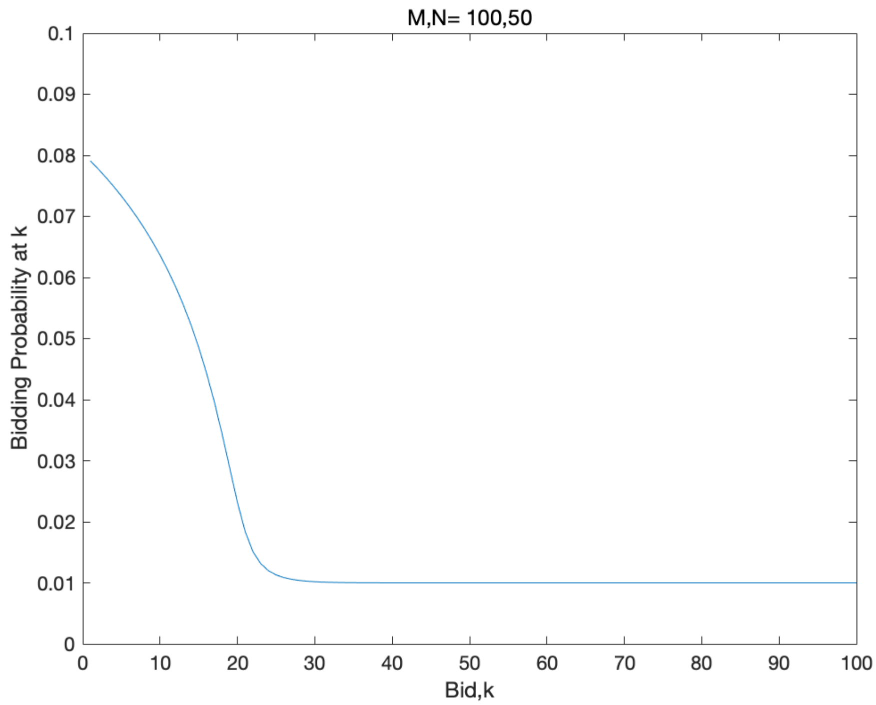

Figure 53: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

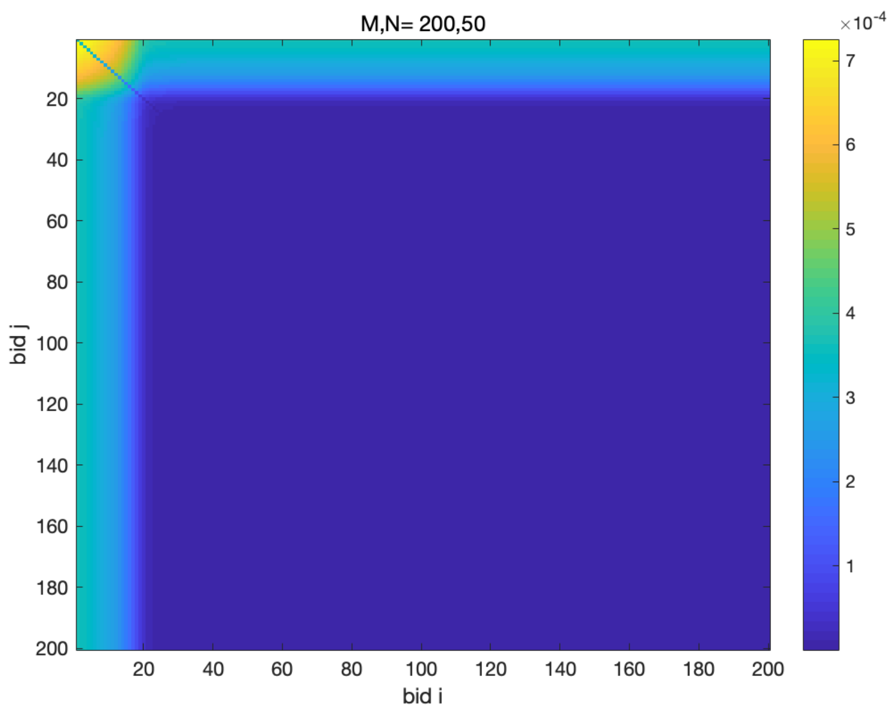

Figure 54: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

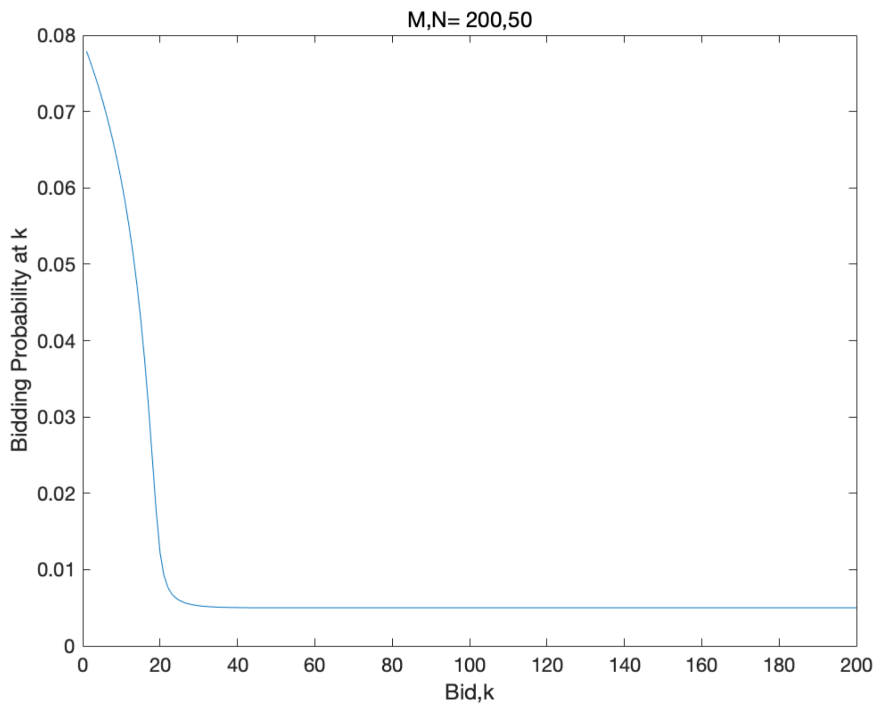

Figure 55: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

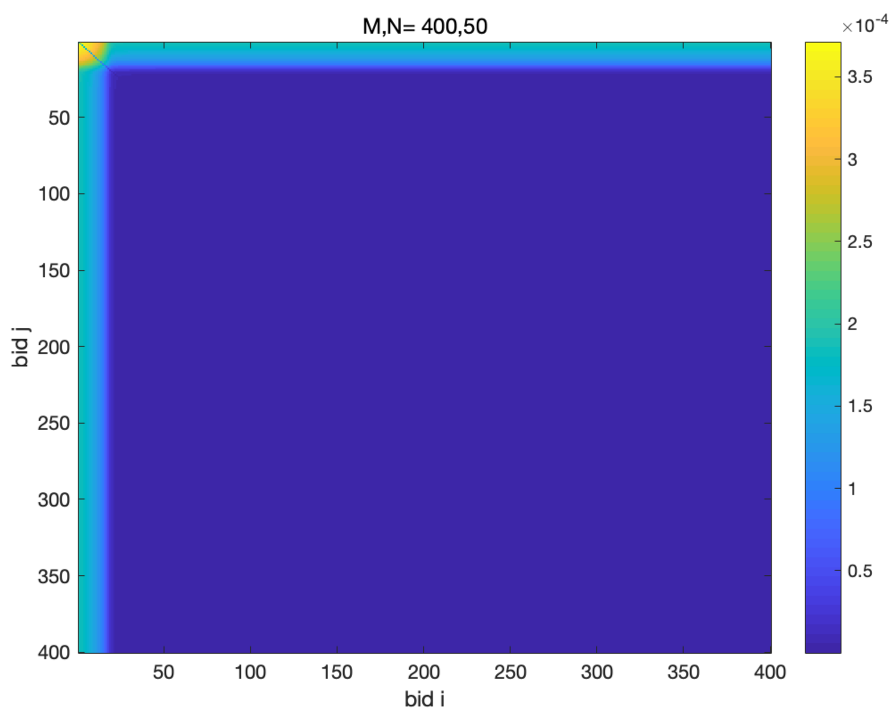

Figure 56: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

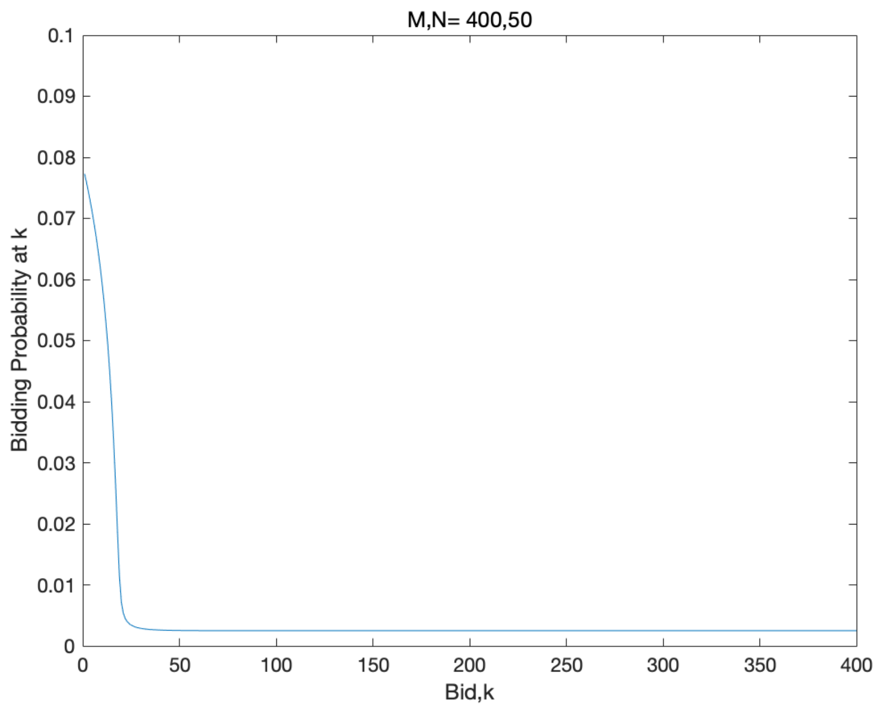

Figure 57: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

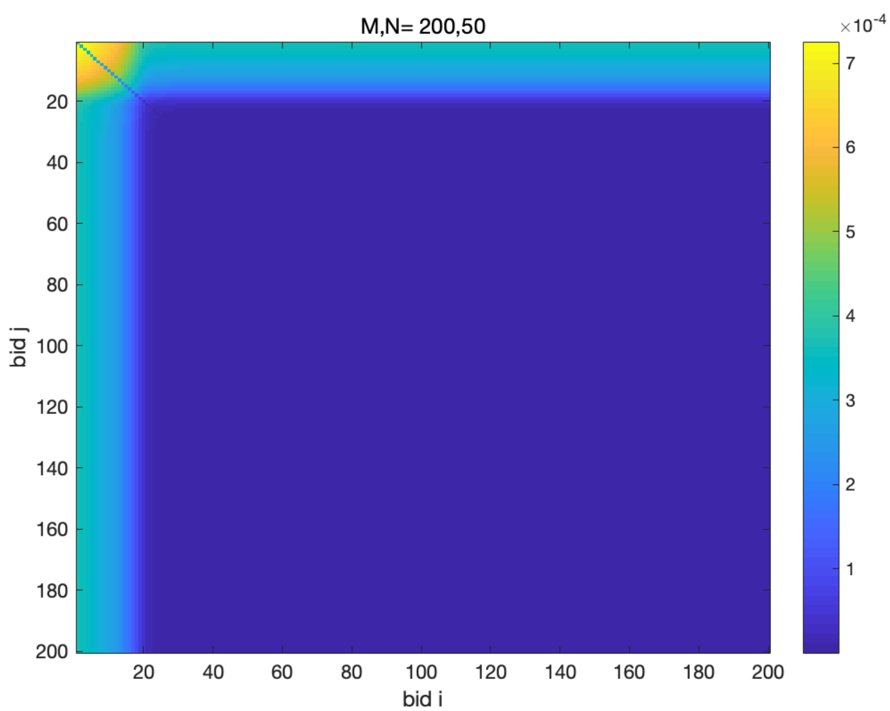

Figure 58: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

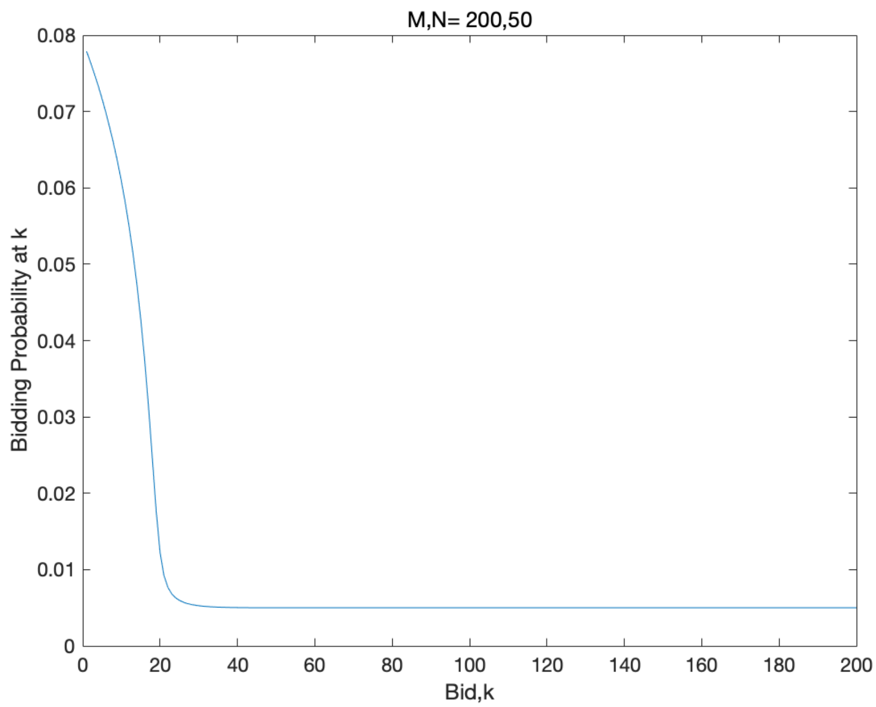

Figure 59: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

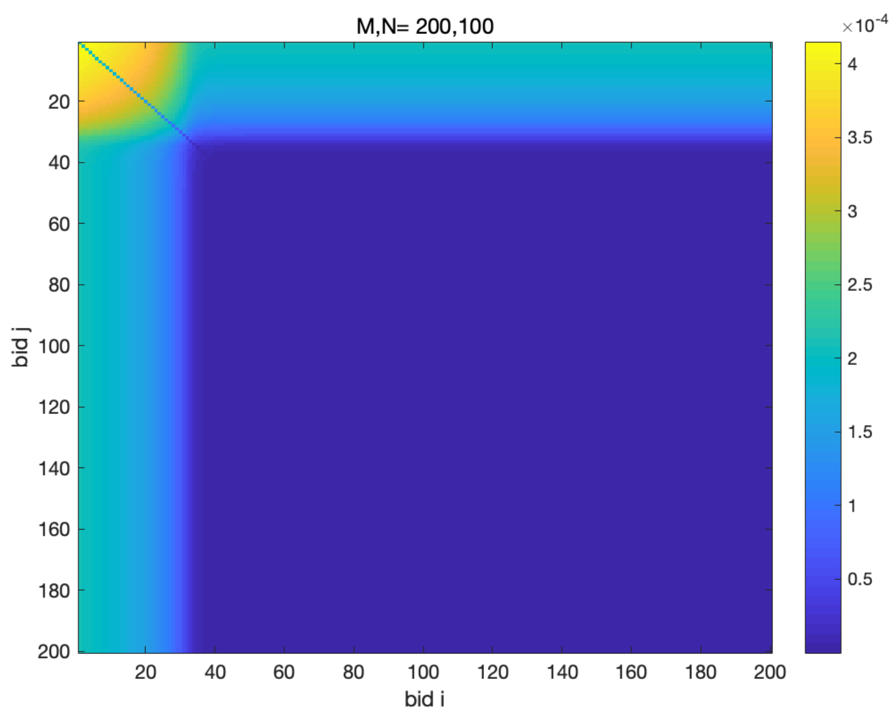

Figure 60: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

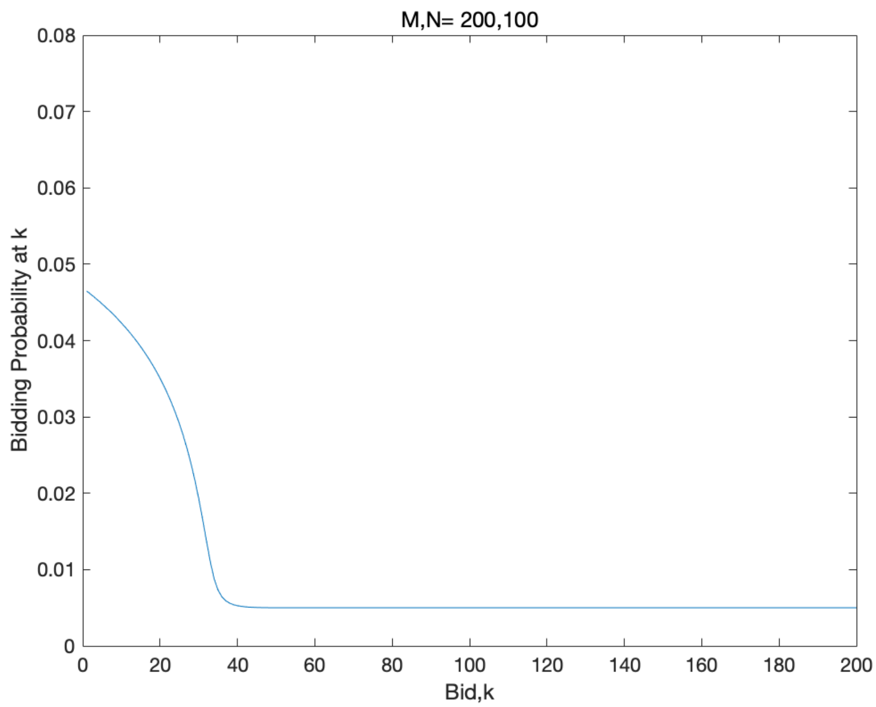

Figure 61: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

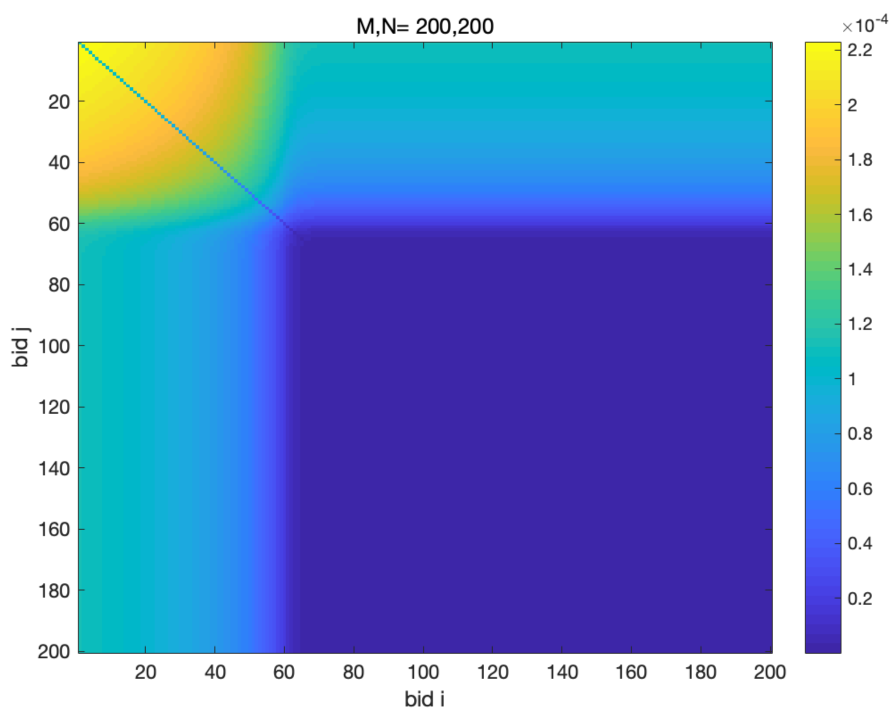

Figure 62: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

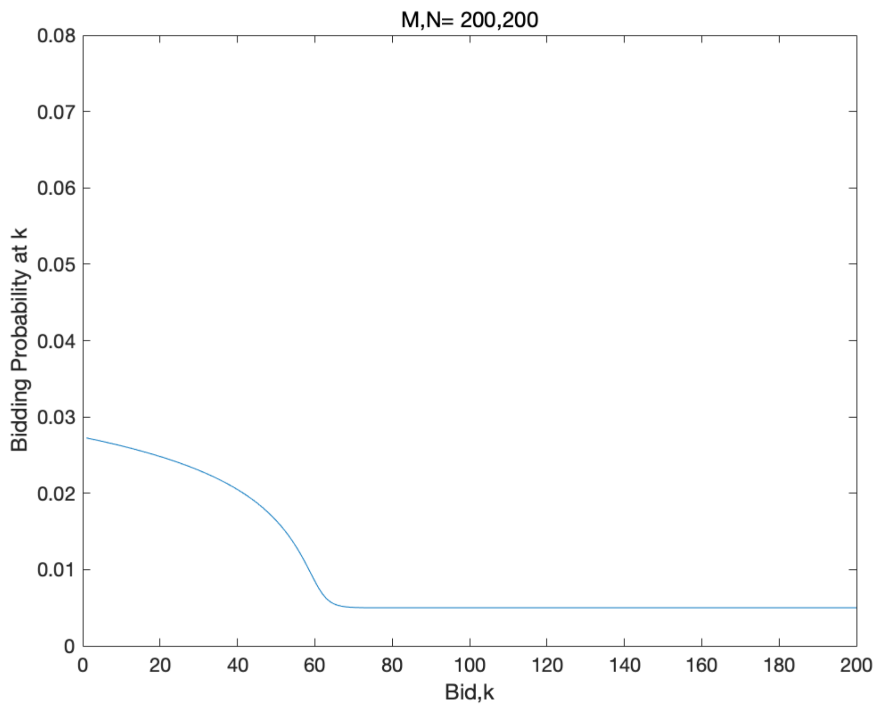

Figure 63: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

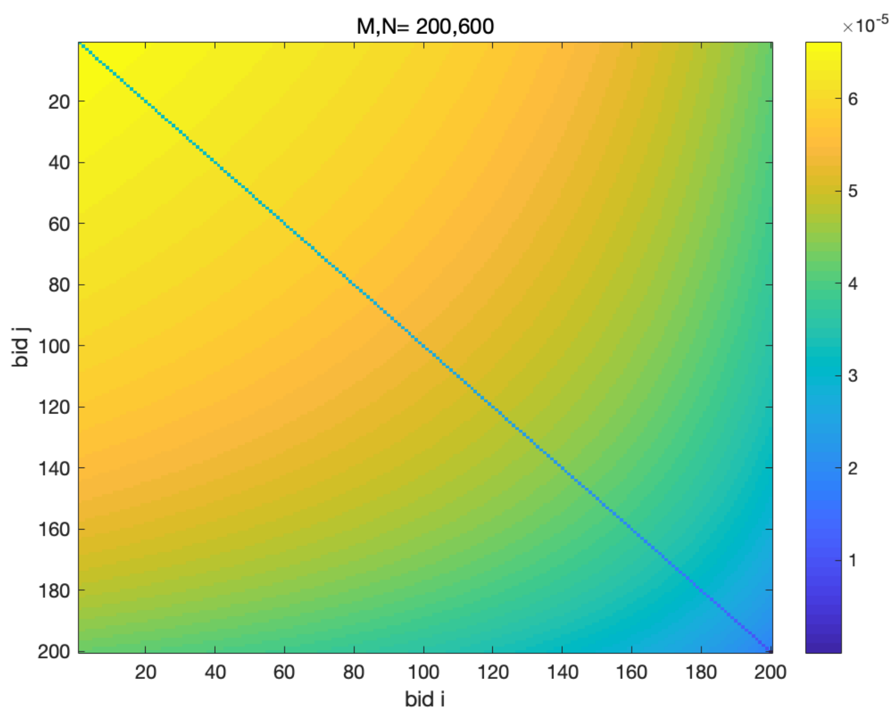

Figure 64: Parameters are labeled as title. Colored graphs show the image with scaled colors of P. Both x-axis and y-axis are labeled as bids.

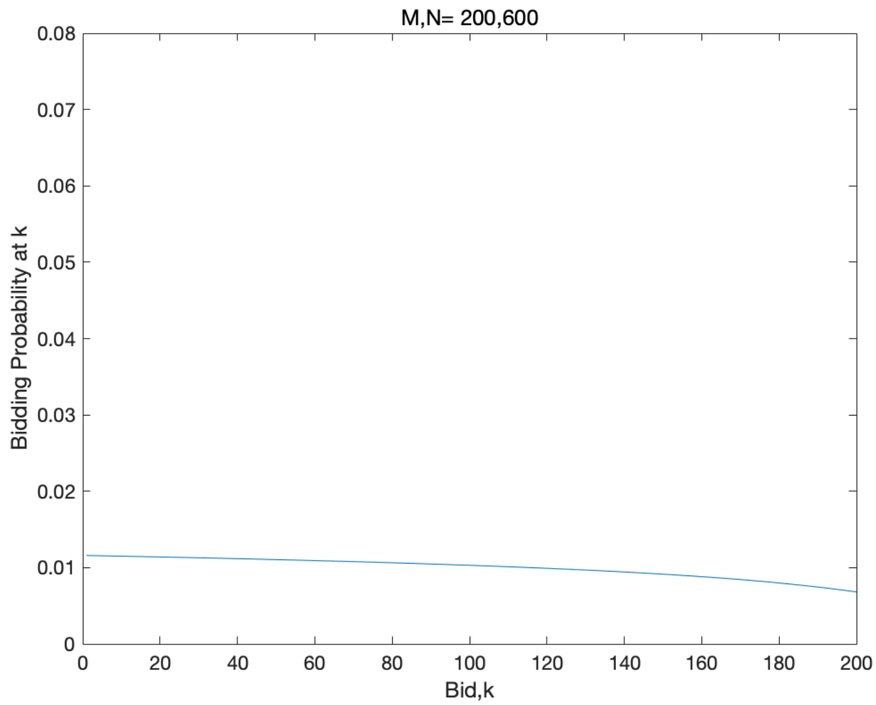

Figure 65: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

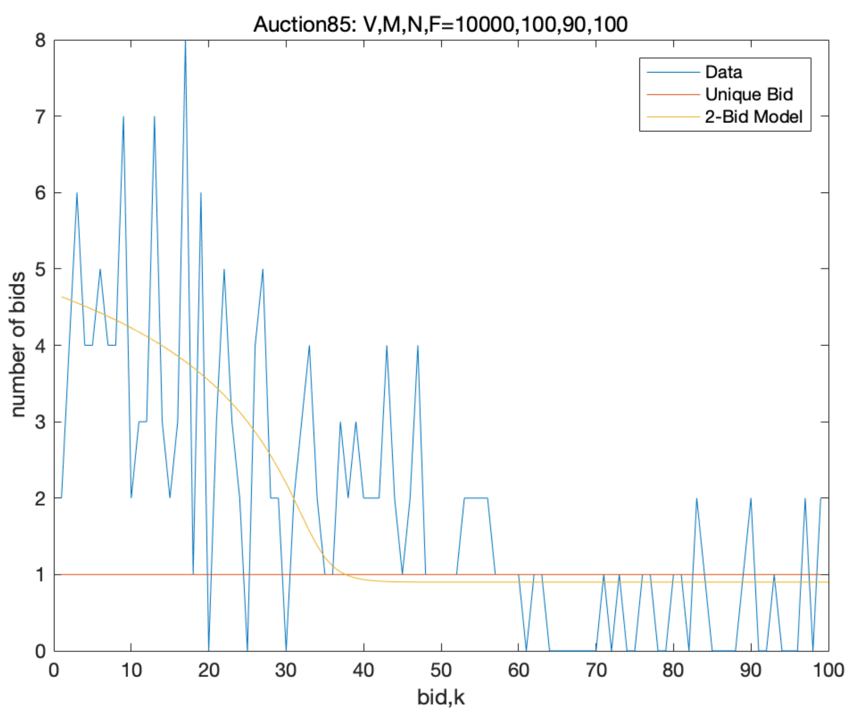

Figure 66: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the total number of bids at k.

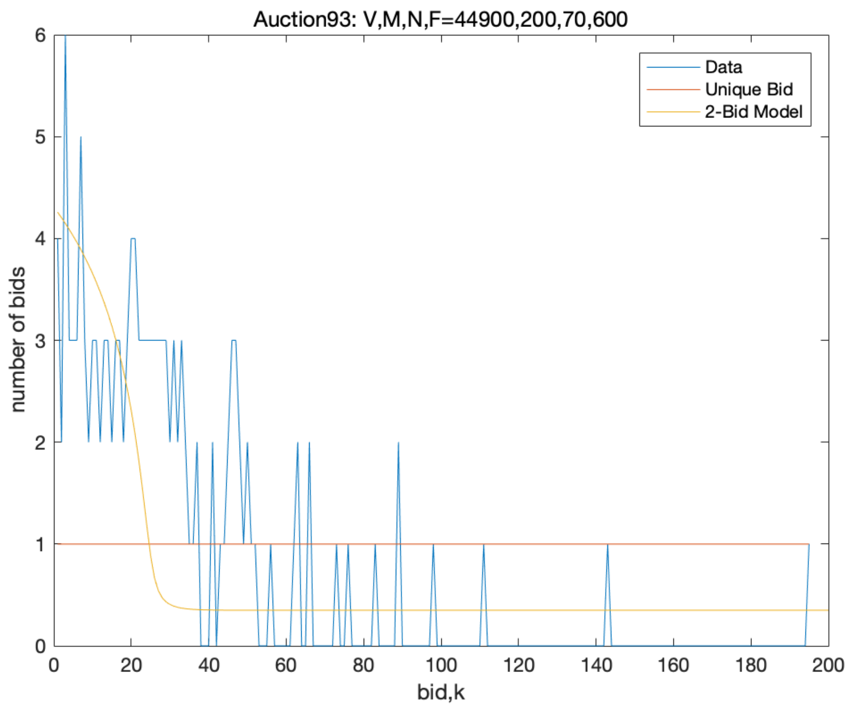

Figure 67: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the total number of bids at k.

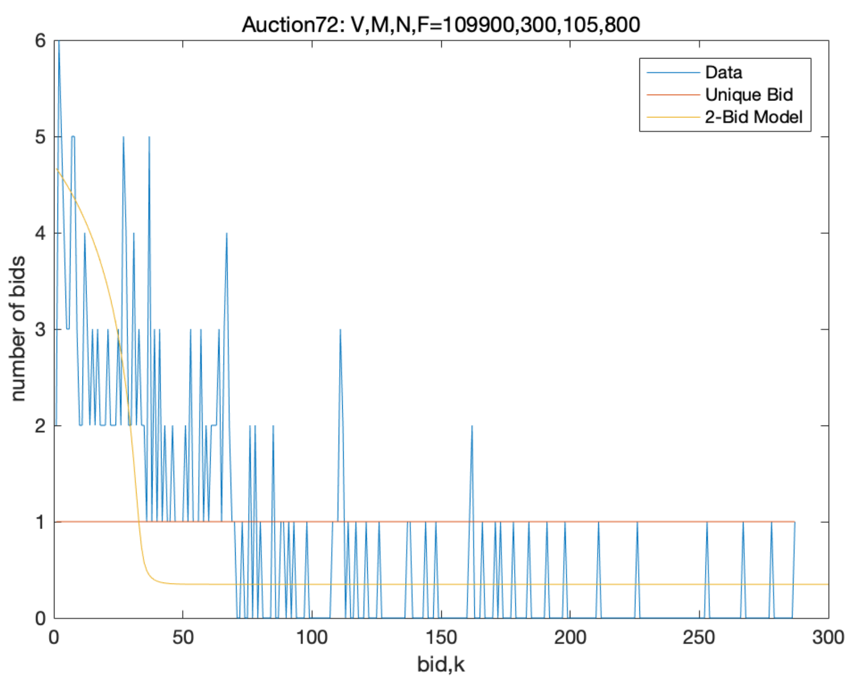

Figure 68: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the total number of bids at k.

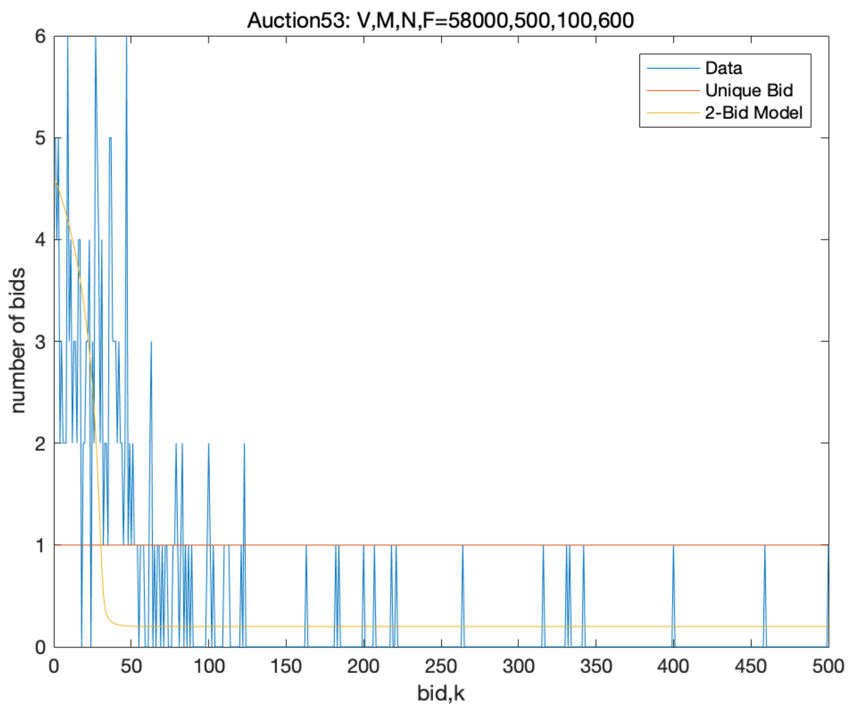

Figure 69: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the total number of bids at k.

Multiple-Bid Probability Matching Model

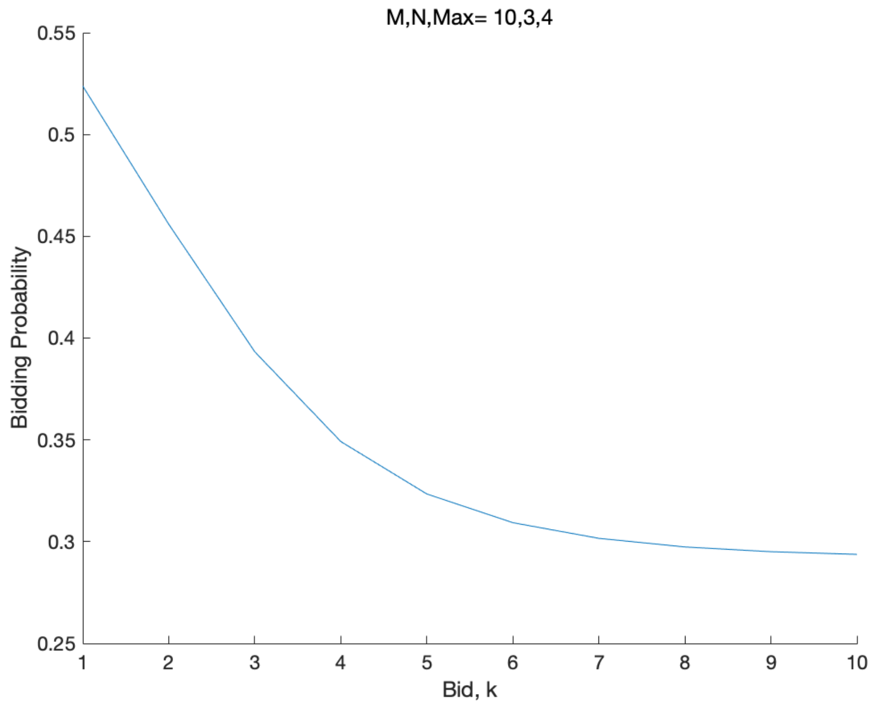

Figure 70: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

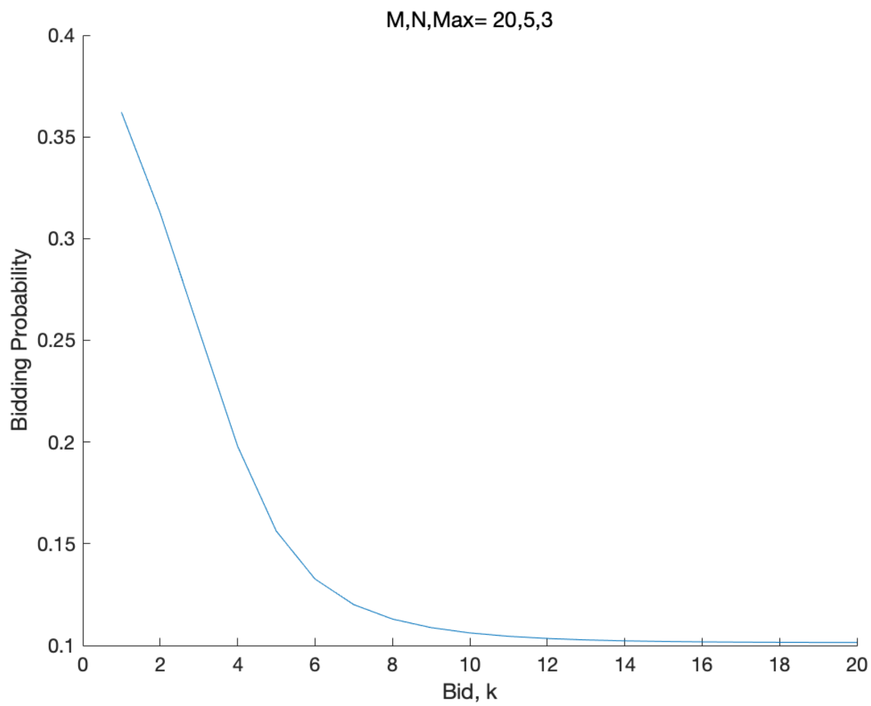

Figure 71: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.

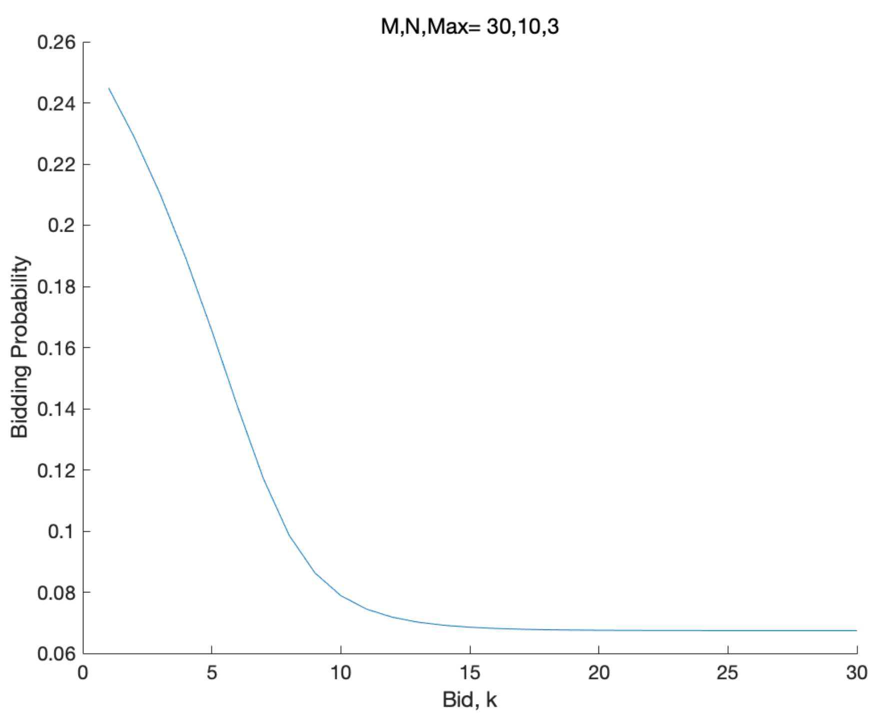

Figure 72: Parameters are labeled as title. The x-axis is labeled as bid and the y-axis is labeled as the bidding probability at k.