Homework 1

1. Suppose $N = 3^p$ with p being a positive integer. For given $u_0, u_1, \cdots, u_{N-1} \in \mathbb{C}$, write down the FFT (fast Fourier transform) algorithm to compute

$$\hat{u}_k = \sum_{j=0}^{N-1} u_j \exp(-\mathrm{i}kx_j), \qquad k = 0, \cdots, N-1.$$

Find the number of additions and multiplications in the algorithm.

2. Let N be an even positive integer. For $u \in L_p^2(0, 2\pi)$, we assume that the Fourier series expansion of u is

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k \exp(\mathrm{i}kx).$$

(a) Define the *Dirichlet kernel*:

$$\mathscr{D}_N(x) = \sum_{k=-N/2}^{N/2} \exp(\mathrm{i}kx).$$

Show that

$$\sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx) = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{D}_N(x-y) u(y) \,\mathrm{d}y.$$

(b) Suppose $u(x) \geqslant 0$ for all $x \in [0, 2\pi)$. Show that for all $x \in [0, 2\pi)$,

$$\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(\mathrm{i}kx) \geqslant 0$$

if the constants σ_k, $k = -N/2, \cdots, N/2$ satisfy

$$\sigma_k = \sigma_{-k}, \qquad \sigma_0 + 2\sum_{k=1}^{N/2} \sigma_k \cos(kx) \geqslant 0, \quad \forall x \in [0, 2\pi). \tag{1}$$

(c) Define the *Fejér kernel*:

$$\mathscr{F}_N(x) = \frac{1}{N/2} \sum_{n=0}^{N/2-1} \mathscr{D}_{2n}(x).$$

Find the coefficients σ_k, $k = -N/2, \cdots, N/2$ such that

$$\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(\mathrm{i}kx) = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{F}_N(x-y) u(y) \,\mathrm{d}y,$$

and show that σ_k satisfies (1).

(d) For $\lambda \in \mathbb{R}\backslash\{0\}$, let
$$\sigma_k = \sinh\left(\lambda\left(1-\frac{k}{N/2}\right)\right) \Big/ \sinh\lambda.$$
Show that σ_k satisfies (1).
[Hint: The corresponding kernel is called *Lorentz kernel.*]

3. For any function $u_N \in \mathscr{T}_N$, define
$$\|u_N\|_p = \left(\int_0^{2\pi} |u_N(x)|^p \,\mathrm{d}x\right)^{1/p}, \quad \text{if } p>1, \qquad \|u_N\|_\infty = \max_{x\in[0,2\pi)} |u_N(x)|.$$
(a) Show that
$$\|u_N\|_\infty \leqslant \left(\frac{N+1}{2\pi}\right)^{1/2} \|u_N\|_2. \tag{2}$$
(b) Let p_0 be an even integer satisfying $p_0 \geqslant p \geqslant 1$. Prove that $u_N^{p_0/2} \in \mathcal{T}_{Np_0/2}$ and use (2) to show
$$\|u_N^{p_0/2}\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|u_N\|_\infty^{(p_0-p)/2}\|u_N\|_p^{p/2}.$$
(c) Show that
$$\|u_N\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p} \|u_N\|_p,$$
and use this inequality to show the more general case:
$$\|u_N\|_q \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p-1/q} \|u_N\|_p, \qquad \text{if } q \geqslant p.$$

4. For $0<s<1$, define the linear operator $\mathscr{L}$ by
$$(\mathscr{L}_s u)(x) = \int_{-\infty}^{+\infty} \frac{u(x)-u(y)}{|x-y|^{2s+1}}\,\mathrm{d}y.$$
(a) Let $v(x) = u(\alpha x)$. Show that
$$(\mathscr{L}_s v)(x) = |\alpha|^{2s}(\mathscr{L}_s u)(\alpha x), \qquad \forall \alpha \in \mathbb{R}.$$
(b) Suppose $u(x)$ is 2π-periodic. Show that $\mathscr{L}_s u$ is also 2π-periodic.
(c) Let $f \in H_p^m(0,2\pi)$ satisfy
$$\int_0^{2\pi} f(x)\,\mathrm{d}x = 0.$$
Describe the Fourier spectral method for solving
$$\mathscr{L}_s u = f, \qquad u \text{ is } 2\pi\text{-periodic}.$$
[Hint: Define the constant $c_s = \displaystyle\int_{\mathbb{R}} |y|^{-(2s+1)}[1-\exp(-\mathrm{i}y)]\,\mathrm{d}y$, and compute $\mathscr{L}_s(\exp(\mathrm{i}kx))$.]