1. Show that the derivatives of Legendre polynomials also satisfy the three-term recurrence relation:

$$L_{n+1}^{(m)}(x) = \alpha_n^{(m)} x L_n^{(m)}(x) - \beta_n^{(m)} L_{n-1}^{(m)}(x).$$

Determine $\alpha_n^{(m)}$, $\beta_n^{(m)}$ and the initial conditions.

**Answer.** The three-term recurrence relation of the Legendre polynomials is

$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x).$$

For $m > 0$, taking the $m$th derivative on both sides of the equality, we obtain

$$L_{n+1}^{(m)}(x) = \frac{2n+1}{n+1} x L_n^{(m)}(x) + \frac{2n+1}{n+1} \cdot m L_n^{(m-1)}(x) - \frac{n}{n+1} L_{n-1}^{(m)}(x).$$

Since

$$L_n(x) = \frac{L'_{n+1}(x) - L'_{n-1}(x)}{2n+1},$$

we can represent the $(m-1)$th derivative of Legendre polynomials using the $m$th derivatives of Legendre polynomials, yielding

$$\begin{aligned} L_{n+1}^{(m)}(x) &= \frac{2n+1}{n+1} x L_n^{(m)}(x) + \frac{2n+1}{n+1} \cdot m \frac{L_{n+1}^{(m)}(x) - L_{n-1}^{(m)}(x)}{2n+1} - \frac{n}{n+1} L_{n-1}^{(m)}(x) \\ &= \frac{m}{n+1} L_{n+1}^{(m)}(x) + \frac{2n+1}{n+1} x L_n^{(m)}(x) - \frac{n+m}{n+1} L_{n-1}^{(m)}(x). \end{aligned}$$

When $m < n+1$, we can regard the equality above as a linear equation of $L_{n+1}^{(m)}$, solving which gives

$$L_{n+1}^{(m)}(x) = \frac{2n+1}{n+1-m} x L_n^{(m)}(x) - \frac{n+m}{n+1-m} L_{n-1}^{(m)}(x).$$

Therefore,

$$\alpha_n^{(m)} = \frac{2n+1}{n+1-m}, \qquad \beta_n^{(m)} = \frac{n+m}{n+1-m}.$$

The initial conditions should be given at $n = m$ and $n = m-1$. When $n = m-1$, it is obvious that

$$L_n^{(m)}(x) = 0.$$

When $n = m$, by Rodrigues' formula,

$$L_n^{(m)}(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}} [(x^2-1)^n] = \frac{(2n)!}{2^n n!} = (2n-1)!!.$$

2. Prove the following differential relations for Chebyshev polynomials:

$$T'_n(x) = 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} \frac{1}{1+\delta_{k0}} T_k(x),$$

$$T''_n(x) = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{n-2} \frac{1}{1+\delta_{k0}} n(n^2-k^2) T_k(x).$$

**Answer.** We first verify the conclusion for $n=0,1,2$:

$$\begin{aligned}
&T_0'(x)=0,\\
&T_1'(x)=1=2\cdot\frac{1}{2}T_0(x),\\
&T_2'(x)=4x=4T_1(x).
\end{aligned}$$

For $n\geqslant 3$, we use the trigonometric definition of Chebyshev polynomials:

$$T_n(x)=\cos(n\arccos x)$$

to obtain

$$\begin{aligned}
\frac{1}{n}T_n'(x)-\frac{1}{n-2}T_{n-2}'(x)&=\frac{1}{\sqrt{1-x^2}}[\sin(n\arccos x)-\sin((n-2)\arccos x)]\\
&=\frac{2}{\sqrt{1-x^2}}\cos((n-1)\arccos x)\sin(\arccos x)\\
&=2\cos((n-1)\arccos x)=2T_{n-1}(x).
\end{aligned}$$

Thus, by recursion, we get

$$\begin{aligned}
\frac{1}{n}T_n'(x)&=\frac{1}{n-2}T_{n-2}'(x)+2T_{n-1}(x)\\
&=\frac{1}{n-4}T_{n-4}'(x)+2T_{n-3}(x)+2T_{n-1}(x)\\
&=\cdots=\frac{1}{n_0}T_{n_0}'(x)+2[T_{n_0+1}(x)+\cdots+T_{n-3}(x)+T_{n-1}(x)],
\end{aligned}$$

where $n_0=1$ if $n$ is odd and $n_0=2$ if $n$ is even. This shows the expression of $T_n'(x)$.

The expression of the second-order derivative can be derived by applying the formula for the first-order derivative:

$$\begin{aligned}
T_n''(x)&=2n\sum_{\substack{k=0\\k+n\text{ odd}}}^{n-1}\frac{1}{1+\delta_{k0}}T_k'(x)=2n\sum_{\substack{k=0\\k+n\text{ odd}}}^{n-1}\frac{2k}{1+\delta_{k0}}\sum_{\substack{j=0\\k+j\text{ odd}}}^{k-1}\frac{1}{1+\delta_{j0}}T_j(x)\\
&=2n\sum_{\substack{j=0\\n+j\text{ even}}}^{n-2}\frac{1}{1+\delta_{j0}}\left(\sum_{\substack{k=j+1\\k+n\text{ odd}}}^{n-1}\frac{2k}{1+\delta_{k0}}\right)T_j(x)\\
&=2n\sum_{\substack{j=0\\n+j\text{ even}}}^{n-2}\frac{2}{1+\delta_{j0}}\left(\sum_{\substack{k=j+1\\k+n\text{ odd}}}^{n-1}k\right)T_j(x)\\
&=2n\sum_{\substack{j=0\\n+j\text{ even}}}^{n-2}\frac{2}{1+\delta_{j0}}\cdot\frac{(n+j)}{2}\cdot\frac{(n-j)}{2}T_j(x)\\
&=\sum_{\substack{j=0\\j+n\text{ even}}}^{n-2}\frac{1}{1+\delta_{j0}}n(n^2-j^2)T_j(x).
\end{aligned}$$

3. Let $\omega(x)=(1-x^2)^{-1/2}$. For any $f\in L^2_\omega(-1,1)$, define its projection $\pi_N f$ by

$$(\pi_N f)(x)=\sum_{n=0}^{N}\hat{f}_nT_n(x),$$

where

$$\hat{f}_n = \frac{2}{(1+\delta_{0n})\pi} \int_{-1}^{1} f(x) T_n(x) \omega(x)\, \mathrm{d}x.$$

Prove the estimation for $\|f - \pi_N f\|_{1,\omega}$ by the following steps:

a) Use the Sturm-Liouville equation to show that

$$\|f - \pi_N f\|_{0,\omega} \lesssim N^{-r} \|f\|_{r,\omega}$$

if $r$ is positive and even.

**Answer.** Define the Sturm-Liouville operator

$$\mathcal{L} f(x) = -\sqrt{1-x^2}\, \frac{\mathrm{d}}{\mathrm{d}x}\Big[\sqrt{1-x^2}\, f'(x)\Big].$$

Then Chebyshev polynomials satisfy

$$\mathcal{L} T_n = n^2 T_n.$$

For any $f \in H^2_\omega(-1,1)$,

$$\begin{aligned}
\|\mathcal{L} f\|^2_{L^2_\omega} &= \int_{-1}^{1} \frac{[\mathcal{L} f(x)]^2}{\sqrt{1-x^2}}\, \mathrm{d}x = \int_{-1}^{1} \sqrt{1-x^2} \left( \frac{\mathrm{d}}{\mathrm{d}x}\Big[\sqrt{1-x^2}\, f'(x)\Big] \right)^2 \mathrm{d}x \\
&= \int_{-1}^{1} \sqrt{1-x^2} \left( \sqrt{1-x^2}\, f''(x) - \frac{x}{\sqrt{1-x^2}} f'(x) \right)^2 \mathrm{d}x \\
&= \int_{-1}^{1} \left( (1-x^2)^{3/2} [f''(x)]^2 + \frac{x^2}{\sqrt{1-x^2}} [f'(x)]^2 - 2x\sqrt{1-x^2}\, f'(x) f''(x) \right) \mathrm{d}x \\
&\leqslant \int_{-1}^{1} \left( (1-x^2)^{3/2} [f''(x)]^2 + x^2 \sqrt{1-x^2} [f''(x)]^2 \right) \mathrm{d}x \\
&\quad + \int_{-1}^{1} \left( \frac{x^2}{\sqrt{1-x^2}} [f'(x)]^2 + \sqrt{1-x^2} [f'(x)]^2 \right) \mathrm{d}x \\
&\leqslant \int_{-1}^{1} \frac{[f''(x)]^2 + [f'(x)]^2}{\sqrt{1-x^2}}\, \mathrm{d}x \leqslant \|f\|^2_{2,\omega}.
\end{aligned}$$

Therefore, $\|\mathcal{L}^{r/2} f\|_{L^2_\omega} \lesssim \|f\|_{r,\omega}$. We are now ready to prove the error estimation:

$$\begin{aligned}
\|f - \pi_N f\|^2_{0,\omega} &= \frac{\pi}{2} \sum_{n=N+1}^{+\infty} |\hat{f}_n|^2 = \frac{\pi}{2} \sum_{n=N+1}^{+\infty} \left| \frac{(f, T_n)_\omega}{(T_n, T_n)_\omega} \right|^2 \\
&= \frac{2}{\pi} \sum_{n=N+1}^{+\infty} |(f, T_n)_\omega|^2 = \frac{2}{\pi} \sum_{n=N+1}^{+\infty} \frac{1}{n^4} |(f, \mathcal{L} T_n)_\omega|^2 \\
&= \frac{2}{\pi} \sum_{n=N+1}^{+\infty} \frac{1}{n^4} |(\mathcal{L} f, T_n)_\omega|^2 \leqslant \frac{2}{\pi} \cdot \frac{1}{N^4} \sum_{n=N+1}^{+\infty} |(\mathcal{L} f, T_n)_\omega|^2 \\
&\leqslant \frac{2}{\pi} \cdot \frac{1}{N^8} \sum_{n=N+1}^{+\infty} |(\mathcal{L}^2 f, T_n)_\omega|^2 \leqslant \cdots \leqslant \frac{2}{\pi} \cdot \frac{1}{N^{2r}} \sum_{n=N+1}^{+\infty} |(\mathcal{L}^{r/2} f, T_n)_\omega|^2 \\
&\leqslant \frac{2}{\pi} \cdot \frac{1}{N^{2r}} \sum_{n=0}^{+\infty} |(\mathcal{L}^{r/2} f, T_n)_\omega|^2 = \frac{1}{N^{2r}} \|\mathcal{L}^{r/2} f\|^2_{0,\omega} \lesssim N^{-2r} \|f\|^2_{r,\omega}.
\end{aligned}$$

Taking square roots on both sides of the inequality proves the error estimation.

b) Prove the inverse inequality:

$$\|p\|_{r,\omega} \lesssim N^{2r} \|p\|_{0,\omega}, \qquad \forall p \in P_N,$$

where $r \in \mathbb{N}$.

**Answer.** Assume

$$p(x) = \sum_{n=0}^{N} \hat{p}_n T_n(x).$$

Its derivative is

$$\begin{aligned}
p'(x) &= \sum_{n=0}^{N} \hat{p}_n T_n'(x) = \sum_{n=0}^{N} 2n\hat{p}_n \sum_{\substack{k=0\\k+n \text{ odd}}}^{n-1} \frac{1}{1+\delta_{k0}} T_k(x),\\
&= \sum_{k=0}^{N-1} \frac{2}{1+\delta_{k0}} \left( \sum_{\substack{n=k+1\\k+n \text{ odd}}}^{N} n\hat{p}_n \right) T_k(x).
\end{aligned}$$

The $L^2$ norm of $p'$ can be estimated by

$$\begin{aligned}
\|p'\|_{0,\omega}^2 &= \sum_{k=0}^{N-1} \left(\frac{2}{1+\delta_{k0}}\right)^2 \left( \sum_{\substack{n=k+1\\k+n \text{ odd}}}^{N} n\hat{p}_n \right)^2 \|T_k\|_{0,\omega}^2\\
&\lesssim \sum_{k=0}^{N-1} \left( \sum_{\substack{n=k+1\\k+n \text{ odd}}}^{N} n\hat{p}_n \right)^2 \lesssim \sum_{k=0}^{N-1} \left( \sum_{\substack{n=k+1\\k+n \text{ odd}}}^{N} n^2 \right) \left( \sum_{\substack{n=k+1\\k+n \text{ odd}}}^{N} |\hat{p}_n|^2 \right)\\
&\lesssim N^4 \|p\|_{0,\omega}^2.
\end{aligned}$$

Therefore, $\|p\|_{1,\omega} \lesssim N^2 \|p\|_{0,\omega}^2$. It can be obtained by recursion that

$$\|p\|_{r,\omega} \lesssim N^{2r} \|p\|_{0,\omega}^2.$$

c) Show that

$$\|\pi_N(\partial_x f) - \partial_x(\pi_N f)\|_{0,\omega} \leqslant N^{3/2-r} \|f\|_{r,\omega}$$

if $r$ is positive and odd.

**Answer.** By straightforward calculation, we have the following expressions for $\pi_N(\partial_x f)$ and $\partial_x(\pi_N f)$:

$$\begin{aligned}
\pi_N(\partial_x f) &= \sum_{k=0}^{N} \frac{2}{1+\delta_{k0}} \left( \sum_{\substack{n=k+1\\k+n \text{ odd}}}^{+\infty} n\hat{f}_n \right) T_k(x),\\
\partial_x(\pi_N f) &= \sum_{k=0}^{N-1} \frac{2}{1+\delta_{k0}} \left( \sum_{\substack{n=k+1\\k+n \text{ odd}}}^{N} n\hat{f}_n \right) T_k(x).
\end{aligned}$$

For simplicity, we let

$$\hat{f}_k^{(1)} = \frac{2}{1+\delta_{k0}} \left( \sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{+\infty} n\hat{f}_n \right).$$

The difference between $\pi_N(\partial_x f)$ and $\partial_x(\pi_N f)$ can be represented by

$$\begin{aligned}
\pi_N(\partial_x f) - \partial_x(\pi_N f) &= \sum_{k=0}^{N} \frac{2}{1+\delta_{k0}} \left( \sum_{\substack{n=N+1 \\ k+n \text{ odd}}}^{+\infty} n\hat{f}_n \right) T_k(x) \\
&= \sum_{k=0}^{N} \frac{1}{1+\delta_{k0}} \hat{f}^{(1)}_{N+\varepsilon_{N,k}} T_k(x),
\end{aligned}$$

where

$$\varepsilon_{N,k} = \begin{cases} 1, & \text{if } N+k \text{ is odd}, \\ 0, & \text{if } N+k \text{ is even}. \end{cases}$$

The coefficient $\hat{f}^{(1)}_{N+\varepsilon_{N,k}}$ can be estimated by

$$\begin{aligned}
\left|\hat{f}^{(1)}_{N+\varepsilon_{N,k}}\right| &= \frac{2}{\pi} |(\partial_x f, T_{N+\varepsilon_{N,k}})_\omega| = \frac{2}{\pi(N+\varepsilon_{N,k})^{r-1}} |(\mathcal{L}^{(r-1)/2}\partial_x f, T_{N+\varepsilon_{N,k}})_\omega| \\
&\lesssim N^{1-r} \|\mathcal{L}^{(r-1)/2}\partial_x f\|_{0,\omega} \lesssim N^{1-r} \|\partial_x f\|_{r-1,\omega} \lesssim N^{1-r} \|f\|_{r,\omega}.
\end{aligned}$$

Thus, $\pi_N(\partial_x f) - \partial_x(\pi_N f)$ can be bounded as

$$\begin{aligned}
\|\pi_N(\partial_x f) - \partial_x(\pi_N f)\|^2_{0,\omega} &= \frac{2}{\pi} \sum_{k=0}^{N} \left| \frac{1}{1+\delta_{k0}} \hat{f}^{(1)}_{N+\varepsilon_{N,k}} \right|^2 \\
&\lesssim \sum_{k=0}^{N} (N^{1-r} \|f\|^2_{r,\omega}) \lesssim N^{3-2r} \|f\|^2_{r,\omega}.
\end{aligned}$$

Taking square roots on both sides completes the proof.

d) Show that

$$\|f - \pi_N f\|_{1,\omega} \lesssim N^{3/2-r} \|f\|_{r,\omega}$$

if $r$ is positive and odd.

**Answer.**

$$\begin{aligned}
\|f - \pi_N f\|_{1,\omega} &\lesssim \|f - \pi_N f\|_{0,\omega} + \|\partial_x f - \partial_x(\pi_N f)\|_{0,\omega} \\
&\lesssim N^{-(r-1)} \|f\|_{r-1,\omega} + \|\partial_x f - \pi_N(\partial_x f)\|_{0,\omega} + \|\partial_x(\pi_N f) - \pi_N(\partial_x f)\|_{0,\omega} \\
&\lesssim N^{-(r-1)} \|f\|_{r,\omega} + N^{-(r-1)} \|f\|_{r,\omega} + N^{3/2-r} \|f\|_{r,\omega} \\
&\lesssim N^{3/2-r} \|f\|_{r,\omega}.
\end{aligned}$$

4. Let $\omega(x) = (1-x^2)^{-1/2}$. Assume $u \in H^r_\omega(-1,1)$ with $r$ being a positive odd integer and $u(-1) = u(1) = 0$. Define

$$\begin{aligned}
p(x) &= \int_{-1}^{x} (\pi_N u')(y)\,\mathrm{d}y, \\
p^*(x) &= \int_{-1}^{x} \left[ (\pi_N u')(y) - \frac{1}{2} p(1) \right] \mathrm{d}y.
\end{aligned}$$

Show that $p^*(-1) = p^*(1) = 0$ and

$$|u - p^*|_{1,\omega} \lesssim N^{1-r} \|u\|_{r,\omega}.$$

**Answer.** It is obvious that $p^*(-1) = 0$. The value of $p^*(1)$ can be calculated by

$$p^*(1) = \int_{-1}^{1} \left[ (\pi_N u')(y) - \frac{1}{2} p(1) \right] \mathrm{d}y = \int_{-1}^{1} (\pi_N u')(y) \, \mathrm{d}y - p(1) = p(1) - p(1) = 0.$$

We now estimate $u - p^*$:

$$\begin{aligned} |u - p^*|_{1,\omega} &= \|\partial_x u - \partial_x p^*\|_{0,\omega} = \left\| u' - \pi_N u' + \frac{1}{2} p(1) \right\|_{0,\omega} \\ &\leqslant \|u' - \pi_N u'\|_{0,\omega} + \frac{\sqrt{\pi}}{2} |p(1)| \lesssim N^{1-r} \|u\|_{r,\omega} + |p(1)|. \end{aligned}$$

Since $u(-1) = u(1) = 0$, we have

$$\int_{-1}^{1} u'(x) \, \mathrm{d}x = 0.$$

Therefore,

$$\begin{aligned} |p(1)| &= \left| \int_{-1}^{1} (\pi_N u')(y) \, \mathrm{d}y \right| = \left| \int_{-1}^{1} (u' - \pi_N u')(y) \, \mathrm{d}y \right| \\ &= \left| \int_{-1}^{1} \frac{(u' - \pi_N u')(y) \cdot \sqrt{1 - y^2}}{\sqrt{1 - y^2}} \, \mathrm{d}y \right| \leqslant \frac{\pi}{2} \|u' - \pi_N u'\|_{0,\omega} \lesssim N^{1-r} \|u\|_{r,\omega}, \end{aligned}$$

which completes the proof.