1. Suppose $N = 3^p$ with p being a positive integer. For given $u_0, u_1, \cdots, u_{N-1} \in \mathbb{C}$, write down the FFT (fast Fourier transform) algorithm to compute

$$\hat{u}_k = \sum_{j=0}^{N-1} u_j \exp(-\mathrm{i}kx_j), \qquad k = 0, \cdots, N-1.$$

Find the number of additions and multiplications in the algorithm.

Answer. Let A_p be the number of additions and M_p be the number of multiplications. When $p = 1$, we have

$$\begin{aligned}
\hat{u}_0 &= u_0 + u_1 + u_2, \\
\hat{u}_1 &= u_0 + \exp\left(-\mathrm{i}\frac{2\pi}{3}\right)u_1 + \exp\left(-\mathrm{i}\frac{4\pi}{3}\right)u_2, \\
\hat{u}_2 &= u_0 + \exp\left(-\mathrm{i}\frac{4\pi}{3}\right)u_1 + \exp\left(-\mathrm{i}\frac{8\pi}{3}\right)u_2.
\end{aligned}$$

Therefore,

$$A_1 = 6, \qquad M_1 = 4.$$

For $p > 1$, we rewrite $\hat{u}_k$ as

$$\begin{aligned}
\hat{u}_k &= \sum_{j=0}^{N/3-1} u_{3j} \exp(-\mathrm{i}kx_{3j}) + \sum_{j=0}^{N/3-1} u_{3j+1} \exp(-\mathrm{i}kx_{3j+1}) + \sum_{j=0}^{N/3-1} u_{3j+2} \exp(-\mathrm{i}kx_{3j+2}) \\
&= \sum_{j=0}^{N/3-1} u_{3j} \exp(-\mathrm{i}kx_{3j}) + \exp\left(-\frac{2\pi \mathrm{i}k}{3}\right) \sum_{j=0}^{N/3-1} u_{3j+1} \exp(-\mathrm{i}kx_{3j}) \\
&\quad + \exp\left(-\frac{4\pi \mathrm{i}k}{3}\right) \sum_{j=0}^{N/3-1} u_{3j+2} \exp(-\mathrm{i}kx_{3j+2}).
\end{aligned}$$

Let

$$\begin{aligned}
\hat{u}_k^{(1)} &= \sum_{j=0}^{N/3-1} u_{3j} \exp(-\mathrm{i}kx_{3j}), \\
\hat{u}_k^{(2)} &= \sum_{j=0}^{N/3-1} u_{3j+1} \exp(-\mathrm{i}kx_{3j}), \\
\hat{u}_k^{(3)} &= \sum_{j=0}^{N/3-1} u_{3j+2} \exp(-\mathrm{i}kx_{3j}).
\end{aligned}$$

It holds that

$$\hat{u}_k^{(1)} = \hat{u}_{k+N/3}^{(1)} = \hat{u}_{k+2N/3}^{(1)}, \quad \hat{u}_k^{(2)} = \hat{u}_{k+N/3}^{(2)} = \hat{u}_{k+2N/3}^{(2)}, \quad \hat{u}_k^{(3)} = \hat{u}_{k+N/3}^{(3)} = \hat{u}_{k+2N/3}^{(3)},$$

which allows us to compute $\hat{u}_k$ by

$$\begin{aligned}
\hat{u}_k &= \hat{u}_k^{(1)} + \exp\left(-\frac{2\pi \mathrm{i}k}{3}\right)\hat{u}_k^{(2)} + \exp\left(-\frac{4\pi \mathrm{i}k}{3}\right)\hat{u}_k^{(3)}, \qquad k = 0, \cdots, N/3-1, \\
\hat{u}_k &= \hat{u}_{k-N/3}^{(1)} + \exp\left(-\frac{2\pi \mathrm{i}k}{3}\right)\hat{u}_{k-N/3}^{(2)} + \exp\left(-\frac{4\pi \mathrm{i}k}{3}\right)\hat{u}_{k-N/3}^{(3)}, \quad k = N/3, \cdots, 2N/3-1, \\
\hat{u}_k &= \hat{u}_{k-2N/3}^{(1)} + \exp\left(-\frac{2\pi \mathrm{i}k}{3}\right)\hat{u}_{k-2N/3}^{(2)} + \exp\left(-\frac{4\pi \mathrm{i}k}{3}\right)\hat{u}_{k-2N/3}^{(3)}, \quad k = 2N/3, \cdots, N-1.
\end{aligned}$$

The number of additions and multiplications satisfy

$$A_p = 3A_{p-1} + 2N, \qquad M_p = 3M_{p-1} + 2N.$$

This yields the general formulae:

$$A_p = 2p \cdot 3^p, \qquad M_p = 2\left(p - \frac{1}{3}\right) \cdot 3^p.$$

2. Let N be an even integer. For $u \in L_p^2(0, 2\pi)$, we assume that the Fourier series expansion of u is

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k \exp(\mathrm{i}kx).$$

a) Define the Dirichlet kernel:

$$\mathcal{D}_N(x) = \sum_{k=-N/2}^{N/2} \exp(\mathrm{i}kx).$$

Show that

$$\sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{D}_N(x-y) u(y)\, \mathrm{d}y.$$

Answer. This can be shown by straightforward calculation:

$$\begin{aligned}
\sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx) &= \sum_{k=-N/2}^{N/2} \left(\frac{1}{2\pi} \int_0^{2\pi} u(y) \exp(-\mathrm{i}ky)\, \mathrm{d}y \right) \exp(\mathrm{i}kx) \\
&= \sum_{k=-N/2}^{N/2} \frac{1}{2\pi} \int_0^{2\pi} u(y) \exp(\mathrm{i}k(x-y))\, \mathrm{d}y \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=-N/2}^{N/2} \exp(\mathrm{i}k(x-y)) \right) u(y)\, \mathrm{d}y.
\end{aligned}$$

b) Suppose $u(x) \geqslant 0$ for all $x \in [0, 2\pi)$. Show that for all $x \in [0, 2\pi)$,

$$\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(\mathrm{i}kx) \geqslant 0$$

if the constants σ_k, $k = -N/2, \cdots, N/2$ satisfy

$$\sigma_k = \sigma_{-k}, \qquad \sigma_0 + 2\sum_{k=1}^{N/2} \sigma_k \cos(kx) \geqslant 0, \quad \forall x \in [0, 2\pi). \tag{1}$$

Answer. By $\sigma_k = \sigma_{-k}$, we get

$$
\begin{aligned}
&\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(\mathrm{i}kx) \\
&= \sum_{k=-N/2}^{N/2} \sigma_k \left(\frac{1}{2\pi} \int_0^{2\pi} u(y) \exp(-\mathrm{i}ky)\, \mathrm{d}y \right) \exp(\mathrm{i}kx) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=-N/2}^{N/2} \sigma_k \exp(\mathrm{i}k(x-y)) \right) u(y)\, \mathrm{d}y \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\sigma_0 + \sum_{k=1}^{N/2} \sigma_k [\exp(\mathrm{i}k(x-y)) + \exp(-\mathrm{i}k(x-y))] \right) u(y)\, \mathrm{d}y \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\sigma_0 + 2\sum_{k=1}^{N/2} \sigma_k \cos(k(x-y)) \right) u(y)\, \mathrm{d}y.
\end{aligned}
$$

This integral is obviously nonnegative due to (1).

c) Define the Fejer kernel:

$$
\mathcal{F}_N(x) = \frac{1}{N/2} \sum_{n=0}^{N/2-1} \mathcal{D}_{2n}(x).
$$

Find the coefficients σ_k, $k = -N/2, \cdots, N/2$ such that

$$
\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(\mathrm{i}kx) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}_N(x-y) u(y)\, \mathrm{d}y,
$$

and show that σ_k satisfies (1).

Answer. Using the result of (a), we have

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}_N(x-y) u(y)\, \mathrm{d}y &= \frac{1}{N/2} \sum_{n=0}^{N/2-1} \frac{1}{2\pi} \int_0^{2\pi} \mathcal{D}_{2n}(x-y) u(y)\, \mathrm{d}y \\
&= \frac{1}{N/2} \sum_{n=0}^{N/2-1} \sum_{k=-n}^{n} \hat{u}_k \exp(\mathrm{i}kx) \\
&= \frac{1}{N/2} \sum_{k=-(N/2-1)}^{N/2-1} \sum_{n=|k|}^{N/2-1} \hat{u}_k \exp(\mathrm{i}kx) \\
&= \sum_{k=-(N/2-1)}^{N/2-1} \left(1 - \frac{|k|}{N/2} \right) \hat{u}_k \exp(\mathrm{i}kx) \\
&= \sum_{k=-N/2}^{N/2} \left(1 - \frac{|k|}{N/2} \right) \hat{u}_k \exp(\mathrm{i}kx).
\end{aligned}
$$

Therefore,

$$
\sigma_k = 1 - \frac{|k|}{N/2}.
$$

We now check condition (1). It is obvious that $\sigma_k = \sigma_{-k}$. To verify the inequality in (1), we first notice that the left-hand side of the inequality is just the Fejer kernel:

$$
\begin{aligned}
\sigma_0 + 2\sum_{k=1}^{N/2} \sigma_k \cos(kx) &= \sum_{k=-N/2}^{N/2} \left(1 - \frac{|k|}{N/2}\right)\cos(kx) \\
&= \sum_{k=-N/2}^{N/2} \left(1 - \frac{|k|}{N/2}\right)\exp(\mathrm{i}kx) \\
&= \frac{1}{N/2} \sum_{k=-N/2}^{N/2} \sum_{n=|k|+1}^{N/2} \exp(\mathrm{i}kx) \\
&= \frac{1}{N/2} \sum_{n=0}^{N/2-1} \sum_{k=-n}^{n} \exp(\mathrm{i}kx) = \mathcal{F}_N(x).
\end{aligned}
$$

To show its positivity, we need the following representation of the Dirichlet kernel:

$$
\begin{aligned}
D_{2n}(x) &= \sum_{k=-n}^{n} \exp(\mathrm{i}kx) \\
&= \exp(-\mathrm{i}nx)\frac{\exp(\mathrm{i}(2n+1)x) - 1}{\exp(\mathrm{i}x) - 1} \\
&= \frac{\exp(\mathrm{i}(n+1/2)x) - \exp(-\mathrm{i}(n+1/2)x)}{\exp(\mathrm{i}x/2) - \exp(-\mathrm{i}x/2)} = \frac{\sin((n+1/2)x)}{\sin(x/2)} \\
&= \frac{\sin((n+1/2)x)\sin(x/2)}{\sin^2(x/2)} = \frac{\cos(nx) - \cos((n+1)x)}{\sin^2(x/2)}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\mathcal{F}_N(x) &= \frac{1}{N/2} \sum_{n=0}^{N/2-1} \mathcal{D}_{2n}(x) \\
&= \frac{1}{N/2} \sum_{n=0}^{N/2-1} \frac{\cos(nx) - \cos((n+1)x)}{\sin^2(x/2)} \\
&= \frac{1}{N/2} \frac{1 - \cos(Nx/2)}{\sin^2(x/2)} \geqslant 0.
\end{aligned}
$$

d) For $\lambda \in \mathbb{R} \setminus \{0\}$, let

$$
\sigma_k = \sinh\left(\lambda\left(1 - \frac{|k|}{N/2}\right)\right) \Big/ \sinh\lambda.
$$

Show that σ_k satisfies (1).

Answer. Let $a_\nu = \exp(-2\lambda\nu/N)$. Then for $k > 0$,

$$
\begin{aligned}
&\sum_{\nu=0}^{N/2-1-k} a_\nu a_{\nu+k} \\
= &\sum_{\nu=0}^{N/2-1-k} \exp(-2\lambda\nu/N)\exp(-2\lambda(\nu+k)/N) \\
= &\sum_{\nu=0}^{N/2-1-k} \exp\left(-\lambda\frac{2\nu+k}{N/2}\right) = \exp\left(-\frac{\lambda k}{N/2}\right)\frac{\exp(-2\lambda(1-2k/N)) - 1}{\exp(-4\lambda/N) - 1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\exp(-2\lambda)-1}{\exp(-4\lambda/N)-1}\frac{\exp(-2\lambda(1-k/N))-\exp(-2\lambda k/N)}{\exp(-2\lambda)-1}\\
&= \frac{\exp(-2\lambda)-1}{\exp(-4\lambda/N)-1}\frac{\exp(-\lambda(1-2k/N))-\exp(\lambda(1-2k/N))}{\exp(-\lambda)-\exp(\lambda)}\\
&= \frac{\exp(-2\lambda)-1}{\exp(-4\lambda/N)-1}\sigma_k.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sigma_0+2\sum_{k=1}^{N/2}\sigma_k\cos(kx)\\
&= \sigma_0+2\sum_{k=1}^{N/2}\sigma_k\exp(\mathrm{i}kx)\\
&= \frac{\exp(-4\lambda/N)-1}{\exp(-2\lambda)-1}\left(\sum_{\nu=0}^{N/2-1}a_\nu a_\nu+2\sum_{k=1}^{N/2}\sum_{\nu=0}^{N/2-1-k}a_\nu a_{\nu+k}\exp(\mathrm{i}kx)\right)\\
&= \frac{\exp(-4\lambda/N)-1}{\exp(-2\lambda)-1}\left(\sum_{\nu=0}^{N/2-1}a_\nu a_\nu+2\sum_{\nu=0}^{N/2-2}\sum_{\mu=\nu+1}^{N/2-1}a_\nu a_\mu\exp(\mathrm{i}(\mu-\nu)\,x)\right)\\
&= \frac{\exp(-4\lambda/N)-1}{\exp(-2\lambda)-1}\left(\sum_{\nu=0}^{N/2-1}a_\nu a_\nu+2\sum_{\nu=0}^{N/2-2}\sum_{\mu=\nu+1}^{N/2-1}a_\mu\exp(\mathrm{i}\mu x)\overline{a_\nu\exp(\mathrm{i}\nu x)}\right)\\
&= \frac{\exp(-4\lambda/N)-1}{\exp(-2\lambda)-1}\left|\sum_{\nu=0}^{N/2-1}a_\nu\exp(\mathrm{i}\nu x)\right|^2\geqslant 0.
\end{aligned}
$$

3. For any function $u_N\in\mathcal{T}_N$, define

$$
\|u_N\|_p=\left(\int_0^{2\pi}|u_N(x)|^p\,\mathrm{d}x\right)^{1/p},\qquad \|u_N\|_\infty=\max_{x\in[0,2\pi)}|u_N(x)|.
$$

a) Show that

$$
\|u_N\|_\infty\leqslant\left(\frac{N+1}{2\pi}\right)^{1/2}\|u_N\|_2. \tag{2}
$$

Answer. Suppose

$$
u_N(x)=\sum_{k=-N/2}^{N/2}\hat{u}_k\exp(\mathrm{i}kx)
$$

with $\hat{u}_{N/2}=\hat{u}_{-N/2}$. Then by Cauchy-Schwarz inequality,

$$
\begin{aligned}
\|u_N\|_\infty&\leqslant\sum_{k=-N/2}^{N/2}|\hat{u}_k|\\
&\leqslant\left(2\pi\sum_{k=-N/2}^{N/2}|\hat{u}_k|^2\right)^{1/2}\left(\frac{1}{2\pi}\sum_{k=-N/2}^{N/2}1\right)^{1/2}=\left(\frac{N+1}{2\pi}\right)^{1/2}\|u_N\|_2.
\end{aligned}
$$

b) Let p_0 be an even integer satisfying $p_0 \geqslant p \geqslant 1$. Prove that $u_N^{p_0/2} \in \mathcal{T}_{Np_0/2}$ and use (2) to show

$$\left\|u_N^{p_0/2}\right\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|u_N\|_\infty^{(p_0-p)/2} \|u_N\|_p^{p/2}.$$

Answer. By straightforward calculation, we get

$$\begin{aligned}
[u_N(x)]^{p_0/2} &= \left(\sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx)\right)^{p_0/2} \\
&= \sum_{k=-Np_0/4}^{Np_0/4} \sum_{k_1+\cdots+k_{p_0/2}=k} \hat{u}_{k_1}\cdots\hat{u}_{k_{p_0/2}} \exp(\mathrm{i}kx).
\end{aligned}$$

When $k = \pm Np_0/2$, each k_s must be $\pm N/2$. By $\hat{u}_{N/2} = \hat{u}_{-N/2}$, we see that

$$\hat{u}_{k_1}\cdots\hat{u}_{k_{p_0/2}} = \hat{u}_{-k_1}\cdots\hat{u}_{-k_{p_0/2}} \qquad \text{if } k_1+\cdots+k_{p_0/2} = Np_0/2.$$

Hence, the function $u_N^{p_0/2}$ lies in $\mathcal{T}_{Np_0/2}$. Now we can use (2) to obtain

$$\begin{aligned}
\left\|u_N^{p_0/2}\right\|_\infty &\leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left\|u_N^{p_0/2}\right\|_2 \\
&= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\int_0^{2\pi} |u_N(x)|^{p_0}\,\mathrm{d}x\right)^{1/2} \\
&= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\int_0^{2\pi} |u_N(x)|^{p_0-p} |u_N(x)|^p\,\mathrm{d}x\right)^{1/2} \\
&\leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\|u_N\|_\infty^{p_0-p} \int_0^{2\pi} |u_N(x)|^p\,\mathrm{d}x\right)^{1/2} \\
&= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|u_N\|_\infty^{(p_0-p)/2} \|u_N\|_p^{p/2}.
\end{aligned}$$

c) Show that

$$\|u_N\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p} \|u_N\|_p,$$

and use this inequality to show the more general case:

$$\|u_N\|_q \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p-1/q} \|u_N\|_p, \qquad \text{if } q \geqslant p.$$

Answer. By the result of (b), we have

$$\|u_N\|_\infty^{p_0/2} = \left\|u_N^{p_0/2}\right\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|u_N\|_\infty^{(p_0-p)/2} \|u_N\|_p^{p/2},$$

from which we obtain

$$\|u_N\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p} \|u_N\|_p. \tag{3}$$

Now we estimate $\|u_N\|_q$:

$$\begin{aligned}\|u_N\|_q &= \left(\int_0^{2\pi} |u_N(x)|^q \,\mathrm{d}x\right)^{1/q} \\ &= \left(\int_0^{2\pi} |u_N(x)|^{q-p}|u_N(x)|^p \,\mathrm{d}x\right)^{1/q} \\ &\leqslant \left(\|u_N\|_\infty^{q-p}\int_0^{2\pi} |u_N(x)|^p \,\mathrm{d}x\right)^{1/q} = (\|u_N\|_\infty^{q-p}\|u_N\|_p^p)^{1/q}.\end{aligned}$$

To proceed, we use (3) to get

$$\|u_N\|_q \leqslant \left[\left(\frac{Np_0/2+1}{2\pi}\right)^{(q-p)/p}\|u_N\|_p^{q-p}\|u_N\|_p^p\right]^{1/q} \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p-1/q}\|u_N\|_p.$$

4. For $0<s<1$, define the linear operator $\mathcal{L}$ by

$$(\mathcal{L}_s u)(x) = \int_{-\infty}^{+\infty}\frac{u(x)-u(y)}{|x-y|^{2s+1}}\,\mathrm{d}y.$$

a) Let $v(x)=u(\alpha x)$. Show that

$$(\mathcal{L}_s v)(x) = |\alpha|^{2s}(\mathcal{L}_s u)(\alpha x), \qquad \forall \alpha \in \mathbb{R}.$$

Answer. When $\alpha \neq 0$,

$$\begin{aligned}(\mathcal{L}_s v)(x) &= \int_{-\infty}^{+\infty}\frac{u(\alpha x)-u(\alpha y)}{|x-y|^{2s+1}}\,\mathrm{d}y \\ &= \frac{1}{|\alpha|}\int_{-\infty}^{+\infty}\frac{u(\alpha x)-u(z)}{|x-z/\alpha|^{2s+1}}\,\mathrm{d}z \\ &= |\alpha|^{2s}\int_{-\infty}^{+\infty}\frac{u(\alpha x)-u(z)}{|\alpha x-z|^{2s+1}}\,\mathrm{d}z = |\alpha|^{2s}(\mathcal{L}_s u)(\alpha x).\end{aligned}$$

When $\alpha=0$, the function $v(x)$ is a constant. Therefore $(\mathcal{L}_s v)(x)=0=|\alpha|^{2s}(\mathcal{L}_s u)(\alpha x)$.

b) Suppose $u(x)$ is 2π-periodic. Show that $\mathcal{L}_s u$ is also 2π-periodic.

Answer. When $u(x)$ is 2π-periodic,

$$\begin{aligned}(\mathcal{L}_s u)(x+2\pi) &= \int_{-\infty}^{+\infty}\frac{u(x+2\pi)-u(y)}{|x+2\pi-y|^{2s+1}}\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty}\frac{u(x+2\pi)-u(y+2\pi)}{|x-y|^{2s+1}}\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty}\frac{u(x)-u(y)}{|x-y|^{2s+1}}\,\mathrm{d}y = (\mathcal{L}_s u)(x).\end{aligned}$$

Therefore, $u(x+2\pi)$ is also periodic.

c) Let $f \in H_p^m(0,2\pi)$ satisfy

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = 0.$$

Describe the Fourier spectral method for solving

$$\mathcal{L}_s u = f, \qquad u \text{ is } 2\pi\text{-periodic.}$$

Answer. We consider the Fourier Galerkin method, which requires us to find $u_N \in \mathcal{T}_N$ such that

$$P_N(\mathcal{L}_s u_N) = P_N f := \sum_{k=-N/2}^{N/2} \hat{f}_k \exp(\mathrm{i}kx).$$

Suppose

$$u_N(x) = \sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx).$$

Then $\mathcal{L}_s u_N$ can be calculated by

$$\begin{aligned}
(\mathcal{L}_s u_N)(x) &= \sum_{k=-N/2}^{N/2} \hat{u}_k \int_{-\infty}^{+\infty} \frac{\exp(\mathrm{i}kx) - \exp(\mathrm{i}ky)}{|x-y|^{2s+1}} \,\mathrm{d}y \\
&= \sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx) \int_{-\infty}^{+\infty} \frac{1 - \exp(-\mathrm{i}k(x-y))}{|x-y|^{2s+1}} \,\mathrm{d}y \\
&= \sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx) \cdot |k|^{2s+1} \int_{-\infty}^{+\infty} \frac{1 - \exp(-\mathrm{i}k(x-y))}{|k(x-y)|^{2s+1}} \,\mathrm{d}y \\
&= \sum_{k=-N/2}^{N/2} \hat{u}_k \exp(\mathrm{i}kx) \cdot |k|^{2s} \int_{-\infty}^{+\infty} \frac{1 - \exp(-\mathrm{i}y)}{|y|^{2s+1}} \,\mathrm{d}y \\
&= \sum_{k=-N/2}^{N/2} |k|^{2s} c_s \hat{u}_k \exp(\mathrm{i}kx).
\end{aligned}$$

Therefore, we have $P_N(\mathcal{L}_s u_N) = \mathcal{L}_s u_N$, and $\hat{u}_k$ can be solved by

$$|k|^{2s} c_s \hat{u}_k = \hat{f}_k, \qquad k = -N/2, \cdots, N/2.$$

The solution is

$$\hat{u}_0 = 0, \qquad \hat{u}_k = \frac{1}{c_s |k|^{2s}} \hat{f}_k, \qquad k \neq 0.$$