Li Xuanguang A0154735B, HW 1

1. Suppose $N = 3^p$ with p being a positive integer. For given $u_0, u_1, \cdots, u_{N-1} \in \mathbb{C}$, write down the FFT (fast Fourier transform) algorithm to compute

$$\hat{u}_k = \sum_{j=0}^{N-1} u_j \exp(-ikx_j), \qquad k = 0, \cdots, N-1.$$

Find the number of additions and multiplications in the algorithm.

Q1. To transform u_j to $\hat{u}_k$, we modify the discrete Fourier transform (DFT) for $N=3^p$

$$\hat{u}_k = \sum_{j=0}^{N-1} u_j \exp\left(-ik\cdot\frac{2\pi j}{N}\right)$$

$$= \sum_{j=0}^{N/3-1} u_{3j} \exp\left(-ik\cdot\frac{2\pi\cdot 3j}{N}\right) + \sum_{j=0}^{N/3-1} u_{3j+1} \exp\left(-ik\cdot\frac{2\pi\cdot(3j+1)}{N}\right) + \sum_{j=0}^{N/3-1} u_{3j+2} \exp\left(-ik\cdot\frac{2\pi\cdot(3j+2)}{N}\right)$$, breaking into equal parts

$$= \sum_{j=0}^{N/3-1} u_{3j} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right) + \exp\left(-ik\frac{2\pi}{N}\right)\sum_{j=0}^{N/3-1} u_{3j+1} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right) + \exp\left(-ik\frac{2\pi\cdot 2}{N}\right)\sum_{j=0}^{N/3-1} u_{3j+2} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right)$$

Let $u_k^{(1)} = \sum_{j=0}^{N/3-1} u_{3j} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right)$,

$u_k^{(2)} = \sum_{j=0}^{N/3-1} u_{3j+1} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right)$,

$u_k^{(3)} = \sum_{j=0}^{N/3-1} u_{3j+2} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right)$

Then $\hat{u}_k = u_k^{(1)} + \exp\left(-ik\frac{2\pi}{N}\right)u_k^{(2)} + \exp\left(-ik\frac{2\pi\cdot 2}{N}\right)u_k^{(3)}$

$$u_{k+N/3}^{(1)} = \sum_{j=0}^{N/3-1} u_{3j} \exp\left(-i(k+N/3)\cdot\frac{2\pi\cdot j}{N/3}\right)$$

$$= \sum_{j=0}^{N/3-1} u_{3j} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right)\exp(-2\pi ij)$$

$$= \sum_{j=0}^{N/3-1} u_{3j} \exp\left(-ik\cdot\frac{2\pi\cdot j}{N/3}\right) = u_k^{(1)}$$

Likewise, $u_{k+N/3}^{(2)} = u_k^{(2)}$ and $u_{k+N/3}^{(3)} = u_k^{(3)}$

Note, $\hat{u}_k = u_k^{(1)} + \exp\left(-ik\frac{2\pi}{N}\right)u_k^{(2)} + \exp\left(-ik\frac{4\pi}{N}\right)u_k^{(3)}$

$$\hat{u}_{k+N/3} = u_k^{(1)} + \exp\left(-i(k+N/3)\frac{2\pi}{N}\right)u_k^{(2)} + \exp\left(-i(k+N/3)\frac{4\pi}{N}\right)u_k^{(3)}$$

$$= u_k^{(1)} + \exp\left(-ik\frac{2\pi}{N}\right)\exp\left(-i\frac{2}{3}\pi\right)u_k^{(2)} + \exp\left(-ik\frac{4\pi}{N}\right)\exp\left(-i\frac{4}{3}\pi\right)u_k^{(3)}$$

$$\hat{u}_{k+2N/3} = u_k^{(1)} + \exp\left(-i\left(k+\frac{2N}{3}\right)\frac{2\pi}{N}\right)u_k^{(2)} + \exp\left(-i\left(k+\frac{2N}{3}\right)\frac{4\pi}{2}\right)u_k^{(3)}$$

$$= u_k^{(1)} + \exp\left(-ik\frac{2\pi}{N}\right)\exp\left(-i\frac{4}{3}\pi\right)u_k^{(2)} + \exp\left(-ik\frac{4\pi}{N}\right)\exp\left(-i\frac{8}{3}\pi\right)u_k^{(3)}$$

$$= u_k^{(1)} + \exp\left(-ik\frac{2\pi}{N}\right)\exp\left(-i\frac{4}{3}\pi\right)u_k^{(2)} + \exp\left(-ik\frac{4\pi}{N}\right)\exp\left(-i\frac{2}{3}\pi\right)u_k^{(3)}$$

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Find the number of additions and multiplications in the algorithm.

Q1. (Cont.)

Algorithm (Pseudocode with Python syntax)

```
fft(x):

    n = len(x)
    if n == 1: # Base case
        return x

    # Roots of unity
    w_n = exp(i2π/n)
    w = 1+0i

    # Extract the indices
    u_1 = fft(x[::3]) # start at 0, step size 3, we get u0, u3, u6, ...
    u_2 = fft(x[1::3]) # start at 1, step size 3, we get u1, u4, u7, ...
    u_3 = fft(x[2::3]) # start at 2, step size 3, we get u2, u5, u8, ...

    u = [0]*n # to store the results

    # Based on û_k, û_{k+N/3}, û_{k+2N/3} calculated earlier
    for k in range(0, n/3):
        u[k] = u_1[k] + w*u_2[k] + w**2*u_3[k] # û_k = u_k^(1) + w u_k^(2) + w^2 u_k^(3)
        u[k+n/3] = u_1[k] + exp(-i(2/3)π)*w*u_2[k] + exp(-i(4/3)π)*w**2*u_3[k] # û_{k+N/3} = u_k^(1) + e^(.) w u_k^(2) + e^(.) w^2 u_k^(3)
        u[k+2n/3] = u_1[k] + exp(-i(4/3)π)*w*u_2[k] + exp(-i(2/3)π)*w**2*u_3[k] # û_{k+2N/3} + u_k^(1) + e^(.) w u_k^(2) + e^(.) w^2 u_k^(3)
        w = w* w_n # as first k=0, we take default value w=1+0i. For later k, we multiply by roots of unity

    return u
```

The total number of additions is $N\log_3 N$, total number of multiplications is $\frac{1}{3}N\log_3(\frac{N}{3})$.

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2. Let N be an even positive integer. For $u \in L^2_p(0, 2\pi)$, we assume that the Fourier series expansion of u is

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k \exp(ikx).$$

(a) Define the *Dirichlet kernel*:

$$\mathscr{D}_N(x) = \sum_{k=-N/2}^{N/2} \exp(ikx).$$

Show that

$$\sum_{k=-N/2}^{N/2} \hat{u}_k \exp(ikx) = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{D}_N(x-y) u(y)\, dy.$$

Q2. (a) $\frac{1}{2\pi}\int_0^{2\pi} D_N(x-y)u(y)\,dy = \frac{1}{2\pi}\sum_{k=-N/2}^{N/2} \int_0^{2\pi} \exp(ik(x-y))u(y)\,dy$

$= \frac{1}{2\pi}\sum_{k=-N/2}^{N/2} \exp(ikx)\int_0^{2\pi}\exp(-iky)u(y)\,dy$

$= \frac{1}{2\pi}\sum_{k=-N/2}^{N/2} \exp(ikx)\, 2\pi\hat{u}_k$

$= \sum_{k=-N/2}^{N/2} \hat{u}_k \exp(ikx)$

(b) Suppose $u(x) \geqslant 0$ for all $x \in [0, 2\pi)$. Show that for all $x \in [0, 2\pi)$,

$$\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(ikx) \geqslant 0$$

if the constants σ_k, $k = -N/2, \cdots, N/2$ satisfy

$$\sigma_k = \sigma_{-k}, \qquad \sigma_0 + 2\sum_{k=1}^{N/2} \sigma_k \cos(kx) \geqslant 0, \quad \forall x \in [0, 2\pi). \tag{1}$$

Q2. (b) As $u(x) \geq 0$ $\forall x \in [0, 2\pi)$, then $u(x)$ is real, implying that $\hat{u}_k = \overline{\hat{u}_{-k}}$

$\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(ikx) = \sigma_0\hat{u}_0 + \sum_{k=1}^{N/2}\sigma_k(\hat{u}_k\exp(ikx) + \hat{u}_{-k}\exp(-ikx))$ (as $\sigma_k = \sigma_{-k}$)

$= \sigma_0\hat{u}_0 + 2\sum_{k=1}^{N/2}\sigma_k\hat{u}_k\cos(kx)$

$= \frac{1}{2\pi}\int_0^{2\pi} \sigma_0 u(y) + 2\sum_{k=1}^{N/2}\sigma_k u(y)\cos(k(y-x))\,dy$

$= \frac{1}{2\pi}\int_0^{2\pi} u(y)\left[\sigma_0 + 2\sum_{k=1}^{N/2}\sigma_k\cos(k(y-x))\right]dy$

≥ 0 as $\sigma_0 + 2\sum_{k=1}^{N/2}\sigma_k\cos(k(y-x)) \geq 0$

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(c) Define the *Fejér kernel*:

$$\mathscr{F}_N(x) = \frac{1}{N/2} \sum_{n=0}^{N/2-1} \mathscr{D}_{2n}(x).$$

Find the coefficients σ_k, $k = -N/2, \cdots, N/2$ such that

$$\sum_{k=-N/2}^{N/2} \sigma_k \hat{u}_k \exp(ikx) = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{F}_N(x-y) u(y)\, dy,$$

and show that σ_k satisfies (1).

Q2. (c) $\frac{1}{2\pi}\int_0^{2\pi} F_n(x-y)u(y)\,dy = \frac{1}{2\pi}\frac{1}{N/2}\sum_{n=0}^{N/2-1}\int_0^{2\pi} D_{2n}(x-y)u(y)\,dy$

$= \frac{1}{N\pi}\sum_{n=0}^{N/2-1}\sum_{k=-n}^{n}\int_0^{2\pi}\exp(ik(x-y))u(y)\,dy$

$= \frac{1}{N\pi}\sum_{n=0}^{N/2-1}\sum_{k=-n}^{n}\exp(ikx)\int_0^{2\pi}\exp(-iky)u(y)\,dy$

$= \frac{1}{N\pi}\sum_{n=0}^{N/2-1}\sum_{k=-n}^{n}\exp(ikx)\,2\pi\hat{u}_k$

$= \frac{2}{N}\sum_{n=0}^{N/2-1}\sum_{k=-n}^{n}\hat{u}_k\exp(ikx)$

$= \frac{2}{N}\left(\hat{u}_{-N/2+1}\exp(-i(N/2-1)x) + \ldots + \frac{N}{2}\hat{u}_0 + \ldots + \hat{u}_{N/2-1}\exp(i(N/2-1)x)\right)$

$= \sum_{k=-N/2}^{N/2}\frac{(N/2)-|k|}{(N/2)}\hat{u}_k\exp(ikx)$, where $\sigma_k = \frac{(N/2)-|k|}{(N/2)}$

Note that $\sigma_k = \frac{(N/2)-|k|}{(N/2)} = \frac{(N/2)-|-k|}{(N/2)} = \sigma_{-k}$

Hence $\sum_{k=-N/2}^{N/2}\sigma_k\cos(kx) = \sum_{k=-N/2}^{N/2}\sigma_k\frac{1}{2}(\exp(ikx)+\exp(-ikx))$

$= \sum_{k=-N/2}^{N/2}\frac{1}{2}\sigma_k\exp(ikx) + \sum_{k=-N/2}^{N/2}\frac{1}{2}\sigma_k\exp(-ikx)$

$= \sum_{k=-N/2}^{N/2}\frac{1}{2}\sigma_k\exp(ikx) + \sum_{k=-N/2}^{N/2}\frac{1}{2}\sigma_k\exp(ikx)$ (reversing the indices, and $\sigma_k = \sigma_{-k}$)

$= \sum_{k=-N/2}^{N/2}\sigma_k\exp(ikx)$

$= \frac{1}{N/2}\left(\exp(-i(N/2+1)x) + \ldots + \frac{N}{2} + \cdots + \exp(i(N/2+1)x)\right)$

$= \frac{1}{N/2}\sum_{n=0}^{N/2-1}\sum_{k=-n}^{n}\exp(ikx)$

$= \frac{1}{N/2}\sum_{n=0}^{N/2-1}D_{2n}(x)$

$= F_{N/2}(x)$

$= \frac{1}{N/2}\left(\frac{1-\cos(N/2\,x)}{1-\cos x}\right)$ (using alternative closed-form representation of Fejer kernel)

≥ 0

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(d) For $\lambda \in \mathbb{R}\backslash\{0\}$, let

$$\sigma_k = \sinh\left(\lambda\left(1 - \frac{k}{N/2}\right)\right) \Big/ \sinh\lambda.$$

Show that σ_k satisfies (1).
[Hint: The corresponding kernel is called *Lorentz kernel*.]

Q2. (d) From (c), we have shown that $\frac{N/2-|k|}{N/2} = 1-\frac{|k|}{N/2}$ is a even function

As $1-\frac{k}{N/2} = 0, \frac{1}{N/2}, \ldots, 1, \ldots, \frac{N/2-1}{N/2}, 2$ for $k = N/2, \ldots, -N/2$

we can use the transformation $2-\frac{2|k|}{N/2}$ to get the same values.

Note, $2-2\frac{|k|}{N/2} = 2\left(1-\frac{|k|}{N/2}\right)$ is an even function as $1-\frac{|k|}{N/2}$ is an even function.

Then, $\sigma_k = \sinh\left(\lambda\left(2\left(1-\frac{|k|}{N/2}\right)\right)\right)/\sinh\lambda$

$= \sinh\left(\lambda\left(2\left(1-\frac{|-k|}{N/2}\right)\right)\right)/\sinh\lambda$

$= \sigma_{-k}$

Also, $\sum_{k=-N/2}^{N/2} \sigma_k \cos(kx) = \sum_{k=-N/2}^{N/2} \sigma_k \exp(ikx)$ (from part (c)).

For $k=-N/2, \ldots, -1$,

$\sinh\left(\lambda\left(1-\frac{|k|}{N/2}\right)\right) = \sinh\left(\lambda\left(1+\frac{k}{N/2}\right)\right) = \frac{1}{2}\left[\exp\left(\lambda\left(1+\frac{k}{N/2}\right)\right) - \exp\left(-\lambda\left(1+\frac{k}{N/2}\right)\right)\right]$

For $k=0$, $\sigma_0 = \sinh(\lambda)/\sinh(\lambda) = 1$

For $k=1,\ldots,N/2$,

$\sinh\left(\lambda\left(1-\frac{|k|}{N/2}\right)\right) = \frac{1}{2}\left[\exp\left(\lambda\left(1-\frac{k}{N/2}\right)\right) - \exp\left(\lambda\left(\frac{k}{N/2}-1\right)\right)\right]$

Hence $\sum_{k=-N/2}^{N/2} \sigma_k \exp(ikx) = \frac{1}{2}\frac{1}{\sinh\lambda}\left[\exp(\lambda)\sum_{k=-N/2}^{N/2}\exp(ikx)\exp\left(-\frac{2\lambda}{N}|k|\right) - \exp(-\lambda)\sum_{k=-N/2}^{N/2}\exp(ikx)\exp\left(\frac{2\lambda}{N}|k|\right)\right] = \frac{T_n}{2\sinh\lambda}$

For $\lambda > 0$, $\exp\left(\frac{2\lambda}{N}|k|\right) \leq \exp\left(\frac{2\lambda}{N}\frac{N}{2}\right) = \exp(\lambda)$

$\exp\left(-\frac{2\lambda}{N}|k|\right) \geq \exp\left(-\frac{2\lambda}{N}\frac{N}{2}\right) = \exp(-\lambda)$

Hence $T_n \geq \exp(\lambda)\exp(-\lambda)\sum_{k=-N/2}^{N/2}\exp(ikx) - \exp(\lambda)\exp(-\lambda)\sum_{k=-N/2}^{N/2}\exp(ikx) = 0$

$\Rightarrow \sum_{k=-N/2}^{N/2}\sigma_k\exp(ikx) = \frac{T_n}{2\sinh\lambda} \geq 0$

?

For $\lambda < 0$, $\exp\left(\frac{2\lambda}{N}|k|\right) \leq \exp\left(\frac{2\lambda}{N}\cdot 0\right) = 1$

$\exp\left(-\frac{2\lambda}{N}|k|\right) \geq \exp\left(-\frac{2\lambda}{N}\cdot 0\right) = 1$

Hence $T_n \geq \sum_{k=-N/2}^{N/2}\exp(ikx)(\exp(\lambda) - \exp(-\lambda)) = 2\sinh(\lambda)\sum_{k=-N/2}^{N/2}\exp(ikx)$

$\Rightarrow \sum_{k=-N/2}^{N/2}\sigma_k\exp(ikx) = \frac{T_n}{2\sinh\lambda} \geq \sum_{k=-N/2}^{N/2}\exp(ikx) \geq 0$

Thus, $\sum_{k=-N/2}^{N/2}\sigma_k\cos(kx) \geq 0$

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3. For any function $u_N \in \mathscr{T}_N$, define

$$\|u_N\|_p = \left(\int_0^{2\pi} |u_N(x)|^p \, dx\right)^{1/p}, \quad \text{if } p > 1, \qquad \|u_N\|_\infty = \max_{x\in[0,2\pi)} |u_N(x)|.$$

(a) Show that

$$\|u_N\|_\infty \leqslant \left(\frac{N+1}{2\pi}\right)^{1/2} \|u_N\|_2. \tag{2}$$

Q3. (a) Let $u_N = \sum_{k=-N/2}^{N/2} \hat{u}_k \exp(ikx)$, $\hat{u}_k = \hat{u}_{-k}$

Then $\|u_N\|_\infty \leq \sum_{k=-N/2}^{N/2} |\hat{u}_k| \cdot 1$ (By triangle inequality)

$\leq \left(\sum_{k=-N/2}^{N/2} |\hat{u}_k|^2 dx\right)^{1/2} \left(\sum_{k=-N/2}^{N/2} 1\right)^{1/2}$

$= \left(\frac{1}{2\pi}\int_0^{2\pi} |u_N(x)|^2 dx\right)^{1/2} \cdot (N+1)^{1/2}$

$= \left(\frac{N+1}{2\pi}\right)^{1/2} \|u_N\|_2$

(b) Let p_0 be an even integer satisfying $p_0 \geqslant p \geqslant 1$. Prove that $u_N^{p_0/2} \in \mathcal{T}_{Np_0/2}$ and use (2) to show

$$\|u_N^{p_0/2}\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|u_N\|_\infty^{(p_0-p)/2} \|u_N\|_p^{p/2}.$$

Q3. (b) $u_N^{p_0/2} = \left[\sum_{k=-N/2}^{N/2} \hat{u}_k \exp(ikx)\right]^{p_0/2}$, $\hat{u}_k = \hat{u}_{-k}$

$= \sum_{k=-Np_0/4}^{Np_0/4} \hat{V}_k \exp(ikx)$, $\hat{V}_{Np_0/4} = \hat{u}_k^{p_0/2} = \hat{u}_{-k}^{p_0/2} = \hat{V}_{-Np_0/2}$

Hence $u_N^{p_0/2} \in T_{Np_0/2}$

$\|u_N^{p_0/2}\|_\infty \leq \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|u_N^{p_0/2}\|_2$ (by part (a))

$= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\int_0^{2\pi} |u_N^{p_0/2}|^2 dx\right)^{1/2}$

$= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\int_0^{2\pi} |u_N^{p_0}| dx\right)^{1/2}$

$= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\int_0^{2\pi} |u_N^{p_0-p}||u_N^p| dx\right)^{1/2}$

$\leq \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\max|u_N^{p_0-p}|\right)^{1/2} \left(\int_0^{2\pi} |u_N^p| dx\right)^{1/2}$

$= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \left(\max|u_N|^{p_0-p}\right)^{1/2} \left(\int_0^{2\pi} |u_N|^p dx\right)^{1/2}$

$= \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|u_N\|_\infty^{(p_0-p)/2} \|u_N\|_p^{p/2}$

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(c) Show that

$$\|u_N\|_\infty \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p} \|u_N\|_p,$$

and use this inequality to show the more general case:

$$\|u_N\|_q \leqslant \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p-1/q} \|u_N\|_p, \qquad \text{if } q \geqslant p.$$

Q3. (c) $\|U_N^{p_0/2}\|_\infty \le \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|U_N\|_\infty^{(p_0-p)/2} \|U_N\|_p^{p/2}$ (by part (b))

As $\|U_N^k\|_\infty = \max|U_N^k| = \max|U_N|^k = \|U_N\|_\infty^k$ for $k \ge 1$

Hence $\|U_N^{p/2}\|_\infty = \|U_N\|_\infty^{p/2} \le \left(\frac{Np_0/2+1}{2\pi}\right)^{1/2} \|U_N\|_p^{p/2}$

$\Rightarrow \|U_N\|_\infty \le \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p} \|U_N\|_p$

$\|U_N\|_q^q = \int_0^{2\pi} |U_N|^q \cdot 1\, dx$

$\le \left(\int_0^{2\pi} |U_N|^p\, dx\right)^{q/p} \left(\int_0^{2\pi} dx\right)^{1-(q/p)}$ (by Hölder inequality)

$= \|U_N\|_p^q\, (2\pi)^{1-(q/p)}$

$= \|U_N\|_p^q \left(\frac{1}{2\pi}\right)^{(q/p)-1}$

Hence $\|U_N\|_q \le \|U_N\|_p \left(\frac{1}{2\pi}\right)^{1/p-1/q}$

$\le \|U_N\|_p \left(\frac{Np_0/2+1}{2\pi}\right)^{1/p-1/q}$

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4. For $0 < s < 1$, define the linear operator $\mathscr{L}$ by

$$(\mathscr{L}_s u)(x) = \int_{-\infty}^{+\infty} \frac{u(x)-u(y)}{|x-y|^{2s+1}}\,\mathrm{d}y.$$

(a) Let $v(x) = u(\alpha x)$. Show that

$$(\mathscr{L}_s v)(x) = |\alpha|^{2s}(\mathscr{L}_s u)(\alpha x), \qquad \forall \alpha \in \mathbb{R}.$$

Q4. (a) $(\mathscr{L}_s v)(x) = \int_{-\infty}^{+\infty} \frac{v(x)-v(y)}{|x-y|^{2s+1}}\,dy$

$= \int_{-\infty}^{+\infty} \frac{u(\alpha x)-u(\alpha y)}{|x-y|^{2s+1}}\,dy$

$= |\alpha|^{2s} \int_{-\infty}^{+\infty} \frac{u(\alpha x)-u(\alpha y)}{|\alpha x-\alpha y|^{2s+1}}\,d\alpha y$

$= |\alpha|^{2s} \int_{-\infty}^{+\infty} \frac{u(z)-u(w)}{|z-w|^{2s+1}}\,dw$, where $z=\alpha x, w=\alpha y$

$= |\alpha|^{2s} (\mathscr{L}_s u)(\alpha x)$

(b) Suppose $u(x)$ is 2π-periodic. Show that $\mathscr{L}_s u$ is also 2π-periodic.

Q4. (b) As $u(x)$ is 2π-periodic, then $u(x+2\pi) = u(x)$

$(\mathscr{L}_s u)(x+2\pi) = \int_{-\infty}^{+\infty} \frac{u(x+2\pi)-u(y)}{|x+2\pi-y|^{2s+1}}\,dy$

$= \int_{-\infty}^{+\infty} \frac{u(x)-u(y)}{|x+2\pi-y|^{2s+1}}\,dy$, as $u(x+2\pi)=u(x)$

$= \int_{-\infty}^{+\infty} \frac{u(x)-u(y)}{|x-y|^{2s+1}}\,dy$, as absolute value function is symmetric around 0

$= (\mathscr{L}_s u)(x)$

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(c) Let $f \in H_p^m(0, 2\pi)$ satisfy

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = 0.$$

Describe the Fourier spectral method for solving

$$\mathscr{L}_s u = f, \qquad u \text{ is } 2\pi\text{-periodic.}$$

[Hint: Define the constant $c_s = \displaystyle\int_{\mathbb{R}} |y|^{-(2s+1)}[1-\exp(-iy)]\,\mathrm{d}y$, and compute $\mathscr{L}_s(\exp(ikx))$.]

Q4.(c) Let $c_s = \int_{\mathbb{R}} |y|^{-(2s+1)}[1-\exp(-iy)]\,dy$

Then $\mathcal{L}_s(\exp(ikx)) = |k|^{2s} \int_{\mathbb{R}} \frac{\exp(ikx)-\exp(iy)}{|kx-y|^{2s+1}}\,dy$ by property of linear operator

$= |k|^{2s} \exp(ikx) \int_{\mathbb{R}} \frac{1-\exp(-i(kx-y))}{|kx-y|^{2s+1}}\,dy$

$= |k|^{2s} \exp(ikx) \int_{\mathbb{R}} |y|^{-(2s+1)} [1-\exp(iy)]\,dy$, integration by substitution

$= |k|^{2s} \exp(ikx)\, c_s$

Using projection method, we know

$$P_n u = \sum_{k=-N/2}^{N/2} \bar{u}_k \exp(ikx)$$

$$P_n f = \sum_{k=-N/2}^{N/2} \bar{f}_k \exp(ikx)$$

Then $P_n \mathcal{L}_N u = P_n f$

$$\sum_{k=-N/2}^{N/2} \bar{u}_k \mathcal{L}_s(\exp(ikx)) = \sum_{k=-N/2}^{N/2} \bar{f}_k \exp(ikx)$$

$$\Rightarrow \sum_{k=-N/2}^{N/2} \bar{u}_k |k|^{2s} \exp(ikx)\, c_s = \sum_{k=-N/2}^{N/2} \bar{f}_k \exp(ikx)$$

As $\int_0^{2\pi} f(x)\,dx = 0$, this means $\bar{f}_0 = 0$, hence $\bar{u}_0 = 0$.

$\Rightarrow \bar{u}_k = \frac{\bar{f}_k}{|k|^{2s} c_s}$ where $k = -N/2, \dots, -1, 1, \dots N/2$

Thus, $\bar{u}_k = \begin{cases} 0 & \text{if } k = 0 \\ \frac{\bar{f}_k}{|k|^{2s} c_s} & \text{if } k = -N/2, \dots, -1, 1, \dots, N/2 \end{cases}$

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