MA5251 Homework 2

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April 5, 2023

Q1. From lecture notes, we have the three-term recurrence relation

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x)$$

Differentiating once and by chain rule, we get

$$(n+1)L_{n+1}^{(1)}(x) = (2n+1)L_n(x) + (2n+1)xL_n^{(1)}(x) - nL_{n-1}^{(1)}(x)$$

Differentiating again, we get

$$\begin{aligned}(n+1)L_{n+1}^{(2)}(x) &= (2n+1)L_n^{(1)}(x) + (2n+1)L_n^{(1)}(x) + (2n+1)xL_n^{(2)}(x) - nL_{n-1}^{(2)}(x) \\ &= 2(2n+1)L_n^{(1)}(x) + (2n+1)xL_n^{(2)}(x) - nL_{n-1}^{(2)}(x)\end{aligned}$$

Hence, by the m-th differentiation, we get

$$(n+1)L_{n+1}^{(m)}(x) = m(2n+1)L_n^{(m-1)}(x) + (2n+1)xL_n^{(m)}(x) - nL_{n-1}^{(m)}(x)$$

From the second Proposition in Properties of Derivatives, Lecture 14 Slide 10, we can see that

$$L_{n+1}^{(1)}(x) - L_{n-1}^{(1)}(x) = (2n+1)L_n(x)$$

Differentiating $m-1$ times, this implies

$$L_{n+1}^{(m)}(x) - L_{n-1}^{(m)}(x) = (2n+1)L_n^{(m-1)}(x)$$

Substituting into the differentiated three-term recurrence relation, we get

$$(n+1)L_{n+1}^{(m)}(x) = m[L_{n+1}^{(m)}(x) - L_{n-1}^{(m)}(x)] + (2n+1)xL_n^{(m)}(x) - nL_{n-1}^{(m)}(x)$$

Thus we get the three-term recurrence relation

$$\begin{aligned}(n-m+1)L_{n+1}^{(m)}(x) &= (2n+1)xL_n^{(m)}(x) - (n+m)L_{n-1}^{(m)}(x) \\ L_{n+1}^{(m)}(x) &= \frac{2n+1}{n-m+1}xL_n^{(m)}(x) - \frac{n+m}{n-m+1}L_{n-1}^{(m)}(x)\end{aligned}$$

where $\alpha_n^{(m)} = \frac{2n+1}{n-m+1}$ and $\beta_n^{(m)} = \frac{n+m}{n-m+1}$
For the initial condition, if $n < m$, then $L_n^{(m)} = 0$.
The first term is where $n = m$, then $L_m^{(m)} = (2m)!a_m = \frac{(2m)!}{2^m m!}$.
The next term is where $n = m+1$, then $L_{m+1}^{(m)} = a_{m+1}[(2m+2)!x - (2m+1)!(m+1)] = \frac{(2m+1)!}{2^m m!}x - \frac{(2m+1)!}{2^{m+1}m!}$.

Q2. We prove both relations with induction.
From Lecture 14 Chebyshev Polynomials , we have the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.
Let $\theta = \arccos(x)$. Then $T_n(x) = \cos(n\theta)$, and this implies $\cos((n+1)\theta) = 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta)$

Taking the derivative of $T_{n+1}(x)$ and $T_{n-1}(x)$,

$$\begin{aligned} T_{n+1}^{(1)}(x) &= -\theta^{(1)}(n+1)\sin((n+1)\theta) \\ T_{n-1}^{(1)}(x) &= -\theta^{(1)}(n-1)\sin((n-1)\theta) \end{aligned}$$

Hence we can prove the relation

$$\begin{aligned} \frac{1}{n+1}T_{n+1}^{(1)}(x) - \frac{1}{n-1}T_{n-1}^{(1)}(x) &= -\theta^{(1)}[\sin((n+1)\theta) - \sin((n-1)\theta)] \\ &= -\theta^{(1)}[2\cos(n\theta)\sin(\theta)] \\ &= \frac{\sin\theta}{\sqrt{1-(\cos\theta)^2}}2\cos(n\theta) \\ &= 2T_n(x) \end{aligned}$$

From Lecture 14 Chebyshev Polynomials, we have the inner product

$$(T_n(x), T_m(x)) = \int_{-1}^{1} T_m(x)T_n(x)\omega(x)\ dx = \frac{\pi}{2}c_n\delta_{mn}$$

where $c_n = 1 + \delta_{n0}$. Hence, we can derive the following results:

$$\begin{aligned} (T_n(x), T_0(x)) &= \frac{\pi}{2}(1+\delta_{n0})\delta_{n0}, \quad n \geq 1 \\ (T_0(x), T_0(x)) &= \frac{\pi}{2}(1+\delta_{00})\delta_{00}, \quad n \geq 1 \end{aligned}$$

For the first differential relation, we prove by induction.

$$T_2^{(1)} = 2 \times 2T_1 = 2 \cdot 2 \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{1} \frac{1}{c_k}T_k(x)$$

Assume $n = N \geq 2$. Then

$$T_N^{(1)}(x) = 2N \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{N-1} \frac{1}{c_k}T_k(x)$$

For $n = N+1$, we use the result from the relation proved earlier.

$$\begin{aligned} \frac{1}{N+1}T_{N+1}^{(1)}(x) &= 2T_N(x) + \frac{1}{N-1}T_{N-1}^{(1)}(x) \\ &= 2T_N(x) + 2 \sum_{\substack{k=0 \\ k+N-1 \text{ odd}}}^{N-1} \frac{1}{c_k}T_k(x) \\ &= 2T_N(x) + 2 \sum_{\substack{k=0 \\ k+N+1 \text{ odd}}}^{N-1} \frac{1}{c_k}T_k(x) \qquad , k+N-1 \text{ odd} \Rightarrow k+N+1 \text{ odd} \\ &= 2 \sum_{\substack{k=0 \\ k+N+1 \text{ odd}}}^{N} \frac{1}{c_k}T_k(x) \end{aligned}$$

Hence by induction,

$$T_n^{(1)}(x) = 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{N-1} \frac{1}{1+\delta_{k0}}T_k(x)$$

For the second differential relation, we prove by induction.

$$T_2^{(2)} = 4 = \frac{1}{2} \cdot 2 \cdot (4-0) T_0 = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{0} \frac{n}{c_k}(n^2 - k^2) T_k(x)$$

Assume $n = N \geq 2$. Then

$$T_N^{(2)}(x) = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{N-2} \frac{n}{c_k}(n^2 - k^2) T_k(x)$$

For $n = N+1$, we use the result from the relation proved earlier.

$$\begin{aligned}
\frac{1}{N+1} T_{N+1}^{(2)}(x) &= \left(2T_N(x) + \frac{1}{N-1} T_{N-1}^{(1)}(x) \right)^{(1)} \\
&= 2T_N^{(1)}(x) + \frac{1}{N-1} T_{N-1}^{(2)}(x) \\
&= 4N \sum_{\substack{k=0 \\ k+N \text{ odd}}}^{N-1} \frac{1}{1+\delta_{k0}} T_k(x) + \frac{1}{N-1} \sum_{\substack{k=0 \\ k+N-1 \text{ even}}}^{N-3} \frac{N-1}{c_k}((N-1)^2 - k^2) T_k(x) \\
&= \frac{1}{c_{N-1}} T_{N-1}(x) \cdot 4N + \sum_{\substack{k=0 \\ k+N-3 \text{ even}}}^{N-3} \frac{1}{c_k}((N-1)^2 - k^2 + 4N) T_k(x) \\
&= \frac{1}{c_{N-1}} T_{N-1}(x) \cdot [(N+1)^2 - (N-1)^2] + \sum_{\substack{k=0 \\ k+N+1 \text{ even}}}^{N-1} \frac{1}{c_k}((N+1)^2 - k^2) T_k(x) \\
&= \sum_{\substack{k=0 \\ k+N+1 \text{ even}}}^{N-1} \frac{1}{c_k}((N+1)^2 - k^2) T_k(x)
\end{aligned}$$

Hence by induction,

$$T_n^{(2)}(x) = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{n-2} \frac{1}{1+\delta_{k0}} n(n^2 - k^2) T_k(x)$$

Q3. (a) Define the Sturm-Liouville operator

$$\mathcal{L}f(x) = -\sqrt{1-x^2}\frac{d}{dx}[\sqrt{1-x^2}f'(x)]$$

According to Sturm-Liouville equation, we have

$$\mathcal{L}T_n(x) = n^2 T_n(x)$$

We also have

$$(\mathcal{L}f, g) = (f, \mathcal{L}g)$$

By completeness of basis $\{T_n(x)\}_{n\in\mathbb{N}}$ in $L^2_\omega(-1,1)$ space, we can expand any function $f \in L^2_\omega(-1,1)$:

$$f(x) = \sum_{n=0}^{+\infty} \hat{f}_n T_n(x)$$

Note

$$\begin{aligned}
\|f - \pi_N f\|^2_{0,\omega} &= \left\| \sum_{n=N+1}^{+\infty} \hat{f}_n T_n(x) \right\|^2_{0,\omega} \\
&= \sum_{n=N+1}^{+\infty} \frac{\pi}{2} \left| \hat{f}_n \right|^2 \quad (\text{as } \|T_n(x)\|^2_{0,\omega} = \frac{\pi}{2}(1+\delta_{k0})) \\
&= \sum_{n=N+1}^{+\infty} \frac{2}{\pi} \left| \int_{-1}^{1} f(x) T_n(x) \omega(x)\ dx \right|^2 \\
&= \frac{2}{\pi} \sum_{n=N+1}^{+\infty} |(f, T_n)_\omega|^2 \\
&= \frac{2}{\pi} \sum_{n=N+1}^{+\infty} \left| \frac{1}{n^2}(f, \mathcal{L}T_n)_\omega \right|^2 \\
&= \frac{2}{\pi} \sum_{n=N+1}^{+\infty} \left| -\frac{1}{n^2}(\mathcal{L}f, T_n)_\omega \right|^2 \\
&= \frac{2}{\pi} \sum_{n=N+1}^{+\infty} \left| \frac{1}{n^r}(\mathcal{L}^{r/2}f, T_n)_\omega \right|^2 \quad (\text{Assume } r \text{ is even}) \\
&\le \frac{2}{\pi}\frac{1}{N^{2r}} \sum_{n=N+1}^{+\infty} \left| (\mathcal{L}^{r/2}f, T_n)_\omega \right|^2 \\
&\le \frac{1}{N^{2r}} \left[\frac{1}{\pi} \left| (\mathcal{L}^{r/2}f, T_n)_\omega \right|^2 + \sum_{n=1}^{+\infty} \frac{2}{\pi} \left| (\mathcal{L}^{r/2}f, T_n)_\omega \right|^2 \right] \\
&\le \frac{1}{N^{2r}} \left\| \mathcal{L}^{r/2} f \right\|^2_{0,\omega} \\
&\lesssim N^{-2r} \|f\|^2_{r,\omega}
\end{aligned}$$

Hence, $\|f - \pi_N f\|_{0,\omega} \lesssim N^{-r} \|f\|_{r,\omega}$.

Q3. (b) $\forall p \in P_N$, the projection on Chebyshev Polynomial is $p(x) = \sum_{n=0}^{N} \hat{p}_n T_n(x)$.

The derivative is then $p^{(1)}(x) = \sum_{n=0}^{N} \hat{p}_n^{(1)} T_n^{(1)}(x)$, where $\hat{p}_n^{(1)} = \frac{2}{\pi(1+\delta_{0n})} \int_{-1}^{1} p^{(1)}(x) T_n(x)\omega(x)\ dx$.

The seminorm of $p(x)$ is defined as $|p|_1^2 = \sum_{n=0}^{N} \left|\hat{p}_n^{(1)}\right|^2 \cdot \|T_n\|_{L^2_\omega}^2$.

From the derivative $p^{(1)}(x)$ and differential relation of $T_n^{(1)}(x)$ from Q2, we can see that

$$\hat{p}_n^{(1)} = \frac{2}{\pi(1+\delta_{0n})} \int_{-1}^{1} \left(\sum_{j=0}^{N} \hat{p}_j\right) \left(2j \sum_{\substack{i=0 \\ i+j \text{ odd}}}^{j-1} \frac{1}{1+\delta_{i0}} T_i(x)\right) T_n(x)\omega(x)\ dx$$

As $\{T_i(x)\}_{i\in\mathbb{N}}$ are orthogonal in $L^2_\omega(-1,1)$, if $i = n$, then the integral will be nonzero, thus $j \geq n+1$ and $j+n$ is odd. Thus

$$\begin{aligned}
\hat{p}_n^{(1)} &= \frac{2}{\pi(1+\delta_{0n})} \left(\sum_{\substack{j=n+1 \\ j+n \text{ odd}}}^{N} \hat{p}_j^{(1)} \cdot 2j \cdot \frac{1}{1+\delta_{n0}} \cdot \|T_n\|_{L^2_\omega}^2\right) \\
&= \frac{4}{\pi(1+\delta_{0n})^2} \left(\sum_{\substack{j=n+1 \\ j+n \text{ odd}}}^{N} \hat{p}_j \cdot j\right) \cdot \|T_n\|_{L^2_\omega}^2
\end{aligned}$$

Thus we can see

$$\begin{aligned}
|p|_1^2 &= \sum_{n=0}^{N} \left|\hat{p}_n^{(1)}\right|^2 \cdot \|T_n\|_{L^2_\omega}^2 \\
&= \sum_{n=0}^{N} \left(\frac{4}{\pi(1+\delta_{0n})^2}\right)^2 \left(\sum_{\substack{j=n+1 \\ j+n \text{ odd}}}^{N} \hat{p}_j \cdot j\right)^2 \cdot \|T_n\|_{L^2_\omega}^6 \quad \text{by earlier calculation} \\
&= \sum_{n=0}^{N} \left(\sum_{\substack{j=n+1 \\ j+n \text{ odd}}}^{N} \hat{p}_j \cdot j\right)^2 \cdot \|T_n\|_{L^2_\omega}^2 \quad (\text{as } \|T_n\|_{L^2_\omega}^2 = \frac{\pi}{2}(1+\delta_{0n})) \\
&\leq \sum_{n=0}^{N} \left(\sum_{j=0}^{N} \hat{p}_j \cdot j\right)^2 \cdot \|T_n\|_{L^2_\omega}^2 \\
&\leq \sum_{n=0}^{N} \left(\sum_{j=0}^{N} \hat{p}_j^2\right) \left(\sum_{j=0}^{N} \cdot j\right) \cdot \|T_n\|_{L^2_\omega}^2 \quad \text{by Cauchy-Schwarz} \\
&\leq N^4 \sum_{n=0}^{N} \left(\sum_{j=0}^{N} \hat{p}_j^2\right) \cdot \|T_n\|_{L^2_\omega}^2 \quad \text{by Cauchy-Schwarz} \\
&\lesssim N^4 \|p\|_{L^2_\omega}^2
\end{aligned}$$

Hence, $|p|_1 \lesssim N^2 \|p\|_{L^2_\omega}$.
Note that for $n \in \mathbb{N}$,

$$\begin{aligned}
|p|_n &= \left|p^{(1)}\right|_n - 1 = \left|p^{(2)}\right|_n - 2 = \cdots = \left|p^{(n-1)}\right|_1 \lesssim N^2 \left\|p^{n-1}\right\|_{L^2_\omega} \\
\left\|p^{(n-1)}\right\|_{L^2_\omega} &= \left|p^{(n-2)}\right|_1 \lesssim N^2 \left\|p^{(n-2)}\right\|_{L^2_\omega}
\end{aligned}$$

Thus by these relations, we would have the error estimate

$$\|p\|_{r,\omega} \lesssim N^{2r} \|p\|_{L^2_\omega} = N^{2r} \|p\|_{0,w\omega} \quad \forall p \in P_N$$

Q3. (c) From Q2, we have that $T_n^{(1)}(x) = 2n \sum\limits_{\substack{k=0 \\ k+n \text{ odd}}}^{N-1} \frac{1}{1+\delta_{k0}} T_k(x)$.

From part (a), we also have the expansion of f as $f(x) = \sum\limits_{n=0}^{+\infty} \hat{f}_n T_n(x)$. Hence

$$\begin{aligned}
\partial_x f &= \sum_{n=1}^{+\infty} \hat{f}_n T_n^{(1)}(x) \\
&= \sum_{n=1}^{+\infty} \hat{f}_n \cdot 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} \frac{1}{1+\delta_{k0}} T_k(x) \\
&= \sum_{k=0}^{+\infty} \frac{1}{1+\delta_{k0}} \left(\sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n \right) T_k(x)
\end{aligned}$$

Then

$$\pi_N(\partial_x f) = \sum_{k=0}^{N} \frac{1}{1+\delta_{k0}} \left(\sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n \right) T_k(x)$$

$$\begin{aligned}
\partial_x(\pi_N f) &= \sum_{n=1}^{N} \hat{f}_n T_n^{(1)}(x) \\
&= \sum_{n=1}^{N} \hat{f}_n \cdot 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} \frac{1}{1+\delta_{k0}} T_k(x) \\
&= \sum_{k=0}^{N-1} \frac{1}{1+\delta_{k0}} \left(\sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{N} 2n \cdot \hat{f}_n \right) T_k(x)
\end{aligned}$$

Thus we can see that

$$\begin{aligned}
\pi_N(\partial_x f) - \partial_x(\pi_N f) &= \sum_{k=0}^{N} \frac{1}{1+\delta_{k0}} \left(\sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n \right) T_k(x) - \sum_{k=0}^{N-1} \frac{1}{1+\delta_{k0}} \left(\sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{N} 2n \cdot \hat{f}_n \right) T_k(x) \\
&= \sum_{k=0}^{N} \frac{1}{1+\delta_{k0}} \left(\sum_{\substack{n=N+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n \right) T_k(x)
\end{aligned}$$

Note that

$$\begin{cases}
\sum\limits_{\substack{n=N+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n = \sum\limits_{\substack{n=N+1 \\ N+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n = \hat{f}_N^{(1)} & \text{if } N+n \text{ is even} \\
\sum\limits_{\substack{n=N+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n = \sum\limits_{\substack{n=N+2 \\ N+1+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n = \hat{f}_{N+1}^{(1)} & \text{if } N+n \text{ is odd}
\end{cases}$$

Thus,

$$
\begin{aligned}
\|\pi_N(\partial_x f) - \partial_x(\pi_N f)\|_{L^2_\omega}^2 &= \sum_{k=0}^{N} \left(\frac{1}{1+\delta_{k0}} \sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n \right)^2 \cdot \frac{\pi}{2}(1+\delta_{k0}) \\
&= \frac{\pi}{2} \sum_{k=0}^{N} \frac{1}{1+\delta_{k0}} \left(\sum_{\substack{n=k+1 \\ k+n \text{ odd}}}^{+\infty} 2n \cdot \hat{f}_n \right)^2 \\
&= \begin{cases} \frac{\pi}{2} \sum\limits_{k=0}^{N} \frac{1}{1+\delta_{k0}} \left(\hat{f}_N^{(1)} \right)^2 & \text{if } k+n \text{ is even} \\ \frac{\pi}{2} \sum\limits_{k=0}^{N} \frac{1}{1+\delta_{k0}} \left(\hat{f}_{N+1}^{(1)} \right)^2 & \text{if } k+n \text{ is odd} \end{cases}
\end{aligned}
$$

To estimate the sum, note that

$$
\begin{aligned}
\left| \hat{f}_N^{(1)} \right|^2 &= \left[\left(\frac{\pi}{2}(1+\delta_{N0}) \right)^{-1} \int_{-1}^{1} \partial_x f \cdot T_N(x)\ dx \right]^2 \\
&= \frac{4}{\pi^2} \left\| (\partial_x f, T_N) \right\|_{L^2_\omega}^2 \\
&= \frac{4}{\pi^2} \left\| -\frac{1}{N^2} (\partial_x f, \mathcal{L} T_N) \right\|_{L^2_\omega}^2 \quad \text{(By Sturm-Liouville Operator)} \\
&\lesssim N^{-4} \left\| (\mathcal{L}(\partial_x f), T_n) \right\|_{L^2_\omega}^2 \\
&\lesssim N^{-4(r-1)/2} \left\| (\mathcal{L}^{(r-1)/2}(\partial_x f), T_n) \right\|_{L^2_\omega}^2 \\
&\lesssim N^{-4(r-1)/2} \left\| \mathcal{L}^{(r-1)/2}(\partial_x f) \right\|_{L^2_\omega}^2 \left\| T_n \right\|_{L^2_\omega}^2 \quad \text{(By Cauchy-Schwarz Inequality)} \\
&\lesssim N^{-4(r-1)/2} \left\| \mathcal{L}^{(r-1)/2}(\partial_x f) \right\|_{L^2_\omega}^2 \left(\frac{\pi}{2} \right)^2 \\
&\lesssim N^{-2r+2} \left\| f \right\|_{r,\omega}^2
\end{aligned}
$$

Similarly,

$$
\left| \hat{f}_{N+1}^{(1)} \right|^2 \lesssim N^{-2r+2} \left\| f \right\|_{r,\omega}^2
$$

Thus,

$$
\begin{aligned}
\|\pi_N(\partial_x f) - \partial_x(\pi_N f)\|_{L^2_\omega}^2 &\lesssim \frac{\pi}{2} \left(\sum_{k=0}^{N} \frac{1}{1+\delta_{k0}} \right) N^{-2r+2} \left\| f \right\|_{r,\omega}^2 \\
&\lesssim N^{3-2r} \left\| f \right\|_{r,\omega}^2
\end{aligned}
$$

Therefore,

$$
\|\pi_N(\partial_x f) - \partial_x(\pi_N f)\|_{L^2_\omega} \lesssim N^{3/2-r} \left\| f \right\|_{r,\omega}
$$

Q3. (d) Note that

$$
\begin{aligned}
\|f-\pi_N f\|_{1,\omega}^2 &= \|f-\pi_N f\|_{L^2_\omega}^2 + |f-\pi_N f|_1^2 \\
&= \|f-\pi_N f\|_{L^2_\omega}^2 + \|\partial_x f - \partial_x(\pi_N f)\|_{L^2_\omega}^2 \\
&\leq \|f-\pi_N f\|_{L^2_\omega}^2 + \|\partial_x f - \pi_N(\partial_x f)\|_{L^2_\omega}^2 + \|\pi_N(\partial_x f) - \partial_x(\pi_N f)\|_{L^2_\omega}^2 \\
&\lesssim N^{-2r}\|f\|_{r,\omega}^2 + N^{-2(r-1)}\|\partial_x f\|_{(r-1),\omega}^2 + N^{3-2r}\|f\|_{r,\omega}^2 \\
&\lesssim N^{3-2r}\|f\|_{r,\omega}^2
\end{aligned}
$$

Hence,

$$\|f-\pi_N f\|_{1,\omega} \lesssim N^{3/2-r}\|f\|_{r,\omega}$$

Q4. Note that

$$
\begin{aligned}
p^*(-1) &= \int_{-1}^{-1} \pi_N u' - \frac{1}{2}p(1)\ dy = 0 \\
p^*(1) &= \int_{1}^{-1} \pi_N u' - \frac{1}{2}p(1)\ dy \\
&= \int_{-1}^{1} \pi_N u'\ dy = 0
\end{aligned}
$$

Hence, $p^*(-1) = p^*(1) = 0$.
Note that

$$
\begin{aligned}
(u' - p^*)' &= u' - \pi_N u' - \frac{1}{2}p(1) \\
&= u' - \pi_N u' - \frac{1}{2}\int_{-1}^{1} \pi_N u' - u'\ dy
\end{aligned}
$$

As $u(-1) = u(1) = 0$, then $\int_{-1}^{1} u'\ dy = 0$. Thus,

$$
\begin{aligned}
|u - p^*|_{1,\omega} &\lesssim \|u' - \pi_N u'\|_{0,\omega} \\
&\lesssim N^{-r} \|u'\|_{r,\omega}
\end{aligned}
$$

From Q3(a) result, where $\|f - \pi_N f\|_{0,\omega} \lesssim N^{-r} \|f\|_{r,\omega}$, r is even, we can then show that

$$
|u - p^*|_{1,\omega} \lesssim N^{1-r} \|u\|_{r,\omega}
$$

where r is odd.