Homework 2

1. Show that the derivatives of Legendre polynomials also satisfy the three-term recurrence relation:
$$L_{n+1}^{(m)}(x) = \alpha_n^{(m)} x L_n^{(m)}(x) - \beta_n^{(m)} L_{n-1}^{(m)}(x).$$
Determine $\alpha_n^{(m)}$, $\beta_n^{(m)}$ and the initial conditions.

2. Prove the following differential relations for Chebyshev polynomials:
$$T_n'(x) = 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} \frac{1}{1+\delta_{k0}} T_k(x),$$
$$T_n''(x) = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{n-2} \frac{1}{1+\delta_{k0}} n(n^2-k^2) T_k(x).$$

3. Let $\omega(x) = (1-x^2)^{-1/2}$. For any $f \in L_\omega^2(-1,1)$, define its projection $\pi_N f$ by
$$(\pi_N f)(x) = \sum_{n=0}^{N} \hat{f}_n T_n(x),$$
where
$$\hat{f}_n = \frac{2}{(1+\delta_{0n})\pi} \int_{-1}^{1} f(x) T_n(x) \omega(x) \, \mathrm{d}x$$
Prove the estimation for $\|f - \pi_N f\|_{1,\omega}$ by the following steps:

   (a) Use the Sturm-Liouville equation to show that
   $$\|f - \pi_N f\|_{0,\omega} \lesssim N^{-r} \|f\|_{r,\omega}$$
   if r is positive and even.

   (b) Prove the inverse inequality:
   $$\|p\|_{r,\omega} \lesssim N^{2r} \|p\|_{0,\omega}, \qquad \forall p \in P_N,$$
   where $r \in \mathbb{N}$.

   (c) Show that
   $$\|\pi_N(\partial_x f) - \partial_x(\pi_N f)\|_{0,\omega} \lesssim N^{3/2-r} \|f\|_{r,\omega}$$
   if r is positive and odd.

(d) Show that

$$\|f - \pi_N f\|_{1,\omega} \lesssim N^{3/2-r} \|f\|_{r,\omega}$$

if r is positive and odd.

4. Let $\omega(x) = (1-x^2)^{-1/2}$. Assume $u \in H^r_\omega(-1,1)$ with r being a positive odd integer and $u(-1) = u(1) = 0$. Define

$$\begin{aligned} p(x) &= \int_{-1}^{x} (\pi_N u')(y)\, \mathrm{d}y, \\ p^*(x) &= \int_{-1}^{x} \left[(\pi_N u')(y) - \frac{1}{2} p(1) \right] \mathrm{d}y. \end{aligned}$$

Show that $p^*(-1) = p^*(1) = 0$ and

$$|u - p^*|_{1,\omega} \lesssim N^{1-r} \|u\|_{r,\omega}.$$