

STA414

Lecture Notes

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1 Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

If we make natural numbers \mathbb{N} closed under subtraction, we obtain

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$$

If we take the closure of \mathbb{Z} under division by non-zero numbers, we obtain

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1 \right\}$$

Remark 1.1. $(m, n) = 1$ means that if $d \in \mathbb{N}$ divides both m and n , then $d = 1$.

Theorem 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for contradiction that there are $m \in \mathbb{Z}, n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m, n) = 1$.

Then $m^2 = 2n^2$ so that m^2 is an even complete square.

Suppose $m = p_1 \dots p_r$ where p_i s are prime numbers. Then $2n^2 = m^2 = p_1^2 \dots p_r^2 \implies p_i^2 = 2^2$.

Then $4|m^2$ and $2|n^2$, so n has to be even. Therefore both m and n are even.

Then $2|m$ and $2|n$, which leads to a contradiction that if $d \in \mathbb{N}$ divides both m and n , then $d = 1$. ■

1.2 Preliminaries

Definition 1.1 (set). A set is any collection of objects.

Definition 1.2 (function). Given two sets A and B , a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B . In this case, we write $(f : A \rightarrow B)$. It is the set of pairs $(A, B) \in A \times B$ s.t.

1. If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.
2. For all $x \in A$, there is some $y \in B$ s.t. $f(x) = y$.

The set A is said to be the domain of f . The range of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

Example 1.1 (absolute value function). For every x ,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem 1.2 (triangle inequality).

$$|x + y| \leq |x| + |y|$$

Proof.

$$\begin{aligned}
 (x+y)^2 &= x^2 + y^2 + 2xy \\
 &\leq |x|^2 + |y|^2 + 2|x||y| \\
 &= (|x| + |y|)^2 \\
 \implies |x+y| &= \sqrt{(x+y)^2} \\
 &\leq \sqrt{(|x| + |y|)^2} \\
 &= ||x| + |y|| \\
 &= |x| + |y|
 \end{aligned}$$

■

Definition 1.3 (maximum and minimum). Assume set $X \subseteq \mathbb{R}$. Then the maximum (minimum) of X is an element $a \in X$ s.t. for all $x \in X, x \leq a$ ($x \geq a$).

Definition 1.4 (least upper bound / supremum). The least upper bound of X (denoted by $\sup(X)$) is a real number $a \in \mathbb{R}$ s.t.

1. For all $x \in X, x \leq a$ (this means that a is an upper bound for X)
2. If b is an upper bound for X , then $a \leq b$

Example 1.2.

$$\begin{aligned}
 \max([0, 1]) &= 1 \\
 \min([0, 1]) &= 0 \\
 \sup((0, 1)) &= 1 \\
 \sup(\mathbb{R}), \sup(\mathbb{N}) &DNE
 \end{aligned}$$

1.3 The axiom of completeness

Definition 1.5 (initial segment). $X \subseteq \mathbb{Q}$ is said to be an initial segment if

1. $X \neq \emptyset$
2. For all $x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$.
3. $X \neq \mathbb{Q}$

Definition 1.6 (real numbers). $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$

Properties of \mathbb{R} :

1. \mathbb{R} is an **ordered field**
2. ???

Lemma 1.1 (supremum). Suppose $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A . If $\forall \epsilon > 0, \exists a \in A, a + \epsilon > s$, then $s = \sup(A)$

Proof. (\Leftarrow) Assume for contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$.

Let $\epsilon = \frac{s-t}{2}$. Obviously $\epsilon > 0$.

But then $\forall a \in A, a + \epsilon \leq t + \epsilon < s$, which is a contradiction.

(\Rightarrow) Assume for contradiction that $\epsilon_0 > 0$ and $\forall a \in A, a + \epsilon \leq S$

Then $\forall a \in A, a \leq S - \epsilon_0$.

So $s - \epsilon_0$ is an upper bound for A , which is a contradiction that $a + \epsilon > s$. ■

Theorem 1.3 (the axiom of completeness). If $X \subset \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let Ax be the initial segment of \mathbb{Q} corresponding to x .

Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \bigcup_{x \in X} Ax$. Note that $\bigcup_{x \in X} Ax$ is an initial segment of \mathbb{Q} . Then $\sup(\bigcup_{x \in X} Ax)$ is $\sup(X)$. ■

1.4 Consequences of Completeness

Definition 1.7 (nested sequence of sets). Assume $\langle A_n : n \in \mathbb{N} \rangle$ is a sequence of sets.

$\langle A_n : n \in \mathbb{N} \rangle$ is said to be nested if

$$A_{n+1} \subseteq A_n$$

Theorem 1.4 (Nested Interval Property). Assume $\langle I_n : n \in \mathbb{N} \rangle$ is a nested sequence of closed intervals of \mathbb{R} . Then

$$\bigcap_n I_n \neq \emptyset$$

Proof. Let $[a_n, b_n] = I_n$ where $a_n, b_n \in \mathbb{R}$.

Since $\langle I_n | n \in \mathbb{N} \rangle$ is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad (\dagger)$$

for all $n \in \mathbb{N}$

Let $A = \{a_n : n \in \mathbb{N}\}$.

Note that b_1 is an upper bound for A . So A has a supremum in \mathbb{R} .

We claim that $\sup(A) \in \bigcap_n I_n$.

By (\dagger) , for all $n \in \mathbb{N}, \sup(A) \leq b_n$

Obviously, for all $n \in \mathbb{N}, \sup(A) \geq a_n$

So $\forall n \in \mathbb{N}, a_n \leq \sup(A) \leq b_n$.

Therefore $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$. ■

Example 1.3.

$$\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$$

$$\bigcap_{n \in \mathbb{N}} [0, \frac{1}{n}] = \{0\}$$

Theorem 1.5 (Archimedian Property). 1. For every $y \in \mathbb{R}$, there is $n \in \mathbb{N}$ s.t. $y \leq n$.

2. For every $y > 0$, there is $n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$.

Proof. (1) Assume for contradiction that \mathbb{N} is bounded in \mathbb{R} .

Let $\alpha = \sup(\mathbb{N})$. Then there is a natural number $n \in \mathbb{N}$ s.t. $n > \alpha - 1$.

But then $n + 1 > (\alpha - 1) + 1 = \alpha$, which is a natural number greater than α , contradiction.

(2) Exercise. ■

Theorem 1.6 (density of \mathbb{Q} in \mathbb{R}). For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Proof. Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, $1 < nb - na$.

Let $m \in \mathbb{Z}$ s.t. $na < m < nb$.

Then $a < \frac{m}{n} < b$.

Pick $r = \frac{m}{n}$ and we are done. ■

1.5 Cardinality

“The size of a set”

1.5.1 1-1 Correspondence

Definition 1.8 (one-to-one and onto). A function $f : A \rightarrow B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B . The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Proposition 1.1. If $f : A \rightarrow B$ and $g : B \rightarrow C$ is 1-1, then $g \circ f : A \rightarrow C$ is 1-1.

Remark 1.2. If a function $f : A \rightarrow B$ is both 1-1 and onto, then there is a 1-1 correspondence between two sets.

Definition 1.9 (the same cardinality). The set A has the same cardinality as B if there exists $f : A \rightarrow B$ that is 1-1 and onto. In this case, we write $A \sim B$.

Proposition 1.2. If $A \sim B$, $B \sim C$, then $A \sim C$

Proposition 1.3. If $\text{Card}(A) \leq \text{Card}(B) \leq \text{Card}(C)$, then $\text{Card}(A) \leq \text{Card}(C)$

1.5.2 Countable Sets

A set A is countable if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.

Theorem 1.7. The set \mathbb{Q} is countable.

Proof. Set $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \left\{ \pm \frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}$$

e.g. $A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\}$, $A_3 = \left\{ \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1} \right\}$

N :	1	2	3	4	5	6	7	8	9	10	11	12	...
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	
Q :	0	$\frac{1}{1}$	$-\frac{1}{1}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{2}{1}$	$-\frac{2}{1}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{3}{1}$	$-\frac{3}{1}$	$\frac{1}{4}$...
	$\underbrace{\hspace{1.5cm}}_{A_1}$		$\underbrace{\hspace{1.5cm}}_{A_2}$		$\underbrace{\hspace{2.5cm}}_{A_3}$			$\underbrace{\hspace{2.5cm}}_{A_4}$					

The above correspondence is onto because every rational number appears in the correspondence exactly once. The above correspondence is 1-1 because A_N were constructed to be disjoint so that no rational number appears twice. ■

Theorem 1.8. The set \mathbb{R} is uncountable.

Proof. Assume for contradiction that there does exist a bijection function $f : \mathbb{N} \rightarrow \mathbb{R}$. Let $x_1 = f(1), x_2 = f(2)$ and so on. Then since f is onto, can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\} \quad (1)$$

and be confident that every real number appears somewhere on the list.

We will now use the Nested Interval Property to produce a real number that is not there. Let I_1 be a closed interval that does not contain x_1 . given an interval I_n , construct I_{n+1} to satisfy $I_{n+1} \subseteq I_n$ and $x_{n+1} \notin I_{n+1}$.

If x_{n_0} is some real number from the list in (1), then we have $x_{n_0} \notin I_{n_0}$, and it follows that

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$$

Since we are assuming that the list in (1) contains every real number, then

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

However, the NIP asserts that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, which is a contradiction. ■

Theorem 1.9. If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.10. (i) If A_1, A_2, \dots, A_m are countable sets, then the union $A_1 \cup A_2 \cup \dots \cup A_m$ is countable.

(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Theorem 1.11. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

1.6 Cantor's Theorem

Notation 1.1. Given a set A , the power set $P(A)$ refers to the collection of all subsets of A .

Theorem 1.12 (Cantor's Theorem). Given any set A , there does not exist a function $f : A \rightarrow P(A)$ that is onto.

Proof. Assume, for contradiction, that $f : A \rightarrow P(A)$ is onto. For each element $a \in A$, $f(a)$ is a particular subset of A . The assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$. To arrive at a contradiction, we will produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.

Construct B using the following rule. For each element $a \in A$, consider the subset $f(a)$. This subset of A may contain the element a or it may not. This depends on the function f . If $f(a)$ does not contain a , then we include a in our set B : Let

$$B = \{a \in A : a \notin f(a)\}$$

Since we have assumed that our function $f : A \rightarrow P(A)$ is onto, it must be that $B = f(a')$ for some $a' \in A$.

Case 1 $a' \in B$

Then $a' \notin f(a') = B$, a contradiction.

Case 2 $a' \notin B$

Then $a' \in f(a') = B$, a contradiction. ■

Theorem 1.13 (Schröder-Bernstein Theorem). If there are 1-1 functions $f : A \rightarrow B$ and $h : B \rightarrow A$, then there is a bijection $g : A \rightarrow B$.

Proof. **Claim:** the statement of the theorem is equivalent to the following:
If $B \subseteq A$ and $f : A \rightarrow B$ is 1-1, then there is a bijection $g : A \rightarrow B$. (*)

proof of claim: theorem \implies (*):

Take $h : X \rightarrow Y$ with $h(x) = x$, then $X \subseteq Y$.

(*) \implies theorem:

Let $f : A \rightarrow B$ and $h : B \rightarrow A$ be 1-1 functions, as in the theorem. We need to show that there is bijection $g : A \rightarrow B$.

Notice that $A \subseteq h(B)$ and $h \circ f : A \rightarrow h(B)$ is a 1-1 function. So by (*), there is a bijection $g_0 : A \rightarrow h(B)$.

But $h : B \rightarrow h(B)$ is also a bijection. So $g = h^{-1} \circ g_0 : A \rightarrow B$ is a bijection (using the fact that bijections are closed under compositions).

Now it suffices to prove (*).

Assume set $X \subseteq Y$ and $f : Y \rightarrow X$. Let $W = \bigcup_{n=0}^{\infty} f^n(Y \setminus X)$.

Define $g : Y \rightarrow X$ by:

- If $y \in W$, then $g(y) = f(y)$
- If $y \in Z := Y \setminus W$, then $g(y) = y$

We need to show that $g : Y \rightarrow X$ is a well-defined bijection.

Since f is 1-1, for all $m < n$, $f^m(Y \setminus X) \cap f^n(Y \setminus X) = \emptyset$

Note that

$$\begin{aligned}
 Y \setminus W &= Y \setminus \bigcup_{n=0}^{\infty} f^n(Y \setminus X) \\
 &= [Y \setminus (Y \setminus X)] \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \\
 &= X \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X)
 \end{aligned}$$

Therefore for all $y \in Y, g(y) \in X$.

(Show g is 1-1) Now assume $y_1, y_2 \in Y$ and $g(y_1) = g(y_2)$. We show that $y_1 = y_2$.

Case 1 $y_1, y_2 \in W$

Then $g(y_1) = g(y_2) \implies f(y_1) = f(y_2) \implies y_1 = y_2$.

Case 2 $y_1 \in W$ but $y_2 \in Y \setminus W$

Then $g(y_1) = g(y_2) \implies f(y_1) = y_2$

Note that if $y_1 \in W$, then for some $n \geq 0, y_1 \in f^n(Y \setminus X)$

Then $y_2 \in f^{n+1}(Y \setminus X) \subseteq W$

So $y_2 \in W$, which leads to a contradiction.

Case 3 y_1, y_2 are both in $Z := Y \setminus W$

Then $g(y_1) = g(y_2) \implies y_1 = y_2$.

Therefore by case 1,2,3, g is 1-1.

(Show g is onto) Let $x \in X$. We need to find $y \in Y$ s.t. $g(y) = x$.

If $x \in Z$, take $y = x$.

If $x \in \bigcup_{n=1}^{\infty} f^n(Y \setminus X)$, then fix $n \in \mathbb{N}$ s.t. $x \in f^n(Y \setminus X)$.

But $f^n(Y \setminus X) = f(f^{n-1}(Y \setminus X))$

Pick $y \in f^{n-1}(Y \setminus X)$ s.t. $f(y) = x$.

Then $y \in W$ and $g(y) = x$. Therefore g is onto. ■

2 Metric Spaces and the Baire Category Theorem

Definition 2.1 (metric and metric space). Given a set X , a function $d : X \times X \rightarrow \mathbb{R}$ is a metric on X if for all $x, y \in X$:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. for all $z \in X, d(x, y) \leq d(x, z) + d(z, y)$

A metric space is a set X together with a metric d .

Example 2.1. The set \mathbb{R} considered with $d : \mathbb{R}^2 \rightarrow [0, \infty), (x, y) \mapsto |x - y|$ is a metric space.

Example 2.2. In general, \mathbb{R}^n considered with the Euclidean distance is a metric space.

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

Example 2.3. Let x be a set. The discrete metric d on X is defined by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Fact If (X, d) is a metric space, $d'(x, y) = \max\{1, d(x, y)\}$ for all $x, y \in X$, then (X, d') is also a metric space.

Example 2.4. Let $X = \{f : A \rightarrow \mathbb{R}\}$

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in A\}$$

if the supremum exists.

2.1 Basic Definitions

Definition 2.2. Let (X, d) be a metric space. A sequence $(X_n) \subseteq X$ converges to an element $x \in X$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies d(x_n, x) < \epsilon$.

Key property: If $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} x_n = y$, then $x = y$.

Proof. WTS $d(x, y) = 0$

Let $\epsilon > 0$. We will show that $d(x, y) < \epsilon$.

Since $\lim_{n \rightarrow \infty} x_n = x$, then $\exists N_1, \forall n \geq N_1, d(x_n, x) < \frac{\epsilon}{2}$

Since $\lim_{n \rightarrow \infty} x_n = y$, then $\exists N_2, \forall n \geq N_2, d(x_n, y) < \frac{\epsilon}{2}$

Take $n \geq \max(N_1, N_2)$, then $d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. ■

Proposition 2.1. Suppose (X, d) is a metric space, (X, τ) is a topological space, and $F \subseteq X$. If $\lim_{n \rightarrow \infty} x_n = x, (x_n) \subseteq F$ and F is closed, then $x \in F$.

Proof. Suppose $x \notin F$, i.e., $x \in X \setminus F$.

Since F is closed, then $X \setminus F$ is open, so there is $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq X \setminus F$.

Let N be such that $\forall n \geq N, d(x_n, x) < \epsilon$.

Then $x_n \in B_\epsilon(x)$, which implies that $(x_n) \subseteq X \setminus F$, a contradiction. ■

Proposition 2.2. Suppose (X, d) is a metric space and $F \subseteq X$. If F is not closed, then there exists $(x_n) \subseteq F$ and $x \notin F$ s.t. $\lim_{n \rightarrow \infty} x_n = x$.

Proof. If F is not closed, then $X \setminus F$ is not open, so there is $x \in X \setminus F$ s.t. $B_\epsilon(x) \not\subseteq X \setminus F$ for all $\epsilon > 0$.

Take $x_n \in B_{1/n}(x) \setminus (X \setminus F) = B_{1/n}(x) \cap F$ for each $n \in \mathbb{N}$, then $(x_n) \subseteq F$ and $\lim_{n \rightarrow \infty} x_n = x$. ■

Definition 2.3 (Cauchy sequence). A sequence (x_n) in a metric space (X, d) is a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}, m, n \geq N \implies d(x_m, x_n) < \epsilon$.

Proposition 2.3. A convergent sequence is Cauchy.

Proof. Let (x_n) be a convergent sequence, so that $\lim_{n \rightarrow \infty} x_n = x$. To check (x_n) is Cauchy, let $\epsilon > 0$. We need to find N s.t. $\forall m, n \geq N, d(x_n, x_m) < \epsilon$.

Apply $\lim_{n \rightarrow \infty} x_n = x$ to $\frac{\epsilon}{2}$, we get N s.t. $\forall n \geq N, d(x, x_n) < \frac{\epsilon}{2}$.

Notice that N works for Cauchy:

Take $m, n \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Remark 2.1. When $X = \mathbb{R}$ with the usual metric, A Cauchy sequence is convergent (the converse is true).

In general not true. For example, $X = \mathbb{R} \setminus \{0\}, d(x, y) = |x - y|, (x_n) = \frac{1}{n}$. ■

Definition 2.4 (completeness of metric spaces). A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X .

Example 2.5. $\mathbb{R}, d(x, y) = |x - y|$

Example 2.6. $(X, d), d$ discrete metric.

Example 2.7. $C[0, 1], d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| = \|f - g\|_\infty$

Example 2.8. $(\mathbb{N}^{\mathbb{N}}, d), d((x_n), (y_n)) = \frac{1}{\min\{n: x_n \neq y_n\}}$
where $\mathbb{N}^{\mathbb{N}} = \{x : \mathbb{N} \rightarrow \mathbb{N}\}$.

Definition 2.5. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \rightarrow Y$ is continuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0, d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$.

2.2 Topology on Metric Spaces

Definition 2.6 (ϵ -neighbourhood). Given $\epsilon > 0$ and an element x in the metric space (X, d) , the ϵ -neighbourhood of x is the set $V_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$

Definition 2.7 (compactness). A subset K of a metric space (X, d) is compact if every sequence in K has a convergent subsequence that converges to a limit in K .

Definition 2.8 (closure and interior). Given a subset E of a metric space (X, d) , the closure \bar{E} is the union of E together with its limit points. The interior of E is denoted by E° and is defined as

$$E^\circ = \{x \in E : \exists V_\epsilon(x) \subseteq E\}$$

Remark 2.2. $(X, \tau), \tau \subseteq P(X), E \subseteq X$

1. \bar{E} = minimal closed superset of $E = \bigcap \{H : H \text{ closed}, H \supseteq E\}$
2. E° = maximal open subset of $E = \bigcup \{U : U \text{ open}, U \subseteq E\}$

Example 2.9. (X, d) is a metric space, τ_d is the topology determined by d : $U \in \tau_d$ iff $\forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$
 $F \subseteq X$

$$\bar{F} = \{x \in X : \forall \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} \quad (2)$$

$$= \bigcap \{H : H \text{ closed } H \supseteq F\} \quad (3)$$

$$= \left\{ \lim_{n \rightarrow \infty} x_n : (x_n) \subseteq F, \lim_{n \rightarrow \infty} x_n \text{ exists} \right\} \quad (4)$$

$$F^\circ = \{x \in X : \exists \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} \quad (5)$$

$$= \bigcup \{B_\epsilon(x) : \epsilon > 0, x \in F, B_\epsilon(x) \subseteq F\} \quad (6)$$

$$(7)$$

Definition 2.9 (density). A set $A \subseteq X$ is dense in the metric space (X, d) if $\bar{A} = X$. A subset E of a metric space (X, d) is nowhere-dense in X if \bar{E}° is empty.

2.3 Baire's Theorem

Definition 2.10 (nowhere-dense). A set E is nowhere-dense if \bar{E} contains no nonempty open intervals.

Theorem 2.1 (Baire's Theorem). The set of real numbers \mathbb{R} cannot be written as the countable union of nowhere-dense sets.

2.4 The Baire Category Theorem

Theorem 2.2. Let (X, d) be a complete metric space, and let $\{O_n\}$ be a countable collection of dense, open subsets of X . Then, $\bigcap_{n=1}^{\infty} O_n$ is not empty.

Proof. ■

Theorem 2.3 (Baire Category Theorem). A complete metric space is not the union of a countable collection of nowhere-dense sets.

Proof. ■

Theorem 2.4. The set

$$D = \{f \in C[0, 1] : f'(x) \text{ exists for some } x \in [0, 1]\}$$

is a set of first category in $C[0, 1]$.

2.5 Topology of (X, d)

Definition 2.11 (open ball). An open ball with radius r and center x is

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

Definition 2.12 (open set). A set $U \subseteq X$ is open iff

$$\forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Example 2.10. $B_\epsilon(x)$ is open.

Proof. Fix $x \in X$ and $\epsilon > 0$. We want to show: $\forall y \in B_\epsilon(x), \exists \delta > 0 \text{ s.t. } B_\delta(y) \subseteq B_\epsilon(x)$. Take $y \in B_\epsilon(x)$, then $d(x, y) < \epsilon$. Take $\delta = \epsilon - d(x, y) > 0$. Take any $z \in B_\delta(y)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \epsilon - d(x, y) = \epsilon$$

Thus $z \in B_\epsilon(x)$ so $B_\delta(y) \subseteq B_\epsilon(x)$. ■

Definition 2.13 (topological space). A topological space is a pair (X, τ) , where X is a set and τ a subset of the power set of X which we call open such that

1. $\emptyset, X \in \tau$
2. $U_1, \dots, U_n \in \tau \implies \bigcap_{i=1}^n U_i \in \tau$
3. $U_1, \dots, U_n \in \tau \implies \bigcup_{i=1}^n U_i \in \tau$

Example 2.11. $(X, \{\emptyset, X\})$

Example 2.12. $(X, P(X))$ is a discrete topological space, where $P(X)$ is the power set of X .

Example 2.13. Given (X, d) a metric space, define τ_d : a set $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Then τ_d is a topology.

Proof. (1) First, $\emptyset, X \in \tau_d$ since $\forall x \in \emptyset, B_1(x) \subseteq \emptyset$ and $\forall x \in X, B_1(x) \subseteq X$.

Then suppose $U_1, \dots, U_n \in \tau_d$.

(2) we want to show:

$$U = \bigcap_{i=1}^n U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Since $x \in U$, then $\forall i = 1, \dots, n, x \in U_i : \exists \epsilon_i > 0 \text{ s.t. } B_{\epsilon_i}(x) \subseteq U_i$.

Take $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$, thus $B_\epsilon(x) \subseteq U_i \forall i$. Hence $B_\epsilon(x) \subseteq U_i \subseteq U$.

(3) We also want to show:

$$\bigcup_{i=1}^n U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Let $x \in U$, then there is some U_i s.t. $x \in U_i$. Since $U_i \in \tau_d$, then $\exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U_i \subseteq U$. Therefore, τ_d is a topology. ■

Definition 2.14. A **subset** F of a topological space (X, τ) is closed if $X \setminus F$ is open.

Property 2.1. Given a topological space (X, τ) and a subset F of it, we have:

1. \emptyset, X are closed
2. If F_1, \dots, F_n are closed, then $\bigcup_{i=1}^n F_i$ is closed
3. If F_1, \dots, F_n are closed, then $\bigcap_{i=1}^n F_i$ is closed

Definition 2.15 (topological closure and interior). Given a topological space (X, τ) , where $\tau \subseteq P(X)$, and a set $F \subseteq X$, the topological closure of F is the minimal closed superset of F , i.e.,

$$\bar{F} = \bigcap \{H : H \text{ is closed, } H \supseteq F\}$$

The interior of F is the maximal open subset of F , i.e.,

$$F^\circ = \bigcap \{U : U \text{ is open, } U \subseteq F\}$$

Example 2.14. Given (X, d) a metric space, define $\tau_d : \text{a set } U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Suppose $F \subseteq X$, then

$$\bar{F} = \{x \in X : \forall \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} = \{\lim_{n \rightarrow \infty} x_n : (x_n) \subseteq F, \lim_{n \rightarrow \infty} x_n \text{ exists}\}$$

3 Sequences and Series

3.1 The Limit of a Sequence

Definition 3.1 (sequence). A sequence is a function whose domain is \mathbb{N}