STA414
Lecture Notes

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Contents

1 Chapter 1: Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

If we make natural numbers $\mathbb{N}$ closed under subtraction, we obtain

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$$

If we take the closure of $\mathbb{Z}$ under division by non-zero numbers, we obtain

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1\}$$

Remark 1.1. $(m, n) = 1$ means that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$.

Theorem 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for contradiction that there are $m \in \mathbb{Z}.n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m, n) = 1$.
Then $m^2 = 2n^2$ so that m^2 is an even complete square.
Suppose $m = p_1 \dots p_r$ where p_is are prime numbers. Then $2n^2 = m^2 = p_1^2 \dots p_r^2 \implies p_i^2 = 2^2$.
Then $4|m^2$ and $2|n^2$, so n has to be even. Therefore both m and n are even.
Then $2|m$ and $2|n$, which leads to a contradiction that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$. ∎

1.2 Preliminaries

Definition 1.1 (set)**.** A set is any collection of objects.

Definition 1.2 (function)**.** Given two sets A and B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $(f : A \to B)$. It is the set of pairs $(A, B) \in A \times B$ s.t.

1. If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.
2. For all $x \in A$, there is some $y \in B$ s.t. $f(x) = y$.

The set A is said to be the domain of f. The range of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

Example 1.1 (absolute value function)**.** For every x,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem 1.2 (triangle inequality)**.**

$$|x + y| \leq |x| + |y|$$

Proof.

$$\begin{aligned} (x+y)^2 &= x^2 + y^2 + 2xy \\ &\leq |x|^2 + |y|^2 + 2|x||y| \\ &= (|x| + |y|)^2 \\ \implies |x + y| &= \sqrt{(x+y)^2} \\ &\leq \sqrt{(|x| + |y|)^2} \\ &= ||x| + |y|| \\ &= |x| + |y| \end{aligned}$$

∎

Definition 1.3 (maximum and minimum)**.** Assume set $X \subseteq \mathbb{R}$. Then the maximum (minimum) of X is an element $a \in X$ s.t. for all $x \in X, x \leq a (x \geq a)$.

Definition 1.4 (least upper bound / supremum)**.** The least upper bound of X (denoted by $\sup(X)$) is a real number $a \in \mathbb{R}$ s.t.

1. For all $x \in X, x \leq a$ (this means that a is an upper bound for X)
2. If b is an upper bound for X, then $a \leq b$

Example 1.2.

$$\max([0,1]) = 1$$
$$\min([0,1]) = 0$$
$$\sup((0,1)) = 1$$
$$\sup(\mathbb{R}), \sup(\mathbb{N})\, DNE$$

1.3 The axiom of completeness

Definition 1.5 (initial segment)**.** $X \subseteq \mathbb{Q}$ is said to be an initial segment if

1. $X \neq \emptyset$
2. For all $x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$.
3. $X \neq \mathbb{Q}$

Definition 1.6 (real numbers)**.** $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$
Properties of $\mathbb{R}$:

1. $\mathbb{R}$ is an ordered field
2. ???

Lemma 1.1 (supremum)**.** Suppose $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A. If $\forall \epsilon > 0, \exists a \in A, a + \epsilon > s$, then $s = \sup(A)$

Proof. ($\Longleftarrow$) Assume for contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$.
Let $\epsilon = \frac{s-t}{2}$. Obviously $\epsilon > 0$.
But then $\forall a \in A, a + \epsilon \leq t + \epsilon < s$, which is a contradiction.
($\Longrightarrow$) Assume for contradiction that $\epsilon_0 > 0$ and $\forall a \in A, a + \epsilon \leq S$)
Then $\forall a \in A, a \leq S - \epsilon_0$.
So $s - \epsilon_0$ is an upper bound for A, which is a contradiction that $a + \epsilon > s$. ■

Theorem 1.3 (the axiom of completeness)**.** If $X \subset \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let Ax be the initial segment of $\mathbb{Q}$ corresponding to x.
Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \bigcup_{x \in X} Ax$. Note that $\bigcup_{x \in X} Ax$ is an initial segment of $\mathbb{Q}$. Then $\sup(\bigcup_{x \in X} Ax)$ is $\sup(X)$. ■

1.4 Consequences of Completeness

Definition 1.7 (nested sequence of sets)**.** Assume $\langle A_n : n \in \mathbb{N}\rangle$ is a sequence of sets.
$\langle A_n : n \in \mathbb{N}\rangle$ is said to be nested if

$$A_{n+1} \subseteq A_n$$

Theorem 1.4 (Nested Interval Property)**.** Assume $\langle I_n : n \in \mathbb{N}\rangle$ is a nested sequence of closed intervals of $\mathbb{R}$. Then

$$\bigcap_n I_n \neq \emptyset$$

Proof. Let $[a_n, b_n] = I_n$ where $a_n, b_n \in \mathbb{R}$.
Since $\langle I_n | n \in \mathbb{N}\rangle$ is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad (\dagger)$$

for all $n \in \mathbb{N}$
Let $A = \{a_n : n \in \mathbb{N}\}$.
Note that b_1 is an upper bound for A. So A has a supremum in $\mathbb{R}$.
We claim that $\sup(A) \in \bigcap_n I_n$.
By $(\dagger)$, for all $n \in \mathbb{N}, \sup(A) \leq b_n$
Obviously, for all $n \in \mathbb{N}, \sup(A) \geq a_n$
So $\forall n \in \mathbb{N}, a_n \leq \sup(A) \leq b_n$.
Therefore $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$. ∎

Example 1.3.

$$\bigcap_{n\in\mathbb{N}} (0, \frac{1}{n}) = \emptyset$$

$$\bigcap_{n\in\mathbb{N}} [0, \frac{1}{n}] = \{0\}$$

Theorem 1.5 (Archimedian Property)**.** 1. For every $y \in \mathbb{R}$, there is $n \in \mathbb{N}$ s.t. $y \leq n$.

2. For every $y > 0$, there is $n \in \mathbb{N} s.t. \frac{1}{n} < y$.

Proof. (1) Assume for contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$.
Let $\alpha = \sup(\mathbb{N})$. Then there is a natural number $n \in \mathbb{N}$ s.t. $n > \alpha - 1$.
But then $n + 1 > (\alpha - 1) + 1 = \alpha$, which is a natural number greater than α, contradiction.
(2) Exercise. ∎

Theorem 1.6 (density of $\mathbb{Q}$ in $\mathbb{R}$)**.** For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Proof. Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a, 1 < nb - na$.
Let $m \in \mathbb{Z}$ s.t. $na < m < nb$.
Then $a < \frac{m}{n} < b$.
Pick $r = \frac{m}{n}$ and we are done. ∎

1.5 Cardinality

"The size of a set"

1.5.1 1-1 Correspondence

Definition 1.8 (one-to-one and onto)**.** A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Remark 1.2. If a function $f : A \to B$ is both 1-1 and onto, then there is a 1-1 correspondence between two sets.

Definition 1.9 (the same cardinality)**.** The set A has the same cardinality as B if there exists $f : A \to B$ that is $1 - 1$ and onto. In this case, we write $A \sim B$.

1.5.2 Countable Sets

A set A is countable if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.

Theorem 1.7. The set $\mathcal{Q}$ is countable.

Proof. Set $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with} p + q = n\}$$

e.g. $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}, A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$

$$\begin{array}{lccccccccccccc}
\mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbf{Q}: & 0 & \frac{1}{1} & -\frac{1}{1} & \frac{1}{2} & -\frac{1}{2} & \frac{2}{1} & -\frac{2}{1} & \frac{1}{3} & -\frac{1}{3} & \frac{3}{1} & -\frac{3}{1} & \frac{1}{4} & \cdots \\
 & \underbrace{\quad}_{A_1} & \multicolumn{2}{c}{\underbrace{\qquad\quad}_{A_2}} & \multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{A_3}} & \multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{A_4}} & &
\end{array}$$

The above correspondence is onto because every rational number appears in the correspondence exactly once. The above correspondence is 1-1 because A_N were constructed to be disjoint so that no rational number appears twice. ■

Theorem 1.8. The set $\mathbb{R}$ is uncountable.

Proof. Assume for contradiction that there does exist a bijection function $f : \mathbb{N} \to \mathbb{R}$.
Let $x_1 = f(1), x_2 = f(2)$ and so on. Then since f is onto, can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\} \tag{1}$$

and be confident that every real number appears somewhere on the list.
We will now use the Nested Interval Property to produce a real number that is not there. Let I_1 be a closed interval that does not contain x_1. given an interval I_n, construct I_{n+1} to satisfy $I_{n+1} \subseteq I_n$ and $x_{n+1} \notin I_{n+1}$. If x_{n_0} is some real number from the list in (1), then we have $x_{n_0} \notin I_{n_0}$, and it follows that

$$x_{n_0} \notin \cap_{n=1}^{\infty} I_n$$

Since we are assuming that the list in (1) contains every real number, then

$$\cap_{n=1}^{\infty} I_n = \emptyset$$

However, the NIP asserts that $\cap_{n=1}^{\infty} I_n \neq \emptyset$, which is a contradiction. ■

Theorem 1.9. If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.10. (i) If $A_1, A_2, \ldots, A_m$ are countable sets, then the union $A_1 \cup A_2 \cup \ldots \cup A_m$ is countable.
(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty}$ is countable.

Theorem 1.11 (Schröder)**.**