MAT337
Lecture Notes

Yuchen Wang

February 22, 2020

Contents

1 Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

If we make natural numbers $\mathbb{N}$ closed under subtraction, we obtain

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$$

If we take the closure of $\mathbb{Z}$ under division by non-zero numbers, we obtain

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m,n) = 1\}$$

Remark 1.1. $(m,n) = 1$ means that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$.

Theorem 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for contradiction that there are $m \in \mathbb{Z}.n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m,n) = 1$.
Then $m^2 = 2n^2$ so that m^2 is an even complete square.
Suppose $m = p_1 \ldots p_r$ where p_is are prime numbers. Then $2n^2 = m^2 = p_1^2 \ldots p_r^2 \implies p_i^2 = 2^2$.
Then $4|m^2$ and $2|n^2$, so n has to be even. Therefore both m and n are even.
Then $2|m$ and $2|n$, which leads to a contradiction that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$. ■

1.2 Preliminaries

Definition 1.1 (set)**.** A set is any collection of objects.

Definition 1.2 (function)**.** Given two sets A and B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $(f : A \to B)$. It is the set of pairs $(A, B) \in A \times B$ s.t.

1. If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.
2. For all $x \in A$, there is some $y \in B$ s.t. $f(x) = y$.

The set A is said to be the domain of f. The range of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

Example 1.1 (absolute value function)**.** For every x,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem 1.2 (triangle inequality)**.**

$$|x + y| \leq |x| + |y|$$

Proof.

$$\begin{aligned}
(x+y)^2 &= x^2 + y^2 + 2xy \\
&\leq |x|^2 + |y|^2 + 2|x||y| \\
&= (|x| + |y|)^2 \\
\implies |x+y| &= \sqrt{(x+y)^2} \\
&\leq \sqrt{(|x|+|y|)^2} \\
&= ||x| + |y|| \\
&= |x| + |y|
\end{aligned}$$

■

Definition 1.3 (maximum and minimum)**.** Assume set $X \subseteq \mathbb{R}$. Then the maximum (minimum) of X is an element $a \in X$ s.t. for all $x \in X, x \leq a (x \geq a)$.

Definition 1.4 (least upper bound / supremum)**.** The least upper bound of X (denoted by $\sup(X)$) is a real number $a \in \mathbb{R}$ s.t.

1. For all $x \in X, x \leq a$ (this means that a is an upper bound for X)
2. If b is an upper bound for X, then $a \leq b$

Example 1.2.

$$\begin{aligned}
\max([0,1]) &= 1 \\
\min([0,1]) &= 0 \\
\sup((0,1)) &= 1 \\
\sup(\mathbb{R}), \sup(\mathbb{N})\ &DNE
\end{aligned}$$

1.3 The axiom of completeness

Definition 1.5 (initial segment)**.** $X \subseteq \mathbb{Q}$ is said to be an initial segment if

1. $X \neq \emptyset$
2. For all $x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$.
3. $X \neq \mathbb{Q}$

Definition 1.6 (real numbers)**.** $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$

Lemma 1.1 (supremum)**.** Suppose $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A. If $\forall \epsilon > 0, \exists a \in A, a + \epsilon > s$, then $s = \sup(A)$

Proof. ($\Longleftarrow$) Assume for contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$.
Let $\epsilon = \frac{s-t}{2}$. Obviously $\epsilon > 0$.
But then $\forall a \in A, a + \epsilon \leq t + \epsilon < s$, which is a contradiction.
($\Longrightarrow$) Assume for contradiction that $\epsilon_0 > 0$ and $\forall a \in A, a + \epsilon \leq S$)
Then $\forall a \in A, a \leq S - \epsilon_0$.
So $s - \epsilon_0$ is an upper bound for A, which is a contradiction that $a + \epsilon > s$. ■

Theorem 1.3 (the axiom of completeness)**.** If $X \subset \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let Ax be the initial segment of $\mathbb{Q}$ corresponding to x.
Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \bigcup_{x\in X} Ax$. Note that $\bigcup_{x\in X} Ax$ is an initial segment of $\mathbb{Q}$. Then $\sup(\bigcup_{x\in X} Ax)$ is $\sup(X)$. ■

1.4 Consequences of Completeness

Definition 1.7 (nested sequence of sets)**.** Assume $\langle A_n : n \in \mathbb{N}\rangle$ is a sequence of sets.
$\langle A_n : n \in \mathbb{N}\rangle$ is said to be nested if

$$A_{n+1} \subseteq A_n$$

Theorem 1.4 (Nested Interval Property)**.** Assume $\langle I_n : n \in \mathbb{N}\rangle$ is a nested sequence of closed intervals of $\mathbb{R}$. Then

$$\bigcap_n I_n \neq \emptyset$$

Proof. Let $[a_n, b_n] = I_n$ where $a_n, b_n \in \mathbb{R}$.
Since $\langle I_n | n \in \mathbb{N}\rangle$ is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad (\dagger)$$

for all $n \in \mathbb{N}$
Let $A = \{a_n : n \in \mathbb{N}\}$.
Note that b_1 is an upper bound for A. So A has a supremum in $\mathbb{R}$.
We claim that $\sup(A) \in \bigcap_n I_n$.
By $(\dagger)$, for all $n \in \mathbb{N}, \sup(A) \leq b_n$
Obviously, for all $n \in \mathbb{N}, \sup(A) \geq a_n$
So $\forall n \in \mathbb{N}, a_n \leq \sup(A) \leq b_n$.
Therefore $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$. ■

Example 1.3.

$$\bigcap_{n\in\mathbb{N}} (0, \frac{1}{n}) = \emptyset$$

$$\bigcap_{n\in\mathbb{N}} [0, \frac{1}{n}] = \{0\}$$

Theorem 1.5 (Archimedian Property)**.**
1. For every $y \in \mathbb{R}$, there is $n \in \mathbb{N}$ s.t. $y \leq n$.
2. For every $y > 0$, there is $n \in \mathbb{N} s.t. \frac{1}{n} < y$.

Proof. (1) Assume for contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$.
Let $\alpha = \sup(\mathbb{N})$. Then there is a natural number $n \in \mathbb{N}$ s.t. $n > \alpha - 1$.
But then $n + 1 > (\alpha - 1) + 1 = \alpha$, which is a natural number greater than α, contradiction.
(2) Exercise. ■

Theorem 1.6 (density of $\mathbb{Q}$ in $\mathbb{R}$)**.** For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Proof. Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a, 1 < nb - na$.
Let $m \in \mathbb{Z}$ s.t. $na < m < nb$.
Then $a < \frac{m}{n} < b$.
Pick $r = \frac{m}{n}$ and we are done. ■

1.5 Cardinality

"The size of a set"

1.5.1 1-1 Correspondence

Definition 1.8 (one-to-one and onto)**.** A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Proposition 1.1. If $f : A \to B$ and $g : B \to C$ is 1-1, then $g \circ f : A \to C$ is 1-1.

Remark 1.2. If a function $f : A \to B$ is both 1-1 and onto, then there is a 1-1 correspondence between two sets.

Definition 1.9 (the same cardinality)**.** The set A has the same cardinality as B if there exists $f : A \to B$ that is $1 - 1$ and onto. In this case, we write $A \sim B$.

Proposition 1.2. If $A \sim B, B \sim C$, then $A \sim C$

Proposition 1.3. If $Card(A) \leq Card(B) \leq Card(C)$, then $Card(A) \leq Card(C)$

1.5.2 Countable Sets

A set A is countable if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.

Theorem 1.7. The set $\mathcal{Q}$ is countable.

Proof. Set $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with} p + q = n\}$$

e.g. $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}, A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$

$$\begin{array}{lcccccccccccccc}
\mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbf{Q}: & 0 & \frac{1}{1} & -\frac{1}{1} & \frac{1}{2} & -\frac{1}{2} & \frac{2}{1} & -\frac{2}{1} & \frac{1}{3} & -\frac{1}{3} & \frac{3}{1} & -\frac{3}{1} & \frac{1}{4} & \cdots
\end{array}$$

$$\underbrace{0}_{A_1}\ \underbrace{\tfrac{1}{1}\ \ -\tfrac{1}{1}}_{A_2}\ \underbrace{\tfrac{1}{2}\ \ -\tfrac{1}{2}\ \ \tfrac{2}{1}\ \ -\tfrac{2}{1}}_{A_3}\ \underbrace{\tfrac{1}{3}\ \ -\tfrac{1}{3}\ \ \tfrac{3}{1}\ \ -\tfrac{3}{1}}_{A_4}$$

The above correspondence is onto because every rational number appears in the correspondence exactly once. The above correspondence is 1-1 because A_N were constructed to be disjoint so that no rational number appears twice. ■

Theorem 1.8. The set $\mathbb{R}$ is uncountable.

Proof. Assume for contradiction that there does exist a bijection function $f : \mathbb{N} \to \mathbb{R}$.
Let $x_1 = f(1), x_2 = f(2)$ and so on. Then since f is onto, can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\} \tag{1}$$

and be confident that every real number appears somewhere on the list.
We will now use the Nested Interval Property to produce a real number that is not there. Let I_1 be a closed interval that does not contain x_1. given an interval I_n, construct I_{n+1} to satisfy $I_{n+1} \subseteq I_n$ and $x_{n+1} \notin I_{n+1}$.
If x_{n_0} is some real number from the list in (1), then we have $x_{n_0} \notin I_{n_0}$, and it follows that

$$x_{n_0} \notin \cap_{n=1}^{\infty} I_n$$

Since we are assuming that the list in (1) contains every real number, then

$$\cap_{n=1}^{\infty} I_n = \emptyset$$

However, the NIP asserts that $\cap_{n=1}^{\infty} I_n \neq \emptyset$, which is a contradiction. ■

Theorem 1.9. If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.10. (i) If $A_1, A_2, \ldots, A_m$ are countable sets, then the union $A_1 \cup A_2 \cup \ldots \cup A_m$ is countable.
(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is countable.

Theorem 1.11. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

1.6 Cantor's Theorem

Notation 1.1. Given a set A, the power set $P(A)$ refers to the collection of all subsets of A.

Theorem 1.12 (Cantor's Theorem)**.** Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

Proof. Assume, for contradiction, that $f : A \to P(A)$ is onto. For each element $a \in A$, $f(a)$ is a particular subset of A. The assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$. To arrive at a contradiction, we will produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.
Construct B using the following rule. For each element $a \in A$, consider the subset $f(a)$. This subset of A may contain the element a or it may not. This depends on the function f. If $f(a)$ does not contain a, then we include a in our set B: Let

$$B = \{a \in A : a \notin f(a)\}$$

Since we have assumed that our function $f : A \to P(A)$ is onto, it must be that $B = f(a')$ for some $a' \in A$.
Case 1 $a' \in B$
Then $a' \notin f(a') = B$, a contradiction.
Case 2 $a' \notin B$
Then $a' \in f(a') = B$, a contradiction. ■

Theorem 1.13 (Schröder-Bernstein Theorem)**.** If there are 1-1 functions $f : A \to B$ and $h : B \to A$, then there is a bijection $g : A \to B$.

Proof. **Claim:** the statement of the theorem is equivalent to the following:
If $B \subseteq A$ and $f : A \to B$ is $1-1$, then there is a bijection $g : A \to B$. (*)

proof of claim: theorem $\implies$ (*):
Take $h : X \to Y$ with $h(x) = x$, then $X \subseteq Y$.
(*) $\implies$ theorem:
Let $f : A \to B$ and $h : B \to A$ be 1-1 functions, as in the theorem. We need to show that there is bijection $g : A \to B$.
Notice that $A \subseteq h(B)$ and $h \circ f : A \to h(B)$ is a 1-1 function. So by (*), there is a bijection $g_0 : A \to h(B)$.
But $h : B \to h(B)$ is also a bijection. So $g = h^{-1} \circ g_0 : A \to B$ is a bijection (using the fact that bijections are closed under compositions).

Now it suffices to prove (*).
Assume set $X \subseteq Y$ and $f : Y \to X$. Let $W = \bigcup_{n=0}^{\infty} f^n(Y \setminus X)$.
Define $g : Y \to X$ by:

- If $y \in W$, then $g(y) = f(y)$
- If $y \in Z := Y \setminus W$, then $g(y) = y$

We need to show that $g : Y \to X$ is a well-defined bijection.
Since f is 1-1, for all $m < n$, $f^m(Y \setminus X) \cap f^n(Y \setminus X) = \emptyset$
Note that

$$\begin{aligned} Y \setminus W &= Y \setminus \bigcup_{n=0}^{\infty} f^n(Y \setminus X) \\ &= [Y \setminus (Y \setminus X)] \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \\ &= X \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \end{aligned}$$

Therefore for all $y \in Y, g(y) \in X$.
(Show g is 1-1) Now assume $y_1, y_2 \in Y$ and $g(y_1) = g(y_2)$. We show that $y_1 = y_2$.
Case 1 $y_1, y_2 \in W$
Then $g(y_1) = g(y_2) \implies f(y_1) = f(y_2) \implies y_1 = y_2$.
Case 2 $y_1 \in W$ but $y_2 \in Y \setminus W$
Then $g(y_1) = g(y_2) \implies f(y_1) = y_2$
Note that if $y_1 \in W$, then for some $n \geq 0, y_1 \in f^n(Y \setminus X)$
Then $y_2 \in f^{n+1}(Y \setminus X) \subseteq W$
So $y_2 \in W$, which leads to a contradiction.
Case 3 y_1, y_2 are both in $Z := Y \setminus W$
Then $g(y_1) = g(y_2) \implies y_1 = y_2$.

Therefore by case 1,2,3, g is 1-1.
(Show g is onto) Let $x \in X$. We need to find $y \in Y$ s.t. $g(y) = X$.
If $x \in Z$, take $y = x$.
If $x \in \bigcup_{n=1}^{\infty} f^n(Y \setminus X)$, then fix $n \in \mathbb{N}$ s.t. $x \in f^n(Y \setminus X)$.
But $f^n(Y \setminus X) = f(f^{n-1}(Y \setminus X))$
Pick $y \in f^{n-1}(Y \setminus X)$ s.t. $f(y) = x$.
Then $y \in W$ and $g(y) = x$. Therefore g is onto. ■

2 Metric Spaces and the Baire Category Theorem

2.1 Basic Definitions

Definition 2.1 (metric and metric space)**.** Given a set X, a function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y \in X$:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. for all $z \in X, d(x, y) \leq d(x, z) + d(z, y)$

A metric space is a set X together with a metric d.

Example 2.1. The set $\mathbb{R}$ considered with $d : \mathbb{R}^2 \to [0, \infty), (x, y) \mapsto |x - y|$ is a metric space.

Example 2.2. In general, $\mathbb{R}^n$ considered with the Euclidean distance is a metric space.

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Example 2.3. Let x be a set. The discrete metric d on X is defined by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Fact If (X, d) is a metric space, $d'(x, y) = \max\{1, d(x, y)\}$ for all $x, y \in X$, then (X, d') is also a metric space.

Example 2.4. Let $X = \{f : A \to \mathbb{R}\}$

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in A\}$$

if the supremum exists.

Definition 2.2. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is continuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0, d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$.

2.2 Topology on Metric Spaces

Definition 2.3 (open ball)**.** An open ball (or ϵ-neighbourhood) with radius r and center x is

$$B_r(x) = \{y \in X : d(x,y) < r\}$$

Definition 2.4 (open set)**.** A set $U \subseteq X$ is open iff

$$\forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Example 2.5. $B_\epsilon(x)$ is open.

Proof. Fix $x \in X$ and $\epsilon > 0$. We want to show: $\forall y \in B_\epsilon(x), \exists \delta > 0$ s.t. $B_\delta(y) \subseteq B_\epsilon(x)$.
Take $y \in B_\epsilon(x)$, then $d(x,y) < \epsilon$. Take $\delta = \epsilon - d(x,y) > 0$. Take any $z \in B_\delta(y)$, we have

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \epsilon - d(x,y) = \epsilon$$

Thus $z \in B_\epsilon(x)$ so $B_\delta(y) \subseteq B_\epsilon(x)$. ■

Definition 2.5 (topological space)**.** A topological space is a pair (X, τ), where X is a set and τ a subset of the power set of X which we call open such that

1. $\emptyset, X \in \tau$
2. $U_1, \ldots, U_n \in \tau \implies \bigcap_{i=1}^n U_i \in \tau$
3. $U_1, \ldots, U_n \in \tau \implies \bigcup_{i=1}^n U_i \in \tau$

Example 2.6. $(X, \{\emptyset, X\})$

Example 2.7. $(X, P(X))$ is a discrete topological space, where $P(X)$ is the power set of X.

Example 2.8. Given (X, d) a metric space, define τ_d : a set $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Then τ_d is a topology.

Proof. (1) First, $\emptyset, X \in \tau_d$ since $\forall x \in \emptyset, B_1(x) \subseteq \emptyset$ and $\forall x \in X, B_1(x) \subseteq X$.
Then suppose $U_1, \ldots, U_n \in \tau_d$.
(2) we want to show:

$$U = \bigcap_{i=1}^n U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Since $x \in U$, then $\forall i = 1, \ldots, n, x \in U_i : \exists \epsilon_i > 0$ s.t. $B_{\epsilon_i}(x) \subseteq U_i$.
Take $\epsilon = \min\limits_{1 \leq i \leq n} \epsilon_i$, thus $B_\epsilon(x) \subseteq U_i \, \forall i$. Hence $B_\epsilon(x) \subseteq U_i \subseteq U$.
(3) We also want to show:

$$\bigcup_{i=1}^n U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Let $x \in U$, then there is some U_i s.t. $x \in U_i$. Since $U_i \in \tau_d$, then $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U_i \subseteq U$.
Therefore, τ_d is a topology. ■

Definition 2.6. A subset F of a topological space (X, τ) is closed if $X \setminus F$ is open.

Property 2.1. Given a topological space (X, τ) and a subset F of it, we have:

1. $\emptyset, X$ are closed
2. If $F_1, \ldots, F_n$ are closed, then $\bigcup_{i=1}^n F_i$ is closed
3. If $F_1, \ldots, F_n$ are closed, then $\bigcap_{i=1}^n F_i$ is closed

Definition 2.7 (topological closure and interior)**.** Given a topological space (X, τ), where $\tau \subseteq P(X)$, and a set $F \subseteq X$, the topological closure of F is the minimal closed superset of F, i.e.,

$$\bar{F} = \bigcap \{H : H \text{ is closed}, H \supseteq F\}$$

The interior of F is the maximal open subset of F, i.e.,

$$F^\circ = \bigcap \{U : U \text{ is open}, U \subseteq F\}$$

Example 2.9. Given (X, d) a metric space, define τ_d : a set $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Suppose $F \subseteq X$, then

$$\bar{F} = \{x \in X : \forall \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} = \{\lim_{n\to\infty} x_n : (x_n) \subseteq F, \lim_{n\to\infty} x_n \text{ exists}\}$$

and

$$F^\circ = \{x \in X : \exists \epsilon > 0, B_\epsilon(x) \subseteq F\} = \bigcup \{B_\epsilon(x) : \epsilon > 0, x \in F, B_\epsilon(x) \subseteq F\}$$

2.3 Compactness and Bolzano-Weierstrass Theorem

Definition 2.8 (compactness)**.** A subset K of a metric space (X, d) is compact if every sequence in K has a convergent subsequence that converges to a limit in K.

Example 2.10. $(\mathbb{R}, |x - y|)$ is not compact (e.g. $(x_n) = n$)

Example 2.11. $([0, 1], |x - y|)$ is compact.

Property 2.2. If (X, d) is compact, then it is bounded, i.e. $\exists M$ s.t. $x, y \in X, d(x, y) \leq M$.

Property 2.3. If $Y \subseteq X, (X, d)$ is a metric space, and (Y, d) is compact, then Y is closed in X.

Property 2.4. If $K_1 \supseteq K_2 \supseteq \ldots$ are compact and nonempty subsets of X, then $K = \bigcap_{n=1}^\infty K_n$ is compact and nonempty.

Theorem 2.1 (Bolzano-Weierstrass theorem)**.** A subset Y of $\mathbb{R}$ is compact iff closed and bounded.

Remark 2.1. The theorem is true for $\mathbb{R}^n$ but is false for infinite dimension.

Theorem 2.2 (Heine-Borel Theorem)**.** A subset Y of a metric space (X, d) is compact if every open cover $Y \subseteq \bigcup_{i \in I} U_i$ has a finite subcover $Y \subseteq \bigcup_{l=1}^n U_{i_l}$.

2.4 Perfect Sets

Definition 2.9 (perfect set)**.** Let (X, d) be a metric space. $P \subseteq X$ is perfect if it is closed, nonempty, and for every open $U \subseteq X, U \cap P$ is not empty and has at least two elements.

Example 2.12. $S = [0, 1] \cup \{\frac{3}{2}\} \cup [2, 3]$ is not perfect.

Property 2.5. Perfect subsets P of a complete metric space are not countable.

Example 2.13 (Cantor set)**.** Let C_0 be the closed interval $[0, 1]$, and define C_1 to be the set that results when the open middle third is removed; that is,

$$C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

Now construct C_2 in a similar way by removing the open middle third of each of the two components of C_1:

$$C_2 = ([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}]) \cup ([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1])$$

Continue this process inductively. For each $n = 0, 1, 2, \ldots$, we get a set C_n consisting of 2^n closed intervals each having length $(\frac{1}{3})^n$. Finally, we define the Cantor set C to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$

Remark 2.2.
- Since we are always removing open middle thirds, then at each stage, endpoints are never removed. Thus, C at least contains the endpoints of all of the intervals that make up each of the sets C_n.
- The Cantor set has zero length.
- The Cantor set is uncountable, with cardinality equal to the cardinality of $\mathbb{R}$.

2.5 Separated and Connected Sets

Definition 2.10 (separated sets)**.** Let (X, d) be a metric space, $A \neq \emptyset, B \subseteq X$. A and B are separated if $\bar{A} \cap B = \bar{B} \cap A = \emptyset$.

Definition 2.11 (connected sets)**.** A set $C \subseteq X$ is connected if for every decomposition $C = A \cup B$ s.t. $A, B \neq \emptyset$, A and B are not separated, i.e. $\bar{A} \cap B \neq \emptyset$ or $\bar{B} \cap A \neq \emptyset$.

Property 2.6. $C \subseteq \mathbb{R}$ is connected iff

$$\forall a, b \in C, [a, b] \subseteq C$$

Proof. Let $C = A \cup B, a_0 \in A, b_0 \in B, a_0 < b_0$. We define $I_0 = [a_0, b_0], c_0 = \frac{a_0+b_0}{2}$. Define $I_1 = [a_0, c_0], \ldots$. We have $x \in \bar{A} \cap B$ or $\bar{B} \cap A$. ■ Is this complete?

2.6 Baire's Theorem

Definition 2.12 (density)**.** A set $A \subseteq X$ is dense in the metric space (X, d) if $\bar{A} = X$. A subset E of a metric space (X, d) is nowhere-dense in X if $\bar{E}^\circ$ is empty.

Theorem 2.3 (Baire's Theorem)**.** The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

2.7 The Baire Category Theorem

Theorem 2.4. Let (X, d) be a complete metric space, and let $\{O_n\}$ be a countable collection of dense, open subsets of X. Then, $\bigcap_{n=1}^{\infty}\{O_n\}$ is not empty.

Proof. ■

Theorem 2.5 (Baire Category Theorem)**.** A complete metric space is not the union of a countable collection of nowhere-dense sets.

Proof. ■

Theorem 2.6. The set

$$D = \{f \in C[0,1] : f'(x) \text{ exists for some } x \in [0,1]\}$$

is a set of first category in $C[0,1]$.

3 Sequences and Series

3.1 The Limit of a Sequence

Definition 3.1 (sequence)**.** A <u>sequence</u> is a function whose domain is $\mathbb{N}$.

Definition 3.2. Let (X, d) be a metric space. A sequence $(X_n) \subseteq X$ <u>converges</u> to an element $x \in X$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies d(x_n, x) < \epsilon$.

Key property: If $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} x_n = y$, then $x = y$.

Proof. WTS $d(x, y) = 0$
Let $\epsilon > 0$. We will show that $d(x, y) < \epsilon$.
Since $\lim\limits_{n\to\infty} x_n = x$, then $\exists N_1, \forall n \geq N_1, d(x_n, x) < \frac{\epsilon}{2}$
Since $\lim\limits_{n\to\infty} x_n = y$, then $\exists N_2, \forall n \geq N_2, d(x_n, y) < \frac{\epsilon}{2}$
Take $n \geq \max(N_1, N_2)$, then $d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. ■

Proposition 3.1. Suppose (X, d) is a metric space, (X, τ) is a topological space, and $F \subseteq X$. If $\lim\limits_{n\to\infty} x_n = x$, $(x_n) \subseteq F$ and F is closed, then $x \in F$.

Proof. Suppose $x \notin F$, i.e., $x \in X \setminus F$.
Since F is closed, then $X \setminus F$ is open, so there is $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq X \setminus F$.
Let N be such that $\forall n \geq N, d(x_n, x) < \epsilon$.
Then $x_n \in B_\epsilon(x)$, which implies that $(x_n) \subseteq X \setminus F$, a contradiction. ■

Proposition 3.2. Suppose (X, d) is a metric space and $F \subseteq X$. If F is not closed, then there exists $(x_n) \subseteq F$ and $x \notin F$ s.t. $\lim\limits_{n\to\infty} x_n = x$.

Proof. If F is not closed, then $X \setminus F$ is not open, so there is $x \in X \setminus F$ s.t. $B_\epsilon(x) \nsubseteq X \setminus F$ for all $\epsilon > 0$.
Take $x_n \in B_{1/n}(x) \setminus (X \setminus F) = B_{1/n}(x) \cap F$ for each $n \in \mathbb{N}$, then $(x_n) \subseteq F$ and $\lim\limits_{n\to\infty} x_n = x$.
■

Definition 3.3 (Cauchy sequence)**.** A sequence (x_n) in a metric space (x_n) in a metric space (X, d) is a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}, m, n \geq N \implies d(x_m, x_n) < \epsilon$.

Proposition 3.3. A convergent sequence is Cauchy.

Proof. Let (x_n) be a convergent sequence, so that $\lim\limits_{n\to\infty} x_n = x$. To check (x_n) is Cauchy, let $\epsilon > 0$. We need to find N s.t. $\forall m, n \geq N, d(x_n, x_m) < \epsilon$.
Apply $\lim\limits_{n\to\infty} x_n = x$ to $\frac{\epsilon}{2}$, we get N s.t. $\forall n \geq N, d(x, x_n) < \frac{\epsilon}{2}$.
Notice that N works for Cauchy:
Take $m, n \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Remark 3.1. When $X = \mathbb{R}$ with the usual metric, A Cauchy sequence is convergent (the converse is true).
In general not true. For example, $X = \mathbb{R} \setminus \{0\}, d(x, y) = |x - y|, (x_n) = \frac{1}{n}$.

■

Definition 3.4 (completeness of metric spaces)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Example 3.1. $\mathbb{R}, d(x, y) = |x - y|$

Example 3.2. $(X, d), d$ discrete metric.

Example 3.3. $C[0, 1], d(f, g) = \sup\limits_{x\in[0,1]} |f(x) - g(x)| = ||f - g||_\infty$

Example 3.4. $(\mathbb{N}^{\mathbb{N}}, d), d((x_n), (y_n)) = \frac{1}{\min\{n : x_n \neq y_n\}}$
where $\mathbb{N}^{\mathbb{N}} = \{x : \mathbb{N} \to \mathbb{N}\}$.

Definition 3.5 (monotone sequence)**.** $(x_n) \subseteq \mathbb{R}$ is monotone if either $x_n \leq x_m, n \leq m$, or $x_n \geq x_m, n \leq m$.

Theorem 3.1 (Monotone Subsequence Theorem)**.** Every sequence $(x_n) \subseteq \mathbb{R}$ has a monotone subsequence.

prove this

Fact 3.1. If $a_n \leq b_n$ for all n, $a = \lim\limits_{n\to\infty} a_n, b = \lim\limits_{n\to\infty} b_n$, then

$$a \leq b$$

Proof. Suppose for contradiction that $a > b$. Let $\epsilon = \frac{a-b}{2}$.
We know $\exists N_1$ s.t. $a_n \in B_\epsilon(a)$ for $n \geq N_1$ and $\exists N_2$ s.t. $b_n \in B_\epsilon(b)$ for $n \geq N_2$. Take $n > \max(N_1, N_2)$, then we have

$$b_n < \frac{a+b}{2} < a_n$$

which is a contradiction. ■

Theorem 3.2 (Algebraic limit theorem)**.** Suppose $a = \lim\limits_{n\to\infty} a_n, b = \lim\limits_{n\to\infty} b_n$, then:

1. $a + b = \lim_{n\to\infty}(a_n + b_n)$
2. $ab = \lim_{n\to\infty} a_n b_n$
3. $\frac{a}{b} = \lim_{n\to\infty} \frac{a_n}{b_n}$, and $b \neq 0$.

Fact 3.2. Monotone bounded sequence (x_n) converges to its supremum or infimum.

Proof. We only prove the supremum case.
Fix $\epsilon > 0$, let $s = \sup\{x_n : n \in \mathbb{N}\}$. We have $s - \epsilon < s$ and thus $s - \epsilon$ is not an upper bound of (x_n). Therefore, there is N s.t. $x_N > s - \epsilon$.
Take $n \geq N$, then we have

$$x_n \geq x_N > s - \epsilon$$

Therefore, we have $|x_n - s| < \epsilon$. ■

Definition 3.6 (limit supremum)**.** We define

$$\limsup_{n\to\infty} x_n = \inf\{y_m : m \in \mathbb{N}\}$$

where $y_m = \sup\{x_n : n \geq m\}$.
Alternatively,

$$\limsup_{n\to\infty} x_n = \lim_{m\to\infty} \sup_{n\geq m} x_n$$

Definition 3.7 (limit infimum)**.**

$$\liminf_{n\to\infty} x_n = \sup\{z_m : m \in \mathbb{N}\}$$

where $z_m = \inf\{x_n : n \geq m\}$.
Alternatively,

$$\liminf_{n\to\infty} x_n = \lim_{m\to\infty} \inf_{n\geq m} x_n$$

3.2 Series

Definition 3.8. We define

$$S_n = \sum_{k=1}^{n} a_k, \quad \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$$

We call $\sum_{k=1}^{\infty} a_k$ a <u>summable series</u> if the limit exists, i.e.,

$$\exists A, \forall \epsilon > 0, \exists N s.t. \forall n \geq N, |S_n - A| < \epsilon$$

Property 3.1 (Cauchy criterion for series)**.** $\sum_{k=1}^{\infty}$ is <u>summable</u> iff

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, |S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

Corollary 3.1. If $\sum_{k=1}^{\infty} a_k$ is summable, then $|a_k| \to 0$.

Proof. We have $|a_k| = |s_k - s_{k-1}| < \epsilon$ for $k > N$. ■

Example 3.5. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is summable.

Proof.

$$\begin{aligned}
S_m &= 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{m^2} \\
&< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \ldots + \frac{1}{m(m-1)} \\
&= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{m-1} - \frac{1}{m}) \\
&= 1 + 1 - \frac{1}{m} \\
&< 2
\end{aligned}$$

■

Example 3.6. $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k} &= (1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots \\
&= 1 + (1/2) + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \ldots \\
&= 1 + 1/2 + 1/2 + 1/2 + \ldots \\
\to \infty
\end{aligned}$$

■

Theorem 3.3 (Algebraic limit theorem for series)**.** Suppose $\sum_{k=1}^{\infty} a_k = A, \sum_{k=1}^{\infty} b_k = B, c \in \mathbb{R}$, then

1. $\sum_{k=1}^{\infty} ca_k = cA$
2. $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

Proof. (1) We want to show $\forall \epsilon > 0, \exists N$ s.t. $\forall n \geq N, |\sum_{k=1}^{\infty} ca_k - cA| < \epsilon$.
We know $\forall \epsilon_0 > 0, \exists N_{\epsilon_0}$ s.t. $\forall n \geq N_{\epsilon_0}, |\sum_{k=1}^{\infty} a_k - A| < \epsilon_0$.
Take $\epsilon_0 = \frac{\epsilon}{|c|}$, then we have

$$\left| \sum_{k=1}^{\infty} ca_k - cA \right| = |c| \left| \sum_{k=1}^{\infty} a_k - A \right| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$$

■

Property 3.2 (Order comparison test)**.** Suppose $b_k \geq a_k \geq 0, \forall k$.

1. If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.
2. If $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$.

Definition 3.9 (geometric series)**.** We call a series a geometric series if it is of the form

$$\sum_{k=1}^{\infty} ar^k$$

Note that the geometric series converges to $\frac{a}{1-r}$ whenever $r^m \to 0$ iff $|r| < 1$.

Definition 3.10 (absolutely convergence)**.** $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k| < \infty$.

Definition 3.11 (conditionally convergence)**.** $\sum_{k=1}^{\infty} a_k$ is conditionally convergent if $\sum_{k=1}^{\infty} a_k < \infty$, but $\sum_{k=1}^{\infty} |a_k| = \infty$

Example 3.7 (alternating series)**.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} < \infty$ but $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$

Property 3.3 (Absolute convergence test)**.** If $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.

Proof. We use Cauchy criterion for $\sum_{k=1}^{\infty} a_k$: we want to show

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

Let $\epsilon > 0$.
Since $\sum_{k=1}^{\infty} |a_k| < \infty$, then we know that $\exists N$ s.t. $\forall n \geq m \geq N$,

$$\left|\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|\right| < \epsilon$$

Then

$$\begin{aligned}
\left|\sum_{k=m+1}^{n} a_k\right| &= \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right| \\
&\leq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k| \\
&\leq \left|\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|\right| \\
&< \epsilon
\end{aligned}$$

■

Property 3.4 (Alternating series test)**.** Suppose $a_1 \geq a_2 \geq \ldots \geq 0$, $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k < \infty$.

Proof. We want to show $\{S_n\} = \{\sum_{k=1}^{n}(-1)^{k+1}a_k\}$ is Cauchy:

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n \geq N, |S_n - S_m| < \epsilon$$

Let $\epsilon > 0$.
Suppose $n > m$, then $|S_n - S_m| = |a_{m+1} - a_{m+2} + \ldots + (-1)^{n-m+1}a_n|$.
Since (a_n) is a non-negative decreasing sequence, then

$$\begin{aligned} a_{m+1} - a_{m+2} + \ldots + (-1)^{n-m-1}a_n &= a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \ldots \\ &\leq a_{m+1} \end{aligned}$$

Since $\lim\limits_{k\to\infty} a_k = 0, \exists N$ s.t. $\forall m+1 \geq N, a_{m+1} < \epsilon$.
Thus $0 \leq |S_n - S_m| \leq a_{m+1} < \epsilon$. ∎

Property 3.5 (Ratio test)**.** Given $\sum_{k=1}^{\infty} a_k$ s.t. $a_k \neq 0$ for all k.
If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, then $\sum_{k=1}^{\infty} |a_k| < \infty$

Proof. Define $S := \{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n}\right| \geq r'\}$, then S contains finitely many elements of $\mathbb{N}$. (If S were to be infinite set, if we take $\epsilon = r' - r$, then $\left|\frac{a_{n+1}}{a_n}\right| - r \geq r' - r$ for infinitely many terms which contradicts that r is the point of convergence.
Therefore, $S' = \{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n} < r'\right|$ contains all but finitely many elements of $\mathbb{N}$. Let $N = 1 + \max S$, then $\forall n \geq N, \left|\frac{a_{n+1}}{a_n} < r'\right| < r' \implies |a_{n+1}| < r'|a_n|$.
Since $0 < r' < 1, \sum_{n=1}^{\infty}(r')^n$ converges which implies $|a_N|\sum_{n=1}^{\infty}(r')^n$ converges. We have $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n| < C + |a_N|\sum_{n=N+1}^{\infty}(r')^{n-N}$ converges, by comparison test. Hence $\sum_{n=1}^{\infty}|a_n|$ converges. ∎

understand the last two lines of the proof

Definition 3.12 (rearrangement)**.** Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a rearrangement of $\sum_{k=1}^{\infty} a_k$ if $\forall n, !\exists k$ s.t. $b_k = a_n$.

4 Functional Limits and Continuity

4.1 Functional Limits

Definition 4.1. Let $A \subseteq \mathbb{R}, a \in \overline{A \setminus \{a\}}$ (a is an accumulation point of A). Let $f : A \to \mathbb{R}$, define $\lim\limits_{x\to a} f(x) = L$ iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Property 4.1 (Sequential criterion for functional limits)**.** $a \in \overline{A \setminus \{a\}}, f : A \to \mathbb{R}$. The following are equivalent:

1. $\lim\limits_{x\to a} f(x) = L$
2. $\forall (x_n) \subseteq A \setminus \{a\}, x_n \to a \implies f(x_n) \to L$

Proof. We prove (1) $\implies$ (2):
Assume $\lim_{x\to a} f(x) = L$, take arbitrary $(x_n) \subseteq A \setminus \{a\}$ s.t. $x_n \to a$.
Let $\epsilon > 0$, then $\exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.
Also, $\exists N s.t. n \geq N \implies |x_n - a| < \delta$.
Therefore, if $|x_n - a| < \delta$, then $|f(x_n) - L| < \epsilon$. ■

Theorem 4.1 (Algebraic limit theorem for functional limits)**.** Suppose $f, g : A \to \mathbb{R}, a \in \overline{A \setminus \{a\}}$.
Suppose $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = M$. Then we have

1. $\lim_{x\to a} cf(x) = cL$
2. $\lim_{x\to a} (f(x) + g(x)) = L + M$
3. $\lim_{x\to a} (f(x)g(x)) = LM$
4. $\lim_{x\to a} (\frac{f(x)}{g(x)}) = \frac{L}{M}$ when $M \neq 0$.

Property 4.2 (Divergence criterion)**.** Suppose $f : A \to \mathbb{R}, a \in \overline{A \setminus \{a\}}$ $\lim_{x\to a} f(x)$ does not exist if there are two sequences $(x_n), (y_n) \subseteq A \setminus \{a\}$ s.t. $x_n \to a, y_n \to a, \lim_{n\to\infty} f(x_n) = L, \lim_{n\to\infty} f(y_n) = M$ exist but $L \neq M$.

Example 4.1. Let $A = \mathbb{R}^+, a = 0, f(x) = \sin(\frac{1}{x})$¿
Let $a_n = \frac{1}{2n\pi}, b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$.
Then we have $a_n, b_n \to a$. Besides, $\lim_{n\to\infty} f(a_n) = 0, \lim_{n\to\infty} f(b_n) = 1$. Hence $\lim_{x\to 0^+} \sin(\frac{1}{x}$ does not exist.

Definition 4.2. Suppose $f : A \to \mathbb{R}, a \in A \setminus \{a\}$. We define $\lim_{x\to a} f(x) = \infty$ iff

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$$

Definition 4.3. We define $\lim_{x\to\infty} f(x) = L$ iff

$$\forall \epsilon > 0, \exists M > 0 \text{ s.t. } x > M \implies |f(x) - L| < \epsilon$$

4.2 Continuous Functions

Definition 4.4 (continuity)**.** Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. $f : X \to Y$ is continuous at $a \in X$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } x \in B_\delta^X(a) \implies f(x) \in B_\epsilon^Y(f(a))$$

Remark 4.1. Note that for $X = Y = \mathbb{R}, d(x, y) = |x - y|$, so that we can write

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| > \epsilon$$

i.e.

$$\lim_{x\to a} f(x) = f(a)$$

Definition 4.5 (continuous function). $f : X \to Y$ is continuous if it is continuous at every point $a \in X$.

Property 4.3. The following are equivalent:

1. f is continuous at a
2. $\lim_{x \to a} f(x) = f(a)$
3. $\forall (x_n) \subseteq A, x_n \to a \implies f(x_n) \to f(a)$.

Corollary 4.1. f is discontinuous at a if there is a sequence $(x_n) \to a$ s.t. $\lim_{n \to \infty} f(x_n) \neq f(a)$.

Remark 4.2. Note that we may have $\lim_{x \to a} f(x)$ exists but f is discontinuous at a.

Theorem 4.2 (Algebraic continuous theorem). Suppose $f, g : A \to \mathbb{R}$ are continuous at $a \in A, c \in \mathbb{R}$. We have

1. $cf(x)$ is continuous at a
2. $f(x) \pm g(x)$ is continuous at a
3. $f(x)g(x)$ is continuous at a
4. $\frac{f(x)}{g(x)}$ is continuous at a if $g(a) \neq 0$

Theorem 4.3. Suppose $f : A \to B \subseteq \mathbb{R}, g : B \to \mathbb{R}$.
$(g \circ f)(x) = g(f(x))$ is continuous at $a \in A$ whenever f is continuous at a and g is continuous at $f(a)$.

Theorem 4.4. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces, $f : X \to Y$ is continuous.
If $K \subseteq X$ is compact, so is its image $f[K] = \{f(x) : x \in K\}$.

Theorem 4.5. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces.
$f^{-1}(F)$ is closed in X whenever $F \subseteq Y$ is closed in Y.

Theorem 4.6 (Extreme Value Theorem). If $f : K \to \mathbb{R}$ is continuous, K is compact, then $\exists x_1, x_2 \in K$ s.t. $\forall x \in K$,

$$f(x_1) \leq f(x) \leq f(x_2)$$

Proof. Let $H = f[K] = \{f(x) : x \in K\} \subseteq \mathbb{R}$, which is compact. Since compact subsets of $\mathbb{R}$ are bounded, then let $y_2 = \sup(H)$.
We have $y \leq y_2$ for all $y \in H$ and $\forall \epsilon > 0, \exists y \in H$ s.t. $y_2 - \epsilon < y \leq y_2$.
Take $\epsilon = \frac{1}{n}$, then we have some $z_n \in H$ s.t. $y_2 - \frac{1}{n} < z_n \leq y_2$.
Now we find $a_n \in k$ s.t. $f(a_n) = z_n, n = 1, 2, \ldots$
By theorem, we have $a_{n_k} \to x_2$, then $f(x_2) = \lim_{k \to \infty} f(a_{n_k}) = y_2$. ■

Which theorem?