Math Notes

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1 Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

2 Trigonometric Formulas

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\cos^2 a = \frac{1 + \cos 2a}{2}$$

$$\sin^2 a = \frac{1 - \cos 2a}{2}$$

3 Arc functions

Name	Usual notation	Definition	Domain	Range
arcsine	$y = \arcsin(x)$	$x = \sin(y)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
arccosine	$y = \arccos(x)$	$x = \cos(y)$	$[-1, 1]$	$[0, \pi]$
arctangent	$y = \arctan(x)$	$x = \tan(y)$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

4 Cross Product

Definition In 3-dimensional Euclidean space only, the cross product of vectors **a** and **b** is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Remark ”xia, dafan, shang”

As a Determinant

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\ &= \begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i} - \begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\mathbf{j} + \begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}\end{aligned}$$

Properties

1. $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$
2. $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|sin\theta$. This says that the length $\mathbf{a}\times\mathbf{b}$ equals the area of the parallelogram generated by $\mathbf{a}$ and $\mathbf{b}$.
3. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
4. $(c_1\mathbf{a}_1 + c_2\mathbf{a}_2)\times\mathbf{b} = c_1\mathbf{a}_1\times\mathbf{b} + c_2\mathbf{a}_2\times\mathbf{b}$
5. $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ and $\mathbf{j}\times\mathbf{k} = \mathbf{i}$ and $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
6. Not associative: $(a\times b)\times c \neq a\times(b\times c)$

5 Derivative Formulas

1. $\frac{d}{dx}\log_a x = \frac{1}{x\cdot ln(a)}$
2. $[u(x)^{v(x)}]' = u(x)^{v(x)}\cdot[\ln(u(x))\cdot v(x)]'$

6 Common Taylor Series

Definition 6.1 (Taylor Series)**.** The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Or equivalently,

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Axiom 6.1.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{2}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{3}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{4}$$

Remark 6.1. Take the anti-derivative of (4) to get the Taylor polynomial of $\ln(1-x)$.

7 ε definition of supremum and infimum

Definition Let S be a nonempty subset of the real numbers that is bounded above. The upper bound u is said to be the supremum of S iff

$$\forall \varepsilon > 0, \exists x \in S, u - \varepsilon < x$$

Definition Let S be a nonempty subset of the real numbers that is bounded below. The lower bound w is said to be the infimum of S iff

$$\forall \varepsilon > 0, \exists x \in S, x < w + \varepsilon$$

8 Even and Odd functions

8.1 Profucts

The product of two even functions is an even function.
The product of two odd functions is an even function.
The product of an even function and an odd function is an odd function.
The quotient of two even functions is an even function.

8.2 Odd-Even Decomposition

For any continuous function f, f can be decomposed into the sum of one even function and one odd function:

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$$

We can verify that the first part is even, and the second part is odd.

9 Real roots and Polynomials

Every polynomial $p(x)$ of degree 3 with real coefficients has at least one real root. For $x = A$ sufficiently negative, $p(x) < 0$; for $x = B$ sufficiently positive, $p(x) > 0$, since the degree is odd. Hence, by the intermediate value theorem for continuous functions, there is at least one solution x_0 of $p(x) = 0$ between A and B.

10 Bilinear Map

A **bilinear map** is a function combining elements of two vector spaces to yield an element of a third vector space, and is linear in each of its arguments. In other words, when we hold the first entry of the bilinear map fixed while letting the second entry vary, the result is a linear operator, and similarly for when we hold the second entry fixed.(example: matrix multiplication)

11 Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Notice that in a special case of $\theta = \pi$, the above identity is a.k.a Euler's Identity($e^{i\pi} + 1 = 0$). Considering any $z \in \mathbb{C}$, to derive the above identity, we have the following

$$\begin{aligned} z &= |z|e^{i\theta} \\ z^n &= |z|^n \left(e^{i\theta}\right)^n \\ &= |z|^n e^{i\theta n} \\ &= |z|^n (\cos n\theta + i\sin n\theta) \end{aligned}$$

notice that we can now interchange, as we please, $\cos\theta + i\sin\theta$ with $e^{i\theta}$.

11.1 n-th Roots of Unity

Suppose $a \in \mathbb{C}$ and $t^n = a$ Then

$$\begin{aligned} t^n &= a \\ t^n &= |a|e^{2\pi k i} \\ t &= e^{\frac{2\pi i k}{n}} \\ k &= 0, 1, 2, \ldots, n-1 \end{aligned}$$

We can form these roots into equal partitions of the unit circle.

12 Definition of Natural Number

Definition 12.1.

$$e := \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

Definition 12.2.

$$e^k := \lim_{n\to\infty} (1 + \frac{k}{n})^n$$

13 Common Integrals

$$\int_{-\infty}^{\infty} e^{-(\frac{x-b}{a})^2}\, dx = a\sqrt{\pi} \tag{5}$$