STA447
Lecture Notes

Yuchen Wang

February 24, 2020

Contents

1 Markov Chain Probabilities

Notation 1.1.

$$P(X_{n+1} = j | X_n = i) = p_{ij}$$

Definition 1.1 (Markov chain)**.** A (discrete time, discrete space, time homogeneous) Markov chain is specified by three ingredients:

- A state space S, any non-empty finite or countable set.
- Initial probabilities $\{v_i\}_{i \in S}$, where v_i is the probability of starting at i (at time 0). (So $v_i \geq 0$ and $\sum_i v_i = 1$)
- Transition probabilities $\{p_{ij}\}_{i,j \in S}$, where p_{ij} is the probability of jumping to j if you start at i. (So $p_{ij} \geq 0$, and $\sum_j p_{ij} = 1$ for all i)

Remark 1.1 (Markov property)**.**

$$P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n) = P(X_{n+1} = j | X_n = i_n) = p_{i_n j}$$

i.e. The probabilities at time $n+1$ depend only on the state at time n.

Remark 1.2.

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = v_{i_0} p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{n-1} i_n}$$

1.1 Markov Chain examples

Example 1.1 (the Frog Walk)**.** Let $X_n :=$ pad index the frog is at after n steps.

$$\begin{aligned} S &= \{1, 2, 3, \ldots, 20\} \\ v_{20} &= 1, v_i = 0 \, \forall i \neq 20 \\ p_{ij} &= \begin{cases} \frac{1}{3}, & |j - i| \leq 1 \text{ or } |j - i| = 19 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Example 1.2 (Bernoulli process)**.**

$$\begin{aligned} S &= \{1, 2, 3, \ldots\} \\ v_0 &= 1, v_i = 0 \, \forall i \neq 0 \\ p_{ij} &= \begin{cases} p, & j = i + 1 \\ 1 - p, & j = i \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $0 < p < 1$.

Example 1.3 (Simple random walk (s.r.w.))**.** Let $X_n :=$ net gain (in dollars) after n bets

$$
\begin{aligned}
S &= \{0, 1, 2, 3, \dots\} \\
v_a &= 1, v_i = 0\,\forall i \neq a \\
p_{ij} &= \begin{cases} p, & j = i+1 \\ 1-p, & j = i-1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

where $0 < p < 1, a \in \mathbb{Z}$.
Special case: When $p = 1/2$, call it simple symmetric random walk.

Example 1.4 (Ehrenfest's Urn)**.** Let $X_n :=$ # balls in Urn 1 at time n.
We have d balls in total, divided into two urns. At each time, we choose one of the d balls uniformly at random, and move it to the other urn.

$$
\begin{aligned}
S &= \{1, 2, 3, \dots, d\} \\
v_a &= 1, v_i = 0\,\forall i \neq a \\
p_{ij} &= \begin{cases} (d-i)/d, & j = i+1 \\ i/d, & j = i-1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

1.2 Elementary Computations

Notation 1.2.

$$\mu_i^{(n)} := P(X_n = i)$$

Notation 1.3.

$$
\begin{aligned}
m &:= |S| && \text{(the number of elements in S, could be infinity)} \\
\mu^{(n)} &= (\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)}, \dots) && (m \times 1) \\
v &= (v_1, v_2, v_3, \dots) && (m \times 1) \\
P = (p_{ij}) &= \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & \\ & \ddots & & \\ p_{m1} & \cdots & & p_{mm} \end{pmatrix} && (m \times m \text{ matrix})
\end{aligned}
$$

Fact 1.1.

$$
\begin{aligned}
\mu^{(1)} &= vP = \mu^{(0)} P \\
\mu^{(n)} &= vP^n = \mu^{(0)} P^n
\end{aligned}
$$

Notation 1.4.

$$p_{ij}^{(n)} := P(X_n = j, X_0 = i) = P(X_{m+n} = j | X_m = i) \qquad \text{(for any } m \in \mathbb{N}\text{)}$$

Fact 1.2.

$$\sum_{j \in S} p_{ij}^{(n)} = 1$$
$$p_{ij}^{(1)} = p_{ij}$$
$$P^{(n)} = P^n \qquad \text{(for all } n \in \mathbb{N}\text{)}$$

Notation 1.5.

$$P^0 := I$$
$$P^{(0)} := I$$
$$p_{ij}^{(0)} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 1.1 (Chapman-Kolmogorov equations)**.**

$$p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}$$
$$P_{ij}^{(m+s+n)} = \sum_{k \in S} \sum_{l \in S} p_{ik}^{(m)} p_{kl}^{(s)} p_{lj}^{(n)}$$

Matrix form:

$$P^{(m+n)} = P^{(m)} P^{(n)}$$
$$P^{(m+s+n)} = P^{(m)} P^{(s)} P^{(n)}$$

Theorem 1.2 (Chapman-Kolmogorov Inequality)**.**

$$p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)} \qquad \text{(for all } k \in S\text{)}$$
$$P_{ij}^{(m+s+n)} \geq p_{ik}^{(m)} p_{kl}^{(s)} p_{lj}^{(n)} \qquad \text{(for any } k, l \in S\text{)}$$

1.3 Recurrence and Transience

Notation 1.6.

$$P_i(\ldots) \equiv P(\ldots | X_0 = i)$$
$$E_i(\ldots) \equiv E(\ldots | X_0 = i)$$
$$N(i) = \#\{n \geq 1 : X_n = i\}$$

(total number of times that the chain hits i, not counting time 0)

Definition 1.2 (return probability)**.** Let f_{ij} be the return probability from i to j.

$$f_{ij} := P_i(X_n = j \text{ for some } n \geq 1) \equiv P_i(N(j) \geq 1)$$

Fact 1.3.

$$\begin{align} 1 - f_{ij} &= P_i(X_n \neq j \text{ for all } n \geq 1) \tag{1} \\ P_i(N(i) \geq k) &= (f_{ii})^k \tag{2} \\ P_i(N(j) \geq k) &= f_{ij}(f_{jj})^{k-1} \tag{3} \\ f_{ik} &\geq f_{ij} f_{jk} \tag{4} \end{align}$$

Fact 1.4. $f_{ij} > 0$ iff $\exists m \geq 1$ with $p_{ij}^{(m)} > 0$, i.e., there is some time m for which it is possible to get from i to j in m steps.

Definition 1.3 (recurrent and transient states)**.** A state i of a Markov chain is recurrent if $f_{ii} = 1$. Otherwise, i is transient if $f_{ii} < 1$.

Proposition 1.1. If Z is a non-negative integer, then

$$E(Z) = \sum_{k=1}^{\infty} P(Z \geq k)$$

Theorem 1.3 (Recurrent State Theorem)**.** As follows

- State i is recurrent $\iff$ $P_i(N(i) = \infty) = 1$ $\iff$ $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$
- State i is transient $\iff$ $P_i(N(i) = \infty) = 0$ $\iff$ $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$

Proof.

$$\begin{align} P_i(N(i) = \infty) &= \lim_{k \to \infty} P_i(N(i) \geq k) && \text{(by continuity of probabilities)} \\ &= \lim_{k \to \infty} (f_{ii})^k && (P_i(N(i) \geq k) = (f_{ii})^k) \\ &= \begin{cases} 1, & f_{ii} = 1 \\ 0, & f_{ii} < 1 \end{cases} \end{align}$$

■

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{ii}^{(n)} &= \sum_{n=1}^{\infty} P_i(X_n = i) \\
&= \sum_{n=1}^{\infty} E_i(\mathbb{1}\{X_n = i\}) \\
&= E_i(\sum_{n=1}^{\infty} \mathbb{1}\{X_n = i\}) \\
&= E_i(N(i)) \\
&= \sum_{k=1}^{\infty} P_i(N(i) \geq k) && \text{(by proposition 1.1)} \\
&= \sum_{k=1}^{\infty} (f_{ii})^k \\
&= \begin{cases} \infty, & f_{ii} = 1 \\ \frac{f_{ii}}{1-f_{ii}} < \infty, & f_{ii} < 1 \end{cases}
\end{aligned}$$

Example 1.5 (simple random walk)**.** If $p = 1/2$ then $\forall i, f_{ii} = 1$. If $p \neq 1/2$, then $\forall i, f_{ii} < 1$

Proof. Consider state 0. We need to check if $\sum_{n=1}^{\infty} p_{00}^{(n)} = \infty$.
If n is odd, then $p_{00}^{(n)} = 0$.
If n is even, $p_{00}^{(n)} = P(\frac{n}{2}$ heads and $\frac{n}{2}$ tails on first n tosses).
This is a Binomial(n, p) distribution, so

$$\begin{aligned}
p_{00}^{(n)} &= \binom{n}{n/2} p^{n/2}(1-p)^{n/2} \\
&= \frac{n!}{[(n/2)!]^2} p^{n/2}(1-p)^{n/2} \\
&= \frac{(n/e)^n \sqrt{2\pi n}}{[(n/2e)^{n/2}\sqrt{2\pi n/2}]^2} p^{n/2}(1-p)^{n/2} && \text{(Sirling's approximation)} \\
&= [4p(1-p)]^{n/2}\sqrt{2/\pi n}
\end{aligned}$$

Case 1: If $p = 1/2$, then $4p(1-p) = 1$, so

$$\begin{aligned}
\sum_{n=1} \infty p_{00}^{(n)} &= \sum_{n=2,4,6,\ldots} \sqrt{2/\pi n} \\
&= \sqrt{2/\pi} \sum_{n=2,4,6,\ldots} n^{-1/2} \\
&= \sqrt{2/\pi} \sum_{n=1}^{\infty} 2k^{-1/2} \\
&= \infty
\end{aligned}$$

Therefore, state 0 is recurrent.
Case 2: If $p \neq 1/2$, then $4p(1-p) < 1$, so

$$\begin{aligned}
\sum_{n=1} \infty p_{00}^{(n)} &= \sum_{n=2,4,6,\dots} [4p(1-p)]^{n/2} \sqrt{2/\pi n} \\
&< \sum_{n=2,4,6,\dots} [4p(1-p)]^{n/2} && \text{(Geometric Series)} \\
&= \frac{4p(1-p)}{1-4p(1-p)} \\
&< \infty
\end{aligned}$$

Therefore, the state 0 is transient.
The same exact calculation applies to any other state i. ■

Theorem 1.4 (f-Expansion)**.**

$$f_{ij} = p_{ij} + \sum_{k \in S, k \neq j} p_{ik} f_{kj}$$

Proof.

$$\begin{aligned}
f_{ij} &= P_i(\exists n \geq 1 : X_n = j) \\
&= \sum_{k \in S} P_i(X_1 = k, \exists n \geq 1 : X_n = j) \\
&= P_i(X_1 = j, \exists n \geq 1 : X_n = j) + \sum_{k \neq j} P_i(X_1 = k, \exists n \geq 1 : X_n = j) \\
&= P_i(X_1 = j) P_i(\exists n \geq 1 : X_n = j | X_1 = j) + \sum_{k \neq j} P_i(X_1 = k) P_i(\exists n \geq 1 : X_n = j | X_1 = k) \\
&= p_{ij}(1) + \sum_{k \neq j} p_{ik}(f_{kj})
\end{aligned}$$

■

Remark 1.3. The f-Expansion shows that $f_{ij} \geq p_{ij}$.

Remark 1.4. It essentially follows from logical reasoning: from i, to get to j eventually, we have to either jump to j immediately (with probability p_{ij}), or jump to some other state k (with probability p_{ik}) and then get to j eventually (with probability p_{kj})

1.4 Communicating States and Irreducibility

Definition 1.4 (communicating states)**.** State i communicates with state j, written $i \to j$, if $f_{ij} > 0$.

Remark 1.5. i.e. if it is possible to get from i to j.

Notation 1.7. Write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$.

Definition 1.5 (irreducibility)**.** A Markov chain is irreducible if $i \to j$ for all $i, j \in S$, i.e., if $f_{ij} > 0$ for all $i, j \in S$. Otherwise, the chain is reducible.

Lemma 1.1 (Sum Lemma)**.** If $i \to k$, and $l \to j$, and $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$, then $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$

Proof. Since $i \to k$, and $l \to j$, there exists $m, r \geq 1$ s.t. $p_{ik}^{(m)} > 0$ and $p_{lj}^{(r)} > 0$.
By the Chapman-Kolmogorov inequality,

$$p_{ij}^{(m+s+r)} \geq p_{ij}^{(m)} p_{kl}^{(s)} p_{lj}^{(r)}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{ij}^{(n)} &\geq \sum_{n=m+r+1}^{\infty} p_{ij}^{(n)} \\
&= \sum_{s=1}^{\infty} p_{ij}^{(m+s+r)} && (s = n - m - r) \\
&\geq \sum_{s=1}^{\infty} p_{ij}^{(m)} p_{kl}^{(s)} p_{lj}^{(r)} \\
&= \underbrace{p_{ij}^{(m)}}_{+} \underbrace{p_{lj}^{(r)}}_{+} \underbrace{\sum_{s=1}^{\infty} p_{kl}^{(s)}}_{=\infty} \\
&= \infty
\end{aligned}$$

■

Corollary 1.1 (Sum Corollary)**.** If $i \leftrightarrow k$, then i is recurrent iff k is recurrent.

Proof. Setting $j = i$ and $l = k$ in the Sum Lemma: If $i \leftrightarrow k$, then $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty \iff \sum_{n=1}^{\infty} p_{kk}^{(n)} = \infty$. ■

Theorem 1.5 (Cases Theorem)**.** For an irreducible Markov chain, either

- (a) $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for all $i, j \in S$, and all states are recurrent (recurrent Markov chain); or
- (b) $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$ for all $i, j \in S$, and all states are transient (transient Markov chain).

Theorem 1.6 (Finite Space Theorem)**.** An irreducible Markov chain on a finite state space always falls into case (a), i.e., $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for all $i, j \in S$, and all states are recurrent.

Proof. Choose any state $i \in S$. We have

$$\begin{aligned}
\sum_{j \in S} \sum_{n=1}^{\infty} p_{ij}^{(n)} &= \sum_{n=1}^{\infty} \sum_{j \in S} p_{ij}^{(n)} && \text{(exchanging the sums)} \\
&= \sum_{n=1}^{\infty} 1 \\
&= \infty
\end{aligned}$$

Then if S is finite, it follows that there must exist at least one $j \in S$ with $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$. So we must be in case (a).

■

Notation 1.8. For $i \neq j$, let H_{ij} be the event that the chain hits the state i before returning to j, i.e.,

$$H_{ij} = \{\exists n \in \mathbb{N} : X_n = i, \text{ but } X_m \neq j \text{ for } 1 \leq m \leq n-1\}$$

Lemma 1.2 (Hit Lemma)**.** If $j \to i$ with $j \neq i$, then $P_j(H_{ij}) > 0$.

Proof. Since $j \to i$, there is some possible path from j to i. i.e., there is $m \in \mathbb{N}$ and $x_0, x_1, \ldots, x_m$ with $x_0 = j$ and $x_m = i$ and $p_{x_r x_{r+1}} > 0$ for all $0 \leq r \leq m-1$.
Let $S = \max\{r : x_r = j\}$ be the last time this path hits j.
Then $x_S, x_{S+1}, \ldots, x_m$ is a possible path which goes from j to i without first returning to j.
Hence $P_j(H_{ij}) \geq P(x_0, x_1, \ldots, x_m) = p_{x_S x_{S+1}} p_{x_{S+1} x_{S+2}} \ldots p_{x_{m-1} x_m} > 0$

■

Remark 1.6. If it is possible to get from j to i at all, then it is possible to get from j to i without first returning to j.
Intuitively obvious: If there is some path from j to i, then the final part of the path (starting with the last time it visits i) is a possible path from j to i which does not return to j.

Lemma 1.3 (f-Lemma)**.** If $j \to i$ and $f_{jj} = 1$, then $f_{ij} = 1$

Proof. If $i = j$ it is trivial, so assume $i \neq j$.
Since $j \to i$, we have $P_j(H_{ij}) > 0$ by the Hit Lemma.
But one way to never return to j is to first hit i and then from i never return to j:

$$P_j(\text{never return to } j) \geq P_j(H_{ij}) P_i(\text{never return to } j)$$

Therefore

$$1 - f_{jj} \geq P_j(H_{ij})(1 - f_{ij})$$

Since $f_{jj} = 1$, then $\underbrace{P_j(H_{ij})}_{>0}(1 - f_{ij}) = 0$
Hence $f_{ij} = 1$.

■

Lemma 1.4 (Infinite Returns Lemma)**.** For an irreducible Markov chain, if it is recurrent, then

$$P_i(N(j) = \infty) = 1$$

for all $i, j \in S$.
But if it transient, then $P_i(N(j) = \infty) = 0$ for all $i, j \in S$.

Proof. Let $i, j \in S$. If the chain is recurrent, then $f_{ij} = f_{jj} = 1$ by the f-Lemma. Then

$$\begin{aligned} P_i(N(j) = \infty) &= \lim_{k\to\infty} P_i(N(j) \geq k) \\ &= \lim_{k\to\infty} f_{ij}(f_{jj})^{k-1} \\ &= \lim_{k\to\infty} (1)(1)^{k-1} \\ &= 1 \end{aligned}$$

If the chain is transient, then $f_{jj} < 1$, then

$$\begin{aligned} P_i(N(j) = \infty) &= \lim_{k\to\infty} P_i(N(j) \geq k) \\ &= \lim_{k\to\infty} f_{ij}(f_{jj})^{k-1} \\ &= \lim_{k\to\infty} (1)(f_{jj})^{k-1} \\ &= 0 \end{aligned}$$

■

Theorem 1.7 (Recurrence Equivalence Theorem)**.** If a chain is irreducible, then the following are equivalent (and all correspond to case (a)):

1. There are $k, l \in S$ with $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$.
2. For all $i, j \in S$, we have $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.
3. There is $k \in S$ with $f_{kk} = 1$, i.e. k is recurrent.
4. For all $j \in S$, we have $f_{jj} = 1$, i.e. all states are recurrent.
5. For all $i, j \in S$, we have $f_{ij} = 1$.
6. There are $k, l \in S$ with $P_k(N(l) = \infty) = 1$.
7. For all $i, j \in S$, we have $P_i(N(j) = \infty) = 1$.

Proof. Follow from results that we have already proven

- $1 \implies 2$: Sum Lemma.
- $2 \implies 4$: Recurrent State Theorem (with $i = j$).
- $4 \implies 5$: f-Lemma.
- $5 \implies 3$: immediate.
- $3 \implies 1$: Recurrent State Theorem (with $l = k$).
- $4 \implies 7$: Infinite Returns Lemma.

- 7 $\implies$ 6: Immediate.
- 6 $\implies$ 3: Recurrent State Theorem (with $l = k$).

■

Theorem 1.8 (Transience Equivalence Theorem)**.** If a chain is irreducible, then the following are equivalent (and all correspond to case (b)):

1. There are $k, l \in S$ with $\sum_{n=1}^{\infty} p_{kl}^{(n)} < \infty$.
2. For all $i, j \in S$, we have $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$.
3. For all $k \in S$, we have $f_{kk} < 1$, i.e. k is transient.
4. There is $j \in S$ with $f_{jj} < 1$, i.e. some state is recurrent.
5. There are $i, j \in S$ with $f_{ij} < 1$.
6. For all $k, l \in S, P_k(N(l) = \infty) = 0$.
7. There are $i, j \in S$ with $P_i(N(j) = \infty) = 0$.

Remark 1.7 (closed subset note)**.** Suppose a chain is reducible, but it has a closed subset $C \subseteq S$ (i.e. $p_{ij} = 0$ for $i \in C$ and $j \notin C$) on which it is irreducible (i.e. $i \to j$ for all $i, j \in C$). Then, the Recurrence Equivalence Theorem and other results about irreducible chains still apply to the chain when restricted to C.

Proposition 1.2. For simple random walk with $p > 1/2, f_{ij} = 1$ whenever $j > i$. (Similarly, if $p < 1/2$ and $j < i$, then $f_{ij} = 1$.)

Proof. Let $X_0 = 0$, and $Z_n = X_n - X_{n-1}$ for $n = 1, 2, \ldots$, so that $X_n = \sum_{i=1}^{n} Z_i$.
Since Z_ns iid with $P(Z_n = 1) = p$ and $P(Z_n = -1) = 1 - p$, then by Law of Large Numbers,

$$\lim_{n\to\infty} \frac{1}{n}(Z_1 + Z_2 + \ldots + Z_n) \overset{p}{=} E(Z_1) = p(1) + (1-p)(-1) = 2p - 1 > 0$$

$$\begin{aligned}
\implies \infty &= \lim_{n\to\infty} (Z_1 + Z_2 + \ldots + Z_n) \\
&= \lim_{n\to\infty} X_n - X_0 \\
&= \lim_{n\to\infty} X_n
\end{aligned}$$

But if $i < j$, then to go from i to ∞, the chain must pass through j, so $f_{ij} = 1$. ■

2 Markov Chain Convergence

2.1 Stationary Distributions

Definition 2.1 (stationary distributions)**.** If π is a probability distribution on S (i.e. $\pi_i \geq 0$ for all $i \in S$, and $\sum_{i \in S} \pi_i = 1$), then π is stationary for a Markov chain with transition probabilities (p_{ij}) if $\sum_{i \in S} \pi_i p_{ij} = \pi_j$ for all $j \in S$ (or $\pi P = \pi$, in matrix notation).

Remark 2.1. Intuitively, π being stationary means if the chain starts with probabilities $\{\pi_i\}$, then it will keep the same probabilities one time unit later.

Definition 2.2 (doubly stochastic)**.** A Markov Chain is doubly stochastic if in addition to the usual condition that $\sum_{j \in S} p_{ij} = 1$ for all $i \in S$, $\sum_{i \in S} p_{ij} = 1$ for all $j \in S$.

Remark 2.2. This holds for the Frog Example.

Proposition 2.1. If a Markov chain with states S satisfies $|S| < \infty$ and is doubly stochastic, then the uniform distribution on S is a stationary distribution.

Proof. Let $\{\pi_i\}$ be a distribution such that $\pi_i = \frac{1}{|S|}$.
Then

$$\begin{aligned}
\sum_{i \in S} \pi_i p_{ij} &= \sum_{i \in S} \frac{1}{|S|} p_{ij} \\
&= \frac{1}{|S|} \sum_{i \in S} p_{ij} \\
&= \frac{1}{|S|}(1) && \text{(doubly stochastic)} \\
&= \frac{1}{|S|} \\
&= \pi_j
\end{aligned}$$

Then $\{\pi_i\}$ is stationary. ∎

2.2 Searching for Stationary

Definition 2.3 (reversibility)**.** A Markov chain is reversible (or time reversible, or satisfies detailed balance) with respect to a probability distribution $\{\pi_i\}$ if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

Proposition 2.2. If a chain is reversible with respect to π, then π is a stationary distribution.

Proof. Reversibility means $\pi_i p_{ij} = \pi_j p_{ji}$, so then for $j \in S$,

$$\sum_{i \in S} \pi_i p_{ij} = \sum_{i \in S} \pi_j p_{ji} = \pi_j \sum_{i \in S} p_{ji} = \pi_j(1) = \pi_j$$

∎

Lemma 2.1 (M-test)**.** Let $\{x_{nk}\}_{n,k \in \mathbb{N}}$ be a collection of real numbers. Suppose that $\lim\limits_{n \to \infty} x_{nk}$ exists for each fixed $k \in \mathbb{N}$. Suppose further that $\sum_{k=1}^{\infty} \sup\limits_{n} |x_{nk}| < \infty$. Then $\lim\limits_{n \to \infty} \sum_{k=1}^{\infty} x_{nk} = \sum_{k=1}^{\infty} \lim\limits_{n \to \infty} x_{nk}$.

Proposition 2.3 (Vanishing Probabilities Proposition)**.** If a Markov chain's transition probabilities satisfy that $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$, then the chain does not have a stationary distribution.

Proof. Suppose for contradiction that there is a stationary distribution π. Then we would have $\pi_j = \sum_{i\in S} \pi_i p_{ij}^{(n)}$ for any n, so

$$\pi_j = \lim_{n\to\infty} \pi_j = \lim_{n\to\infty} \sum_{i\in S} \pi_i p_{ij}^{(n)}$$

$$\begin{aligned}
\pi_j &= \lim_{n\to\infty} \pi_j \\
&= \lim_{n\to\infty} \sum_{i\in S} \pi_i p_{ij}^{(n)} \\
&= \sum_{i\in S} \lim_{n\to\infty} \pi_i p_{ij}^{(n)} && \text{(exchange the sum and the limit, which is valid by M-test)} \\
&= \sum_{i\in S} \pi_i \lim_{n\to\infty} p_{ij}^{(n)} \\
&= \sum_{i\in S} 0 \\
&= 0
\end{aligned}$$

So we would have $\pi_j = 0$ for all j. But this means that $\sum_j \pi_j = 0$, which is a contradiction. ■

Lemma 2.2 (Vanishing Lemma)**.** If a Markov chain has some $k, l \in S$ with $\lim_{n\to\infty} p_{kl}^{(n)} = 0$, then for any $i, j \in S$ with $k \to i$ and $j \to l$, $\lim_{n\to\infty} p_{ij}^{(n)} = 0$.

Proof. Since $k \to i$ and $j \to l$, we can find $r, s \in \mathbb{N}$ with $p_{ki}^{(r)} > 0$ and $p_{jl}^{(s)} > 0$. Then by the Chapman-Kolmogorov Inequality,

$$p_{kl}^{(r+n+s)} \geq p_{ki}^{(r)} p_{ij}^{(n)} p_{jl}^{(s)}$$

Hence

$$p_{ij}^{(n)} \leq p_{kl}^{(r+n+s)} / p_{ki}^{(r)} p_{jl}^{(s)}$$

But the assumptions imply that

$$\lim_{n\to\infty} \left[p_{kl}^{(r+n+s)} / p_{ki}^{(r)} p_{jl}^{(s)} \right] = 0$$

Hence

$$\begin{aligned}
&0 \leq \lim_{n\to\infty} p_{ij}^{(n)} \leq 0 \\
&\implies \lim_{n\to\infty} p_{ij}^{(n)} = 0
\end{aligned}$$

■

Corollary 2.1 (Vanishing Together Corollary)**.** For an irreducible Markov chain, either

1. $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$, or

2. $\lim_{n\to\infty} p_{ij}^{(n)} \neq 0$ for all $i, j \in S$

Corollary 2.2 (Vanishing Probabilities Corollary)**.** If an irreducible Markov chain's transition probabilities satisfy that $\lim_{n\to\infty} p_{kl}^{(n)} = 0$ for some $k, l \in S$, then the chain does not have a stationary distribution.

Lemma 2.3. If the x_ns are non-negative, and $\sum_{n=1}^{\infty} x_n < \infty$, then $\lim_{n\to\infty} x_n = 0$.

Corollary 2.3 (Transient Not Stationary Corollary)**.** A Markov chain which is irreducible and transient cannot have a stationary distribution.

Proof. If a chain is irreducible and transient, then by the Transience Equivalence Theorem, $\sum_{n=1}^{\infty} < \infty$ for all $i, j \in S$. Hence $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$.
Thus by the Vanishing Probabilities Corollary, there is no stationary distribution.

■

2.3 Obstacles to Convergence

Definition 2.4 (period)**.** The period of a state i is the greatest common divisor (gcd) of the set $\{n \geq 1 : p_{ii}^{(n)} > 0\}$, i.e. the largest number m such that all the values of n with $p_{ii}^{(n)} > 0$ are all integer multiples of m. If the period of each state is 1, we say the chain is aperiodic; otherwise we say the chain is periodic.

Remark 2.3. Intuitively, the period of a state i is the pattern of returning to i from i. e.g. If the period of i is 2, then it is only possible to get from i to i in an even numbers of steps.

Fact 2.1. If state i has period t, and $p_{ii}^{(m)} > 0$, then m is an integer multiple of t, i.e., t divides m.

Fact 2.2. If $p_{ii} > 0$, then the period of state i is 1.

Fact 2.3. If $p_{ii}^{(n)} > 0$ and $p_{ii}^{(n+1)} > 0$, then the period of state i is 1.

Lemma 2.4 (Equal Periods Lemma)**.** If $i \leftrightarrow j$, then the periods of i and of j are equal.

Proof. Let the periods of i and j be t_i and t_j. Since $i \leftrightarrow j$, we can find $r, s \in \mathbb{N}$ with $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$. Then

$$p_{ii}^{(r+s)} \geq p_{ij}^{(r)} p_{ji}^{(s)} > 0$$

Therefore by Fact 2.1, t_i divides $r + s$.
Suppose now that $p_{jj}^{(n)} > 0$. Then

$$p_{ii}^{(r+n+s)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)} > 0$$

So t_i divides $r+n+s$.
Since t_i divides both $r+n+s$ and $r+s$, then it must divide n as well.
Since this is true for any n with $p_{jj}^{(n)} > 0$, it follows that t_i is a common divisor of $\{n \in \mathbb{N} : p_{jj}^{(n)} > 0\}$.
But t_j is the greatest such common divisor, so $t_j \geq t_i$.
Similarly we can show that $t_i \geq t_j$, so we have $t_i = t_j$.

■

Corollary 2.4 (Equal Periods Corollary)**.** If a chain is irreducible, then all states have the same period.

Corollary 2.5. If a chain is irreducible and $p_{ii} > 0$ for some state i, then the chain is aperiodic.

2.4 Convergence Theorem

Theorem 2.1 (Markov Chain Convergence Theorem)**.** If a Markov chain is irreducible, aperiodic, and has a stationary distribution $\{\pi_i\}$, then $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$, and $\lim_{n\to\infty} P(X_n = j) = \pi_j$ for any initial probabilities $\{v_i\}$.

Theorem 2.2 (Stationary Recurrence Theorem)**.** If chain irreducible and has a stationary distribution, then it is recurrent.

Proof. The Transient Not Stationary Corollary says that a chain cannot be irreducible, transient and have a stationary distribution.
Therefore, if a chain is irreducible and has a stationary distribution, then it cannot be transient, i.e. it must be recurrent. ■

Lemma 2.5 (Number Theory Lemma)**.** If a set A of positive integers is non-empty, and satisfies additivity, and $gcd(A) = 1$, then there is some $n_0 \in \mathbb{N}$ s.t. for all $n \geq n_0$ we have $n \in A$ i.e. the set A includes all of the integers $n_0, n_0+1, n_0+2, \ldots$

Proposition 2.4. If a state i has $f_{ii} > 0$ and is aperiodic, then there is $n_0(i) \in \mathbb{N}$ such that $p_{ii}^{(n)} > 0$ for all $n \geq n_0(i)$

Proof. Let $A = \{n \geq 1 : p_{ii}^{(n)} > 0\}$. Since $f_{ii} > 0$, then A is not empty.
If $m, n \in A$, then

$$p_{ii}^{(m+n)} \geq p_{ii}^{(m)} p_{ii}^{(n)} > 0$$

So $m+n \in A$, which shows that A satisfies additivity. Also $gcd(A) = 1$ since the state i is aperiodic. Hence from the Number Theory Lemma, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $n \in A$ i.e. $p_{ii}^{(n)} > 0$.

■

Corollary 2.6. If a chain is irreducible and aperiodic, then for any states $i, j \in S$, there is $n_0(i,j) \in \mathbb{N}$ s.t. $p_{ij}^{(n)} > 0$ for all $n \geq n_0(i,j)$

Proof. Find $n_0(i)$ as in Proposition 2.3, and find $m \in \mathbb{N}$ with $p_{ij}^{(m)} > 0$.
Then let $n_0(i,j) = n_0(i) + m$
Then if $n \geq n_0(i,j)$, then $n - m \geq n_0(i)$, so $p_{ij}^{(n)} \geq p_{ii}^{(n-m)} p_{ij}^{(m)} > 0$. ∎

Lemma 2.6 (Markov Forgetting Lemma)**.** If a Markov chain is irreducible and aperiodic, and has stationary distribution $\{\pi_i\}$, then for all $i, j, k \in S$,

$$\lim_{n\to\infty} \left| p_{ik}^{(n)} - p_{jk}^{(n)} \right| = 0$$

Remark 2.4. Intuitively, after a long time n, the chain "forgets" whether it started from state i or from state j.

Proof. ∎ long

Proof of Markov Chain Convergence Theorem long

Corollary 2.7. If a chain is irreducible, then it has at most one stationary distribution.

Proof. By Markov Chain Convergence Theorem, any stationary distribution that ie has must be equal to $\lim_{n\to\infty} P(X_n = j)$, so it is unique. ∎

Definition 2.5 (convergence in distribution)**.**

$$\forall a < b, \underset{n\to\infty}{P}(a < X_n < b) = P(a < X < b)$$

Definition 2.6 (weak convergence)**.**

$$\forall \epsilon > 0, \lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

Remark 2.5. This is "converge in probability".

Definition 2.7 (strong convergence)**.**

$$P(\lim_{n\to\infty} X_n = X) = 1$$

Remark 2.6. This is "converge almost surely".

Remark 2.7. Strong convergence implies weak convergence.

Proposition 2.5. If $\{X_n\}$ is a simple symmetric random walk, then the absolute values $|X_n|$ converge weakly to