MAT337
Lecture Notes

Yuchen Wang

March 26, 2020

Contents

1 Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

If we make natural numbers $\mathbb{N}$ closed under subtraction, we obtain

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$$

If we take the closure of $\mathbb{Z}$ under division by non-zero numbers, we obtain

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1\}$$

Remark 1.1. $(m, n) = 1$ means that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$.

Theorem 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for contradiction that there are $m \in \mathbb{Z}.n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m, n) = 1$.
Then $m^2 = 2n^2$ so that m^2 is an even complete square.
Suppose $m = p_1 \ldots p_r$ where p_is are prime numbers. Then $2n^2 = m^2 = p_1^2 \ldots p_r^2 \implies p_i^2 = 2^2$.
Then $4|m^2$ and $2|n^2$, so n has to be even. Therefore both m and n are even.
Then $2|m$ and $2|n$, which leads to a contradiction that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$. ∎

1.2 Preliminaries

Definition 1.1 (set)**.** A <u>set</u> is any collection of objects.

Definition 1.2 (function)**.** Given two sets A and B, a <u>function</u> from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $(f : A \to B)$. It is the set of pairs $(A, B) \in A \times B$ s.t.

1. If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.
2. For all $x \in A$, there is some $y \in B$ s.t. $f(x) = y$.

The set A is said to be the <u>domain</u> of f. The <u>range</u> of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

Example 1.1 (absolute value function)**.** For every x,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem 1.2 (triangle inequality)**.**

$$|x + y| \leq |x| + |y|$$

Proof.

$$
\begin{aligned}
(x+y)^2 &= x^2 + y^2 + 2xy \\
&\leq |x|^2 + |y|^2 + 2|x||y| \\
&= (|x|+|y|)^2 \\
\implies |x+y| &= \sqrt{(x+y)^2} \\
&\leq \sqrt{(|x|+|y|)^2} \\
&= ||x|+|y|| \\
&= |x|+|y|
\end{aligned}
$$

■

Definition 1.3 (maximum and minimum)**.** Assume set $X \subseteq \mathbb{R}$. Then the maximum (minimum) of X is an element $a \in X$ s.t. for all $x \in X, x \leq a (x \geq a)$.

Definition 1.4 (least upper bound / supremum)**.** The least upper bound of X (denoted by $\sup(X)$) is a real number $a \in \mathbb{R}$ s.t.

1. For all $x \in X, x \leq a$ (this means that a is an upper bound for X)
2. If b is an upper bound for X, then $a \leq b$

Example 1.2.

$$
\begin{aligned}
\max([0,1]) &= 1 \\
\min([0,1]) &= 0 \\
\sup((0,1)) &= 1 \\
\sup(\mathbb{R}), \sup(\mathbb{N})\, &DNE
\end{aligned}
$$

1.3 The axiom of completeness

Definition 1.5 (initial segment)**.** $X \subseteq \mathbb{Q}$ is said to be an initial segment if

1. $X \neq \emptyset$
2. For all $x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$.
3. $X \neq \mathbb{Q}$

Alternative definition: Let $(A, \leq)$ be a well-ordered set. Then the set

$$\{a \in A : a < k\}$$

for some $k \in A$ is called an initial segment of A.

Definition 1.6 (real numbers)**.** $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$

Lemma 1.1 (supremum)**.** Suppose $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A. If $\forall \epsilon > 0, \exists a \in A, a + \epsilon > s$, then $s = \sup(A)$

Proof. ($\Longleftarrow$) Assume for contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$.
Let $\epsilon = \frac{s-t}{2}$. Obviously $\epsilon > 0$.
But then $\forall a \in A, a + \epsilon \leq t + \epsilon < s$, which is a contradiction.
($\Longrightarrow$) Assume for contradiction that $\epsilon_0 > 0$ and $\forall a \in A, a + \epsilon \leq S$)
Then $\forall a \in A, a \leq S - \epsilon_0$.
So $s - \epsilon_0$ is an upper bound for A, which is a contradiction that $a + \epsilon > s$. ■

Theorem 1.3 (the Axiom of Completeness)**.** If $X \subset \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let Ax be the initial segment of $\mathbb{Q}$ corresponding to x.
Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \underset{x \in X}{\cup} Ax$. Note that $\underset{x \in X}{\cup} Ax$ is an initial segment of $\mathbb{Q}$. Then $\sup(\underset{x \in X}{\cup} Ax)$ is $\sup(X)$. ■

1.4 Consequences of Completeness

Definition 1.7 (nested sequence of sets)**.** Assume $\langle A_n : n \in \mathbb{N} \rangle$ is a sequence of sets.
$\langle A_n : n \in \mathbb{N} \rangle$ is said to be <u>nested</u> if

$$A_{n+1} \subseteq A_n$$

Theorem 1.4 (Nested Interval Property)**.** Assume $\langle I_n : n \in \mathbb{N} \rangle$ is a nested sequence of closed intervals of $\mathbb{R}$. Then

$$\underset{n}{\cap} I_n \neq \emptyset$$

Proof. Let $[a_n, b_n] = I_n$ where $a_n, b_n \in \mathbb{R}$.
Since $\langle I_n | n \in \mathbb{N} \rangle$ is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad (\dagger)$$

for all $n \in \mathbb{N}$
Let $A = \{a_n : n \in \mathbb{N}\}$.
Note that b_1 is an upper bound for A. So A has a supremum in $\mathbb{R}$.
We claim that $\sup(A) \in \underset{n}{\cap} I_n$.
By ($\dagger$), for all $n \in \mathbb{N}, \sup(A) \leq b_n$
Obviously, for all $n \in \mathbb{N}, \sup(A) \geq a_n$
So $\forall n \in \mathbb{N}, a_n \leq \sup(A) \leq b_n$.
Therefore $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$. ■

Example 1.3.

$$\underset{n \in \mathbb{N}}{\cap} (0, \frac{1}{n}) = \emptyset$$

$$\underset{n \in \mathbb{N}}{\cap} [0, \frac{1}{n}] = \{0\}$$

Theorem 1.5 (Archimedian Property)**.** We have

1. For every $y \in \mathbb{R}$, there is $n \in \mathbb{N}$ s.t. $y \leq n$.

2. For every $y > 0$, there is $n \in \mathbb{N} s.t. \frac{1}{n} < y$.

Proof. (1) Assume for contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$.
Let $\alpha = \sup(\mathbb{N})$. Then there is a natural number $n \in \mathbb{N}$ s.t. $n > \alpha - 1$.
But then $n + 1 > (\alpha - 1) + 1 = \alpha$, which is a natural number greater than α, contradiction.
(2) Exercise. ■

Theorem 1.6 (density of $\mathbb{Q}$ in $\mathbb{R}$)**.** For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Proof. Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a, 1 < nb - na$.
Let $m \in \mathbb{Z}$ s.t. $na < m < nb$.
Then $a < \frac{m}{n} < b$.
Pick $r = \frac{m}{n}$ and we are done. ■

1.5 Cardinality

"The size of a set"

1.5.1 1-1 Correspondence

Definition 1.8 (one-to-one and onto)**.** A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Proposition 1.1. If $f : A \to B$ and $g : B \to C$ is 1-1, then $g \circ f : A \to C$ is 1-1.

Remark 1.2. If a function $f : A \to B$ is both 1-1 and onto, then there is a 1-1 correspondence between two sets.

Definition 1.9 (the same cardinality)**.** The set A has the same cardinality as B if there exists $f : A \to B$ that is $1 - 1$ and onto. In this case, we write $A \sim B$.

Proposition 1.2. If $A \sim B, B \sim C$, then $A \sim C$

Proposition 1.3. If $Card(A) \leq Card(B) \leq Card(C)$, then $Card(A) \leq Card(C)$

1.5.2 Countable Sets

A set A is countable if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.

Theorem 1.7. The set $\mathcal{Q}$ is countable.

Proof. Set $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with} p + q = n\}$$

e.g. $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}, A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$

$$\begin{array}{cccccccccccccc}
\mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbf{Q}: & 0 & \frac{1}{1} & -\frac{1}{1} & \frac{1}{2} & -\frac{1}{2} & \frac{2}{1} & -\frac{2}{1} & \frac{1}{3} & -\frac{1}{3} & \frac{3}{1} & -\frac{3}{1} & \frac{1}{4} & \cdots
\end{array}$$

$$\underbrace{0}_{A_1}\ \underbrace{\tfrac{1}{1}\ \ -\tfrac{1}{1}}_{A_2}\ \underbrace{\tfrac{1}{2}\ \ -\tfrac{1}{2}\ \ \tfrac{2}{1}\ \ -\tfrac{2}{1}}_{A_3}\ \underbrace{\tfrac{1}{3}\ \ -\tfrac{1}{3}\ \ \tfrac{3}{1}\ \ -\tfrac{3}{1}}_{A_4}\ \tfrac{1}{4}\ \cdots$$

The above correspondence is onto because every rational number appears in the correspondence exactly once. The above correspondence is 1-1 because A_N were constructed to be disjoint so that no rational number appears twice. ■

Theorem 1.8. The set $\mathbb{R}$ is uncountable.

Proof. Assume for contradiction that there does exist a bijection function $f : \mathbb{N} \to \mathbb{R}$.
Let $x_1 = f(1), x_2 = f(2)$ and so on. Then since f is onto, can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\} \tag{1}$$

and be confident that every real number appears somewhere on the list.
We will now use the Nested Interval Property to produce a real number that is not there. Let I_1 be a closed interval that does not contain x_1. given an interval I_n, construct I_{n+1} to satisfy $I_{n+1} \subseteq I_n$ and $x_{n+1} \notin I_{n+1}$.
If x_{n_0} is some real number from the list in (1), then we have $x_{n_0} \notin I_{n_0}$, and it follows that

$$x_{n_0} \notin \cap_{n=1}^{\infty} I_n$$

Since we are assuming that the list in (1) contains every real number, then

$$\cap_{n=1}^{\infty} I_n = \emptyset$$

However, the NIP asserts that $\cap_{n=1}^{\infty} I_n \neq \emptyset$, which is a contradiction. ■

Theorem 1.9. If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.10. We have
(i) If $A_1, A_2, \ldots, A_m$ are countable sets, then the union $A_1 \cup A_2 \cup \ldots \cup A_m$ is countable.
(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is countable.

Theorem 1.11. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

1.6 Cantor's Theorem

Notation 1.1. Given a set A, the power set $P(A)$ refers to the collection of all subsets of A.

Theorem 1.12 (Cantor's Theorem)**.** Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

Proof. Assume, for contradiction, that $f : A \to P(A)$ is onto. For each element $a \in A$, $f(a)$ is a particular subset of A. The assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$. To arrive at a contradiction, we will produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.
Construct B using the following rule. For each element $a \in A$, consider the subset $f(a)$. This subset of A may contain the element a or it may not. This depends on the function f. If $f(a)$ does not contain a, then we include a in our set B: Let

$$B = \{a \in A : a \notin f(a)\}$$

Since we have assumed that our function $f : A \to P(A)$ is onto, it must be that $B = f(a')$ for some $a' \in A$.
Case 1 $a' \in B$
Then $a' \notin f(a') = B$, a contradiction.
Case 2 $a' \notin B$
Then $a' \in f(a') = B$, a contradiction. ■

Theorem 1.13 (Schröder-Bernstein Theorem)**.** If there are 1-1 functions $f : A \to B$ and $h : B \to A$, then there is a bijection $g : A \to B$.

Proof. **Claim:** the statement of the theorem is equivalent to the following:
If $B \subseteq A$ and $f : A \to B$ is $1-1$, then there is a bijection $g : A \to B$. (*)

proof of claim: theorem $\implies$ (*):
Take $h : X \to Y$ with $h(x) = x$, then $X \subseteq Y$.
(*) $\implies$ theorem:
Let $f : A \to B$ and $h : B \to A$ be 1-1 functions, as in the theorem. We need to show that there is bijection $g : A \to B$.
Notice that $A \subseteq h(B)$ and $h \circ f : A \to h(B)$ is a 1-1 function. So by (*), there is a bijection $g_0 : A \to h(B)$.
But $h : B \to h(B)$ is also a bijection. So $g = h^{-1} \circ g_0 : A \to B$ is a bijection (using the fact that bijections are closed under compositions).

Now it suffices to prove (*).
Assume set $X \subseteq Y$ and $f : Y \to X$. Let $W = \bigcup_{n=0}^{\infty} f^n(Y \setminus X)$.
Define $g : Y \to X$ by:

- If $y \in W$, then $g(y) = f(y)$
- If $y \in Z := Y \setminus W$, then $g(y) = y$

We need to show that $g : Y \to X$ is a well-defined bijection.
Since f is 1-1, for all $m < n$, $f^m(Y \setminus X) \cap f^n(Y \setminus X) = \emptyset$

Note that

$$\begin{aligned} Y \setminus W &= Y \setminus \bigcup_{n=0}^{\infty} f^n(Y \setminus X) \\ &= [Y \setminus (Y \setminus X)] \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \\ &= X \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \end{aligned}$$

Therefore for all $y \in Y, g(y) \in X$.
(Show g is 1-1) Now assume $y_1, y_2 \in Y$ and $g(y_1) = g(y_2)$. We show that $y_1 = y_2$.
Case 1 $y_1, y_2 \in W$
Then $g(y_1) = g(y_2) \implies f(y_1) = f(y_2) \implies y_1 = y_2$.
Case 2 $y_1 \in W$ but $y_2 \in Y \setminus W$
Then $g(y_1) = g(y_2) \implies f(y_1) = y_2$
Note that if $y_1 \in W$, then for some $n \geq 0, y_1 \in f^n(Y \setminus X)$
Then $y_2 \in f^{n+1}(Y \setminus X) \subseteq W$
So $y_2 \in W$, which leads to a contradiction.
Case 3 y_1, y_2 are both in $Z := Y \setminus W$
Then $g(y_1) = g(y_2) \implies y_1 = y_2$.
Therefore by case 1,2,3, g is 1-1.
(Show g is onto) Let $x \in X$. We need to find $y \in Y$ s.t. $g(y) = X$.
If $x \in Z$, take $y = x$.
If $x \in \bigcup_{n=1}^{\infty} f^n(Y \setminus X)$, then fix $n \in \mathbb{N}$ s.t. $x \in f^n(Y \setminus X)$.
But $f^n(Y \setminus X) = f(f^{n-1}(Y \setminus X))$
Pick $y \in f^{n-1}(Y \setminus X)$ s.t. $f(y) = x$.
Then $y \in W$ and $g(y) = x$. Therefore g is onto. ∎

2 Sequences and Series

2.1 The Limit of a Sequence

Definition 2.1 (sequence)**.** A <u>sequence</u> is a function whose domain is $\mathbb{N}$.

Definition 2.2. Let (X, d) be a metric space. A sequence $(X_n) \subseteq X$ <u>converges</u> to an element $x \in X$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies d(x_n, x) < \epsilon$.

Key property: If $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$, then $x = y$.

Proof. WTS $d(x, y) = 0$
Let $\epsilon > 0$. We will show that $d(x, y) < \epsilon$.
Since $\lim_{n\to\infty} x_n = x$, then $\exists N_1, \forall n \geq N_1, d(x_n, x) < \frac{\epsilon}{2}$
Since $\lim_{n\to\infty} x_n = y$, then $\exists N_2, \forall n \geq N_2, d(x_n, y) < \frac{\epsilon}{2}$
Take $n \geq \max(N_1, N_2)$, then $d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. ∎

Proposition 2.1. Suppose (X, d) is a metric space, (X, τ) is a topological space, and $F \subseteq X$. If $\lim_{n\to\infty} x_n = x$, $(x_n) \subseteq F$ and F is closed, then $x \in F$.

Proof. Suppose $x \notin F$, i.e., $x \in X \setminus F$.
Since F is closed, then $X \setminus F$ is open, so there is $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq X \setminus F$.
Let N be such that $\forall n \geq N, d(x_n, x) < \epsilon$.
Then $x_n \in B_\epsilon(x)$, which implies that $(x_n) \subseteq X \setminus F$, a contradiction. ■

Proposition 2.2. Suppose (X, d) is a metric space and $F \subseteq X$. If F is not closed, then there exists $(x_n) \subseteq F$ and $x \notin F$ s.t. $\lim_{n\to\infty} x_n = x$.

Proof. If F is not closed, then $X \setminus F$ is not open, so there is $x \in X \setminus F$ s.t. $B_\epsilon(x) \not\subseteq X \setminus F$ for all $\epsilon > 0$.
Take $x_n \in B_{1/n}(x) \setminus (X \setminus F) = B_{1/n}(x) \cap F$ for each $n \in \mathbb{N}$, then $(x_n) \subseteq F$ and $\lim_{n\to\infty} x_n = x$. ■

Definition 2.3 (Cauchy sequence)**.** A sequence (x_n) in a metric space (x_n) in a metric space (X, d) is a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}, m, n \geq N \implies d(x_m, x_n) < \epsilon$.

Proposition 2.3. A convergent sequence is Cauchy.

Proof. Let (x_n) be a convergent sequence, so that $\lim_{n\to\infty} x_n = x$. To check (x_n) is Cauchy, let $\epsilon > 0$. We need to find N s.t. $\forall m, n \geq N, d(x_n, x_m) < \epsilon$.
Apply $\lim_{n\to\infty} x_n = x$ to $\frac{\epsilon}{2}$, we get N s.t. $\forall n \geq N, d(x, x_n) < \frac{\epsilon}{2}$.
Notice that N works for Cauchy:
Take $m, n \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Remark 2.1. When $X = \mathbb{R}$ with the usual metric, A Cauchy sequence is convergent (the converse is true).
In general not true. For example, $X = \mathbb{R} \setminus \{0\}, d(x, y) = |x - y|, (x_n) = \frac{1}{n}$.
■

Definition 2.4 (monotone sequence)**.** $(x_n) \subseteq \mathbb{R}$ is monotone if either $x_n \leq x_m, n \leq m$, or $x_n \geq x_m, n \leq m$.

Theorem 2.1 (Monotone Subsequence Theorem)**.** Every sequence $(x_n) \subseteq \mathbb{R}$ has a monotone subsequence.

prove this

Fact 2.1. If $a_n \leq b_n$ for all n, $a = \lim_{n\to\infty} a_n, b = \lim_{n\to\infty} b_n$, then

$$a \leq b$$

Proof. Suppose for contradiction that $a > b$. Let $\epsilon = \frac{a-b}{2}$.
We know $\exists N_1$ s.t. $a_n \in B_\epsilon(a)$ for $n \geq N_1$ and $\exists N_2$ s.t. $b_n \in B_\epsilon(b)$ for $n \geq N_2$. Take $n > \max(N_1, N_2)$, then we have

$$b_n < \frac{a+b}{2} < a_n$$

which is a contradiction. ■

Theorem 2.2 (Algebraic limit theorem)**.** Suppose $a = \lim_{n\to\infty} a_n, b = \lim_{n\to\infty} b_n$, then:

1. $a + b = \lim_{n\to\infty}(a_n + b_n)$
2. $ab = \lim_{n\to\infty} a_n b_n$
3. $\frac{a}{b} = \lim_{n\to\infty} \frac{a_n}{b_n}$, and $b \neq 0$.

Fact 2.2. Monotone bounded sequence (x_n) converges to its supremum or infimum.

Proof. We only prove the supremum case.
Fix $\epsilon > 0$, let $s = \sup\{x_n : n \in \mathbb{N}\}$. We have $s - \epsilon < s$ and thus $s - \epsilon$ is not an upper bound of (x_n). Therefore, there is N s.t. $x_N > s - \epsilon$.
Take $n \geq N$, then we have

$$x_n \geq x_N > s - \epsilon$$

Therefore, we have $|x_n - s| < \epsilon$. ■

Definition 2.5 (limit supremum)**.** We define

$$\limsup_{n\to\infty} x_n = \inf\{y_m : m \in \mathbb{N}\}$$

where $y_m = \sup\{x_n : n \geq m\}$.
Alternatively,

$$\limsup_{n\to\infty} x_n = \lim_{m\to\infty} \sup_{n\geq m} x_n$$

Definition 2.6 (limit infimum)**.**

$$\liminf_{n\to\infty} x_n = \sup\{z_m : m \in \mathbb{N}\}$$

where $z_m = \inf\{x_n : n \geq m\}$.
Alternatively,

$$\liminf_{n\to\infty} x_n = \lim_{m\to\infty} \inf_{n\geq m} x_n$$

2.2 Series

Definition 2.7. We define

$$S_n = \sum_{k=1}^{n} a_k, \quad \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$$

We call $\sum_{k=1}^{\infty} a_k$ a <u>summable series</u> if the limit exists, i.e.,

$$\exists A, \forall \epsilon > 0, \exists N s.t. \forall n \geq N, |S_n - A| < \epsilon$$

Property 2.1 (Cauchy criterion for series)**.** $\sum_{k=1}^{\infty}$ is <u>summable</u> iff

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, |S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

Corollary 2.1. If $\sum_{k=1}^{\infty} a_k$ is summable, then $|a_k| \to 0$.

Proof. We have $|a_k| = |s_k - s_{k-1}| < \epsilon$ for $k > N$. ∎

Example 2.1. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is summable.

Proof.

$$\begin{aligned}
S_m &= 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{m^2} \\
&< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \ldots + \frac{1}{m(m-1)} \\
&= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{m-1} - \frac{1}{m}) \\
&= 1 + 1 - \frac{1}{m} \\
&< 2
\end{aligned}$$

∎

Example 2.2. $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k} &= (1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots \\
&= 1 + (1/2) + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \ldots \\
&= 1 + 1/2 + 1/2 + 1/2 + \ldots \\
\to \infty
\end{aligned}$$

∎

Theorem 2.3 (Algebraic limit theorem for series)**.** Suppose $\sum_{k=1}^{\infty} a_k = A, \sum_{k=1}^{\infty} b_k = B, c \in \mathbb{R}$, then

1. $\sum_{k=1}^{\infty} ca_k = cA$
2. $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

Proof. (1) We want to show $\forall \epsilon > 0, \exists N$ s.t. $\forall n \geq N, |\sum_{k=1}^{\infty} ca_k - cA| < \epsilon$.
We know $\forall \epsilon_0 > 0, \exists N_{\epsilon_0}$ s.t. $\forall n \geq N_{\epsilon_0}, |\sum_{k=1}^{\infty} a_k - A| < \epsilon_0$.
Take $\epsilon_0 = \frac{\epsilon}{|c|}$, then we have

$$\left| \sum_{k=1}^{\infty} ca_k - cA \right| = |c| \left| \sum_{k=1}^{\infty} a_k - A \right| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$$

∎

Property 2.2 (Order comparison test)**.** Suppose $b_k \geq a_k \geq 0, \forall k$.

1. If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.
2. If $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$.

Definition 2.8 (geometric series)**.** We call a series a geometric series if it is of the form

$$\sum_{k=1}^{\infty} ar^k$$

Note that the geometric series converges to $\frac{a}{1-r}$ whenever $r^m \to 0$ iff $|r| < 1$.

Definition 2.9 (absolutely convergence)**.** $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k| < \infty$.

Definition 2.10 (conditionally convergence)**.** $\sum_{k=1}^{\infty} a_k$ is conditionally convergent if $\sum_{k=1}^{\infty} a_k < \infty$, but $\sum_{k=1}^{\infty} |a_k| = \infty$

Example 2.3 (alternating series)**.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} < \infty$ but $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$

Property 2.3 (Absolute convergence test)**.** If $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.

Proof. We use Cauchy criterion for $\sum_{k=1}^{\infty} a_k$: we want to show

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

Let $\epsilon > 0$.
Since $\sum_{k=1}^{\infty} |a_k| < \infty$, then we know that $\exists N$ s.t. $\forall n \geq m \geq N$,

$$\left|\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|\right| < \epsilon$$

Then

$$\begin{aligned}
\left|\sum_{k=m+1}^{n} a_k\right| &= \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right| \\
&\leq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k| \\
&\leq \left|\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|\right| \\
&< \epsilon
\end{aligned}$$

■

Property 2.4 (Alternating series test)**.** Suppose $a_1 \geq a_2 \geq \ldots \geq 0$, $\lim\limits_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty}(-1)^{k+1} a_k < \infty$.

Proof. We want to show $\{S_n\} = \{\sum_{k=1}^{n}(-1)^{k+1}a_k\}$ is Cauchy:

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n \geq N, |S_n - S_m| < \epsilon$$

Let $\epsilon > 0$.
Suppose $n > m$, then $|S_n - S_m| = |a_{m+1} - a_{m+2} + \ldots + (-1)^{n-m+1}a_n|$.
Since (a_n) is a non-negative decreasing sequence, then

$$\begin{aligned} a_{m+1} - a_{m+2} + \ldots + (-1)^{n-m-1}a_n &= a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \ldots \\ &\leq a_{m+1} \end{aligned}$$

Since $\lim_{k\to\infty} a_k = 0, \exists N$ s.t. $\forall m + 1 \geq N, a_{m+1} < \epsilon$.
Thus $0 \leq |S_n - S_m| \leq a_{m+1} < \epsilon$. ■

Property 2.5 (Ratio test)**.** Given $\sum_{k=1}^{\infty} a_k$ s.t. $a_k \neq 0$ for all k.
If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, then $\sum_{k=1}^{\infty} |a_k| < \infty$

Proof. Define $S := \{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n}\right| \geq r'\}$, then S contains finitely many elements of $\mathbb{N}$. (If S were to be infinite set, if we take $\epsilon = r' - r$, then $\left|\frac{a_{n+1}}{a_n}\right| - r \geq r' - r$ for infinitely many terms which contradicts that r is the point of convergence.
Therefore, $S' = \{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n} < r'\right|$ contains all but finitely many elements of $\mathbb{N}$. Let $N = 1 + \max S$, then $\forall n \geq N, \left|\frac{a_{n+1}}{a_n} < r'\right| < r' \implies |a_{n+1}| < r'|a_n|$.
Since $0 < r' < 1, \sum_{n=1}^{\infty}(r')^n$ converges which implies $|a_N|\sum_{n=1}^{\infty}(r')^n$ converges. We have $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n| < C + |a_N|\sum_{n=N+1}^{\infty}(r')^{n-N}$ converges, by comparison test. Hence $\sum_{n=1}^{\infty}|a_n|$ converges. ■

understand the last two lines of the proof

Definition 2.11 (rearrangement)**.** Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a rearrangement of $\sum_{k=1}^{\infty} a_k$ if $\forall n, !\exists k$ s.t. $b_k = a_n$.

3 Metric Spaces and the Baire Category Theorem

3.1 Basic Definitions

Definition 3.1 (metric and metric space)**.** Given a set X, a function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y \in X$:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. for all $z \in X, d(x, y) \leq d(x, z) + d(z, y)$

A metric space is a set X together with a metric d.

Example 3.1. The set $\mathbb{R}$ considered with $d : \mathbb{R}^2 \to [0, \infty), (x, y) \mapsto |x - y|$ is a metric space.

Example 3.2. In general, $\mathbb{R}^n$ considered with the Euclidean distance is a metric space.

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Example 3.3. Let x be a set. The discrete metric d on X is defined by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Fact If (X, d) is a metric space, $d'(x, y) = \max\{1, d(x, y)\}$ for all $x, y \in X$, then (X, d') is also a metric space.

Example 3.4. Let $X = \{f : A \to \mathbb{R}\}$

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in A\}$$

if the supremum exists.

Definition 3.2. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is continuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0, d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$.

3.2 Topology on Metric Spaces

Definition 3.3 (open ball)**.** An open ball (or ϵ-neighbourhood) with radius r and center x is

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

Definition 3.4 (open set)**.** A set $U \subseteq X$ is open iff

$$\forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Example 3.5. $B_\epsilon(x)$ is open.

Proof. Fix $x \in X$ and $\epsilon > 0$. We want to show: $\forall y \in B_\epsilon(x), \exists \delta > 0$ s.t. $B_\delta(y) \subseteq B_\epsilon(x)$.
Take $y \in B_\epsilon(x)$, then $d(x, y) < \epsilon$. Take $\delta = \epsilon - d(x, y) > 0$. Take any $z \in B_\delta(y)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \epsilon - d(x, y) = \epsilon$$

Thus $z \in B_\epsilon(x)$ so $B_\delta(y) \subseteq B_\epsilon(x)$. ■

Definition 3.5 (topological space)**.** A topological space is a pair (X, τ), where X is a set and τ a subset of the power set of X which we call open such that

1. $\emptyset, X \in \tau$
2. $U_1, \ldots, U_n \in \tau \implies \bigcap_{i=1}^n U_i \in \tau$
3. $U_1, \ldots, U_n \in \tau \implies \bigcup_{i=1}^n U_i \in \tau$

Example 3.6. $(X, \{\emptyset, X\})$

Example 3.7. $(X, P(X))$ is a discrete topological space, where $P(X)$ is the power set of X.

Example 3.8. Given (X, d) a metric space, define τ_d : a set $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Then τ_d is a topology.

Proof. (1) First, $\emptyset, X \in \tau_d$ since $\forall x \in \emptyset, B_1(x) \subseteq \emptyset$ and $\forall x \in X, B_1(x) \subseteq X$.
Then suppose $U_1, \ldots, U_n \in \tau_d$.
(2) we want to show:

$$U = \bigcap_{i=1}^{n} U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Since $x \in U$, then $\forall i = 1, \ldots, n, x \in U_i : \exists \epsilon_i > 0$ s.t. $B_{\epsilon_i}(x) \subseteq U_i$.
Take $\epsilon = \min\limits_{1 \le i \le n} \epsilon_i$, thus $B_\epsilon(x) \subseteq U_i \, \forall i$. Hence $B_\epsilon(x) \subseteq U_i \subseteq U$.
(3) We also want to show:

$$\bigcup_{i=1}^{n} U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Let $x \in U$, then there is some U_i s.t. $x \in U_i$. Since $U_i \in \tau_d$, then $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U_i \subseteq U$.
Therefore, τ_d is a topology. ■

Definition 3.6. A subset F of a topological space (X, τ) is closed if $X \setminus F$ is open.

Property 3.1. Given a topological space (X, τ) and a subset F of it, we have:

1. $\emptyset, X$ are closed
2. If $F_1, \ldots, F_n$ are closed, then $\bigcup_{i=1}^n F_i$ is closed
3. If $F_1, \ldots, F_n$ are closed, then $\bigcap_{i=1}^n F_i$ is closed

Definition 3.7 (topological closure and interior)**.** Given a topological space (X, τ), where $\tau \subseteq P(X)$, and a set $F \subseteq X$, the topological closure of F is the minimal closed superset of F, i.e.,

$$\bar{F} = \bigcap \{H : H \text{ is closed}, H \supseteq F\}$$

The interior of F is the maximal open subset of F, i.e.,

$$F^\circ = \bigcap \{U : U \text{ is open}, U \subseteq F\}$$

Example 3.9. Given (X, d) a metric space, define τ_d : a set $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Suppose $F \subseteq X$, then

$$\bar{F} = \{x \in X : \forall \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} = \{\lim_{n \to \infty} x_n : (x_n) \subseteq F, \lim_{n \to \infty} x_n \text{ exists}\}$$

and

$$F^\circ = \{x \in X : \exists \epsilon > 0, B_\epsilon(x) \subseteq F\} = \bigcup \{B_\epsilon(x) : \epsilon > 0, x \in F, B_\epsilon(x) \subseteq F\}$$

3.3 Compactness and Bolzano-Weierstrass Theorem

Definition 3.8 (compactness)**.** A subset K of a metric space (X, d) is compact if every sequence in K has a convergent subsequence that converges to a limit in K.

Example 3.10. $(\mathbb{R}, |x-y|)$ is not compact (e.g. $(x_n) = n$)

Example 3.11. $([0,1], |x-y|)$ is compact.

Property 3.2. If (X, d) is compact, then it is bounded, i.e. $\exists M$ s.t. $x, y \in X, d(x, y) \leq M$.

Property 3.3. If $Y \subseteq X, (X, d)$ is a metric space, and (Y, d) is compact, then Y is closed in X.

Property 3.4. If $K_1 \supseteq K_2 \supseteq \dots$ are compact and nonempty subsets of X, then $K = \bigcap_{n=1}^{\infty} K_n$ is compact and nonempty.

Theorem 3.1 (Bolzano-Weierstrass theorem)**.** A subset Y of $\mathbb{R}$ is compact iff closed and bounded.
Alternative formation: Every bounded subsequence contains a convergent subsequence.

Remark 3.1. The theorem is true for $\mathbb{R}^n$ but is false for infinite dimension.

Theorem 3.2 (Heine-Borel Theorem)**.** Let K be a subset of a metric space (X, d). The following statements are equivalent:

1. K is compact.
2. K is closed and bounded.
3. Every open cover $K \subseteq \bigcup_{i \in I} U_i$ for K has a finite subcover $K \subseteq \bigcup_{l=1}^{n} U_{i_l}$.

3.4 Completeness of Metric Spaces

Definition 3.9 (completeness of metric spaces)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Example 3.12. $\mathbb{R}, d(x, y) = |x-y|$

Example 3.13. $(X, d), d$ discrete metric.

Example 3.14. $C[0,1], d(f, g) = \sup\limits_{x \in [0,1]} |f(x) - g(x)| = ||f-g||_{\infty}$

Example 3.15. $(\mathbb{N}^{\mathbb{N}}, d), d((x_n), (y_n)) = \frac{1}{\min\{n : x_n \neq y_n\}}$
where $\mathbb{N}^{\mathbb{N}} = \{x : \mathbb{N} \to \mathbb{N}\}$.

3.5 Perfect Sets

Definition 3.10 (perfect set)**.** Let (X, d) be a metric space. $P \subseteq X$ is perfect if it is closed, nonempty, and for every open $U \subseteq X, U \cap P$ is not empty and has at least two elements.

Example 3.16. $S = [0, 1] \cup \{\frac{3}{2}\} \cup [2, 3]$ is not perfect.

Property 3.5. Perfect subsets P of a complete metric space are not countable.

Example 3.17 (Cantor set)**.** Let C_0 be the closed interval $[0, 1]$, and define C_1 to be the set that results when the open middle third is removed; that is,

$$C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

Now construct C_2 in a similar way by removing the open middle third of each of the two components of C_1:

$$C_2 = ([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}]) \cup ([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1])$$

Continue this process inductively. For each $n = 0, 1, 2, \ldots$, we get a set C_n consisting of 2^n closed intervals each having length $(\frac{1}{3})^n$. Finally, we define the Cantor set C to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$

Remark 3.2. As follows

- Since we are always removing open middle thirds, then at each stage, endpoints are never removed. Thus, C at least contains the endpoints of all of the intervals that make up each of the sets C_n.
- The Cantor set has zero length.
- The Cantor set is uncountable, with cardinality equal to the cardinality of $\mathbb{R}$.

3.6 Separated and Connected Sets

Definition 3.11 (separated sets)**.** Let (X, d) be a metric space, $A \neq \emptyset, B \subseteq X$. A and B are separated if $\bar{A} \cap B = \bar{B} \cap A = \emptyset$.

Definition 3.12 (connected sets)**.** A set $C \subseteq X$ is connected if for every decomposition $C = A \cup B$ s.t. $A, B \neq \emptyset$, A and B are not separated, i.e. $\bar{A} \cap B \neq \emptyset$ or $\bar{B} \cap A \neq \emptyset$.

Property 3.6. $C \subseteq \mathbb{R}$ is connected iff

$$\forall a, b \in C, [a, b] \subseteq C$$

Proof. Let $C = A \cup B, a_0 \in A, b_0 \in B, a_0 < b_0$. We define $I_0 = [a_0, b_0], c_0 = \frac{a_0+b_0}{2}$. Define $I_1 = [a_0, c_0], \ldots$. We have $x \in \bar{A} \cap B$ or $\bar{B} \cap A$. ∎

Is this complete?

3.7 Baire's Theorem

Definition 3.13 (dense)**.** A set $A \subseteq X$ is dense in the metric space (X, d) if $\bar{A} = X$.

Definition 3.14 (nowhere-dense)**.** A subset E of a metric space (X, d) is nowhere-dense in X if $\bar{E}^{\circ}$ is empty.
i.e., A nowhere-dense set of a metric space is a set whose closure has empty interior.

Remark 3.3. It is a set whose elements are not tightly clustered anywhere.

Example 3.18. $\mathbb{Z}$ is nowhere-dense in $\mathbb{R}$.

Example 3.19. $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is nowhere-dense in $\mathbb{R}$.
$\bar{S} = S \cup \{0\}$, which has empty interior.

Theorem 3.3 (Baire's Theorem)**.** The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

Remark 3.4. Baire's Theorem asserts that the only way to make $\mathbb{R}$ from a countable union of arbitrary sets is for the closure of at least one of these sets to contain an interval.

3.8 The Baire Category Theorem

Theorem 3.4. Let (X, d) be a complete metric space, and let $\{O_n\}$ be a countable collection of dense, open subsets of X. Then, $\bigcap_{n=1}^{\infty}\{O_n\}$ is not empty. prove this

Theorem 3.5 (Baire Category Theorem)**.** A complete metric space cannot be written as the countable union of nowhere-dense sets. prove this

Remark 3.5. This result is called the Baire Category Theorem because it creates two categories of size for subsets in a metric space:

1. A set of "first category" is one that can be written as a countable union of nowhere-dense sets. These are the small, intuitively "thin" subsets of a metric space.

2. If our metric space is complete, then it is necessarily of "second category", meaning it cannot be written as a countable union of nowhere-dense sets.

Theorem 3.6. The set

$$D = \{f \in C[0, 1] : f'(x) \text{ exists for some } x \in [0, 1]\}$$

is a set of first category in $C[0, 1]$.

4 Functional Limits and Continuity

4.1 Functional Limits

Definition 4.1. Let $A \subseteq \mathbb{R}, a \in \overline{A \setminus \{a\}}$ (a is an accumulation point of A). Let $f : A \to \mathbb{R}$, define $\lim\limits_{x \to a} f(x) = L$ iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Property 4.1 (Sequential criterion for functional limits)**.** $a \in \overline{A \setminus \{a\}}, f : A \to \mathbb{R}$. The following are equivalent:

1. $\lim_{x \to a} f(x) = L$
2. $\forall (x_n) \subseteq A \setminus \{a\}, x_n \to a \implies f(x_n) \to L$

Proof. We prove (1) $\implies$ (2):
Assume $\lim_{x \to a} f(x) = L$, take arbitrary $(x_n) \subseteq A \setminus \{a\}$ s.t. $x_n \to a$.
Let $\epsilon > 0$, then $\exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.
Also, $\exists N$ s.t. $n \geq N \implies |x_n - a| < \delta$.
Therefore, if $|x_n - a| < \delta$, then $|f(x_n) - L| < \epsilon$. ∎

Theorem 4.1 (Algebraic Limit Theorem for functional limits)**.** Suppose $f, g : A \to \mathbb{R}, a \in \overline{A \setminus \{a\}}$.
Suppose $\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$. Then we have

1. $\lim_{x \to a} cf(x) = cL$
2. $\lim_{x \to a} (f(x) + g(x)) = L + M$
3. $\lim_{x \to a} (f(x)g(x)) = LM$
4. $\lim_{x \to a} (\frac{f(x)}{g(x)}) = \frac{L}{M}$ when $M \neq 0$.

Property 4.2 (Divergence criterion)**.** Suppose $f : A \to \mathbb{R}, a \in \overline{A \setminus \{a\}}$ $\lim_{x \to a} f(x)$ does not exist if there are two sequences $(x_n), (y_n) \subseteq A \setminus \{a\}$ s.t. $x_n \to a, y_n \to a$, $\lim_{n \to \infty} f(x_n) = L$, $\lim_{n \to \infty} f(y_n) = M$ exist but $L \neq M$.

Example 4.1. Let $A = \mathbb{R}^+, f(x) = \sin(\frac{1}{x})$. Let $a_n = \frac{1}{2n\pi}, b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$.
Then we have $a_n, b_n \to 0$. Besides, $\lim_{n \to \infty} f(a_n) = 0$, $\lim_{n \to \infty} f(b_n) = 1$. Hence $\lim_{x \to 0^+} \sin(\frac{1}{x})$ does not exist.

Definition 4.2. Suppose $f : A \to \mathbb{R}, x \in A \setminus \{a\}$. We define $\lim_{x \to a} f(x) = \infty$ iff

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$$

4.2 Continuous Functions

Definition 4.3 (continuity)**.** Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. $f : X \to Y$ is continuous at $a \in X$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } x \in B_\delta^X(a) \implies f(x) \in B_\epsilon^Y(f(a))$$

Remark 4.1. Note that for $X = Y = \mathbb{R}, d(x, y) = |x - y|$, so that we can write

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

i.e.

$$\lim_{x \to a} f(x) = f(a)$$

Definition 4.4 (continuous function)**.** $f : X \to Y$ is continuous if it is continuous at every point $a \in X$.

Property 4.3. The following are equivalent:

1. f is continuous at a
2. $\lim_{x \to a} f(x) = f(a)$
3. $\forall (x_n) \subseteq A, x_n \to a \implies f(x_n) \to f(a)$.

Corollary 4.1. f is discontinuous at a if there is a sequence $(x_n) \to a$ s.t. $\lim_{n\to\infty} f(x_n) \neq f(a)$.

Remark 4.2. Note that we may have $\lim_{x \to a} f(x)$ exists but f is discontinuous at a.

Theorem 4.2 (Algebraic Continuous Theorem)**.** Suppose $f, g : A \to \mathbb{R}$ are continuous at $a \in A, c \in \mathbb{R}$. We have

1. $cf(x)$ is continuous at a
2. $f(x) \pm g(x)$ is continuous at a
3. $f(x)g(x)$ is continuous at a
4. $\frac{f(x)}{g(x)}$ is continuous at a if $g(a) \neq 0$

Theorem 4.3. Suppose $f : A \to B \subseteq \mathbb{R}, g : B \to \mathbb{R}$.
$(g \circ f)(x) = g(f(x))$ is continuous at $a \in A$ whenever f is continuous at a and g is continuous at $f(a)$.

Theorem 4.4. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces and $f : X \to Y$ is continuous.
If $K \subseteq X$ is compact, then its image $f[K] = \{f(x) : x \in K\}$ is compact.

Theorem 4.5. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. If $F \subseteq Y$ is closed in Y, then $f^{-1}(F)$ is closed in X.

Theorem 4.6 (Extreme Value Theorem)**.** If $f : K \to \mathbb{R}$ is continuous, K is compact, then $\exists x_1, x_2 \in K$ s.t. $\forall x \in K$,

$$f(x_1) \leq f(x) \leq f(x_2)$$

Proof. Let $H = f[K] = \{f(x) : x \in K\} \subseteq \mathbb{R}$, which is compact. Since compact subsets of $\mathbb{R}$ are bounded, then let $y_2 = \sup(H)$.
We have $y \leq y_2$ for all $y \in H$ and $\forall \epsilon > 0, \exists y \in H$ s.t. $y_2 - \epsilon < y \leq y_2$.
Take $\epsilon = \frac{1}{n}$, then we have some $z_n \in H$ s.t. $y_2 - \frac{1}{n} < z_n \leq y_2$.
Now we find $a_n \in k$ s.t. $f(a_n) = z_n, n = 1, 2, \ldots$
By theorem, we have $a_{n_k} \to x_2$, then $f(x_2) = \lim_{k\to\infty} f(a_{n_k}) = y_2$. ■

Which theorem?