MAT337
Lecture Notes

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April 6, 2020

Contents

1 Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

If we make natural numbers $\mathbb{N}$ closed under subtraction, we obtain

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$$

If we take the closure of $\mathbb{Z}$ under division by non-zero numbers, we obtain

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m,n) = 1\}$$

Remark 1.1. $(m,n) = 1$ means that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$.

Theorem 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for contradiction that there are $m \in \mathbb{Z}.n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m,n) = 1$.
Then $m^2 = 2n^2$ so that m^2 is an even complete square.
Suppose $m = p_1 \dots p_r$ where p_is are prime numbers. Then $2n^2 = m^2 = p_1^2 \dots p_r^2 \implies p_i^2 = 2^2$.
Then $4|m^2$ and $2|n^2$, so n has to be even. Therefore both m and n are even.
Then $2|m$ and $2|n$, which leads to a contradiction that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$. ■

1.2 Preliminaries

Definition 1.1 (set)**.** A set is any collection of objects.

Definition 1.2 (function)**.** Given two sets A and B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $(f : A \to B)$. It is the set of pairs $(A, B) \in A \times B$ s.t.

1. If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.
2. For all $x \in A$, there is some $y \in B$ s.t. $f(x) = y$.

The set A is said to be the domain of f. The range of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

Example 1.1 (absolute value function)**.** For every x,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem 1.2 (triangle inequality)**.**

$$|x + y| \leq |x| + |y|$$

Proof.

$$\begin{aligned} (x+y)^2 &= x^2 + y^2 + 2xy \\ &\leq |x|^2 + |y|^2 + 2|x||y| \\ &= (|x|+|y|)^2 \\ \implies |x+y| &= \sqrt{(x+y)^2} \\ &\leq \sqrt{(|x|+|y|)^2} \\ &= ||x|+|y|| \\ &= |x|+|y| \end{aligned}$$

■

Definition 1.3 (maximum and minimum)**.** Assume set $X \subseteq \mathbb{R}$. Then the maximum (minimum) of X is an element $a \in X$ s.t. for all $x \in X, x \leq a (x \geq a)$.

Definition 1.4 (least upper bound / supremum)**.** The least upper bound of X (denoted by $\sup(X)$) is a real number $a \in \mathbb{R}$ s.t.

1. For all $x \in X, x \leq a$ (this means that a is an upper bound for X)
2. If b is an upper bound for X, then $a \leq b$

Example 1.2.

$$\begin{aligned} \max([0,1]) &= 1 \\ \min([0,1]) &= 0 \\ \sup((0,1)) &= 1 \\ \sup(\mathbb{R}), \sup(\mathbb{N}) \, DNE \end{aligned}$$

1.3 The axiom of completeness

Definition 1.5 (initial segment)**.** $X \subseteq \mathbb{Q}$ is said to be an initial segment if

1. $X \neq \emptyset$
2. For all $x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$.
3. $X \neq \mathbb{Q}$

Alternative definition: Let $(A, \leq)$ be a well-ordered set. Then the set

$$\{a \in A : a < k\}$$

for some $k \in A$ is called an initial segment of A.

Definition 1.6 (real numbers)**.** $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$

Lemma 1.1 (supremum)**.** Suppose $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A. If $\forall \epsilon > 0, \exists a \in A, a + \epsilon > s$, then $s = \sup(A)$

Proof. ($\Longleftarrow$) Assume for contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$.
Let $\epsilon = \frac{s-t}{2}$. Obviously $\epsilon > 0$.
But then $\forall a \in A, a + \epsilon \leq t + \epsilon < s$, which is a contradiction.
($\Longrightarrow$) Assume for contradiction that $\epsilon_0 > 0$ and $\forall a \in A, a + \epsilon \leq S)$
Then $\forall a \in A, a \leq S - \epsilon_0$.
So $s - \epsilon_0$ is an upper bound for A, which is a contradiction that $a + \epsilon > s$. ■

Theorem 1.3 (the Axiom of Completeness)**.** If $X \subset \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let Ax be the initial segment of $\mathbb{Q}$ corresponding to x.
Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \bigcup_{x \in X} Ax$. Note that $\bigcup_{x \in X} Ax$ is an initial segment of $\mathbb{Q}$. Then $\sup(\bigcup_{x \in X} Ax)$ is $\sup(X)$. ■

1.4 Consequences of Completeness

Definition 1.7 (nested sequence of sets)**.** Assume $\langle A_n : n \in \mathbb{N} \rangle$ is a sequence of sets.
$\langle A_n : n \in \mathbb{N} \rangle$ is said to be nested if

$$A_{n+1} \subseteq A_n$$

Theorem 1.4 (Nested Interval Property)**.** Assume $\langle I_n : n \in \mathbb{N} \rangle$ is a nested sequence of closed intervals of $\mathbb{R}$. Then

$$\bigcap_n I_n \neq \emptyset$$

Proof. Let $[a_n, b_n] = I_n$ where $a_n, b_n \in \mathbb{R}$.
Since $\langle I_n | n \in \mathbb{N} \rangle$ is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad (\dagger)$$

for all $n \in \mathbb{N}$
Let $A = \{a_n : n \in \mathbb{N}\}$.
Note that b_1 is an upper bound for A. So A has a supremum in $\mathbb{R}$.
We claim that $\sup(A) \in \bigcap_n I_n$.
By ($\dagger$), for all $n \in \mathbb{N}, \sup(A) \leq b_n$
Obviously, for all $n \in \mathbb{N}, \sup(A) \geq a_n$
So $\forall n \in \mathbb{N}, a_n \leq \sup(A) \leq b_n$.
Therefore $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$. ■

Example 1.3.

$$\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$$

$$\bigcap_{n \in \mathbb{N}} [0, \frac{1}{n}] = \{0\}$$

Theorem 1.5 (Archimedian Property)**.** We have

1. For every $y \in \mathbb{R}$, there is $n \in \mathbb{N}$ s.t. $y \leq n$.

2. For every $y > 0$, there is $n \in \mathbb{N} s.t. \frac{1}{n} < y$.

Proof. (1) Assume for contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$.
Let $\alpha = \sup(\mathbb{N})$. Then there is a natural number $n \in \mathbb{N}$ s.t. $n > \alpha - 1$.
But then $n + 1 > (\alpha - 1) + 1 = \alpha$, which is a natural number greater than α, contradiction.
(2) Exercise. ■

Theorem 1.6 (density of $\mathbb{Q}$ in $\mathbb{R}$)**.** For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Proof. Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a, 1 < nb - na$.
Let $m \in \mathbb{Z}$ s.t. $na < m < nb$.
Then $a < \frac{m}{n} < b$.
Pick $r = \frac{m}{n}$ and we are done. ■

1.5 Cardinality

"The size of a set"

1.5.1 1-1 Correspondence

Definition 1.8 (one-to-one and onto)**.** A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Proposition 1.1. If $f : A \to B$ and $g : B \to C$ is 1-1, then $g \circ f : A \to C$ is 1-1.

Remark 1.2. If a function $f : A \to B$ is both 1-1 and onto, then there is a 1-1 correspondence between two sets.

Definition 1.9 (the same cardinality)**.** The set A has the same cardinality as B if there exists $f : A \to B$ that is $1 - 1$ and onto. In this case, we write $A \sim B$.

Proposition 1.2. If $A \sim B, B \sim C$, then $A \sim C$

Proposition 1.3. If $Card(A) \leq Card(B) \leq Card(C)$, then $Card(A) \leq Card(C)$

1.5.2 Countable Sets

A set A is countable if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.

Theorem 1.7. The set $\mathcal{Q}$ is countable.

Proof. Set $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \{\pm \frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with} p + q = n\}$$

e.g. $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}, A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$

$$\begin{array}{rcccccccccccccc}
\mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbf{Q}: & \underbrace{0}_{A_1} & \multicolumn{2}{c}{\underbrace{\tfrac{1}{1} \quad -\tfrac{1}{1}}_{A_2}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{2} \quad -\tfrac{1}{2} \quad \tfrac{2}{1} \quad -\tfrac{2}{1}}_{A_3}} & \multicolumn{4}{c}{\underbrace{\tfrac{1}{3} \quad -\tfrac{1}{3} \quad \tfrac{3}{1} \quad -\tfrac{3}{1}}_{A_4}} & \tfrac{1}{4} & \cdots
\end{array}$$

The above correspondence is onto because every rational number appears in the correspondence exactly once. The above correspondence is 1-1 because A_N were constructed to be disjoint so that no rational number appears twice. ∎

Theorem 1.8. The set $\mathbb{R}$ is uncountable.

Proof. Assume for contradiction that there does exist a bijection function $f : \mathbb{N} \to \mathbb{R}$.
Let $x_1 = f(1), x_2 = f(2)$ and so on. Then since f is onto, can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\} \tag{1}$$

and be confident that every real number appears somewhere on the list.
We will now use the Nested Interval Property to produce a real number that is not there. Let I_1 be a closed interval that does not contain x_1. given an interval I_n, construct I_{n+1} to satisfy $I_{n+1} \subseteq I_n$ and $x_{n+1} \notin I_{n+1}$.
If x_{n_0} is some real number from the list in (1), then we have $x_{n_0} \notin I_{n_0}$, and it follows that

$$x_{n_0} \notin \cap_{n=1}^{\infty} I_n$$

Since we are assuming that the list in (1) contains every real number, then

$$\cap_{n=1}^{\infty} I_n = \emptyset$$

However, the NIP asserts that $\cap_{n=1}^{\infty} I_n \neq \emptyset$, which is a contradiction. ∎

Theorem 1.9. If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.10. We have
(i) If $A_1, A_2, \ldots, A_m$ are countable sets, then the union $A_1 \cup A_2 \cup \ldots \cup A_m$ is countable.
(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is countable.

Theorem 1.11. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

1.6 Cantor's Theorem

Notation 1.1. Given a set A, the power set $P(A)$ refers to the collection of all subsets of A.

Theorem 1.12 (Cantor's Theorem)**.** Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

Proof. Assume, for contradiction, that $f : A \to P(A)$ is onto. For each element $a \in A$, $f(a)$ is a particular subset of A. The assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$. To arrive at a contradiction, we will produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.
Construct B using the following rule. For each element $a \in A$, consider the subset $f(a)$. This subset of A may contain the element a or it may not. This depends on the function f. If $f(a)$ does not contain a, then we include a in our set B: Let

$$B = \{a \in A : a \notin f(a)\}$$

Since we have assumed that our function $f : A \to P(A)$ is onto, it must be that $B = f(a')$ for some $a' \in A$.
Case 1 $a' \in B$
Then $a' \notin f(a') = B$, a contradiction.
Case 2 $a' \notin B$
Then $a' \in f(a') = B$, a contradiction. ■

Theorem 1.13 (Schröder-Bernstein Theorem)**.** If there are 1-1 functions $f : A \to B$ and $h : B \to A$, then there is a bijection $g : A \to B$.

Proof. **Claim:** the statement of the theorem is equivalent to the following:
If $B \subseteq A$ and $f : A \to B$ is $1 - 1$, then there is a bijection $g : A \to B$. (*)

proof of claim: theorem $\implies$ (*):
Take $h : X \to Y$ with $h(x) = x$, then $X \subseteq Y$.
(*) $\implies$ theorem:
Let $f : A \to B$ and $h : B \to A$ be 1-1 functions, as in the theorem. We need to show that there is bijection $g : A \to B$.
Notice that $A \subseteq h(B)$ and $h \circ f : A \to h(B)$ is a 1-1 function. So by (*), there is a bijection $g_0 : A \to h(B)$.
But $h : B \to h(B)$ is also a bijection. So $g = h^{-1} \circ g_0 : A \to B$ is a bijection (using the fact that bijections are closed under compositions).

Now it suffices to prove (*).
Assume set $X \subseteq Y$ and $f : Y \to X$. Let $W = \bigcup_{n=0}^{\infty} f^n(Y \setminus X)$.
Define $g : Y \to X$ by:

- If $y \in W$, then $g(y) = f(y)$
- If $y \in Z := Y \setminus W$, then $g(y) = y$

We need to show that $g : Y \to X$ is a well-defined bijection.
Since f is 1-1, for all $m < n$, $f^m(Y \setminus X) \cap f^n(Y \setminus X) = \emptyset$

Note that

$$\begin{aligned} Y \setminus W &= Y \setminus \bigcup_{n=0}^{\infty} f^n(Y \setminus X) \\ &= [Y \setminus (Y \setminus X)] \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \\ &= X \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \end{aligned}$$

Therefore for all $y \in Y, g(y) \in X$.
(Show g is 1-1) Now assume $y_1, y_2 \in Y$ and $g(y_1) = g(y_2)$. We show that $y_1 = y_2$.
Case 1 $y_1, y_2 \in W$
Then $g(y_1) = g(y_2) \implies f(y_1) = f(y_2) \implies y_1 = y_2$.
Case 2 $y_1 \in W$ but $y_2 \in Y \setminus W$
Then $g(y_1) = g(y_2) \implies f(y_1) = y_2$
Note that if $y_1 \in W$, then for some $n \geq 0, y_1 \in f^n(Y \setminus X)$
Then $y_2 \in f^{n+1}(Y \setminus X) \subseteq W$
So $y_2 \in W$, which leads to a contradiction.
Case 3 y_1, y_2 are both in $Z := Y \setminus W$
Then $g(y_1) = g(y_2) \implies y_1 = y_2$.
Therefore by case 1,2,3, g is 1-1.
(Show g is onto) Let $x \in X$. We need to find $y \in Y$ s.t. $g(y) = X$.
If $x \in Z$, take $y = x$.
If $x \in \bigcup_{n=1}^{\infty} f^n(Y \setminus X)$, then fix $n \in \mathbb{N}$ s.t. $x \in f^n(Y \setminus X)$.
But $f^n(Y \setminus X) = f(f^{n-1}(Y \setminus X))$
Pick $y \in f^{n-1}(Y \setminus X)$ s.t. $f(y) = x$.
Then $y \in W$ and $g(y) = x$. Therefore g is onto. ∎

2 Sequences and Series

2.1 The Limit of a Sequence

Definition 2.1 (sequence)**.** A sequence is a function whose domain is $\mathbb{N}$.

Definition 2.2. Let (X, d) be a metric space. A sequence $(X_n) \subseteq X$ converges to an element $x \in X$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies d(x_n, x) < \epsilon$.

Key property: If $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} x_n = y$, then $x = y$.

Proof. WTS $d(x, y) = 0$
Let $\epsilon > 0$. We will show that $d(x, y) < \epsilon$.
Since $\lim\limits_{n\to\infty} x_n = x$, then $\exists N_1, \forall n \geq N_1, d(x_n, x) < \frac{\epsilon}{2}$
Since $\lim\limits_{n\to\infty} x_n = y$, then $\exists N_2, \forall n \geq N_2, d(x_n, y) < \frac{\epsilon}{2}$
Take $n \geq \max(N_1, N_2)$, then $d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. ∎

Proposition 2.1. Suppose (X, d) is a metric space, (X, τ) is a topological space, and $F \subseteq X$. If $\lim\limits_{n\to\infty} x_n = x$, $(x_n) \subseteq F$ and F is closed, then $x \in F$.

Proof. Suppose $x \notin F$, i.e., $x \in X \setminus F$.
Since F is closed, then $X \setminus F$ is open, so there is $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq X \setminus F$.
Let N be such that $\forall n \geq N, d(x_n, x) < \epsilon$.
Then $x_n \in B_\epsilon(x)$, which implies that $(x_n) \subseteq X \setminus F$, a contradiction. ∎

Proposition 2.2. Suppose (X, d) is a metric space and $F \subseteq X$. If F is not closed, then there exists $(x_n) \subseteq F$ and $x \notin F$ s.t. $\lim\limits_{n\to\infty} x_n = x$.

Proof. If F is not closed, then $X \setminus F$ is not open, so there is $x \in X \setminus F$ s.t. $B_\epsilon(x) \not\subseteq X \setminus F$ for all $\epsilon > 0$.
Take $x_n \in B_{1/n}(x) \setminus (X \setminus F) = B_{1/n}(x) \cap F$ for each $n \in \mathbb{N}$, then $(x_n) \subseteq F$ and $\lim\limits_{n\to\infty} x_n = x$.
∎

Definition 2.3 (Cauchy sequence)**.** A sequence (x_n) in a metric space (x_n) in a metric space (X, d) is a <u>Cauchy sequence</u> if $\forall \epsilon > 0, \exists N \in \mathbb{N}, m, n \geq N \implies d(x_m, x_n) < \epsilon$.

Proposition 2.3. A convergent sequence is Cauchy.

Proof. Let (x_n) be a convergent sequence, so that $\lim\limits_{n\to\infty} x_n = x$. To check (x_n) is Cauchy, let $\epsilon > 0$. We need to find N s.t. $\forall m, n \geq N, d(x_n, x_m) < \epsilon$.
Apply $\lim\limits_{n\to\infty} x_n = x$ to $\frac{\epsilon}{2}$, we get N s.t. $\forall n \geq N, d(x, x_n) < \frac{\epsilon}{2}$.
Notice that N works for Cauchy:
Take $m, n \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Remark 2.1. When $X = \mathbb{R}$ with the usual metric, A Cauchy sequence is convergent (the converse is true).
In general not true. For example, $X = \mathbb{R} \setminus \{0\}, d(x, y) = |x - y|, (x_n) = \frac{1}{n}$.
∎

Definition 2.4 (monotone sequence)**.** $(x_n) \subseteq \mathbb{R}$ is <u>monotone</u> if either $x_n \leq x_m, n \leq m$, or $x_n \geq x_m, n \leq m$.

Theorem 2.1 (Monotone Subsequence Theorem)**.** Every sequence $(x_n) \subseteq \mathbb{R}$ has a monotone subsequence.

prove this

Fact 2.1. If $a_n \leq b_n$ for all n, $a = \lim\limits_{n\to\infty} a_n, b = \lim\limits_{n\to\infty} b_n$, then

$$a \leq b$$

Proof. Suppose for contradiction that $a > b$. Let $\epsilon = \frac{a-b}{2}$.
We know $\exists N_1$ s.t. $a_n \in B_\epsilon(a)$ for $n \geq N_1$ and $\exists N_2$ s.t. $b_n \in B_\epsilon(b)$ for $n \geq N_2$. Take $n > \max(N_1, N_2)$, then we have

$$b_n < \frac{a+b}{2} < a_n$$

which is a contradiction. ∎

Theorem 2.2 (Algebraic limit theorem)**.** Suppose $a = \lim_{n\to\infty} a_n, b = \lim_{n\to\infty} b_n$, then:

1. $a + b = \lim_{n\to\infty}(a_n + b_n)$
2. $ab = \lim_{n\to\infty} a_n b_n$
3. $\frac{a}{b} = \lim_{n\to\infty}\frac{a_n}{b_n}$, and $b \neq 0$.

Fact 2.2. Monotone bounded sequence (x_n) converges to its supremum or infimum.

Proof. We only prove the supremum case.
Fix $\epsilon > 0$, let $s = \sup\{x_n : n \in \mathbb{N}\}$. We have $s - \epsilon < s$ and thus $s - \epsilon$ is not an upper bound of (x_n). Therefore, there is N s.t. $x_N > s - \epsilon$.
Take $n \geq N$, then we have

$$x_n \geq x_N > s - \epsilon$$

Therefore, we have $|x_n - s| < \epsilon$. ■

Definition 2.5 (limit supremum)**.** We define

$$\limsup_{n\to\infty} x_n = \inf\{y_m : m \in \mathbb{N}\}$$

where $y_m = \sup\{x_n : n \geq m\}$.
Alternatively,

$$\limsup_{n\to\infty} x_n = \lim_{m\to\infty}\sup_{n\geq m} x_n$$

Definition 2.6 (limit infimum)**.**

$$\liminf_{n\to\infty} x_n = \sup\{z_m : m \in \mathbb{N}\}$$

where $z_m = \inf\{x_n : n \geq m\}$.
Alternatively,

$$\liminf_{n\to\infty} x_n = \lim_{m\to\infty}\inf_{n\geq m} x_n$$

2.2 Series

Definition 2.7. We define

$$S_n = \sum_{k=1}^{n} a_k, \quad \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k$$

We call $\sum_{k=1}^{\infty} a_k$ a <u>summable series</u> if the limit exists, i.e.,

$$\exists A, \forall\epsilon > 0, \exists N s.t. \forall n \geq N, |S_n - A| < \epsilon$$

Property 2.1 (Cauchy criterion for series)**.** $\sum_{k=1}^{\infty}$ is <u>summable</u> iff

$$\forall\epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, |S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

Corollary 2.1. If $\sum_{k=1}^{\infty} a_k$ is summable, then $|a_k| \to 0$.

Proof. We have $|a_k| = |s_k - s_{k-1}| < \epsilon$ for $k > N$. ■

Example 2.1. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is summable.

Proof.

$$\begin{aligned}
S_m &= 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{m^2} \\
&< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \ldots + \frac{1}{m(m-1)} \\
&= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{m-1} - \frac{1}{m}) \\
&= 1 + 1 - \frac{1}{m} \\
&< 2
\end{aligned}$$

■

Example 2.2. $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

Proof. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k} &= (1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \ldots \\
&= 1 + (1/2) + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \ldots \\
&= 1 + 1/2 + 1/2 + 1/2 + \ldots \\
\to \infty
\end{aligned}$$

■

Theorem 2.3 (Algebraic limit theorem for series)**.** Suppose $\sum_{k=1}^{\infty} a_k = A, \sum_{k=1}^{\infty} b_k = B, c \in \mathbb{R}$, then

1. $\sum_{k=1}^{\infty} ca_k = cA$
2. $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

Proof. (1) We want to show $\forall \epsilon > 0, \exists N$ s.t. $\forall n \geq N, |\sum_{k=1}^{\infty} ca_k - cA| < \epsilon$.
We know $\forall \epsilon_0 > 0, \exists N_{\epsilon_0}$ s.t. $\forall n \geq N_{\epsilon_0}, |\sum_{k=1}^{\infty} a_k - A| < \epsilon_0$.
Take $\epsilon_0 = \frac{\epsilon}{|c|}$, then we have

$$\left| \sum_{k=1}^{\infty} ca_k - cA \right| = |c| \left| \sum_{k=1}^{\infty} a_k - A \right| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$$

■

Property 2.2 (Order comparison test)**.** Suppose $b_k \geq a_k \geq 0, \forall k$.

1. If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.
2. If $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$.

Definition 2.8 (geometric series)**.** We call a series a geometric series if it is of the form

$$\sum_{k=1}^{\infty} ar^k$$

Note that the geometric series converges to $\frac{a}{1-r}$ whenever $r^m \to 0$ iff $|r| < 1$.

Definition 2.9 (absolutely convergence)**.** $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k| < \infty$.

Definition 2.10 (conditionally convergence)**.** $\sum_{k=1}^{\infty} a_k$ is conditionally convergent if $\sum_{k=1}^{\infty} a_k < \infty$, but $\sum_{k=1}^{\infty} |a_k| = \infty$

Example 2.3 (alternating series)**.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} < \infty$ but $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$

Property 2.3 (Absolute convergence test)**.** If $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.

Proof. We use Cauchy criterion for $\sum_{k=1}^{\infty} a_k$: we want to show

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

Let $\epsilon > 0$.
Since $\sum_{k=1}^{\infty} |a_k| < \infty$, then we know that $\exists N$ s.t. $\forall n \geq m \geq N$,

$$\left|\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|\right| < \epsilon$$

Then

$$\begin{aligned}\left|\sum_{k=m+1}^{n} a_k\right| &= \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right| \\ &\leq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k| \\ &\leq \left|\sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|\right| \\ &< \epsilon\end{aligned}$$

■

Property 2.4 (Alternating series test)**.** Suppose $a_1 \geq a_2 \geq \ldots \geq 0$, $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k < \infty$.

Proof. We want to show $\{S_n\} = \{\sum_{k=1}^{n}(-1)^{k+1}a_k\}$ is Cauchy:

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n \geq N, |S_n - S_m| < \epsilon$$

Let $\epsilon > 0$.
Suppose $n > m$, then $|S_n - S_m| = |a_{m+1} - a_{m+2} + \ldots + (-1)^{n-m+1}a_n|$.
Since (a_n) is a non-negative decreasing sequence, then

$$\begin{aligned} a_{m+1} - a_{m+2} + \ldots + (-1)^{n-m-1}a_n &= a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \ldots \\ &\leq a_{m+1} \end{aligned}$$

Since $\lim\limits_{k\to\infty} a_k = 0, \exists N$ s.t. $\forall m + 1 \geq N, a_{m+1} < \epsilon$.
Thus $0 \leq |S_n - S_m| \leq a_{m+1} < \epsilon$.

■

Property 2.5 (Ratio test)**.** Given $\sum_{k=1}^{\infty} a_k$ s.t. $a_k \neq 0$ for all k.
If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, then $\sum_{k=1}^{\infty} |a_k| < \infty$

Proof. Define $S := \{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n}\right| \geq r'\}$, then S contains finitely many elements of $\mathbb{N}$. (If S were to be infinite set, if we take $\epsilon = r' - r$, then $\left|\frac{a_{n+1}}{a_n}\right| - r \geq r' - r$ for infinitely many terms which contradicts that r is the point of convergence.
Therefore, $S' = \{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n} < r'\right|$ contains all but finitely many elements of $\mathbb{N}$. Let $N = 1 + \max S$, then $\forall n \geq N, \left|\frac{a_{n+1}}{a_n} < r'\right| < r' \implies |a_{n+1}| < r'|a_n|$.
Since $0 < r' < 1, \sum_{n=1}^{\infty}(r')^n$ converges which implies $|a_N|\sum_{n=1}^{\infty}(r')^n$ converges. We have $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n| < C + |a_N|\sum_{n=N+1}^{\infty}(r')^{n-N}$ converges, by comparison test. Hence $\sum_{n=1}^{\infty}|a_n|$ converges. ■

understand the last two lines of the proof

Definition 2.11 (rearrangement)**.** Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a rearrangement of $\sum_{k=1}^{\infty} a_k$ if $\forall n, !\exists k$ s.t. $b_k = a_n$.

3 Metric Spaces and the Baire Category Theorem

3.1 Basic Definitions

Definition 3.1 (metric and metric space)**.** Given a set X, a function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y \in X$:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. for all $z \in X, d(x, y) \leq d(x, z) + d(z, y)$

A metric space is a set X together with a metric d.

Example 3.1. The set $\mathbb{R}$ considered with $d : \mathbb{R}^2 \to [0, \infty), (x, y) \mapsto |x - y|$ is a metric space.

Example 3.2. In general, $\mathbb{R}^n$ considered with the Euclidean distance is a metric space.

$$d(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

Example 3.3. Let x be a set. The discrete metric d on X is defined by

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Fact If (X,d) is a metric space, $d'(x,y) = \max\{1, d(x,y)\}$ for all $x, y \in X$, then (X,d') is also a metric space.

Example 3.4. Let $X = \{f : A \to \mathbb{R}\}$

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in A\}$$

if the supremum exists.

Definition 3.2. Let (X,d_1) and (Y,d_2) be metric spaces. A function $f : X \to Y$ is continuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0, d_1(x,y) < \delta \implies d_2(f(x), f(y)) < \epsilon$.

3.2 Topology on Metric Spaces

Definition 3.3 (open ball)**.** An open ball (or ϵ-neighbourhood) with radius r and center x is

$$B_r(x) = \{y \in X : d(x,y) < r\}$$

Definition 3.4 (open set)**.** A set $U \subseteq X$ is open iff

$$\forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Example 3.5. $B_\epsilon(x)$ is open.

Proof. Fix $x \in X$ and $\epsilon > 0$. We want to show: $\forall y \in B_\epsilon(x), \exists \delta > 0$ s.t. $B_\delta(y) \subseteq B_\epsilon(x)$.
Take $y \in B_\epsilon(x)$, then $d(x,y) < \epsilon$. Take $\delta = \epsilon - d(x,y) > 0$. Take any $z \in B_\delta(y)$, we have

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \epsilon - d(x,y) = \epsilon$$

Thus $z \in B_\epsilon(x)$ so $B_\delta(y) \subseteq B_\epsilon(x)$. ■

Definition 3.5 (topological space)**.** A topological space is a pair (X,τ), where X is a set and τ a subset of the power set of X which we call open such that

1. $\emptyset, X \in \tau$
2. $U_1, \ldots, U_n \in \tau \implies \bigcap_{i=1}^{n} U_i \in \tau$
3. $U_1, \ldots, U_n \in \tau \implies \bigcup_{i=1}^{n} U_i \in \tau$

Example 3.6. $(X, \{\emptyset, X\})$

Example 3.7. $(X, P(X))$ is a discrete topological space, where $P(X)$ is the power set of X.

Example 3.8. Given (X, d) a metric space, define τ_d : a set $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Then τ_d is a topology.

Proof. (1) First, $\emptyset, X \in \tau_d$ since $\forall x \in \emptyset, B_1(x) \subseteq \emptyset$ and $\forall x \in X, B_1(x) \subseteq X$.
Then suppose $U_1, \ldots, U_n \in \tau_d$.
(2) we want to show:

$$U = \bigcap_{i=1}^{n} U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Since $x \in U$, then $\forall i = 1, \ldots, n, x \in U_i : \exists \epsilon_i > 0$ s.t. $B_{\epsilon_i}(x) \subseteq U_i$.
Take $\epsilon = \min_{1 \le i \le n} \epsilon_i$, thus $B_\epsilon(x) \subseteq U_i \, \forall i$. Hence $B_\epsilon(x) \subseteq U_i \subseteq U$.
(3) We also want to show:

$$\bigcup_{i=1}^{n} U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Let $x \in U$, then there is some U_i s.t. $x \in U_i$. Since $U_i \in \tau_d$, then $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U_i \subseteq U$.
Therefore, τ_d is a topology. ■

Definition 3.6. A subset F of a topological space (X, τ) is closed if $X \setminus F$ is open.

Property 3.1. Given a topological space (X, τ) and a subset F of it, we have:

1. $\emptyset, X$ are closed
2. If $F_1, \ldots, F_n$ are closed, then $\bigcup_{i=1}^n F_i$ is closed
3. If $F_1, \ldots, F_n$ are closed, then $\bigcap_{i=1}^n F_i$ is closed

Definition 3.7 (topological closure and interior)**.** Given a topological space (X, τ), where $\tau \subseteq P(X)$, and a set $F \subseteq X$, the topological closure of F is the minimal closed superset of F, i.e.,

$$\bar{F} = \bigcap \{H : H \text{ is closed}, H \supseteq F\}$$

The interior of F is the maximal open subset of F, i.e.,

$$F^\circ = \bigcap \{U : U \text{ is open}, U \subseteq F\}$$

Example 3.9. Given (X, d) a metric space, define τ_d : a set $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$. Suppose $F \subseteq X$, then

$$\bar{F} = \{x \in X : \forall \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} = \{\lim_{n\to\infty} x_n : (x_n) \subseteq F, \lim_{n\to\infty} x_n \text{ exists}\}$$

and

$$F^\circ = \{x \in X : \exists \epsilon > 0, B_\epsilon(x) \subseteq F\} = \bigcup \{B_\epsilon(x) : \epsilon > 0, x \in F, B_\epsilon(x) \subseteq F\}$$

3.3 Compactness and Bolzano-Weierstrass Theorem

Definition 3.8 (compactness)**.** A subset K of a metric space (X, d) is compact if every sequence in K has a convergent subsequence that converges to a limit in K.

Example 3.10. $(\mathbb{R}, |x - y|)$ is not compact (e.g. $(x_n) = n$)

Example 3.11. $([0, 1], |x - y|)$ is compact.

Property 3.2. If (X, d) is compact, then it is bounded, i.e. $\exists M$ s.t. $x, y \in X, d(x, y) \leq M$.

Property 3.3. If $Y \subseteq X, (X, d)$ is a metric space, and (Y, d) is compact, then Y is closed in X.

Property 3.4. If $K_1 \supseteq K_2 \supseteq \ldots$ are compact and nonempty subsets of X, then $K = \bigcap_{n=1}^{\infty} K_n$ is compact and nonempty.

Theorem 3.1 (Bolzano-Weierstrass theorem)**.** A subset Y of $\mathbb{R}$ is compact iff closed and bounded.
Alternative formation: Every bounded subsequence contains a convergent subsequence.

Remark 3.1. The theorem is true for $\mathbb{R}^n$ but is false for infinite dimension.

Theorem 3.2 (Heine-Borel Theorem)**.** Let K be a subset of a metric space (X, d). The following statements are equivalent:

1. K is compact.
2. K is closed and bounded.
3. Every open cover $K \subseteq \bigcup_{i \in I} U_i$ for K has a finite subcover $K \subseteq \bigcup_{l=1}^{n} U_{i_l}$.

3.4 Completeness of Metric Spaces

Definition 3.9 (completeness of metric spaces)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Example 3.12. $\mathbb{R}, d(x, y) = |x - y|$

Example 3.13. $(X, d), d$ discrete metric.

Example 3.14. $C[0, 1], d(f, g) = \sup\limits_{x \in [0,1]} |f(x) - g(x)| = ||f - g||_{\infty}$

Example 3.15. $(\mathbb{N}^{\mathbb{N}}, d), d((x_n), (y_n)) = \frac{1}{\min\{n : x_n \neq y_n\}}$
where $\mathbb{N}^{\mathbb{N}} = \{x : \mathbb{N} \to \mathbb{N}\}$.

3.5 Perfect Sets

Definition 3.10 (perfect set)**.** Let (X, d) be a metric space. $P \subseteq X$ is perfect if it is closed, nonempty, and for every open $U \subseteq X, U \cap P$ is not empty and has at least two elements.

Example 3.16. $S = [0, 1] \cup \{\frac{3}{2}\} \cup [2, 3]$ is not perfect.

Property 3.5. Perfect subsets P of a complete metric space are not countable.

Example 3.17 (Cantor set)**.** Let C_0 be the closed interval $[0, 1]$, and define C_1 to be the set that results when the open middle third is removed; that is,

$$C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

Now construct C_2 in a similar way by removing the open middle third of each of the two components of C_1:

$$C_2 = ([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}]) \cup ([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1])$$

Continue this process inductively. For each $n = 0, 1, 2, \ldots$, we get a set C_n consisting of 2^n closed intervals each having length $(\frac{1}{3})^n$. Finally, we define the Cantor set C to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$

Remark 3.2. As follows

- Since we are always removing open middle thirds, then at each stage, endpoints are never removed. Thus, C at least contains the endpoints of all of the intervals that make up each of the sets C_n.
- The Cantor set has zero length.
- The Cantor set is uncountable, with cardinality equal to the cardinality of $\mathbb{R}$.

3.6 Separated and Connected Sets

Definition 3.11 (separated sets)**.** Let (X, d) be a metric space, $A \neq \emptyset, B \subseteq X$. A and B are separated if $\bar{A} \cap B = \bar{B} \cap A = \emptyset$.

Definition 3.12 (connected sets)**.** A set $C \subseteq X$ is connected if for every decomposition $C = A \cup B$ s.t. $A, B \neq \emptyset$, A and B are not separated, i.e. $\bar{A} \cap B \neq \emptyset$ or $\bar{B} \cap A \neq \emptyset$.

Property 3.6. $C \subseteq \mathbb{R}$ is connected iff

$$\forall a, b \in C, [a, b] \subseteq C$$

Proof. Let $C = A \cup B, a_0 \in A, b_0 \in B, a_0 < b_0$. We define $I_0 = [a_0, b_0], c_0 = \frac{a_0+b_0}{2}$. Define $I_1 = [a_0, c_0], \ldots$. We have $x \in \bar{A} \cap B$ or $\bar{B} \cap A$. ■

Is this complete?

3.7 Baire's Theorem

Definition 3.13 (dense)**.** A set $A \subseteq X$ is dense in the metric space (X, d) if $\bar{A} = X$.

Definition 3.14 (nowhere-dense)**.** A subset E of a metric space (X, d) is nowhere-dense in X if $\bar{E}^\circ$ is empty.
i.e., A nowhere-dense set of a metric space is a set whose closure has empty interior.

Remark 3.3. It is a set whose elements are not tightly clustered anywhere.

Example 3.18. $\mathbb{Z}$ is nowhere-dense in $\mathbb{R}$.

Example 3.19. $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is nowhere-dense in $\mathbb{R}$.
$\bar{S} = S \cup \{0\}$, which has empty interior.

Theorem 3.3 (Baire's Theorem)**.** The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

Remark 3.4. Baire's Theorem asserts that the only way to make $\mathbb{R}$ from a countable union of arbitrary sets is for the closure of at least one of these sets to contain an interval.

3.8 The Baire Category Theorem

Theorem 3.4. Let (X, d) be a complete metric space, and let $\{O_n\}$ be a countable collection of dense, open subsets of X. Then, $\bigcap_{n=1}^{\infty}\{O_n\}$ is not empty.

prove this

Theorem 3.5 (Baire Category Theorem)**.** A complete metric space cannot be written as the countable union of nowhere-dense sets.

prove this

Remark 3.5. This result is called the Baire Category Theorem because it creates two categories of size for subsets in a metric space:

1. A set of "first category" is one that can be written as a countable union of nowhere-dense sets. These are the small, intuitively "thin" subsets of a metric space.

2. If our metric space is complete, then it is necessarily of "second category", meaning it cannot be written as a countable union of nowhere-dense sets.

Theorem 3.6. The set

$$D = \{f \in C[0,1] : f'(x) \text{ exists for some } x \in [0,1]\}$$

is a set of first category in $C[0,1]$.

4 Functional Limits and Continuity

4.1 Functional Limits

Definition 4.1. Let $A \subseteq \mathbb{R}, a \in \overline{A \setminus \{a\}}$ (a is an accumulation point of A). Let $f : A \to \mathbb{R}$, define $\lim\limits_{x \to a} f(x) = L$ iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Property 4.1 (Sequential criterion for functional limits)**.** $a \in \overline{A \setminus \{a\}}, f : A \to \mathbb{R}$. The following are equivalent:

1. $\lim_{x \to a} f(x) = L$
2. $\forall (x_n) \subseteq A \setminus \{a\}, x_n \to a \implies f(x_n) \to L$

Proof. We prove (1) $\implies$ (2):
Assume $\lim_{x \to a} f(x) = L$, take arbitrary $(x_n) \subseteq A \setminus \{a\}$ s.t. $x_n \to a$.
Let $\epsilon > 0$, then $\exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.
Also, $\exists N$ s.t. $n \geq N \implies |x_n - a| < \delta$.
Therefore, if $|x_n - a| < \delta$, then $|f(x_n) - L| < \epsilon$. ∎

Theorem 4.1 (Algebraic Limit Theorem for functional limits)**.** Suppose $f, g : A \to \mathbb{R}, a \in \overline{A \setminus \{a\}}$.
Suppose $\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$. Then we have

1. $\lim_{x \to a} cf(x) = cL$
2. $\lim_{x \to a} (f(x) + g(x)) = L + M$
3. $\lim_{x \to a} (f(x)g(x)) = LM$
4. $\lim_{x \to a} (\frac{f(x)}{g(x)}) = \frac{L}{M}$ when $M \neq 0$.

Property 4.2 (Divergence criterion)**.** Suppose $f : A \to \mathbb{R}, a \in \overline{A \setminus \{a\}}$ $\lim_{x \to a} f(x)$ does not exist if there are two sequences $(x_n), (y_n) \subseteq A \setminus \{a\}$ s.t. $x_n \to a, y_n \to a$, $\lim_{n \to \infty} f(x_n) = L$, $\lim_{n \to \infty} f(y_n) = M$ exist but $L \neq M$.

Example 4.1. Let $A = \mathbb{R}^+, f(x) = \sin(\frac{1}{x})$. Let $a_n = \frac{1}{2n\pi}, b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$.
Then we have $a_n, b_n \to 0$. Besides, $\lim_{n \to \infty} f(a_n) = 0, \lim_{n \to \infty} f(b_n) = 1$. Hence $\lim_{x \to 0^+} \sin(\frac{1}{x})$ does not exist.

Definition 4.2. Suppose $f : A \to \mathbb{R}, x \in A \setminus \{a\}$. We define $\lim_{x \to a} f(x) = \infty$ iff

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$$

4.2 Continuous Functions

Definition 4.3 (continuity)**.** Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. $f : X \to Y$ is continuous at $a \in X$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } x \in B_\delta^X(a) \implies f(x) \in B_\epsilon^Y(f(a))$$

Remark 4.1. Note that for $X = Y = \mathbb{R}, d(x, y) = |x - y|$, so that we can write

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

i.e.

$$\lim_{x \to a} f(x) = f(a)$$

Definition 4.4 (continuous function)**.** $f : X \to Y$ is continuous if it is continuous at every point $a \in X$.

Property 4.3. The following are equivalent:

1. f is continuous at a
2. $\lim_{x \to a} f(x) = f(a)$
3. $\forall (x_n) \subseteq A, x_n \to a \implies f(x_n) \to f(a)$.

Corollary 4.1. f is discontinuous at a if there is a sequence $(x_n) \to a$ s.t. $\lim_{n \to \infty} f(x_n) \neq f(a)$.

Remark 4.2. Note that we may have $\lim_{x \to a} f(x)$ exists but f is discontinuous at a.

Theorem 4.2 (Algebraic Continuity Theorem)**.** Suppose $f, g : A \to \mathbb{R}$ are continuous at $a \in A, c \in \mathbb{R}$. We have

1. $cf(x)$ is continuous at a
2. $f(x) \pm g(x)$ is continuous at a
3. $f(x)g(x)$ is continuous at a
4. $\frac{f(x)}{g(x)}$ is continuous at a if $g(a) \neq 0$

Theorem 4.3. Suppose $f : A \to B \subseteq \mathbb{R}, g : B \to \mathbb{R}$.
$(g \circ f)(x) = g(f(x))$ is continuous at $a \in A$ whenever f is continuous at a and g is continuous at $f(a)$.

Theorem 4.4. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces and $f : X \to Y$ is continuous.
If $K \subseteq X$ is compact, then its image $f[K] = \{f(x) : x \in K\}$ is compact.

Theorem 4.5. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. If $F \subseteq Y$ is closed in Y, then $f^{-1}(F)$ is closed in X.

Theorem 4.6 (Extreme Value Theorem)**.** If $f : K \to \mathbb{R}$ is continuous, K is compact, then $\exists x_1, x_2 \in K$ s.t. $\forall x \in K$,

$$f(x_1) \leq f(x) \leq f(x_2)$$

Proof. Let $H = f[K] = \{f(x) : x \in K\} \subseteq \mathbb{R}$, which is compact. Since compact subsets of $\mathbb{R}$ are bounded, then let $y_2 = \sup(H)$.
We have $y \leq y_2$ for all $y \in H$ and $\forall \epsilon > 0, \exists y \in H$ s.t. $y_2 - \epsilon < y \leq y_2$.
Take $\epsilon = \frac{1}{n}$, then we have some $z_n \in H$ s.t. $y_2 - \frac{1}{n} < z_n \leq y_2$.
as Now we find $a_n \in k$ s.t. $f(a_n) = z_n, n = 1, 2, \ldots$
By theorem, we have $a_{n_k} \to x_2$, then $f(x_2) = \lim_{k \to \infty} f(a_{n_k}) = y_2$. ■

Which theorem?

4.3 Continuous Functions on Compact Sets

4.3.1 Uniform Continuity

Definition 4.5 (uniform continuity)**.** We say function $f : A \to \mathbb{R}$ is uniformly continuous on A if

$$\forall \epsilon > 0, \exists \delta > 0, x, y \in A \wedge |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Example 4.2. $f(x) = x^2$ is not uniformly continuous.

Proof. WTS $\exists \epsilon > 0, \forall \delta > 0, \exists x, y \in \mathbb{R}$ s.t. $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.
Let $\epsilon = 1, \delta > 0$.
Choose $y = x + \frac{1}{2}\delta$, so that $|x - y| < \delta$.

$$\begin{aligned} f(y) - f(x) &= y^2 - x^2 \\ &= (x + \frac{1}{2}\delta)^2 - x^2 \\ &= x^2 + \delta x + \frac{1}{4}\delta^2 - x^2 \\ &= \delta x + \frac{1}{4}\delta^2 \end{aligned}$$

If $x > \frac{1}{\delta}$, then $f(y) - f(x) > 1$. ∎

Property 4.4 ()**.** Function $f : A \to \mathbb{R}$ fails to be uniformly continuous iff $\exists \epsilon_0 > 0, \exists (x_n), (y_n) \subseteq A$ s.t. $\lim\limits_{n\to\infty} |x_n - y_n| = 0 \wedge \forall n, |f(x_n) - f(y_n)| \geq \epsilon_0$.

Proof. ($\Leftarrow$) Obvious.
($\Rightarrow$) Assume f is not uniformly continuous.
Then $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0, \exists x_n, y_n \in \mathbb{R}$ s.t. $|x_n - y_n| < \delta$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.
Then this is true for $\delta \in \mathbb{N}$ as well.
For each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$, and pick x_n, y_n as above. Then it is obvious that $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n)| \geq \epsilon_0$. ∎

Property 4.5 (Continuous functions on compact sets are uniformly continuous)**.** Assume $f : K \to \mathbb{R}$ is continuous and K is compact, then f is uniformly continuous on K.

Proof. Assume for a contradiction that $f : K \to \mathbb{R}$ is continuous and K is compact, but f is not uniformly continuous. Then by Property 4.4, $\exists \epsilon_0 > 0, (x_n), (y_n) \subseteq K$ s.t. $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n)| \geq \epsilon_0$.
Since K is compact, then (x_n) has a subsequence (x_{n_k}) s.t. $x_{n_k} \to x \in K$.
Moreover, (y_{n_k}) has a subsequence $(y_{n_{k_m}})$ s.t. $y_{n_{k_m}} \to y \in K$.
Let $x'_m = x_{n_{k_m}}, y'_m = y_{n_{k_m}}$, then $x'_m \to x, y'_m \to y$.
Since $\lim\limits_{m\to\infty} |x'_m - y'_m| = 0$, thus $x = y$.

Then

$$\begin{aligned} &|f(x'_m) - f(y'_m)| \geq \epsilon_0 \\ \implies \lim_{m\to\infty} &|f(x'_m) - f(y'_m)| \geq \epsilon_0 \\ \implies &|f(x) - f(y)| \geq \epsilon_0 \\ \implies &0 \geq \epsilon_0 \end{aligned}$$

which is a contradiction. ■

Definition 4.6. A function $f : A \to \mathbb{R}$ is said to be Lipschitz if $\exists M \in \mathbb{N}$ s.t. $\forall x \neq y \in A$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| < M$$

Property 4.6. Lipschitz functions are uniformly continuous.

Proof. Let $f : A \to \mathbb{R}$ be Lipschitz on A. Then for every $\epsilon > 0$, take $\delta < \frac{\epsilon}{M}$.
Then if $|x - y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &< M|x - y| \\ &< M\frac{\epsilon}{M} \\ &= \epsilon \end{aligned}$$

So f is uniformly continuous. ■

Remark 4.3. The converse does not hold.

Property 4.7 (Continuous image of connected sets is connected)**.** If $f : E \to \mathbb{R}$ is continuous and E is connected, then $f(E)$ is connected.

4.4 Sets of Discontinuity

Let $f : \mathbb{R} \to \mathbb{R}, D_f = \{x \in \mathbb{R} : f \text{ is not continuous at } x\}$.

Example 4.3 ($D_f = \emptyset$)**.** f is continuous

Example 4.4 ($D_f = \mathbb{R}$)**.** $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Example 4.5. Given a countable set $A = \{a_1, \ldots\}$, define $f(a_n) := \frac{1}{n}$ and $f(x) = 0, \forall x \notin A$. Then we have $D_f = A$.

Fact 4.1. There is no $f : \mathbb{R} \to \mathbb{R}$ s.t. $D_f = \mathbb{R} \setminus \mathbb{Q}$.

Definition 4.7 (F_σ-set)**.** A subset F of $\mathbb{R}$ is a F_σ-set if $F = \bigcup_{n=1}^\infty F_n$ s.t. F_n is closed for all n.

Definition 4.8 (α-continuity)**.** Let $\alpha > 0, f : \mathbb{R} \to \mathbb{R}, a \in \mathbb{R}$. f is α-continuous at a if

$$\exists \delta > 0 \text{ s.t. } x, y \in (a-\delta, a+\delta) \implies |f(x) - f(y)| < \alpha$$

Note that f is continuous at a iff f is α-continuous at a for all $a > 0$.

Property 4.8. For every $f : \mathbb{R} \to \mathbb{R}$, the set D_f is F_σ-set of $\mathbb{R}$.

red parts

Definition 4.9. Let $f : \mathbb{R} \to \mathbb{R}$.
f is removable discontinuous if $\lim_{x\to a} f(x)$ exists but does not equal $f(a)$.
f has a jump at a if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.
If $\lim_{x\to a} f(x)$ does not exist for other reasons, we say f is essential discontinuous.

Definition 4.10 (monotonicity)**.** $f : \mathbb{R} \to \mathbb{R}$ is monotone if either $x \leq y \implies f(x) \leq f(y)$ or $x \leq y \implies f(x) \geq f(y)$.

Property 4.9. Discontinuity of a monotone function f is a jump. Moreover, D_f is countable.

5 the Derivative

5.1 Derivatives and the Intermediate Value Property

Definition 5.1 (derivative)**.** Let $f : \mathbb{R} \to \mathbb{R}, c \in \mathbb{R}$. Define the derivative of f at c:

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

If $f'(c)$ exists, we say that f is differentiable at c. If $f'(a)$ exists for all $a \in \mathbb{R}$, we say that g is differentiable on $\mathbb{R}$.

Property 5.1. If f is differentiable at c, then f is continuous at c.

Proof. We have

$$\lim_{x\to c} (f(x) - f(c)) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) = f'(c) \cdot 0 = 0$$

■

Theorem 5.1 (Algebraic Differentiability Theorem)**.** Suppose f, g are differentiable, $a, c \in \mathbb{R}$. We have

1. $(cf)'(a) = cf'(a)$
2. $(f+g)'(a) = f'(a) + g'(a)$
3. $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$
4. $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$

Theorem 5.2 (Chain Rule)**.** Let $f : A \to B, g : B \to \mathbb{R}, f(A) \subseteq B$ so that $g \circ f$ is defined. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Theorem 5.3 (Interior Extremum Theorem)**.** If f is differentiable on (a, b), f attains maximum at some $c \in (a, b)$, then $f'(c) = 0$.

Proof. We have

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

and

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

then $f'(c) = 0$. ∎

Theorem 5.4 (Darboux's Theorem)**.** If f is differentiable on $[a, b]$ and $f'(a) < \alpha < f'(b)$ or $f'(a) > \alpha > f'(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = \alpha$.

5.2 the Mean Value Theorems

Theorem 5.5 (Rolle's Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

Proof. By EVT, since f is continuous on a compact set, then f attains a maximum and a minimum. If both extremums occur at the endpoints, then f is necessarily a constant function and $f'(x) = 0$ on (a, b).
If either the maximum or minimum occurs at some point $c \in (a, b)$, then it follows from the Interior Extremum Theorem that $f'(c) = 0$. ∎

Theorem 5.6 (Mean Value Theorem)**.** If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Consider

$$d(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right]$$

We know d is continuous on $[a, b]$ and differentiable on (a, b). Also, $d(a) = d(b) = 0$.
By Rolle's Theorem, $\exists c \in (a, b)$ s.t. $d'(c) = 0 \implies f'(c) = \frac{f(b)-f(a)}{b-a}$. ∎

Corollary 5.1. If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Proof. Assume $x, y \in (a, b)$ and $x < y$. We set $c \in (x, y)$, then by Mean Value Theorem,

$$0 = f'(c) = \frac{f(y) - f(x)}{y - x} \implies f(y) - f(x) = 0$$

∎

Corollary 5.2. If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + c$ for some $c \in \mathbb{R}$.

Proof. Apply the previous corollary to the function $h(x) = f(x) - g(x)$. ■

Theorem 5.7 (Generalized Mean Value Theorem)**.** If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t.

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If g' is never zero on (a, b), then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Apply the Mean Value Theorem to the function $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. ■

Theorem 5.8 (L'Hospital's Rule: 0/0 case)**.** Suppose f, g are continuous on I with $a \in I$ and are differentiable on $I \setminus \{a\}$. If $f(a) = g(a) = 0$ and $\forall x \neq a, g'(x) \neq 0$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

Proof. Since $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$, then for all $\epsilon > 0, \exists \delta > 0$ s.t.

$$x \in (a - \delta, a + \delta) \implies \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon$$

By the Generalized Mean Value Theorem, for every $y \in (a, a + \delta), \exists x \in (a, y)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(y)}{g(y)}$$

and thus

$$\left| \frac{f(y)}{g(y)} - L \right| = \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon$$

■

Theorem 5.9 (L'Hospital's Rule: ∞/∞ case)**.** Suppose f, g are differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim\limits_{x \to a} g(x) = \infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

6 Sequences and Series of Functions

6.1 Uniform Convergence of a Sequence of Functions

Definition 6.1 (pointwise convergence)**.** For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. If $\forall x \in A, f_n(x) \to f(x)$ for some function f, then sequence (f_n) of functions converges pointwise on A to f.
We can write $f_n \to f, \lim f_n = f$, or $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

Example 6.1. Consider $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \frac{x^2 + nx}{n}$$

We can compute

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + nx}{n} = \lim_{n\to\infty} \frac{x^2}{n} + x = x$$

Thus, (f_n) converges pointwise to $f(x) = x$ on $\mathbb{R}$.

Example 6.2. Consider $f_n : [0, 1] \to \mathbb{R}$

$$f_n(x) = x^n$$

If $0 \leq x < 1, x^n \to 0$. If $x = 1, x^n \to 1$. It follows that $f_n \to f$ pointwise on $[0, 1]$ where

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Note that pointwise convergent sequence of continuous functions may converge to a non-continuous function.

Definition 6.2 (uniformly convergence)**.** Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$, then (f_n) converges uniformly on A to a limit function f defined on A if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq N, \forall x \in A, |f(x) - f_n(x)| < \epsilon$$

Remark 6.1. This is a **stronger** notion of convergence.

Example 6.3. Consider $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \frac{x^2 + nx}{n}$$

which converges pointwise on $\mathbb{R}$ to $f(x) = x$. But the convergence is not uniform, since

$$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \frac{x^2}{n}$$

In order to force $|f_n(x) - f(x)| < \epsilon$, we need $N < \frac{x^2}{\epsilon}$. Although it is possible to do for each $x \in \mathbb{R}$, there is no way to choose a single value of N that will work for all values of x at the same time.
On the other hand, we can show that $f_n \to f$ uniformly on the set $[-b, b]$.

Property 6.1 (Cauchy Criterion for Uniform Convergence)**.** A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A iff

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in A, \forall m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$$

Theorem 6.1 (Continuous Limit Theorem)**.** Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof. Let $\epsilon > 0$ and fix $c \in A$. Choose N s.t.

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}, \forall x \in A$$

Since f_N is continuous, then $\exists \delta > 0$ s.t.

$$|x - c| < \delta \implies |f_N(x) - f_N(c)| < \frac{\epsilon}{3}$$

Thus,

$$\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(x)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

∎

Hence f is continuous at $c \in A$.

Property 6.2. (Algebraic Limit Theorem for Uniform Convergence) Suppose $(f_n), (g_n)$ are uniformly convergent on A, then

1. $(cf_n + g_n)$ is uniformly convergent on A
2. If $\exists M > 0$ s.t. $|f_n| \leq M$ and $|g_n| \leq M$, then $(f_n g_n)$ is uniformly convergent.

Proof. (1) Obvious.
(2) Let $\epsilon > 0$. Since $(f_n), (g_n)$ are uniformly convergent on A, then $\exists N$ s.t. $\forall m, n \geq N, |f_n(x) - f_m(x)| < \frac{\epsilon}{2M}$ and $g_n(x) - g_m(x) < \frac{\epsilon}{2M}$. Using Cauchy criterion, we have

$$\begin{aligned}
|f_m(x)g_m(x) - f_n(x)g_n(x)| &= |f_m(x)g_m(x) - f_m(x)g_n(x) + f_m(x)g_n(x) - f_n(x)g_n(x)| \\
&\leq |f_m(x)||g_m(x) - g_n(x)| + |g_n(x)||f_m(x) - f_n(x)| \\
&\leq M(|g_m(x) - g_n(x)| + |f_m(x) - f_n(x)| \\
&< M(\frac{\epsilon}{M}) \\
&= \epsilon
\end{aligned}$$

So $(f_n g_n)$ is uniformly convergent. ∎

6.2 Uniform Convergence and Differentiation

Theorem 6.2 (Differentiable Limit Theorem)**.** Let $f_n \to f$ pointwisely on $[a, b]$ and assume each f_n is differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then the function f is differentiable and $f' = g$.

Theorem 6.3. Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume (f_n') converges uniformly on $[a, b]$. If $\exists x_0 \in [a, b]$ s.t. $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$.

Theorem 6.4 (stronger form of Differentiable Limit Theorem)**.** Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume (f_n') converges uniformly on $[a, b]$ to a function g. If $\exists x_0 \in [a, b]$ s.t. $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$. Moreover, the limit function $f = \lim f_n$ is differentiable and $f' = g$.

6.3 Series of Functions

Definition 6.3 (pointwise convergence)**.** For each $n \in \mathbb{N}$, let f_n and f be functions defined on a set $A \subseteq \mathbb{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \dots$$

<u>converges pointwise</u> on A to $f(x)$ if the sequence $s_k(x)$ of partial sums defined by

$$s_k(x) = f_1(x) + f_2(x) + \ldots + f_k(x)$$

converges pointwise to $f(x)$.

Definition 6.4 (uniform convergence)**.** The series <u>converges uniformly</u> on A to f if the sequence $s_k(x)$ converges uniformly on A to $f(x)$.
In either case, we write $f = \sum_{n=1}^{\infty} f_n$ or $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

Theorem 6.5 (Term-by-term Continuity Theorem)**.** Let f_n be continuous functions defined on a set $A \subseteq \mathbb{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f. Then, f is continuous on A.

Proof. Apply the Continuous Limit Theorem 6.1 to the partial sums $s_k = f_1 + f_2 + \ldots + f_k$. ■

Theorem 6.6 (Term-by-term Differentiability Theorem)**.** Let f_n be differentiable functions defined on an interval A, and assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to a limit $g(x)$ on A. If there exists a point $x_0 \in [a, b]$ where $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{and} \quad f'(x) = g(x)$$

Proof. Apply the stronger form of the Differentiable Limit Theorem 6.4 to the partial sums $s_k = f_1 + f_2 + \ldots + f_k$. ■

Theorem 6.7 (Cauchy Criterion for Uniform Convergence of Series)**.** A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$ if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > m \geq N, \forall x \in A, |s_n - s_m| = \left| \sum_{i=m+1}^{n} f_i(x) \right| < \epsilon$$

Remark 6.2. The benefit of the Cauchy Criterion is that it does not depend on the value of the limit.

Corollary 6.1 (Weierstrass M-Test)**.** For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying that

$$\sup_{x \in A} |f_n(x)| \leq M_n$$

If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Proof. Let $\epsilon > 0$. Choose N that satisfies the Cauchy Criterion. Let $m > n \geq N$.
Then by Cauchy Criterion for Uniform Convergence of Series,

$$M_{m+1} + \ldots + M_n < \epsilon$$

Then for $n > m \geq N$ and all $x \in A$,

$$\begin{aligned} |f_{m+1}(x) + \ldots + f_n(x)| &\leq |f_{m+1}(x)| + \ldots + |f_n(x)| \\ &\leq M_{m+1} + \ldots + M_n \\ &< \epsilon \end{aligned}$$

■

Remark 6.3. The reverse is not true.

Example 6.4. If $f_n(x) = (-1)^n \frac{1}{n}$, then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent, but the M-test fails because if $M_n = \frac{1}{n}$ (the smallest M_n possible), then $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Corollary 6.2. If $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$, then the sequence (f_n) converges uniformly on A to 0.

Proof. WTS $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall x \in A, |f_n(x)| < \epsilon$.
Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} f_n$ converges uniformly, then by Cauchy Criterion,

$$\exists N \in \mathbb{N} \text{ s.t. } \forall n > m \geq N, \forall x \in A, |f_{m+1}(x) + \ldots + f_n(x)| < \epsilon$$

Let $n = m + 1$, then

$$|f_n(x)| < \epsilon$$

as wanted. ■

Corollary 6.3. Suppose $\forall n \in \mathbb{N}, \forall x \in A, g_n(x) \geq f_n(x) \geq 0$. If $\sum_{n=1}^{\infty} g_n$ converge uniformly on A, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Proof. Let $\epsilon > 0$. Apply Cauchy Criterion for $\sum_{n=1}^{\infty} g_n$, we get $N \in \mathbb{N}$ s.t. for $n > m \geq N$ and $x \in A$,

$$\begin{aligned} |f_{m+1}(x) + \ldots + f_n(x)| &= f_m(x) + \ldots + f_n(x) \\ &\leq g_{m+1}(x) + \ldots + g_n(x) \\ &= |g_{m+1}(x) + \ldots + g_n(x)| \\ &< \epsilon \end{aligned}$$

So $\sum_{n=1}^{\infty} f_n$ converges uniformly on A. ■

6.4 Power Series

Theorem 6.8. If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.

Proof. If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then $(a_n x_0^n)$ is bounded and $\to 0$.
Let $M > 0$ be s.t. $|a_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. If $x \in \mathbb{R}$ satisfies $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n$$

But $|x/x_0| < 1$, so the geometric series

$$\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$$

is convergent. By the Comparison Test, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. ■

Theorem 6.9. If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.

Proof. For $n \in \mathbb{N}$, let $M_n = |a_n| \cdot |x_0|^n$.
Note that $\sup_{x \in [-c,c]} |a_n x^n| \leq |a_n| \cdot |x_0|^n = M_n$.
Since $\sum_{n=0}^{\infty} M_n$ is convergent by assumption, then by Weierstrass M-Test 6.1, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$. ■

Remark 6.4. If the power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$ converges conditionally at $x = R$, then it is possible for it to diverge when $x = -R$.

Example 6.5.

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

Lemma 6.1 (Abel's Lemma)**.** Let b_n satisfy $b_1 \geq b_2 \geq b_3 \geq \ldots \geq 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded. In other words, assume there exists $A > 0$ such that

$$|a_1 + a_2 + \ldots + a_n| \leq A$$

for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots + a_n b_n| \leq A b_1$$

Proof.

$$\begin{aligned}
\left|\sum_{k=1}^{n} a_k b_k\right| &= \left|s_n b_{n+1} + \sum_{k=1}^{n} s_k (b_k - b_{k+1})\right| && \text{by summation-by-parts formula} \\
&\leq |s_n b_{n+1}| + \left|\sum_{k=1}^{n} s_k (b_k - b_{k+1})\right| && \text{by Triangle Inequality} \\
&\leq A b_{n+1} + \sum_{k=1}^{n} A(b_k - b_{k+1}) \\
&= A b_{n+1} + (A b_1 - A b_{n+1}) \\
&= A b_1
\end{aligned}$$

■

Theorem 6.10 (Abel's Theorem)**.** Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. A similar result holds if the series converges at $x = -R$.

Proof. To set the stage for Abel's Lemma 6.1, we first write

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n) \left(\frac{x}{R}\right)^n$$

Let $\epsilon > 0$. Since we are assuming that $\sum_{n=0}^{\infty} a_n R^n$ converges, then by the Cauchy Criterion for Uniform Convergence of Series 6.7, $\exists N \in \mathbb{N}$ s.t. if $n > m \geq N$, then

$$|a_{m+1} R^{m+11} + a_{m+2} R^{m+2} + \ldots + a_n R^n| < \epsilon$$

Now for any fixed $m \in \mathbb{N}$, we apply Abel's Lemma 6.1 to the sequence $\sum_{i=1}^{\infty} a_{m+i} R^{m+i}$. Since $x \in [0, R]$, then we have

$$\left(\frac{x}{R}\right)^{m+1} \geq \left(\frac{x}{R}\right)^{m+2} \geq \ldots \geq 0$$

Then

$$\left|(a_{m+1} R^{m+1}) \left(\frac{x}{R}\right)^{m+1} + (a_{m+2} R^{m+2}) \left(\frac{x}{R}\right)^{m+2} + \ldots + (a_n R^n) \left(\frac{x}{R}\right)^{n}\right| \leq \epsilon \left(\frac{x}{R}\right)^{m+1} \leq \epsilon$$

Therefore the series converges uniformly on the interval $[0, R]$. ■

Theorem 6.11. If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on closed intervals in $(-R, R)$. prove this

Theorem 6.12. Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on an interval $A \subseteq \mathbb{R}$. The function f is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$. The derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Moreover, f is infinitely differentiable on $(-R, R)$, and the successive derivatives can be obtained via term-by-term differentiation of the appropriate series:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)x^{n-k}$$

Corollary 6.4. If $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$ exist and equal for all $x \in (-R, R)$, then it must be the case that $a_n = b_n$ for all $n \in \mathbb{N}$.

7 The Riemann Integral

7.1 The Definition of the Riemann Integral

7.1.1 Partitions, Upper Sums, and Lower Sums

Definition 7.1 (partition)**.** A partition P of $[a, b]$ is a finite set of points from $[a, b]$ that includes both a and b. The notational convention is to always list the points of a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ in increasing order; thus

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

Definition 7.2 (lower sum and upper sum)**.** For each subinterval $[x_{k-1}, x_k]$ of P, let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The lower sum of f with respect to P is given by

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

Likewise, we define the upper sum of f with respect to P by

$$U(f, P) \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

Fact 7.1. For a particular partition P, it is clear that $U(f, P) \geq L(f, P)$.

Definition 7.3 (refinement)**.** A partition Q is a refinement of a partition P if Q contains all of the points of P; that is, if $P \subseteq Q$.

Lemma 7.1. If $P \subseteq Q$, then $L(f,P) \leq L(f,Q)$, and $U(f,P) \geq U(f,Q)$.

Proof. Consider what happens when we refine P by adding a single point z to some subinterval $[x_{k-1}, x_k]$ of P. Focusing on the lower sum, we have

$$\begin{aligned} m_k(x_k - x_{k-1}) &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m_k'(x_k - z) + m_k^{kk}(z - x_{k-1}) \end{aligned}$$

where

$$m_k' = \inf\{f(x) : x \in [z, x_k]\} \text{ and } m_k^{''} = \inf\{f(x) : x \in [x_{k-1}, z]\}$$

are each necessarily as large or larger than m_k.
By induction, we have $L(f,P) \leq L(f,Q)$, and an analogous argument holds for the upper sums. ∎

Lemma 7.2. If P_1 and P_2 are any two partitions of $[a,b]$, then $L(f,P_1) \leq U(f,P_2)$.

Proof. Let $Q = P_1 \cup P_2$. Because $P_1 \subseteq Q$ and $P_2 \subseteq Q$, it follows that

$$L(f,P_1) \leq L(f,Q) \leq U(f,Q) \leq U(f,P_2)$$

∎

7.1.2 Integrability

Definition 7.4 (upper integral and lower integral)**.** Let $\mathcal{P}$ be the collection of all possible partitions of the interval $[a,b]$. The upper integral of f is defined to be

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}$$

Similarly, we define the lower integral of f by

$$L(f) = \sup\{L(f,P) : P \in \mathcal{P}\}$$

Lemma 7.3. For any bounded function f on $[a,b]$, it is always the case that

$$U(f) \geq L(f)$$

Definition 7.5 (Riemann Integrability)**.** A bounded function f defined on the interval $[a,b]$ is Riemann-integrable if $U(f) = L(f)$. In this case, we define $\int_a^b f$ or $\int_a^b f(x)\,dx$ to be this common value; namely,

$$\int_a^b f = U(f) = L(f)$$

7.1.3 Criteria for Integrability

Theorem 7.1 (Integrability Criterion)**.** A bounded function f is integrable on $[a,b]$ if and only if, for every $\epsilon > 0, \exists$ a partition P_ϵ of $[a,b]$ such that

$$U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$$

prove this

Theorem 7.2. If f is continuous on $[a,b]$, then it is integrable.

prove this

Definition 7.6 (tagged partition)**.** A tagged partition $(P,\{c_k\})$ is one where in addition to a partition P we choose a sampling point c_k in each of the subintervals$[x_{k-1},x_k]$.

$$P = [x_0, x_1, \ldots, x_n]$$

$$c_k \in [x_{k-1}, x_k], \quad 0 < k \leq n$$

Definition 7.7 (Riemann sum)**.**

$$R(f,P,\{c_k\}) = \sum_{k=1}^{n} f(c_k)\cdot(x_k - x_{k-1})$$

Definition 7.8 (Riemann's Original Definition of the Integral)**.** A bounded function f is integrable on $[a,b]$ with $\int_a^b f = A$ if for all $\epsilon > 0, \exists\delta > 0$ such that for any tagged partition $(P,\{c_k\})$ satisfying $\delta x_k < \delta$ for all k¡ it follows that

$$|R(f,P,\{c_k\}) - A| < \epsilon$$

Remark 7.1. This definition is equivalent to our definition.

7.2 Integrating Functions with Discontinuities

Fact 7.2. Suppose two functions $f,g : [a,b] \to \mathbb{R}$ are both bounded. and f is integrable. Suppose there are finitely many points $y_1, y_2, \ldots, y_l \in [a,b]$ s.t. $f(x) = g(x)$ for $x \neq y_k$ for $k = 1, 2, \ldots, l$.
Then g is integrable and

$$\int_a^b g = \int_a^b f$$

prove this

Theorem 7.3. If $f : [a,b] \to \mathbb{R}$ is bounded, and f is integrable on $[c,b]$ for all $c \in (a,b)$, then f is integrable on $[a,b]$. An analogous result holds at the other endpoint.

prove this

Example 7.1 (Dirichlet's function)**.**

$$g(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ 0 & \text{for } x \text{ irrational} \end{cases}$$

If P is some partition of $[0,1]$, then the density of the rationals in $\mathbb{R}$ implies that every subinterval of P will contain a point where $g(x) = 1$ as well as a point where $g(y) = 0$.. It follows that $U(g,P) = 1$ and $L(g,P) = 0$. Because this is the case for every partition P, we see that $U(f) = 1, L(f) = 0$. The two are not equal, so we conclude that Dirichlet's function is not integrable.

7.3 Properties of the Integral

Theorem 7.4. Assume $f : [a,b] \to \mathbb{R}$ is bounded, and let $c \in (a,b)$. Then, f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and $[c,b]$. In this case, we have

$$\int_a^b f = \int_a^c f + \int_c^b f$$

prove this

Theorem 7.5. Assume f and g are integrable functions on the interval $[a,b]$.

1. The function $f+g$ is integrable on $[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.
2. For $k \in \mathbb{R}$, the function is kf is integrable with $\int_a^b kf = k\int_a^b f$.
3. If $m \leq f(x) \leq M$ on $[a,b]$, then $m(b-a) \leq \int_a^b f \leq M(b-a)$.
4. If $f(x) \leq g(x)$ on $[a,b]$, then $\int_a^b f \leq \int_a^b g$.
5. If $f(x) \leq g(x)$ on $[a,b]$, then $\int_a^b f \leq \int_a^b g$.
6. The function $|f|$ is integrable an $|\int_a^b f| \leq \int_a^b |f|$.

Definition 7.9. If f is integrable on the interval $[a,b]$, define

$$\int_b^a f = -\int_a^b f$$

Also for $c \in [a,b]$, define

$$\int_c^c f = 0$$

Fact 7.3. If $f : [a,b] \to \mathbb{R}$ is integrable, then

$$|f|(x) = |f(x)|$$

is also integrable, and

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

prove this

7.3.1 Uniform Convergence and Integration

Theorem 7.6 (Integrable Limit Theorem)**.** Assume that $f_n \to f$ uniformly on $[a,b]$ and that each f_n is integrable. Then, f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

prove this

7.4 The Fundamental Theorem of Calculus