STA414

Lecture Notes

Yuchen Wang

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Contents

1 Chapter 1: Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

If we make natural numbers $\mathbb{N}$ closed under subtraction, we obtain

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$$

If we take the closure of $\mathbb{Z}$ under division by non-zero numbers, we obtain

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1\}$$

Remark 1.1. $(m, n) = 1$ means that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$.

Theorem 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for contradiction that there are $m \in \mathbb{Z}.n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m, n) = 1$.
Then $m^2 = 2n^2$ so that m^2 is an even complete square.
Suppose $m = p_1 \ldots p_r$ where p_is are prime numbers. Then $2n^2 = m^2 = p_1^2 \ldots p_r^2 \implies p_i^2 = 2^2$.
Then $4|m^2$ and $2|n^2$, so n has to be even. Therefore both m and n are even.
Then $2|m$ and $2|n$, which leads to a contradiction that if $d \in \mathbb{N}$ divides both m and n, then $d = 1$. ■

1.2 Preliminaries

Definition 1.1 (set)**.** A <u>set</u> is any collection of objects.

Definition 1.2 (function)**.** Given two sets A and B, a <u>function</u> from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $(f : A \to B)$. It is the set of pairs $(A, B) \in A \times B$ s.t.

1. If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.

2. For all $x \in A$, there is some $y \in B$ s.t. $f(x) = y$.

The set A is said to be the <u>domain</u> of f. The <u>range</u> of f is not necessarily equal to B but refers to the subset of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

Example 1.1 (absolute value function)**.** For every x,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Theorem 1.2 (triangle inequality)**.**

$$|x + y| \leq |x| + |y|$$

Proof.

$$\begin{aligned}
(x+y)^2 &= x^2 + y^2 + 2xy \\
&\leq |x|^2 + |y|^2 + 2|x||y| \\
&= (|x| + |y|)^2 \\
\implies |x+y| &= \sqrt{(x+y)^2} \\
&\leq \sqrt{(|x|+|y|)^2} \\
&= ||x| + |y|| \\
&= |x| + |y|
\end{aligned}$$

■

Definition 1.3 (maximum and minimum)**.** Assume set $X \subseteq \mathbb{R}$. Then the maximum (minimum) of X is an element $a \in X$ s.t. for all $x \in X, x \leq a (x \geq a)$.

Definition 1.4 (least upper bound / supremum)**.** The least upper bound of X (denoted by $\sup(X)$) is a real number $a \in \mathbb{R}$ s.t.

1. For all $x \in X, x \leq a$ (this means that a is an upper bound for X)
2. If b is an upper bound for X, then $a \leq b$

Example 1.2.

$$\begin{aligned}
\max([0,1]) &= 1 \\
\min([0,1]) &= 0 \\
\sup((0,1)) &= 1 \\
\sup(\mathbb{R}), \sup(\mathbb{N})\ &DNE
\end{aligned}$$

1.3 The axiom of completeness

Definition 1.5 (initial segment)**.** $X \subseteq \mathbb{Q}$ is said to be an initial segment if

1. $X \neq \emptyset$
2. For all $x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$.
3. $X \neq \mathbb{Q}$

Definition 1.6 (real numbers)**.** $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$
Properties of $\mathbb{R}$:

1. $\mathbb{R}$ is an ordered field
2. ???

Lemma 1.1 (supremum)**.** Suppose $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A. If $\forall \epsilon > 0, \exists a \in A, a + \epsilon > s$, then $s = \sup(A)$

Proof. ($\Longleftarrow$) Assume for contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$.
Let $\epsilon = \frac{s-t}{2}$. Obviously $\epsilon > 0$.
But then $\forall a \in A, a + \epsilon \leq t + \epsilon < s$, which is a contradiction.
($\Longrightarrow$) Assume for contradiction that $\epsilon_0 > 0$ and $\forall a \in A, a + \epsilon \leq S$)
Then $\forall a \in A, a \leq S - \epsilon_0$.
So $s - \epsilon_0$ is an upper bound for A, which is a contradiction that $a + \epsilon > s$. ■

Theorem 1.3 (the axiom of completeness)**.** If $X \subset \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let Ax be the initial segment of $\mathbb{Q}$ corresponding to x.
Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \underset{x \in X}{\cup} Ax$. Note that $\underset{x \in X}{\cup} Ax$ is an initial segment of $\mathbb{Q}$. Then $\sup(\underset{x \in X}{\cup} Ax)$ is $\sup(X)$. ■

1.4 Consequences of Completeness

Definition 1.7 (nested sequence of sets)**.** Assume $\langle A_n : n \in \mathbb{N} \rangle$ is a sequence of sets.
$\langle A_n : n \in \mathbb{N} \rangle$ is said to be <u>nested</u> if

$$A_{n+1} \subseteq A_n$$

Theorem 1.4 (Nested Interval Property)**.** Assume $\langle I_n : n \in \mathbb{N} \rangle$ is a nested sequence of closed intervals of $\mathbb{R}$. Then

$$\underset{n}{\cap} I_n \neq \emptyset$$

Proof. Let $[a_n, b_n] = I_n$ where $a_n, b_n \in \mathbb{R}$.
Since $\langle I_n | n \in \mathbb{N} \rangle$ is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad (\dagger)$$

for all $n \in \mathbb{N}$
Let $A = \{a_n : n \in \mathbb{N}\}$.
Note that b_1 is an upper bound for A. So A has a supremum in $\mathbb{R}$.
We claim that $\sup(A) \in \underset{n}{\cap} I_n$.
By ($\dagger$), for all $n \in \mathbb{N}, \sup(A) \leq b_n$
Obviously, for all $n \in \mathbb{N}, \sup(A) \geq a_n$
So $\forall n \in \mathbb{N}, a_n \leq \sup(A) \leq b_n$.
Therefore $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$. ■

Example 1.3.

$$\underset{n \in \mathbb{N}}{\cap} (0, \frac{1}{n}) = \emptyset$$

$$\underset{n \in \mathbb{N}}{\cap} [0, \frac{1}{n}] = \{0\}$$

Theorem 1.5 (Archimedian Property)**.**

1. For every $y \in \mathbb{R}$, there is $n \in \mathbb{N}$ s.t. $y \leq n$.
2. For every $y > 0$, there is $n \in \mathbb{N} s.t. \frac{1}{n} < y$.

Proof. (1) Assume for contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$.
Let $\alpha = \sup(\mathbb{N})$. Then there is a natural number $n \in \mathbb{N}$ s.t. $n > \alpha - 1$.

But then $n+1 > (\alpha - 1) + 1 = \alpha$, which is a natural number greater than α, contradiction.
(2) Exercise. ■

Theorem 1.6 (density of $\mathbb{Q}$ in $\mathbb{R}$)**.** For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.

Proof. Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a, 1 < nb - na$.
Let $m \in \mathbb{Z}$ s.t. $na < m < nb$.
Then $a < \frac{m}{n} < b$.
Pick $r = \frac{m}{n}$ and we are done. ■

1.5 Cardinality

"The size of a set"

1.5.1 1-1 Correspondence

Definition 1.8 (one-to-one and onto)**.** A function $f : A \to B$ is one-to-one (1-1) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is onto if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.

Proposition 1.1. If $f : A \to B$ and $g : B \to C$ is 1-1, then $g \circ f : A \to C$ is 1-1.

Remark 1.2. If a function $f : A \to B$ is both 1-1 and onto, then there is a 1-1 correspondence between two sets.

Definition 1.9 (the same cardinality)**.** The set A has the same cardinality as B if there exists $f : A \to B$ that is $1-1$ and onto. In this case, we write $A \sim B$.

Proposition 1.2. If $A \sim B, B \sim C$, then $A \sim C$

Proposition 1.3. If $Card(A) \leq Card(B) \leq Card(C)$, then $Card(A) \leq Card(C)$

1.5.2 Countable Sets

A set A is countable if $\mathbb{N} \sim A$. An infinite set that is not countable is called an uncountable set.

Theorem 1.7. The set $\mathcal{Q}$ is countable.

Proof. Set $A_1 = \{0\}$ and for each $n \geq 2$, let A_n be the set given by

$$A_n = \{\pm\frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with} p + q = n\}$$

e.g. $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}, A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$

$$\begin{array}{lccccccccccccc}
\mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbf{Q}: & 0 & \frac{1}{1} & -\frac{1}{1} & \frac{1}{2} & -\frac{1}{2} & \frac{2}{1} & -\frac{2}{1} & \frac{1}{3} & -\frac{1}{3} & \frac{3}{1} & -\frac{3}{1} & \frac{1}{4} & \cdots
\end{array}$$

$$\underbrace{0}_{A_1}\ \underbrace{\tfrac{1}{1}\ \ -\tfrac{1}{1}}_{A_2}\ \underbrace{\tfrac{1}{2}\ \ -\tfrac{1}{2}\ \ \tfrac{2}{1}\ \ -\tfrac{2}{1}}_{A_3}\ \underbrace{\tfrac{1}{3}\ \ -\tfrac{1}{3}\ \ \tfrac{3}{1}\ \ -\tfrac{3}{1}}_{A_4}$$

The above correspondence is onto because every rational number appears in the correspondence exactly once. The above correspondence is 1-1 because A_N were constructed to be disjoint so that no rational number appears twice. ■

Theorem 1.8. The set $\mathbb{R}$ is uncountable.

Proof. Assume for contradiction that there does exist a bijection function $f : \mathbb{N} \to \mathbb{R}$.
Let $x_1 = f(1), x_2 = f(2)$ and so on. Then since f is onto, can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\} \tag{1}$$

and be confident that every real number appears somewhere on the list.
We will now use the Nested Interval Property to produce a real number that is not there. Let I_1 be a closed interval that does not contain x_1. given an interval I_n, construct I_{n+1} to satisfy $I_{n+1} \subseteq I_n$ and $x_{n+1} \notin I_{n+1}$.
If x_{n_0} is some real number from the list in (1), then we have $x_{n_0} \notin I_{n_0}$, and it follows that

$$x_{n_0} \notin \cap_{n=1}^{\infty} I_n$$

Since we are assuming that the list in (1) contains every real number, then

$$\cap_{n=1}^{\infty} I_n = \emptyset$$

However, the NIP asserts that $\cap_{n=1}^{\infty} I_n \neq \emptyset$, which is a contradiction. ■

Theorem 1.9. If $A \subseteq B$ and B is countable, then A is either countable or finite.

Theorem 1.10. (i) If $A_1, A_2, \ldots, A_m$ are countable sets, then the union $A_1 \cup A_2 \cup \ldots \cup A_m$ is countable.
(ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is countable.

Theorem 1.11. The open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable.

1.6 Cantor's Theorem

Notation 1.1. Given a set A, the power set $P(A)$ refers to the collection of all subsets of A.

Theorem 1.12 (Cantor's Theorem)**.** Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

Proof. Assume, for contradiction, that $f : A \to P(A)$ is onto. For each element $a \in A$, $f(a)$ is a particular subset of A. The assumption that f is onto means that every subset of A appears as $f(a)$ for some $a \in A$.
To arrive at a contradiction, we will produce a subset $B \subseteq A$ that is not equal to $f(a)$ for any $a \in A$.
Construct B using the following rule. For each element $a \in A$, consider the subset $f(a)$. This subset of A may contain the element a or it may not. This depends on the function f. If $f(a)$ does not contain a, then we include a in our set B: Let

$$B = \{a \in A : a \notin f(a)\}$$

Since we have assumed that our function $f : A \to P(A)$ is onto, it must be that $B = f(a')$ for some $a' \in A$.

Case 1 $a' \in B$
Then $a' \notin f(a') = B$, a contradiction.
Case 2 $a' \notin B$
Then $a' \in f(a') = B$, a contradiction. ■

Theorem 1.13 (Schröder-Bernstein Theorem)**.** If there are 1-1 functions $f : A \to B$ and $h : B \to A$, then there is a bijection $g : A \to B$.

Proof. **Claim:** the statement of the theorem is equivalent to the following:
If $B \subseteq A$ and $f : A \to B$ is $1-1$, then there is a bijection $g : A \to B$. (*)

proof of claim: theorem $\implies$ (*):
Take $h : X \to Y$ with $h(x) = x$, then $X \subseteq Y$.
(*) $\implies$ theorem:
Let $f : A \to B$ and $h : B \to A$ be 1-1 functions, as in the theorem. We need to show that there is bijection $g : A \to B$.
Notice that $A \subseteq h(B)$ and $h \circ f : A \to h(B)$ is a 1-1 function. So by (*), there is a bijection $g_0 : A \to h(B)$. But $h : B \to h(B)$ is also a bijection. So $g = h^{-1} \circ g_0 : A \to B$ is a bijection (using the fact that bijections are closed under compositions).

Now it suffices to prove (*).
Assume set $X \subseteq Y$ and $f : Y \to X$. Let $W = \bigcup_{n=0}^{\infty} f^n(Y \setminus X)$.
Define $g : Y \to X$ by:

- If $y \in W$, then $g(y) = f(y)$
- If $y \in Z := Y \setminus W$, then $g(y) = y$

We need to show that $g : Y \to X$ is a well-defined bijection.
Since f is 1-1, for all $m < n$, $f^m(Y \setminus X) \cap f^n(Y \setminus X) = \emptyset$
Note that

$$\begin{aligned} Y \setminus W &= Y \setminus \bigcup_{n=0}^{\infty} f^n(Y \setminus X) \\ &= [Y \setminus (Y \setminus X)] \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \\ &= X \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \end{aligned}$$

Therefore for all $y \in Y, g(y) \in X$.
(Show g is 1-1) Now assume $y_1, y_2 \in Y$ and $g(y_1) = g(y_2)$. We show that $y_1 = y_2$.
Case 1 $y_1, y_2 \in W$
Then $g(y_1) = g(y_2) \implies f(y_1) = f(y_2) \implies y_1 = y_2$.
Case 2 $y_1 \in W$ but $y_2 \in Y \setminus W$
Then $g(y_1) = g(y_2) \implies f(y_1) = y_2$
Note that if $y_1 \in W$, then for some $n \geq 0, y_1 \in f^n(Y \setminus X)$

Then $y_2 \in f^{n+1}(Y \setminus X) \subseteq W$
So $y_2 \in W$, which leads to a contradiction.
Case 3 y_1, y_2 are both in $Z := Y \setminus W$
Then $g(y_1) = g(y_2) \implies y_1 = y_2$.
Therefore by case 1,2,3, g is 1-1.
(Show g is onto) Let $x \in X$. We need to find $y \in Y$ s.t. $g(y) = X$.
If $x \in Z$, take $y = x$.
If $x \in \bigcup_{n=1}^{\infty} f^n(Y \setminus X)$, then fix $n \in \mathbb{N}$ s.t. $x \in f^n(Y \setminus X)$.
But $f^n(Y \setminus X) = f(f^{n-1}(Y \setminus X))$
Pick $y \in f^{n-1}(Y \setminus X)$ s.t. $f(y) = x$.
Then $y \in W$ and $g(y) = x$. Therefore g is onto. ∎

2 Sequences and Series

2.1 The Limit of a Sequence

Definition 2.1 (sequence)**.** A sequence is a function whose domain is $\mathbb{N}$

3 Metric Spaces and the Baire Category Theorem

Definition 3.1 (metric and metric space)**.** Given a set X, a function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y \in X$:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. for all $z \in X, d(x, y) \leq d(x, z) + d(z, y)$

A metric space is a set X together with a metric d.

Example 3.1. The set $\mathbb{R}$ considered with $d : \mathbb{R}^2 \to [0, \infty), (x, y) \mapsto |x - y|$ is a metric space.

Example 3.2. In general, $\mathbb{R}^n$ considered with the Euclidean distance is a metric space.

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Example 3.3. Let x be a set. The discrete metric d on X is defined by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Fact If (X, d) is a metric space, $d'(x, y) = \max\{1, d(x, y)\}$ for all $x, y \in X$, then (X, d') is also a metric space.

Example 3.4. Let $X = \{f : A \to \mathbb{R}\}$

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in A\}$$

if the supremum exists.

3.1 Basic Definitions

Definition 3.2. Let (X, d) be a metric space. A sequence $(X_n) \subseteq X$ converges to an element $x \in X$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies d(x_n, x) < \epsilon$.

Key property: If $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} x_n = y$, then $x = y$.

Proof. WTS $d(x, y) = 0$
Let $\epsilon > 0$. We will show that $d(x, y) < \epsilon$.
Since $\lim\limits_{n\to\infty} x_n = x$, then $\exists N_1, \forall n \geq N_1, d(x_n, x) < \frac{\epsilon}{2}$
Since $\lim\limits_{n\to\infty} x_n = y$, then $\exists N_2, \forall n \geq N_2, d(x_n, y) < \frac{\epsilon}{2}$
Take $n \geq \max(N_1, N_2)$, then $d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. ■

Proposition 3.1. Suppose (X, d) is a metric space, (X, τ) is a topological space, and $F \subseteq X$. If $\lim\limits_{n\to\infty} x_n = x$, $(x_n) \subseteq F$ and F is closed, then $x \in F$.

Proof. Suppose $x \notin F$, i.e., $x \in X \setminus F$.
Since F is closed, then $X \setminus F$ is open, so there is $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq X \setminus F$.
Let N be such that $\forall n \geq N, d(x_n, x) < \epsilon$.
Then $x_n \in B_\epsilon(x)$, which implies that $(x_n) \subseteq X \setminus F$, a contradiction. ■

Proposition 3.2. Suppose (X, d) is a metric space and $F \subseteq X$. If F is not closed, then there exists $(x_n) \subseteq F$ and $x \notin F$ s.t. $\lim\limits_{n\to\infty} x_n = x$.

Proof. If F is not closed, then $X \setminus F$ is not open, so there is $x \in X \setminus F$ s.t. $B_\epsilon(x) \nsubseteq X \setminus F$ for all $\epsilon > 0$.
Take $x_n \in B_{1/n}(x) \setminus (X \setminus F) = B_{1/n}(x) \cap F$ for each $n \in \mathbb{N}$, then $(x_n) \subseteq F$ and $\lim\limits_{n\to\infty} x_n = x$. ■

Definition 3.3 (Cauchy sequence)**.** A sequence (x_n) in a metric space (x_n) in a metric space (X, d) is a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}, m, n \geq N \implies d(x_m, x_n) < \epsilon$.

Proposition 3.3. A convergent sequence is Cauchy.

Proof. Let (x_n) be a convergent sequence, so that $\lim\limits_{n\to\infty} x_n = x$. To check (x_n) is Cauchy, let $\epsilon > 0$. We need to find N s.t. $\forall m, n \geq N, d(x_n, x_m) < \epsilon$.
Apply $\lim\limits_{n\to\infty} x_n = x$ to $\frac{\epsilon}{2}$, we get N s.t. $\forall n \geq N, d(x, x_n) < \frac{\epsilon}{2}$.
Notice that N works for Cauchy:
Take $m, n \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Remark 3.1. When $X = \mathbb{R}$ with the usual metric, A Cauchy sequence is convergent (the converse is true).
In general not true. For example, $X = \mathbb{R} \setminus \{0\}, d(x,y) = |x-y|, (x_n) = \frac{1}{n}$.

■

Definition 3.4 (completeness of metric spaces)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Example 3.5. $\mathbb{R}, d(x,y) = |x-y|$

Example 3.6. $(X,d), d$ discrete metric.

Example 3.7. $C[0,1], d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)| = ||f-g||_\infty$

Example 3.8. $(\mathbb{N}^\mathbb{N}, d), d((x_n),(y_n)) = \frac{1}{\min\{n : x_n \neq y_n\}}$
where $\mathbb{N}^\mathbb{N} = \{x : \mathbb{N} \to \mathbb{N}\}$.

Definition 3.5. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is continuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0, d_1(x,y) < \delta \implies d_2(f(x), f(y)) < \epsilon$.

3.2 Topology on Metric Spaces

Definition 3.6 (ϵ-neighbourhood)**.** Given $\epsilon > 0$ and an element x in the metric space (X, d), the ϵ-neighbourhood of x is the set $V_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$

Definition 3.7 (compactness)**.** A subset K of a metric space (X, d) is compact if every sequence in K has a convergent subsequence that converges to a limit in K.

Definition 3.8 (closure and interior)**.** Given a subset E of a metric space (X, d), the closure $\bar{E}$ is the union of E together with its limit points. The interior of E is denoted by E° and is defined as

$$E^\circ = \{x \in E : \exists V_\epsilon(x) \subseteq E\}$$

Remark 3.2. $(X, \tau), \tau \subseteq P(X), E \subseteq X$

1. $\bar{E} =$ minimal closed superset of $E = \bigcap\{H : H \text{ closed }, H \supseteq E\}$
2. $E^\circ =$ maximal open subset of $E = \bigcup\{U : U \text{ open }, U \subseteq E\}$

Example 3.9. (X, d) is a metric space, τ_d is the topology determined by d: $U \in \tau_d$ iff $\forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$
$F \subseteq X$

$$\begin{aligned} \bar{F} &= \{x \in X : \forall \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} && (2) \\ &= \bigcap\{H : H \text{ closed } H \supseteq F\} && (3) \\ &= \{\lim_{n\to\infty} x_n : (x_n) \subseteq F, \lim_{n\to\infty} x_n \text{ exists }\} && (4) \end{aligned}$$

$$F^\circ = \{x \in X : \exists \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} \tag{5}$$
$$= \bigcup \{B_\epsilon(x) : \epsilon > 0, x \in F, B_\epsilon(x) \subseteq F\} \tag{6}$$
$$\tag{7}$$

Definition 3.9 (density)**.** A set $A \subseteq X$ is dense in the metric space (X, d) if $\bar{A} = X$. A subset E of a metric space (X, d) is nowhere-dense in X if $\bar{E}^\circ$ is empty.

3.3 Baire's Theorem

Definition 3.10 (nowhere-dense)**.** A set E is nowhere-dense if $\bar{E}$ contains no nonempty open intervals.

Theorem 3.1 (Baire's Theorem)**.** The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

3.4 The Baire Category Theorem

Theorem 3.2. Let (X, d) be a complete metric space, and let $\{O_n\}$ be a countable collection of dense, open subsets of X. Then, $\bigcap_{n=1}^\infty \{O_n\}$ is not empty.

Proof. ■

Theorem 3.3 (Baire Category Theorem)**.** A complete metric space is not the union of a countable collection of nowhere-dense sets.

Proof. ■

Theorem 3.4. The set

$$D = \{f \in C[0, 1] : f'(x) \text{ exists for some } x \in [0, 1]\}$$

is a set of first category in $C[0, 1]$.

3.5 Topology of (X, d)

Definition 3.11 (topological space)**.** A topological space is a pair (X, τ), where X is a set and τ a subset of the power set of X which we call open such that

1. $\emptyset, X \in \tau$
2. $U_1, \ldots, U_n \in \tau \implies \bigcap_{i=1}^n U_i \in \tau$
3. $U_1, \ldots, U_n \in \tau \implies \bigcup_{i=1}^n U_i \in \tau$

Example 3.10. $(X, \{\emptyset, X\})$

Definition 3.12. (X, τ) is a discrete topological space iff $\tau = P(X)$.

Definition 3.13. A subset F of a topological space (X, τ) is closed if $X \setminus F$ is open.
Properties:

1. $\emptyset, X$ are closed
2. If $F_1, \ldots, F_n$ are closed, then $\bigcup_{i=1}^n F_i$ is closed
3. If $F_1, \ldots, F_n$ are closed, then $\bigcap_{i=1}^n F_i$ is closed