STA447
Lecture Notes

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Contents

1 Preliminary

Proposition 1.1. If Z is a non-negative-integer-valued random variable, then

$$E(Z) = \sum_{k=1}^{\infty} P(Z \geq k)$$

Proof.

$$\begin{aligned}
\sum_{k=1} P(Z \geq k) &= \sum_{k=1}^{\infty} [P(Z = k) + P(Z = k+1) + \dots] \\
&= [P(Z = 1) + P(Z = 2) + P(Z = 3) + \dots] \\
&\quad + [P(Z = 2) + P(Z = 3) + P(Z = 4) + \dots] \\
&\quad + [P(Z = 3) + P(Z = 4) + P(Z = 5) + \dots] \\
&\quad + \dots \\
&= P(Z = 1) + 2P(Z = 2) + 3P(Z = 3) + \dots \\
&= \sum_{l=1}^{\infty} lP(Z = l) \\
&= E(Z)
\end{aligned}$$

■

Fact 1.1.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty \iff p \leq 1$$

Fact 1.2. If the x_ns are non-negative, and $\sum_{n=1}^{\infty} x_n < \infty$, then

$$\lim_{n \to \infty} x_n = 0$$

Definition 1.1 (bounded random variable)**.** X is a <u>bounded random variable</u> if there is $M < \infty$ with $P(|X| \leq M) = 1$, i.e. if it is always in some interval $[-M, M]$ for some finite number M.

Definition 1.2 (finite random variable)**.** X is a <u>finite random variable</u> if $P(|X| \leq \infty) = 1$, i.e., if $P(|X| = \infty) = 0$, i.e. if it always takes on finite values.

Definition 1.3 (finite expectation)**.** A random variable X has <u>finite expectation</u> if $E|X| < \infty$; this is also sometimes called <u>integrable</u>.

Fact 1.3. Bounded $\Longrightarrow$ finite expectation.

Fact 1.4. Unbounded $\Longrightarrow$ infinite expectation.

Fact 1.5. Finite expectation $\Longrightarrow$ finite.

Theorem 1.1 (Law of Total Expectation)**.** If X and Y are discrete random variables, then

$$E(X) = \sum_y P(Y = y)E(X|Y = y)$$

i.e. we can compute $E(X)$ by averaging conditional expectations.

prove this

Theorem 1.2 (Double-expectation formula)**.**

$$E[E(X|Y)] = E(X)$$

i.e. the random variable $E(X|Y)$ equals X on average.

Proof. Since $E(X|Y)$ is equal to $E(X|Y = y)$ with probability $Y = y$, we compute that

$$E[E(X|Y)] = \sum_y P(Y = y)E(X|Y = y) = E(X)$$

which the results follows from Double-expectation formula1.2. ∎

Theorem 1.3 (Dominated Convergence Theorem)**.** If $\lim_{n\to\infty} X_n = X$, and there is some random variable Y with $E|Y| < \infty$ and $|X_n| \leq Y$ for all n, then

$$\lim_{n\to\infty} E(X_n) = E(X)$$

Definition 1.4 (weak convergence)**.** X_n converge to X weakly if

$$\forall \epsilon > 0, \lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

Definition 1.5 (strong convergence)**.** X_n converge to X strongly if

$$P(\lim_{n\to\infty} X_n = X) = 1$$

Theorem 1.4 (Law of Large Numbers)**.** If the sequence $\{X_n\}$ is i.i.d. with common mean m, then the sequence $\frac{1}{n}\sum_{i=1}^n X_i$ converges to m (both weakly and strongly), i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_i = m \quad w.p.1$$

If time, please finish reading the preliminary, which I found useful. –Mar 9

2 Markov Chain Probabilities

Notation 2.1.

$$P(X_{n+1} = j|X_n = i) = p_{ij}$$

Definition 2.1 (Markov chain)**.** A (discrete time, discrete space, time homogeneous) Markov chain is specified by three ingredients:

- A state space S, any non-empty finite or countable set.
- Initial probabilities $\{v_i\}_{i \in S}$, where v_i is the probability of starting at i (at time 0). (So $v_i \geq 0$ and $\sum_i v_i = 1$)
- Transition probabilities $\{p_{ij}\}_{i,j \in S}$, where p_{ij} is the probability of jumping to j if you start at i. (So $p_{ij} \geq 0$, and $\sum_j p_{ij} = 1$ for all i)

Remark 2.1 (Markov property)**.**

$$P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) = P(X_{n+1} = j | X_n = i_n) = p_{i_n j}$$

i.e. The probabilities at time $n+1$ depend only on the state at time n.

Remark 2.2.

$$P(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = v_{i_0} p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n}$$

2.1 Markov Chain examples

Example 2.1 (the Frog Walk)**.** Let $X_n :=$ pad index the frog is at after n steps.

$$\begin{aligned} S &= \{1, 2, 3, \dots, 20\} \\ v_{20} &= 1, v_i = 0 \, \forall i \neq 20 \\ p_{ij} &= \begin{cases} \frac{1}{3}, & |j-i| \leq 1 \text{ or } |j-i| = 19 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Example 2.2 (Bernoulli process)**.**

$$\begin{aligned} S &= \{1, 2, 3, \dots\} \\ v_0 &= 1, v_i = 0 \, \forall i \neq 0 \\ p_{ij} &= \begin{cases} p, & j = i+1 \\ 1-p, & j = i \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $0 < p < 1$.

Example 2.3 (Simple random walk (s.r.w.))**.** Let $X_n :=$ net gain (in dollars) after n bets

$$\begin{aligned} S &= \{0, 1, 2, 3, \dots\} \\ v_a &= 1, v_i = 0 \, \forall i \neq a \\ p_{ij} &= \begin{cases} p, & j = i+1 \\ 1-p, & j = i-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $0 < p < 1, a \in \mathbb{Z}$.
Special case: When $p = 1/2$, call it simple symmetric random walk.

Example 2.4 (Ehrenfest's Urn)**.** Let $X_n :=$ # balls in Urn 1 at time n.
We have d balls in total, divided into two urns. At each time, we choose one of the d balls uniformly at random, and move it to the other urn.

$$\begin{aligned} S &= \{1, 2, 3, \dots, d\} \\ v_a &= 1, v_i = 0 \, \forall i \neq a \\ p_{ij} &= \begin{cases} (d-i)/d, & j = i+1 \\ i/d, & j = i-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

2.2 Elementary Computations

Notation 2.2.

$$\mu_i^{(n)} := P(X_n = i)$$

Notation 2.3.

$$\begin{aligned} m &:= |S| && \text{(the number of elements in S, could be infinity)} \\ \mu^{(n)} &= (\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)}, \dots) && (m \times 1) \\ v &= (v_1, v_2, v_3, \dots) && (m \times 1) \\ P = (p_{ij}) &= \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & \\ & \ddots & & \\ p_{m1} & \dots & & p_{mm} \end{pmatrix} && (m \times m \text{ matrix}) \end{aligned}$$

Fact 2.1.

$$\begin{aligned} \mu^{(1)} &= vP = \mu^{(0)} P \\ \mu^{(n)} &= vP^n = \mu^{(0)} P^n \end{aligned}$$

Notation 2.4.

$$p_{ij}^{(n)} := P(X_n = j, X_0 = i) = P(X_{m+n} = j | X_m = i) \qquad \text{(for any } m \in \mathbb{N})$$

Fact 2.2.

$$\begin{aligned} \sum_{j \in S} p_{ij}^{(n)} &= 1 \\ p_{ij}^{(1)} &= p_{ij} \\ P^{(n)} &= P^n && \text{(for all } n \in \mathbb{N}) \end{aligned}$$

Notation 2.5.

$$P^0 := I$$
$$P^{(0)} := I$$
$$p_{ij}^{(0)} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.1 (Chapman-Kolmogorov equations)**.**

$$p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}$$
$$P_{ij}^{(m+s+n)} = \sum_{k \in S} \sum_{l \in S} p_{ik}^{(m)} p_{kl}^{(s)} p_{lj}^{(n)}$$

Matrix form:

$$P^{(m+n)} = P^{(m)} P^{(n)}$$
$$P^{(m+s+n)} = P^{(m)} P^{(s)} P^{(n)}$$

Theorem 2.2 (Chapman-Kolmogorov Inequality)**.**

$$p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)} \qquad \text{(for all } k \in S\text{)}$$
$$P_{ij}^{(m+s+n)} \geq p_{ik}^{(m)} p_{kl}^{(s)} p_{lj}^{(n)} \qquad \text{(for any } k, l \in S\text{)}$$

2.3 Recurrence and Transience

Notation 2.6.

$$P_i(\dots) \equiv P(\dots | X_0 = i)$$
$$E_i(\dots) \equiv E(\dots | X_0 = i)$$
$$N(i) = \#\{n \geq 1 : X_n = i\}$$
$$\text{(total number of times that the chain hits } i\text{, not counting time 0)}$$

Definition 2.2 (return probability)**.** Let f_{ij} be the return probability from i to j.

$$f_{ij} := P_i(X_n = j \text{ for some } n \geq 1) \equiv P_i(N(j) \geq 1)$$

Fact 2.3.

$$1 - f_{ij} = P_i(X_n \neq j \text{ for all } n \geq 1) \tag{1}$$
$$P_i(N(i) \geq k) = (f_{ii})^k \tag{2}$$
$$P_i(N(j) \geq k) = f_{ij}(f_{jj})^{k-1} \tag{3}$$
$$f_{ik} \geq f_{ij} f_{jk} \tag{4}$$

Fact 2.4. $f_{ij} > 0$ iff $\exists m \geq 1$ with $p_{ij}^{(m)} > 0$, i.e., there is some time m for which it is possible to get from i to j in m steps.

Definition 2.3 (recurrent and transient states)**.** A state i of a Markov chain is recurrent if $f_{ii} = 1$. Otherwise, i is transient if $f_{ii} < 1$.

Proposition 2.1. If Z is a non-negative integer, then

$$E(Z) = \sum_{k=1}^{\infty} P(Z \geq k)$$

Theorem 2.3 (Recurrent State Theorem)**.** As follows

- State i is recurrent $\iff P_i(N(i) = \infty) = 1 \iff \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$
- State i is transient $\iff P_i(N(i) = \infty) = 0 \iff \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$

Proof.

$$\begin{aligned}
P_i(N(i) = \infty) &= \lim_{k\to\infty} P_i(N(i) \geq k) && \text{(by continuity of probabilities)} \\
&= \lim_{k\to\infty} (f_{ii})^k && (P_i(N(i) \geq k) = (f_{ii})^k) \\
&= \begin{cases} 1, & f_{ii} = 1 \\ 0, & f_{ii} < 1 \end{cases}
\end{aligned}$$

∎

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{ii}^{(n)} &= \sum_{n=1}^{\infty} P_i(X_n = i) \\
&= \sum_{n=1}^{\infty} E_i(\mathbb{1}\{X_n = i\}) \\
&= E_i(\sum_{n=1}^{\infty} \mathbb{1}\{X_n = i\}) \\
&= E_i(N(i)) \\
&= \sum_{k=1}^{\infty} P_i(N(i) \geq k) && \text{(by proposition 1.1)} \\
&= \sum_{k=1}^{\infty} (f_{ii})^k \\
&= \begin{cases} \infty, & f_{ii} = 1 \\ \frac{f_{ii}}{1-f_{ii}} < \infty, & f_{ii} < 1 \end{cases}
\end{aligned}$$

Example 2.5 (simple random walk)**.** If $p = 1/2$ then $\forall i, f_{ii} = 1$. If $p \neq 1/2$, then $\forall i, f_{ii} < 1$

Proof. Consider state 0. We need to check if $\sum_{n=1}^{\infty} p_{00}^{(n)} = \infty$.
If n is odd, then $p_{00}^{(n)} = 0$.
If n is even, $p_{00}^{(n)} = P(\frac{n}{2}$ heads and $\frac{n}{2}$ tails on first n tosses$)$.
This is a Binomial(n, p) distribution, so

$$\begin{aligned} p_{00}^{(n)} &= \binom{n}{n/2} p^{n/2}(1-p)^{n/2} \\ &= \frac{n!}{[(n/2)!]^2} p^{n/2}(1-p)^{n/2} \\ &= \frac{(n/e)^n\sqrt{2\pi n}}{[(n/2e)^{n/2}\sqrt{2\pi n/2}]^2} p^{n/2}(1-p)^{n/2} && \text{(Sirling's approximation)} \\ &= [4p(1-p)]^{n/2}\sqrt{2/\pi n} \end{aligned}$$

Case 1: If $p = 1/2$, then $4p(1-p) = 1$, so

$$\begin{aligned} \sum_{n=1} \infty p_{00}^{(n)} &= \sum_{n=2,4,6,\ldots} \sqrt{2/\pi n} \\ &= \sqrt{2/\pi} \sum_{n=2,4,6,\ldots} n^{-1/2} \\ &= \sqrt{2/\pi} \sum_{n=1}^{\infty} 2k^{-1/2} \\ &= \infty \end{aligned}$$

Therefore, state 0 is recurrent.
Case 2: If $p \neq 1/2$, then $4p(1-p) < 1$, so

$$\begin{aligned} \sum_{n=1} \infty p_{00}^{(n)} &= \sum_{n=2,4,6,\ldots} [4p(1-p)]^{n/2}\sqrt{2/\pi n} \\ &< \sum_{n=2,4,6,\ldots} [4p(1-p)]^{n/2} && \text{(Geometric Series)} \\ &= \frac{4p(1-p)}{1-4p(1-p)} \\ &< \infty \end{aligned}$$

Therefore, the state 0 is transient.
The same exact calculation applies to any other state i. ■

Theorem 2.4 (f-Expansion)**.**

$$f_{ij} = p_{ij} + \sum_{k \in S, k \neq j} p_{ik} f_{kj}$$

Proof.

$$\begin{aligned}
f_{ij} &= P_i(\exists n \geq 1 : X_n = j) \\
&= \sum_{k \in S} P_i(X_1 = k, \exists n \geq 1 : X_n = j) \\
&= P_i(X_1 = j, \exists n \geq 1 : X_n = j) + \sum_{k \neq j} P_i(X_1 = k, \exists n \geq 1 : X_n = j) \\
&= P_i(X_1 = j) P_i(\exists n \geq 1 : X_n = j | X_1 = j) + \sum_{k \neq j} P_i(X_1 = k) P_i(\exists n \geq 1 : X_n = j | X_1 = k) \\
&= p_{ij}(1) + \sum_{k \neq j} p_{ik}(f_{kj})
\end{aligned}$$

■

Remark 2.3. The f-Expansion shows that $f_{ij} \geq p_{ij}$.

Remark 2.4. It essentially follows from logical reasoning: from i, to get to j eventually, we have to either jump to j immediately (with probability p_{ij}), or jump to some other state k (with probability p_{ik}) and then get to j eventually (with probability p_{kj})

2.4 Communicating States and Irreducibility

Definition 2.4 (communicating states)**.** State i communicates with state j, written $i \to j$, if $f_{ij} > 0$.

Remark 2.5. i.e. if it is possible to get from i to j.

Notation 2.7. Write $i \leftrightarrow j$ if both $i \to j$ and $j \to i$.

Definition 2.5 (irreducibility)**.** A Markov chain is irreducible if $i \to j$ for all $i, j \in S$, i.e., if $f_{ij} > 0$ for all $i, j \in S$. Otherwise, the chain is reducible.

Lemma 2.1 (Sum Lemma)**.** If $i \to k$, and $l \to j$, and $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$, then $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$

Proof. Since $i \to k$, and $l \to j$, there exists $m, r \geq 1$ s.t. $p_{ik}^{(m)} > 0$ and $p_{lj}^{(r)} > 0$.
By the Chapman-Kolmogorov inequality,

$$p_{ij}^{(m+s+r)} \geq p_{ij}^{(m)} p_{kl}^{(s)} p_{lj}^{(r)}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} p_{ij}^{(n)} &\geq \sum_{n=m+r+1}^{\infty} p_{ij}^{(n)} \\
&= \sum_{s=1}^{\infty} p_{ij}^{(m+s+r)} && (s = n - m - r) \\
&\geq \sum_{s=1}^{\infty} p_{ij}^{(m)} p_{kl}^{(s)} p_{lj}^{(r)} \\
&= \underbrace{p_{ij}^{(m)}}_{+} \underbrace{p_{lj}^{(r)}}_{+} \underbrace{\sum_{s=1}^{\infty} p_{kl}^{(s)}}_{=\infty} \\
&= \infty
\end{aligned}$$

■

Corollary 2.1 (Sum Corollary)**.** If $i \leftrightarrow k$, then i is recurrent iff k is recurrent.

Proof. Setting $j = i$ and $l = k$ in the Sum Lemma: If $i \leftrightarrow k$, then $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty \iff \sum_{n=1}^{\infty} p_{kk}^{(n)} = \infty$. ■

Theorem 2.5 (Cases Theorem)**.** For an irreducible Markov chain, either

- (a) $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for all $i, j \in S$, and all states are recurrent (recurrent Markov chain); or
- (b) $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$ for all $i, j \in S$, and all states are transient (transient Markov chain).

Theorem 2.6 (Finite Space Theorem)**.** An irreducible Markov chain on a finite state space always falls into case (a), i.e., $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for all $i, j \in S$, and all states are recurrent.

Proof. Choose any state $i \in S$. We have

$$\begin{aligned}
\sum_{j \in S} \sum_{n=1}^{\infty} p_{ij}^{(n)} &= \sum_{n=1}^{\infty} \sum_{j \in S} p_{ij}^{(n)} && \text{(exchanging the sums)} \\
&= \sum_{n=1}^{\infty} 1 \\
&= \infty
\end{aligned}$$

Then if S is finite, it follows that there must exist at least one $j \in S$ with $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$. So we must be in case (a).

■

Notation 2.8. For $i \neq j$, let H_{ij} be the event that the chain hits the state i before returning to j, i.e.,

$$H_{ij} = \{\exists n \in \mathbb{N} : X_n = i,\ \text{but } X_m \neq j \text{ for } 1 \leq m \leq n-1\}$$

Lemma 2.2 (Hit Lemma)**.** If $j \to i$ with $j \neq i$, then $P_j(H_{ij}) > 0$.

Proof. Since $j \to i$, there is some possible path from j to i. i.e., there is $m \in \mathbb{N}$ and $x_0, x_1, \ldots, x_m$ with $x_0 = j$ and $x_m = i$ and $p_{x_r x_{r+1}} > 0$ for all $0 \leq r \leq m-1$.
Let $S = \max\{r : x_r = j\}$ be the last time this path hits j.
Then $x_S, x_{S+1}, \ldots, x_m$ is a possible path which goes from j to i without first returning to j.
Hence $P_j(H_{ij}) \geq P(x_0, x_1, \ldots, x_m) = p_{x_S x_{S+1}} p_{x_{S+1} x_{S+2}} \cdots p_{x_{m-1} x_m} > 0$

■

Remark 2.6. If it is possible to get from j to i at all, then it is possible to get from j to i without first returning to j.
Intuitively obvious: If there is some path from j to i, then the final part of the path (starting with the last time it visits i) is a possible path from j to i which does not return to j.

Lemma 2.3 (f-Lemma)**.** If $j \to i$ and $f_{jj} = 1$, then $f_{ij} = 1$

Proof. If $i = j$ it is trivial, so assume $i \neq j$.
Since $j \to i$, we have $P_j(H_{ij}) > 0$ by the Hit Lemma.
But one way to never return to j is to first hit i and then from i never return to j:

$$P_j(\text{never return to } j) \geq P_j(H_{ij}) P_i(\text{never return to } j)$$

Therefore

$$1 - f_{jj} \geq P_j(H_{ij})(1 - f_{ij})$$

Since $f_{jj} = 1$, then $\underbrace{P_j(H_{ij})}_{>0}(1 - f_{ij}) = 0$

Hence $f_{ij} = 1$.

■

Lemma 2.4 (Infinite Returns Lemma)**.** For an irreducible Markov chain, if it is recurrent, then

$$P_i(N(j) = \infty) = 1$$

for all $i, j \in S$.
But if it transient, then $P_i(N(j) = \infty) = 0$ for all $i, j \in S$.

Proof. Let $i, j \in S$. If the chain is recurrent, then $f_{ij} = f_{jj} = 1$ by the f-Lemma.
Then

$$\begin{aligned} P_i(N(j) = \infty) &= \lim_{k \to \infty} P_i(N(j) \geq k) \\ &= \lim_{k \to \infty} f_{ij}(f_{jj})^{k-1} \\ &= \lim_{k \to \infty} (1)(1)^{k-1} \\ &= 1 \end{aligned}$$

If the chain is transient, then $f_{jj} < 1$, then

$$\begin{aligned} P_i(N(j) = \infty) &= \lim_{k\to\infty} P_i(N(j) \geq k) \\ &= \lim_{k\to\infty} f_{ij}(f_{jj})^{k-1} \\ &= \lim_{k\to\infty} (1)(f_{jj})^{k-1} \\ &= 0 \end{aligned}$$

■

Theorem 2.7 (Recurrence Equivalence Theorem)**.** If a chain is irreducible, then the following are equivalent (and all correspond to case (a)):

1. There are $k, l \in S$ with $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$.
2. For all $i, j \in S$, we have $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.
3. There is $k \in S$ with $f_{kk} = 1$, i.e. k is recurrent.
4. For all $j \in S$, we have $f_{jj} = 1$, i.e. all states are recurrent.
5. For all $i, j \in S$, we have $f_{ij} = 1$.
6. There are $k, l \in S$ with $P_k(N(l) = \infty) = 1$.
7. For all $i, j \in S$, we have $P_i(N(j) = \infty) = 1$.

Proof. Follow from results that we have already proven

- $1 \implies 2$: Sum Lemma.
- $2 \implies 4$: Recurrent State Theorem (with $i = j$).
- $4 \implies 5$: f-Lemma.
- $5 \implies 3$: immediate.
- $3 \implies 1$: Recurrent State Theorem (with $l = k$).
- $4 \implies 7$: Infinite Returns Lemma.
- $7 \implies 6$: Immediate.
- $6 \implies 3$: Recurrent State Theorem (with $l = k$).

■

Theorem 2.8 (Transience Equivalence Theorem)**.** If a chain is irreducible, then the following are equivalent (and all correspond to case (b)):

1. There are $k, l \in S$ with $\sum_{n=1}^{\infty} p_{kl}^{(n)} < \infty$.

2. For all $i, j \in S$, we have $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$.
3. For all $k \in S$, we have $f_{kk} < 1$, i.e. k is transient.
4. There is $j \in S$ with $f_{jj} < 1$, i.e. some state is recurrent.
5. There are $i, j \in S$ with $f_{ij} < 1$.
6. For all $k, l \in S$, $P_k(N(l) = \infty) = 0$.
7. There are $i, j \in S$ with $P_i(N(j) = \infty) = 0$.

Remark 2.7 (closed subset note)**.** Suppose a chain is reducible, but it has a closed subset $C \subseteq S$ (i.e. $p_{ij} = 0$ for $i \in C$ and $j \notin C$) on which it is irreducible (i.e. $i \to j$ for all $i, j \in C$). Then, the Recurrence Equivalence Theorem and other results about irreducible chains still apply to the chain when restricted to C.

Proposition 2.2. For simple random walk with $p > 1/2$, $f_{ij} = 1$ whenever $j > i$. (Similarly, if $p < 1/2$ and $j < i$, then $f_{ij} = 1$.)

Proof. Let $X_0 = 0$, and $Z_n = X_n - X_{n-1}$ for $n = 1, 2, \ldots$, so that $X_n = \sum_{i=1}^{n} Z_i$.
Since Z_ns iid with $P(Z_n = 1) = p$ and $P(Z_n = -1) = 1 - p$, then by Law of Large Numbers,

$$\lim_{n\to\infty} \frac{1}{n}(Z_1 + Z_2 + \ldots + Z_n) \overset{p}{=} E(Z_1) = p(1) + (1-p)(-1) = 2p - 1 > 0$$

$$\begin{aligned}\implies \infty &= \lim_{n\to\infty} (Z_1 + Z_2 + \ldots + Z_n) \\ &= \lim_{n\to\infty} X_n - X_0 \\ &= \lim_{n\to\infty} X_n\end{aligned}$$

But if $i < j$, then to go from i to ∞, the chain must pass through j, so $f_{ij} = 1$. ■

3 Markov Chain Convergence

3.1 Stationary Distributions

Definition 3.1 (stationary distributions)**.** If π is a probability distribution on S (i.e. $\pi_i \geq 0$ for all $i \in S$, and $\sum_{i \in S} \pi_i = 1$), then π is stationary for a Markov chain with transition probabilities (p_{ij}) if $\sum_{i \in S} \pi_i p_{ij} = \pi_j$ for all $j \in S$ (or $\pi P = \pi$, in matrix notation).

Remark 3.1. Intuitively, π being stationary means if the chain starts with probabilities $\{\pi_i\}$, then it will keep the same probabilities one time unit later.

Definition 3.2 (doubly stochastic)**.** A Markov Chain is doubly stochastic if in addition to the usual condition that $\sum_{j \in S} p_{ij} = 1$ for all $i \in S$, $\sum_{i \in S} p_{ij} = 1$ for all $j \in S$.

Remark 3.2. This holds for the Frog Example.

Proposition 3.1. If a Markov chain with states S satisfies $|S| < \infty$ and is doubly stochastic, then the uniform distribution on S is a stationary distribution.

Proof. Let $\{\pi_i\}$ be a distribution such that $\pi_i = \frac{1}{|S|}$.
Then

$$\begin{aligned}\sum_{i\in S}\pi_i p_{ij} &= \sum_{i\in S}\frac{1}{|S|}p_{ij}\\ &= \frac{1}{|S|}\sum_{i\in S}p_{ij}\\ &= \frac{1}{|S|}(1) && \text{(doubly stochastic)}\\ &= \frac{1}{|S|}\\ &= \pi_j\end{aligned}$$

Then $\{\pi_i\}$ is stationary. ∎

3.2 Searching for Stationary

Definition 3.3 (reversibility)**.** A Markov chain is reversible (or time reversible, or satisfies detailed balance) with respect to a probability distribution $\{\pi_i\}$ if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

Proposition 3.2. If a chain is reversible with respect to π, then π is a stationary distribution.

Proof. Reversibility means $\pi_i p_{ij} = \pi_j p_{ji}$, so then for $j \in S$,

$$\sum_{i\in S}\pi_i p_{ij} = \sum_{i\in S}\pi_j p_{ji} = \pi_j\sum_{i\in S}p_{ji} = \pi_j(1) = \pi_j$$

∎

Lemma 3.1 (M-test)**.** Let $\{x_{nk}\}_{n,k\in\mathbb{N}}$ be a collection of real numbers. Suppose that $\lim_{n\to\infty} x_{nk}$ exists for each fixed $k \in \mathbb{N}$. Suppose further that $\sum_{k=1}^{\infty}\sup_n |x_{nk}| < \infty$. Then $\lim_{n\to\infty}\sum_{k=1}^{\infty} x_{nk} = \sum_{k=1}^{\infty}\lim_{n\to\infty} x_{nk}$.

Proposition 3.3 (Vanishing Probabilities Proposition)**.** If a Markov chain's transition probabilities satisfy that $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$, then the chain does not have a stationary distribution.

Proof. Suppose for contradiction that there is a stationary distribution π. Then we would have $\pi_j = \sum_{i\in S}\pi_i p_{ij}^{(n)}$ for any n, so

$$\pi_j = \lim_{n\to\infty}\pi_j = \lim_{n\to\infty}\sum_{i\in S}\pi_i p_{ij}^{(n)}$$

$$\begin{aligned}
\pi_j &= \lim_{n\to\infty} \pi_j \\
&= \lim_{n\to\infty} \sum_{i\in S} \pi_i p_{ij}^{(n)} \\
&= \sum_{i\in S} \lim_{n\to\infty} \pi_i p_{ij}^{(n)} && \text{(exchange the sum and the limit, which is valid by M-test)} \\
&= \sum_{i\in S} \pi_i \lim_{n\to\infty} p_{ij}^{(n)} \\
&= \sum_{i\in S} 0 \\
&= 0
\end{aligned}$$

So we would have $\pi_j = 0$ for all j. But this means that $\sum_j \pi_j = 0$, which is a contradiction.

■

Lemma 3.2 (Vanishing Lemma)**.** If a Markov chain has some $k, l \in S$ with $\lim_{n\to\infty} p_{kl}^{(n)} = 0$, then for any $i, j \in S$ with $k \to i$ and $j \to l$, $\lim_{n\to\infty} p_{ij}^{(n)} = 0$.

Proof. Since $k \to i$ and $j \to l$, we can find $r, s \in \mathbb{N}$ with $p_{ki}^{(r)} > 0$ and $p_{jl}^{(s)} > 0$. Then by the Chapman-Kolmogorov Inequality,

$$p_{kl}^{(r+n+s)} \geq p_{ki}^{(r)} p_{ij}^{(n)} p_{jl}^{(s)}$$

Hence

$$p_{ij}^{(n)} \leq p_{kl}^{(r+n+s)} / p_{ki}^{(r)} p_{jl}^{(s)}$$

But the assumptions imply that

$$\lim_{n\to\infty} \left[p_{kl}^{(r+n+s)} / p_{ki}^{(r)} p_{jl}^{(s)} \right] = 0$$

Hence

$$\begin{aligned}
&0 \leq \lim_{n\to\infty} p_{ij}^{(n)} \leq 0 \\
&\implies \lim_{n\to\infty} p_{ij}^{(n)} = 0
\end{aligned}$$

■

Corollary 3.1 (Vanishing Together Corollary)**.** For an irreducible Markov chain, either

1. $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$, or
2. $\lim_{n\to\infty} p_{ij}^{(n)} \neq 0$ for all $i, j \in S$

Corollary 3.2 (Vanishing Probabilities Corollary)**.** If an irreducible Markov chain's transition probabilities satisfy that $\lim_{n\to\infty} p_{kl}^{(n)} = 0$ for some $k, l \in S$, then the chain does not have a stationary distribution.

Lemma 3.3. If the x_ns are non-negative, and $\sum_{n=1}^{\infty} x_n < \infty$, then $\lim_{n\to\infty} x_n = 0$.

Corollary 3.3 (Transient Not Stationary Corollary)**.** A Markov chain which is irreducible and transient cannot have a stationary distribution.

Proof. If a chain is irreducible and transient, then by the Transience Equivalence Theorem, $\sum_{n=1}^{\infty} < \infty$ for all $i, j \in S$. Hence $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$.
Thus by the Vanishing Probabilities Corollary, there is no stationary distribution. ∎

3.3 Obstacles to Convergence

Definition 3.4 (period)**.** The period of a state i is the greatest common divisor (gcd) of the set $\{n \geq 1 : p_{ii}^{(n)} > 0\}$, i.e. the largest number m such that all the values of n with $p_{ii}^{(n)} > 0$ are all integer multiples of m. If the period of each state is 1, we say the chain is aperiodic; otherwise we say the chain is periodic.

Remark 3.3. Intuitively, the period of a state i is the pattern of returning to i from i. e.g. If the period of i is 2, then it is only possible to get from i to i in an even numbers of steps.

Fact 3.1. If state i has period t, and $p_{ii}^{(m)} > 0$, then m is an integer multiple of t, i.e., t divides m.

Fact 3.2. If $p_{ii} > 0$, then the period of state i is 1.

Fact 3.3. If $p_{ii}^{(n)} > 0$ and $p_{ii}^{(n+1)} > 0$, then the period of state i is 1.

Lemma 3.4 (Equal Periods Lemma)**.** If $i \leftrightarrow j$, then the periods of i and of j are equal.

Proof. Let the periods of i and j be t_i and t_j. Since $i \leftrightarrow j$, we can find $r, s \in \mathbb{N}$ with $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$. Then

$$p_{ii}^{(r+s)} \geq p_{ij}^{(r)} p_{ji}^{(s)} > 0$$

Therefore by Fact 2.1, t_i divides $r + s$.
Suppose now that $p_{jj}^{(n)} > 0$. Then

$$p_{ii}^{(r+n+s)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)} > 0$$

So t_i divides $r + n + s$.
Since t_i divides both $r + n + s$ and $r + s$, then it must divide n as well.
Since this is true for any n with $p_{jj}^{(n)} > 0$, it follows that t_i is a common divisor of $\{n \in \mathbb{N} : p_{jj}^{(n)} > 0\}$.
But t_j is the greatest such common divisor, so $t_j \geq t_i$.
Similarly we can show that $t_i \geq t_j$, so we have $t_i = t_j$. ∎

Corollary 3.4 (Equal Periods Corollary)**.** If a chain is irreducible, then all states have the same period.

Corollary 3.5. If a chain is irreducible and $p_{ii} > 0$ for some state i, then the chain is aperiodic.

3.4 Convergence Theorem

Theorem 3.1 (Markov Chain Convergence Theorem)**.** If a Markov chain is irreducible, aperiodic, and has a stationary distribution $\{\pi_i\}$, then $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$, and $\lim_{n\to\infty} P(X_n = j) = \pi_j$ for any initial probabilities $\{v_i\}$.

Theorem 3.2 (Stationary Recurrence Theorem)**.** If chain irreducible and has a stationary distribution, then it is recurrent.

Proof. The Transient Not Stationary Corollary says that a chain cannot be irreducible, transient and have a stationary distribution.
Therefore, if a chain is irreducible and has a stationary distribution, then it cannot be transient, i.e. it must be recurrent. ■

Lemma 3.5 (Number Theory Lemma)**.** If a set A of positive integers is non-empty, and satisfies additivity, and $gcd(A) = 1$, then there is some $n_0 \in \mathbb{N}$ s.t. for all $n \geq n_0$ we have $n \in A$ i.e. the set A includes all of the integers $n_0, n_0 + 1, n_0 + 2, \ldots$

Proposition 3.4. If a state i has $f_{ii} > 0$ and is aperiodic, then there is $n_0(i) \in \mathbb{N}$ such that $p_{ii}^{(n)} > 0$ for all $n \geq n_0(i)$

Proof. Let $A = \{n \geq 1 : p_{ii}^{(n)} > 0\}$. Since $f_{ii} > 0$, then A is not empty.
If $m, n \in A$, then

$$p_{ii}^{(m+n)} \geq p_{ii}^{(m)} p_{ii}^{(n)} > 0$$

So $m + n \in A$, which shows that A satisfies additivity. Also $gcd(A) = 1$ since the state i is aperiodic. Hence from the Number Theory Lemma, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $n \in A$ i.e. $p_{ii}^{(n)} > 0$.

■

Corollary 3.6. If a chain is irreducible and aperiodic, then for any states $i, j \in S$, there is $n_0(i, j) \in \mathbb{N}$ s.t. $p_{ij}^{(n)} > 0$ for all $n \geq n_0(i, j)$

Proof. Find $n_0(i)$ as in Proposition 2.3, and find $m \in \mathbb{N}$ with $p_{ij}^{(m)} > 0$.
Then let $n_0(i, j) = n_0(i) + m$
Then if $n \geq n_0(i, j)$, then $n - m \geq n_0(i)$, so $p_{ij}^{(n)} \geq p_{ii}^{(n-m)} p_{ij}^{(m)} > 0$. ■

Lemma 3.6 (Markov Forgetting Lemma)**.** If a Markov chain is irreducible and aperiodic, and has stationary distribution $\{\pi_i\}$, then for all $i, j, k \in S$,

$$\lim_{n\to\infty} \left| p_{ik}^{(n)} - p_{jk}^{(n)} \right| = 0$$

Remark 3.4. Intuitively, after a long time n, the chain “forgets” whether it started from state i or from state j.

Proof. ■ long

Proof of Markov Chain Convergence Theorem long

Corollary 3.7. If a chain is irreducible, then it has at most one stationary distribution.

Proof. By Markov Chain Convergence Theorem, any stationary distribution that ie has must be equal to $\lim_{n\to\infty} P(X_n = j)$, so it is unique. ■

Definition 3.5 (convergence in distribution)**.**

$$\forall a < b, \underset{n\to\infty}{P}(a < X_n < b) = P(a < X < b)$$

Definition 3.6 (weak convergence)**.**

$$\forall \epsilon > 0, \lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

Remark 3.5. This is "converge in probability".

Definition 3.7 (strong convergence)**.**

$$P(\lim_{n\to\infty} X_n = X) = 1$$

Remark 3.6. This is "converge almost surely".

Remark 3.7. Strong convergence implies weak convergence, and weak convergence implies convergence in distribution.

Proposition 3.5. If $\{X_n\}$ is a simple symmetric random walk, then the absolute values $|X_n|$ converge weakly to positive infinity. prove this

3.5 Periodic Convergence

Theorem 3.3 (Periodic Convergence Theorem)**.** Suppose a Markov chain is irreducible, with period $b \geq 2$, and stationary distribution $\{\pi_i\}$. Then for all $i, j \in S$,

$$\lim_{n\to\infty} \frac{1}{b}[p_{ij}^{(n)} + \ldots + p_{ij}^{(n+b-1)}] = \pi_j$$

and

$$\lim_{n\to\infty} \frac{1}{b}(P[X_n = j] + P[X_{n+1} = j] + \ldots + P[X_{n+b-1} = j]) = \pi_j$$

and also

$$\lim_{n\to\infty} \frac{1}{b} P(X_n = j \text{ or } X_{n+1} = j \text{ or } \ \ldots \ \text{ or } X_{n+b-1} = j) = \pi_j$$

Theorem 3.4 (Average Probability Convergence)**.** If a Markov chain is irreducible with stationary distribution $\{\pi_i\}$ (whether periodic or not), then

$$\forall i, j \in S, \lim_{n\to\infty} \frac{1}{n}[p_{ij}^{(1)} + p_{ij}^{(2)} + \ldots + p_{ij}^{(n)}] = \pi_j$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} p_{ij}^{(l)} = \pi_j$$

prove this

Corollary 3.8 (Unique Stationary Corollary)**.** If Markov chain P is irreducible (whether periodic or not), then it has at most **one** stationary distribution.

3.6 Application - Markov Chain Monte Carlo Algorithms

Let S be any contiguous subset of $\mathbb{Z}$.
e.g. $S = \{1, 2, 3\}$, or $S = \{-5, -4, \ldots, 17\}$, or $S = \mathbb{N}$.
Let $\{\pi_i\}$ be any probability distribution on S. Assume for simplicity that $\pi_i > 0$ for all $i \in S$.
Suppose we want to sample from π, i.e., create a random variable X with $P(X = i) \approx \pi_i$ for all $i \in S$.

Metropolis Algorithm Let

$$p_{i,i+1} = \frac{1}{2}\min(1, \frac{\pi_{i+1}}{\pi_i})$$

$$p_{i,i-1} = \frac{1}{2}\min(1, \frac{\pi_{i-1}}{\pi_i})$$

and

$$p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}$$

Fact 3.4. This chain have $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$.

Proof.

$$\pi_i p_{i,i+1} = \pi_i \frac{1}{2}\min(1, \frac{\pi_{i+1}}{\pi_i}) = \frac{1}{2}\min(\pi_i, \pi_{i+1})$$
$$\pi_{i+1} p_{i+1,i} = \pi_{i+1} \frac{1}{2}\min(1, \frac{\pi_i}{\pi_{i+1}}) = \frac{1}{2}\min(\pi_{i+1}, \pi_i)$$
$$\implies \pi_i p_{ij} = \pi_j p_{ji} \text{ if } j = i + 1, \text{ hence for all } i, j \in S$$

Therefore, the chain is reversible w.r.t. $\{\pi_i\}$. So $\{\pi_i\}$ is stationary.
Also, the chain is easily checked to be irreducible and aperiodic.
Then by Markov Chain Convergence Theorem, $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$, and $\lim_{n\to\infty} P[X_n = j] = \pi_j$, for all i, j and v. ∎

Hence, for "large enough" n, X_n is approximately a sample from π.

3.7 Application - Random Walks on Graphs

Let V be a non-empty finite or countable set. Let $w : V \times V \to [0, \infty)$ be a symmetric weight function so that $w(u, v) = w(v, u)$. (usual unweighted case: $w(u, v) = 1$ if there is an edge between u and v, otherwise $w(u, v) = 0$).
Let $d(u) = \sum_{v\in V} w(u, v)$ be the degree of the vertex u. Assume that $d(u) > 0$ for all $u \in V$ (for example, by giving any isolated point a self-edge).

Definition 3.8 ((simple) random walk on the (undirected) graph)**.** Given a vertex set V with symmetric weights w, the (simple) random walk on the (undirected) graph (V, w) is the Markov chain with state space $S = V$ and transition probabilities $p_{uv} = \frac{w(u,v)}{d(u)}$ for all $u, v \in V$.

Remark 3.8. It follows that

$$\sum_{v\in V} p_{uv} = \frac{\sum_{v\in V} w(u,v)}{\sum_{v\in V} w(u,v)} = 1$$

Remark 3.9. The most common case is where each $w(u,v) = 0$ or 1, so from u, the chain moves to one of the $d(u)$ vertices connected to u with equal probability.

Theorem 3.5 (Graph Stationary Distribution)**.** Consider a random walk on a graph V with degrees $d(u)$. Assume that Z is finite. Then if $\pi_u = \frac{d(u)}{Z}$, then π is a stationary distribution for this walk.

Theorem 3.6 (Graph Convergence Theorem)**.** For a random walk on a connected non-bipartite graph, if $Z < \infty$, then $\lim_{n\to\infty} p_{uv}^{(n)} = \frac{d(v)}{Z}$ for all $u, v \in V$, and $\lim_{n\to\infty} P[X_n = v] = \frac{d(v)}{Z}$ (for any initial probabilities).

prove this

Theorem 3.7 (Graph Average Convergence)**.** For a random walk on any connected graph with $Z < \infty$ (whether bipartite or not), for all $u, v \in V$,

$$\lim_{n\to\infty} \frac{1}{2}[p_{uv}^{(n)} + p_{uv}^{(n+1)}] = \frac{d(v)}{Z}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} p_{uv}^{(l)} = \frac{d(v)}{Z}$$

prove this

3.8 Application - Gambler's Ruin

Consider the following gambling game:
Let $0 < a < c$ be integers, and let $0 < p < 1$. Suppose player A starts with a dollars, player B starts with $c - a$ dollars, and they repeatedly bet. At each bet, A wins \$1 from B with probability p, or B wins \$1 from A with probability $1 - p$.
If X_n is the amount of money that A has at time n, then clearly $X_0 = a$, and $\{X_n\}$ follows a simple random walk.
Let $T_i = \inf\{n \geq 0 : X_n = i\}$ be the first time A has i dollars.

The Gambler's Ruin question What is $P_a(T_c < T_0)$, i.e., what is the probability that A reaches c dollars before losing all their money?
Answer: Define $s(a) := P_a(T_c < T_0)$, so that the probability we want to find is a function of the player's initial fortune a. Clearly $s(0) = 0$ and $s(c) = 1$.
For $1 \leq a \leq c - 1$, we have

$$\begin{aligned}
s(a) &= P_a(T_c < T_0) \\
&= P_a(T_c < T_0, X_0 + 1) + P_a(T_c < T_0, X_1 = X_0 - 1) \\
&\qquad\qquad (A \text{ either wins or loses \$1 on the first bet}) \\
&= P(X_1 = X_0 + 1)P_a(T_c < T_0 | X_1 = X_0 + 1) + P(X_1 = X_0 - 1)P_a(T_c < T_0 | X_1 = X_0 - 1) \\
&= ps(a+1) + (1-p)s(a-1)
\end{aligned}$$

This gives $c-1$ equations for the $c-1$ unknowns, which can be solved by simple algebra:

$$\begin{aligned} ps(a) + (1-p)s(a) &= ps(a+1) + (1-p)s(a-1) \qquad \text{(re-arranging)} \\ \implies s(a+1) - s(a) &= \frac{1-p}{p}[s(a) - s(a-1)] \end{aligned}$$

Suppose $s(1) = x$ for some $x \in \mathbb{R}$, then

$$\begin{aligned} s(1) - s(0) &= x \\ s(2) - s(1) &= \frac{1-p}{p}[s(1) - s(0)] = \frac{1-p}{p}x \\ s(3) - s(2) &= \frac{1-p}{p}[s(2) - s(1)] = \left(\frac{1-p}{p}\right)^2 x \\ \implies s(a+1) - s(a) &= \left(\frac{1-p}{p}\right)^a x \qquad \text{(for } 1 \leq a \leq c\text{)} \\ \implies s(a) &= s(a) - s(0) \\ &= [s(a) - s(a-1)] + [s(a-1) - s(a-2)] + \ldots + [s(1) - s(0)] \\ &= \left[\left(\frac{1-p}{p}\right)^{a-1} + \left(\frac{1-p}{p}\right)^{a-2} + \ldots + \left(\frac{1-p}{p}\right) + 1\right] x \\ &= \begin{cases} \left[\frac{(\frac{1-p}{p})^a - 1}{(\frac{1-p}{p}) - 1}\right] x, & p \neq \frac{1}{2} \\ ax, & p = \frac{1}{2} \end{cases} \end{aligned}$$

Since $s(c) = 1$, we can solve for x:

$$x = \begin{cases} \frac{(\frac{1-p}{p}) - 1}{(\frac{1-p}{p})^c - 1}, & p \neq \frac{1}{2} \\ \frac{1}{c}, & p = \frac{1}{2} \end{cases}$$

We then obtain our final **Gambler's Ruin formula**:

$$s(a) = \begin{cases} \frac{(\frac{1-p}{p})^a - 1}{(\frac{1-p}{p})^c - 1}, & p \neq \frac{1}{2} \\ \frac{a}{c}, & p = \frac{1}{2} \end{cases}$$

Remark 3.10. We will sometimes write $s(a)$ as $s_{c,p}(a)$, to show the explicit dependence on c and p.

Example 3.1. $c = 10,000, a = 9,700, p = 0.5$, then

$$s(a) = a/c = 0.97$$

Example 3.2. $c = 10,000, a = 9,700, p = 0.49$, then

$$s(a) \approx \frac{1}{163,000}$$

Proposition 3.6 ()**.** Let $T = \min(T_0, T_c)$ be the time when the Gambler's Ruin game ends. Then $P(T > mc) \le (1-p^c)^m$ where $m \in \mathbb{Z}^+$ and $P(T = \infty) = 0$, and $\mathbb{E}[T] < \infty$.

Proof. (1) If the player ever wins c bets in a row, then the game must be over.
Then if $T > mc$, then the player has failed to win c bets in a row, despite having m independent attempts to do so.
But the probability of winning c bets in a row is p^c. So the probability of failing to win c bets in a row is $1-p^c$. Therefore the probability of failing on m independent attempts is $(1-p^c)^m$, as claimed.
(2) Then by continuity of probabilities,

$$P(T = \infty) = \lim_{m\to\infty} P(T > mc) \le \lim_{m\to\infty} (1-p^c)^m = 0$$

(3) We have

$$\begin{aligned}
E(T) &= \sum_{i=1}^{\infty} P(T \ge i) \\
&\le \sum_{i=0}^{\infty} P(T \ge i) \\
&= P(T \ge 0) + P(T \ge 1) + P(T \ge 2) + P(T \ge 3) + P(T \ge 4) + \dots \\
&\le P(T \ge 0) + P(T \ge 0) + \dots + P(T \ge 0) + P(T \ge c) + P(T \ge c) + \dots \\
&= \sum_{j=0}^{\infty} cP(T \ge cj) \\
&\le \sum_{j=0}^{\infty} c(1-p^c)^j \\
&= \frac{c}{1-(1-p^c)} \\
&= \frac{c}{p^c} < \infty
\end{aligned}$$

■

Remark 3.11. This says that, with probability 1 the Gambler's Ruin game must eventually end, and the time it takes to end has finite expected value.

3.9 Mean Recurrence Times

Definition 3.9 (mean recurrence time)**.** The mean recurrence time of a state i is

$$m_i = E_i(\inf\{n \ge 1 : X_n = i\}) = E_i(\tau_i)$$

where $\tau_i = \inf\{n \ge 1 : X_n = i\}$

Remark 3.12. That is, m_i is the expected value of the time to return from i back to i.

Definition 3.10 (positive recurrence and null recurrence)**.** A state is positive recurrent if $m_i < \infty$. It is null recurrent if it is recurrent but $m_i = \infty$.

Theorem 3.8 (Recurrence Time Theorem)**.** For an irreducible Markov chain, either

1. $m_i < \infty$ for all $i \in S$, and there is a unique stationary distribution given by $\pi_i = 1/m_i$; or

2. $m_i = \infty$ for all $i \in S$, and there is no stationary distribution.

Proposition 3.7. An irreducible Markov chain on a finite state space S always falls into case (i) above:
$m_i < \infty$ for all $i \in S$, and there is a unique stationary distribution given by $\pi_i = 1/m_i$.

Remark 3.13. The converse is false: There could be an example that has infinite state space $S = \mathbb{N}$, but still has a stationary distribution, so it falls into case (i).

3.10 Application - Sequence Waiting Times

Problem Suppose we repeatedly flip a fair coin and get Heads(H) or Tails(T) independently each time with probability 1/2 each. Let τ be the first time the sequence HTH is completed. What is $E[\tau]$?
To find $E[\tau]$, we can use Markov chains.
Let X_n be the partial amount of the desired sequence (HTH) that the chain has "achieved so far" after n flips. Then we always have $X_\tau = 3$, since we "win" upon reaching state 3. Assume we "start over" right after we win ($X_{\tau+1} = 1$ if flip $(\tau + 1)$ is Heads, otherwise $X_{\tau+1} = 0$). Also, we take $X_0 = 0$, i.e., at the beginning we have not achieved any of the sequence.

Here, $\{X_n\}$ is a Markov chain with state space $S = \{0, 1, 2, 3\}$ and $P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}$.

The mean waiting time of HTH is thus equal to the mean recurrence time of state 3.
Using the equation $\pi P = \pi$, it can be computed that the stationary distribution is $(0.3, 0.4, 0.2, 0.1)$. Therefore, by the Recurrence Time Theorem, the mean time to return from state 3 to state 3 (has the same probability as going from state 0 to state 3) is $1/\pi_3 = 10$.

4 Martingales

Roughly speaking, martingales are stochastic processes which "stays the same on average".

4.1 Martingale Definitions

For a formal definition, let $\{X_n\}_{n=0}^{\infty}$ be a sequence of random variables. We assume throughout that random variables X_n have **finite expectation** (or are **integrable**): $E|X_n| < \infty \quad \forall n$.

Definition 4.1 (Martingale)**.** A sequence $\{X_n\}_{n=0}^{\infty}$ is a martingale if for all n,

$$E(X_{n+1}|X_0, \ldots, X_n) = X_n$$

Remark 4.1. No matter what has happened so far, the average of the next value will be equal to the most recent one.

Special case: Markov chain If the sequence $\{X_n\}$ is a Markov chain, then we have

$$\begin{aligned} E[X_{n+1}|X_0 = i_0, \dots, X_n = i_n] &= \sum_{j\in S} jP[X_{n+1}|X_0 = i_0, \dots, X_n = i_n] \\ &= \sum_{j} jP[X_{n+1}|X_n = i_n] \\ &= \sum_{j} jp_{i_n,j} \end{aligned}$$

To be a martingale, this value must equal i_n. That is, a Markov chain (with $E|X_n| < \infty$) is a martingale if

$$\sum_{j\in S} jp_{ij} = i$$

for all $i \in S$.

Example 4.1 (simple symmetric random walk)**.** Let $\{X_n\}$ be s.s.r.w. with $p = 1/2$. We always have $|X_n| \le n$, so $E|X_n| \le n < \infty$, so there is no problem with finite expectations. For all $i \in S$, we compute that $\sum_{j\in S} jp_{ij} = (i+1)(1/2)+(i-1)(1/2) = i$, so s.s.r.w. is indeed a martingale.

Proposition 4.1. If $\{X_n\}$ is a martingale, then by the Law of Total Expectation,

$$\begin{gathered} E(X_{n+1}) = E[E(X_{n+1}|X_0, X_1, \dots, X_n)] = E(X_n) \\ \implies E(X_n) = E(X_0) \quad \forall n \end{gathered}$$

This is not surprising, since martingales stay the same on average. However, this is not a sufficient condition for $\{X_n\}$ to be a martingale.

4.2 Stopping Times

We often want to consider $E(X_T)$ for a random time T. We need to prevent the random time T from looking into the future of the process, before deciding whether to stop.

Definition 4.2 (stopping time)**.** A non-negative-integer-valued random variable T is a stopping time for $\{X_n\}$ if the event $\{T = n\}$ is determined by $X_0, X_1, \dots, X_n$, i.e. if the indicator function $\mathbb{1}\{T = n\}$ is a function of $X_0, X_1, \dots, X_n$.

Remark 4.2. Intuitively, this definition says that a stopping time T must decide whether to stop at time n based solely on what has happened up to time n, without first looking into the future.

Example 4.2. valid stopping times:
$T = 5, T = \inf\{n \ge 0 : X_n = 5\}, T = \inf\{n \ge 0 : X_n = 0 \lor X_n = c\}, T = \inf\{n \ge 2 : X_{n-2} = 5\}$ not valid stopping time: $T = \inf\{n \ge 0 : X_{n+1} = 5\}$ (since it looks into the future)

Lemma 4.1 (Optional Stopping Lemma)**.** If $\{X_n\}$ is a martingale, and T is a stopping time which is bounded (i.e., $\exists M < \infty$ with $P(T \le M) = 1$), then

$$E(X_T) = E(X_0)$$

prove this!

Example 4.3. Consider s.s.r.w. with $X_0 = 0$, and let

$$T = \min\{10^{12}, \inf\{n \geq 0 : X_n = -5\}$$

Then T is a bounded stopping time. Hence by the Optional Stopping Lemma,

$$E(X_T) = E(X_0) = E(0) = 0$$

But near always, we will have $X_T = -5$.
By the Law of Total Expectation,

$$\begin{aligned} 0 &= E(X_T) \\ &= \underbrace{P(X_T = -5)}_{\approx 1}\underbrace{E(X_T|X_T = -5)}_{=-5} + \underbrace{P(X_T \neq -5)}_{\approx 0}\underbrace{E(X_T|X_T \neq -5)}_{huge} \end{aligned}$$

Theorem 4.1 (Optional Stopping Theorem)**.** If $\{X_n\}$ is a martingale with stopping time T, and $P(T < \infty) = 1$, and $E|X_T| < \infty$, and $\lim\limits_{n\to\infty} E(X_n \mathbb{1}\{T > n\}) = 0$, then

$$E(X_T) = E(X_0)$$

Proof. For each $m \in \mathbb{N}$, let $S_m = \min\{T, m\}$, so that S_m is a bounded stopping time.
Then by Optional Stopping Lemma, $E(X_{S_m}) = E(X_0)$ (for any m).
Then for any m,

$$\begin{aligned} X_{S_m} &= X_{\min(T,m)} \\ &= X_T \mathbb{1}\{T \leq m\} + X_m \mathbb{1}\{T > m\} \\ &= X_T(1 - \mathbb{1}\{T > m\}) + X_m \mathbb{1}\{T > m\} \\ &= X_T - X_T \mathbb{1}\{T > m\} + X_m \mathbb{1}\{T > m\} \\ \implies X_T &= X_{S_m} + X_T \mathbb{1}\{T > m\} - X_m \mathbb{1}\{T > m\} \\ \implies E(X_T) &= E(X_{S_m}) + E(X_T \mathbb{1}\{T > m\}) - E(X_m \mathbb{1}\{T > m\}) \\ &= E(X_0) + E(X_T \mathbb{1}\{T > m\}) - E(X_m \mathbb{1}\{T > m\}) \end{aligned}$$

Take $m \to \infty$. Since $P(T < \infty) = 1$, we have $\mathbb{1}\{T > m\} \to 0$.
Since $E|X_T| < \infty$ and $\mathbb{1}\{T > m\} \to 0$, we have

$$\lim_{m\to\infty} E(X_T \mathbb{1}\{T > m\}) = 0$$

by the Dominated Convergence Theorem 1.3
Also, $\lim\limits_{m\to\infty} E(X_m \mathbb{1}\{T > m\}) = 0$ by assumption.
Hence $E(X_T) \to E(X_0)$, i.e. $E(X_T) = E(X_0)$. ■

Corollary 4.1 (Optional Stopping Corollary)**.** If $\{X_n\}$ is a martingale with stopping time T, which is "bounded up to time T" (i.e., $\exists M < \infty$ with $P(|X_n|\mathbb{1}\{n \leq T\} \leq M) = 1$ for all n), and $P(T < \infty) = 1$, then

$$E(X_T) = E(X_0)$$

Proof. It follows that, $P(|X_T| \leq M) = 1$.
Hence, $E|X_T| \leq M < \infty$.
Also,

$$\begin{aligned} |E(X_n \mathbb{1}\{T > n\})| &\leq E(|X_n| \mathbb{1}\{T > n\}) \\ &= E(|X_n| \mathbb{1}\{n \leq T\} \mathbb{1}\{T > n\}) \\ &\leq E(M \mathbb{1}\{T > n\}) \\ &= MP(T > n) \to 0 \qquad \text{(Since } P(T < \infty) = 1\text{)} \end{aligned}$$

Hence the result follows from the Optional Stopping Theorem. ■

Example 4.4 (Gambler's Ruin problem - $p = 1/2$)**.** Let $T = \inf\{n \geq 0 : X_n \vee X_n = c\}$ be the time when the game ends. Then $P(T < \infty) = 1$ by Proposition 3.6. Also, if the game has not yet ended, i.e. $n \leq T$, then X_n must be between 0 and c. Hence $|X_n| \mathbb{1}\{n \leq T\} \leq c < \infty$ for all $n \leq T$.
So by the Optional Stopping Corollary 4.1, $E(X_T) = cs(a) + 0(1 - s(a) = E(X_0) = a \implies s(a) = a/c$.

Example 4.5 (Gambler's Ruin problem - $p \neq 1/2$)**.** Then $\{X_n\}$ is not a martingale since

$$\sum_j j p_{ij} = p(i+1) + (1-p)(i-1) = i + 2p - 1 \neq i$$

Instead we use a trick: Let $Y_n := \left(\frac{1-p}{p}\right)^{X_n}$, then $\{Y_n\}$ is also a Markov chain, and

$$\begin{aligned} E(Y_{n+1} | Y_0, Y_1, \ldots, Y_n) &= p \left(\frac{1-p}{p}\right)^{X_n + 1} + (1-p) \left(\frac{1-p}{p}\right)^{X_n - 1} \\ &= p \left[Y_n \left(\frac{1-p}{p}\right)\right] + (1-p) \left[Y_n / \left(\frac{1-p}{p}\right)\right] \\ &= Y_n(1-p) + Y_n(p) \\ &= Y_n \end{aligned}$$

So $\{Y_n\}$ is a martingale.
Again, $P(T < \infty) = 1$ by Proposition 3.6.
Also, $|Y_n| \mathbb{1}\{n \leq T\} \leq \max\left(\left(\frac{1-p}{p}\right)^0, \left(\frac{1-p}{p}\right)^c\right) := M < \infty$ for all n. Hence by the Optional Stopping Corollary 4.1,

$$E(Y_T) = s(a) \left(\frac{1-p}{p}\right)^c + [1 - s(a)](1) = E(Y_0) = \left(\frac{1-p}{p}\right)^a$$

$$\implies s(a) = \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1}$$

4.3 Wald's Theorem

Suppose $X_n = a + Z_1 + \ldots + Z_n$, where $\{Z_i\}$ are i.i.d. with finite mean m. Let T be a stopping time for $\{X_n\}$ which has finite mean, i.e. $E(T) < \infty$. Then

$$E(X_T) = a + mE(T)$$

Property 4.1 (Special case: $m = 0$)**.** Then $\{X_n\}$ is a martingale, and Optional Stopping Theorem 4.1 says that $E(X_T) = a = E(X_0)$.

prove this!

Corollary 4.2. If $\{X_n\}$ is Gambler's Ruin with $p \neq 1/2$, and $T = \inf\{n \geq 0 : X_n = 0 \vee X_n = c\}$, then

$$E(T) = \frac{1}{2p-1}\left(c\frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1} - a\right)$$

Proof. We again apply Wald's Theorem:
Here $Z_i = +1$ if you win the ith bet, otherwise $Z_i = -1$. So

$$m = E(Z_i) = p(1) + (1-p)(-1) = 2p - 1$$

Also, $E(T) < \infty$ by Proposition 3.6. Then by Wald's Theorem,

$$\begin{aligned} E(X_T) &= a + mE(T) \\ &= cs(a) + 0(1 - s(a)) \\ &= c\frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1} \\ \implies E(T) &= \frac{1}{m}(E(X_T) - a) \\ &= \frac{1}{2p-1}\left(c\frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1} - a\right) \end{aligned}$$

■

Lemma 4.2. Let $X_n = a + Z_1 + \ldots + Z_n$, where $\{Z_i\}$ are i.i.d. with mean 0 and variance $v < \infty$. Let $Y_n = (X_n - a)^2 - nv = (Z_1 + \ldots + Z_n)^2 - nv$. Then $\{Y_n\}$ is a martingale. prove this!

Corollary 4.3. If $\{X_n\}$ is Gambler's Ruin with $p = 1/2$, and $T = \inf\{n \geq 0 : X_n = 0 \vee X_n = c\}$, then

$$E(T) = Var(X_T) = a(c - a)$$

prove this!

4.4 Application - Sequence Waiting Times

Suppose at each time n, a new "player" appears, and bets \$1 on heads, then if they win they bet \$2 on tails, then if they win again they bet \$4 on heads. Each player stops betting as soon as they either lose once (and hence are down a total of \$1), or win three bets in a row (and hence are up a total of \$7.
Let X_n be the total amount won by all the betters by time n. Then since the bets were fair, $\{X_n\}$ is a martingale with stopping time τ.

4.5 Martingale Convergence Theorem

Suppose $\{X_n\}$ is a martingale. Then $\{X_n\}$ could have infinite fluctuations in both directions, as we have seen for s.s.r.w.; Or $\{X_n\}$ could converge with probability 1 to a fixed (perhaps random) value.

Example 4.6. Let $\{X_n\}$ be Gambler's Ruin with $p = 1/2$, where we stop as soon as we either win or lose. Then $X_n \to X$ with probability 1, where $P(X = c) = a/c$ and $P(X = 0) = 1-a/c$.

Example 4.7. Let $\{X_n\}$ be a Markov chain on $S = \{2^m : m \in \mathbb{Z}\}$, with $X_0 = 1$, and $p_{i,2i} = 1/2$ and $p_{i,i/2} = 2/3$ for $i \in S$. This is a martingale, since $\sum_j jp_{ij} = (2i)(1/3)+(i/2)(2/3) = i$. Let $Y_n = \log_2 X_n$. Then $Y_0 = 0$, and $\{Y_n\}$ is s.r.w. with $p = 1/3$, $Y_n \to -\infty$ w.p. 1 by the Law of Large Numbers 1.4. Hence, $X_n = 2^{Y_n} \to 2^{-\infty} = 0$ w.p. 1.

Theorem 4.2 (Martingale Convergence Theorem)**.** Any non-negative martingale $\{X_n\}$ ($X_n \geq 0$) which is bounded below (i.e. $X_n \geq c$ for all n, for some finite number c), or is bounded above (i.e. $X_n \leq c$ for all n, for some finite number c), converges w.p. 1 to some random variable X.

Remark 4.3. The intuition behind this theorem is:

1. Since the martingale is bounded on one side, it cannot "spread out" forever.
2. Since it is a martingale, it cannot "drift" in a positive or negative direction.
3. So it has somewhere to go, and eventually has to stop somewhere.

Remark 4.4. If $\{X_n\}$ is not non-negative, then if $X_n \geq c$, then $\{X_n - c\}$ is a non-negative martingale, or if $X_n \leq c$, then $\{-X_n + c\}$ is a non-negative martingale, and in either case the non-negative martingale converges iff $\{X_n\}$ converges.

4.6 Application - Branching Processes

Definition 4.3 (offspring distribution)**.** Let μ be any prob dist on $\{0, 1, 2, \ldots\}$, the offspring distribution.
Let X_n be the number of individuals at time n. Start with $X_0 = a$ individuals. Assume $0 < a < \infty$, $Z_{n,i} \overset{i.i.d.}{\sim} \mu(i)$.
(i.e., Each of the X_n individuals at time n has a random number of offspring under the distribution μ). Then

$$X_{n+1} = Z_{n,1} + Z_{n,2} + \ldots + Z_{n,X_n}$$

Here $\{X_n\}$ is a Markov chain, on the state space $\{0, 1, 2, \ldots\}$.

Transition probabilties If X_n ever reaches 0, then it stays there forever: $p_{0j} = 0 \quad \forall j \geq 0$. This is called extinction.
Also, $p_{ij} = (\mu * \mu * \ldots * \mu)(j)$, a convolution of i copies of μ.

Theorem 4.3. Let $m = \sum_i \mu(i)$ be the mean of μ, which is called the reproductive number. If $m < 1$, then $E(X_n) \to 0$, and $P(X_n = 0) \to 1$.

Proof. Assume $0 < m < \infty$. Then

$$E(X_{n+1}|X_0, \ldots, X_n) = E(Z_{n,1} + Z_{n,2} + \ldots + Z_{n,X_n}|X_0, \ldots, X_n) = mX_n$$

$$\implies E(X_n) = m^n E(X_0) = m^n a < \infty$$

So if $m < 1$, then $E(X_n) = am^n \to 0$. Then we have

$$\begin{aligned} E(X_n) &= \sum_{k=0}^{\infty} kP(X_n = k) \\ &\geq \sum_{k=1}^{\infty} P(X_n = k) \\ &= P(X_n \geq 1) \\ \implies P(X_n \geq 1) &\leq E(X_n) = am^n \to 0 \\ \implies P(X_n = 0) &\to 1 \end{aligned}$$

■

Fact 4.1. Let $m = \sum_i \mu(i)$ be the mean of μ, which is called the reproductive number. If $m > 1$, then $E(X_n) \to \infty$, $P(X_n \to \infty) > 0$ and $P(X_n \to 0) > 0$, i.e., we have possible extinction but also possible flourishing.

Theorem 4.4. Let $m = \sum_i \mu(i)$ be the mean of μ, which is called the reproductive number. If $m = 1$, and μ is non-degenerate (i.e. $\mu(1) < 1$, so that μ is not a constant), then $\{X_n\} \to 0$ w.p. 1.

Proof. If $m = 1$, then $E(X_n) = E(X_0) = a$ for all n. Then $E(X_{n+1}|X_0, \ldots, X_n) = mX_n - X_n$, so $\{X_n\}$ is a non-negative martingale.
Hence by the Martingale Convergence Theorem 4.2, we must have $X_n \to X$ for some random variable X. This could only happen if

1. $\mu(1) = 1$; or
2. $X = 0$

■

4.7 Application - Stock Options (Discrete)

In mathematical finance, it is common to model the price of one share of some stock as a random process.
For now, we work in discrete time, and suppose that X_n is the price of one share of the stock at each date n. If you buy the stock, then the situation is clear: if X_n increases then you will make a profit, but if X_n decreases then you will suffer a loss.

Definition 4.4 (stock option)**.** A stock option is the option to buy one share of the stock for some fixed strike price K at some fixed future strike date (time) $S > 0$. If at the strike time S, the stock price X_s is less than the strike price K, then the option would not be exercised, and would thus be worth exactly zero. If the stock price X_S is more than K, then the option would be exercised to obtain a stock worth X_S for a price of just K, for a net profit of $X_S - K$. Hence at time S, the stock option is worth $\max(0, X_S - K)$.

Remark 4.5. At time 0, X_S is an unknown quantity. The fair price of a stock option is defined to be the no-arbitrage price, i.e., the price for the option which makes it impossible to make a guaranteed profit through any combination of buying or selling the option, and buying and selling the stock. At time 0, what is the fair price (no-arbitrage price of the stock option?

Example 4.8 (naive example)**.** Suppose that at time 0, you buy x stock shares (for \$100 each), and y option shares (for \$$c$ each) where $x, y \in \mathbb{R}$ (negative values indicates selling).
Then if the stock goes up to \$130, you make \$30 on each stock share and \$$(20 - c)$ on each option share for a total profit of $30x + (20 - c)y$.
But if the stock goes down to \$80, you lose \$20 on each stock share and \$$c$ on each option share for a total profit of $-20x - cy$.
To attempt to make a guaranteed profit, we could make these two different total profit amounts equal to each other $\implies y = (-5/2)x$, profits $= (5/2)(c - 8)x$.
If $c > 8$, then you buy $x > 0$ stock shares and $y = (-5/2)x < 0$ option shares and make a guaranteed profit of $(5/2)(c - 8)x > 0$.
If $c < 8$, then you buy $x < 0$ stock shares and $y = (-5/2)x < 0$ option shares and make a guaranteed profit of $(5/2)(c - 8)(-x) > 0$.
But if $c = 8$, then profits $= 0$.
In summary, there is no arbitrage iff $c = 8$.

Example 4.9. Suppose we assign the new probabilities $P(X_s = 80) = 3/5$ and $P(X_S = 130) = 2/5$. Then the stock price is a martingale since $E(X_S) = (3/5)80 + (2/5)(130) = 100 =$ initial price. The option price is a martingale since $option_value = (3/5)0 + (2/5)(130 - 110) = 8 = c =$ initial price.
Then the fair price is the martingale expected value, 8.

Theorem 4.5 (Martingale Pricing Principle)**.** The fair price of an option is equal to its expected value (worth) under the martingale probabilities.

Proposition 4.2. Suppose a stock price at time 0 equals $X_0 = a$, and at strike date $S > 0$ equals either $X_s = d$ (down) or $X_s = u$ (up), where $d < a < u$. Then if $d < K < u$ then at time 0, the fair (no-arbitrage) price of an option to buy the stock at time S for strike price K is equal to $(a - d)(u - K)/(u - d)$.

prove this!

Proof. **Profit Computation**
Suppose you buy x shares of the stock for \$$a$ per stock, plus y shares of the option for \$$c$ per share.
Then if the stock goes up to $X_S = u$, your profits is $x(u-a)+y(u-K-c)$. If the stock goes down to $X_S = d$, your profit is $x(d-a)+y(-c)$.
These are equal if $x(d-u) = y(u-K) \iff y = \frac{-x(u-d)}{u-K}$.
If there is no arbitrage, then your guaranteed profit is 0, which equals

$$x(d-a) - yc = x(d-a) + \frac{xc(u-d)}{u-K} = 0 \implies c = \frac{(a-d)(u-K)}{u-d}$$

■

Proof. **Martingale Pricing Principle**
We need to find martingale probabilities $q_1 = P(X_S = d)$ and $q_2 = P(X_S = u)$ to make the stock price a martingale. So we need that

$$\begin{aligned}
dq_1 + uq_2 &= a \\
\implies d_1 + u(1-q_1) &= a \\
\implies (d-u)q_1 + u &= a \\
\implies q_1 = \frac{u-a}{u-d}, &\quad q_2 = 1 - q_1 = \frac{a-d}{u-d}
\end{aligned}$$

Then by the Martingale Pricing Principle, the fair price of the option is the martingale expectation of the option's worth, which equals

$$q_1(0) + q_2(u-K) = \frac{(a-d)(u-K)}{u-d}$$

■

5 Continuous Processes

So far, we have mostly considered discrete processes, where the time is indexed by non-negative integers, and the process takes on a finite or countable number of different values.
We now consider various generalizations of this to continuous time and/or space. We begin with a continuous generalization of symmetric simple random walk, called Brownian motion.

5.1 Brownian Motion

Let $\{X_n\}_{n=0}^{\infty}$ be a symmetric simple random walk with $X_0 = 0$.
We have

$$X_n = \begin{cases} Z_1 + Z_2 + \ldots + Z_n = X_n + Z_n, & n \geq 1 \\ 0, & n = 0 \end{cases}$$

where $\{Z_i\}$ are i.i.d. with $P(Z_i = +1) = P(Z_i = -1) = \frac{1}{2}$.
Let M be a large integer, and let $\{Y_t^{(M)}\}$ be like $\{X_n\}$, except with time sped up by a factor of M, and space shrunk down by a factor of $\sqrt{M}$. We have $Y_0^{(M)} = 0$ and

$$Y^{(M)}_{\frac{i+1}{M}} = Y^{(M)}_{\frac{i}{M}} + \frac{1}{\sqrt{M}} Z_{i+1}$$

Fill in $\{Y_t^{(M)}\}_{t\geq 0}$ by linear interpolation.
<u>Brownian motion</u> $\{B_t\}_{t\geq 0}$ is (intuitively) the limit as $M \to \infty$ of $\{Y_t^{(M)}\}$.
Since $Y_0^{(M)} = 0$ for all M, also $B_0 = 0$. Also, note that $Y_t^{(M)} = \frac{1}{\sqrt{M}}(Z_1 + Z_2 + \ldots + Z_{tM})$ (at least if $tM \in \mathbb{Z}$, otherwise within $\mathcal{O}(1/\sqrt{M})$, which)