

# MAT337

## Lecture Notes

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## 1 Real Numbers

### 1.1 Discussion: The Irrationality of $\sqrt{2}$

If we make natural numbers  $\mathbb{N}$  closed under subtraction, we obtain

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$$

If we take the closure of  $\mathbb{Z}$  under division by non-zero numbers, we obtain

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1 \right\}$$

**Remark 1.1.**  $(m, n) = 1$  means that if  $d \in \mathbb{N}$  divides both  $m$  and  $n$ , then  $d = 1$ .

**Theorem 1.1.** There is no  $r \in \mathbb{Q}$  s.t.  $r^2 = 2$ .

*Proof.* Assume for contradiction that there are  $m \in \mathbb{Z}, n \in \mathbb{N}$  s.t.  $\frac{m}{n} = \sqrt{2}$  and  $(m, n) = 1$ .

Then  $m^2 = 2n^2$  so that  $m^2$  is an even complete square.

Suppose  $m = p_1 \dots p_r$  where  $p_i$ s are prime numbers. Then  $2n^2 = m^2 = p_1^2 \dots p_r^2 \implies p_i^2 = 2^2$ .

Then  $4|m^2$  and  $2|n^2$ , so  $n$  has to be even. Therefore both  $m$  and  $n$  are even.

Then  $2|m$  and  $2|n$ , which leads to a contradiction that if  $d \in \mathbb{N}$  divides both  $m$  and  $n$ , then  $d = 1$ . ■

### 1.2 Preliminaries

**Definition 1.1** (set). A set is any collection of objects.

**Definition 1.2** (function). Given two sets  $A$  and  $B$ , a function from  $A$  to  $B$  is a rule or mapping that takes each element  $x \in A$  and associates with it a single element of  $B$ . In this case, we write  $(f : A \rightarrow B)$ . It is the set of pairs  $(A, B) \in A \times B$  s.t.

1. If  $(x, y_1) \in f$  and  $(x, y_2) \in f$ , then  $y_1 = y_2$ .
2. For all  $x \in A$ , there is some  $y \in B$  s.t.  $f(x) = y$ .

The set  $A$  is said to be the domain of  $f$ . The range of  $f$  is not necessarily equal to  $B$  but refers to the subset of  $B$  given by  $\{y \in B : y = f(x) \text{ for some } x \in A\}$ .

**Example 1.1** (absolute value function). For every  $x$ ,

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

**Theorem 1.2** (triangle inequality).

$$|x + y| \leq |x| + |y|$$

*Proof.*

$$\begin{aligned}
 (x + y)^2 &= x^2 + y^2 + 2xy \\
 &\leq |x|^2 + |y|^2 + 2|x||y| \\
 &= (|x| + |y|)^2 \\
 \implies |x + y| &= \sqrt{(x + y)^2} \\
 &\leq \sqrt{(|x| + |y|)^2} \\
 &= ||x| + |y|| \\
 &= |x| + |y|
 \end{aligned}$$

■

**Definition 1.3** (maximum and minimum). Assume set  $X \subseteq \mathbb{R}$ . Then the maximum (minimum) of  $X$  is an element  $a \in X$  s.t. for all  $x \in X, x \leq a$  ( $x \geq a$ ).

**Definition 1.4** (least upper bound / supremum). The least upper bound of  $X$  (denoted by  $\sup(X)$ ) is a real number  $a \in \mathbb{R}$  s.t.

1. For all  $x \in X, x \leq a$  (this means that  $a$  is an upper bound for  $X$ )
2. If  $b$  is an upper bound for  $X$ , then  $a \leq b$

**Example 1.2.**

$$\begin{aligned}
 \max([0, 1]) &= 1 \\
 \min([0, 1]) &= 0 \\
 \sup((0, 1)) &= 1 \\
 \sup(\mathbb{R}), \sup(\mathbb{N}) &DNE
 \end{aligned}$$

### 1.3 The axiom of completeness

**Definition 1.5** (initial segment).  $X \subseteq \mathbb{Q}$  is said to be an initial segment if

1.  $X \neq \emptyset$
2. For all  $x, y \in \mathbb{Q}$ , if  $x < y$  and  $y \in X$ , then  $x \in X$ .
3.  $X \neq \mathbb{Q}$

**Alternative definition:** Let  $(A, \leq)$  be a well-ordered set. Then the set

$$\{a \in A : a < k\}$$

for some  $k \in A$  is called an initial segment of  $A$ .

**Definition 1.6** (real numbers).  $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$

**Lemma 1.1** (supremum). Suppose  $A \subseteq \mathbb{R}$  and  $s \in \mathbb{R}$  is an upper bound for  $A$ . If  $\forall \epsilon > 0, \exists a \in A, a + \epsilon > s$ , then  $s = \sup(A)$

*Proof.* ( $\Leftarrow$ ) Assume for contradiction that  $t \in \mathbb{R}$  is an upper bound for  $A$  and  $t < s$ .

Let  $\epsilon = \frac{s-t}{2}$ . Obviously  $\epsilon > 0$ .

But then  $\forall a \in A, a + \epsilon \leq t + \epsilon < s$ , which is a contradiction.

( $\Rightarrow$ ) Assume for contradiction that  $\epsilon_0 > 0$  and  $\forall a \in A, a + \epsilon \leq S$

Then  $\forall a \in A, a \leq S - \epsilon_0$ .

So  $s - \epsilon_0$  is an upper bound for  $A$ , which is a contradiction that  $a + \epsilon > s$ . ■

**Theorem 1.3** (the Axiom of Completeness). If  $X \subset \mathbb{R}$  is bounded above, then  $X$  has a least upper bound.

*Proof.* For  $x \in X$ , let  $Ax$  be the initial segment of  $\mathbb{Q}$  corresponding to  $x$ .

Since  $X$  is bounded above, pick  $b \in \mathbb{R}$  s.t.  $\forall x \in X, x < b$ . Then  $b \notin \bigcup_{x \in X} Ax$ . Note that  $\bigcup_{x \in X} Ax$  is an initial segment of  $\mathbb{Q}$ . Then  $\sup(\bigcup_{x \in X} Ax)$  is  $\sup(X)$ . ■

## 1.4 Consequences of Completeness

**Definition 1.7** (nested sequence of sets). Assume  $\langle A_n : n \in \mathbb{N} \rangle$  is a sequence of sets.  $\langle A_n : n \in \mathbb{N} \rangle$  is said to be nested if

$$A_{n+1} \subseteq A_n$$

**Theorem 1.4** (Nested Interval Property). Assume  $\langle I_n : n \in \mathbb{N} \rangle$  is a nested sequence of **closed intervals of  $\mathbb{R}$** . Then

$$\bigcap_n I_n \neq \emptyset$$

*Proof.* Let  $[a_n, b_n] = I_n$  where  $a_n, b_n \in \mathbb{R}$ .

Since  $\langle I_n | n \in \mathbb{N} \rangle$  is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad (\dagger)$$

for all  $n \in \mathbb{N}$

Let  $A = \{a_n : n \in \mathbb{N}\}$ .

Note that  $b_1$  is an upper bound for  $A$ . So  $A$  has a supremum in  $\mathbb{R}$ .

We claim that  $\sup(A) \in \bigcap_n I_n$ .

By  $(\dagger)$ , for all  $n \in \mathbb{N}, \sup(A) \leq b_n$

Obviously, for all  $n \in \mathbb{N}, \sup(A) \geq a_n$

So  $\forall n \in \mathbb{N}, a_n \leq \sup(A) \leq b_n$ .

Therefore  $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$ . ■

**Example 1.3.**

$$\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset$$

$$\bigcap_{n \in \mathbb{N}} [0, \frac{1}{n}] = \{0\}$$

**Theorem 1.5** (Archimedian Property). We have

1. For every  $y \in \mathbb{R}$ , there is  $n \in \mathbb{N}$  s.t.  $y \leq n$ .

2. For every  $y > 0$ , there is  $n \in \mathbb{N}$  s.t.  $\frac{1}{n} < y$ .

*Proof.* (1) Assume for contradiction that  $\mathbb{N}$  is bounded in  $\mathbb{R}$ .

Let  $\alpha = \sup(\mathbb{N})$ . Then there is a natural number  $n \in \mathbb{N}$  s.t.  $n > \alpha - 1$ .

But then  $n + 1 > (\alpha - 1) + 1 = \alpha$ , which is a natural number greater than  $\alpha$ , contradiction.

(2) Exercise. ■

**Theorem 1.6** (density of  $\mathbb{Q}$  in  $\mathbb{R}$ ). For every two real numbers  $a$  and  $b$  with  $a < b$ , there exists a rational number  $r$  satisfying  $a < r < b$ .

*Proof.* Let  $n \in \mathbb{N}$  s.t.  $\frac{1}{n} < b - a$ ,  $1 < nb - na$ .

Let  $m \in \mathbb{Z}$  s.t.  $na < m < nb$ .

Then  $a < \frac{m}{n} < b$ .

Pick  $r = \frac{m}{n}$  and we are done. ■

## 1.5 Cardinality

“The size of a set”

### 1.5.1 1-1 Correspondence

**Definition 1.8** (one-to-one and onto). A function  $f : A \rightarrow B$  is one-to-one (1-1) if  $a_1 \neq a_2$  in  $A$  implies that  $f(a_1) \neq f(a_2)$  in  $B$ . The function  $f$  is onto if, given any  $b \in B$ , it is possible to find an element  $a \in A$  for which  $f(a) = b$ .

**Proposition 1.1.** If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  is 1-1, then  $g \circ f : A \rightarrow C$  is 1-1.

**Remark 1.2.** If a function  $f : A \rightarrow B$  is both 1-1 and onto, then there is a 1-1 correspondence between two sets.

**Definition 1.9** (the same cardinality). The set  $A$  has the same cardinality as  $B$  if there exists  $f : A \rightarrow B$  that is 1-1 and onto. In this case, we write  $A \sim B$ .

**Proposition 1.2.** If  $A \sim B$ ,  $B \sim C$ , then  $A \sim C$

**Proposition 1.3.** If  $\text{Card}(A) \leq \text{Card}(B) \leq \text{Card}(C)$ , then  $\text{Card}(A) \leq \text{Card}(C)$

### 1.5.2 Countable Sets

A set  $A$  is countable if  $\mathbb{N} \sim A$ . An infinite set that is not countable is called an uncountable set.

**Theorem 1.7.** The set  $\mathbb{Q}$  is countable.

*Proof.* Set  $A_1 = \{0\}$  and for each  $n \geq 2$ , let  $A_n$  be the set given by

$$A_n = \left\{ \pm \frac{p}{q} : \text{where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}$$

e.g.  $A_2 = \left\{ \frac{1}{1}, \frac{-1}{1} \right\}$ ,  $A_3 = \left\{ \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1} \right\}$

<b>N :</b>	1	2	3	4	5	6	7	8	9	10	11	12	...
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	
<b>Q :</b>	0	$\frac{1}{1}$	$-\frac{1}{1}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{2}{1}$	$-\frac{2}{1}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{3}{1}$	$-\frac{3}{1}$	$\frac{1}{4}$	...
	$\underbrace{\hspace{1.5cm}}_{A_1}$		$\underbrace{\hspace{1.5cm}}_{A_2}$		$\underbrace{\hspace{2.5cm}}_{A_3}$			$\underbrace{\hspace{3.5cm}}_{A_4}$					

The above correspondence is onto because every rational number appears in the correspondence exactly once. The above correspondence is 1-1 because  $A_N$  were constructed to be disjoint so that no rational number appears twice. ■

**Theorem 1.8.** The set  $\mathbb{R}$  is uncountable.

*Proof.* Assume for contradiction that there does exist a bijection function  $f : \mathbb{N} \rightarrow \mathbb{R}$ . Let  $x_1 = f(1), x_2 = f(2)$  and so on. Then since  $f$  is onto, can write

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\} \quad (1)$$

and be confident that every real number appears somewhere on the list.

We will now use the Nested Interval Property to produce a real number that is not there. Let  $I_1$  be a closed interval that does not contain  $x_1$ . given an interval  $I_n$ , construct  $I_{n+1}$  to satisfy  $I_{n+1} \subseteq I_n$  and  $x_{n+1} \notin I_{n+1}$ .

If  $x_{n_0}$  is some real number from the list in (1), then we have  $x_{n_0} \notin I_{n_0}$ , and it follows that

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$$

Since we are assuming that the list in (1) contains every real number, then

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

However, the NIP asserts that  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ , which is a contradiction. ■

**Theorem 1.9.** If  $A \subseteq B$  and  $B$  is countable, then  $A$  is either countable or finite.

**Theorem 1.10.** We have

- (i) If  $A_1, A_2, \dots, A_m$  are countable sets, then the union  $A_1 \cup A_2 \cup \dots \cup A_m$  is countable.
- (ii) If  $A_n$  is a countable set for each  $n \in \mathbb{N}$ , then  $\bigcup_{n=1}^{\infty} A_n$  is countable.

**Theorem 1.11.** The open interval  $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$  is uncountable.

## 1.6 Cantor's Theorem

**Notation 1.1.** Given a set  $A$ , the power set  $P(A)$  refers to the collection of all subsets of  $A$ .

**Theorem 1.12** (Cantor's Theorem). Given any set  $A$ , there does not exist a function  $f : A \rightarrow P(A)$  that is onto.

*Proof.* Assume, for contradiction, that  $f : A \rightarrow P(A)$  is onto. For each element  $a \in A$ ,  $f(a)$  is a particular subset of  $A$ . The assumption that  $f$  is onto means that every subset of  $A$  appears as  $f(a)$  for some  $a \in A$ . To arrive at a contradiction, we will produce a subset  $B \subseteq A$  that is not equal to  $f(a)$  for any  $a \in A$ .

Construct  $B$  using the following rule. For each element  $a \in A$ , consider the subset  $f(a)$ . This subset of  $A$  may contain the element  $a$  or it may not. This depends on the function  $f$ . If  $f(a)$  does not contain  $a$ , then we include  $a$  in our set  $B$ : Let

$$B = \{a \in A : a \notin f(a)\}$$

Since we have assumed that our function  $f : A \rightarrow P(A)$  is onto, it must be that  $B = f(a')$  for some  $a' \in A$ .

**Case 1**  $a' \in B$

Then  $a' \notin f(a') = B$ , a contradiction.

**Case 2**  $a' \notin B$

Then  $a' \in f(a') = B$ , a contradiction. ■

**Theorem 1.13** (Schröder-Bernstein Theorem). If there are 1-1 functions  $f : A \rightarrow B$  and  $h : B \rightarrow A$ , then there is a bijection  $g : A \rightarrow B$ .

*Proof.* **Claim:** the statement of the theorem is equivalent to the following:

If  $B \subseteq A$  and  $f : A \rightarrow B$  is 1-1, then there is a bijection  $g : A \rightarrow B$ . (\*)

**proof of claim:** theorem  $\implies$  (\*):

Take  $h : X \rightarrow Y$  with  $h(x) = x$ , then  $X \subseteq Y$ .

(\*)  $\implies$  theorem:

Let  $f : A \rightarrow B$  and  $h : B \rightarrow A$  be 1-1 functions, as in the theorem. We need to show that there is bijection  $g : A \rightarrow B$ .

Notice that  $A \subseteq h(B)$  and  $h \circ f : A \rightarrow h(B)$  is a 1-1 function. So by (\*), there is a bijection  $g_0 : A \rightarrow h(B)$ .

But  $h : B \rightarrow h(B)$  is also a bijection. So  $g = h^{-1} \circ g_0 : A \rightarrow B$  is a bijection (using the fact that bijections are closed under compositions).

Now it suffices to prove (\*).

Assume set  $X \subseteq Y$  and  $f : Y \rightarrow X$ . Let  $W = \bigcup_{n=0}^{\infty} f^n(Y \setminus X)$ .

Define  $g : Y \rightarrow X$  by:

- If  $y \in W$ , then  $g(y) = f(y)$
- If  $y \in Z := Y \setminus W$ , then  $g(y) = y$

We need to show that  $g : Y \rightarrow X$  is a well-defined bijection.

Since  $f$  is 1-1, for all  $m < n$ ,  $f^m(Y \setminus X) \cap f^n(Y \setminus X) = \emptyset$



Note that

$$\begin{aligned}
 Y \setminus W &= Y \setminus \bigcup_{n=0}^{\infty} f^n(Y \setminus X) \\
 &= [Y \setminus (Y \setminus X)] \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X) \\
 &= X \setminus \bigcup_{n=1}^{\infty} f^n(Y \setminus X)
 \end{aligned}$$

Therefore for all  $y \in Y, g(y) \in X$ .

(Show  $g$  is 1-1) Now assume  $y_1, y_2 \in Y$  and  $g(y_1) = g(y_2)$ . We show that  $y_1 = y_2$ .

**Case 1**  $y_1, y_2 \in W$

Then  $g(y_1) = g(y_2) \implies f(y_1) = f(y_2) \implies y_1 = y_2$ .

**Case 2**  $y_1 \in W$  but  $y_2 \in Y \setminus W$

Then  $g(y_1) = g(y_2) \implies f(y_1) = y_2$

Note that if  $y_1 \in W$ , then for some  $n \geq 0, y_1 \in f^n(Y \setminus X)$

Then  $y_2 \in f^{n+1}(Y \setminus X) \subseteq W$

So  $y_2 \in W$ , which leads to a contradiction.

**Case 3**  $y_1, y_2$  are both in  $Z := Y \setminus W$

Then  $g(y_1) = g(y_2) \implies y_1 = y_2$ .

Therefore by case 1,2,3,  $g$  is 1-1.

(Show  $g$  is onto) Let  $x \in X$ . We need to find  $y \in Y$  s.t.  $g(y) = x$ .

If  $x \in Z$ , take  $y = x$ .

If  $x \in \bigcup_{n=1}^{\infty} f^n(Y \setminus X)$ , then fix  $n \in \mathbb{N}$  s.t.  $x \in f^n(Y \setminus X)$ .

But  $f^n(Y \setminus X) = f(f^{n-1}(Y \setminus X))$

Pick  $y \in f^{n-1}(Y \setminus X)$  s.t.  $f(y) = x$ .

Then  $y \in W$  and  $g(y) = x$ . Therefore  $g$  is onto. ■

## 2 Sequences and Series

### 2.1 The Limit of a Sequence

**Definition 2.1** (sequence). A sequence is a function whose domain is  $\mathbb{N}$ .

**Definition 2.2.** Let  $(X, d)$  be a metric space. A sequence  $(X_n) \subseteq X$  converges to an element  $x \in X$  if  $\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies d(x_n, x) < \epsilon$ .

**Key property:** If  $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} x_n = y$ , then  $x = y$ .

*Proof.* WTS  $d(x, y) = 0$

Let  $\epsilon > 0$ . We will show that  $d(x, y) < \epsilon$ .

Since  $\lim_{n \rightarrow \infty} x_n = x$ , then  $\exists N_1, \forall n \geq N_1, d(x_n, x) < \frac{\epsilon}{2}$

Since  $\lim_{n \rightarrow \infty} x_n = y$ , then  $\exists N_2, \forall n \geq N_2, d(x_n, y) < \frac{\epsilon}{2}$

Take  $n \geq \max(N_1, N_2)$ , then  $d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ . ■

**Proposition 2.1.** Suppose  $(X, d)$  is a metric space,  $(X, \tau)$  is a topological space, and  $F \subseteq X$ . If  $\lim_{n \rightarrow \infty} x_n = x$ ,  $(x_n) \subseteq F$  and  $F$  is closed, then  $x \in F$ .

*Proof.* Suppose  $x \notin F$ , i.e.,  $x \in X \setminus F$ .

Since  $F$  is closed, then  $X \setminus F$  is open, so there is  $\epsilon > 0$  s.t.  $B_\epsilon(x) \subseteq X \setminus F$ .

Let  $N$  be such that  $\forall n \geq N, d(x_n, x) < \epsilon$ .

Then  $x_n \in B_\epsilon(x)$ , which implies that  $(x_n) \subseteq X \setminus F$ , a contradiction. ■

**Proposition 2.2.** Suppose  $(X, d)$  is a metric space and  $F \subseteq X$ . If  $F$  is not closed, then there exists  $(x_n) \subseteq F$  and  $x \notin F$  s.t.  $\lim_{n \rightarrow \infty} x_n = x$ .

*Proof.* If  $F$  is not closed, then  $X \setminus F$  is not open, so there is  $x \in X \setminus F$  s.t.  $B_\epsilon(x) \not\subseteq X \setminus F$  for all  $\epsilon > 0$ .

Take  $x_n \in B_{1/n}(x) \setminus (X \setminus F) = B_{1/n}(x) \cap F$  for each  $n \in \mathbb{N}$ , then  $(x_n) \subseteq F$  and  $\lim_{n \rightarrow \infty} x_n = x$ . ■

**Definition 2.3** (Cauchy sequence). A sequence  $(x_n)$  in a metric space  $(X, d)$  is a Cauchy sequence if  $\forall \epsilon > 0, \exists N \in \mathbb{N}, m, n \geq N \implies d(x_m, x_n) < \epsilon$ .

**Proposition 2.3.** A convergent sequence is Cauchy.

*Proof.* Let  $(x_n)$  be a convergent sequence, so that  $\lim_{n \rightarrow \infty} x_n = x$ . To check  $(x_n)$  is Cauchy, let  $\epsilon > 0$ . We need to find  $N$  s.t.  $\forall m, n \geq N, d(x_n, x_m) < \epsilon$ .

Apply  $\lim_{n \rightarrow \infty} x_n = x$  to  $\frac{\epsilon}{2}$ , we get  $N$  s.t.  $\forall n \geq N, d(x, x_n) < \frac{\epsilon}{2}$ .

Notice that  $N$  works for Cauchy:

Take  $m, n \geq N$ , then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

**Remark 2.1.** When  $X = \mathbb{R}$  with the usual metric, A Cauchy sequence is convergent (the converse is true).

In general not true. For example,  $X = \mathbb{R} \setminus \{0\}, d(x, y) = |x - y|, (x_n) = \frac{1}{n}$ . ■

**Definition 2.4** (monotone sequence).  $(x_n) \subseteq \mathbb{R}$  is monotone if either  $x_n \leq x_m, n \leq m$ , or  $x_n \geq x_m, n \leq m$ .

**Theorem 2.1** (Monotone Subsequence Theorem). Every sequence  $(x_n) \subseteq \mathbb{R}$  has a monotone subsequence.

prove this

**Fact 2.1.** If  $a_n \leq b_n$  for all  $n$ ,  $a = \lim_{n \rightarrow \infty} a_n, b = \lim_{n \rightarrow \infty} b_n$ , then

$$a \leq b$$

*Proof.* Suppose for contradiction that  $a > b$ . Let  $\epsilon = \frac{a-b}{2}$ .

We know  $\exists N_1$  s.t.  $a_n \in B_\epsilon(a)$  for  $n \geq N_1$  and  $\exists N_2$  s.t.  $b_n \in B_\epsilon(b)$  for  $n \geq N_2$ . Take  $n > \max(N_1, N_2)$ , then we have

$$b_n < \frac{a+b}{2} < a_n$$

which is a contradiction. ■

**Theorem 2.2** (Algebraic limit theorem). Suppose  $a = \lim_{n \rightarrow \infty} a_n, b = \lim_{n \rightarrow \infty} b_n$ , then:

1.  $a + b = \lim_{n \rightarrow \infty} (a_n + b_n)$
2.  $ab = \lim_{n \rightarrow \infty} a_n b_n$
3.  $\frac{a}{b} = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ , and  $b \neq 0$ .

**Fact 2.2.** Monotone bounded sequence  $(x_n)$  converges to its supremum or infimum.

*Proof.* We only prove the supremum case.

Fix  $\epsilon > 0$ , let  $s = \sup\{x_n : n \in \mathbb{N}\}$ . We have  $s - \epsilon < s$  and thus  $s - \epsilon$  is not an upper bound of  $(x_n)$ . Therefore, there is  $N$  s.t.  $x_N > s - \epsilon$ .

Take  $n \geq N$ , then we have

$$x_n \geq x_N > s - \epsilon$$

Therefore, we have  $|x_n - s| < \epsilon$ . ■

**Definition 2.5** (limit supremum). We define

$$\limsup_{n \rightarrow \infty} x_n = \inf\{y_m : m \in \mathbb{N}\}$$

where  $y_m = \sup\{x_n : n \geq m\}$ .

Alternatively,

$$\limsup_{n \rightarrow \infty} x_n = \lim_{m \rightarrow \infty} \sup_{n \geq m} x_n$$

**Definition 2.6** (limit infimum).

$$\liminf_{n \rightarrow \infty} x_n = \sup\{z_m : m \in \mathbb{N}\}$$

where  $z_m = \inf\{x_n : n \geq m\}$ .

Alternatively,

$$\liminf_{n \rightarrow \infty} x_n = \lim_{m \rightarrow \infty} \inf_{n \geq m} x_n$$

## 2.2 Series

**Definition 2.7.** We define

$$S_n = \sum_{k=1}^n a_k, \quad \lim_{n \rightarrow \infty} S_n = \sum_{k=1}^{\infty} a_k$$

We call  $\sum_{k=1}^{\infty} a_k$  a summable series if the limit exists, i.e.,

$$\exists A, \forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq N, |S_n - A| < \epsilon$$

**Property 2.1** (Cauchy criterion for series).  $\sum_{k=1}^{\infty}$  is summable iff

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, |S_n - S_m| = \left| \sum_{k=m+1}^n a_k \right| < \epsilon$$

**Corollary 2.1.** If  $\sum_{k=1}^{\infty} a_k$  is summable, then  $|a_k| \rightarrow 0$ .

*Proof.* We have  $|a_k| = |s_k - s_{k-1}| < \epsilon$  for  $k > N$ . ■

**Example 2.1.**  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  is summable.

*Proof.*

$$\begin{aligned} S_m &= 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{m^2} \\ &< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \dots + \frac{1}{m(m-1)} \\ &= 1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{m-1} - \frac{1}{m}) \\ &= 1 + 1 - \frac{1}{m} \\ &< 2 \end{aligned}$$

**Example 2.2.**  $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

*Proof.* We have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} &= (1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \dots \\ &= 1 + (1/2) + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \dots \\ &= 1 + 1/2 + 1/2 + 1/2 + \dots \\ &\rightarrow \infty \end{aligned}$$

**Theorem 2.3** (Algebraic limit theorem for series). Suppose  $\sum_{k=1}^{\infty} a_k = A$ ,  $\sum_{k=1}^{\infty} b_k = B$ ,  $c \in \mathbb{R}$ , then

1.  $\sum_{k=1}^{\infty} ca_k = cA$
2.  $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

*Proof.* (1) We want to show  $\forall \epsilon > 0, \exists N$  s.t.  $\forall n \geq N, |\sum_{k=1}^{\infty} ca_k - cA| < \epsilon$ .

We know  $\forall \epsilon_0 > 0, \exists N_{\epsilon_0}$  s.t.  $\forall n \geq N_{\epsilon_0}, |\sum_{k=1}^{\infty} a_k - A| < \epsilon_0$ .

Take  $\epsilon_0 = \frac{\epsilon}{|c|}$ , then we have

$$\left| \sum_{k=1}^{\infty} ca_k - cA \right| = |c| \left| \sum_{k=1}^{\infty} a_k - A \right| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$$

**Property 2.2** (Order comparison test). Suppose  $b_k \geq a_k \geq 0, \forall k$ . ■

1. If  $\sum_{k=1}^{\infty} b_k < \infty$ , then  $\sum_{k=1}^{\infty} a_k < \infty$ .
2. If  $\sum_{k=1}^{\infty} a_k = \infty$ , then  $\sum_{k=1}^{\infty} b_k = \infty$ .

**Definition 2.8** (geometric series). We call a series a geometric series if it is of the form

$$\sum_{k=1}^{\infty} ar^k$$

Note that the geometric series converges to  $\frac{a}{1-r}$  whenever  $r^m \rightarrow 0$  iff  $|r| < 1$ .

**Definition 2.9** (absolute convergence).  $\sum_{k=1}^{\infty} a_k$  is absolutely convergent if  $\sum_{k=1}^{\infty} |a_k| < \infty$ .

**Definition 2.10** (conditionally convergence).  $\sum_{k=1}^{\infty} a_k$  is conditionally convergent if  $\sum_{k=1}^{\infty} a_k < \infty$ , but  $\sum_{k=1}^{\infty} |a_k| = \infty$

**Example 2.3** (alternating series).  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} < \infty$  but  $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$

**Property 2.3** (Absolute convergence test). If  $\sum_{k=1}^{\infty} |a_k| < \infty$ , then  $\sum_{k=1}^{\infty} a_k < \infty$ .

*Proof.* We use Cauchy criterion for  $\sum_{k=1}^{\infty} a_k$ : we want to show

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, \left| \sum_{k=m+1}^n a_k \right| < \epsilon$$

Let  $\epsilon > 0$ .

Since  $\sum_{k=1}^{\infty} |a_k| < \infty$ , then we know that  $\exists N$  s.t.  $\forall n \geq m \geq N$ ,

$$\left| \sum_{k=1}^n |a_k| - \sum_{k=1}^m |a_k| \right| < \epsilon$$

Then

$$\begin{aligned} \left| \sum_{k=m+1}^n a_k \right| &= \left| \sum_{k=1}^n a_k - \sum_{k=1}^m a_k \right| \\ &\leq \sum_{k=1}^n |a_k| - \sum_{k=1}^m |a_k| \\ &\leq \left| \sum_{k=1}^n |a_k| - \sum_{k=1}^m |a_k| \right| \\ &< \epsilon \end{aligned}$$

■

**Property 2.4** (Alternating series test). Suppose  $a_1 \geq a_2 \geq \dots \geq 0$ ,  $\lim_{k \rightarrow \infty} a_k = 0$ , then  $\sum_{k=1}^{\infty} (-1)^{k+1} a_k < \infty$ .

*Proof.* We want to show  $\{S_n\} = \{\sum_{k=1}^n (-1)^{k+1} a_k\}$  is Cauchy:

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n \geq N, |S_n - S_m| < \epsilon$$

Let  $\epsilon > 0$ .

Suppose  $n > m$ , then  $|S_n - S_m| = |a_{m+1} - a_{m+2} + \dots + (-1)^{n-m+1} a_n|$ .

Since  $(a_n)$  is a non-negative decreasing sequence, then

$$\begin{aligned} a_{m+1} - a_{m+2} + \dots + (-1)^{n-m-1} a_n &= a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \dots \\ &\leq a_{m+1} \end{aligned}$$

Since  $\lim_{k \rightarrow \infty} a_k = 0, \exists N \text{ s.t. } \forall m+1 \geq N, a_{m+1} < \epsilon$ .

Thus  $0 \leq |S_n - S_m| \leq a_{m+1} < \epsilon$ . ■

**Property 2.5** (Ratio test). Given  $\sum_{k=1}^{\infty} a_k$  s.t.  $a_k \neq 0$  for all  $k$ .

If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1$ , then  $\sum_{k=1}^{\infty} |a_k| < \infty$

*Proof.* Define  $S := \{n \in \mathbb{N} : \left| \frac{a_{n+1}}{a_n} \right| \geq r'\}$ , then  $S$  contains finitely many elements of  $\mathbb{N}$ . (If  $S$  were to be infinite set, if we take  $\epsilon = r' - r$ , then  $\left| \frac{a_{n+1}}{a_n} \right| - r \geq r' - r$  for infinitely many terms which contradicts that  $r$  is the point of convergence.)

Therefore,  $S' = \{n \in \mathbb{N} : \left| \frac{a_{n+1}}{a_n} \right| < r'\}$  contains all but finitely many elements of  $\mathbb{N}$ . Let

$N = 1 + \max S$ , then  $\forall n \geq N, \left| \frac{a_{n+1}}{a_n} \right| < r' \implies |a_{n+1}| < r' |a_n|$ .

Since  $0 < r' < 1$ ,  $\sum_{n=1}^{\infty} (r')^n$  converges which implies  $|a_N| \sum_{n=1}^{\infty} (r')^n$  converges. We have  $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^N |a_n| + \sum_{n=N+1}^{\infty} |a_n| < C + |a_N| \sum_{n=N+1}^{\infty} (r')^{n-N}$  converges, by comparison test. Hence  $\sum_{n=1}^{\infty} |a_n|$  converges. ■

understand the last two lines of the proof

**Definition 2.11** (rearrangement). Let  $\sum_{k=1}^{\infty} a_k$  be a series. A series  $\sum_{k=1}^{\infty} b_k$  is called a rearrangement of  $\sum_{k=1}^{\infty} a_k$  if  $\forall n, \exists k$  s.t.  $b_k = a_n$ .

### 3 Metric Spaces and the Baire Category Theorem

#### 3.1 Basic Definitions

**Definition 3.1** (metric and metric space). Given a set  $X$ , a function  $d : X \times X \rightarrow \mathbb{R}$  is a metric on  $X$  if for all  $x, y \in X$ :

1.  $d(x, y) \geq 0$  with  $d(x, y) = 0$  if and only if  $x = y$ ;
2.  $d(x, y) = d(y, x)$ ;
3. for all  $z \in X, d(x, y) \leq d(x, z) + d(z, y)$

A metric space is a set  $X$  together with a metric  $d$ .

**Example 3.1.** The set  $\mathbb{R}$  considered with  $d : \mathbb{R}^2 \rightarrow [0, \infty), (x, y) \mapsto |x - y|$  is a metric space.

**Example 3.2.** In general,  $\mathbb{R}^n$  considered with the Euclidean distance is a metric space.

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

**Example 3.3.** Let  $X$  be a set. The discrete metric  $d$  on  $X$  is defined by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

**Fact** If  $(X, d)$  is a metric space,  $d'(x, y) = \max\{1, d(x, y)\}$  for all  $x, y \in X$ , then  $(X, d')$  is also a metric space.

**Example 3.4.** Let  $X = \{f : A \rightarrow \mathbb{R}\}$

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in A\}$$

if the supremum exists.

**Definition 3.2.** Let  $(X, d_1)$  and  $(Y, d_2)$  be metric spaces. A function  $f : X \rightarrow Y$  is continuous at  $x \in X$  if  $\forall \epsilon > 0, \exists \delta > 0, d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$ .

### 3.2 Topology on Metric Spaces

**Definition 3.3** (open ball). An open ball (or  $\epsilon$ -neighbourhood) with radius  $r$  and center  $x$  is

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

**Definition 3.4** (open set). A set  $U \subseteq X$  is open iff

$$\forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

**Example 3.5.**  $B_\epsilon(x)$  is open.

*Proof.* Fix  $x \in X$  and  $\epsilon > 0$ . We want to show:  $\forall y \in B_\epsilon(x), \exists \delta > 0$  s.t.  $B_\delta(y) \subseteq B_\epsilon(x)$ . Take  $y \in B_\epsilon(x)$ , then  $d(x, y) < \epsilon$ . Take  $\delta = \epsilon - d(x, y) > 0$ . Take any  $z \in B_\delta(y)$ , we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \epsilon - d(x, y) = \epsilon$$

Thus  $z \in B_\epsilon(x)$  so  $B_\delta(y) \subseteq B_\epsilon(x)$ . ■

**Definition 3.5** (topological space). A topological space is a pair  $(X, \tau)$ , where  $X$  is a set and  $\tau$  a subset of the power set of  $X$  which we call open such that

1.  $\emptyset, X \in \tau$
2.  $U_1, \dots, U_n \in \tau \implies \bigcap_{i=1}^n U_i \in \tau$
3.  $U_1, \dots, U_n \in \tau \implies \bigcup_{i=1}^n U_i \in \tau$

**Example 3.6.**  $(X, \{\emptyset, X\})$

**Example 3.7.**  $(X, P(X))$  is a discrete topological space, where  $P(X)$  is the power set of  $X$ .

**Example 3.8.** Given  $(X, d)$  a metric space, define  $\tau_d$  : a set  $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$ . Then  $\tau_d$  is a topology.

*Proof.* (1) First,  $\emptyset, X \in \tau_d$  since  $\forall x \in \emptyset, B_1(x) \subseteq \emptyset$  and  $\forall x \in X, B_1(x) \subseteq X$ .

Then suppose  $U_1, \dots, U_n \in \tau_d$ .

(2) we want to show:

$$U = \bigcap_{i=1}^n U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Since  $x \in U$ , then  $\forall i = 1, \dots, n, x \in U_i : \exists \epsilon_i > 0 \text{ s.t. } B_{\epsilon_i}(x) \subseteq U_i$ .

Take  $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$ , thus  $B_\epsilon(x) \subseteq U_i \forall i$ . Hence  $B_\epsilon(x) \subseteq U_i \subseteq U$ .

(3) We also want to show:

$$\bigcup_{i=1}^n U_i \in \tau_d \iff \forall x \in U, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U$$

Let  $x \in U$ , then there is some  $U_i$  s.t.  $x \in U_i$ . Since  $U_i \in \tau_d$ , then  $\exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U_i \subseteq U$ . Therefore,  $\tau_d$  is a topology. ■

**Definition 3.6.** A subset  $F$  of a topological space  $(X, \tau)$  is closed if  $X \setminus F$  is open.

**Property 3.1.** Given a topological space  $(X, \tau)$  and a subset  $F$  of it, we have:

1.  $\emptyset, X$  are closed
2. If  $F_1, \dots, F_n$  are closed, then  $\bigcup_{i=1}^n F_i$  is closed
3. If  $F_1, \dots, F_n$  are closed, then  $\bigcap_{i=1}^n F_i$  is closed

**Definition 3.7** (topological closure and interior). Given a topological space  $(X, \tau)$ , where  $\tau \subseteq P(X)$ , and a set  $F \subseteq X$ , the topological closure of  $F$  is the minimal closed superset of  $F$ , i.e.,

$$\bar{F} = \bigcap \{H : H \text{ is closed, } H \supseteq F\}$$

The interior of  $F$  is the maximal open subset of  $F$ , i.e.,

$$F^\circ = \bigcap \{U : U \text{ is open, } U \subseteq F\}$$

**Example 3.9.** Given  $(X, d)$  a metric space, define  $\tau_d$  : a set  $U \in \tau_d \iff \forall x \in U, \exists \epsilon > 0, B_\epsilon(x) \subseteq U$ . Suppose  $F \subseteq X$ , then

$$\bar{F} = \{x \in X : \forall \epsilon > 0, B_\epsilon(x) \cap F \neq \emptyset\} = \{\lim_{n \rightarrow \infty} x_n : (x_n) \subseteq F, \lim_{n \rightarrow \infty} x_n \text{ exists}\}$$

and

$$F^\circ = \{x \in X : \exists \epsilon > 0, B_\epsilon(x) \subseteq F\} = \bigcup \{B_\epsilon(x) : \epsilon > 0, x \in F, B_\epsilon(x) \subseteq F\}$$



### 3.3 Compactness and Bolzano-Weierstrass Theorem

**Definition 3.8** (compactness). A subset  $K$  of a metric space  $(X, d)$  is compact if every sequence in  $K$  has a convergent subsequence that converges to a limit in  $K$ .

**Example 3.10.**  $(\mathbb{R}, |x - y|)$  is not compact (e.g.  $(x_n) = n$ )

**Example 3.11.**  $([0, 1], |x - y|)$  is compact.

**Property 3.2.** If  $(X, d)$  is compact, then it is bounded, i.e.  $\exists M$  s.t.  $x, y \in X, d(x, y) \leq M$ .

**Property 3.3.** If  $Y \subseteq X$ ,  $(X, d)$  is a metric space, and  $(Y, d)$  is compact, then  $Y$  is closed in  $X$ .

**Property 3.4.** If  $K_1 \supseteq K_2 \supseteq \dots$  are compact and nonempty subsets of  $X$ , then  $K = \bigcap_{n=1}^{\infty} K_n$  is compact and nonempty.

**Theorem 3.1** (Bolzano-Weierstrass theorem). A subset  $Y$  of  $\mathbb{R}$  is **compact** iff **closed and bounded**.

**Alternative formation:** Every **bounded** subsequence contains a **convergent subsequence**.

**Remark 3.1.** The theorem is true for  $\mathbb{R}^n$  but is false for infinite dimension.

**Theorem 3.2** (Heine-Borel Theorem). Let  $K$  be a subset of a metric space  $(X, d)$ . The following statements are equivalent:

1.  $K$  is compact.
2.  $K$  is closed and bounded.
3. Every open cover  $K \subseteq \bigcup_{i \in I} U_i$  for  $K$  has a finite subcover  $K \subseteq \bigcup_{i=1}^n U_{i_i}$ .

### 3.4 Completeness of Metric Spaces

**Definition 3.9** (completeness of metric spaces). A metric space  $(X, d)$  is complete if every Cauchy sequence in  $X$  converges to an element of  $X$ .

**Example 3.12.**  $\mathbb{R}, d(x, y) = |x - y|$

**Example 3.13.**  $(X, d), d$  discrete metric.

**Example 3.14.**  $C[0, 1], d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| = \|f - g\|_{\infty}$

**Example 3.15.**  $(\mathbb{N}^{\mathbb{N}}, d), d((x_n), (y_n)) = \frac{1}{\min\{n: x_n \neq y_n\}}$   
where  $\mathbb{N}^{\mathbb{N}} = \{x : \mathbb{N} \rightarrow \mathbb{N}\}$ .

### 3.5 Perfect Sets

**Definition 3.10** (perfect set). Let  $(X, d)$  be a metric space.  $P \subseteq X$  is perfect if it is closed, nonempty, and for every open  $U \subseteq X$ ,  $U \cap P$  is not empty and has at least two elements.

**Example 3.16.**  $S = [0, 1] \cup \{\frac{3}{2}\} \cup [2, 3]$  is not perfect.

**Property 3.5.** Perfect subsets  $P$  of a complete metric space are not countable.

**Example 3.17** (Cantor set). Let  $C_0$  be the closed interval  $[0, 1]$ , and define  $C_1$  to be the set that results when the open middle third is removed; that is,

$$C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

Now construct  $C_2$  in a similar way by removing the open middle third of each of the two components of  $C_1$ :

$$C_2 = ([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}]) \cup ([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1])$$

Continue this process inductively. For each  $n = 0, 1, 2, \dots$ , we get a set  $C_n$  consisting of  $2^n$  closed intervals each having length  $(\frac{1}{3})^n$ . Finally, we define the Cantor set  $C$  to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$

**Remark 3.2.** As follows

- Since we are always removing open middle thirds, then at each stage, endpoints are never removed. Thus,  $C$  at least contains the endpoints of all of the intervals that make up each of the sets  $C_n$ .
- The Cantor set has zero length.
- The Cantor set is uncountable, with cardinality equal to the cardinality of  $\mathbb{R}$ .

### 3.6 Separated and Connected Sets

**Definition 3.11** (separated sets). Let  $(X, d)$  be a metric space,  $A \neq \emptyset, B \subseteq X$ .  $A$  and  $B$  are separated if  $\bar{A} \cap B = \bar{B} \cap A = \emptyset$ .

**Definition 3.12** (connected sets). A set  $C \subseteq X$  is connected if for every decomposition  $C = A \cup B$  s.t.  $A, B \neq \emptyset$ ,  $A$  and  $B$  are not separated, i.e.  $\bar{A} \cap B \neq \emptyset$  or  $\bar{B} \cap A \neq \emptyset$ .

**Property 3.6.**  $C \subseteq \mathbb{R}$  is connected iff

$$\forall a, b \in C, [a, b] \subseteq C$$

*Proof.* Let  $C = A \cup B, a_0 \in A, b_0 \in B, a_0 < b_0$ . We define  $I_0 = [a_0, b_0], c_0 = \frac{a_0 + b_0}{2}$ . Define  $I_1 = [a_0, c_0], \dots$  We have  $x \in \bar{A} \cap B$  or  $\bar{B} \cap A$ . ■

Is this complete?

### 3.7 Baire's Theorem

**Definition 3.13** (dense). A set  $A \subseteq X$  is dense in the metric space  $(X, d)$  if  $\bar{A} = X$ .

**Definition 3.14** (nowhere-dense). A subset  $E$  of a metric space  $(X, d)$  is nowhere-dense in  $X$  if  $\bar{E}^\circ$  is empty.

i.e., A nowhere-dense set of a metric space is a set whose closure has empty interior.

**Remark 3.3.** It is a set whose elements are not tightly clustered anywhere.

**Example 3.18.**  $\mathbb{Z}$  is nowhere-dense in  $\mathbb{R}$ .

**Example 3.19.**  $S = \{\frac{1}{n} : n \in \mathbb{N}\}$  is nowhere-dense in  $\mathbb{R}$ .  
 $\bar{S} = S \cup \{0\}$ , which has empty interior.

**Theorem 3.3** (Baire's Theorem). The set of real numbers  $\mathbb{R}$  cannot be written as the countable union of nowhere-dense sets.

**Remark 3.4.** Baire's Theorem asserts that the only way to make  $\mathbb{R}$  from a countable union of arbitrary sets is for the closure of at least one of these sets to contain an interval.

### 3.8 The Baire Category Theorem

**Theorem 3.4.** Let  $(X, d)$  be a complete metric space, and let  $\{O_n\}$  be a countable collection of dense, open subsets of  $X$ . Then,  $\bigcap_{n=1}^{\infty} \{O_n\}$  is not empty. prove this

**Theorem 3.5** (Baire Category Theorem). A complete metric space cannot be written as the countable union of nowhere-dense sets. prove this

**Remark 3.5.** This result is called the Baire Category Theorem because it creates two categories of size for subsets in a metric space:

1. A set of “first category” is one that can be written as a countable union of nowhere-dense sets. These are the small, intuitively “thin” subsets of a metric space.
2. If our metric space is complete, then it is necessarily of “second category”, meaning it cannot be written as a countable union of nowhere-dense sets.

**Theorem 3.6.** The set

$$D = \{f \in C[0, 1] : f'(x) \text{ exists for some } x \in [0, 1]\}$$

is a set of first category in  $C[0, 1]$ .

## 4 Functional Limits and Continuity

### 4.1 Functional Limits

**Definition 4.1.** Let  $A \subseteq \mathbb{R}$ ,  $a \in \overline{A \setminus \{a\}}$  ( $a$  is an accumulation point of  $A$ ). Let  $f : A \rightarrow \mathbb{R}$ , define  $\lim_{x \rightarrow a} f(x) = L$  iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

**Property 4.1** (Sequential criterion for functional limits).  $a \in \overline{A \setminus \{a\}}$ ,  $f : A \rightarrow \mathbb{R}$ . The following are equivalent:

1.  $\lim_{x \rightarrow a} f(x) = L$
2.  $\forall (x_n) \subseteq A \setminus \{a\}, x_n \rightarrow a \implies f(x_n) \rightarrow L$

*Proof.* We prove (1)  $\implies$  (2):

Assume  $\lim_{x \rightarrow a} f(x) = L$ , take arbitrary  $(x_n) \subseteq A \setminus \{a\}$  s.t.  $x_n \rightarrow a$ .

Let  $\epsilon > 0$ , then  $\exists \delta > 0$  s.t.  $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$ .

Also,  $\exists N$  s.t.  $n \geq N \implies |x_n - a| < \delta$ .

Therefore, if  $|x_n - a| < \delta$ , then  $|f(x_n) - L| < \epsilon$ . ■

**Theorem 4.1** (Algebraic Limit Theorem for functional limits). Suppose  $f, g : A \rightarrow \mathbb{R}, a \in \overline{A \setminus \{a\}}$ .

Suppose  $\lim_{x \rightarrow a} f(x) = L, \lim_{x \rightarrow a} g(x) = M$ . Then we have

1.  $\lim_{x \rightarrow a} cf(x) = cL$
2.  $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$
3.  $\lim_{x \rightarrow a} (f(x)g(x)) = LM$
4.  $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$  when  $M \neq 0$ .

**Property 4.2** (Divergence criterion). Suppose  $f : A \rightarrow \mathbb{R}, a \in \overline{A \setminus \{a\}}$   $\lim_{x \rightarrow a} f(x)$  does not exist if there are two sequences  $(x_n), (y_n) \subseteq A \setminus \{a\}$  s.t.  $x_n \rightarrow a, y_n \rightarrow a, \lim_{n \rightarrow \infty} f(x_n) = L, \lim_{n \rightarrow \infty} f(y_n) = M$  exist but  $L \neq M$ .

**Example 4.1.** Let  $A = \mathbb{R}^+, f(x) = \sin(\frac{1}{x})$ . Let  $a_n = \frac{1}{2n\pi}, b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ .

Then we have  $a_n, b_n \rightarrow 0$ . Besides,  $\lim_{n \rightarrow \infty} f(a_n) = 0, \lim_{n \rightarrow \infty} f(b_n) = 1$ . Hence  $\lim_{x \rightarrow 0^+} \sin(\frac{1}{x})$  does not exist.

**Definition 4.2.** Suppose  $f : A \rightarrow \mathbb{R}, x \in A \setminus \{a\}$ . We define  $\lim_{x \rightarrow a} f(x) = \infty$  iff

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$$

## 4.2 Continuous Functions

**Definition 4.3** (continuity). Suppose  $(X, d_X), (Y, d_Y)$  are metric spaces.  $f : X \rightarrow Y$  is continuous at  $a \in X$  if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } x \in B_\delta^X(a) \implies f(x) \in B_\epsilon^Y(f(a))$$

**Remark 4.1.** Note that for  $X = Y = \mathbb{R}, d(x, y) = |x - y|$ , so that we can write

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

i.e.

$$\lim_{x \rightarrow a} f(x) = f(a)$$

**Definition 4.4** (continuous function).  $f : X \rightarrow Y$  is continuous if it is continuous at every point  $a \in X$ .

**Property 4.3.** The following are equivalent:

1.  $f$  is continuous at  $a$
2.  $\lim_{x \rightarrow a} f(x) = f(a)$
3.  $\forall (x_n) \subseteq A, x_n \rightarrow a \implies f(x_n) \rightarrow f(a)$ .

**Corollary 4.1.**  $f$  is discontinuous at  $a$  if there is a sequence  $(x_n) \rightarrow a$  s.t.  $\lim_{n \rightarrow \infty} f(x_n) \neq f(a)$ .

**Remark 4.2.** Note that we may have  $\lim_{x \rightarrow a} f(x)$  exists but  $f$  is discontinuous at  $a$ .

**Theorem 4.2** (Algebraic Continuity Theorem). Suppose  $f, g : A \rightarrow \mathbb{R}$  are continuous at  $a \in A, c \in \mathbb{R}$ . We have

1.  $cf(x)$  is continuous at  $a$
2.  $f(x) \pm g(x)$  is continuous at  $a$
3.  $f(x)g(x)$  is continuous at  $a$
4.  $\frac{f(x)}{g(x)}$  is continuous at  $a$  if  $g(a) \neq 0$

**Theorem 4.3.** Suppose  $f : A \rightarrow B \subseteq \mathbb{R}, g : B \rightarrow \mathbb{R}$ .

$(g \circ f)(x) = g(f(x))$  is continuous at  $a \in A$  whenever  $f$  is continuous at  $a$  and  $g$  is continuous at  $f(a)$ .

**Theorem 4.4.** Suppose  $(X, d_X), (Y, d_Y)$  are metric spaces and  $f : X \rightarrow Y$  is continuous. If  $K \subseteq X$  is compact, then its image  $f[K] = \{f(x) : x \in K\}$  is compact.

**Theorem 4.5.** Suppose  $(X, d_X), (Y, d_Y)$  are metric spaces. If  $F \subseteq Y$  is closed in  $Y$ , then  $f^{-1}(F)$  is closed in  $X$ .

**Theorem 4.6** (Extreme Value Theorem). If  $f : K \rightarrow \mathbb{R}$  is continuous,  $K$  is compact, then  $\exists x_1, x_2 \in K$  s.t.  $\forall x \in K$ ,

$$f(x_1) \leq f(x) \leq f(x_2)$$

*Proof.* Let  $H = f[K] = \{f(x) : x \in K\} \subseteq \mathbb{R}$ , which is compact. Since compact subsets of  $\mathbb{R}$  are bounded, then let  $y_2 = \sup(H)$ .

We have  $y \leq y_2$  for all  $y \in H$  and  $\forall \epsilon > 0, \exists y \in H$  s.t.  $y_2 - \epsilon < y \leq y_2$ .

Take  $\epsilon = \frac{1}{n}$ , then we have some  $z_n \in H$  s.t.  $y_2 - \frac{1}{n} < z_n \leq y_2$ .

as Now we find  $a_n \in K$  s.t.  $f(a_n) = z_n, n = 1, 2, \dots$

By theorem, we have  $a_{n_k} \rightarrow x_2$ , then  $f(x_2) = \lim_{k \rightarrow \infty} f(a_{n_k}) = y_2$ .

■

Which theorem?

### 4.3 Continuous Functions on Compact Sets

#### 4.3.1 Uniform Continuity

**Definition 4.5** (uniform continuity). We say function  $f : A \rightarrow \mathbb{R}$  is uniformly continuous on  $A$  if

$$\forall \epsilon > 0, \exists \delta > 0, x, y \in A \wedge |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

**Example 4.2.**  $f(x) = x^2$  is not uniformly continuous.

*Proof.* WTS  $\exists \epsilon > 0, \forall \delta > 0, \exists x, y \in \mathbb{R}$  s.t.  $|x - y| < \delta$  and  $|f(x) - f(y)| \geq \epsilon$ .

Let  $\epsilon = 1, \delta > 0$ .

Choose  $y = x + \frac{1}{2}\delta$ , so that  $|x - y| < \delta$ .

$$\begin{aligned} f(y) - f(x) &= y^2 - x^2 \\ &= \left(x + \frac{1}{2}\delta\right)^2 - x^2 \\ &= x^2 + \delta x + \frac{1}{4}\delta^2 - x^2 \\ &= \delta x + \frac{1}{4}\delta^2 \end{aligned}$$

If  $x > \frac{1}{\delta}$ , then  $f(y) - f(x) > 1$ . ■

**Property 4.4** (). Function  $f : A \rightarrow \mathbb{R}$  fails to be uniformly continuous iff  $\exists \epsilon_0 > 0, \exists (x_n), (y_n) \subseteq A$  s.t.  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0 \wedge \forall n, |f(x_n) - f(y_n)| \geq \epsilon_0$ .

*Proof.* ( $\Leftarrow$ ) Obvious.

( $\Rightarrow$ ) Assume  $f$  is not uniformly continuous.

Then  $\exists \epsilon_0 > 0$  s.t.  $\forall \delta > 0, \exists x_n, y_n \in \mathbb{R}$  s.t.  $|x_n - y_n| < \delta$  and  $|f(x_n) - f(y_n)| \geq \epsilon_0$ .

Then this is true for  $\delta \in \mathbb{N}$  as well.

For each  $n \in \mathbb{N}$ , let  $\delta = \frac{1}{n}$ , and pick  $x_n, y_n$  as above. Then it is obvious that  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$  and  $\forall n, |f(x_n) - f(y_n)| \geq \epsilon_0$ . ■

**Property 4.5** (Continuous functions on compact sets are uniformly continuous). Assume  $f : K \rightarrow \mathbb{R}$  is continuous and  $K$  is compact, then  $f$  is uniformly continuous on  $K$ .

*Proof.* Assume for a contradiction that  $f : K \rightarrow \mathbb{R}$  is continuous and  $K$  is compact, but  $f$  is not uniformly continuous. Then by Property 4.4,  $\exists \epsilon_0 > 0, (x_n), (y_n) \subseteq K$  s.t.  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$  and  $\forall n, |f(x_n) - f(y_n)| \geq \epsilon_0$ .

Since  $K$  is compact, then  $(x_n)$  has a subsequence  $(x_{n_k})$  s.t.  $x_{n_k} \rightarrow x \in K$ .

Moreover,  $(y_{n_k})$  has a subsequence  $(y_{n_{km}})$  s.t.  $y_{n_{km}} \rightarrow y \in K$ .

Let  $x'_m = x_{n_{km}}, y'_m = y_{n_{km}}$ , then  $x'_m \rightarrow x, y'_m \rightarrow y$ .

Since  $\lim_{m \rightarrow \infty} |x'_m - y'_m| = 0$ , thus  $x = y$ .

Then

$$\begin{aligned}
 & |f(x'_m) - f(y'_m)| \geq \epsilon_0 \\
 \implies & \lim_{m \rightarrow \infty} |f(x'_m) - f(y'_m)| \geq \epsilon_0 \\
 \implies & |f(x) - f(y)| \geq \epsilon_0 \\
 \implies & 0 \geq \epsilon_0
 \end{aligned}$$

which is a contradiction. ■

**Definition 4.6.** A function  $f : A \rightarrow \mathbb{R}$  is said to be Lipschitz if  $\exists M \in \mathbb{N}$  s.t.  $\forall x \neq y \in A$ ,

$$\left| \frac{f(x) - f(y)}{x - y} \right| < M$$

**Property 4.6.** Lipschitz functions are uniformly continuous.

*Proof.* Let  $f : A \rightarrow \mathbb{R}$  be Lipschitz on  $A$ . Then for every  $\epsilon > 0$ , take  $\delta < \frac{\epsilon}{M}$ . Then if  $|x - y| < \delta$ , then

$$\begin{aligned}
 |f(x) - f(y)| & < M|x - y| \\
 & < M \frac{\epsilon}{M} \\
 & = \epsilon
 \end{aligned}$$

So  $f$  is uniformly continuous. ■

**Remark 4.3.** The converse does not hold.

**Property 4.7** (Continuous image of connected sets is connected). If  $f : E \rightarrow \mathbb{R}$  is continuous and  $E$  is connected, then  $f(E)$  is connected.

#### 4.4 Sets of Discontinuity

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $D_f = \{x \in \mathbb{R} : f \text{ is not continuous at } x\}$ .

**Example 4.3** ( $D_f = \emptyset$ ).  $f$  is continuous

**Example 4.4** ( $D_f = \mathbb{R}$ ).  $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

**Example 4.5.** Given a countable set  $A = \{a_1, \dots\}$ , define  $f(a_n) := \frac{1}{n}$  and  $f(x) = 0, \forall x \notin A$ . Then we have  $D_f = A$ .

**Fact 4.1.** There is no  $f : \mathbb{R} \rightarrow \mathbb{R}$  s.t.  $D_f = \mathbb{R} \setminus \mathbb{Q}$ .

**Definition 4.7** ( $F_\sigma$ -set). A subset  $F$  of  $\mathbb{R}$  is a  $F_\sigma$ -set if  $F = \bigcup_{n=1}^{\infty} F_n$  s.t.  $F_n$  is closed for all  $n$ .

**Definition 4.8** ( $\alpha$ -continuity). Let  $\alpha > 0$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $a \in \mathbb{R}$ .  $f$  is  $\alpha$ -continuous at  $a$  if

$$\exists \delta > 0 \text{ s.t. } x, y \in (a - \delta, a + \delta) \implies |f(x) - f(y)| < \alpha$$

Note that  $f$  is continuous at  $a$  iff  $f$  is  $\alpha$ -continuous at  $a$  for all  $\alpha > 0$ .

**Property 4.8.** For every  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the set  $D_f$  is  $F_\sigma$ -set of  $\mathbb{R}$ .

red parts

**Definition 4.9.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

$f$  is removable discontinuous if  $\lim_{x \rightarrow a} f(x)$  exists but does not equal  $f(a)$ .

$f$  has a jump at  $a$  if  $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ .

If  $\lim_{x \rightarrow a} f(x)$  does not exist for other reasons, we say  $f$  is essential discontinuous.

**Definition 4.10** (monotonicity).  $f : \mathbb{R} \rightarrow \mathbb{R}$  is monotone if either  $x \leq y \implies f(x) \leq f(y)$  or  $x \leq y \implies f(x) \geq f(y)$ .

**Property 4.9.** Discontinuity of a monotone function  $f$  is a jump. Moreover,  $D_f$  is countable.

## 5 the Derivative

### 5.1 Derivatives and the Intermediate Value Property

**Definition 5.1** (derivative). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $c \in \mathbb{R}$ . Define the derivative of  $f$  at  $c$ :

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

If  $f'(c)$  exists, we say that  $f$  is differentiable at  $c$ . If  $f'(a)$  exists for all  $a \in \mathbb{R}$ , we say that  $g$  is differentiable on  $\mathbb{R}$ .

**Property 5.1.** If  $f$  is differentiable at  $c$ , then  $f$  is continuous at  $c$ .

*Proof.* We have

$$\lim_{x \rightarrow c} (f(x) - f(c)) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) = f'(c) \cdot 0 = 0$$

■

**Theorem 5.1** (Algebraic Differentiability Theorem). Suppose  $f, g$  are differentiable,  $a, c \in \mathbb{R}$ . We have

1.  $(cf)'(a) = cf'(a)$
2.  $(f + g)'(a) = f'(a) + g'(a)$
3.  $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$
4.  $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$



**Theorem 5.2** (Chain Rule). Let  $f : A \rightarrow B, g : B \rightarrow \mathbb{R}, f(A) \subseteq B$  so that  $g \circ f$  is defined. If  $f$  is differentiable at  $c$  and  $g$  is differentiable at  $f(c)$ , then  $g \circ f$  is differentiable at  $a$  with

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

**Theorem 5.3** (Interior Extremum Theorem). If  $f$  is differentiable on  $(a, b)$ ,  $f$  attains maximum at some  $c \in (a, b)$ , then  $f'(c) = 0$ .

*Proof.* We have

$$f'(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

and

$$f'(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

then  $f'(c) = 0$ . ■

**Theorem 5.4** (Darboux's Theorem). If  $f$  is differentiable on  $[a, b]$  and  $f'(a) < \alpha < f'(b)$  or  $f'(a) > \alpha > f'(b)$ , then  $\exists c \in (a, b)$  s.t.  $f'(c) = \alpha$ .

## 5.2 the Mean Value Theorems

**Theorem 5.5** (Rolle's Theorem). Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$ , then  $\exists c \in (a, b)$  s.t.  $f'(c) = 0$ .

*Proof.* By EVT, since  $f$  is continuous on a compact set, then  $f$  attains a maximum and a minimum. If both extremums occur at the endpoints, then  $f$  is necessarily a constant function and  $f'(x) = 0$  on  $(a, b)$ .

If either the maximum or minimum occurs at some point  $c \in (a, b)$ , then it follows from the Interior Extremum Theorem that  $f'(c) = 0$ . ■

**Theorem 5.6** (Mean Value Theorem). If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then  $\exists c \in (a, b)$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

*Proof.* Consider

$$d(x) = f(x) - \left[ \left( \frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right]$$

We know  $d$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Also,  $d(a) = d(b) = 0$ .

By Rolle's Theorem,  $\exists c \in (a, b)$  s.t.  $d'(c) = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a}$ . ■

**Corollary 5.1.** If  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable and  $f'(x) = 0$  for all  $x \in (a, b)$ , then  $f$  is constant on  $(a, b)$ .

*Proof.* Assume  $x, y \in (a, b)$  and  $x < y$ . We set  $c \in (x, y)$ , then by Mean Value Theorem,

$$0 = f'(c) = \frac{f(y) - f(x)}{y - x} \implies f(y) - f(x) = 0$$

■

**Corollary 5.2.** If  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable and  $f'(x) = g'(x)$  for all  $x \in (a, b)$ , then  $f(x) = g(x) + c$  for some  $c \in \mathbb{R}$ .

*Proof.* Apply the previous corollary to the function  $h(x) = f(x) - g(x)$ . ■

**Theorem 5.7** (Generalized Mean Value Theorem). If  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then  $\exists c \in (a, b)$  s.t.

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If  $g'$  is never zero on  $(a, b)$ , then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

*Proof.* Apply the Mean Value Theorem to the function  $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$ . ■

**Theorem 5.8** (L'Hospital's Rule:  $0/0$  case). Suppose  $f, g$  are continuous on  $I$  with  $a \in I$  and are differentiable on  $I \setminus \{a\}$ . If  $f(a) = g(a) = 0$  and  $\forall x \neq a, g'(x) \neq 0$ , then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$$

*Proof.* Since  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ , then for all  $\epsilon > 0, \exists \delta > 0$  s.t.

$$x \in (a - \delta, a + \delta) \implies \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon$$

By the Generalized Mean Value Theorem, for every  $y \in (a, a + \delta), \exists x \in (a, y)$  s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(y)}{g(y)}$$

and thus

$$\left| \frac{f(y)}{g(y)} - L \right| = \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon$$

■

**Theorem 5.9** (L'Hospital's Rule:  $\infty/\infty$  case). Suppose  $f, g$  are differentiable on  $(a, b)$  and  $g'(x) \neq 0$  for all  $x \in (a, b)$ . If  $\lim_{x \rightarrow a} g(x) = \infty$  or  $-\infty$ , then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$$

## 6 Sequences and Series of Functions

### 6.1 Uniform Convergence of a Sequence of Functions

**Definition 6.1** (pointwise convergence). For each  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on a set  $A \subseteq \mathbb{R}$ . If  $\forall x \in A, f_n(x) \rightarrow f(x)$  for some function  $f$ , then sequence  $(f_n)$  of functions converges pointwise on  $A$  to  $f$ .

We can write  $f_n \rightarrow f$ ,  $\lim f_n = f$ , or  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

**Example 6.1.** Consider  $f_n : \mathbb{R} \rightarrow \mathbb{R}$

$$f_n(x) = \frac{x^2 + nx}{n}$$

We can compute

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x^2 + nx}{n} = \lim_{n \rightarrow \infty} \frac{x^2}{n} + x = x$$

Thus,  $(f_n)$  converges pointwise to  $f(x) = x$  on  $\mathbb{R}$ .

**Example 6.2.** Consider  $f_n : [0, 1] \rightarrow \mathbb{R}$

$$f_n(x) = x^n$$

If  $0 \leq x < 1$ ,  $x^n \rightarrow 0$ . If  $x = 1$ ,  $x^n \rightarrow 1$ . It follows that  $f_n \rightarrow f$  pointwise on  $[0, 1]$  where

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Note that pointwise convergent sequence of continuous functions may converge to a non-continuous function.

**Definition 6.2** (uniformly convergence). Let  $(f_n)$  be a sequence of functions defined on a set  $A \subseteq \mathbb{R}$ , then  $(f_n)$  converges uniformly on  $A$  to a limit function  $f$  defined on  $A$  if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq N, \forall x \in A, |f(x) - f_n(x)| < \epsilon$$

**Remark 6.1.** This is a **stronger** notion of convergence.

**Example 6.3.** Consider  $f_n : \mathbb{R} \rightarrow \mathbb{R}$

$$f_n(x) = \frac{x^2 + nx}{n}$$

which converges pointwise on  $\mathbb{R}$  to  $f(x) = x$ . But the convergence is not uniform, since

$$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \frac{x^2}{n}$$

In order to force  $|f_n(x) - f(x)| < \epsilon$ , we need  $N < \frac{x^2}{\epsilon}$ . Although it is possible to do for each  $x \in \mathbb{R}$ , there is no way to choose a single value of  $N$  that will work for all values of  $x$  at the same time.

On the other hand, we can show that  $f_n \rightarrow f$  uniformly on the set  $[-b, b]$ .

**Property 6.1** (Cauchy Criterion for Uniform Convergence). A sequence of functions  $(f_n)$  defined on a set  $A \subseteq \mathbb{R}$  converges uniformly on  $A$  iff

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in A, \forall m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$$

**Theorem 6.1** (Continuous Limit Theorem). Let  $(f_n)$  be a sequence of functions defined on  $A \subseteq \mathbb{R}$  that converges uniformly on  $A$  to a function  $f$ . If each  $f_n$  is continuous at  $c \in A$ , then  $f$  is continuous at  $c$ .

*Proof.* Let  $\epsilon > 0$  and fix  $c \in A$ . Choose  $N$  s.t.

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}, \forall x \in A$$

Since  $f_N$  is continuous, then  $\exists \delta > 0$  s.t.

$$|x - c| < \delta \implies |f_N(x) - f_N(c)| < \frac{\epsilon}{3}$$

Thus,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

■

Hence  $f$  is continuous at  $c \in A$ .

**Property 6.2.** (Algebraic Limit Theorem for Uniform Convergence) Suppose  $(f_n), (g_n)$  are uniformly convergent on  $A$ , then

1.  $(cf_n + g_n)$  is uniformly convergent on  $A$
2. If  $\exists M > 0$  s.t.  $|f_n| \leq M$  and  $|g_n| \leq M$ , then  $(f_n g_n)$  is uniformly convergent.

*Proof.* (1) Obvious.

(2) Let  $\epsilon > 0$ . Since  $(f_n), (g_n)$  are uniformly convergent on  $A$ , then  $\exists N$  s.t.  $\forall m, n \geq N, |f_n(x) - f_m(x)| < \frac{\epsilon}{2M}$  and  $|g_n(x) - g_m(x)| < \frac{\epsilon}{2M}$ . Using Cauchy criterion, we have

$$\begin{aligned} |f_m(x)g_m(x) - f_n(x)g_n(x)| &= |f_m(x)g_m(x) - f_m(x)g_n(x) + f_m(x)g_n(x) - f_n(x)g_n(x)| \\ &\leq |f_m(x)||g_m(x) - g_n(x)| + |g_n(x)||f_m(x) - f_n(x)| \\ &\leq M(|g_m(x) - g_n(x)| + |f_m(x) - f_n(x)|) \\ &< M\left(\frac{\epsilon}{M}\right) \\ &= \epsilon \end{aligned}$$

So  $(f_n g_n)$  is uniformly convergent.

■

## 6.2 Uniform Convergence and Differentiation

**Theorem 6.2** (Differentiable Limit Theorem). Let  $f_n \rightarrow f$  pointwisely on  $[a, b]$  and assume each  $f_n$  is differentiable. If  $(f'_n)$  converges uniformly on  $[a, b]$  to a function  $g$ , then the function  $f$  is differentiable and  $f' = g$ .

**Theorem 6.3.** Let  $(f_n)$  be a sequence of differentiable functions defined on  $[a, b]$  and assume  $(f'_n)$  converges uniformly on  $[a, b]$ . If  $\exists x_0 \in [a, b]$  s.t.  $f_n(x_0)$  is convergent, then  $(f_n)$  converges uniformly on  $[a, b]$ .

**Theorem 6.4** (stronger form of Differentiable Limit Theorem). Let  $(f_n)$  be a sequence of differentiable functions defined on  $[a, b]$  and assume  $(f'_n)$  converges uniformly on  $[a, b]$  to a function  $g$ . If  $\exists x_0 \in [a, b]$  s.t.  $f_n(x_0)$  is convergent, then  $(f_n)$  converges uniformly on  $[a, b]$ . Moreover, the limit function  $f = \lim f_n$  is differentiable and  $f' = g$ .

## 6.3 Series of Functions

**Definition 6.3** (pointwise convergence). For each  $n \in \mathbb{N}$ , let  $f_n$  and  $f$  be functions defined on a set  $A \subseteq \mathbb{R}$ . The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \dots$$

converges pointwise on  $A$  to  $f(x)$  if the sequence  $s_k(x)$  of partial sums defined by

$$s_k(x) = f_1(x) + f_2(x) + \dots + f_k(x)$$

converges pointwise to  $f(x)$ .

**Definition 6.4** (uniform convergence). The series converges uniformly on  $A$  to  $f$  if the sequence  $s_k(x)$  converges uniformly on  $A$  to  $f(x)$ .

In either case, we write  $f = \sum_{n=1}^{\infty} f_n$  or  $f(x) = \sum_{n=1}^{\infty} f_n(x)$ .

**Theorem 6.5** (Term-by-term Continuity Theorem). Let  $f_n$  be continuous functions defined on a set  $A \subseteq \mathbb{R}$ , and assume  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$  to a function  $f$ . Then,  $f$  is continuous on  $A$ .

*Proof.* Apply the Continuous Limit Theorem 6.1 to the partial sums  $s_k = f_1 + f_2 + \dots + f_k$ . ■

**Theorem 6.6** (Term-by-term Differentiability Theorem). Let  $f_n$  be differentiable functions defined on an interval  $A$ , and assume  $\sum_{n=1}^{\infty} f'_n(x)$  **converges uniformly** to a limit  $g(x)$  on  $A$ . If there exists a **point**  $x_0 \in [a, b]$  where  $\sum_{n=1}^{\infty} f_n(x_0)$  converges, then the series  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly to a differentiable function  $f(x)$  satisfying  $f'(x) = g(x)$  on  $A$ . In other words,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{and} \quad f'(x) = g(x)$$

*Proof.* Apply the stronger form of the Differentiable Limit Theorem 6.4 to the partial sums  $s_k = f_1 + f_2 + \dots + f_k$ . ■

**Theorem 6.7** (Cauchy Criterion for Uniform Convergence of Series). A series  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A \subseteq \mathbb{R}$  if and only if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > m \geq N, \forall x \in A, |s_n - s_m| = \left| \sum_{i=m+1}^n f_i(x) \right| < \epsilon$$

**Remark 6.2.** The benefit of the Cauchy Criterion is that it does not depend on the value of the limit.

**Corollary 6.1** (Weierstrass M-Test). For each  $n \in \mathbb{N}$ , let  $f_n$  be a function defined on a set  $A \subseteq \mathbb{R}$ , and let  $M_n > 0$  be a real number satisfying that

$$\sup_{x \in A} |f_n(x)| \leq M_n$$

If  $\sum_{n=1}^{\infty} M_n$  converges, then  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$ .

*Proof.* Let  $\epsilon > 0$ . Choose  $N$  that satisfies the Cauchy Criterion. Let  $m > n \geq N$ . Then by Cauchy Criterion for Uniform Convergence of Series,

$$M_{m+1} + \dots + M_n < \epsilon$$

Then for  $n > m \geq N$  and all  $x \in A$ ,

$$\begin{aligned} |f_{m+1}(x) + \dots + f_n(x)| &\leq |f_{m+1}(x)| + \dots + |f_n(x)| \\ &\leq M_{m+1} + \dots + M_n \\ &< \epsilon \end{aligned}$$

■

**Remark 6.3.** The reverse is not true.

**Example 6.4.** If  $f_n(x) = (-1)^n \frac{1}{n}$ , then  $\sum_{n=1}^{\infty} f_n(x)$  is uniformly convergent, but the M-test fails because if  $M_n = \frac{1}{n}$  (the smallest  $M_n$  possible), then  $\sum_{n=1}^{\infty} \frac{1}{n}$  is not convergent.

**Corollary 6.2.** If  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A \subseteq \mathbb{R}$ , then the sequence  $(f_n)$  converges uniformly on  $A$  to 0.

*Proof.* WTS  $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall x \in A, |f_n(x)| < \epsilon$ .

Let  $\epsilon > 0$ . Since  $\sum_{n=1}^{\infty} f_n$  converges uniformly, then by Cauchy Criterion,

$$\exists N \in \mathbb{N} \text{ s.t. } \forall n > m \geq N, \forall x \in A, |f_{m+1}(x) + \dots + f_n(x)| < \epsilon$$

Let  $n = m + 1$ , then

$$|f_n(x)| < \epsilon$$

as wanted. ■

**Corollary 6.3.** Suppose  $\forall n \in \mathbb{N}, \forall x \in A, g_n(x) \geq f_n(x) \geq 0$ . If  $\sum_{n=1}^{\infty} g_n$  converge uniformly on  $A$ , then  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$ .

*Proof.* Let  $\epsilon > 0$ . Apply Cauchy Criterion for  $\sum_{n=1}^{\infty} g_n$ , we get  $N \in \mathbb{N}$  s.t. for  $n > m \geq N$  and  $x \in A$ ,

$$\begin{aligned} |f_{m+1}(x) + \dots + f_n(x)| &= f_m(x) + \dots + f_n(x) \\ &\leq g_{m+1}(x) + \dots + g_n(x) \\ &= |g_{m+1}(x) + \dots + g_n(x)| \\ &< \epsilon \end{aligned}$$

So  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$ . ■

## 6.4 Power Series

**Theorem 6.8.** If a power series  $\sum_{n=0}^{\infty} a_n x^n$  converges at some point  $x_0 \in \mathbb{R}$ , then it converges absolutely for any  $x$  satisfying  $|x| < |x_0|$ .

*Proof.* If  $\sum_{n=0}^{\infty} a_n x_0^n$  converges, then  $(a_n x_0^n)$  is bounded and  $\rightarrow 0$ . Let  $M > 0$  be s.t.  $|a_n x_0^n| \leq M$  for all  $n \in \mathbb{N}$ . If  $x \in \mathbb{R}$  satisfies  $|x| < |x_0|$ , then

$$|a_n x^n| = |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \leq M \left| \frac{x}{x_0} \right|^n$$

But  $|x/x_0| < 1$ , so the geometric series

$$\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$$

is convergent. By the Comparison Test,  $\sum_{n=0}^{\infty} a_n x^n$  converges absolutely. ■

**Theorem 6.9.** If a power series  $\sum_{n=0}^{\infty} a_n x^n$  converges absolutely at a point  $x_0$ , then it converges uniformly on the closed interval  $[-c, c]$ , where  $c = |x_0|$ .

*Proof.* For  $n \in \mathbb{N}$ , let  $M_n = |a_n| \cdot |x_0|^n$ .

Note that  $\sup_{x \in [-c, c]} |a_n x^n| \leq |a_n| \cdot |x_0|^n = M_n$ .

Since  $\sum_{n=0}^{\infty} M_n$  is convergent by assumption, then by Weierstrass M-Test 6.1,  $\sum_{n=0}^{\infty} a_n x^n$  converges uniformly on  $[-c, c]$ . ■

**Remark 6.4.** If the power series  $g(x) = \sum_{n=0}^{\infty} a_n x^n$  converges conditionally at  $x = R$ , then it is possible for it to diverge when  $x = -R$ .

**Example 6.5.**

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

**Lemma 6.1** (Abel's Lemma). Let  $b_n$  satisfy  $b_1 \geq b_2 \geq b_3 \geq \dots \geq 0$ , and let  $\sum_{n=1}^{\infty} a_n$  be a series for which the **partial sums are bounded**. In other words, assume there exists  $A > 0$  such that

$$|a_1 + a_2 + \dots + a_n| \leq A$$

for all  $n \in \mathbb{N}$ . Then for all  $n \in \mathbb{N}$ ,

$$|a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n| \leq A b_1$$

*Proof.*

$$\begin{aligned} \left| \sum_{k=1}^n a_k b_k \right| &= \left| s_n b_{n+1} + \sum_{k=1}^n s_k (b_k - b_{k+1}) \right| && \text{by summation-by-parts formula} \\ &\leq |s_n b_{n+1}| + \left| \sum_{k=1}^n s_k (b_k - b_{k+1}) \right| && \text{by Triangle Inequality} \\ &\leq A b_{n+1} + \sum_{k=1}^n A (b_k - b_{k+1}) \\ &= A b_{n+1} + (A b_1 - A b_{n+1}) \\ &= A b_1 \end{aligned}$$

■

**Theorem 6.10** (Abel's Theorem). Let  $g(x) = \sum_{n=0}^{\infty} a_n x^n$  be a power series that **converges at the point**  $x = R > 0$ . Then the series converges uniformly on the interval  $[0, R]$ . A similar result holds if the series converges at  $x = -R$ .

*Proof.* To set the stage for Abel's Lemma 6.1, we first write

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_n R^n) \left( \frac{x}{R} \right)^n$$

Let  $\epsilon > 0$ . Since we are assuming that  $\sum_{n=0}^{\infty} a_n R^n$  converges, then by the Cauchy Criterion for Uniform Convergence of Series 6.7,  $\exists N \in \mathbb{N}$  s.t. if  $n > m \geq N$ , then

$$|a_{m+1} R^{m+1} + a_{m+2} R^{m+2} + \dots + a_n R^n| < \epsilon$$

Now for any fixed  $m \in \mathbb{N}$ , we apply Abel's Lemma 6.1 to the sequence  $\sum_{i=1}^{\infty} a_{m+i} R^{m+i}$ . Since  $x \in [0, R]$ , then we have

$$\left( \frac{x}{R} \right)^{m+1} \geq \left( \frac{x}{R} \right)^{m+2} \geq \dots \geq 0$$

Then

$$|(a_{m+1} R^{m+1}) \left( \frac{x}{R} \right)^{m+1} + (a_{m+2} R^{m+2}) \left( \frac{x}{R} \right)^{m+2} + \dots + (a_n R^n) \left( \frac{x}{R} \right)^n| \leq \epsilon \left( \frac{x}{R} \right)^{m+1} \leq \epsilon$$

Therefore the series converges uniformly on the interval  $[0, R]$ . ■



**Theorem 6.11.** If  $\sum_{n=0}^{\infty} a_n x^n$  converges for all  $x \in (-R, R)$ , then the differentiated series  $\sum_{n=1}^{\infty} n a_n x^{n-1}$  converges at each  $x \in (-R, R)$  as well. Consequently, the convergence is uniform on closed intervals in  $(-R, R)$ .

prove this

**Theorem 6.12.** Assume  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  converges on an interval  $A \subseteq \mathbb{R}$ . The function  $f$  is continuous on  $A$  and differentiable on any open interval  $(-R, R) \subseteq A$ . The derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Moreover,  $f$  is infinitely differentiable on  $(-R, R)$ , and the successive derivatives can be obtained via term-by-term differentiation of the appropriate series:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)x^{n-k}$$

**Corollary 6.4.** If  $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=0}^{\infty} b_n x^n$  exist and equal for all  $x \in (-R, R)$ , then it must be the case that  $a_n = b_n$  for all  $n \in \mathbb{N}$ .

## 7 The Riemann Integral

### 7.1 The Definition of the Riemann Integral

#### 7.1.1 Partitions, Upper Sums, and Lower Sums

**Definition 7.1** (partition). A partition  $P$  of  $[a, b]$  is a finite set of points from  $[a, b]$  that includes both  $a$  and  $b$ . The notational convention is to always list the points of a partition  $P = \{x_0, x_1, x_2, \dots, x_n\}$  in increasing order; thus

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

**Definition 7.2** (lower sum and upper sum). For each subinterval  $[x_{k-1}, x_k]$  of  $P$ , let

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \text{ and } M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The lower sum of  $f$  with respect to  $P$  is given by

$$L(f, P) = \sum_{k=1}^n m_k (x_k - x_{k-1})$$

Likewise, we define the upper sum of  $f$  with respect to  $P$  by

$$U(f, P) = \sum_{k=1}^n M_k (x_k - x_{k-1})$$

**Fact 7.1.** For a particular partition  $P$ , it is clear that  $U(f, P) \geq L(f, P)$ .

**Definition 7.3** (refinement). A partition  $Q$  is a refinement of a partition  $P$  if  $Q$  contains all of the points of  $P$ ; that is, if  $P \subseteq Q$ .

**Lemma 7.1.** If  $P \subseteq Q$ , then  $L(f, P) \leq L(f, Q)$ , and  $U(f, P) \geq U(f, Q)$ .

*Proof.* Consider what happens when we refine  $P$  by adding a single point  $z$  to some subinterval  $[x_{k-1}, x_k]$  of  $P$ . Focusing on the lower sum, we have

$$\begin{aligned} m_k(x_k - x_{k-1}) &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m'_k(x_k - z) + m''_k(z - x_{k-1}) \end{aligned}$$

where

$$m'_k = \inf\{f(x) : x \in [z, x_k]\} \text{ and } m''_k = \inf\{f(x) : x \in [x_{k-1}, z]\}$$

are each necessarily as large or larger than  $m_k$ .

By induction, we have  $L(f, P) \leq L(f, Q)$ , and an analogous argument holds for the upper sums. ■

**Lemma 7.2.** If  $P_1$  and  $P_2$  are any two partitions of  $[a, b]$ , then  $L(f, P_1) \leq U(f, P_2)$ .

*Proof.* Let  $Q = P_1 \cup P_2$ . Because  $P_1 \subseteq Q$  and  $P_2 \subseteq Q$ , it follows that

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2)$$

■

### 7.1.2 Integrability

**Definition 7.4** (upper integral and lower integral). Let  $\mathcal{P}$  be the collection of all possible partitions of the interval  $[a, b]$ . The upper integral of  $f$  is defined to be

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

Similarly, we define the lower integral of  $f$  by

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$$

**Lemma 7.3.** For any bounded function  $f$  on  $[a, b]$ , it is always the case that

$$U(f) \geq L(f)$$

**Definition 7.5** (Riemann Integrability). A bounded function  $f$  defined on the interval  $[a, b]$  is Riemann-integrable if  $U(f) = L(f)$ . In this case, we define  $\int_a^b f$  or  $\int_a^b f(x) dx$  to be this common value; namely,

$$\int_a^b f = U(f) = L(f)$$

### 7.1.3 Criteria for Integrability

**Theorem 7.1** (Integrability Criterion). A bounded function  $f$  is integrable on  $[a, b]$  if and only if, for every  $\epsilon > 0$ ,  $\exists$  a partition  $P_\epsilon$  of  $[a, b]$  such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

prove this

**Theorem 7.2.** If  $f$  is continuous on  $[a, b]$ , then it is integrable.

prove this

**Definition 7.6** (tagged partition). A tagged partition  $(P, \{c_k\})$  is one where in addition to a partition  $P$  we choose a sampling point  $c_k$  in each of the subintervals  $[x_{k-1}, x_k]$ .

$$P = [x_0, x_1, \dots, x_n]$$

$$c_k \in [x_{k-1}, x_k], \quad 0 < k \leq n$$

**Definition 7.7** (Riemann sum).

$$R(f, P, \{c_k\}) = \sum_{k=1}^n f(c_k) \cdot (x_k - x_{k-1})$$

**Definition 7.8** (Riemann's Original Definition of the Integral). A bounded function  $f$  is integrable on  $[a, b]$  with  $\int_a^b f = A$  if for all  $\epsilon > 0$ ,  $\exists \delta > 0$  such that for any tagged partition  $(P, \{c_k\})$  satisfying  $\delta x_k < \delta$  for all  $k$  it follows that

$$|R(f, P, \{c_k\}) - A| < \epsilon$$

**Remark 7.1.** This definition is equivalent to our definition.

## 7.2 Integrating Functions with Discontinuities

**Fact 7.2.** Suppose two functions  $f, g : [a, b] \rightarrow \mathbb{R}$  are both bounded. and  $f$  is integrable. Suppose there are finitely many points  $y_1, y_2, \dots, y_l \in [a, b]$  s.t.  $f(x) = g(x)$  for  $x \neq y_k$  for  $k = 1, 2, \dots, l$ .

Then  $g$  is integrable and

$$\int_a^b g = \int_a^b f$$

prove this

**Theorem 7.3.** If  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, and  $f$  is integrable on  $[c, b]$  for all  $c \in (a, b)$ , then  $f$  is integrable on  $[a, b]$ . An analogous result holds at the other endpoint.

prove this

**Example 7.1** (Dirichlet's function).

$$g(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ 0 & \text{for } x \text{ irrational} \end{cases}$$

If  $P$  is some partition of  $[0, 1]$ , then the density of the rationals in  $\mathbb{R}$  implies that every subinterval of  $P$  will contain a point where  $g(x) = 1$  as well as a point where  $g(y) = 0$ . It follows that  $U(g, P) = 1$  and  $L(g, P) = 0$ . Because this is the case for every partition  $P$ , we see that  $U(f) = 1, L(f) = 0$ . The two are not equal, so we conclude that Dirichlet's function is **not** integrable.

### 7.3 Properties of the Integral

**Theorem 7.4.** Assume  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, and let  $c \in (a, b)$ . Then,  $f$  is integrable on  $[a, b]$  if and only if  $f$  is integrable on  $[a, c]$  and  $[c, b]$ . In this case, we have

$$\int_a^b f = \int_a^c f + \int_c^b f$$

prove this

**Theorem 7.5.** Assume  $f$  and  $g$  are integrable functions on the interval  $[a, b]$ .

1. The function  $f + g$  is integrable on  $[a, b]$  with  $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ .
2. For  $k \in \mathbb{R}$ , the function  $kf$  is integrable with  $\int_a^b kf = k \int_a^b f$ .
3. If  $m \leq f(x) \leq M$  on  $[a, b]$ , then  $m(b - a) \leq \int_a^b f \leq M(b - a)$ .
4. If  $f(x) \leq g(x)$  on  $[a, b]$ , then  $\int_a^b f \leq \int_a^b g$ .
5. If  $f(x) \leq g(x)$  on  $[a, b]$ , then  $\int_a^b f \leq \int_a^b g$ .
6. The function  $|f|$  is integrable and  $|\int_a^b f| \leq \int_a^b |f|$ .

**Definition 7.9.** If  $f$  is integrable on the interval  $[a, b]$ , define

$$\int_b^a f = - \int_a^b f$$

Also for  $c \in [a, b]$ , define

$$\int_c^c f = 0$$

**Fact 7.3.** If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable, then

$$|f|(x) = |f(x)|$$

is also integrable, and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

prove this

#### 7.3.1 Uniform Convergence and Integration

**Theorem 7.6** (Integrable Limit Theorem). Assume that  $f_n \rightarrow f$  **uniformly** on  $[a, b]$  and that each  $f_n$  is integrable. Then,  $f$  is integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f$$

prove this

## 7.4 The Fundamental Theorem of Calculus

**Theorem 7.7** (Fundamental Theorem of Calculus). We have

1. If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable, and  $F : [a, b] \rightarrow \mathbb{R}$  satisfies  $F'(x) = f(x)$  for all  $x \in [a, b]$ , then

$$\int_a^b f = F(b) - F(a)$$

2. Let  $g : [a, b] \rightarrow \mathbb{R}$  be integrable, and for  $x \in [a, b]$ , define

$$G(x) = \int_a^x g$$

Then  $G$  is continuous on  $[a, b]$ . If  $g$  is continuous at some point  $c \in [a, b]$ , then  $G$  is differentiable at  $c$  and  $G'(c) = g(c)$ .

prove this

**Example 7.2.**  $f(x) = |x|$  on  $[-1, 1]$ .

Define  $F(x) = \int_{-1}^x f(x) dx$ , then  $F$  is continuous and differentiable on  $[-1, 1]$ .

On  $[-1, 0]$ ,  $F(x) = -\frac{1}{2}x^2 + \frac{1}{2}$ .

On  $[0, 1]$ ,  $F(x) = \frac{1}{2}x^2 + \frac{1}{2}$ .

So combining the two, we keep the relationship  $F'(x) = f(x) (= |x|)$ .

**Example 7.3.**  $f : [a, b] \rightarrow \mathbb{R}$  continuous. Let  $F(x) = \int_a^x f : [a, b] \rightarrow \mathbb{R}$ .

We know from FTC that  $F'(x) = f(x)$  for all  $x \in [a, b]$ .

Assume  $F(x) = \int_a^x f = 0$  for all  $x \in [a, b]$ . Then  $f(x) = 0$  for all  $x \in [a, b]$ .

**Example 7.4.** T or F:

1.  $h' = g$  does not imply continuity of  $g$ . (True)
2. If  $g$  is continuous on  $[a, b]$ , then there is a differentiable  $h$  s.t.  $h' = g$ . (True)
3. If  $H(x) = \int_a^x h$  is differentiable at  $c \in (a, b)$ , then  $h$  is continuous at  $c$ . (False)

Counterexample for (3):  $h : [0, 1] \rightarrow \mathbb{R}$ .

$$h(x) = \begin{cases} 0, & x \neq 1/2 \\ 1, & x = 1/2 \end{cases} \implies H(x) = 0$$

**Example 7.5.**  $f_n \rightarrow 0$  pointwise on  $[0, 1]$ , but  $\lim_{n \rightarrow \infty} \int f_n$  does not exist.

$$f_n = \begin{cases} x^n, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

For every  $x \in [0, 1]$ ,  $\int_0^1 f_n = n \rightarrow \infty$ , But  $f_n(x) \rightarrow 0$ .

**Fact 7.4.** If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and there is  $M \in \mathbb{R}$  s.t.,  $|f(x)| \leq M$  for all  $x \in [a, b]$ , then  $f^2(x) = (f(x))^2$  is integrable.