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Synchronization of uncertain fractional-order chaotic systems via a novel adaptive controller

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ABSTRACT

This paper deals with the drive-response synchronization scheme for uncertain fractional-order chaotic systems. Some novel sufficient conditions for chaos synchronization of fractional-order chaotic systems with model uncertainties and external disturbances are derived by using the fractional-order extension of the Lyapunov stability theorem. The designed synchronization are new, simple and yet easily realized experimentally compared with those where complex control functions are used. Simulation results are given for several fractional-order chaotic examples to illustrate the effectiveness of the proposed scheme.

1. Introduction

Chaotic systems, which are characteristic of nonlinear systems, have many complex nonlinear dynamic behaviors, such as being extremely sensitive to tiny variations of the initial conditions, having bounded trajectories in the phase space, etc. After the pioneering work of Pecora and Carroll [1], research efforts have investigated chaos synchronization problems in many chaotic systems [2–6]. Recently, the synchronization of fractional-order chaotic systems has increasingly received attention due to its potential applications in secure communication and control processing. For example, the chaotic dynamics of the fractional-order Genesio–Tesi system has been studied in [7]. Based on stability analysis of the system, a necessary condition of chaos occurrence is obtained. Paper [8] investigates the synchronization between integer-order chaotic systems and a class of fractional-order chaotic systems using the stability theory of fractional-order systems. Based on sliding mode control and the stability theorem, a new method for chaos synchronization between integer-order chaotic systems and a class of fractional-order chaotic system is presented. An adaptive sliding mode technique based on a fractional-order switching-type control law is designed in [9]. A novel fractional-order switching-type control law is proposed to ensure the existence of the sliding motion in finite time. In [10], the authors consider the modified projective synchronization of fractional-order chaotic systems with unknown parameters and disturbance. Based on the stability theory of fractional-order systems, the adaptive controllers and the parameters' updated laws are designed. The robust projective synchronization of a fractional-order chaotic system was investigated in [11]. On the basis of the input-to-state stable (ISS) theory, a single sinusoidal state coupling controller has been derived to achieve projective synchronization of this fractional dynamical system.

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It should be noted that most of the the schemes proposed in the papers [7–11] are only valid for the 3-dimensional fractional order chaotic systems. However, from the point of practical applications, it is desired that the synchronization schemes can be used for more chaotic systems. In addition, most of the previous studies do not take into account the influence of the system parameter's uncertainties and the external uncertainties, which cannot be avoided in real applications. To the best of our knowledge, there are only a few investigations considering the synchronization between n-dimensional fractional order chaotic systems in the presence of system parameter's uncertainties and external uncertainties.

Motivated by the aforementioned discussion, in this paper we investigate the synchronization of an n-dimensional fractional order chaotic system. By using the fractional-order extension of the Lyapunov stability theorem, some new sufficient conditions for the chaos synchronization of fractional-order chaotic systems with model uncertainties and external disturbances are proposed. Numerical simulation results are presented to demonstrate the validity of the proposed method.

This paper is organized as follows. In Section 2, some preliminaries and definitions are presented. Based on the the fractional-order extension of the Lyapunov stability theorem, an adaptive controller is proposed in Section 3 to synchronize uncertain chaotic fractional-order systems. Numerical examples which demonstrate the effectiveness of the proposed approach are shown in Section 4. Some conclusions are drawn in Section 5.

2. Preliminaries

There exist multiple definitions of fractional derivatives, due to its explicit physical interpretation in our paper only the Caputo fractional-order derivative is adopted.

Definition 1 [12]**.** The Caputo fractional-order derivative of a function $f(t)$ with respect to t is defined by

$$ {}_{t_0}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{(\alpha-n+1)}} d\tau, \qquad t > t_0 $$

where α is the order of the fractional derivative, n is an integer satisfying $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α, and $\Gamma(\cdot)$ is the Gamma function which is defined by the integral

$$ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt. $$

Lemma 1 [13]**.** *Let $x(t) \in R$ be a continuous and differentiable function. Then, for any time instant $t \geq t_0$*

$$ \frac{1}{2}\, {}_{t_0}^{c}D_t^{\alpha} x^2 \leq x\, {}_{t_0}^{c}D_t^{\alpha} x, \forall \alpha \in (0, 1] $$

Definition 2 [14]**.** A continuous function α: $[0, t) \to [0, \infty)$ is said to belong to class-K if it is strictly increasing and $\alpha(0) = 0$.

Lemma 2 [15]**.** *Let $x = 0$ be an equilibrium point for the nonautonomous fractional-order system*

$$ D^{\alpha} x(t) = f(t, x) \qquad 0 < \alpha < 1. \tag{1} $$

Assume that there exists a Lyapunov function $V(t, x(t))$ and class-K functions $\beta_i (i = 1, 2, 3)$ satisfying

$$ \beta_1(||x||) \leq V(t, x) \leq \beta_2(||x||) $$

and

$$ D^{\alpha} V(v, x) \leq -\beta_2(||x||). $$

Then the nonlinear fractional-order system (1) is asymptotically stable.

In the following, unless otherwise specified, we use $D^{\alpha}x(t)$ to denote ${}_0^{C}D_t^{\alpha} x(t)$.

3. The chaos synchronization scheme

Consider the following n-dimensional chaotic system:

$$ \begin{cases} D^{\alpha_1} x_1 = f_1(x) + \Delta f_1(x) + d_{11}(t), \\ D^{\alpha_2} x_2 = f_2(x) + \Delta f_2(x) + d_{12}(t), \\ \qquad \vdots \\ D^{\alpha_n} x_n = f_n(x) + \Delta f_n(x) + d_{1n}(t), \end{cases} \tag{2} $$

where α_i $(0 < \alpha_i \leq 1)$ is the order of the fractional derivative, $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ is the state variable, $f_i(x) \in R$ is a nonlinear function of x and t, $\Delta f_i(x) \in R$ is the model uncertainty, and $d_{1i}(t)$ is the external disturbance, $i = 1, 2, \ldots, n$.

Remark 1. System (2) is the general form of a chaotic system, almost all of the fractional order chaos systems can be expressed in the form (2).

Suppose system (2) is the drive system, in order to synchronize system (2) the response system is constructed as

$$\begin{cases} D^{\alpha_1} y_1 = g_1(y) + \Delta g_1(y) + d_{21}(t) + u_1, \\ D^{\alpha_2} y_2 = g_2(y) + \Delta g_2(y) + d_{22}(t) + u_2, \\ \quad \vdots \\ D^{\alpha_n} y_n = g_n(y) + \Delta g_n(y) + d_{2n}(t) + u_n, \end{cases} \tag{3}$$

where α_i $(0 < \alpha_i \le 1)$ is the order of the fractional derivative, $y = (x_1, y_2, \ldots, x_n)^T \in R^n$ is the state variable, $g_i(y) \in R$ is a nonlinear function of x and t, $\Delta g_i(y) \in R$ is the model uncertainty, $d_{2i}(t)$ is the external disturbance, and u_i is the controller to be designed later, $i = 1, 2, \ldots, n$.

In order to observe the synchronization between systems (2) and (3), we define the errors $e_1 = y_1 - x_1, e_2 = y_2 - x_2, \ldots, e_n = y_n - x_n$. Then the dynamics of the error system is determined, directly from subtracting (2) from (3), as follows:

$$\begin{cases} D^{\alpha_1} e_1 = g_1(y) + \Delta g_1(y) + d_{21}(t) - (f_1(x) + \Delta f_1(x) + d_{11}(t)) + u_1, \\ D^{\alpha_2} e_2 = g_2(y) + \Delta g_2(y) + d_{22}(t) - (f_2(x) + \Delta f_2(x) + d_{12}(t)) + u_2, \\ \quad \vdots \\ D^{\alpha_n} e_n = g_n(y) + \Delta g_n(y) + d_{2n}(t) - (f_n(x) + \Delta f_n(x) + d_{1n}(t)) + u_n. \end{cases} \tag{4}$$

It can be easily seen that the synchronization problem between systems (2) and (3) is equivalent to the problem of stabilizing system (4). In order to obtain further results, we need the following assumption:

Assumption 1. Suppose $f_i(x)$, $g_i(y)$, $\Delta f_i(x)$, $\Delta g_i(y)$ and $d_{1i}(t)$, $d_{2i}(t)$ are all bounded.

In general, without a control $(u_i = 0, i = 1, 2, \ldots, n)$ the trajectories of systems (2) and (3) will quickly separate and become unrelated on the condition that the initial values $(x_1(0), x_2(0), \cdots, x_n(0)) \ne (y_1(0), y_2(0), \cdots, y_n(0))$. However, with appropriate control schemes, the two systems will approach synchronization for any initial value.

Theorem 1. *Systems (2) and (3) will approach global and asymptotical synchronization for any initial condition with the following control laws:*

$$u_i = -k_i \mathrm{sign}(e_i), \tag{5}$$

where k_i is the adaptive feedback gain which is updated according to the following adaptive algorithm:

$$D^{\alpha_i} k_i = |e_i|, \qquad i = 1, 2, \ldots, n. \tag{6}$$

Proof. In order to prove that systems (2) and (3) will approach global and asymptotical synchronization we only need to show that $\lim_{t\to+\infty} e_1 = \lim_{t\to+\infty} e_2 = \cdots = \lim_{t\to+\infty} e_n = 0$ for any initial condition.

From the ith equation of system (4), one gets

$$\begin{aligned} D^{\alpha_i} e_i &= g_i(y) + \Delta g_i(y) + d_{2i}(t) - (f_i(x) + \Delta f_i(x) + d_{1i}(t)) + u_i \\ &= -e_i + (e_i + g_i(y) + \Delta g_i(y) + d_{2i}(t) \\ &\quad -(f_i(x) + \Delta f_i(x) + d_{1i}(t))) - k_i \mathrm{sign}(e_i). \end{aligned} \tag{7}$$

It is noted that the trajectories of chaotic systems are bounded, which means that e_i is bounded. According to Assumption 1 it is easy to see that there exists a positive unknown constant l_i such that

$$|e_i + g_i(y) + \Delta g_i(y) + d_{2i}(t) - (f_i(x) + \Delta f_i(x) + d_{1i}(t))| \le l_i.$$

Now, we take the Lyapunov function as

$$V = \frac{1}{2}(e_i^2 + (l_i - k_i)^2). \tag{8}$$

By Lemma 1, the derivative of (8) along the trajectory of system (7) is given as

$$\begin{aligned} D^{\alpha_i} V &\le e_i D^{\alpha_i} e_i - (l_i - k_i) D^{\alpha_i} k_i \\ &= e_i(-e_i + (e_i + g_i(y) + \Delta g_i(y) + d_{2i}(t) - (f_i(x) + \Delta f_i(x) + d_{1i}(t))) - k_i sign(e_i)) \\ &\quad -(l_i - k_i)|e_i| \\ &\le -e_i^2 + |e_i|(l_i - k_i) - |e_i|(l_i - k_i) \\ &= -e_i^2. \end{aligned}$$

In view of Lemma 2, we obtain $\lim_{t\to\infty} e_i = 0, i = 1, 2, \ldots, n$. This completes the proof of Theorem 3. □

Remark 2. It is well known that in practice a parameter mismatch is unavoidable, which can destroy the synchronization of the fractional-order chaotic systems. It is noted that the unknown parameters are bounded, thus we can view the terms that contain the unknown parameters as the model uncertainties or external disturbances, which means that our synchronization scheme is robust against parameter mismatch.

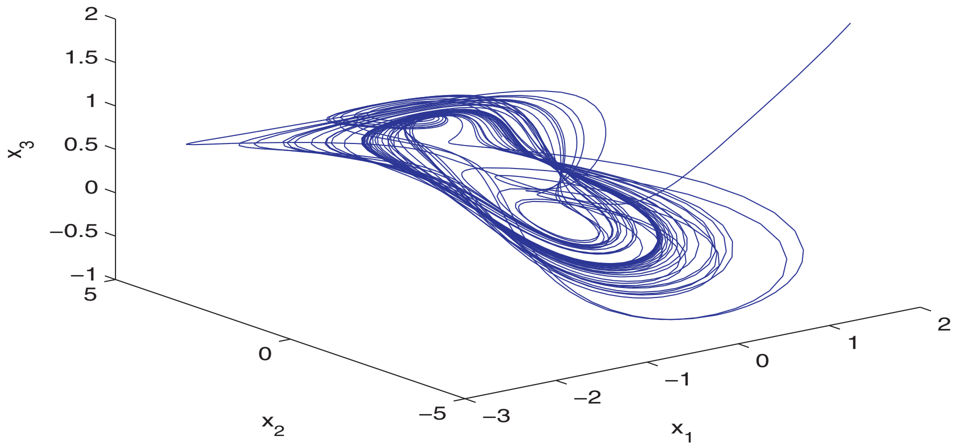

Fig. 1. The chaos attractor of system (9) with $x_1(0) = 2, x_2(0) = -3, x_3(0) = 2$.

4. Numerical simulations

In this section we evaluate the performance of our synchronization scheme by applying the method to two different chaotic systems. Some numerical simulations are presented to show the effectiveness of the proposed scheme. Two cases are discussed in this section. Case 1 is the synchronization between two identical fractional-order chaotic systems while Case 2 is the synchronization between two nonidentical fractional-order chaotic systems. In the following, the orders of the fractional derivative are fixed as $\alpha_1 = 0.995, \alpha_2 = 0.996, \alpha_3 = 0.997$.

Case 1: The synchronization between two identical fractional-order chaotic systems 分数阶金融系统

The model to be investigated in Case 1 was proposed by Chen [16]. The model that describes a fractional-order financial system of three nonlinear differential equations is described by

$$\begin{cases} D^{\alpha_1}x_1 = x_3 + (x_2 - a)x_1, \\ D^{\alpha_2}x_2 = 1 - bx_2 - x_1^2, \\ D^{\alpha_3}x_3 = x_1 - cx_3, \end{cases} \tag{9}$$

where a, b and c are positive real constants. Fig. 1 is the chaos attractor of the financial system (9) for $a = 1, b = 0.1$ and $c = 1$ with $x_1(0) = 2, x_2(0) = -3, x_3(0) = 2$.

Suppose system (9) is affected by model uncertainties and external disturbances. Thus, system (9) with model uncertainties and external disturbances can be rewritten as:

$$\begin{cases} D^{\alpha_1}x_1 = x_3 + (x_2 - a)x_1 + 0.2x_1x_2 + 0.1\cos(t), \\ D^{\alpha_2}x_2 = 1 - bx_2 - x_1^2 + 0.2x_2x_3 + 0.1\sin(t), \\ D^{\alpha_3}x_3 = x_1 - cx_3 + 0.2x_1x_3 + 0.1\cos(t)\sin(t), \end{cases} \tag{10}$$

where $0.2x_1x_2$, $0.2x_2x_3$ and $0.2x_1x_3$ are the model uncertainties, respectively. $0.1\cos(t)$, $0.1\sin(t)$ and $0.1\cos(t)\sin(t)$ are the external disturbances, respectively.

Let system (10) be the drive system, then the controlled response system is given by

$$\begin{cases} D^{\alpha_1}y_1 = y_3 + (y_2 - a)x_1 + 0.2y_1y_2 + 0.1\sin(t) + u_1, \\ D^{\alpha_2}y_2 = 1 - by_2 - y_1^2 + 0.2y_2^2 + 0.1\cos(t) + u_2, \\ D^{\alpha_3}y_3 = y_1 - cy_3 + 0.2y_2y_3 + 0.1\cos(t)^2 + u_3, \end{cases} \tag{11}$$

where $0.2y_1y_2, 0.2y_2^2, 0.2y_2y_3$ are the model uncertainties and $0.1\sin(t)$, $0.\cos(t)$, $0.1\cos(t)^2$ are the external disturbances, u_1, u_2, u_3 are the controllers to be designed later.

In order to achieve synchronization between systems (10) and (11) we take

$$u_i = -k_i \mathrm{sign}(y_i - x_i), \tag{12}$$

where k_i is the adaptive feedback gain which is updated as

$$D^{\alpha_i}k_i = |y_i - x_i|, \quad i = 1, 2, 3. \tag{13}$$

According to Theorem 3, the synchronization between systems (10) and (11) will be achieved. In our numerical simulations, we set $a = 1, b = 0.1, c = 1$. The simulation results with $x_1(0) = 2, x_2(0) = -3, x_3(0) = 2$, $y_1(0) = 1, y_2(0) = 3, y_3(0) = 5$ and $k_1(0) = k_2(0) = k_3(0) = 0$ are given in Figs. 2–4. From Figs. 2–4, one can observe that the errors e_1, e_2, e_3 approach 0 as $t \to +\infty$, respectively, which means that the synchronization between systems (10) and (11) is achieved.

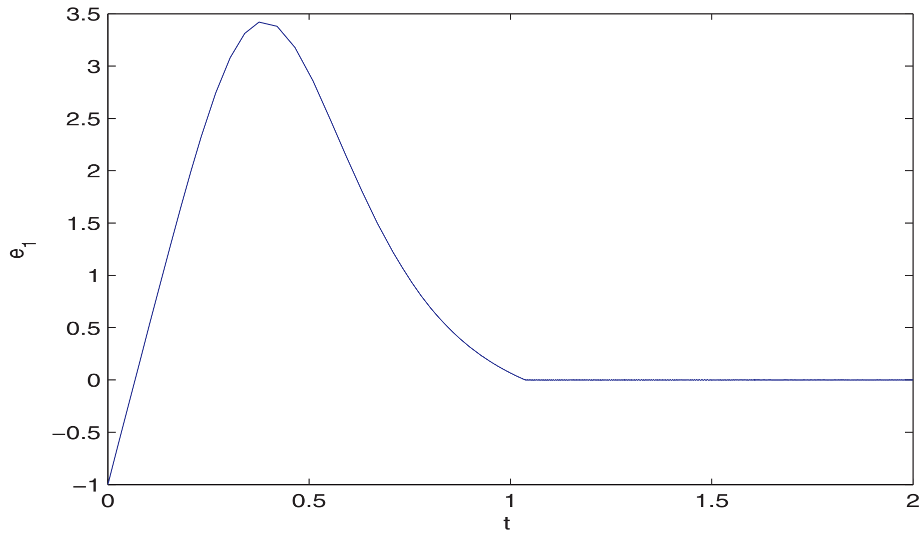

Fig. 2. The synchronization error e_1 between systems (10) and (11).

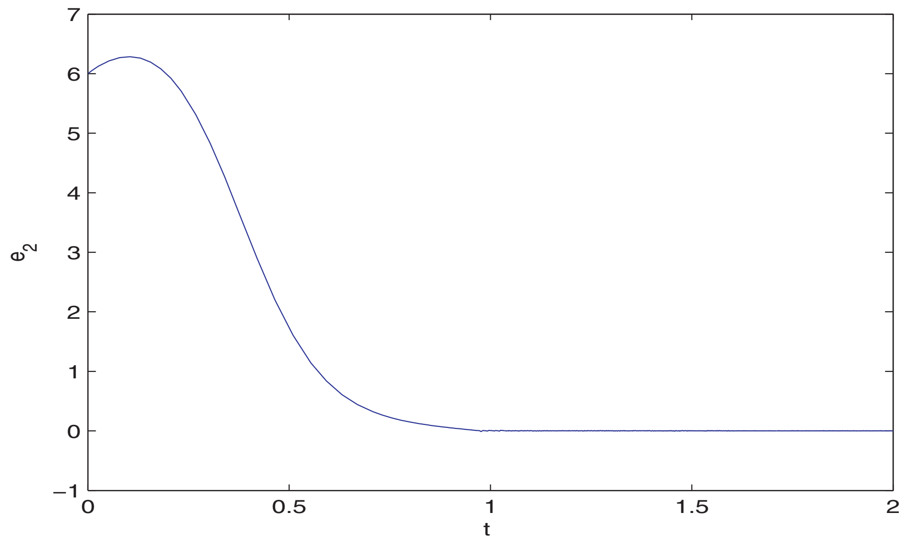

Fig. 3. The synchronization error e_2 between systems (10) and (11).

Remark 3. The control of system (9) has been studied via the sliding model in papers [18–20]. However, to the best of our knowledge, there are little to no articles that have investigated the synchronization of system (9) and our study fills this gap. In addition, it is easy to see that the controllers proposed in [18–20] are more complex than those given in our paper.

Case 2: The synchronization between two nonidentical fractional-order chaotic systems

Consider the fractional-order Sundarapandian–Pehlivan chaotic system [17] described by the following set of differential equations

$$\begin{cases} D^{\alpha_1} x_1 = ax_2 - x_1, \\ D^{\alpha_2} x_2 = -bx_1 - x_3, \\ D^{\alpha_3} x_3 = cx_3 + x_1 x_2^2 - x_1. \end{cases} \tag{14}$$

System (14) is chaotic when $a = 1, b = 0.46, c = 0.46$. Fig. 5 is the chaos attractor of system (14) for $a = 1, b = 0.46, c = 0.46$ with $x_1(0) = 0.2, x_2(0) = -0.3, x_3(0) = 0.5$.

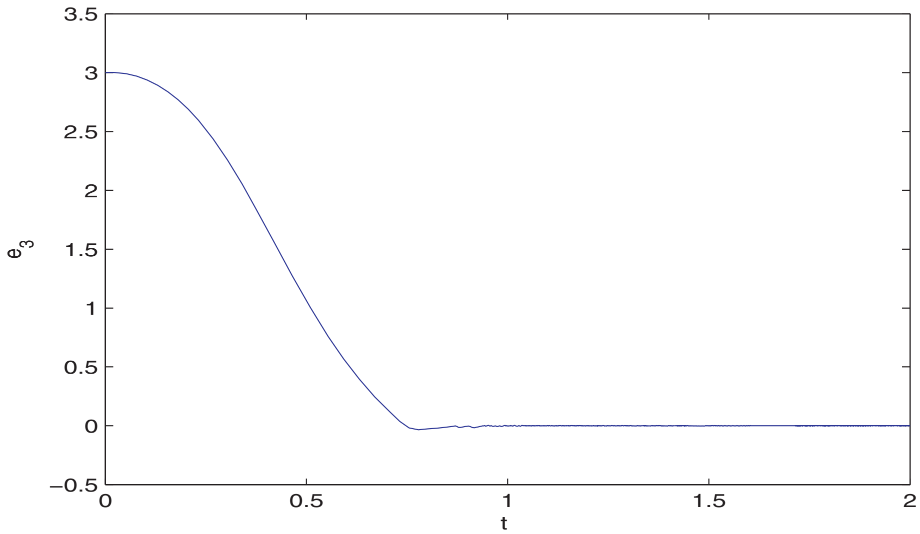

Fig. 4. The synchronization error e_3 between systems (10) and (11).

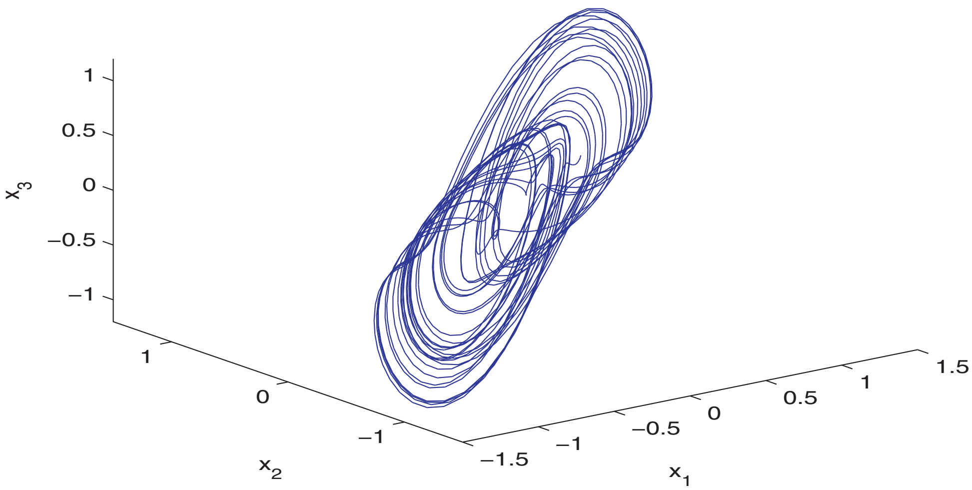

Fig. 5. The chaos attractor of system (14) with $x_1(0) = 0.2, x_2(0) = -0.3, x_3(0) = 0.5$.

Suppose system (10) is the drive system, based on system (14) the controlled response system is:

$$\begin{cases} D^{\alpha_1} y_1 = ay_2 - y_1 + 0.2y_1y_2^2 + 0.1\cos(t)^2 + u_1, \\ D^{\alpha_2} y_2 = -by_1 - y_3 + 0.2y_1^2 + 0.1\sin(t) + u_2, \\ D^{\alpha_3} y_3 = cy_3 + y_1y_2^2 - y_1 + 0.2y_2y_3 + 0.1\cos(t)\sin(t)^2 + u_3, \end{cases} \tag{15}$$

Based on Theorem 1, we take

$$u_i = -k_i \mathrm{sign}(y_i - x_i), \tag{16}$$

where k_i is the adaptive feedback gain which is updated as

$$D^{\alpha_i} k_i = |y_i - x_i|, \qquad i = 1, 2, 3. \tag{17}$$

According to Theorem 1, the synchronization between systems (10) and (15) will be achieved. In this simulation process, the initial conditions for the drive and response systems are: $x_1(0) = 2, x_2(0) = -3, x_3(0) = 2,\ y_1(0) = 1, y_2(0) = 3, y_3(0) = -2$ and $k_1(0) = k_2(0) = k_3(0) = 0$. The simulation results are presented in Figs. 6–8.

From Figs. 6–8, one can see that the synchronization errors converge asymptotically to zero after a short transient time, which implies that the synchronization between the drive system (10) and the response system (15) is reached. The

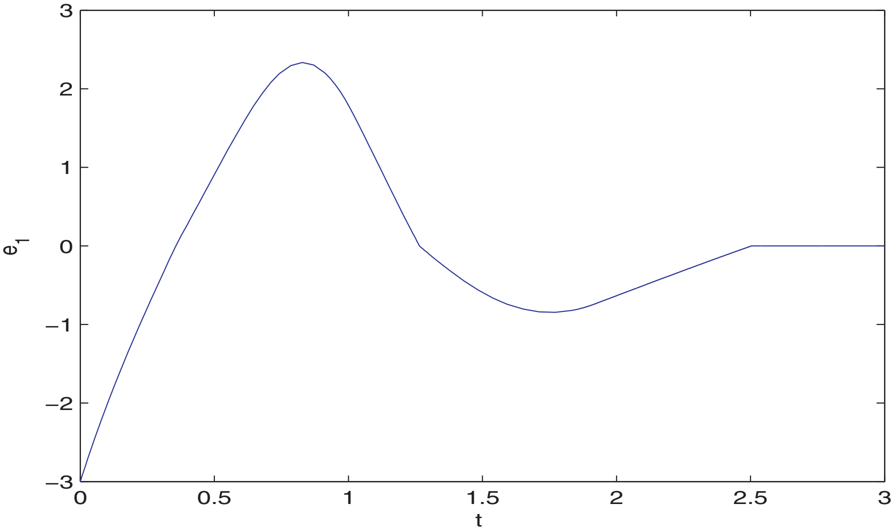

Fig. 6. The synchronization error e_1 between systems (10) and (15).

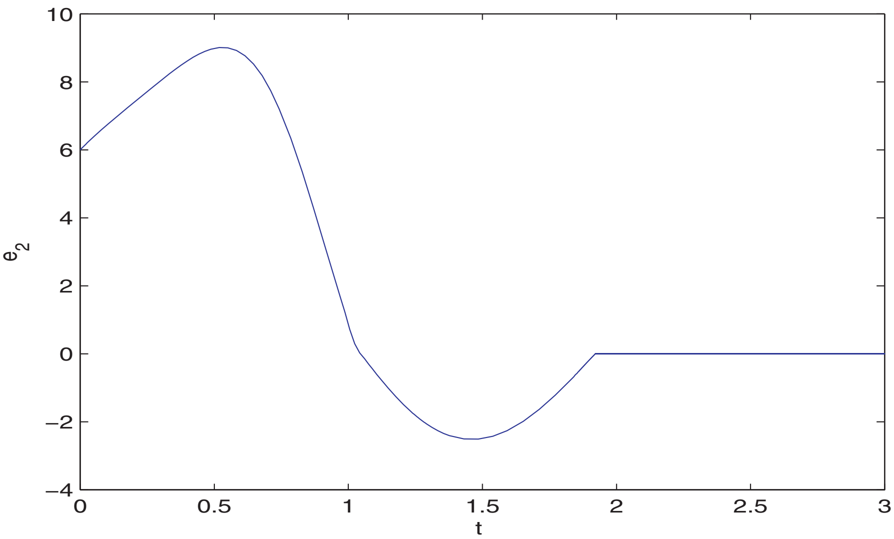

Fig. 7. The synchronization error e_2 between systems (10) and (15).

simulation results show that the obtained theoretic results are feasible and efficient for synchronizing uncertain fractional-order systems.

Remark 4. The reduced-order observer-based synchronization of the integer-order chaotic system (14) is studied in [17]. As far as we know, no studies have considered the synchronization of the fractional-order chaotic system (14), which means that our study is significant.

5. Conclusions

In this study, the problem of chaos synchronization is investigated via the adaptive control approach. New results for achieving robust chaos synchronization of n-dimensional fractional-order chaotic systems under the presence of system uncertainties and external disturbances are presented. Our designed synchronization schemes possess the following advantages: (1) Our method can be used not only for some special chaotic systems but also for general chaotic systems; (2) No knowledge of the bounds of the model uncertainties and external disturbances is required in advance; (3) The designed

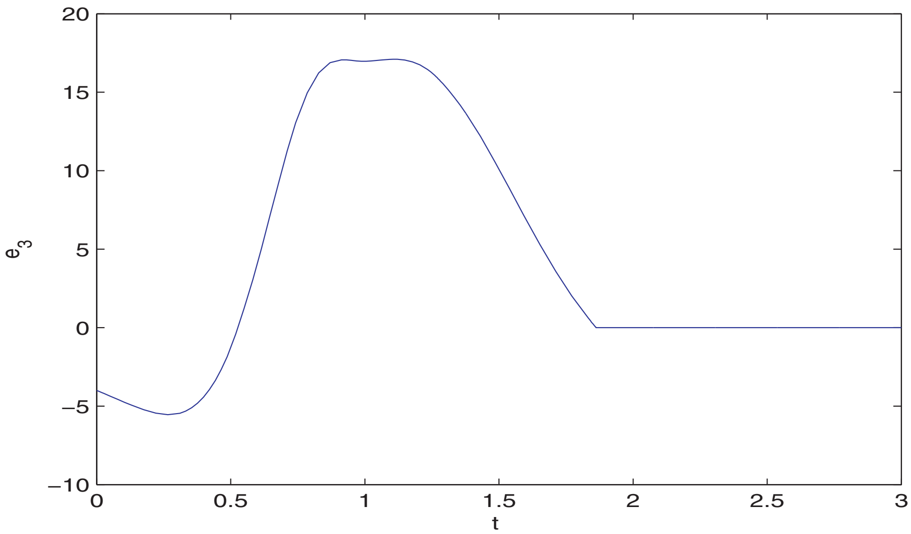

Fig. 8. The synchronization error e_3 between systems (10) and (15).

synchronization is new, simple and yet easily realized experimentally compared with those where complex control functions are used.

Acknowledgments

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