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Highlights

- The main variables and parameters in the chaotic systems are complex.
- The scaling function matrix from CMFPS is a complex-valued diagonal one.
- A set of sufficient criteria for achieving CMFPS and control are derived.
- The parameter update laws for estimating unknown complex parameters are given.
- Do not separate the real and imaginary parts of the parameters or variables.

Complex modified function projective synchronization of complex chaotic systems with known and unknown complex parameters

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Abstract

Much progress has been made in the research of modified function projective synchronization (MFPS) for real (complex) chaotic systems with real parameters. The scaling functions are always chosen as real-valued ones in previous MFPS schemes for chaotic systems evolving in the same or inverse directions simultaneously. However, MFPS with different complex-valued scaling functions (CMFPS) has not been previously reported, where complex-variable chaotic (hyperchaotic) systems (CVCSs) evolve in different directions with a time-dependent intersection angle. Therefore, CMFPS is discussed for CVCSs with known and unknown complex parameters in this paper. By constructing appropriate Lyapunov functions defined on complex field, and employing adaptive control technique, a set of simple and practical sufficient conditions for achieving CMFPS are derived, and complex update laws for estimating unknown parameters are also given. The corresponding theoretical proofs and computer simulations are worked out to demonstrate the effectiveness and feasibility of the proposed schemes.

Keywords: Complex-variable chaotic (hyperchaotic) system, complex parameter, complex scaling function, complex update law, modified function projective synchronization

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1. Introduction

The Lorenz system with real variables [1], envisaged as a model for convection between parallel plates, has been a paradigm of chaos. In 1982, the Lorenz system in the complex domain was unexpectedly discovered from laser physics and baroclinic instability of the geophysical flows [2-5], and is described by

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - xz - ay, \\ \dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - bz, \end{cases} \tag{1}$$

where the Rayleigh number r and parameter a are complex numbers defined by $r = r_1 + jr_2, a = 1 - j\delta$, $j = \sqrt{-1}$ is the imaginary unit, and σ, b, r_1, r_2, δ are real and positive, a dot indicates time t derivative and an overbar denotes complex conjugate variable. The complex variables x, y and real variable z of the system (1) have relations with electric field and the atomic polarization amplitudes and the population inversion in a ring laser system of two-level atoms respectively. The chaotic attractor is plotted in Figure.1, which is the solution of system (1). In addition, the complex-variable chaotic systems (CVCSs) arises in various important fields such as fluids [3,5], superconductors, plasma physics [6] and electromagnetic fields [7] and secure communications [8].

In recent years, the stabilization and synchronization of CVCSs have triggered a great deal of interest among scientist for the wide scope of applications. For example, chaotic synchronization was achieved by using the active control technique for two identical Lorenz CVCSs [9] in the form as

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = r_1x - xz - y, \\ \dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - bz, \end{cases} \tag{2}$$

where σ, r_1, b are real and positive parameters, x, y are complex variables and z is real variable. Global synchronization are introduced for so-called Chen and Lü CVCSs [10]. Module-phase synchronization was proposed for a class of complex neural networks by employing adaptive feedback control [11]. Projective synchronization (PS) has been especially deeply studied because it can be used to obtain faster communication with its proportional feature and the unpredictability of the scaling factor can additionally enhance the security of communication [12, 13]. Modified function projective

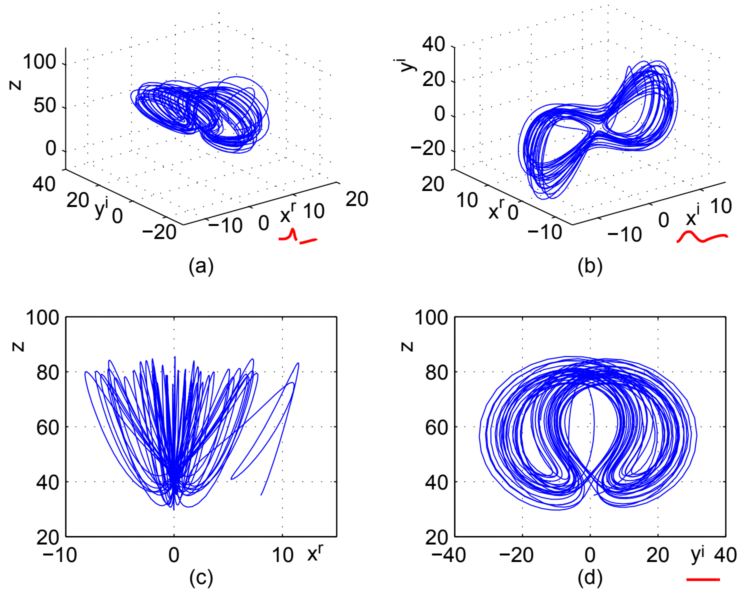

Figure 1: Phase portraits of the attractors for Lorenz CVCSs (1) with complex parameters $\sigma = 2$, $r = 60 + 0.02j$, $a = 1 - 0.06j$, $b = 0.8$ and initial values $x(0) = 8 + 13j, y(0) = 16 + j, z(0) = 35$: (a) on (x^r, y^i, z) space; (b) on $(x^i, x^r, y^i))$ space; (c) on (x^r, z) plane; (d) on (y^i, z) plane.

synchronization (MFPS) was presented for general uncertain real [14, 15] and complex [16, 17, 18] chaotic systems. MFPS associates with the function projective synchronization (FPS) and the modified one (MPS), and the drive system synchronizes the response system with regard to scaling functions $\delta_i(t), i = 1, 2, \cdots, n$.

It is emphasized that, all of the scaling functions in the the above mentioned MFPS [14-18] are supposed to be real-valued, in other words, the drive and response systems evolve in the same or inverse direction simultaneously. In many real applications, however, complex-valued scaling functions are more common, the drive and response complex systems may evolve in different directions with a constant even time-dependent intersection angle. As an example, $z = \rho e^{j\eta t} y$, where $\rho e^{j\eta t} = \rho(\cos \eta t + j \sin \eta t)$ is complex scaling function, z and y represent the complex state variables of drive and response systems, respectively, $\rho > 0$ represents the zoom rate, ηt represents

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the time-dependent rotate angle, $\eta \in [0, 2\pi)$. What's more, due to its more unpredictable than real-valued scaling functions (i.e. $\rho e^{j\eta t}$ at $\eta = 0$ or $\eta = \pi$) and complex-valued scaling factors (i.e. $\rho e^{j\eta t}$ at $t = constant$), the complex-valued scaling functions will greatly increase the complexity and diversity of the synchronization, and the different versions of function synchronization may provide a higher versatility and better security. Consequently, FPS with complex scaling function (CFPS) was presented for coupled CVCSs with known parameters and its applications in secure communication in 2013 [19]. However, MFPS with complex-valued scaling functions (CMFPS), as more general synchronization, has not been previously reported.

Another motivation should be noted, which has rarely been previously discussed, however. Although much effort has been devoted for chaotic synchronization, the aforementioned synchronization schemes had been considered for real systems [13, 14, 15] or complex ones with real parameters [8-12,16-19] such as system (2), which is embedded in Lorenz model (1) with complex parameters. It is clear that there are essential differences between real-variable chaotic systems and complex-variable ones. As usual, properties and conclusions of real-variable chaotic systems cannot be simply extended to that of CVCSs. Moreover, as described previously in the references [2-7], the complex parameters in CVCSs follow from purely physical consideration, for example, complex parameters r and a in system (1) arise due to the weak dispersive effects, so the chaotic synchronization for CVCSs with complex parameters is a topic of both theoretical and practical interests. Very recently, Liu *et al.* achieved complex modified projective synchronization (CMPS) [20], complex modified hybrid projective synchronization (CMHPS) [21], and complex modified hybrid function projective synchronization (CMHFPS) [22] for CVCSs, and gave several illustrative examples concerning complex parameters. However, as far as the authors know, there are no relevant results for CMFPS of identical and nonidentical CVCSs with complex parameters in the exiting literatures.

In practical application, parameter fluctuations and unavoidable uncertainties commonly exist in CVCSs as a result of external disturbances. As mentioned in [14-18,20,22], the well-known adaptive control still is an effective method to achieve synchronization of chaotic systems with unknown parameters. However, how to achieve CMFPS between identical and nonidentical CVCSs with uncertain complex parameters via the adaptive control?

Inspired by the above discussion, in this paper, we consider a class of chaotic (hyperchaotic) systems where both parameters and variables are complex-valued, and discuss CMFPS for complex systems with known and unknown parameters.

In detail, the distinguishing feature of this paper are refined as follows.

First, the systems taken into consideration in synchronization are remarkably more general than those in the closely related researches [9,10,12,14-19]. As described in references [20, 21, 22], a chaotic system with complex variables and with complex parameters produces more complex and unpredictable signals.

Second, this work can be perceived as a continuation of chaos synchronization in the previous literatures. The scaling functions are chosen as complex-valued ones in CMFPS for CVCSs, which are different from both real-valued scaling functions in [14-18] and real-valued (complex-valued) scaling constant factors in [8-10,12,13,20], the response system can asymptotically synchronize drive system in different directions with a time-dependent intersection angle. As a generalization of synchronization, depending on the form of the scaling function matrix, CMFPS covers MFPS [16], CFPS [19], FPS, CMPS [20], CPS [8], and extend recently previous works.

Third, from the technical perspective, quite different from general synchronization scheme [8-19], we construct appropriate Lyapunov functions dependent on complex variables, and sufficient criteria on CMFPS are derived in complex space. As pointed out in references [20, 21, 22], because of the complexity and the difficulties in constructing general quadratic Lyapunov functions defined on complex field, the synchronization of the given CVCSs in [8-12, 16-19] was achieved by separating imaginary and real parts of complex variables or complex parameters. In point of fact, this kind of synchronization still is that of real chaotic system. Furthermore, compared with all the existing control methodologies, the outstanding merits of the proposed control scheme are presented as follows. The designed adaptive controller guarantees that synchronization is achieved between identical and nonidentical CVCSs with complex parameters in the sense of CMFPS, and all of unknown complex parameters in both drive and response systems are identified by virtue of the complex update laws.

The remainder of this paper is structured as follows. We begin with the definition of CMFPS for CVCSs with complex parameters in Section 2, and draw on five sufficient criteria on CMFPS and two sufficient criteria on control for CVCSs with known and unknown parameters in Section 3. Section 4 is devoted to simulation. The proposed adaptive CMFPS schemes are achieved between two identical Lorenz CVCSs with full unknown complex parameters as well as drive Lü CVCSs with full unknown real parameters and response Lorenz CVCSs with full unknown complex parameters, respectively. The last section provides discussion and conclusions.

Notation 1. $\mathbb{C}^n$ stands for n dimensional complex vector space. If $\mathbf{z} \in \mathbb{C}^n$ is a complex vector, then $\mathbf{z} = \mathbf{z}^r + j\mathbf{z}^i$, $j = \sqrt{-1}$ is the imaginary unit, superscripts r and i stand for the real and imaginary parts of $\mathbf{z}$, respectively, $\mathbf{z}^{\mathrm{H}}$, $\mathbf{z}^{\mathrm{T}}$ are the conjugate transpose and transpose of $\mathbf{z}$, respectively, and $\|\mathbf{z}\|$ implies the 2-norm of $\mathbf{z}$. If z is a complex scalar, $|z|$ indicates the modulus of z and $\bar{z}$ is the conjugate of z, while $\mathbf{M}^{\mathrm{H}}(t)$ is the conjugate transpose of $\mathbf{M}(t)$, provided that $\mathbf{M}(t)$ is a complex matrix. $\hat{\mathbf{\Theta}}$ is the estimation of the unknown parameter vector $\mathbf{\Theta}$.

2. The definition of CMFPS and problem descriptions

First, the following general n-dimensional drive and response CVCSs with the controller are considered as

$$\dot{\mathbf{z}}(t) = \mathbf{\Upsilon}(\mathbf{z},t), \tag{3}$$

$$\dot{\mathbf{y}}(t) = \mathbf{\Psi}(\mathbf{y},t) + \mathbf{u}(\mathbf{z},\mathbf{y},t), \tag{4}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_n)^T \in \mathbb{C}^n$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} \in \mathbb{C}^n$ are complex state vectors, and $\mathbf{u}(\mathbf{z},\mathbf{y},t)$ is the control vector to be determined.

Next, the definition of CMFPS of CVCSs with complex parameters is introduced as follows.

Definition 1. For the drive system (3) and response system (4) with any initial conditions $(\mathbf{y}(0),\mathbf{z}(0)) \in \mathbb{C}^n \times \mathbb{C}^n$, it is said to achieve CMFPS with $\mathbf{\Lambda}(t)$ between $\mathbf{y}(t)$ and $\mathbf{z}(t)$, if there exists a controller $\mathbf{u}(\mathbf{z},\mathbf{y},t)$ such that all trajectories $(\mathbf{z}(t),\mathbf{y}(t))$ in the systems (3) and (4) approach the manifold $\mathbf{E} = \{(\mathbf{y}(t),\mathbf{z}(t)) : \mathbf{y}(t) = \mathbf{\Lambda}(t)\mathbf{z}(t)\}$ as $t \to +\infty$. That is,

$$\lim_{t\to+\infty} \|\mathbf{y}(t) - \mathbf{\Lambda}(t)\mathbf{z}(t)\| = 0,$$

where $\mathbf{\Lambda}(t) = diag\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)\}$, and $\lambda_l(t) = {\lambda_l}^r(t) + j{\lambda_l}^i(t), l = 1, 2, \cdots n$ should be continuously differential function with bounded norm.

Remark 1. The matrix $\mathbf{\Lambda}(t) \in \mathbb{C}^{n\times n}$ is called a complex scaling function matrix. Several kinds of function synchronization are special cases of CMFPS, such as MFPS, CFPS, FPS, see in Table 1.

Settings the Matrix $\Lambda(t)$	Type of Synchronization
$\Lambda(t) = diag\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)\} \in \mathbb{C}^{n\times n}$	CMFPS
$\Lambda(t) = diag\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)\} \in \mathbb{R}^{n\times n}$	MFPS
$\Lambda(t) = diag\{\lambda(t), \lambda(t), \cdots, \lambda(t)\} \in \mathbb{C}^{n\times n}$	CFPS
$\Lambda(t) = diag\{\lambda(t), \lambda(t), \cdots, \lambda(t)\} \in \mathbb{R}^{n\times n}$	FPS

还可以可以继续扩展，反同步，完全同步，稳定到平衡点...

Table 1: Types of function synchronization

Remark 2. If $\mathbf{\Lambda}(t) = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \in \mathbb{C}^{n\times n}$, where $\lambda_l, (l = 1, 2, \cdots, n)$ are complex numbers, then CMFPS becomes CMPS for complex dynamical systems with complex parameters [21]. What's more, if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, then CMPS becomes CPS of CVCSs with complex parameters.

Therefore, this new type of complex synchronization CMFPS is considered as a generalization of MFPS, CFPS, FPS, CMPS, CPS that have appeared in the recent literatures.

The general n-dimensional drive CVCSs with complex parameters is written as

$$\dot{\mathbf{z}}(t) = \mathbf{\Upsilon}(\mathbf{z},t) = \mathbf{R}(\mathbf{z})\mathbf{\Theta} + \mathbf{h}(\mathbf{z}), \tag{5}$$

where $\mathbf{z} = \mathbf{z}^r + j\mathbf{z}^i \in \mathbb{C}^n$ and $\mathbf{z}^r = (z_1^r, z_2^r, \cdots, z_n^r)^{\mathrm{T}}$, $\mathbf{z}^i = (z_1^i, z_2^i, \cdots, z_n^i)^{\mathrm{T}}$. $\mathbf{\Theta} = (\theta_1, \theta_2, \cdots, \theta_s)^{\mathrm{T}} \in \mathbb{C}^s$ is a $s \times 1$ complex parameter vector, $\mathbf{R}(\mathbf{z})$ is a $n\times s$ complex matrix and its elements are functions of complex state variables, and $\mathbf{h} : \mathbb{C}^n \to \mathbb{C}^n$ is a complex nonlinear function vector. On the other hand, the n-dimensional response CVCSs with the controller is depicted as

$$\dot{\mathbf{y}}(t) = \mathbf{\Psi}(\mathbf{y},t) + \mathbf{u} = \mathbf{Q}(\mathbf{y})\mathbf{\Xi} + \mathbf{q}(\mathbf{y}) + \mathbf{u}, \tag{6}$$

where $\mathbf{y} = \mathbf{y}^r + j\mathbf{y}^i \in \mathbb{C}^n$ and $\mathbf{y}^r = (y_1^r, y_2^r, \cdots, y_n^r)^{\mathrm{T}}$, $\mathbf{y}^i = (y_1^i, y_2^i, \cdots, y_n^i)^{\mathrm{T}}$. $\mathbf{\Xi} = (\xi_1, \xi_2, \cdots, \xi_m)^{\mathrm{T}} \in \mathbb{C}^m$ is a $m \times 1$ complex parameter vector, $\mathbf{Q}(\mathbf{y})$ is a $n \times m$ complex matrix and its elements are functions of complex state variables, and $\mathbf{q} = (q_1, q_2, \cdots, q_n)^T$ is a $n \times 1$ vector of complex nonlinear function and $\mathbf{u} = \mathbf{u}^r + j\mathbf{u}^i \in \mathbb{C}^n$ is the control input.

If CMFPS error is defined as

$$\mathbf{e}(t) = \mathbf{e}^r(t) + j\mathbf{e}^i(t) = \mathbf{y}(t) - \mathbf{\Lambda}(t)\mathbf{z}(t), \tag{7}$$

where $\mathbf{e}^r = (e_1^r, e_2^r, \cdots, e_n^r)^{\mathrm{T}}$, $\mathbf{e}^i = (e_1^i, e_2^i, \cdots, e_n^i)^{\mathrm{T}} \in \mathbb{R}^n$, then the essential goal of this paper is to design complex update laws of unknown parameter vectors and adaptive controller $\mathbf{u} = \mathbf{u}^r + j\mathbf{u}^i \in \mathbb{C}^n$ such that synchronization error tends to zero, i.e.

$$\lim_{t\to+\infty} \|\mathbf{e}(t)\| = \lim_{t\to+\infty} \|\mathbf{y}(t) - \mathbf{\Lambda}(t)\mathbf{z}(t)\| = 0. \tag{8}$$

Remark 3. The class described by (5) includes several well-known CVCSs, such as Lorenz CVCSs (1) and (2), Chen CVCSs, Lü CVCSs, complex Van der Pol oscillator, Duffing CVCSs, and Lorenz CVCSs, Lü CVCSs, etc.

Remark 4. In many previous works, the numbers of variables and parameters are equal in CVCSs. In this paper, the condition is not required, namely, $n \neq s$ in system (5) and $n \neq m$ in system (6). 变量的个数与参数的个数不相同

3. Main results

3.1. Adaptive CMFPS schemes of CVCSs with full unknown complex parameters

Theorem 1. *For given complex scaling function matrix $\mathbf{\Lambda}(t)$ and any initial conditions $\mathbf{y}(0), \mathbf{z}(0)$, if the adaptive controller is designed as*

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$$\mathbf{u} = -\mathbf{Q}(\mathbf{y})\hat{\mathbf{\Xi}} - \mathbf{q}(\mathbf{y}) + \dot{\mathbf{\Lambda}}(t)\mathbf{z}(t) + \mathbf{\Lambda}(t)(\mathbf{R}(\mathbf{z}(t))\hat{\mathbf{\Theta}} + \mathbf{h}(\mathbf{z}(t))) - \mathbf{K}\mathbf{e}, \tag{9}$$

and the complex update laws of complex parameters are chosen as

$$\begin{cases} \dot{\hat{\mathbf{\Theta}}} = \dot{\hat{\mathbf{\Theta}}}^r + j\dot{\hat{\mathbf{\Theta}}}^i = -(\mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t)))^{\mathrm{H}}\ \mathbf{e}, \\ \dot{\hat{\mathbf{\Xi}}} = \dot{\hat{\mathbf{\Xi}}}^r + j\dot{\hat{\mathbf{\Xi}}}^i = \mathbf{Q}^{\mathrm{H}}(\mathbf{y})\ \mathbf{e}, \end{cases} \tag{10}$$

where $\mathbf{K} = diag\{k_1, k_2, \cdots, k_n\}$ is real positive definite matrix, $k_i, (i = 1, 2, \cdots, n)$ are coupling strengths, then adaptive CMFPS between the response system (6) and drive system (5) is achieved asymptotically, where (10) is called complex update laws of unknown parameter vectors $\mathbf{\Theta}$ and $\mathbf{\Xi}$. Syn..

Proof. Insertion of (5), (6) and (9) into (7) gives

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \dot{\mathbf{y}}(t) - (\dot{\mathbf{\Lambda}}(t)\mathbf{z}(t) + \mathbf{\Lambda}(t)\dot{\mathbf{z}}(t)) \\ &= \mathbf{Q}(\mathbf{y})\mathbf{\Xi} + \mathbf{q}(\mathbf{y}) - \mathbf{Q}(\mathbf{y})\hat{\mathbf{\Xi}} - \mathbf{q}(\mathbf{y}) + \dot{\mathbf{\Lambda}}(t)\mathbf{z}(t) \\ &\quad + \mathbf{\Lambda}(t)(\mathbf{R}(\mathbf{z}(t))\hat{\mathbf{\Theta}} + \mathbf{h}(\mathbf{z}(t))) - \mathbf{K}\mathbf{e} \\ &\quad - \dot{\mathbf{\Lambda}}(t)\mathbf{z}(t) - \mathbf{\Lambda}(\mathbf{t})(\mathbf{R}(\mathbf{z}(t))\mathbf{\Theta} + \mathbf{h}(\mathbf{z}(t))) \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t))(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) - \mathbf{Q}(\mathbf{y})(\hat{\mathbf{\Xi}} - \mathbf{\Xi}) - \mathbf{K}\mathbf{e} \\
&= \mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t))\tilde{\mathbf{\Theta}} - \mathbf{Q}(\mathbf{y})\tilde{\mathbf{\Xi}} - \mathbf{K}\mathbf{e}, \qquad (11)
\end{aligned}
$$

where $\tilde{\mathbf{\Theta}} = \hat{\mathbf{\Theta}} - \mathbf{\Theta}$ and $\tilde{\mathbf{\Xi}} = \hat{\mathbf{\Xi}} - \mathbf{\Xi}$ are the parameter error vectors, respectively.

Introducing the following Lyapunov function candidate defined on the complex field in the form as

$$
V(\mathbf{e}, \tilde{\mathbf{\Theta}}, \tilde{\mathbf{\Xi}}, t) = \frac{1}{2}(\mathbf{e}^{\mathrm{H}}\mathbf{e} + \tilde{\mathbf{\Theta}}^{\mathrm{H}}\tilde{\mathbf{\Theta}} + \tilde{\mathbf{\Xi}}^{\mathrm{H}}\tilde{\mathbf{\Xi}}), \qquad (12)
$$

from the complex update laws (10) and $\dot{\tilde{\mathbf{\Theta}}} = \dot{\hat{\mathbf{\Theta}}}$, $\dot{\tilde{\mathbf{\Xi}}} = \dot{\hat{\mathbf{\Xi}}}$, the time derivative of the $V(t)$ along the trajectories of the errors system (11) reads

$$
\begin{aligned}
\dot{V}(t) &= \frac{1}{2}[(\dot{\mathbf{e}})^{\mathrm{H}}\mathbf{e} + \mathbf{e}^{\mathrm{H}}(\dot{\mathbf{e}}) + \dot{\tilde{\mathbf{\Theta}}}^{\mathrm{H}}\tilde{\mathbf{\Theta}} + \tilde{\mathbf{\Theta}}^{\mathrm{H}}\dot{\tilde{\mathbf{\Theta}}} + \dot{\tilde{\mathbf{\Xi}}}^{\mathrm{H}}\tilde{\mathbf{\Xi}} + \tilde{\mathbf{\Xi}}^{\mathrm{H}}\dot{\tilde{\mathbf{\Xi}}}] \\
&= \frac{1}{2}[(\mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t))\tilde{\mathbf{\Theta}} - \mathbf{Q}(\mathbf{y})\tilde{\mathbf{\Xi}} - \mathbf{K}\mathbf{e})^{\mathrm{H}}\mathbf{e} \\
&\quad + \mathbf{e}^{\mathrm{H}}(\mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t))\tilde{\mathbf{\Theta}} - \mathbf{Q}(\mathbf{y})\tilde{\mathbf{\Xi}} - \mathbf{K}\mathbf{e}) \\
&\quad + ((-\mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t)))^{\mathrm{H}}\ \mathbf{e} + \tilde{\mathbf{\Theta}}^{\mathrm{H}}\ ((-\mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t)))^{\mathrm{H}}\ \mathbf{e} \\
&\quad + (\mathbf{Q}^{\mathrm{H}}(\mathbf{y})\ \mathbf{e} + \tilde{\mathbf{\Xi}}^{\mathrm{H}}(\mathbf{Q}^{\mathrm{H}}(\mathbf{y})\mathbf{e}] \\
&= -\mathbf{e}^{\mathrm{H}}\mathbf{K}\mathbf{e} \\
&\leq -k_{min}\|\mathbf{e}\|^2, \qquad (13)
\end{aligned}
$$

where $k_{min} = min(k_1, k_2, \cdots, k_n)$. This implies that $V(t)$ is bounded. On the other hand, from (13), we have

$$
\begin{aligned}
\int_0^t \mathbf{e}^{\mathrm{H}}\mathbf{e}dt &= \int_0^t \|\mathbf{e}\|^2 dt \\
&\leq (V(0) - V(t))/k_{min} \\
&\leq V(0)/k_{min}. \qquad (14)
\end{aligned}
$$

Thus, it is obvious that $\mathbf{e}(t) \in L_2$. So it can be concluded that $\mathbf{e}(t), \tilde{\mathbf{\Theta}}, \tilde{\mathbf{\Xi}} \in L_\infty$. And from (11), we can obtain that $\dot{\mathbf{e}}(t)$ exists and $\dot{\mathbf{e}}(t) \in L_\infty$.

According to the Barbalat's Lemma [23], when $t \to \infty$, we have $\mathbf{e}(t) \to 0$. So far, one arrives at adaptive CMFPS with desired complex scaling function matrix $\mathbf{\Lambda}(t)$ between two nonidentical CVCSs (5) and (6) by using the controller (9) and complex update laws (10). □

Theorem 2. *Suppose the structure of systems (5) and (6) is identical, i.e.* $\mathbf{Q}(\cdot) = \mathbf{R}(\cdot)$, $\mathbf{q}(\cdot) = \mathbf{h}(\cdot)$, *and* $\mathbf{\Theta} = \mathbf{\Xi}$. *Then, for given complex scaling* *function matrix* $\mathbf{\Lambda}(t)$ *and any initial conditions* $\mathbf{y}(0), \mathbf{z}(0)$*, if the adaptive controller is designed as*

$$\mathbf{u} = (\mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t)) - \mathbf{R}(\mathbf{y}))\hat{\mathbf{\Theta}} + \mathbf{\Lambda}(t)\mathbf{h}(\mathbf{z}(t)) - \mathbf{h}(\mathbf{y}) + \dot{\mathbf{\Lambda}}(t)\mathbf{z}(t) - \mathbf{K}\mathbf{e}, \quad (15)$$

and the complex parameter update law is chosen as

$$\dot{\hat{\mathbf{\Theta}}} = \dot{\hat{\mathbf{\Theta}}}^r + j\dot{\hat{\mathbf{\Theta}}}^i = (\mathbf{Q}(\mathbf{y}) - \mathbf{\Lambda}(t)\mathbf{Q}(\mathbf{z}(t)))^{\mathrm{H}}\,\mathbf{e} \quad (16)$$

where $\mathbf{K} = diag\{k_1, k_2, \cdots, k_n\}$ *is real positive definite matrix,* $k_i, (i = 1, 2, \cdots, n)$ *are the control strengths, then adaptive CMFPS between the response system (6) and drive system (5) is achieved asymptotically.*

Proof. Introducing the following Lyapunov function candidate as

$$V(\mathbf{e}, \tilde{\mathbf{\Theta}}, t) = \frac{1}{2}(\mathbf{e}^{\mathrm{H}}\mathbf{e} + \tilde{\mathbf{\Theta}}^{\mathrm{H}}\tilde{\mathbf{\Theta}}), \quad (17)$$

where $\tilde{\mathbf{\Theta}} = \hat{\mathbf{\Theta}} - \mathbf{\Theta}$ is the parameter error vector. Then it is similar to the proof in Theorem 1 and thus is omitted. □

3.2. CMFPS schemes of CVCSs with partially known and full known complex parameters

Theorem 3. *Suppose the parameter* $\mathbf{\Xi}$ *of the response system (6) is known a priori. Then, for given complex scaling function matrix* $\mathbf{\Lambda}(t)$ *and any initial conditions* $\mathbf{y}(0), \mathbf{z}(0)$*, if the adaptive controller is designed as*

5(9),13

$$\mathbf{u} = -\mathbf{Q}(\mathbf{y})\mathbf{\Xi} - \mathbf{q}(\mathbf{y}) + \dot{\mathbf{\Lambda}}(t)\mathbf{z}(t) + \mathbf{\Lambda}(t)(\mathbf{R}(\mathbf{z}(t))\hat{\mathbf{\Theta}} + \mathbf{h}(\mathbf{z}(t))) - \mathbf{K}\mathbf{e}, \quad (18)$$

and the complex parameter update law is chosen as

$$\dot{\hat{\mathbf{\Theta}}} = \dot{\hat{\mathbf{\Theta}}}^r + j\dot{\hat{\mathbf{\Theta}}}^i = -(\mathbf{\Lambda}(t)\mathbf{R}(\mathbf{z}(t)))^{\mathrm{H}}\,\mathbf{e} \quad (19)$$

where $\mathbf{K} = diag\{k_1, k_2, \cdots, k_n\}$ *is real positive definite matrix,* $k_i, (i = 1, 2, \cdots, n)$ *are coupling strengths, then adaptive CMFPS between the response system (6) and drive system (5) is achieved asymptotically.*

Proof. It is similar to the proof in Theorem 2 and thus is omitted. □

Theorem 4. *Suppose the parameter* $\boldsymbol{\Theta}$ *of the drive system (5) is known a priori. Then, for given complex scaling function matrix* $\boldsymbol{\Lambda}(t)$ *and any initial conditions* $\mathbf{y}(0), \mathbf{z}(0)$*, if the adaptive controller is designed as*

$$\mathbf{u} = -\mathbf{Q}(\mathbf{y})\hat{\boldsymbol{\Xi}} - \mathbf{q}(\mathbf{y}) + \dot{\boldsymbol{\Lambda}}(t)\mathbf{z}(t) + \boldsymbol{\Lambda}(t)(\mathbf{R}(\mathbf{z}(t))\boldsymbol{\Theta} + \mathbf{h}(\mathbf{z}(t))) - \mathbf{K}\mathbf{e}, \tag{20}$$

and the complex parameter update law is chosen as

$$\dot{\hat{\boldsymbol{\Xi}}} = \dot{\hat{\boldsymbol{\Xi}}}^{r} + j\dot{\hat{\boldsymbol{\Xi}}}^{i} = \mathbf{Q}^{\mathrm{H}}(\mathbf{y})\ \mathbf{e} \tag{21}$$

where $\mathbf{K} = diag\{k_1, k_2, \cdots, k_n\}$ *is real positive definite matrix,* $k_i, (i = 1, 2, \cdots, n)$ *are coupling strengths, then adaptive CMFPS between the response system (6) and drive system (5) is achieved asymptotically.*

Proof. Introducing the following Lyapunov function candidate as

$$V(\mathbf{e}, \tilde{\boldsymbol{\Xi}}, t) = \frac{1}{2}(\mathbf{e}^{\mathrm{H}}\mathbf{e} + \tilde{\boldsymbol{\Xi}}^{\mathrm{H}}\tilde{\boldsymbol{\Xi}}). \tag{22}$$

where $\tilde{\boldsymbol{\Xi}} = \hat{\boldsymbol{\Xi}} - \boldsymbol{\Xi}$ is the parameter error vector. Then it is similar to the proof in Theorem 1 and thus is omitted. □

Theorem 5. *Suppose both* $\boldsymbol{\Xi}$ *and* $\boldsymbol{\Theta}$ *of the systems (5) and (6) are full known a priori. Then, for given complex scaling function matrix* $\boldsymbol{\Lambda}(t)$ *and any initial conditions* $\mathbf{y}(0), \mathbf{z}(0)$*, if the controller is designed as*

$$\mathbf{u} = -\mathbf{Q}(\mathbf{y})\boldsymbol{\Xi} - \mathbf{q}(\mathbf{y}) + \dot{\boldsymbol{\Lambda}}(t)\mathbf{z}(t) + \boldsymbol{\Lambda}(t)(\mathbf{R}(\mathbf{z}(t))\boldsymbol{\Theta} + \mathbf{h}(\mathbf{z}(t))) - \mathbf{K}\mathbf{e}, \tag{23}$$

与前同

where $\mathbf{K} = diag\{k_1, k_2, \cdots, k_n\}, k_i > 0, (i = 1, 2, \cdots, n)$ *are coupling strengths, then CMFPS between the response system (6) and drive system (5) is achieved asymptotically.*

Proof. Introducing the following Lyapunov function candidate as

$$V(\mathbf{e}, t) = \frac{1}{2}\mathbf{e}^{\mathrm{H}}\mathbf{e}.$$

Then it is similar to the proof in Theorem 1 and thus is omitted. □

Remark 5. Unlike the schemes proposed in the literature [8-19], we aim at CMFPS scheme of CVCSs with unknown complex parameters, and design complex update laws of unknown parameters and adaptive controller in complex space, thus the results of this paper are more applicable and representative.

Remark 6. Note that the complex scaling function matrix $\mathbf{\Lambda}(t)$ has no effect on the time derivative $\dot{V}(t)$, thus one can adjust complex scaling function matrix arbitrarily without altering the control robustness. Hence, the complex scaling function matrix can be used as information signal for the communication scheme. In particular, when $\mathbf{\Lambda}(t)$ is real, Theorem 1-5 are also applied to achieve MFPS with real-valued scaling function matrix of CVCSs with complex parameters.

Remark 7. If either of the two matrices $\mathbf{\Theta}$, $\mathbf{\Xi}$ is complex parameter vector, then Theorem 1-5 can also be applied to realize CMFPS of CVCSs with partly complex parameters. In particular, if both $\mathbf{\Theta}$ and $\mathbf{\Xi}$ are real parameter vector, Theorem 1-5 are also applied to achieve MFPS of real-variable chaotic systems or CVCSs with real parameters in references [12-16,19]. However, it's clear that the CMFPS problem for CVCSs with complex parameters cannot be solved by using the method in references [12-16,19].

Remark 8. If $\mathbf{\Lambda}(t) = 0$ in Definition 1, CMFPS degenerates to the control of CVCSs with complex parameters.

The following corollaries are easily obtained from Theorem 4-5, and their proofs are omitted.

Corollary 1. *For the complex system (6) with any initial conditions* $\mathbf{y}(0)$*, if the adaptive controller is designed as*

$$\mathbf{u} = -\mathbf{Q}(\mathbf{y})\hat{\mathbf{\Xi}} - \mathbf{q}(\mathbf{y}) - \mathbf{K}\mathbf{e}, \tag{24}$$

and the complex parameter update law is chosen as

$$\dot{\hat{\mathbf{\Xi}}} = \dot{\hat{\mathbf{\Xi}}}^{r} + j\dot{\hat{\mathbf{\Xi}}}^{i} = \mathbf{Q}^{\mathrm{H}}(\mathbf{y})\ \mathbf{e} \tag{25}$$

where $\mathbf{K} = diag\{k_1, k_2, \cdots, k_n\}$ *is real positive definite matrix,* $k_i, (i = 1, 2, \cdots, n)$ *are coupling strengths,* $\hat{\mathbf{\Xi}}$ *is the estimation of the unknown parameter vector* $\mathbf{\Xi}$*, then the equilibrium point of system (6) is asymptotically stable.*

Corollary 2. *Suppose* $\mathbf{\Theta}$ *of the system (6) is known a priori. For any initial conditions* $\mathbf{w}(0)$*, if the controller is designed as*

$$\mathbf{u} = -\mathbf{Q}(\mathbf{y})\mathbf{\Xi} - \mathbf{q}(\mathbf{y}) - \mathbf{K}\mathbf{e}, \tag{26}$$

where $\mathbf{K} = diag\{k_1, k_2, \cdots, k_n\}$ *is real positive definite matrix,* $k_i > 0, (i = 1, 2, \cdots, n)$ *are coupling strengths, then the equilibrium point of system (6) is asymptotically stable.*

4. Numerical examples

Throughout this section, in order to verify the effectiveness and feasibility of the proposed synchronization scheme in section 3, examples are given for two kinds of cases: adaptive CMFPS of two identical and nonidentical CVCSs with full unknown complex parameters, respectively.

针对两个不相同的混沌系统

4.1. Adaptive CMFPS of two nonidentical CVCSs with full unknown complex (real) parameters

In order to illustrate adaptive CMFPS for two nonidentical systems, we assume that Lü CVCSs with full unknown real parameters [10] drives Lorenz CVCSs (1) with full unknown complex parameters [1]. Therefore, the drive system is described by

$$\begin{cases} \dot{z}_1 &= \theta_1(z_2 - z_1), \\ \dot{z}_2 &= \theta_2 z_2 - z_1 z_3, \\ \dot{z}_3 &= -\theta_3 z_3 + (1/2)(\bar{z}_1 z_2 + z_1 \bar{z}_2), \end{cases} \tag{27}$$

where $z_1 = z_1^r + jz_1^i, z_2 = z_2^r + jz_2^i$ are complex state variables and z_3 is real state variable, $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3)^{\mathrm{T}}$ is unknown real parameter vector.

The response CVCSs with the controller is described by

$$\begin{cases} \dot{w}_1 &= \xi_1(w_2 - w_1) + u_1, \\ \dot{w}_2 &= \xi_2 w_1 - \xi_3 w_2 - w_1 w_3 + u_2, \\ \dot{w}_3 &= -\xi_4 w_3 + \frac{1}{2}(\bar{w}_1 w_2 + w_1 \bar{w}_2) + u_3, \end{cases} \tag{28}$$

where $y_1 = y_1^r + jy_1^i, y_2 = y_2^r + jy_2^i$ are complex state variables, and y_3 is a real state variable, $\mathbf{\Xi} = (\xi_1, \xi_2, \xi_3, \xi_4)^{\mathrm{T}}$ is unknown complex parameter vector.

The controller is designed according to (9) in Theorem 1 as follows,

$$\begin{cases} u_1 &= -\hat{\xi}_1(y_2(t) - y_1(t)) + \dot{\lambda}_1(t)z_1(t) + \lambda_1(t)\hat{\theta}_1(z_2(t) - z_1(t)), \\ u_2 &= -\hat{\xi}_2 y_1(t) + \hat{\xi}_3 y_2(t) + y_1(t)y_3(t) + \dot{\lambda}_2(t)z_2(t) \\ & \quad +\lambda_2(t)(\hat{\theta}_2 z_2(t) - z_1(t)z_3(t)) - k_2 e_2, \\ u_3 &= \hat{\xi}_4 y_3(t) - \frac{1}{2}(\bar{y}_1(t)y_2(t) + y_1(t)\bar{y}_2(t)) + \dot{\lambda}_3(t)z_3(t) \\ & \quad +\lambda_3(t)[-\hat{\theta}_3 z_3(t) + \frac{1}{2}(\bar{z}_1(t)z_2(t) + z_1(t)\bar{z}_2(t))] - k_3 e_3, \end{cases} \tag{29}$$

and the parameter update laws are given according to (10) as

$$\dot{\hat{\mathbf{\Theta}}} = \begin{pmatrix} -\bar{\lambda}_1(t)(\bar{z}_2(t) - \bar{z}_1(t))e_1 \\ -\bar{\lambda}_2(t)\bar{z}_2(t)e_2 \\ \bar{\lambda}_3(t)z_3(t)e_3 \end{pmatrix}, \quad \dot{\hat{\mathbf{\Xi}}} = \begin{pmatrix} (\bar{y}_2(t) - \bar{y}_1(t))e_1 \\ \bar{y}_1(t)e_2 \\ -\bar{y}_2(t)e_2 \\ -y_3(t)e_3 \end{pmatrix}. \tag{30}$$

The complex scaling function matrix is selected as

$$\mathbf{\Lambda}(t) = \begin{pmatrix} 0.4\exp(j\pi t/4) & 0 & 0 \\ 0 & 1.2\exp(j\pi t/12) & 0 \\ 0 & 0 & 1.8 + \sin t \end{pmatrix}. \tag{31}$$

In the numerical simulations, unknown parameters are specified as $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3)^{\mathrm{T}} = (40, 22, 5)^{\mathrm{T}}$ and $\mathbf{\Xi} = (\xi_1, \xi_2, \xi_3, \xi_4)^{\mathrm{T}} = (2, 60 + 0.02j, 1 - 0.06j, 0.8)^{\mathrm{T}}$, respectively, to ensure that both drive system and response system exhibit chaotic attractors. The initial conditions of drive system (27) and response system (28) are employed as $\mathbf{z(0)} = (1+2j, 2+3j, 5)^{\mathrm{T}}$, $\mathbf{y}(\mathbf{0}) = (8+13j,\ 16+j,\ 35)^{\mathrm{T}}$. The initial values of estimated parameters are $\hat{\mathbf{\Theta}}(0) = (\hat{\theta}_1(0),\ \hat{\theta}_2(0),\ \hat{\theta}_3(0))^{\mathrm{T}} = (15,\ 10,\ 30{+}3j)^{\mathrm{T}}$, $\hat{\mathbf{\Xi}}(0) = (\hat{\xi}_1(0),\ \hat{\xi}_2(0),\ \hat{\xi}_3(0),\ \hat{\xi}_4(0))^{\mathrm{T}} = (4 + j,\ 20 + 0.1j,\ 3 - 0.1j,\ 1 - j)^{\mathrm{T}}$. The control gain matrix is chosen as $\mathbf{K} = diag\{3, 18, 12\}$. The error of adaptive CMFPS converges asymptotically to zero as demonstrated in Figure.2. Time evolution of parameter estimations are shown in Figure.3 and Figure.4, respectively.

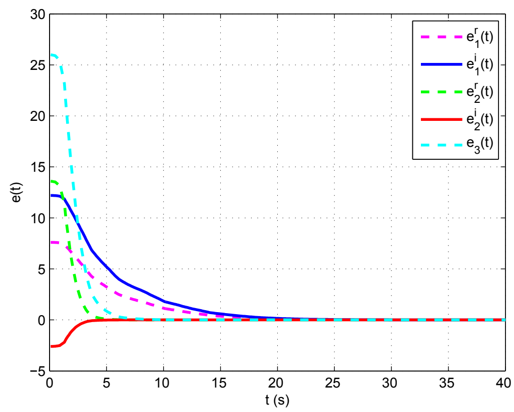

Figure 2: The dynamic of adaptive CMFPS error between Lü CVCSs (27) and Lorenz CVCSs (28) with the controller (29), complex parameter update laws (30), and complex scaling function matrix (31). Here $e_1 = y_1 - 0.4\exp(j\pi t/4)z_1, e_2 = y_2 - 1.2\exp(j\pi t/12)z_2$, and $e_3 = y_3 - (1.8 + \sin t)z_3$.

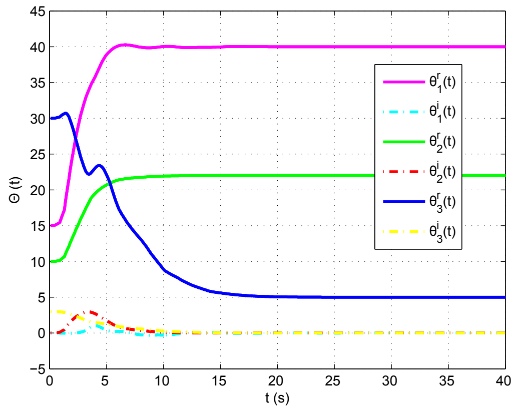

Figure 3: Time evolution of parameter estimation of unknown real parameter vector $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3)^{\mathrm{T}}$ for the systems (27) when complex scaling functions are $0.4\exp(j\pi t/4)$, $1.2\exp(j\pi t/12)$, and $1.8 + \sin t$.

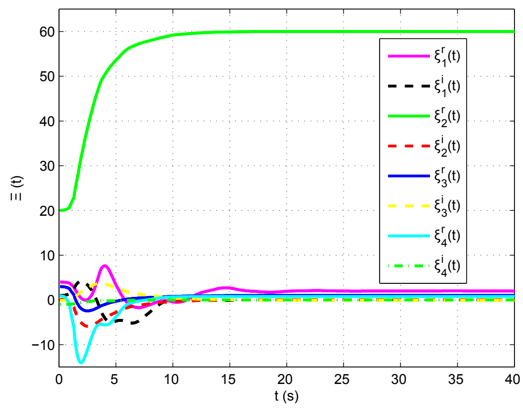

Figure 4: Time evolution of parameter estimation of unknown complex parameter vector $\mathbf{\Xi} = (\xi_1, \xi_2, \xi_3, \xi_4)^{\mathrm{T}}$ for the system (28) when complex scaling functions are $0.4\exp(j\pi t/4)$, $1.2\exp(j\pi t/12)$, and $1.8 + \sin t$.

As expected, the above results demonstrate that adaptive CMFPS has been achieved between drive Lü CVCSs (27) with full unknown real parameters and response Lorenz CVCSs (28) with full unknown complex parameters and all of unknown parameters in both drive and response systems are identified successfully with the designed complex controller (29) and the complex parameter updated laws (30).

带有完全未知的复值参数

4.2. Adaptive CMFPS of two identical CVCSs with full unknown complex parameters

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In order to illustrate adaptive CMFPS for two identical systems, we take Lorenz CVCSs (1) with full unknown complex parameters as drive and response systems. Therefore, Lorenz CVCSs (28) is taken as response system and the drive system is described by

$$\left\{\begin{array}{rcl} \dot{z}_1 & = & \theta_1(z_2 - z_1), \\ \dot{z}_2 & = & \theta_2 z_1 - \theta_3 z_2 - z_1 z_3, \\ \dot{z}_3 & = & -\theta_4 z_3 + \frac{1}{2}(\bar{z}_1 z_2 + z_1 \bar{z}_2), \end{array}\right. \tag{32}$$

where $z_1 = z_1^r + j z_1^i$, $z_2 = z_2^r + j z_2^i$ are complex state variables and z_3 is a real state variable, $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4)^{\mathrm{T}} = \mathbf{\Xi}$ is unknown complex parameter vector.

The controller is designed according to (15) in Theorem 2 as follows,

$$\left\{\begin{array}{rcl} u_1 & = & [\lambda_1(t)(z_2(t) - z_1(t)) - (y_2(t) - y_1(t))]\hat{\theta}_1 + \dot{\lambda}_1(t) z_1(t) - k_1 e_1, \\ u_2 & = & [\lambda_2(t) z_1(t) - y_1(t)]\hat{\theta}_2 + [-\lambda_2(t) z_2(t) + y_2(t)]\hat{\theta}_3 \\ & & +y_1(t) y_3(t) - \lambda_2(t) z_1(t) z_3(t)) + \dot{\lambda}_2(t) z_2(t) - k_2 e_2, \\ u_3 & = & [-\lambda_3(t) z_3(t) + y_3(t)]\hat{\theta}_4 + \dot{\lambda}_3(t) z_3(t) - k_3 e_3, \\ & & +\frac{1}{2}[\lambda_3(t)(\bar{z}_1(t) z_2(t) + z_1(t)\bar{z}_2(t)) - (\bar{y}_1(t) y_2(t) + y_1(t)\bar{y}_2(t))], \end{array}\right. \tag{33}$$

and the complex parameter update law is given according to (16) as

$$\dot{\hat{\mathbf{\Theta}}} = \begin{pmatrix} [(\bar{y}_2(t) - \bar{y}_1(t)) - \bar{\lambda}_1(t)(\bar{z}_2(t) - \bar{z}_1(t))]e_1 - \gamma_{\theta_1}(\hat{\theta}_1 - \theta_1) \\ (\bar{y}_1(t) - \bar{\lambda}_2(t)\bar{z}_1(t))e_2 - \gamma_{\theta_2}(\hat{\theta}_2 - \theta_2) \\ (-\bar{y}_2(t) + \bar{\lambda}_2(t)\bar{z}_2(t))e_2 - \gamma_{\theta_3}(\hat{\theta}_3 - \theta_3) \\ (-y_3(t) + \bar{\lambda}_3(t) z_3(t))e_3 - \gamma_{\theta_4}(\hat{\theta}_4 - \theta_4) \end{pmatrix}. \tag{34}$$

The complex scaling function matrix is selected as

$$\mathbf{\Lambda}(t) = \begin{pmatrix} 0.2\exp(j\pi t/2) & 0 & 0 \\ 0 & \exp(j\pi t/10) & 0 \\ 0 & 0 & 1.2 + 0.5\cos 2t \end{pmatrix}. \tag{35}$$

In the numerical simulations, the value of unknown parameter vector is chosen as $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4)^{\mathrm{T}} = (2, 60 + 0.02j, 1 - 0.06j, 0.8)^{\mathrm{T}}$. The initial conditions of drive system (32) and response system (28) are employed as $\mathbf{z}(\mathbf{0}) = (2 + 0.02j, 1 + 0.2j, -1)^{\mathrm{T}}$, $\mathbf{y}(\mathbf{0}) = (8 + 13j, 16 + j, 35)^{\mathrm{T}}$. The initial value of estimated parameter is $\hat{\mathbf{\Theta}}(0) = (\hat{\theta}_1(0), \hat{\theta}_2(0), \hat{\theta}_3(0), \hat{\theta}_4(0))^{\mathrm{T}} = (1-3j,\ 1-2j,\ 2+j,\ 1+j)^{\mathrm{T}}$. The control gain matrix is $\mathbf{K} = diag\{5, 20, 12\}$. The error of adaptive CMFPS converges asymptotically to zero as demonstrated in Figure.5. Time evolution of parameter estimation is shown in Figure.6.

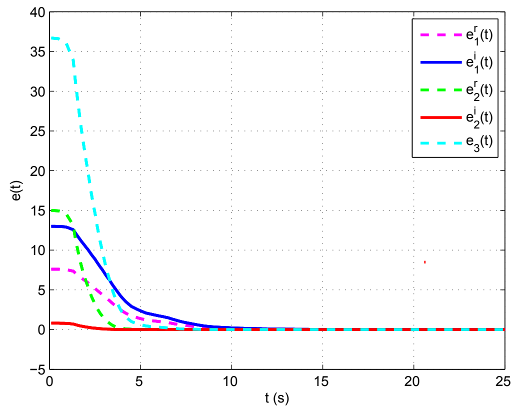

Figure 5: The dynamic of adaptive CMFPS error between two identical Lorenz CVCSs (32) and (28) with the controller (33), complex parameter update law (34), and complex scaling function matrix (35). Here $e_1 = y_1 - 0.2\exp(j\pi t/2)z_1, e_2 = y_2 - \exp(j\pi t/10)z_2$, and $e_3 = y_3 - (1.2 + 0.5\cos 2t)z_3$.

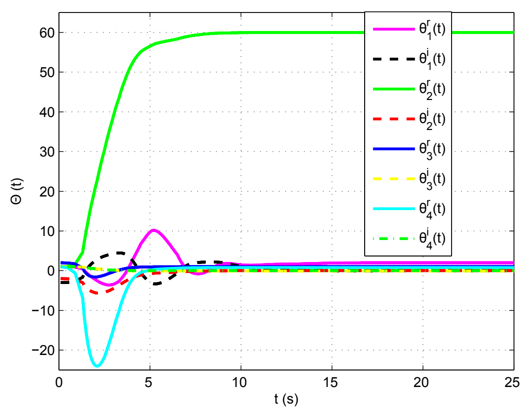

Figure 6: The identification process of unknown complex parameter vector $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4)^{\mathrm{T}}$ for the systems (32) and (28) when complex scaling functions are $0.2\exp(j\pi t/2)$, $\exp(j\pi t/10)$, and $1.2 + 0.5\cos 2t$.

As expected, the above results demonstrate that adaptive CMFPS has been achieved between two identical Lorenz CVCSs (32) and (28) and all of unknown complex parameters are identified successfully with the designed controller (33) and the complex parameter update law (34).

Remark 9. Note that z_3, y_3 are real variables in CVCSs (27), (32) and (28), the real-valued scaling function $\lambda_3(t)$ is chosen in the matrices (31) and (35) to make $e_3 = y_3 - \lambda_3(t)z_3$ real for the convenience of reasonable discussion. Therefore, we do not need consider the design of the imaginary part of the controller u_3.

5. Discussion and conclusions

In this paper, we consider a class of chaotic (hyperchaotic) systems where the main variables and parameters are taken to be complex, and propose general schemes to achieve CMFPS for the drive and response systems with known and unknown parameters. In complex space, with the proposed scheme, the response system become a projection of drive system with a desired complex-valued scaling function matrix via the complex controller and complex parameter update laws.

Compared with the previously reported schemes, we avoid dividing the complex variables or complex parameters into real and imaginary parts. It is necessary noting that sufficient conditions on adaptive CMFPS and parameter identification are derived by constructing appropriate Lyapunov functions dependent on complex variables, and employing adaptive control technique. We hope the performed work will serve as a guideline for further studies in synchronization of CVCSs in complex space.

What's more, CMFPS establishes a link between CVCSs. In many real-world applications, it is extremely obvious and common that the relationship between two identical or nonidentical CVCSs is described as complex functions. It means the scaling function matrix is composed of different complex-valued functions, which increases the complexity and scope of the synchronization and directs high security and large variety of secure communications.

Simulation results indicate that the proposed method is efficient and has wide applicability to n-dimensional CVCSs with certain and uncertain complex parameters.

Finally, CMFPS of complex network in complex field have not been previously reported, we will consider the topic in the near future.

Acknowledgments

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