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Research paper

Finite-time stability and synchronization of memristor-based fractional-order fuzzy cellular neural networks☆

Mingwen Zheng[a,b], Lixiang Li[c,*], Haipeng Peng[c], Jinghua Xiao[a,d], Yixian Yang[c], Yanping Zhang[b], Hui Zhao[a]

[a] School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China
[b] School of Science, Shandong University of Technology, Zibo 255000, China
[c] Information Security Center, State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China
[d] State Key Laboratory of Information Photonics and Optical Communications, Beijing University of Posts and Telecommunications, Beijing 100876, China

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ABSTRACT

This paper mainly studies the finite-time stability and synchronization problems of memristor-based fractional-order fuzzy cellular neural network (MFFCNN). Firstly, we discuss the existence and uniqueness of the Filippov solution of the MFFCNN according to the Banach fixed point theorem and give a sufficient condition for the existence and uniqueness of the solution. Secondly, a sufficient condition to ensure the finite-time stability of the MFFCNN is obtained based on the definition of finite-time stability of the MFFCNN and Gronwall–Bellman inequality. Thirdly, by designing a simple linear feedback controller, the finite-time synchronization criterion for drive-response MFFCNN systems is derived according to the definition of finite-time synchronization. These sufficient conditions are easy to verify. Finally, two examples are given to show the effectiveness of the proposed results.

1. Introduction

In 1996, T Yang and L B Yang proposed the fuzzy cellular neural network (FCNN) for the first time and pointed out that it has a wide range of applications in image processing and fuzzy recognition [1,2]. Different from the traditional cellular neural network structure, the FCNN adds fuzzy logic (fuzzy AND and fuzzy OR) to its structure and maintains the local connection between cells. With the further study of the FCNN, some scholars have expanded its applications in more fields, such as association memory [3], optimization calculation [4], etc. It is well known that the stability is a prerequisite for ensuring the performance of FCNN when applied in these areas. Therefore, in the theoretical research on the FCNN, such as the existence, stability of solutions and synchronization have been widely concerned [1,5–13]. Li et al. investigated existence, uniqueness and the global asymptotic stability of fuzzy cellular neural networks with leakage delay, time-varying delays and continuously distributed delays, and some sufficient conditions are derived to ensure global asymptotic stability of the equilibrium point by using Lyapunov approach and the linear matrix inequality (LMI) method [8]. Bao et al. considered

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* Corresponding author.
E-mail address: lixiang@bupt.edu.cn (L. Li).

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the existence, uniqueness and global robust exponential stability of the solution of interval fuzzy Cohen–Grossberg neural network with piecewise arguments, and new theoretical results of exponential stability are derived based on comparison principle [9]. Ratnavelu et al. studied the synchronization of fuzzy bidirectional associative memory neural network with mixed delays, and based on Lyapunov–Krasovskii functional and the LMI method, some sufficient conditions are obtained to ensure synchronization between master-slave systems [11]. We can see from these references that most of the research on stability and synchronization of fuzzy neural network is based on the Lyapunov stability theorem, but it is well known that the suitable Lyapunov function is not easy to be designed in most cases.

Due to the infinite memory property and hereditary property of fractional-order calculus, fractional-order systems have gained wide attention in the fields of mechanics, viscoelasticity, electrical circuits and engineering control [14–16]. Especially, Yang et al. made a lot of foundation work in the local fractional calculus, which made the theory of fractional calculus more perfect [17–19]. Recent studies show that fractional-order systems can provide a fundamental and general computation for single neurons, which helps to the information processing, stimulus anticipation and frequency-independent phase shifts of oscillatory neuronal firing [20]. The theory of fractional-order calculus has been a powerful tool in modeling and analyzing many scientific phenomena. Naturally, in order to make neural networks have better performance, many scholars incorporated the fractional calculus into neural networks to form a class of fractional-order neural network. Since Arena et al. firstly studied the dynamical characteristics of fractional-order cellular neural networks and found some interesting phenomena [21], fractional-order neural networks have attracted the considerable attention of scholars [22–30]. The memristor, which was first postulated by Chua in 1971 [31] and was successfully developed by the researchers at the Hewlett–Packard lab [32], its resistance value can vary with the current passing through it. The changing characteristics of the resistance is similar to the neuronal synapse when stimulated by the biological electrical signals. Meanwhile, because of its nanometer size, nonvolatile and pinched hysteresis characteristics, the memristor becomes the best component used to simulate the synapse so far. As a result, the researchers use the memristor instead of the resistors in traditional artificial neural network, thus forming memristor-based neural network. Because of the particularity of the fractional-order calculus and memristor, the dynamical behaviors such as stability and synchronization, which combine the memristor and the fractional-order neural network have been studied extensively [33–38]. However, the fuzzy neural network combined with the fractional order (abbreviated as the FFNN), or combined with the memristor (abbreviated as the MFNN), or simultaneously combined with the fractional order and memristor (abbreviated as the MFFCNN), is still in a initial stage. Liu et al. studied the robust adaptive lag synchronization of uncertain MFNN with time-varying delays, and obtained some sufficient conditions to ensure the lag synchronization of master-slave systems by utilizing Lyapunov function method and an adaptive controller [39]. Zhang et al. investigated the uniform stability problem in the sense of mean square of stochastic the FFNN with delay based on the stochastic analysis theory and the Banach fixed point principle [40]. Unfortunately, so far we have not found the relevant research results on the MFFCNN.

Stability analysis is one of the main research contents of neural network dynamics. Synchronization between complex networks has been widely used in many fields, such as secure communication, image processing, and so on. The synchronization of the drive-response systems can eventually be translated into the stability analysis of the error system. In addition, compared with the asymptotic stability or synchronization, the finite-time stability or synchronization of neural networks is paid more and more attention by researchers because of its practicality [34,35,41–45]. In the literature on the finite-time stability or synchronization in relation to the memristor, fractional order, or fuzzy neural networks, Bai et al. discussed the finite-time stability problem of discrete-time fuzzy Hopfield neural networks and proposed a sufficient condition by the concept of finite-time stability of fuzzy neural network, Lyapunov method and LMI [41]; The authors used similar method to study the finite-time synchronization problem of fuzzy cellular nerual networks with time-varying delays[42–44]; Zheng et al. investigated the finite-time stability and synchronization problem of memristor-based fractional-order Cohen–Grossberg neural network and derived some new sufficient conditions using set-valued map, differential inclusions, Gronwall inequality and a simple linear feedback controller [34]. But, there are few researches about the MFFCNN.

Inspired by the above discussion, considering that fractional-order calculus, memristor, and fuzzy logic may lead to more complex dynamic behaviors of neural networks, the main objective of this paper is to analysis the finite-time stability and synchronization problems of the MFFCNN in the sense of Filippov with the help of set-valued map, differential inclusions, Banach fixed point theorem, Gronwall–Bellman inequality and the definitions of finite-time stability and synchronization. The main contributions of this paper are summarized as follows.

(1) We study the finite-time stability and synchronization of the MFFCNN for the first time. By using the set-valued map and differential inclusions theory, the MFFCNN with discontinuous right-hand side is transformed into ordinary differential equations. Then we prove the existence and uniqueness of MFFCNN's solution under certain conditions by means of Banach fixed point theorem.

(2) We give the definitions of finite-time stability of the MFFCNN and synchronization of drive-response MFFCNN systems. According to these two definitions, some easy verifiable sufficient conditions for ensuring that the finite-time stability of the MFFCNN and the finite-time synchronization of drive-response the MFFCNN systems are obtained. Furthermore, our results can be easily extended to the memristor-based fractional-order neural networks without fuzzy logic.

(3) Two simulation examples are shown to illustrate the correctness of the main results.

The rest of this paper is organized as follows. we introduce the definition of the Caputo fractional calculus, the MFFCNN model, the definition of finite-time stability and synchronization and some lemmas in Section 2. In Section 3, we prove the existence of the solution of the MFFCNN, and deduce the sufficient conditions for the finite-time stability of the MFFCNN and the finite-time synchronization of the drive-response MFFCNN. Two numerical examples show the correctness of our main results in Section 4. At last, we conclude this paper and prospect for the future work in Section 5.

Notations: Throughout this paper, $\mathbb{R}$ and $\mathbb{R}^n$ represent the set of real numbers and the n-dimensional Euclidean space. $\mathbb{N}^+$ denotes the positive integer. $co[\cdot,\cdot]$ denotes the closure of the convex hull of interval $[\cdot,\cdot]$. Given any $x=(x_1,x_2,\dots,x_n)^T\in\mathbb{R}^n$, the vector norm is defined as $\|x(t)\|=\sqrt{\sum_{i=1}^n \sup_{t\in(0,T]}\{e^{-2t}|x_i(t)|^2\}}$; the matrix norm is defined as $\|A\|=\max_{1\le j\le n}\sum_{i=1}^n |a_{ij}|$.

2. Preliminaries and network model

In this section, we will firstly give the definition of fractional calculus, the mathematical model of the MFFCNN, some assumptions and lemmas.

2.1. Caputo fractional calculus

In order to describe the MFFCNN model, we will introduce the definition of the Caputo fractional calculus.

Definition 1. [46] Let $f(t)\in\mathbb{C}^{n+1}([t_0,+\infty],\mathbb{R})$, the left-sided Caputo fractional-order derivative of order α for $f(t)$ is defined as

$$ {}_{t_0}^{C}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, $$

where $t\ge t_0$ and $n\in\mathbb{N}^+$ is a positive integer such that $n-1<\alpha<n$. $\Gamma(\cdot)$ is the Gamma function, which is defined by $\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt$. Especially, when $0<\alpha<1$, we have

$$ {}_{t_0}^{C}D_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau. $$

Definition 2. [46] Let $f(t):[t_0,\infty]\to\mathbb{R}$ be an integrable function. The Caputo fractional integral of order α for $f(t)$ is defined as

$$ {}_{t_0}^{C}D^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau, $$

where $\alpha>0$.

For the sake of simplicity, we use D^{α} and $D^{-\alpha}$ to represent ${}_{t_0}^{C}D_t^{\alpha}$ and ${}_{t_0}^{C}D^{-\alpha}$, respectively.

The following lemma about the Caputo fractional derivative and integral will be used in the derivation of the main conclusions.

Lemma 1. [46] *For the Caputo fractional-order derivative, when $n-1<\alpha<n, n\in\mathbb{N}^+$ we have*

$$ D^{-\alpha}(D^{\alpha})f(t)=f(t)-\sum_{i=1}^{n-1}\frac{f^{(i)}(t_0)}{i!}(t-t_0)^i. $$

Especially, when $0<\alpha<1$, we can obtain

$$ D^{-\alpha}(D^{\alpha})f(t)=-f(t_0)+f(t). $$

2.2. Network model

In this subsection, we will give the mathematical model of the MFFCNN.

Consider the following memristor-based fractional-order fuzzy neural network model

$$ \begin{cases} D^q x_i(t)=-\tilde{d}_i(x_i(t))x_i(t)+\sum\limits_{j=1}^{n}\tilde{a}_{ij}(x_j(t))f_j(x_j(t)) \\ \quad+\sum\limits_{j=1}^{n}b_{ij}v_j+\bigwedge\limits_{j=1}^{n}\alpha_{ij}g_j(x_j(t-\tau))+\bigwedge\limits_{j=1}^{n}T_{ij}v_j+\bigvee\limits_{j=1}^{n}S_{ij}v_j \\ \quad+\bigvee\limits_{j=1}^{n}\beta_{ij}g_j(x_j(t-\tau))+I_i,\ t\in[0,T], \\ \quad x_i(t)=\varphi_i(t),\ t\in[-\tau,0],\ i=1,2,\dots,n, \end{cases} \tag{1} $$

where $0 < q < 1$, n is the number of neurons, x_i, v_i, I_i represent the state, input and bias of the ith neuron; α_{ij}, β_{ij}, T_{ij} and S_{ij} are the elements of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed-forward MIN template, fuzzy feed-forward MAX template, respectively; $\bigvee$ and $\bigwedge$ denote the fuzzy OR and fuzzy AND; $\tilde{d}_i(x_i(t))$ denotes the memristor-based passive decay rate to the state of ith neuron; $\tilde{a}_{ij}(x_j(t))$ is the memristor-based elements of feedback template; b_{ij} is the elements of feed-forward template; $f(\cdot)$, $g(\cdot)$ are the neuron activation functions; τ is the delay.

Based on the memristor model, $\tilde{d}_i(x_i(t))$ and $\tilde{a}_{ij}(x_j(t))$ satisfy

$$\tilde{d}_i(x_i(t)) = \begin{cases} d_i^*, |x_i(t)| \le T_i, \\ d_i^{**}, |x_i(t)| > T_i, \end{cases}$$

$$\tilde{a}_{ij}(x_i(t)) = \begin{cases} a_{ij}^*, |x_i(t)| \le T_i, \\ a_{ij}^{**}, |x_i(t)| > T_i, \end{cases}$$

where $T_i > 0$ is the switching jumps, $d_i^*, d_i^{**}, a_{ij}^*$ and $a_{ij}^{**}, i, j = 1, 2, \ldots, n$ are constants.

Remark 1. From the above analysis, we can see that $\tilde{d}_i(x_i(t)), \tilde{a}_{ij}(x_j(t))$ are changed as the state changes when the memristance is switched. Therefore, the MFFCNN is a switching system with discontinuous right-hand sides. For such systems, we need to use the set-valued map and differential inclusion theory to process them. About the set-valued map and differential inclusion theory, please refer to the literature[47]. In this paper, all the discussions about the solution of the MFFCNN are under the sense of the Filippov framework.

According to the set-valued map and differential inclusion theory, the MFFCNN (1) can be written as the following differential inclusion form

$$\begin{aligned} D^q x_i(t) \in & -co[\underline{d}_i, \overline{d}_i]x_i(t) + \sum_{j=1}^{n} co[\underline{a}_{ij}, \overline{a}_{ij}]f_j(x_j(t)) \\ & + \sum_{j=1}^{n} b_{ij}v_j + \bigwedge_{j=1}^{n} \alpha_{ij}g_j(x_j(t-\tau)) \\ & + \bigwedge_{j=1}^{n} T_{ij}v_j + \bigvee_{j=1}^{n} S_{ij}v_j + \bigvee_{j=1}^{n} \beta_{ij}g_j(x_j(t-\tau)) + I_i, \end{aligned} \tag{2}$$

where $\underline{d}_i = \min\{d_i^*, d_i^{**}\}, \overline{d}_i = \max\{d_i^*, d_i^{**}\},\ \underline{a}_{ij} = \min\{a_{ij}^*, a_{ij}^{**}\}, \overline{a}_{ij} = \max\{a_{ij}^*, a_{ij}^{**}\}$, and

$$co[\underline{d}_i, \overline{d}_i] = \begin{cases} d_i^*, |x_i(t)| < T_i, \\ [\underline{d}_i, \overline{d}_i], |x_i(t)| = T_i, \\ d_i^{**}, |x_i(t)| > T_i, \end{cases}$$

$$co[\underline{a}_{ij}, \overline{a}_{ij}] = \begin{cases} a_{ij}^*, |x_i(t)| < T_i, \\ [\underline{a}_{ij}, \overline{a}_{ij}], |x_i(t)| = T_i, \\ a_{ij}^{**}, |x_i(t)| > T_i, \end{cases}$$

or equivalently, there exist $d_i \in co[\underline{d}_i, \overline{d}_i],\ a_{ij} \in co[\underline{a}_{ij}, \overline{a}_{ij}]$ such that

$$\begin{aligned} D^q x_i(t) = & -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}v_j \\ & + \bigwedge_{j=1}^{n} \alpha_{ij}g_j(x_j(t-\tau)) + \bigwedge_{j=1}^{n} T_{ij}v_j + \bigvee_{j=1}^{n} S_{ij}v_j \\ & + \bigvee_{j=1}^{n} \beta_{ij}g_j(x_j(t-\tau)) + I_i, t \in [0, T], \\ & i, j = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

If we regard the MFFCNN (1) as the drive system, the corresponding response system can be given as following

$$\begin{cases} D^q y_i(t) = -\tilde{d}_i(y_i(t))y_i(t) + \sum_{j=1}^{n} \tilde{a}_{ij}(y_j(t))f_j(y_j(t)) \\ \quad + \sum_{j=1}^{n} b_{ij}\nu_j + \bigwedge_{j=1}^{n} \alpha_{ij}g_j(y_j(t-\tau)) + \bigwedge_{j=1}^{n} T_{ij}\nu_j \\ \quad + \bigvee_{j=1}^{n} S_{ij}\nu_j + \bigvee_{j=1}^{n} \beta_{ij}g_j(y_j(t-\tau)) + I_i + u_i(t), t \in [0,T], \\ \quad y_i(t) = \phi_i(t), t \in [-\tau, 0], i = 1,2,\dots,n, \end{cases} \tag{4}$$

where $u_i(t)$ the finite-time controller to be designed. $\tilde{d}(y_i(t))$ and $\tilde{a}_{ij}(y_j(t))$ satisfy

$$\tilde{d}_i(y_i(t)) = \begin{cases} d_i^*, |y_i(t)| \le T_i, \\ d_i^{**}, |y_i(t)| > T_i, \end{cases}$$

$$\tilde{a}_{ij}(y_i(t)) = \begin{cases} a_{ij}^*, |y_i(t)| \le T_i, \\ a_{ij}^{**}, |y_i(t)| > T_i, \end{cases}$$

where $T_i > 0$ is the switching jumps.

Similarly, according the set-valued map and differential inclusion theories, the response system (4) can be rewritten as the following differential inclusion form

$$\begin{aligned} D^q y_i(t) \in -co[\underline{d}_i, \overline{d}_i]y_i(t) + \sum_{j=1}^{n} co[\underline{a}_{ij}, \overline{a}_{ij}]f_j(y_j(t)) \\ + \sum_{j=1}^{n} b_{ij}\nu_j + \bigwedge_{j=1}^{n} \alpha_{ij}g_j(y_j(t-\tau)) + \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigvee_{j=1}^{n} S_{ij}\nu_j \\ + \bigvee_{j=1}^{n} \beta_{ij}g_j(y_j(t-\tau)) + I_i + u_i(t), \end{aligned} \tag{5}$$

where $\underline{d}_i = \min\{d_i^*, d_i^{**}\}, \overline{d}_i = \max\{d_i^*, d_i^{**}\},\ \underline{a}_{ij} = \min\{a_{ij}^*, a_{ij}^{**}\}, \overline{a}_{ij} = \max\{a_{ij}^*, a_{ij}^{**}\}$, and

$$co[\underline{d}_i, \overline{d}_i] = \begin{cases} d_i^*, |y_i(t)| < T_i, \\ [\underline{d}_i, \overline{d}_i], |y_i(t)| = T_i, \\ d_i^{**}, |y_i(t)| > T_i, \end{cases}$$

$$co[\underline{a}_{ij}, \overline{a}_{ij}] = \begin{cases} a_{ij}^*, |y_i(t)| < T_i, \\ [\underline{a}_{ij}, \overline{a}_{ij}], |y_i(t)| = T_i, \\ a_{ij}^{**}, |y_i(t)| > T_i, \end{cases}$$

or equivalently, there exist $d_i \in co[\underline{d}_i, \overline{d}_i],\ a_{ij} \in co[\underline{a}_{ij}, \overline{a}_{ij}]$ such that

$$\begin{aligned} D^q y_i(t) = -d_i y_i(t) + \sum_{j=1}^{n} a_{ij}f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}\nu_j \\ + \bigwedge_{j=1}^{n} \alpha_{ij}g_j(y_j(t-\tau)) + \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigvee_{j=1}^{n} S_{ij}\nu_j \\ + \bigvee_{j=1}^{n} \beta_{ij}g_j(y_j(t-\tau)) + I_i + u_i(t), t \in [0,T], \\ i, j = 1,2,\dots,n, \end{aligned} \tag{6}$$

The initial condition of the response system (4) is $y_i(t) = \phi_i(t), t \in [-\tau, 0]$.

In order to obtain our main results, the following some definitions, assumptions and lemmas are needed.

Definition 3. [48] The solution $x(t)$ of the MFFCNN given by Eq. (3) is finite-time stable $w.r.t.\{\delta, \varepsilon, t_0, J\}, \delta < \varepsilon, J = [t_0, t_0 + T]$ if and only if $\|\varphi(t)\| < \delta$ implies $\|x(t)\| < \varepsilon,\ \forall t \in J$.

Definition 4. For any two solutions $x(t)$ and $y(t)$ with differential initial conditions $\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T$ and $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T, t \in [-\tau, 0]$, let $e(t) = y(t) - x(t), e(0) = \phi(t) - \varphi(t)$. The MFFCNN (3) is said to be finite-time stable w.r.t. $\{\sigma, \varepsilon, J\}$, $\sigma < \varepsilon, J \in [0, T]$, $\|e(0)\| < \sigma$ implies $\|e(t)\| < \varepsilon$.

Definition 5. The drive system (3) and the response system (4) are said to achieve finite-time synchronization under a suitable controller $u(t)$, if the state error $e(t) = y(t) - x(t)$ can reach stability in finite time, i.e. there exist positive numbers $\{\delta, \varepsilon, J\}$, $\varepsilon > \delta$ and $\|e(t_0)\| = \|\phi(t) - \varphi(t)\| < \delta, t \in J$ implies $\|e(t)\| < \varepsilon$.

Assumption 1. The neuron activation functions $f_i(x)$, $g_i(x)$ satisfy the Lipschitz continuity condition, i.e., there exist positive constants m_i, k_i such that

$$|f_i(y(t)) - f_i(x(t))| \leq m_i|y(t) - x(t)|,$$
$$|g_i(y(t)) - g_i(x(t))| \leq k_i|y(t) - x(t)|,$$

where $x, y \in \mathbb{R}, i = 1, 2, \ldots, n$.

Lemma 2. *[1] Let x_j, y_j be two states of the MFFCNN (3), then we have*

$$\Big|\bigwedge_{j=1}^{n} \alpha_{ij}g_j(y_j) - \bigwedge_{j=1}^{n} \alpha_{ij}g_j(x_j)\Big| \leq \sum_{j=1}^{n} k_j|\alpha_{ij}||y_j - x_j|$$
$$\Big|\bigvee_{j=1}^{n} \beta_{ij}g_j(y_j) - \bigvee_{j=1}^{n} \alpha_{ij}g_j(x_j)\Big| \leq \sum_{j=1}^{n} k_j|\beta_{ij}||y_j - x_j|.$$

Lemma 3. *[49] Suppose $q > 0$, $a(t)$ is nonnegative function locally integrable on $0 \leq t < T \leq +\infty$, and g(t) is a nonnegative, nondecreasing continuous function defined on $[0, T)$, $g(t)$ is bounded. Suppose $u(t)$ is nonnegative and locally integrable on $[0, T)$ with*

$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{q-1}u(s)ds.$$

Then we have

$$u(t) \leq a(t) + \int_0^t \Big[\sum_{i=1}^{\infty} \frac{(g(t)\Gamma(q))^n}{\Gamma(nq)}(t-s)^{nq-1}a(s)\Big]ds$$

If $a(t)$ is a nondecreasing function on $[0, T)$, then we have

$$u(t) \leq a(t)E_q(g(t)\Gamma(q)t^q), t \in [0, T),$$

where $E_q(\cdot)$ is the Mittag–Leffler function defined as $E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)}$.

Lemma 4. *[50] If $|f(\cdot)|^2$, $|g(\cdot)|^2 \in L^1(E)$, then $f(\cdot)g(\cdot) \in L^1(E)$ and*

$$\Big(\int_E |f(x)g(x)|\Big)^2 \leq \int_E |f(x)|^2dx \int_E |g(x)|^2dx$$

Lemma 5. *[51] Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ be nonnegative real numbers. Then for $p > 1$, we have*

$$\Big(\sum_{i=1}^{n} a_i\Big)^p \leq n^{p-1}\sum_{i=1}^{n} a_i^p.$$

3. Main results

In this section, we will firstly obtain the condition for the existence of the solution of the MFFCNN (3), and then determine whether the solution is finite-time stable. Finally, a finite-time controller is designed to derive the sufficient conditions for the drive-response systems (3) and (4) to achieve the finite-time synchronization.

3.1. Existence of Filippov solution of the MFFCNN

Theorem 1. *Given the initial condition $x(t) = \varphi(t), t \in [-\tau, 0]$, the MFFCNN system (1) has a unique solution if Assumption 1 and the following sufficient condition holds*

$$\sum_{i=1}^{n}\Big\{\Big(\sum_{j=1}^{n} a_{ij}m_j - \underline{d}_j\Big)^2 + e^{-2\tau}\sum_{j=1}^{n}\big[(|\alpha_{ij}| + |\beta_{ij}|)k_j\big]^2\Big\} \leq \frac{1}{2} \tag{7}$$

Proof. According to the Lemma 1, the MFFCNN (3) is equivalent to the following Volterra integral equation

$$x_i(t) = \varphi_i(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big[-d_ix_i(s)$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}\nu_j \\
&+\bigwedge_{j=1}^{n} \alpha_{ij} g_j(x_j(s-\tau)) + \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigvee_{j=1}^{n} S_{ij}\nu_j \\
&+\bigvee_{j=1}^{n} \beta_{ij} g_j(x_j(s-\tau)) + I_i\Big]ds, t\in[0,T]
\end{aligned}
\tag{8}
$$

Construct a map $T:\mathbb{R}^n \to \mathbb{R}^n$

$$
\begin{aligned}
T(x_i(t)) = \varphi_i(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big[&-d_i x_i(s) + \sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij}\nu_j \\
&+\bigwedge_{j=1}^{n} \alpha_{ij} g_j(x_j(s-\tau)) + \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigvee_{j=1}^{n} S_{ij}\nu_j \\
&+\bigvee_{j=1}^{n} \beta_{ij} g_j(x_j(s-\tau)) + I_i\Big]ds
\end{aligned}
\tag{9}
$$

Next, we will prove that the map T is a contractive map through Banach fixed point theorem.

Assume that $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ and $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ are two different solutions of the MF-FCNN (3) with the same initial value $\varphi(t_0)$ and let $\epsilon(t) = y(t) - x(t)$, then we have

$$
\begin{aligned}
T(y_i(t)) - T(x_i(t)) = \frac{1}{\Gamma(q)}\int_0^{t_0} (t-s)^{q-1}\Big[&-d_i\epsilon_i(s) + \sum_{j=1}^{n} a_{ij}(f_j(y_j(s)) - f_j(x_j(s))) \\
&+\bigwedge_{j=1}^{n} \alpha_{ij}(g_j(y_j(s-\tau)) - g_j(x_j(s-\tau))) \\
&+\bigvee_{j=1}^{n} \beta_{ij}(g_j(y_j(s-\tau)) - g_j(x_j(s-\tau)))\Big]ds.
\end{aligned}
$$

First taking absolute values on both sides of the equation above, and then according to Lemma 2 and Assumption 1, we get

$$
\begin{aligned}
&|T(y_i(t)) - T(x_i(t))| \le \left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big[-d_i\epsilon_i(s) + \sum_{j=1}^{n} a_{ij} m_j|\epsilon_j(s)| + \sum_{j=1}^{n} k_j|\alpha_{ij}||\epsilon_j(s-\tau)|\right. \\
&\left.+\sum_{j=1}^{n} k_j|\beta_{ij}||\epsilon_j(t-\tau)|\Big]ds\right| \le \left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Bigg[\left(\sum_{j=1}^{n} a_{ij} m_j - \underline{d}_j\right)|\epsilon_j(s)|\right. \\
&\left.+\sum_{j=1}^{n} k_j(|\alpha_{ij}| + |\beta_{ij}|)|\epsilon_j(t-\tau)|\Bigg]ds\right|.
\end{aligned}
\tag{10}
$$

Squaring on the both sides of the inequality (10) and multiply by e^{-2t}, we have

$$
\begin{aligned}
&e^{-2t}|T(y_i(t)) - T(x_i(t))|^2 \le \frac{1}{\Gamma^2(q)}\left\{\int_0^t (t-s)^{q-1}\Bigg[\left(\sum_{j=1}^{n} a_{ij} m_j - \underline{d}_j\right)e^{-t}|\epsilon_j(s)|\right. \\
&\left.+\sum_{j=1}^{n} k_j(|\alpha_{ij}| + |\beta_{ij}|)e^{-t}|\epsilon_j(t-\tau)|\Bigg]ds\right\}^2
\end{aligned}
$$

Based on Lemma 5, we obtain

$$
\begin{aligned}
&e^{-2t}|T(y_i(t)) - T(x_i(t))|^2 \le \frac{2}{\Gamma^2(q)} \times \left[\int_0^t (t-s)^{q-1} e^{-t}\left(\sum_{j=1}^{n} a_{ij} m_j - \underline{d}_j\right)|\epsilon_j(s)|ds\right]^2 + \frac{2}{\Gamma^2(q)} \\
&\times\left[\int_0^t (t-s)^{q-1} e^{-t}\sum_{j=1}^{n} k_j(|\alpha_{ij}| + |\beta_{ij}|)|\epsilon_j(s)|ds\right]^2.
\end{aligned}
\tag{11}
$$

Let $F_1 = [\int_0^t (t-s)^{q-1} e^{-t} (\sum_{j=1}^n a_{ij} m_j - \underline{d}_j) |\epsilon_j(s)| ds]^2$, $F_2 = [\int_0^t (t-s)^{q-1} e^{-t} \sum_{j=1}^n k_j (|\alpha_{ij}| + |\beta_{ij}|) |\epsilon_j(s)| ds]^2$. According to Lemma 4, we have

$$\begin{aligned}
F_1 &= \left[\int_0^t (t-s)^{\frac{q-1}{2}} e^{-\frac{t-s}{2}} \cdot (t-s)^{\frac{q-1}{2}} e^{-\frac{t-s}{2}} e^{-s} \left(\sum_{j=1}^n a_{ij} m_j - \underline{d}_j \right) |\epsilon_j(s)| ds \right]^2 \\
&\le \int_0^t (t-s)^{q-1} e^{-(t-s)} ds \int_0^t (t-s)^{q-1} e^{-(t-s)} \times e^{-2s} \left[\left(\sum_{j=1}^n a_{ij} m_j - \underline{d}_j \right) |\epsilon_j(s)| \right]^2 ds \\
&\le \left[\int_0^t (t-s)^{q-1} e^{-(t-s)} ds \right]^2 \times \left(\sum_{j=1}^n a_{ij} m_j - d_j \right)^2 \sum_{j=1}^n \sup_{t\in[0,T]} \{ e^{-2t} |\epsilon_j(t)|^2 \} \\
&\le \Gamma^2(q) \left(\sum_{j=1}^n a_{ij} m_j - d_j \right)^2 \sum_{j=1}^n \sup_{t\in[0,T]} \{ e^{-2t} |\epsilon_j(t)|^2 \}
\end{aligned}$$

Using the similar approach to F_2, we get

$$\begin{aligned}
F_2 &\le \int_0^t (t-s)^{q-1} e^{-(t-s)} ds \int_0^t (t-s)^{q-1} e^{-(t-s)} e^{-2s} \\
&\left[\sum_{j=1}^n (|\alpha_{ij}| + |\beta_{ij}|) k_j |e_j(s-\tau)| \right]^2 ds \le \Gamma(q) \int_0^\tau (t-s)^{q-1} e^{-(t-s)} e^{-2s} \\
&\times \left[\sum_{j=1}^n (|\alpha_{ij}| + |\beta_{ij}|) k_j |e_j(s-\tau)| \right]^2 ds \le \Gamma(q) \sum_{j=1}^n \left[(|\alpha_{ij}| + |\beta_{ij}|) k_j \right]^2 \\
&\times \sum_{j=1}^n \sup_{t\in[0,T]} e^{-2s} |e_j(s)|^2 \int_0^{t-\tau} (t-s-\tau)^{q-1} e^{-(t-s+\tau)} ds \le e^{-2\tau} \Gamma^2(q) \sum_{j=1}^n \left[(|\alpha_{ij}| + |\beta_{ij}|) k_j \right]^2 \\
&\times \sum_{j=1}^n \sup_{t\in[0,T]} e^{-2s} |e_j(s)|^2
\end{aligned}$$

According to the definition of vector norm and taking F_1, F_2 into the inequality (11), we obtain

$$\begin{aligned}
&\|T(y(t)) - T(x(t))\|^2 = \sum_{i=1}^n \sup_{t\in[0,T]} \{ e^{-2t} |T(y_i) - T(x_i)|^2 \} \\
&\le 2 \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} m_j - \underline{d}_j \right)^2 \sum_{j=1}^n \sup_{t\in[0,T]} \{ e^{-2t} |\epsilon_j(t)|^2 \} \\
&+ 2e^{-2\tau} \sum_{i=1}^n \sum_{j=1}^n \left[\left(|\alpha_{ij}| + |\beta_{ij}| \right) k_j \right]^2 \sum_{j=1}^n \sup_{t\in[0,T]} e^{-2s} |e_j(s)|^2 \\
&\le \delta \|y(t) - x(t)\|^2,
\end{aligned}$$

where $\delta = 2\sum_{i=1}^n \{ (\sum_{j=1}^n a_{ij} m_j - \underline{d}_j)^2 + e^{-2\tau} \sum_{j=1}^n [(|\alpha_{ij}| + |\beta_{ij}|) k_j]^2 \}$. Obviously, we have $\delta \ge 0$.

From the condition (7), we know

$$\|T(y(t)) - T(x(t))\| \le \sqrt{\delta} \|y(t) - x(t)\|.$$

Therefore the map T is a contraction map. According to the Banach fixed point theorem, Eq. (8) has a unique fixed point, so the MFFCNN (3) has a unique solution. Proof finished. □

Remark 2. In the proof of Theorem 1, we mainly use the basic inequality, Cauchy–Schwartz inequality and the property of definite integral to derive the sufficient condition. The proof process does not need to construct the Lyapunov function and is easy to understand.

3.2. Finite-time stability of the MFFCNN

Next, we will derive sufficient conditions for the solution of the MFFCNN (3) to be finite-time stable.

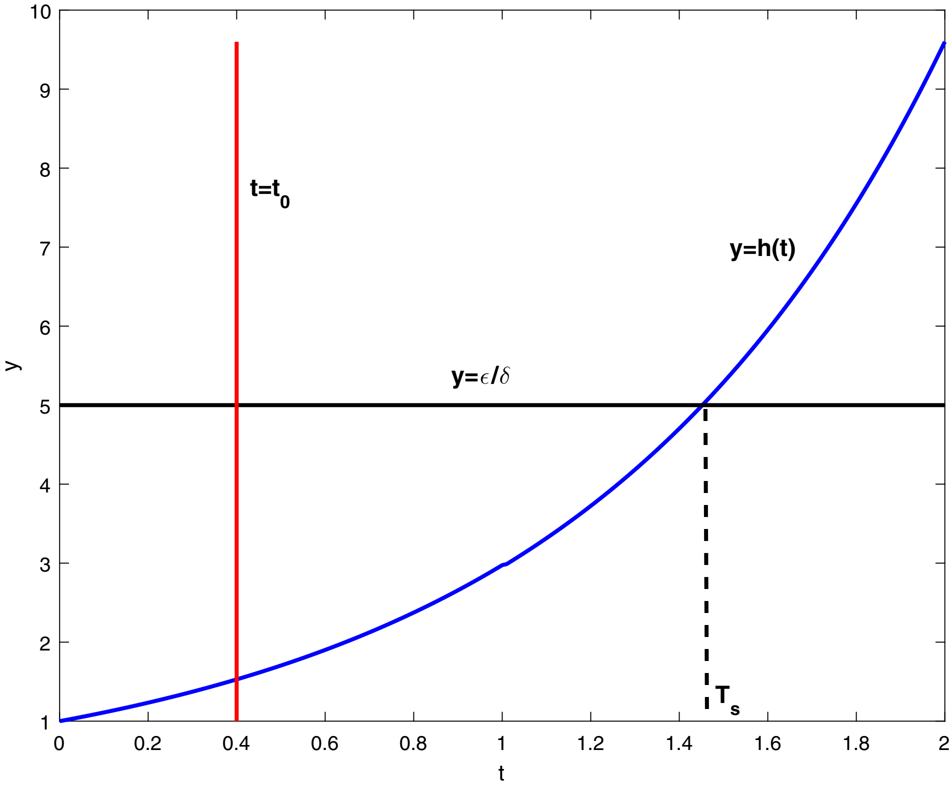

Fig. 1. Diagrammatic sketch of the settling time T_s.

Theorem 2. *Supposing Assumption 1 holds, the MFFCNN (3) is finite-time stable w.r.t.* $\{\sigma, \varepsilon, J\}$, *if the following sufficient condition holds*

$$\left(1 + \frac{\| |\alpha| + |\beta| \| \|\mathcal{K}\|}{\Gamma(q+1)} t^q\right) \times E_q\Big((\|\mathcal{AM} - \mathcal{D}\| + \| |\alpha| + |\beta| \| \|\mathcal{K}\|) t^q\Big) \le \frac{\varepsilon}{\sigma}. \tag{12}$$

where $e(0) = \psi(t) - \varphi(t)$, $\mathcal{A} = (\bar{a}_{ij})_{n\times n}$, $\mathcal{D} = \mathrm{diag}\{\underline{d}_1, \ldots, \underline{d}_n\}$, $\alpha = (\alpha_{ij})_{n\times n}$, $\beta = (\beta_{ij})_{n\times n}$, $\mathcal{M} = \mathrm{diag}\{m_1, m_2, \ldots, m_n\}$, $\mathcal{K} = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}$.

Proof. According to the MFFCNNs (1), (4) and lemma 1, we have

$$\begin{aligned} e_i(t) = \psi_i(t) - \varphi_i(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big[&- d_i e_i(s) + \sum_{j=1}^n a_{ij}(f_j(y_j(s)) - f_j(x_j(s))) \\ &+ \bigwedge_{j=1}^n \alpha_{ij} g_j(y_j(s-\tau)) - \bigwedge_{j=1}^n \alpha_{ij} g_j(y_j(s-\tau)) \\ &+ \bigvee_{j=1}^n \beta_{ij} g_j(y_j(s-\tau)) - \bigvee_{j=1}^n \beta_{ij} g_j(y_j(s-\tau)) \Big] ds \end{aligned} \tag{13}$$

Take the absolute value on both sides of Eq. (13) and using the Lemma 2, we get

$$\begin{aligned} |e_i(t)| \le |\psi_i(t) - \varphi_i(t)| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Bigg[\Bigg(\sum_{j=1}^n a_{ij} m_j - \underline{d}_j \Bigg) |e_j(s)| \\ + \sum_{j=1}^n (|\alpha_{ij}| + |\beta_{ij}|) k_j |e_j(s-\tau)| \Bigg] ds \end{aligned}$$

For the sake of simplicity, the vector form of the above equation is

$$|e(t)| \le |e(0)| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [(\mathcal{AM} - \mathcal{D})|e(s)| + (\alpha + \beta)\mathcal{K}|e(s-\tau)|] ds \tag{14}$$

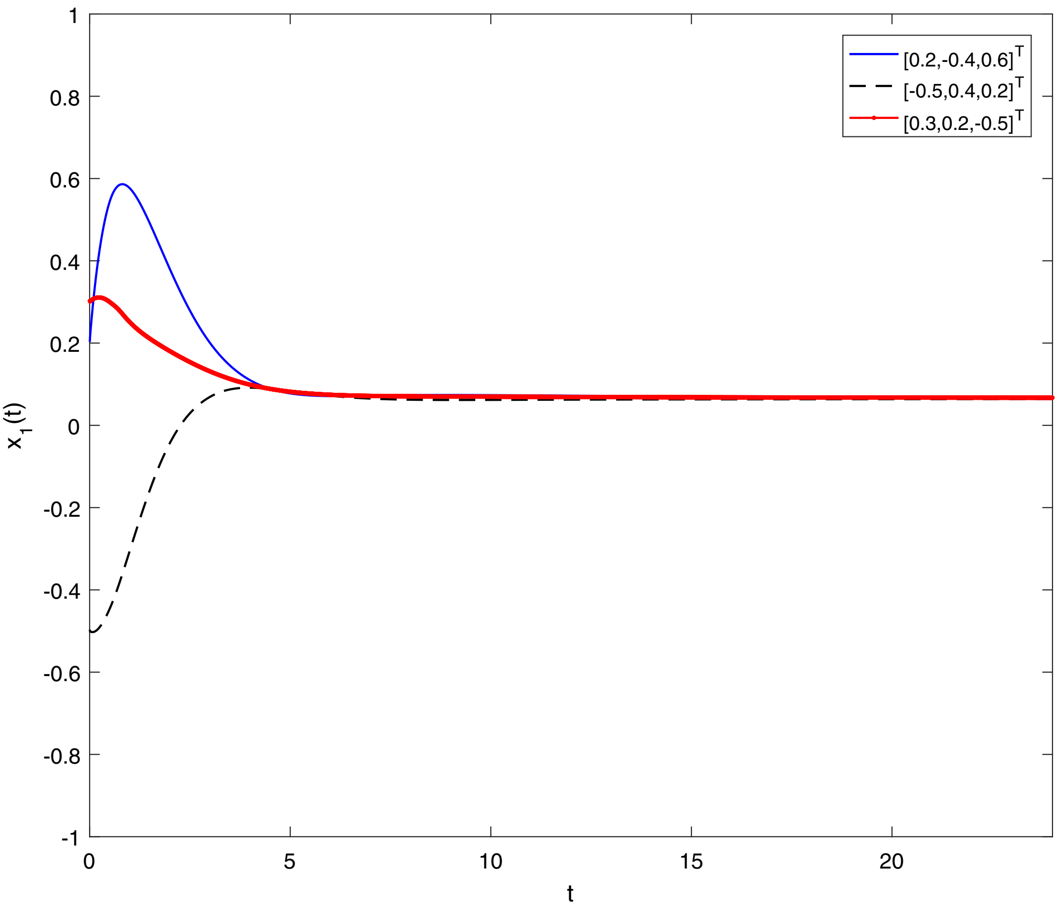

Fig. 2. The first dimensional trajectories of the MFFCNN(22) with the initial values $\varphi(t)=[0.2,-0.4,0.6]^T$, $\varphi(t)=[-0.5,0.4,0.2]^T$, and $\varphi(t)=[0.3,0.2,-0.5]^T$.

Applying the norm on the both sides of Eq. (14), we obtain

$$\begin{aligned}\|e(t)\| &\le \|e(0)\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big[\|\mathcal{AM}-\mathcal{D}\|\|e(s)\| + \||\alpha|+|\beta|\|\|\mathcal{K}\|\|e(s-\tau)\|\Big]ds \\ &\le \|e(0)\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|\mathcal{AM}-\mathcal{D}\|\|e(s)\|ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\||\alpha|+|\beta|\|\|\mathcal{K}\|(\|e(s)\|+\|e(0)\|)ds \\ &= \left(1+\frac{\||\alpha|+|\beta|\|\|\mathcal{K}\|}{\Gamma(q+1)}t^q\right)\|e(0)\| + \frac{\|\mathcal{AM}-\mathcal{D}\|+\||\alpha|+|\beta|\|\|\mathcal{K}\|}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|e(s)\|ds\end{aligned}$$

Let $a(t)=\left(1+\frac{\||\alpha|+|\beta|\|\|\mathcal{N}\|}{\Gamma(q+1)}t^q\right)\|e(0)\|$, $g(t)=\frac{\|\mathcal{AM}-\mathcal{D}\|+\||\alpha|+|\beta|\|\|\mathcal{N}\|}{\Gamma(q)}$. Obviously, $a(t)$ is a nonnegative and nondecreasing function, according to Lemma 3, we get

$$\begin{aligned}\|e(s)\| \le a(t)E_q(g(t)\Gamma(q)t^q) &\le \left(1+\frac{\||\alpha|+|\beta|\|\|\mathcal{K}\|}{\Gamma(q+1)}t^q\right) \\ &\quad\times E_q((\|\mathcal{AM}-\mathcal{D}\|+\||\alpha|+|\beta|\|\|\mathcal{K}\|)t^q)\|e(0)\|\end{aligned}$$

Using the condition (12), we obtain

$$\|e(t)\| < \varepsilon.$$

Therefore the MFFCNN (3) is finite-time stable according the Definition 4. Proof finished. □

If the MFFCNN (3) does not contain fuzzy logic terms, it degrades to a general memristive neural network without delay.

$$D^q x_i(t) = -\tilde{d}(x_i(t))_i x_i(t) + \sum_{j=1}^{n}\tilde{a}_{ij}(x_j(t))f_j(x_j(t))$$

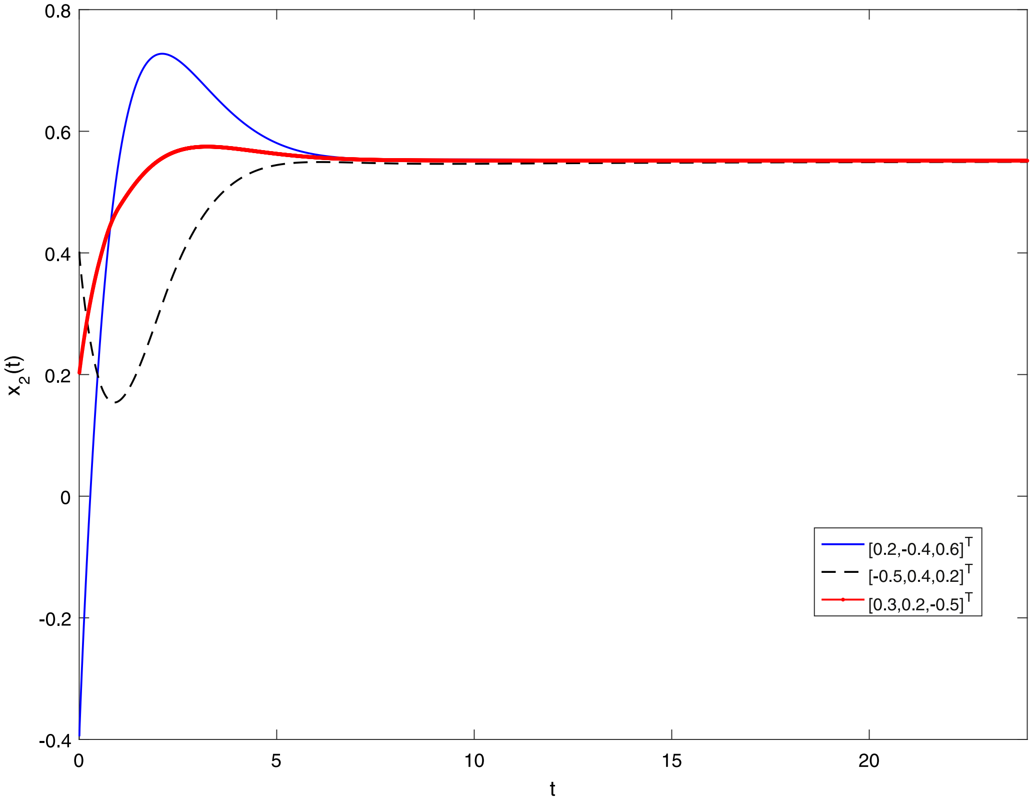

Fig. 3. The second dimensional trajectories of the MFFCNN(22) with the initial values $\varphi(t) = [0.2, -0.4, 0.6]^T$, $\varphi(t) = [-0.5, 0.4, 0.2]^T$, and $\varphi(t) = [0.3, 0.2, -0.5]^T$.

$$+I_i, t \in [0, T], i, j = 1, 2, \ldots, n. \tag{15}$$

From Theorem 2, we easily come to the following corollary.

Corollary 1. *Supposing Assumption 1 holds, the system (15) is finite-time stable w.r.t.* $\{\sigma, \varepsilon, J\}$, *if the following condition holds*

$$E_q\left(\|\mathcal{AM} - \mathcal{D}\| t^q\right) < \frac{\varepsilon}{\sigma}. \tag{16}$$

Remark 3. The condition (16) in Corollary 1 can be seen as a special case of the condition (12) in Theorem 2. Utilizing the condition (12) and the condition (16), we can estimate the settling time T_s. When all the relevant parameters given, we can obtain a T_s by solving the inequality condition (12) or (16). Next, we take Eq. (16) as an example to estimate the T_s. Eq. (16) can be transformed into the following equations:

$$\begin{cases} y = h(t) = E_q\left(\|\mathcal{AM} - \mathcal{D}\| t^q\right), \\ y = \dfrac{\varepsilon}{\delta}, \\ t = t_0. \end{cases}$$

We plot the above equations and mark T_s in the Cartesian coordinate system, as shown in Fig. 1

3.3. Finite-time synchronization of drive-response systems

Finally, we will derive the sufficient conditions to achieve finite-time synchronization under the controller $u(t)$ between the drive MFFCNN system (3) and response MFFCNN system (4). We define the following simple linear feedback controller

$$u_i(t) = -\lambda_i e_i(t), i = 1, 2, \ldots, n, \tag{17}$$

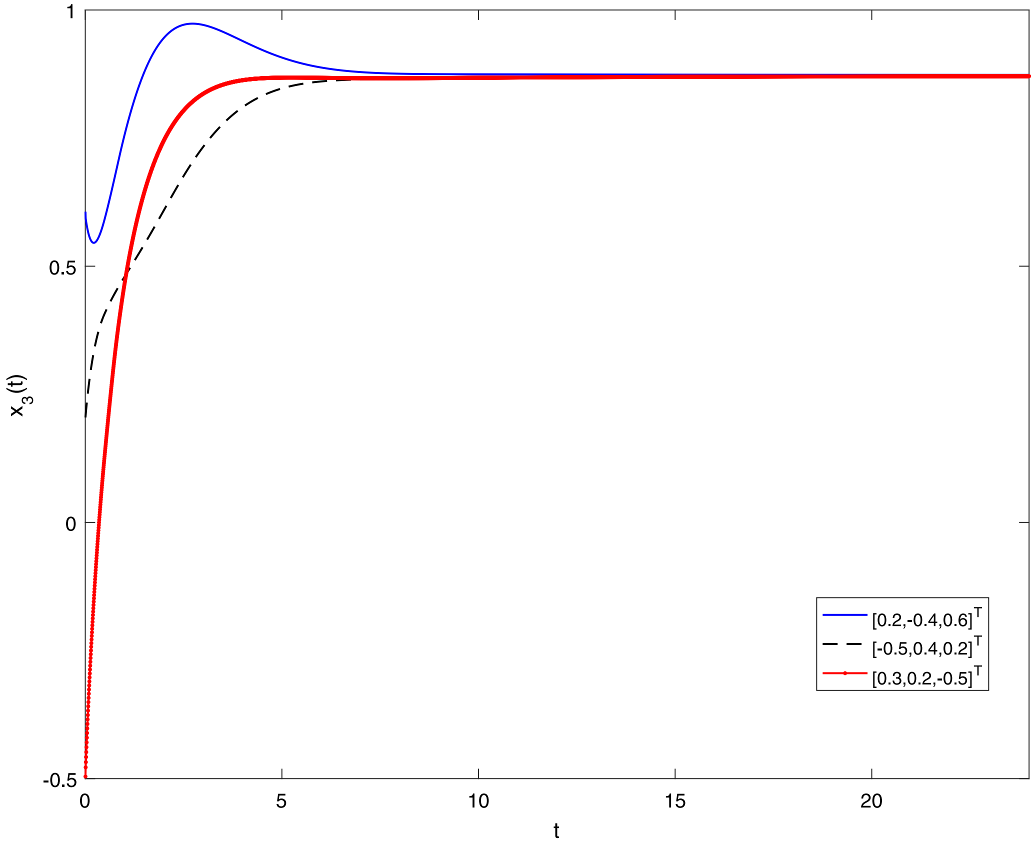

Fig. 4. The third dimensional trajectories of the MFFCNN(22) with the initial values $\varphi(t) = [0.2, -0.4, 0.6]^T$, $\varphi(t) = [-0.5, 0.4, 0.2]^T$, and $\varphi(t) = [0.3, 0.2, -0.5]^T$.

where $\lambda_i > 0$ is the control gain. The error system $e_i(t)$ is defined as

$$\begin{cases} D^q e_i(t) = -(d_i + \lambda_i)e_i(t) + \sum_{j=1}^{n} a_{ij}(f_j(y_j(t)) - f_j(x_j(t))) \\ + \bigwedge_{j=1}^{n} \alpha_{ij} g_j(y_j(t-\tau)) - \bigwedge_{j=1}^{n} \alpha_{ij} g_j(x_j(t-\tau)) \\ + \bigvee_{j=1}^{n} \beta_{ij} g_j(y_j(t-\tau)) - \bigvee_{j=1}^{n} \beta_{ij} g_j(x_j(t-\tau)), t \in [0, T] \\ e_i(t) = \phi_i(t) - \varphi_i(t), t \in [-\tau, 0]. \end{cases} \tag{18}$$

Thus, the finite-time synchronization problem between the drive system (3) and the response system (4) under the controller (17) is transformed into the finite-time stability problem of the error system (18). Similar to Theorem 2, we have the following theorem.

Theorem 3. *Supposing Assumption 1 holds, the drive MFFCNN (3) and the response MFFCNN (4) under the controller (17) can achieve finite-time stable if the following sufficient condition meets.*

$$\left(1 + \frac{\| |\alpha| + |\beta| \| \|\mathcal{K}\|}{\Gamma(q+1)} t^q\right) \times E_q((\|\mathcal{AM} - \mathcal{D} - \Lambda\| + \| |\alpha| + |\beta| \| \|\mathcal{K}\|) t^q) < \frac{\varepsilon}{\sigma}, \tag{19}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *and other parameters are the same as Theorem 2.*

Proof. The proof method is similar to Theorem 2 and is omitted here. □

Remark 4. Although there are many literatures that study the finite-time or exponential stability or synchronization control problem of fuzzy neural networks by the Lyapunov function[41,42,52–54], there is little literature to discuss the stability

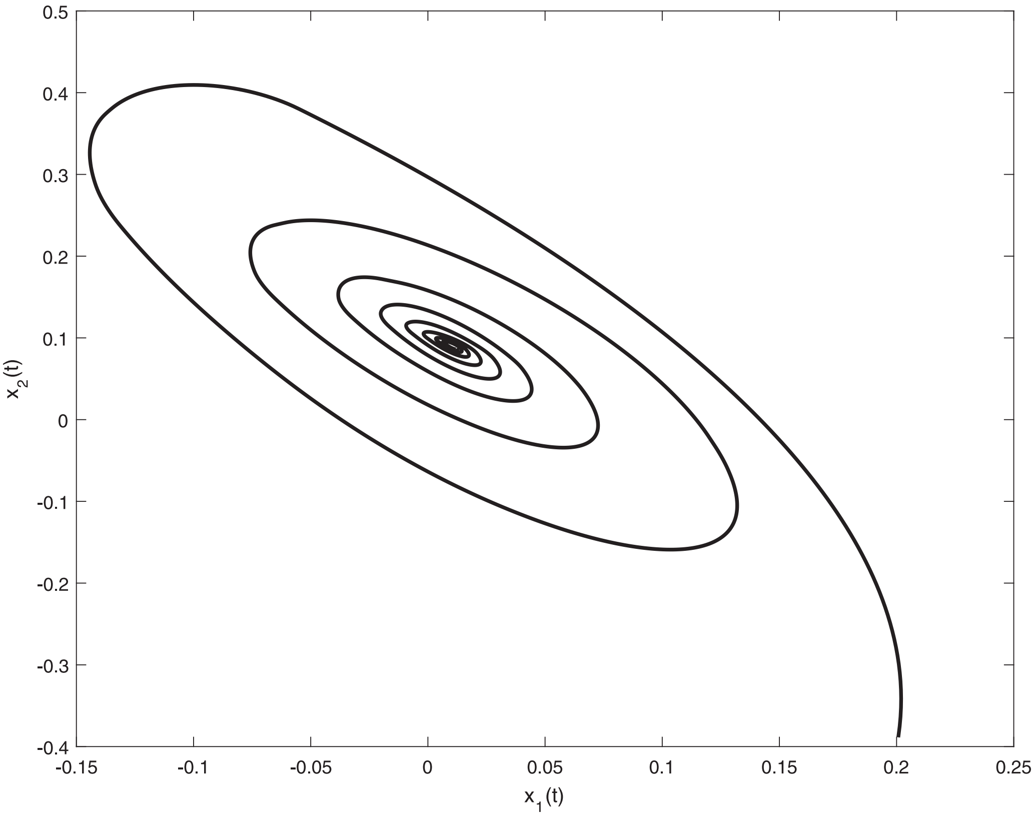

Fig. 5. The phase diagram of the drive the MFFCNN system(23) with the initial value $\varphi(t) = [0.2, -0.4]^T$.

and synchronization of the MFFCNN. Moreover, we are based on the definition of finite-time stability and synchronization to obtain sufficient conditions, which avoid the difficulty finding a suitable Lyapunov function.

Regard the system (15) as the drive system and the corresponding response system without fuzzy logic defined as follows.

$$D^q y_i(t) = -\bar{d}(y_i(t))_i y_i(t) + \sum_{j=1}^{n} \bar{a}_{ij}(y_j(t)) f_j(y_j(t)) + I_i + u_i(t), t \in [0, T], i, j = 1, 2, \ldots, n. \tag{20}$$

Then we have the following corollary.

Corollary 2. *Supposing Assumption 1 holds, the drive system (15) and the response system (20) under the controller (17) can achieve finite-time stable if the following sufficient condition meets.*

$$E_q(\|\mathcal{AM} - \mathcal{D} - \Lambda\| t^q) < \frac{\varepsilon}{\sigma}. \tag{21}$$

4. Numerical simulations

In this section, we will verify the correctness of the main results in this paper through two simulation examples. The modified Adams-Bashforth–Moulton Predictor-Corrector algorithm [55] is used to solve the delay fractional-order differential equations in these two examples.

Example 1. Consider the following 3-D MFFCNN model

$$\begin{aligned} D^q x_i(t) = &-\tilde{d}_i(x_i(t))x_i(t) + \sum_{j=1}^{3} \tilde{a}_{ij}(x_j(t)) f_j(x_j(t)) \\ &+ \sum_{j=1}^{3} b_{ij}\nu_j + \bigwedge_{j=1}^{3} \alpha_{ij} g_j(x_j(t-\tau)) + \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigvee_{j=1}^{3} S_{ij}\nu_j \end{aligned}$$

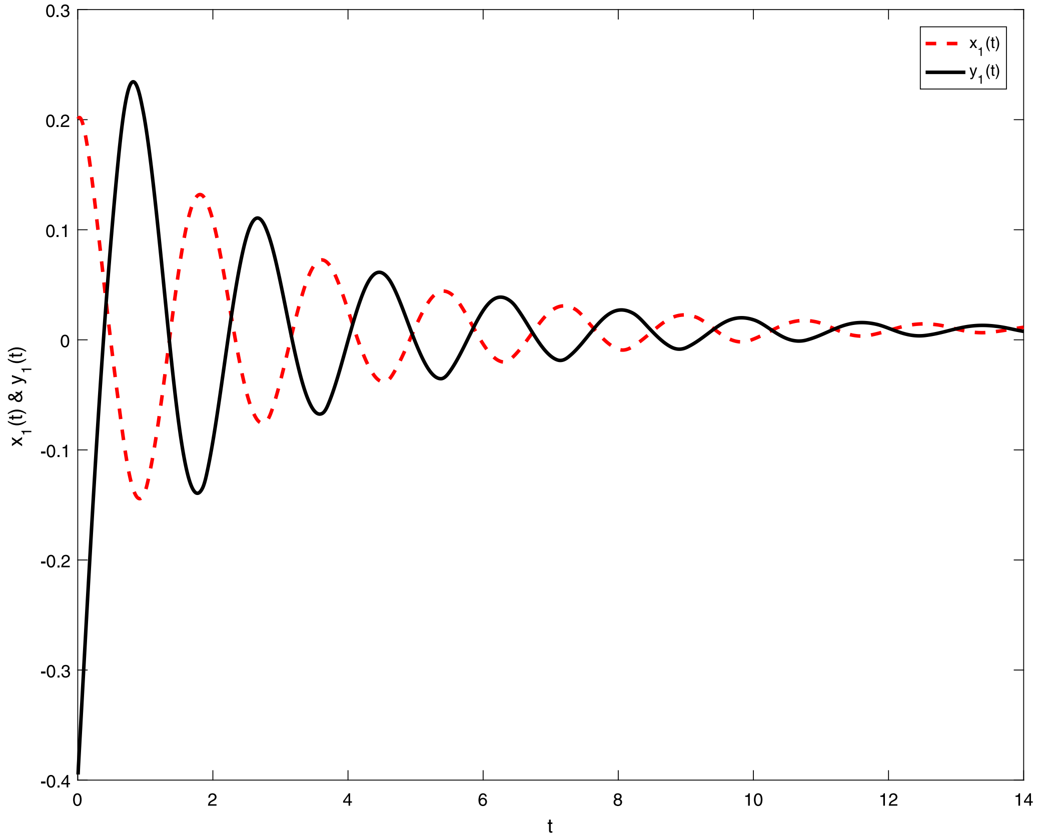

Fig. 6. The first dimensional state trajectories of the drive-response MFFCNN systems(23) and (24) with the initial value $\varphi(t) = [0.2, -0.4]^T, \phi(t) = [-0.4, 0.6]^T$ when no control.

$$
\begin{aligned}
&+\bigvee_{j=1}^{3}\beta_{ij}g_j(x_j(t-\tau))+I_i,\\
&i=1,2,3, t\in[0,T],
\end{aligned}
\tag{22}
$$

where the parameters of Eq. (22) are listed below

$$
\tilde{d}_1(x_1)=\begin{cases}0.1, |x_1|\le 1,\\ 0.2, |x_1|>1.\end{cases}\qquad \tilde{d}_2(x_2)=\begin{cases}0.2, |x_2|\le 1,\\ 0.1, |x_2|>1.\end{cases}
$$

$$
\tilde{d}_3(x_3)=\begin{cases}1.2, |x_3|\le 1,\\ 0.2, |x_3|>1.\end{cases}\qquad \tilde{a}_{11}(x_1)=\begin{cases}0.2, |x_1|\le 1,\\ 0.1, |x_1|>1.\end{cases}
$$

$$
\tilde{a}_{12}(x_1)=\begin{cases}-1.2, |x_1|\le 1,\\ 1, |x_1|>1.\end{cases}\qquad \tilde{a}_{13}(x_1)=\begin{cases}0.1, |x_1|\le 1,\\ 0.2, |x_1|>1.\end{cases}
$$

$$
\tilde{a}_{21}(x_2)=\begin{cases}0.2, |x_2|\le 1,\\ 0.4, |x_2|>1.\end{cases}\qquad \tilde{a}_{22}(x_2)=\begin{cases}-1.2, |x_2|\le 1,\\ 1.0, |x_2|>1.\end{cases}
$$

$$
\tilde{a}_{23}(x_2)=\begin{cases}0.1, |x_2|\le 1,\\ 0.2, |x_2|>1.\end{cases}\qquad \tilde{a}_{31}(x_3)=\begin{cases}0.2, |x_3|\le 1,\\ 0.4, |x_3|>1.\end{cases}
$$

$$
\tilde{a}_{32}(x_3)=\begin{cases}1.1, |x_3|\le 1,\\ 0.2, |x_3|>1.\end{cases}\qquad \tilde{a}_{33}(x_3)=\begin{cases}-0.5, |x_3|\le 1,\\ 0.5, |x_3|>1.\end{cases}
$$

Note: in order to simplify the expression, x_i represents $x_i(t)$ here.

$$
f_i(x)=g_i(x)=0.5(|x+1|-|x-1|),
$$

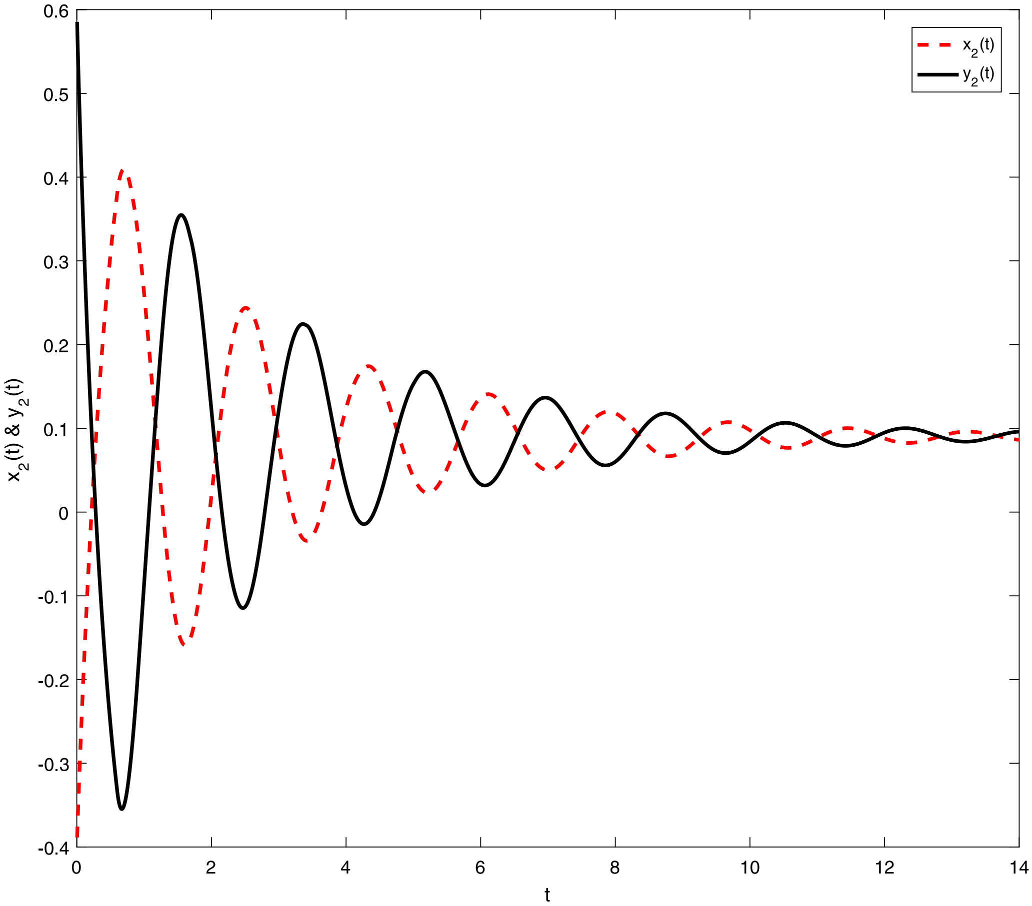

Fig. 7. The second dimensional state trajectories of the drive-response MFFCNN systems(23) and (24) with the initial value $\varphi(t) = [0.2, -0.4]^T, \phi(t) = [-0.4, 0.6]^T$ when no control.

$$\alpha = \begin{bmatrix} -0.1 & -0.01 & 0.1 \\ -0.2 & -0.1 & 0.1 \\ -0.04 & -0.2 & 0.4 \end{bmatrix}; \beta = \begin{bmatrix} -0.1 & -0.01 & 0.3 \\ -0.1 & -0.2 & 0.2 \\ -0.1 & -0.2 & 0.3 \end{bmatrix};$$
$$B = \begin{bmatrix} 0.1 & 0.1 & -0.1 \\ 0.1 & 0.1 & -0.2 \\ 0.2 & 0.1 & 0.2 \end{bmatrix}; T = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.2 & 0.2 & -0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix};$$
$$S = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}; V = [1; 2; 1];$$
$$\tau = 0.5; I = [0; 0; 0],$$

where $\alpha = (\alpha_{ij})_{3\times3}, \beta = (\beta_{ij})_{3\times3}, B = (b_{ij})_{3\times3}, T = (T_{ij})_{3\times3}, S = (S_{ij})_{3\times3}, V = [v1; v2; v3], i, j = 1, 2, 3.$

We first verify the existence of the solution of the MFFCNN (22), i.e. the condition (7) of Theorem 1. Selecting the initial values $\varphi(t) = [0.2, -0.4, 0.6]^T$, $\varphi(t) = [-0.5, 0.4, 0.2]^T$, $\varphi(t) = [0.3, 0.2, -0.5]^T, t \in [-0.5, 0]$, memristor-based weight values $\mathcal{D} = \text{diag}\{0.1, 0.1, 0.2\}, \mathcal{A} = \begin{bmatrix} 0.2 & 1 & 0.2 \\ 0.4 & 1 & 0.2 \\ 0.4 & 1.1 & 0.5 \end{bmatrix}$, and Lipschitz constants $m = [0.2, 0.2, 0.2]^T, k = [0.1, 0.1, 0.1]^T$, respectively, by simple calculations, we obtain $\sum_{i=1}^{n}\{(\sum_{j=1}^{n} a_{ij}m_j - \underline{d}_j)^2 + e^{-2\tau}\sum_{j=1}^{n}[(|\alpha_{ij}| + |\beta_{ij}|)k_j]^2\} = 0.1315 < \frac{1}{2}$. It satisfies the condition (7) of Theorem 1. This shows that the MFFCNN (22) has a unique solution $x^* = [0.0689, 0.5648, 0.8840]^T$.

Then we verify the finite-time stability of the solution x^* of the MFFCNN (3), i.e. the condition (12) in *Theorem 2*. We choose the initial value of the MFFCNN (22) as $\varphi(t) = [0.2, -0.4, 0.6]^T$, and its norm is $\|e(0)\| = \|\varphi(t)\| \approx 0.74$. Take $\sigma = 1 > 0.74$ and $\varepsilon = 10 > \sigma$. By solving the condition (12), the solution of the MFFCNN (22) can achieve finite-time stability when $T_s \geq 4.2968$. Figs. 2–4 show the state trajectory curves.

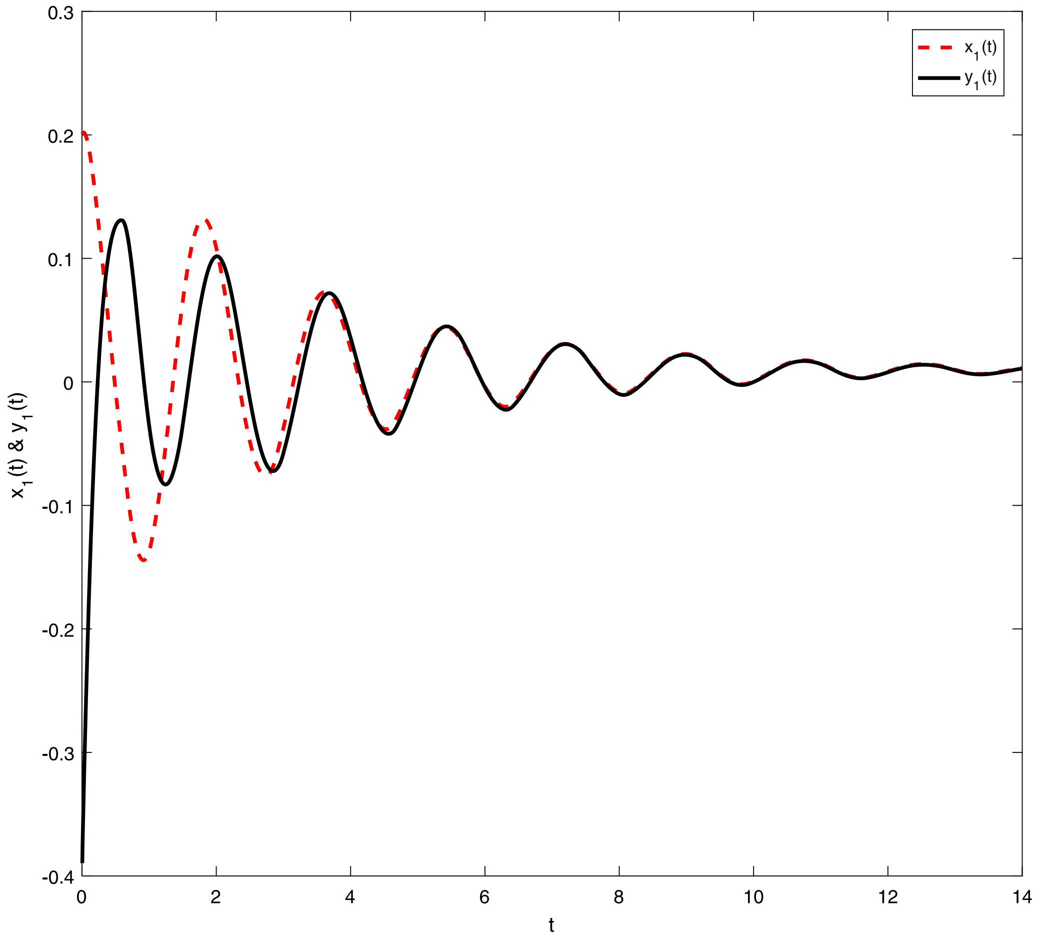

Fig. 8. The first dimensional state trajectories of the drive-response MFFCNN systems(23) and (24) with the initial value $\varphi(t) = [0.2, -0.4]^T, \phi(t) = [-0.4, 0.6]^T$ under control.

Remark 5. In Example 1, we select the initial values of Eq. (22), $\varphi(t) = [0.2, -0.4, 0.6]^T$, $\varphi(t) = [-0.5, 0.4, 0.2]^T$, $\varphi(t) = [0.3, 0.2, -0.5]^T$ as examples to verify the condition (7) in Theorem 1. Figs. 2–4 show that Eq. (22) is eventually stabilized in the point $x^* = [0.0689, 0.5648, 0.8840]^T$ when three different initial values above are taken. Figs. 2–4, respectively illustrate the first, the second and third dimensional state trajectories of Eq. (22).

Example 2. Consider the following 2-D drive-response MFFCNN systems

$$\begin{aligned} D^q x_i(t) = & -\tilde{d}_i(x_i(t))x_i(t) + \sum_{j=1}^{2} \tilde{a}_{ij}(x_j(t))f_j(x_j(t)) \\ & + \sum_{j=1}^{2} b_{ij}v_j + \bigwedge_{j=1}^{2} \alpha_{ij}g_j(x_j(t-\tau)) \\ & + \bigwedge_{j=1}^{n} T_{ij}v_j + \bigvee_{j=1}^{2} S_{ij}v_j + \bigvee_{j=1}^{2} \beta_{ij}g_j(x_j(t-\tau)) + I_i, \\ & i = 1, 2, t \in [0, T], \end{aligned} \tag{23}$$

$$\begin{aligned} D^q y_i(t) = & -\tilde{d}_i(y_i(t))y_i(t) + \sum_{j=1}^{2} \tilde{a}_{ij}(y_j(t))f_j(y_j(t)) \\ & + \sum_{j=1}^{2} b_{ij}v_j + \bigwedge_{j=1}^{2} \alpha_{ij}g_j(y_j(t-\tau)) \end{aligned}$$

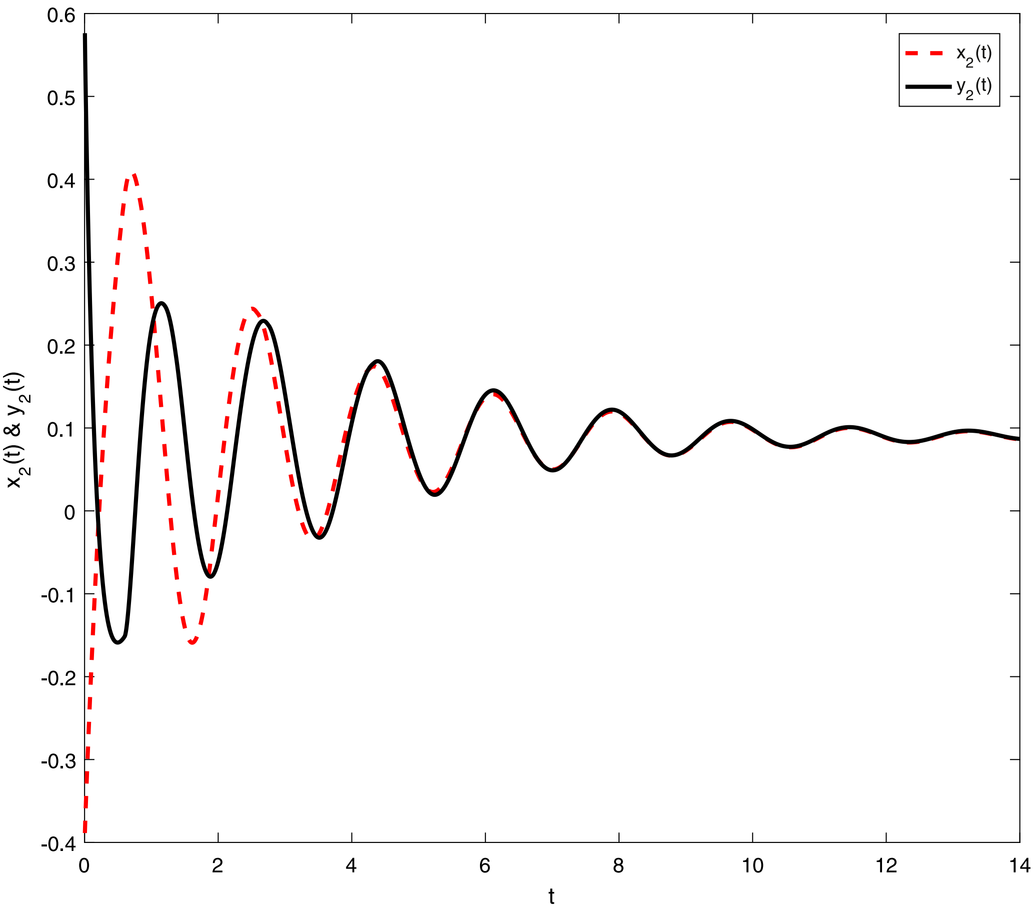

Fig. 9. The second dimensional state trajectories of the drive-response MFFCNN systems(23) and (24) with the initial value $\varphi(t) = [0.2, -0.4]^T, \phi(t) = [-0.4, 0.6]^T$ under control.

$$+\bigwedge_{j=1}^{n} T_{ij}v_j + \bigvee_{j=1}^{2} S_{ij}v_j + \bigvee_{j=1}^{2} \beta_{ij}g_j(y_j(t-\tau)) + I_i + u_i(t),$$
$$i = 1, 2, t \in [0, T], \tag{24}$$

where the parameters of the drive system (23) are as follows

$$\tilde{d}_1(x_1) = \begin{cases} 2.0, |x_1| \le 1, \\ 2.5, |x_1| > 1. \end{cases} \qquad \tilde{d}_2(x_2) = \begin{cases} 0.2, |x_2| \le 1, \\ 0.6, |x_2| > 1. \end{cases}$$
$$\tilde{a}_{11}(x_1) = \begin{cases} -1.8, |x_1| \le 1, \\ -1.5, |x_1| > 1. \end{cases} \qquad \tilde{a}_{12}(x_1) = \begin{cases} -2.0, |x_1| \le 1, \\ 3.5, |x_1| > 1. \end{cases}$$
$$\tilde{a}_{21}(x_2) = \begin{cases} 1.2, |x_2| \le 1, \\ 0.8, |x_2| > 1. \end{cases} \qquad \tilde{a}_{22}(x_2) = \begin{cases} -1.0, |x_2| \le 1, \\ -1.2, |x_2| > 1. \end{cases}$$

$$f_i(x) = g_i(x) = 0.5(|x+1| - |x-1|),$$
$$\alpha = \begin{bmatrix} -1.6 & -0.2 \\ -0.4 & -2.8 \end{bmatrix}; \quad \beta = \begin{bmatrix} -1.2 & -0.4 \\ -0.1 & -2.4 \end{bmatrix};$$
$$B = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}; \quad T = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}; \quad \varphi(t) = [0.2, -0.4]^T,$$
$$S = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}; V = [0.5; 0.5]; \tau = 0.6; I = [0; 0],$$

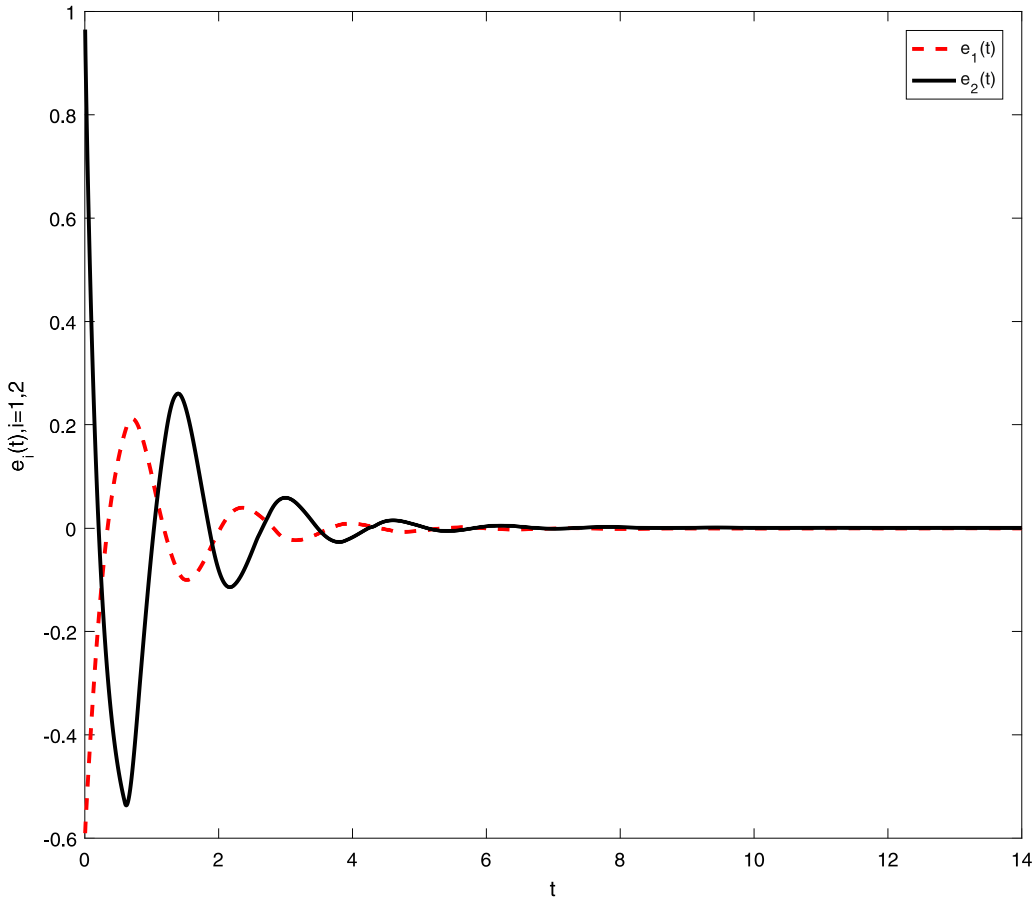

Fig. 10. The error curves of the drive-response MFFCNN systems(23) and (24) with the initial value $\varphi(t) = [0.2, -0.4]^T, \phi(t) = [-0.4, 0.6]^T$ under control.

Fig. 5 shows the phase diagram of the drive MFFCNN system (8).

In the response MFFCNN system (24), let $u_i(t) = -\lambda_i e_i(t)$, where makes $\Lambda = \text{diag}\{1, 1\}, e_i(t) = y_i(t) - x_i(t)$.

Next, we will verify the condition (15) of Theorem 3 and calculate the settling time T_s to achieve synchronization.

We choose the initial value of the response MFFCNN system (24) as $\phi(t) = [-0.4, 0.6]^T$, and the norm of the error system is $\|e(0)\| = \|\phi(t) - \varphi(t)\| \approx 0.721$. Take $m_1 = 0.2, m_2 = 0.2, k_1 = 0.2, k_2 = 0.2,\ \sigma = 1.1 > 0.721$ and $\varepsilon = 100 > \sigma$. By solving the condition (19), the drive-response systems can achieve finite-time synchronization when $T_s \geq 1.1585$.

Figs. 6 and 7 illustrate the state trajectories when no control.

Figs. 8 and 9 depict the state trajectories under control.

The error curve is shown in Fig. 10.

Remark 6. In the two examples above, we use the modified Adams–Bashforth–Moulton Predictor-Corrector algorithm to compute the fractional-order differential equations. We select the computation step length $h = 0.005$.

Remark 7. From the simulation results in above figures, we can see that the real running time after the system stability is larger than the computation time, because the computer spends more time to solve the fractional-order differential equations.

5. Conclusion

The theoretical research on the MFFCNN is still relatively few. It is a challenge to study the MFFCNN, which has more complex dynamic behaviors because of the combination of the memristor, fractional-order calculus and fuzzy logic in the neural network. In this paper, we try to investigate the finite-time stability for the MFFCNN and finite-time synchronization for drive-response MFFNNs. With the help of definitions of finite-time stability and synchronization, Banach fixed point theorem and Gronwall–Bellman inequality, some sufficient conditions are obtained to ensure the existence and the finite-time stability of MFFCNN and the finite-time synchronization of drive-response MFFCNN systems. These conditions are relatively easy to verify. Finally, two numerical examples are presented to show the effectiveness of our theoretical results. In addition, the limitations of this study, suggested improvements and future work are outlined below: (1) The linear feedback controller $u(t)$ has certain hysteresis. Nonlinear feedback or adaptive feedback controller can be considered in the future; (2) How to analyze the dynamic behaviors of the MFFCNN with mixed time-varying delays and stochastic perturbations? (3) Study more advanced and appropriate mathematical theory for the MFFCNN model; (4) Explore the application of the MFFCNN model in practical engineering problems.

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