Finite Time Synchronization For Delayed Fuzzy Inertial Cellular Neural Networks

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***Abstract*—This paper focuses on the finite-time synchronization for a class of fuzzy inertial cellular neural networks (FICNNs) with time-varying delays. First, by constructing a proper variable substitution, the original FICNNs with time-varying delays can be rewritten as first-order differential system. Second, based on Lyapunov functionals, we derive some new and sufficient conditions of finite-time synchronization for the addressed system. We illustrate the effectiveness of the approach through a few examples.**

***Index Terms*—fuzzy inertial neural networks, finite-time synchronization, time-varying delays**

I. Introduction

During the past decades, artificial neural networks have been widely investigated due to their extensive applications in pattern recognition, image processing, optimization solvers, artificial intelligence and so on [1]– [10]. On the other hand, chaos synchronization and control [2]– [4], [11]– [15] is one of the most important dynamics of a designed neural network and have been extensively investigated recently because of their potential application in many fields such as secure communication, biological and chemical reactions, human heartbeat regulation, information science, image processing and other engineering areas [16]. The synchronization problem can be divided into two types of synchronization: one is the infinite time synchronization, the other is the finite-time synchronization. Contrary to the concept of exponential and asymptotic synchronization, the finite-time synchronization requires the master and the slave system remain completely identical after some finite time called the ”settling time”.

In 1996, Yang and Yang studied and suggested fuzzy cellular neural networks (FCNNs) [17], [18] as another type cellular neural networks (CNNs) model, which combined fuzzy operation ”fuzzy OR and fuzzy AND” with CNNs. Recently, the analysis of finite-time synchronization FCNNs with delays has received much attention [19], [20]. In [19], the authors studied the finite-time synchronisation of the FCNNs with time-varying delays. The problem of finite-time synchronisation for a class of fuzzy cellular neural networks with time-varying coefficients and proportional delays has been investigated in [20].

To the best of our knowledge, no major investigation on finite-time synchronisation of FICNNs have been carried out. Motivated by the above, the main purpose of this paper is to study the finite-time synchronisation of FICNNs with time-varying delays. By using Lyapunov functionals and analytical techniques, we obtain the sufficient condition ensuring finite-time synchronisation of class of neural networks.

II. Problem Formulation and Hypotheses

For convenience, let $\mathbb{R}$ denote the set of real numbers. $\mathbb{R}^n$ denotes the set of all n-dimensional real vectors (real numbers).

For any $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $\|x\|$ is the square norm defined by $\|x\| = (\sum_{j=1}^{n} x_i^2)^{\frac{1}{2}}$.

In this paper, the following delayed fuzzy inertial neural networks are considered:

$$\begin{aligned}\frac{d^2x_i(t)}{dt^2} &= -a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{n} c_{ij}f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} d_{ij}f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{n} e_{ij}\nu_j \\ &+ \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigwedge_{j=1}^{n} \alpha_{ij}f_j(x_j(t-\tau_j(t))) \\ &+ \bigvee_{j=1}^{n} \beta_{ij}f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{n} S_{ij}\nu_j + I_i, \quad (1)\end{aligned}$$

where $n \geq 2$, $t \geq t_0$ $i = 1, 2\cdots, n$, where the second derivative is called an inertial term of system (1); $x_i(t)$ denote the state of the i^{th} neuron at time t; $a_i > 0$, $b_i > 0$ are constants; c_{ij} and d_{ij} are elements of feedback templates and e_{ij} is feed-forward template; α_{ij}, β_{ij}, T_{ij} and S_{ij} donate elements of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; $\bigwedge$ denote the fuzzy AND operation and $\bigvee$ is the fuzzy OR operation; ν_i denote

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the input of the i^th neuron; I_i is the bias of the i^th neuron; $f_j(t)$ denote the activation function of j^{th} neuron at time t , $i, j = 1, 2, \cdots, n$; $\tau_j(t) \geq 0$ is the time delay of jth neuron and corresponds to finite speed of axonal signal transmission at time t.

The initial values of system (1) are,

$$x_i(s) = \varphi_i(s), \quad \frac{dx_i(s)}{dt} = \psi_i(s), \quad s \in [-\tau, t_0], \tag{2}$$

for $i = 1, 2, \cdots, n$, where $\tau = \max_{1\leq j\leq n}\{\sup_{t\in\mathbb{R}^+} \tau_j(t)\}$, $\varphi_i(s),\ \psi_i(s) \in C([-\tau, t_0], \mathbb{R}^n)$ with $C([-\tau, t_0], \mathbb{R}^n)$ denotes the Banach space of all continuous functions mapping $[-\tau, t_0]$ into $\mathbb{R}^n$.

To derive the main results, we assume that the following conditions hold:

(H1) The activation functions $f_j(.)$ satisfy the Lipschitz condition, i.e., there exist constant $L_j^f > 0$, such that $|f_j(x)-f_j(y)| \leq L_j^f|x-y|, x, y \in \mathbb{R}$, for $j = 1, 2, \cdots, n$.

(H2) For each $j = 1, 2, \cdots, n$, the activation function $f_j(.)$ is bounded. That is, there exists a positive constant M_j such that,
$|f_j(u)| \leq M_j$, $\forall u \in \mathbb{R}$.

For constant ξ_i, the following transformation is employed:

$$y_i(t) = \frac{dx_i(t)}{dt} + \xi_i x_i(t), \ \ i = 1, 2, \cdots, n,$$

then the inertial neural network (1) can be written as

$$\begin{cases} \frac{dx_i(t)}{dt} &= -\xi_i x_i(t) + y_i(t) \\ \frac{dy_i(t)}{dt} &= -\theta_i y_i(t) + \delta_i x_i(t) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} d_{ij} f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{n} e_{ij}\nu_j \\ &+ \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t-\tau_j(t))) \\ &+ \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{n} S_{ij}\nu_j \\ &+ I_i,\ i = 1, 2, \cdots, n. \end{cases} \tag{3}$$

and the initial conditions can be written as:

$$\begin{cases} x_i(s) &= \varphi_i(s), \\ y_i(s) &= \xi_i\varphi_i(s) + \psi_i(s) = \phi_i(s), \ \ s \in [-\tau, t_0], \end{cases} \tag{4}$$

where $\theta_i = a_i - \xi_i$, $\delta_i = \xi_i\theta_i - b_i$.

In this paper, we will make drive-response chaotic neural networks with delays achieve synchronization in finite time by designing some effective controllers. Based on the concept of drive-response synchronization, the corresponding response system of (3) is given in the following form:

$$\begin{cases} \frac{du_i(t)}{dt} &= -\xi_i u_i(t) + v_i(t) + A_i(t) \\ \frac{dv_i(t)}{dt} &= -\theta_i v_i(t) + \delta_i u_i(t) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} d_{ij} f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{n} e_{ij}\nu_j(t) \\ &+ \bigwedge_{j=1}^{n} T_{ij}\nu_j + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t-\tau_j(t))) \\ &+ \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{n} S_{ij}\nu_j \\ &+ I_i + B_i(t),\ i = 1, 2, \cdots, n, \end{cases} \tag{5}$$

where $A = (A_1, A_2, \cdots, A_n)^T$ and $B = (B_1, B_2, \cdots, B_n)^T$ are the controller that will be appropriately designed for fixed/finite time synchronization objective.

Define error states $e_i(t) = u_i(t) - x_i(t)$ and $\epsilon_i(t) = v_i(t) - y_i(t)$, we can derive the following error system

$$\begin{cases} \frac{de_i(t)}{dt} &= -\xi_i e_i(t) + \epsilon_i(t) + A_i(t) \\ \frac{d\epsilon_i(t)}{dt} &= -\theta_i\epsilon_i(t) + \delta_i e_i(t) + \sum_{j=1}^{n} c_{ij} F_j(e_j(t)) \\ &+ \sum_{j=1}^{n} d_{ij} F_j(e_j(t-\tau_j(t))) \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij} F_j(e_j(t-\tau_j(t))) \\ &+ \bigvee_{j=1}^{n} \beta_{ij} F_j(e_j(t-\tau_j(t))) + B_i(t), \end{cases} \tag{6}$$

where $F_j(e_j(t)) = f_j(u_j(t)) - f_j(x_j(t))$ and $F_j(e_j(t - \tau_j(t))) = f_j(u_j(t-\tau_j(t))) - f_j(x_j(t-\tau_j(t)))$.

Remark 1: Based on Assumption **(H1)** and **(H2)**, we conclude that $F_j(.)$ is bounded and satisfies:
$|F(e_j(t))| \leq L_j^f|e_j(t)|$, $|F_j(.)| \leq K_j$,
where $K_j \geq 0$ is a constant.

III. Definitions and Lemmas

In this section, we introduce some definitions and state some preliminary results

Definition 1: The system (5) is said to be synchronized with (3) in finite-time if, under suitable designed feedback controllers $A_i(t)$ and $B_i(t)$, there exists a constant $T > 0$ such that:
$\lim_{t\to T} \|u_i(t) - x_i(t)\| = 0$, $\lim_{t\to T} \|v_i(t) - y_i(t)\|$
and
$\|u_i(t) - x_i(t)\| \equiv 0$, $\|v_i(t) - y_i(t)\| \equiv 0$ for $t > T$, $i = 1, 2, \cdots, n$.
T (namely, settling time or halting time).

Definition 2: The upper right Dini derivative of a function $V : \mathbb{R}^n \to \mathbb{R}_+$ is indicated with the symbol $D^+V(.)$ and is defined as:
$D^+V(t) = \lim_{h\to 0^+} \sup \frac{1}{h}(V(t+h) - V(t))$.

Lemma 1: [19]Let $x_j,\ y_j,\ \alpha_{ij},\ \beta_{ij} \in \mathbb{R},\ f_j : \mathbb{R} \to \mathbb{R}$ be continuous functions, and $i, j = 1, 2, \cdots, n$, then the following inequalities hold

$$\Big|\bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(y_j)\Big| \leq \sum_{j=1}^{n} |\alpha_{ij}||f_j(x_j) - f_j(y_j)|,$$

$$\Big|\bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(y_j)\Big| \leq \sum_{j=1}^{n} |\beta_{ij}||f_j(x_j) - f_j(y_j)|.$$

Lemma 2: [21] Let $x = (x_1, x_2, \cdots, x_n) \geq 0$, $0 < p \leq 1$ the following inequalitie hold:
$\sum_{i=1}^{n} x_i^p \geq \left(\sum_{i=1}^{n} x_i \right)^p.$

Lemma 3: [14] Assume that a continuous, positive-definite function $V(t)$ satisfies the following inequality

$$\dot{V}(t) \leq -\gamma V^{\eta}(t),\ \ \forall t \geq t_0,\ \ V(t_0) \geq 0,$$

where $\gamma > 0$, $0 < \eta < 1$ are two constants. Then $V(t)$ satisfies

$$V^{1-\eta}(t) \leq V^{1-\eta}(t_0) - \gamma(1-\eta)(t - t_0),\ \ t_0 \leq t \leq T$$

and

$$V(t) = 0,\ \ \forall t \geq T,$$

with T given by

$$T = t_0 + \frac{V^{1-\eta}(t_0)}{\gamma(1-\eta)}$$

IV. Main results

In section IV, we will derive some criteria to guarantee the finite-time synchronization between drive system (3) and response system (5). First, an adaptive feedback controller is defined as follows:

$$\begin{cases} A_i(t) &= -k_{i1} e_i(t) - k_{i2} sign(e_i(t))|e_i(t)|^{\eta} \\ B_i(t) &= -\eta_{i1}\epsilon_i(t) - \eta_{i2} sign(\epsilon_i(t))|\epsilon_i(t)|^{\eta} \\ &- \sum_{j=1}^{n} d_{ij} F_j(e_j(t-\tau_j(t))) \\ &- \bigwedge_{j=1}^{n} \alpha_{ij} F_j(e_j(t-\tau_j(t))) \\ &- \bigvee_{j=1}^{n} \beta_{ij} F_j(e_j(t-\tau_j(t))), \end{cases} \tag{7}$$

where $0 < \eta < 1$, $k_{i1},\ k_{i2},\ \eta_{i1},\ \eta_{i2}$ are the parameters to be designed later.

Theorem 1: Under Assumption **(H1)**, the drive-response systems (3) and (5) will achieve finite-time synchronization under controller (7) if the following conditions hold

$$k_{i1} \geq -\xi_i + \frac{1}{2} + \frac{1}{2}|\delta_i| + \sum_{j=1}^{n} \frac{1}{2}|c_{ji}| l_i, \tag{8}$$

$$\eta_{i1} \geq -\theta_i + \frac{1}{2} + \frac{1}{2}|\delta_i| + \sum_{j=1}^{n} \frac{1}{2}|c_{ij}| l_j, \tag{9}$$

$$k_{i2} > 0,\ \eta_{i2} > 0. \tag{10}$$

Moreover, the finite time t_1 for synchronization satisfies

$$t_1 \leq t_0 + \frac{2}{\Lambda(1-\eta)} V^{\frac{1-\eta}{2}}(t_0) \tag{11}$$

where $V(t_0) = \frac{1}{2}\sum_{i=1}^{n} e_i^2(t_0) + \frac{1}{2}\sum_{i=1}^{n} \epsilon_i^2(t_0)$,
$\Lambda = \min\{k_{i2}, \eta_{i2}\} 2^{\frac{\eta+1}{2}}$.

Proof: Consider the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{n} e_i^2(t) + \frac{1}{2}\sum_{i=1}^{n} \epsilon_i^2(t)$$

Calculating the derivative of $V(t)$ along the solution of system (6) and by using Lemma 1 we have

$$\begin{aligned} V'(t) &= \sum_{i=1}^{n} e_i(t)\dot{e}_i(t) + \sum_{i=1}^{n} \epsilon_i(t)\dot{\epsilon}_i(t) \\ &= \sum_{i=1}^{n} e_i(t)\Big[-\xi_i e_i(t) + \epsilon_i(t) + A_i(t) \Big] \\ &+ \sum_{i=1}^{n} \epsilon_i(t)\Big[-\theta_i \epsilon_i(t) + \delta_i e_i(t) \\ &+ \sum_{j=1}^{n} c_{ij} F_j(e_j(t)) + \sum_{j=1}^{n} d_{ij} F_j(e_j(t-\tau_j(t))) \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij} F_j(e_j(t-\tau_j(t))) \\ &+ \bigvee_{j=1}^{n} \beta_{ij} F_j(e_j(t-\tau_j(t))) + B_i(t) \Big] \\ &\leq \sum_{i=1}^{n} \Big[-\xi_i e_i^2(t) + e_i(t)\epsilon_i(t) - k_{i1} e_i^2(t) \\ &- k_{i2}|e_i(t)|^{\eta+1} \Big] + \sum_{i=1}^{n} \Big[-\theta_i \epsilon_i^2(t) + \delta_i \epsilon_i(t) e_i(t) \\ &+ \sum_{j=1}^{n} c_{ij}\epsilon_i(t) F_j(e_j(t)) \\ &+ \sum_{j=1}^{n} d_{ij}\epsilon_i(t) F_j(e_j(t-\tau_j(t))) \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij}\epsilon_i(t) F_j(e_j(t-\tau_j(t))) \\ &+ \bigvee_{j=1}^{n} \beta_{ij}\epsilon_i(t) F_j(e_j(t-\tau_j(t))) - \eta_{i1}\epsilon_i^2(t) \\ &- \eta_{i2}|\epsilon_i(t)|^{\eta+1} - \sum_{j=1}^{n} d_{ij}\epsilon_i(t) F_j(e_j(q_{ij}t)) \\ &- \bigwedge_{j=1}^{n} \alpha_{ij}(t)\epsilon_i(t) F_j(e_j(t-\tau_j(t))) \\ &- \bigvee_{j=1}^{n} \epsilon_i(t)\beta_{ij}(t) F_j(e_j(t-\tau_j(t))) \Big] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}\Big[-\xi_i e_i^2(t)+e_i(t)\epsilon_i(t)-k_{i1}e_i^2(t)\\
&- k_{i2}|e_i(t)|^{\eta+1}\Big]\\
&+ \sum_{i=1}^{n}\Big[-\theta_i\epsilon_i^2(t)+|\delta_i|\epsilon_i(t)e_i(t)\\
&+ \sum_{j=1}^{n}|c_{ij}||\epsilon_i(t)|l_j|e_j(t)|-\eta_{i1}\epsilon_i^2(t)\\
&- \eta_{i2}|\epsilon_i(t)|^{\eta+1}\Big]. \qquad (12)
\end{aligned}
$$

Using the inequality $2ab \leq (a^2+b^2)$, for all $a,\ b\in\mathbb{R}$, we get

$$
\begin{aligned}
e_i(t)\epsilon_i(t) &\leq \frac{1}{2}e_i^2(t)+\frac{1}{2}\epsilon_i^2(t), \qquad (13)\\
|\delta_i|\epsilon_i(t)e_i(t) &\leq |\delta_i|(\frac{1}{2}e_i^2(t)+\frac{1}{2}\epsilon_i^2(t)), \qquad (14)\\
\sum_{i=1}^{n}\sum_{j=1}^{n}|\epsilon_i(t)||c_{ij}|L_j^f|e_j(t)| &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}|c_{ij}|L_j^f\epsilon_i^2(t)\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}|c_{ij}|L_j^f e_j^2(t)\\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}|c_{ij}|L_j^f\epsilon_i^2(t)\\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}|L_i^f e_i^2(t). \qquad (15)
\end{aligned}
$$

So, along with (12)-(15), we obtain the following estimation for the right-hand side of (12):

$$
\begin{aligned}
V'(t) &\leq \sum_{i=1}^{n}\Big[-\xi_i e_i^2(t)+\frac{1}{2}e_i^2(t)+\frac{1}{2}\epsilon_i^2(t)-k_{i1}e_i^2(t)\\
&- k_{i2}|e_i(t)|^{\eta+1}\Big]+\sum_{i=1}^{n}\Big[-\theta_i\epsilon_i^2(t)\\
&+ |\delta_i|(\frac{1}{2}e_i^2(t)+\frac{1}{2}\epsilon_i^2(t))+\sum_{j=1}^{n}\frac{1}{2}c_{ij}^{+}L_j^f\epsilon_i^2(t)\\
&+ \sum_{j=1}^{n}\frac{1}{2}|c_{ji}|l_i e_i^2(t)-\eta_{i1}\epsilon_i^2(t)-\eta_{i2}|\epsilon_i(t)|^{\eta+1}\Big]\\
&\leq \sum_{i=1}^{n}\Big[-\xi_i+\frac{1}{2}-k_{i1}+\frac{1}{2}|\delta_i|+\sum_{j=1}^{n}\frac{1}{2}|c_{ji}|L_i^f\Big]\\
&\times e_i^2(t)+\sum_{i=1}^{n}\Big[-\theta_i+\frac{1}{2}-\eta_{i1}+\frac{1}{2}|\delta_i|\\
&+ \sum_{j=1}^{n}\frac{1}{2}|c_{ij}|L_j^f\Big]\epsilon_i^2(t)+\sum_{i=1}^{n}\Big[-k_{i2}|e_i(t)|^{\eta+1}\\
&- \eta_{i2}|\epsilon_i(t)|^{\eta+1}\Big] \qquad (16)
\end{aligned}
$$

By using (8)-(10), and Lemma 2, we have

$$
\begin{aligned}
V'(t) &\leq -\min\{k_{i2},\eta_{i2}\}\sum_{i=1}^{n}\Big\{|e_i(t)|^{\eta+1}+|\epsilon_i(t)|^{\eta+1}\Big\}\\
&\leq -\min\{k_{i2},\eta_{i2}\}\sum_{i=1}^{n}\Big\{|e_i(t)|^{2}+|\epsilon_i(t)|^{2}\Big\}^{\frac{\eta+1}{2}}\\
&\leq -\min\{k_{i2},\eta_{i2}\}2^{\frac{\eta+1}{2}}V(t)^{\frac{\eta+1}{2}}\\
&\leq -\Lambda V(t)^{\frac{\eta+1}{2}} \qquad (17)
\end{aligned}
$$

Thus, according to Lemma 3, we obtain

$$
V^{(1-\eta)}(t) \leq V^{1-\eta}(t_0)-\Lambda(1-\eta)(t-t_0),\ \ t_0\leq t\leq t_1
$$

and

$$
V(t)=0,\ \ \forall t\geq t_1,
$$

hence, the finite-time synchronization between system (3) and system (5) is realized. ■

Remark 2: In Theorem 1, by choose a special controller, we achieved the finite-time synchronization between two chaotic fuzzy inertial neural networks with time-varying delays. In addition, the used control law $A_i(t)$ and $B_i(t)$ are not easily applicable, especially if the activation functions f_i in model (1) satisfy some special condition. In this step, we will modify the adaptive laws $B_i(t)$ to make the applicability of our results better.

Suppose that the activation functions f_i are bounded and let

$$
\begin{cases}
A_i(t) &= -k_{i1}e_i(t)-k_{i2}sign(e_i(t))|e_i(t)|^{\eta}\\
B_i(t) &= -\eta_{i1}\epsilon_i(t)-\eta_{i2}sign(\epsilon_i(t))|\epsilon_i(t)|^{\eta}\\
&- \lambda sing(\epsilon_i(t)),
\end{cases} \qquad (18)
$$

where $\lambda>0$.

Theorem 2: Under Assumption **(H1)** and **(H2)**, the drive-response systems (3) and (5) will achieve finite-time synchronization under controller (18) if (8)-(10) are satisfied and

$$
\lambda \geq \sum_{j=1}^{n}K_j(|d_{ij}|+|\alpha_{ij}|+|\beta_{ij}|). \qquad (19)
$$

In addition, the settling time for synchronization is estimated by t_1.

Proof: Let us consider the following Lyapunov functional:

$$
V(t)=\frac{1}{2}\sum_{i=1}^{n}e_i^2(t)+\frac{1}{2}\sum_{i=1}^{n}\epsilon_i^2(t).
$$

Calculating the derivative of $V(t)$ along the trajectory of the error system (6) and using lemma 1 we have

$$\begin{aligned}
V'(t) &= \sum_{i=1}^{n} e_i(t)\Big[-\xi_i e_i(t) + \epsilon_i(t) - k_{i1}e_i(t) \\
&- k_{i2}sign(e_i(t))|e_i(t)|^{\eta}\Big] + \sum_{i=1}^{n}\epsilon_i(t)\Big[-\theta_i\epsilon_i(t) \\
&+ \delta_i e_i(t) + \sum_{j=1}^{n} c_{ij}(t)F_j(e_j(t)) \\
&+ \sum_{j=1}^{n} d_{ij}(t)F_j(e_j(t-\tau_j(t))) \\
&+ \bigwedge_{j=1}^{n}\alpha_{ij}(t)F_j(e_j(t-\tau_j(t))) \\
&+ \bigvee_{j=1}^{n}\beta_{ij}(t)F_j(e_j(t-\tau_j(t))) - \eta_{i1}\epsilon_i(t) \\
&- \eta_{i2}sign(\epsilon_i(t))|\epsilon_i(t)|^{\eta} - \lambda sing(\epsilon_i(t))\Big] \\
&\le \sum_{i=1}^{n}\Big[-\xi_i + \frac{1}{2} - k_{i1} + \frac{1}{2}|\delta_i| + \frac{1}{2}\sum_{j=1}^{n}|c_{ji}|L_i^f\Big] \\
&\times e_i^2(t) + \sum_{i=1}^{n}\Big[-\theta_i + \frac{1}{2} - \eta_{i1} + \frac{1}{2}|\delta_i| \\
&+ \sum_{j=1}^{n}\frac{1}{2}|c_{ij}|L_j^f\Big]\epsilon_i^2(t) + \sum_{i=1}^{n}\Big[-\lambda + \sum_{j=1}^{n}K_j(|d_{ij}| \\
&+ |\alpha_{ij}| + |\beta_{ij}|)\Big]|\epsilon_i(t)| + \sum_{i=1}^{n}\Big[-k_{i2}|e_i(t)|^{\eta+1} \\
&- \eta_{i2}|\epsilon_i(t)|^{\eta+1}\Big] \qquad (20)
\end{aligned}$$

From Lemma 2, we get

$$\begin{aligned}
V'(t) &\le -\min\{k_{i2}, \eta_{i2}\}2^{\frac{\eta+1}{2}}V(t)^{\frac{\eta+1}{2}} \\
&\le -\Lambda V(t)^{\frac{\eta+1}{2}}. \qquad (21)
\end{aligned}$$

By Lemma 3, the master-slave systems (3) and (5) achieve the finite-time synchronization and the settling time is given as t_1. ■

V. Example

In this section, we will present two numerical examples to illustrate the effectiveness of our results.

Example 1: Consider the following FICNNs with time-varying delay:

$$\begin{aligned}
\frac{d^2x_i(t)}{dt^2} &= -a_i\frac{dx_i(t)}{dt} - b_i x_i(t) + \sum_{j=1}^{2}c_{ij}f_j(x_j(t)) \\
&+ \sum_{j=1}^{2}d_{ij}f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{2}e_{ij}\nu_j \\
&+ \bigwedge_{j=1}^{2}T_{ij}\nu_j + \bigwedge_{j=1}^{2}\alpha_{ij}f_j(x_j(t-\tau_j(t))) \\
&+ \bigvee_{j=1}^{2}\beta_{ij}f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{2}S_{ij}\nu_j + I_i, (22)
\end{aligned}$$

where, $i = j = 1, 2$, $a_1 = a_2 = 3$, $b_1 = b_2 = 3.1$, $c_{11} = 1.5$, $c_{12} = -0.4$, $c_{21} = -1$, $c_{22} = -0.3$, $d_{11} = 1.2$, $d_{12} = -0.3$, $d_{21} = 0.7$, $d_{22} = 2$, $\alpha_{11} = 1.5$, $\alpha_{12} = -0.2$, $\alpha_{21} = -1$, $\alpha_{22} = 0.5$, $\beta_{11} = 1.5$, $\beta_{12} = 0.8$, $\beta_{21} = 0.4$, $\beta_{22} = -1$, $\tau_j(t) = |sin(t)|$, $f_i(u) = \tanh(u)$, $L_i^f = 1$, $\xi_i = 0.1$ and $I_i = -\sum_{j=1}^{2}e_{ij}\nu_j - \bigwedge_{j=1}^{2}T_{ij} - \bigvee_{j=1}^{2}S_{ij}\nu_j$.

The fuzzy inertial neural network in (24) with the above conditions exhibits chaotic behavior, which is plotted in Fig.1 It is easy to see that assumptions **(H1)** hold and according to

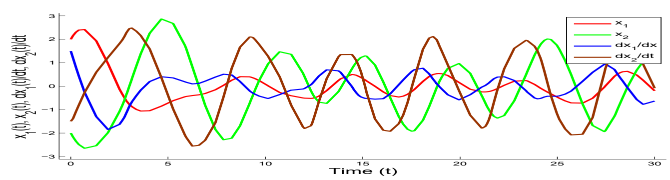

Fig. 1. The chaotic phase of system (24).

the conditions presented in Theorem 1, choose:

$$\begin{aligned}
k_{11} &= 2.03 \ge -\xi_1 + \frac{1}{2} + \frac{1}{2}|\delta_1| + \sum_{j=1}^{n}\frac{1}{2}|c_{j1}|l_1 = 2.03, \\
k_{21} &= 0.93 \ge -\xi_2 + \frac{1}{2} + \frac{1}{2}|\delta_2| + \sum_{j=1}^{n}\frac{1}{2}|c_{j2}|l_2 = 0.93, \\
\eta_{11} &= 1.23 \ge -\theta_1 + \frac{1}{2} + \frac{1}{2}|\delta_1| + \sum_{j=1}^{n}\frac{1}{2}|c_{1j}|l_j = 1.23, \\
\eta_{21} &= 0.93 \ge -\theta_2 + \frac{1}{2} + \frac{1}{2}|\delta_2| + \sum_{j=1}^{n}\frac{1}{2}|c_{2j}|l_j = 0.93,
\end{aligned}$$

$k_{i2} = 0.9 \; > \; 0,\; k_{i2} = 1 > 0,\; \eta_{i2} = 0.8 > 0,\; \eta = 0.6.$

Therefore, the control inputs of the slave system are formulated as follow :

$$
\begin{cases}
A_1(t) &= -2.03e_1(t) - 0.9sign(e_1(t))|e_1(t)|^{0.6}, \\
A_2(t) &= -0.93e_2(t) - 0.9sign(e_2(t))|e_2(t)|^{0.6}, \\
B_1(t) &= -1.23\epsilon_1(t) - 0.8sign(\epsilon_1(t))|\epsilon_1(t)|^{0.6} \\
&- \sum_{j=1}^{2} d_{1j}F_j(e_j(t-\tau_j(t))) \\
&- \bigwedge_{j=1}^{2} \alpha_{1j}F_j(e_j(t-\tau_j(t))) \\
&- \bigvee_{j=1}^{2} \beta_{1j}F_j(e_j(t-\tau_j(t))), \\
B_2(t) &= -0.93\epsilon_2(t) - 0.8sign(\epsilon_2(t))|\epsilon_2(t)|^{0.6} \\
&- \sum_{j=1}^{2} d_{2j}F_j(e_j(t-\tau_j(t))) \\
&- \bigwedge_{j=1}^{2} \alpha_{2j}F_j(e_j(t-\tau_j(t))) \\
&- \bigvee_{j=1}^{2} \beta_{2j}F_j(e_j(t-\tau_j(t))),
\end{cases} \tag{23}
$$

- ✓ Under controller (23) the state trajectories of master/salve system are illustrated in Fig.2 and Fig.3.
- ✓ Under controller (23) and the initial conditions $e_1(0) = -1$, $e_2(0) = -0.8$, $\epsilon_1(0) = -0.6$, $\epsilon_2(0) = -0.4$, the evolution of the synchronization errors is described in Fig.4, which indicates that synchronization can be reached in finite time. Furthermore, the settling time for synchronization is estimated by $t_1 \leq 7.2907$.

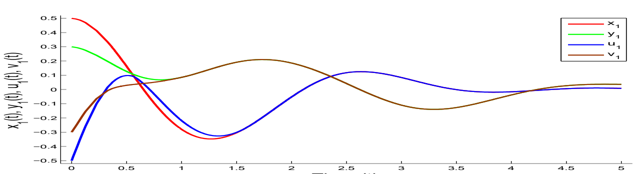

Fig. 2. Synchronization curves of x_1, u_1, y_1 and v_1 under controller (25).

Example 2: Consider the following FICNNs with time-varying delay:

$$
\begin{aligned}
\frac{d^2x_i(t)}{dt^2} &= -a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{2} c_{ij}f_j(x_j(t)) \\
&+ \sum_{j=1}^{2} d_{ij}f_j(x_j(t-\tau_j(t))) + \sum_{j=1}^{2} e_{ij}\nu_j \\
&+ \bigwedge_{j=1}^{2} T_{ij}\nu_j + \bigwedge_{j=1}^{2} \alpha_{ij}f_j(x_j(t-\tau_j(t))) \\
&+ \bigvee_{j=1}^{2} \beta_{ij}f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{2} S_{ij}\nu_j + I_i, \quad (24)
\end{aligned}
$$

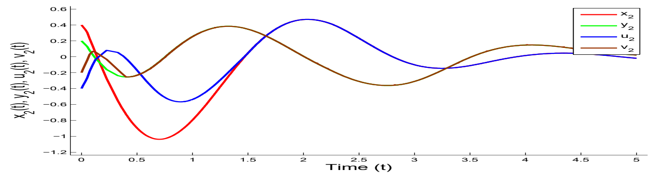

Fig. 3. Synchronization curves of x_2, u_2, y_2 and v_2 under controller (25).

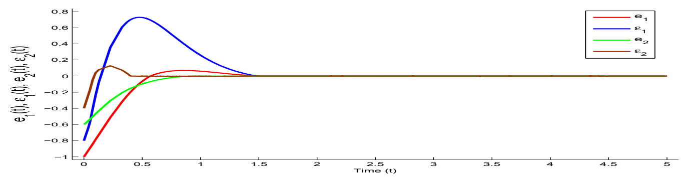

Fig. 4. The evaluation of synchronization error e_1, e_2, ϵ_1 and ϵ_2.

where, $i = j = 1, 2$, $a_1 = a_2 = 2$, $b_1 = b_2 = 2.1$, $c_{11} = -1.8$, $c_{12} = -0.1$, $c_{21} = -2$, $c_{22} = 0.4$, $d_{11} = -1.5$, $d_{12} = -0.5$, $d_{21} = 0.6$, $d_{22} = -2$, $\alpha_{11} = -1.8$, $\alpha_{12} = -0.1$, $\alpha_{21} = -2$, $\alpha_{22} = 0.4$, $\beta_{11} = -1.5$, $\beta_{12} = -0.5$, $\beta_{21} = 0.6$, $\beta_{22} = -2$, $\tau_j(t) = \frac{e^t}{(1+e^t)}$, $f_i(u) = \tanh(u)$, $L_i^f = 1$, $\xi_i = 1$ and $I_i = -\sum_{j=1}^{2} e_{ij}\nu_j - \bigwedge_{j=1}^{2} T_{ij} - \bigvee_{j=1}^{2} S_{ij}\nu_j$.

The numerical simulation of system (24) with above coefficients is represented in Fig.5, which shows that the system has a chaotic attractor with initial values $x_1(0) = 0.5$, $x_2(0) = -0.5$, $\frac{dx_1(t)}{dt} = 0.25$ and $\frac{dx_2(t)}{dt} = -0.25$.

It is easy to check that assumptions **(H1)** and **(H2)** hold and according to the conditions presented in Theorem 2, choose:

$$
\begin{aligned}
k_{11} &= 1.95 \geq -\xi_1 + \frac{1}{2} + \frac{1}{2}|\delta_1| + \sum_{j=1}^{2} \frac{1}{2}|c_{j1}|L_1^f = 1.95, \\
k_{21} &= 0.3 \geq -\xi_2 + \frac{1}{2} + \frac{1}{2}|\delta_2| + \sum_{j=1}^{2} \frac{1}{2}|c_{j2}|L_2^f = 0.3, \\
\eta_{11} &= 1 \geq -\theta_1 + \frac{1}{2} + \frac{1}{2}|\delta_1| + \sum_{j=1}^{2} \frac{1}{2}|c_{1j}|L_j^f = 1,
\end{aligned}
$$

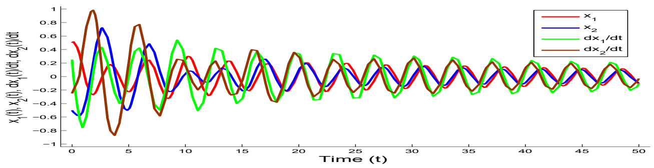

Fig. 5. The chaotic phase of system (24).

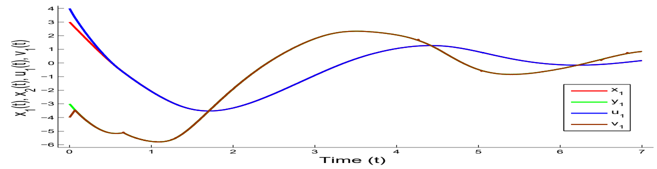

Fig. 6. Synchronization curves of x_1, u_1, y_1 and v_1 under controller (25).

$$
\begin{aligned}
\eta_{21} &= 1.25 \geq -\theta_2 + \frac{1}{2} + \frac{1}{2}|\delta_2| + \sum_{j=1}^{2} \frac{1}{2}|c_{2j}|L_j^f = 1.25, \\
\lambda &= 15.2 \geq \sum_{j=1}^{2} K_j(|d_{1j}| + |\alpha_{1j}| + |\beta_{1j}|) = 11.8, \\
\lambda &= 15.2 \geq \sum_{j=1}^{2} K_j(|d_{2j}| + |\alpha_{2j}| + |\beta_{2j}|) = 15.2, \\
k_{12} &= 1 > 0,\ k_{22} = 1.1 > 0, \eta_{12} = 1 > 0,\ \eta_{22} = 1.2 > 0, \\
\eta &= 0.5,
\end{aligned}
$$

Thus, the control inputs of the slave system are formulated as

$$
\begin{cases}
A_1(t) &= -1.95e_1(t) - sign(e_1(t))|e_1(t)|^{0.5} \\
A_2(t) &= -0.3e_2(t) - 1.1sign(e_2(t))|e_2(t)|^{0.5} \\
B_1(t) &= -\epsilon_1(t) - sign(\epsilon_1(t))|\epsilon_1(t)|^{0.5} \\
 &- 15.2sing(\epsilon_1(t)), \\
B_2(t) &= -1.25\epsilon_2(t) - 1.2sign(\epsilon_1(t))|\epsilon_i(t)|^{0.5} \\
 &- 15.2sing(\epsilon_2(t)),
\end{cases}
\tag{25}
$$

- ✓ Under controller (25) the state trajectories of master/salve system are illustrated in Fig.6 and Fig.7.
- ✓ Under controller (25) and the initial conditions $e_1(0) = e_2(0) = 1$, $\epsilon_1(0) = \epsilon_2(0) = -1$, the evolution of the synchronization errors is described in Fig.8, which indicates that synchronization can be reached in finite time. Furthermore, the settling time for synchronization is estimated by $t_1 \leq 1.6818$.

Remark 3: Comparing with the control methods of [11]–[13], only the asymptotic or pinning synchronization can be realized for inertial neural networks. From the example, it can be seen that the proposed finite-time synchronization control method used in this paper has better convergence property for inertial neural networks. Example can also illustrate that our control method is effective and the conditions are easy to be verified.

VI. Conclusion

In this paper, fuzzy inertial cellular neural networks with time delays have been studied. By using differential inequalities and by constructing Lyapunov functions, some new and

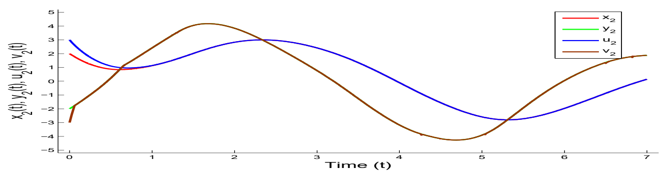

Fig. 7. Synchronization curves of x_2, u_2, y_2 and v_2 under controller (25).

effective criteria are obtained for ensuring the finite-time synchronization result. To the best of our knowledge, this is the first paper to study finite-time synchronization of FICNNs. Finally, a numerical example with their simulations have been given to demonstrate the effectiveness of the theoretical results.

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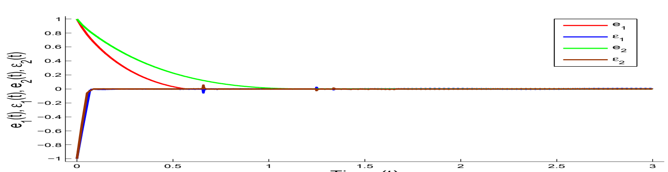

Fig. 8. The evaluation of synchronization error e_1, e_2, ϵ_1 and ϵ_2.

of the paper more accurate.

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