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Finite-time cluster synchronization for a class of fuzzy cellular neural networks via non-chattering quantized controllers

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ABSTRACT

This paper considers the finite-time cluster synchronization (FTCS) of coupled fuzzy cellular neural networks (FCNNs) with Markovian switching topology, discontinuous activation functions, proportional leakage, and time-varying unbounded delays. Novel quantized controllers without the sign function are designed to avoid the chattering and save communication resources. Under the framework of Filippov solution, several sufficient conditions are derived to guarantee the FTCS by constructing new Lyapunov–Krasovskii functionals and utilizing M-matrix methods. The new analytical techniques skillfully overcome the difficulties caused by time-varying delays and cope with the uncertainties of both Filippov solution and Markov jumping, which enable us determine the settling time explicitly. Numerical simulations demonstrate the effectiveness of the theoretical analysis.

1. Introduction

Recently, synchronization of coupled systems, as a typical collective behavior, has been widely investigated due to its theoretical importance and practical applications (Lu & Ho, 2010; Lv, He, Qian, & Cao, 2018; Zeng, Wu, Zhang, Zhong, & Shi, 2018). Until now, various synchronization patterns have been discovered, for example, complete synchronization (Li, Rakkiyappan, & Sakthivel, 2015; Yang & Cao, 2010), generalized synchronization (Yang, Zhu, & Huang, 2011) and cluster synchronization (Cai, Li, Jia, & Liu, 2016; Kang, Qin, Ma, Gao, & Zheng, 2018; Li, Ho, Cao and Lu, 2016; Tseng, 2017). The characteristic of cluster synchronization is that all the nodes in the same cluster should have the same dynamical property, but there is no synchronization among different clusters.

As we all know, neural network (NN) models have been successfully applied in different fields including image processing, classification of pattern, associative memories, etc. It is worth mentioning that, the FCNN model is absolutely a special one, which was first proposed by Yang in Yang, Yang, Wu, and Chua (1996, 1997b). FCNN combines the advantages of the traditional cellular NN and fuzzy set theory and has been proved to be a very useful paradigm for image processing problems (Barbounis & Theocharis, 2007; Du, Yan, & Zhao, 2018; Ratnavelu, Kalpana, Balasubramaniam, Wong, & Raveendran, 2017; Yang, Yang, Wu, & Chua, 1997a). Up to now, many interesting results concerning stabilization (Jia, 2017; Kao, Shi, Xie, & Karimi, 2015; Xu & Li, 2017b) and synchronization of FCNNs (Abdurahman, Jiang, & Teng, 2016; Duan, Fang, & Fu, 2017; Duan, Wei, & Huang, 2018; Gan, Xu, & Yang, 2012; Tang & Fang, 2009; Wang, 2018; Xia, Yang, & Han, 2009; Yan & Lv, 2012; Yang, Yu, Cao, Alsaadi and Hayat, 2017) have been published. However, most of existing results on FCNNs assume that the activation functions are continuous or even Lipschitz continuous (Abdurahman et al., 2016; Gan et al., 2012; Jia, 2017; Kao et al., 2015; Tang & Fang, 2009; Wang, 2018; Xia et al., 2009; Xu & Li, 2017b; Yan & Lv, 2012; Yang and Yu et al., 2017). Such assumptions might not be appropriate when handling dynamical systems possessing high-slope nonlinear elements. Thus, in this case, it is unreasonable to model it by a differential equation with continuous activation functions. In addition, the dynamical systems with discontinuous activation functions have many important applications and are encountered frequently in practice. Therefore, FCNNs with discontinuous activation functions, which are so-called discontinuous FCNN (DFCNN) in this paper, have attracted increasing attention in Duan et al. (2017, 2018). To the best of our knowledge, the results concerning cluster synchronization especially on FTCS of coupled DFCNNs are still few.

Compared with asymptotic and exponential synchronization, convergence speed of finite-time synchronization (FTS) is optimal. Moreover, it has other advantages such as better robustness and disturbance rejection properties (Tang, 1998). Hence, FTS and

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finite-time control techniques were extensively considered in recent works (Abdurahman, Jiang, & Teng, 2015; Cui, Fang, Zhang, & Wang, 2014; Gao, Zhu, Alsaedi, Alsaadi, & Hayat, 2017; Jiang, Wang, Mei, & Shen, 2015; Tang, Ju, & Shen, 2017; Yang, Ho, Lu and Song, 2015; Yang & Lu, 2016; Yang, Wu, & Cao, 2013; Zhang, Yang, Xu, Feng and Li, 2017). Especially, authors in Cui et al. (2014) studied the FTCS problem for a class of Markovian switching complex networks with stochastic perturbations by designing a continuous feedback controller with sign function. Under the framework of Filippov solution, the authors in Yang and Ho et al. (2015) have researched FTCS of a class of T-S fuzzy complex networks with discontinuous nodes via some discontinuous feedback controllers with sign function. The authors in Tang et al. (2017) designed several pinning controllers such that coupled discontinuous Lur'e systems achieved FTCS. It is not difficult to find from Abdurahman et al. (2015), Cui et al. (2014), Gao et al. (2017), Jiang et al. (2015), Tang et al. (2017), Yang and Ho et al. (2015), Yang and Lu (2016), Yang et al. (2013) and Zhang and Yang et al. (2017) that the sign function plays an extremely pivotal role in achieving FTS and dealing with the uncertainty introduced by Filippov solution. While, if the sign function is applied in controller, chattering phenomena are unavoidably introduced to the system state and controlled signals (Yang & Lu, 2016). Such chattering phenomena may cause bad influence on equipments or even damage equipments. On the other hand, the transmitted signals in Abdurahman et al. (2015), Cui et al. (2014), Gao et al. (2017), Jiang et al. (2015), Tang et al. (2017), Yang and Ho et al. (2015), Yang and Lu (2016), Yang et al. (2013) and Zhang and Yang et al. (2017) need to be received by controllers without any errors. Actually, due to the limited bits rate of communication channels and the limited bandwidth, communication constraint cannot be negligible in the problems related to the synchronization and control theory of coupled systems. So, the quantization problems have been paid considerable attention in recent years (Feng, Xiong, Tang, & Yang, 2018; Li, Ho, & Lu, 2013; Lu & Ho, 2011; Xiao, Zhou, & Zhang, 2014; Xu et al., 2017; Zhang, Li, Yang and Yang, 2018; Zhang, Yang, Li, Zhang and Yang, 2018). Although the FTS of complex networks was considered by using quantized control in Xu et al. (2017), the chattering phenomena are inevitable. Up till now, there are still few synchronization results via non-chattering quantized controller, let alone FTS. Thus, it is urgent to design a simple quantized controller to realize FTS without introducing the chattering phenomena.

It is well known that the effect of time-delays on FTS is very difficult to overcome. Time-delays are inevitable in NN models due to finite speeds of switching of amplifiers and transmission of signals in hardware implementation (Li, Ho, & Lu, 2017; Yang et al., 2011). To be noted that, in practical implementation, NNs usually have a spatial nature owing to the presence of an amount of parallel pathways with a variety of axon lengths and sizes, and thus time-delays encountered in NNs are often time-varying, even unbounded in Jia (2017), Li and Cao (2017) and Yang, Song, Liang and He (2015). In the literature, many results on FTS of NNs with different time delays have been published. Without utilizing the traditional finite-time stability theorem in Tang (1998), authors in Li, Yang and Shi (2016), Yang (2014), Yang, Cao, Song, Xu and Feng (2017), Zhang and Yang et al. (2017) and Zhou, Zhang, Yang, Xu, and Feng (2017) investigated FTS of NNs with infinite-time distributed delay, which is one of time-varying and unbounded delays. It is reported in Yang (2014) that the settling time cannot be estimated though the NNs with infinite-time distributed delays can be finite-timely synchronized. Obviously, it is not convenient in practice if the settling time is not available. Recently, another one, called proportional delay, has been also considered for finite-time control (Jia, 2017; Le & Son, 2015; Wang, 2018). By designing continuous controller with sign function and proportional delay terms, several FTS conditions for FCNNs with proportional delays were obtained in Wang (2018). Note that the controller in Wang (2018) is very special and complex since some special terms were designed to compensate the effects of delays and the chattering phenomena were also introduced. Hence, it is difficult or even impossible to be implemented in practice. In addition, time-delays may also appear in the negative feedback term of the neural network models, which are so-called leakage delays and have a great influence on the dynamics of neural networks (Park, Kwon, Ju, Lee, & Cha, 2012). There are different kinds of delays to be introduced into the leakage term, such as constant delay (Zhang, Song and Zhao, 2017), time-varying delay (Xu & Li, 2017a), distributed delay (Peng, 2010) and so on. However, as far as we know, no paper introduces proportional delay into the leakage term, and FTCS of coupled DFCNNs with leakage delay has not been studied in the literature.

On the other hand, coupled systems may experience abrupt changes in their structure caused by component failures or repairs, which may degrade the stability of the coupled systems. Generally, such abrupt changes are governed by Markov chains with finite state space (Davis, 1993; Jiang, Kao, Gao, & Yao, 2017; Li & Rakkiyappan, 2012). In fact, the applications of the Markov jump systems can be found in economic systems, modeling production system, network control systems, and communication systems. Meanwhile, many interesting results concerning synchronization of Markovian coupled NNs have been reported (Cui et al., 2014; Jing, Kao, & Ju, 2018; Mao, 1999; Yang and Cao et al., 2017; Yang, Feng, Feng and Cao, 2017; Yang, Xu, Feng, & Lu, 2018; Zeng et al., 2018). In Yang and Feng et al. (2017) and Yang et al. (2018), based on tracker information, authors studied synchronization of delayed discrete-time and continuous-time NNs with Markovian topology. Investigating FTCS of Markovian coupled DFCNNs with both proportional leakage and time-varying delays via non-chattering quantized control becomes the motivation.

Inspired by the above discussions, this paper investigates FTCS of a class of DFCNNs with proportional leakage delay, time-varying delay, and Markovian topology via non-chattering quantized control. The main contributions are as follows:

- The quantized controllers can save both bits rate of communication and bandwidth. Moreover, the controllers are non-chattering since they abandon the sign function which usually induces chattering.
- New 1-norm-based analytical techniques are developed to cope with the difficulties induced by both leakage delay and proportional delay. The effects of the uncertainties caused by Filippov solutions are overcome at the same time.
- By designing new Lyapunov functionals and utilizing M-matrix methods, novel analytical techniques are established to obtain several sufficient conditions ensuring the FTCS of the considered DFCNNs. The established synchronization criteria do not have any free parameters and can be easily verified.
- Different from the FTS of neural networks with infinite-time distributed delays in Li and Yang et al. (2016), Yang (2014), Yang and Cao et al. (2017), Yang and Song et al. (2015) and Zhou et al. (2017), the settling time in this paper can be explicitly estimated.

The paper is organized as follows. In Section 2, the model of the coupled DFCNNs with Markovian topology, proportional leakage, and time-varying delays is presented. Some useful definitions and lemmas are also given in this section. Section 3 gives some FTCS criteria by strict mathematical proofs. Then two simulation examples are presented to illuminate the effectiveness of the theoretical results in Section 4. Finally, conclusions are given in Section 5.

Notations : $\mathbb{R}$ is the set of real number, $\mathbb{R}^n$ denotes the set of $n \times 1$ real vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices; $\|\cdot\|_1$ is 1-norm of a vector or a matrix, respectively.

Let $X = (x_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ and $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$. $\bigwedge X \circ y = (\bigwedge_{j=1}^n x_{1j}y_j, \bigwedge_{j=1}^n x_{2j}y_j, \ldots, \bigwedge_{j=1}^n x_{nj}y_j)^T$, $\bigvee X \circ y = (\bigvee_{j=1}^n x_{1j}y_j, \bigvee_{j=1}^n x_{2j}y_j, \ldots, \bigvee_{j=1}^n x_{nj}y_j)^T$, where $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR, respectively; $\overline{co}[E]$ is the closure of the convex hull of the set $E \subset \mathbb{R}^n$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition(*i.e.* the filtration contains all $\mathcal{P}$–null sets and is right continuous). $\mathbf{E}(\cdot)$ stands for mathematical expectation operator with respect to the given probability measure $\mathcal{P}$.

2. Model description and preliminaries

Consider the following FCNN with both proportional leakage and time-varying unbounded delays:

$$\begin{cases} \dot{x}(t) = -C\int_{q_1 t}^{t} x(s)\mathrm{d}s + Af(x(t)) + \bigvee S\circ f(x(q_2 t)) \\ \qquad + \bigwedge D\circ f(x(q_2 t)) \\ \qquad + \bigvee R\circ v + \bigwedge H\circ v + J, \quad t > 0, \\ x(0) = \varphi(0), \end{cases} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state vector, n is the number of neurons, $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ is a diagonal matrix with positive entries $c_j > 0, j = 1, 2, \ldots, n$; $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T \in \mathbb{R}^n$ denotes the neuron activation function, $A = (a_{jl})_{n\times n}$ is the connection weight matrix; $S = (s_{jl})_{n\times n}$, $D = (d_{jl})_{n\times n}$, $R = (r_{jl})_{n\times n}$, and $H = (h_{jl})_{n\times n}$ are matrices of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; $v = (v_1, v_2, \ldots, v_n)^T$ and $J = (J_1, J_2, \ldots, J_n)^T$ denote input and bias of neurons respectively; $q_1 \in (0, 1)$ is the proportional factor of leakage delay and $q_2 \in (0, 1)$ is the proportional delay factor; $q_i t = t - (1 - q_i)t$ $(i = 1, 2)$, where $(1 - q_i)t$ $(i = 1, 2)$ correspond to the distributed proportional leakage delay and proportional delay, respectively; $\varphi(0) \in \mathbb{R}^n$ denotes the initial value of $x(t)$.

Remark 1. The proportional delay is unbounded and time-varying and exists in many practical systems. For example, it is usually introduced into Web quality of service routing decision. If these routing algorithms based on neural networks with proportional delays are proposed, they will be the most suitable algorithms corresponding with the actual situation. In this paper, proportional delay is firstly introduced into the leakage term, which is essentially different from the continuously distributed leakage delays in Peng (2010). From the literature (Peng, 2010), one can find that some indispensable conditions on the infinite-time distributed delay such as $\int_0^{+\infty} K_{ij}(u)\mathrm{d}u = k_{ij}$ and $\int_0^{+\infty} uK_{ij}(u)\mathrm{d}u < \infty$ for positive constants k_{ij}, while there is no special condition on the proportional delay.

Different from Abdurahman et al. (2016), Gan et al. (2012), Jia (2017), Kao et al. (2015), Tang and Fang (2009), Xia et al. (2009), Wang (2018), Xu and Li (2017b), Yan and Lv (2012) and Yang and Yu et al. (2017), this paper assumes that the activation function $f(\cdot)$ is discontinuous on $\mathbb{R}^n$. Then, the FCNN (1) becomes a differential equation with discontinuous state on the right-hand side. In this case, the traditional solution of system (1) may not exist. According to the literature (Yang and Song et al., 2015), one can study the dynamical of the FCNN (1) with the help of Filippov solution (Arscott, 1988).

For research purposes, the following assumptions are necessary.

(H_1) For every $i = 1, 2, \ldots, n, f_i : \mathbb{R} \to \mathbb{R}$ is continuous except on a countable set of isolate points $\{\rho_k^i\}$, where there exist finite right and left limits $f_i^+(\rho_k^i)$ and $f_i^-(\rho_k^i)$ respectively. Moreover, f_i has at most a finite number of jump discontinuous in every compact interval of $\mathbb{R}$.

(H_2) For each $i = 1, 2, \ldots, n$, $0 \in F_i(0)$, and there exist nonnegative constants ζ_i and ϱ_i such that $\sup|\varpi_i - \eta_i| \leq \zeta_i|u - v| + \varrho_i$, holds $\forall u, v \in \mathbb{R}$, where $\varpi_i \in F_i(u)$ and $\eta_i \in F_i(v)$ with $F_i(u) = [\min\{f_i^-(u), f_i^+(u)\}, \max\{f_i^-(u), f_i^+(u)\}]$.

Definition 1 (*Arscott, 1988*)**.** The Filippov set-valued map of $f(x)$ at $x \in \mathbb{R}^n$ is defined as follows:

$$F(x) = \bigcap_{\delta>0}\bigcap_{\mu(\Omega)=0} \overline{co}[f(B(x,\delta)\setminus\Omega)].$$

where $B(z, \delta) = \{y : \|y - z\| \leq \delta\}$, $\mu(\Omega)$ is the Lebesgue measure of set Ω, and $\overline{co}[C]$ is the closure of the convex hull of the set C.

Lemma 1. *Suppose that* (H_1)–(H_2) *are satisfied. Then, any initial value for* (1) *has at least one solution* $[x(t), \gamma(t)]$ *defined on* $[q, +\infty)$, *where* $q = \min\{q_1, q_2\}$, $\gamma(t) \in F(x(t))$, .

Proof. Let $z(t) = x(e^t)$ for $t \geq t_0 + \log q$, where $t_0 \geq 0$. Then, by using the approaches explored in Iserles (1993), DFCNN (1) is equivalently transformed into the following delayed DFCNN with time-varying coefficients

$$\begin{cases} \dot{z}(t) = e^t\{-C\int_{t-\tau_1}^{t} e^s z(s)\mathrm{d}s + Af(z(t)) + \bigvee S\circ f(z(t-\tau_2)) \\ \qquad + \bigwedge D\circ f(z(t-\tau_2)) \\ \qquad + \bigvee R\circ v + \bigwedge H\circ v + J\}, \quad t > t_0, \\ z(u) = x(e^u),\ u\in[-\tau, t_0], \end{cases}$$

where $\tau_1 = -\log q_1$, $\tau_2 = -\log q_2$ and $\tau = \max\{\tau_1, \tau_2\}$. Then, one can obtain the conclusion by using similar analysis methods as those in Yang and Song et al. (2015). The proof is completed. ■

In view of Definition 1 and Lemma 1, the DFCNN (1) has at least one Filippov solution on $\mathbb{R}^n$, *i.e.* , there exists a measurable function $\alpha(t) \in F(x(t))$ such that

$$\begin{aligned} \dot{x}(t) = &-C\int_{q_1 t}^{t} x(s)\mathrm{d}s + A\alpha(t) + \bigvee S\circ\alpha(q_2 t) + \bigwedge D\circ\alpha(q_2 t) \\ &+ \bigvee R\circ v + \bigwedge H\circ v + J, \quad t > 0. \end{aligned} \tag{2}$$

Generally, a class of linearly coupled DFCNNs with N identical nodes and Markovian topology can be modeled as:

$$\begin{cases} \dot{x}_i(t) = -C_\nu\int_{q_1 t}^{t} x_i(s)\mathrm{d}s + A_\nu f(x_i(t)) + \bigvee S_\nu\circ f(x_i(q_2 t)) \\ \qquad + \bigwedge D_\nu\circ f(x_i(q_2 t)) + \bigvee R_\nu\circ v_\nu \\ \qquad + \bigwedge H_\nu\circ v_\nu + J_\nu + \sum_{j=1}^N k_{ij}(r_t)\Phi x_j(t) + u_i(r_t, t), \\ \qquad\quad i\in\mathcal{C}_\nu,\ 1\leq\nu\leq\wp,\ t > 0, \\ x_i(0) = \varphi_i(0), \end{cases} \tag{3}$$

where $\mathcal{C}_\nu = \{l_{\nu-1} + 1, l_{\nu-1} + 2, \ldots, l_\nu\}$ denotes the index of set of all nodes in the νth cluster, $l_0 = 0$, $l_\wp = N$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ corresponds to the state vector of the ith node; $C_\nu = \mathrm{diag}(c_1^\nu, c_2^\nu, \ldots, c_n^\nu)$ is a positive definite diagonal matrix, $A_\nu = (a_{ij}^\nu)_{n\times n}$ is the connection weight matrix; $S_\nu = (s_{ij}^\nu)_{n\times n}$, $D_\nu = (d_{ij}^\nu)_{n\times n}$, $R_\nu = (r_{ij}^\nu)_{n\times n}$ and $H_\nu = (h_{ij}^\nu)_{n\times n}$ are matrices of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; $v_\nu(t) = (v_1^\nu, v_2^\nu, \ldots, v_n^\nu)^T$ and $J_\nu = (J_1^\nu, J_2^\nu, \ldots, J_n^\nu)^T$ denote input and bias of neurons respectively; $K(r_t) = (k_{ij}(r_t))_{N\times N}$ describes the linear coupling configuration of the network, $\Phi = \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_n) \in \mathbb{R}^{n\times n}$ represents the inner connecting matrix with $\phi_l \geq 0$, $l = 1, 2, \ldots, n$; $u_i(r_t, t)$ is the controller to be designed. $\{r_t, t \geq 0\}$ is a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t_{\geq 0}}, P)$ and takes values in a finite state space $\mathcal{W} = \{1, 2, \ldots, \omega\}$ with generator $\Pi = (\pi_{ij})_{w\times w}$ given by:

$$\mathrm{Prob} = \{r_{t+\Delta t} = j | r_t = i\}\begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if } i\neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{if } i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t} = 0$. Here, $\pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\pi_{ii} = -\Sigma_{j=1,j\neq i}^{w}\pi_{ij}$. As a standard hypothesis, it is assumed that matrix Π is irreducible.

Definition 2. Matrix $\Upsilon = (\upsilon_{ij})_{M\times M}$ is said to belong to class Q_1, denoted as $\Upsilon \in Q_1$, if $\upsilon_{ij} \geq 0$ for $i \neq j$, and $\upsilon_{ii} = -\sum_{j=1,i\neq j}^{M}\upsilon_{ij}$, $i = 1, 2, \ldots, M$.

(H$_3$) Suppose that the coupling matrix $K(r_t)$ satisfies

$$K(r_t) = \begin{pmatrix} K_{11}(r_t) & K_{12}(r_t) & \cdots & K_{1\wp}(r_t) \\ K_{12}(r_t) & K_{22}(r_t) & \cdots & K_{2\wp}(r_t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{\wp 1}(r_t) & K_{\wp 2}(r_t) & \cdots & K_{\wp\wp}(r_t) \end{pmatrix},$$

where $K_{\nu\nu}(r_t) \in \mathbb{R}^{(l_\nu - l_{\nu-1})\times(l_\nu - l_{\nu-1})}$ belongs to Q_1 and $K_{\nu\hbar}(r_t) \in \mathbb{R}^{(l_\nu - l_{\nu-1})\times(l_\hbar - l_{\hbar-1})}$ is a zero-row-sum matrix, $\nu, \hbar = 1, 2, \ldots, \wp$.

Our objective is to design suitable controllers such that the controlled network (3) can be finite-timely achieved the synchronization state defined by $x_{l_0+1}(t) = x_{l_0+2}(t) = \cdots = x_{l_1}(t) = y_1(t), x_{l_1+1}(t) = x_{l_1+2}(t) = \cdots = x_{l_2}(t) = y_2(t), \ldots, x_{l_{\wp-1}+1}(t) = x_{l_{\wp-1}+2}(t) = \cdots = x_{l_\wp}(t) = y_\wp(t)$, where $y_\nu(t) = (y_{\nu 1}(t), y_{\nu 2}(t), \ldots, y_{\nu n}(t))^T \in \mathbb{R}^n$, $\nu = 1, 2\ldots, \wp$ are defined as:

$$\begin{cases} \dot{y}_\nu(t) = -C_\nu\int_{q_1 t}^{t} y_\nu(s)\mathrm{d}s + A_\nu f(y_\nu(t)) + \bigvee S_\nu \circ f(y_\nu(q_2 t)) \\ \qquad + \bigwedge D_\nu \circ f(y_\nu(q_2 t)) \\ \qquad + \bigvee R_\nu \circ \upsilon_\nu + \bigwedge H_\nu \circ \upsilon_\nu + J_\nu, \quad t > 0, \nu = 1, 2, \ldots, \wp, \\ y_\nu(0) = \psi_\nu(0). \end{cases} \tag{4}$$

Precisely, the synchronization goal is defined as follows.

Definition 3 (Yang and Ho et al., 2015)**.** The coupled DFCNNs (3) is said to realize FTCS if, by adding suitable controllers, there exists a constant $t_1 > 0$ depending on the initial values of (3) such that $\lim_{t\to t_1}\mathbf{E}\{\|x_i(t) - y_\nu(t)\|_1\} = 0$ and $\mathbf{E}\{\|x_i(t) - y_\nu(t)\|_1\} \equiv 0$ for $t > t_1$, $i \in \mathcal{C}_\nu$, $\nu = 1, 2\ldots, \wp$, where t_1 is called the settling time.

Remark 2. Note that the finite-time issue in this paper is essentially different from those considered in Jia (2017) and Le and Son (2015). Actually, although proportional delay is considered in Jia (2017) and Le and Son (2015), the finite-time techniques proposed in Jia (2017) and Le and Son (2015) can only guarantee the boundedness of the solutions in a given time. However, the object of this paper is to control the coupled FCNNs such that the synchronization errors converge to zero in a settling time. Therefore, the analytical methods used in Jia (2017) and Le and Son (2015) cannot be applied to the problem stated in Definition 3.

Lemma 2 (Horn, 1994)**.** *If $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ with $a_{ij} \leq 0 (i \neq j)$, then the following statements are equivalent:*

(1) A is a M-matrix.
(2) A^{-1} exists and all the elements of A^{-1} are nonnegative.
(3) All the eigenvalues of A have positive real parts.

Lemma 3 (Yang & Lu, 2016)**.** *Let $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ with $a_{ij} \leq 0 (i \neq j)$, $\sum_{j=1}^{n} a_{ij} = 0$ and $\kappa = \mathrm{diag}(\kappa_1, \kappa_2, \ldots, \kappa_n)$ with $\kappa_i > 0$, $i, j = 1, 2, \ldots, n$. If A is irreducible, then, for any $\kappa > 0$, $A + \kappa I$ is a non-singular M-matrix.*

3. Main results

In this section, novel non-chattering quantized controllers are designed such that the controlled network (3) achieves cluster synchronization in finite time. Moreover, the upper bounds of the synchronization time are estimated. Note that the proposed control schemes do not use the sign function, which avoid any chattering phenomenon and save communication resources.

For convenience, denote $k_{ij}(r_t) = k_{ijm}$ and $u_i(r_t, t) = u_i^m(t)$ when $m = r_t \in \mathcal{W}$. Let $e_i(t) = x_i(t) - y_\nu(t)$ for $i \in \mathcal{C}_\nu$, $\nu = 1, 2, \ldots, \wp$. It is easy to obtain from (H$_3$) that $\sum_{j\in\mathcal{C}_\nu} k_{ijm}y_\nu(t) = 0$ and $\sum_{\nu=1}^{\wp}\sum_{j\in\mathcal{C}_\nu} k_{ijm}x_j(t) = \sum_{\nu=1}^{\wp}\sum_{j\in\mathcal{C}_\nu} k_{ijm}x_j(t) - \sum_{\nu=1}^{\wp}\sum_{j\in\mathcal{C}_\nu} k_{ijm}y_\nu(t) = \sum_{\nu=1}^{\wp}\sum_{j\in\mathcal{C}_\nu} k_{ijm}e_j(t) = \sum_{j=1}^{N} k_{ijm}e_j(t)$. Then, combining (3) and (4) with Definition 1 and Lemma 1, one has that

$$\begin{cases} \dot{e}_i(t) = -C_\nu\int_{q_1 t}^{t} e_i(s)\mathrm{d}s + A_\nu\gamma_i(t) + \bigvee S_\nu \circ \alpha_i(q_2 t) \\ \qquad - \bigvee S_\nu \circ \beta_\nu(q_2 t) + \bigwedge D_\nu \circ \alpha_i(q_2 t) \\ \qquad - \bigwedge D_\nu \circ \beta_\nu(q_2 t) + \sum_{j=1}^{N} k_{ijm}\Phi e_j(t) + u_i^m(t), \ t > 0, \\ e_i(0) = \varphi_i(0) - \psi_\nu(0), \end{cases} \tag{5}$$

where $i \in \mathcal{C}_\nu$, $\nu = 1, 2, \ldots, \wp, \gamma_i(t) = \alpha_i(t) - \beta_\nu(t), \alpha_i(t) \in F(x_i(t))$, $\beta_\nu(t) \in F(y_\nu(t))$.

Design the following non-chattering quantized controllers:

$$u_{\nu i}^m(t) = \begin{cases} -(\xi_{\nu i}^m + \theta_\nu)h(e_i(t)) - \eta\dfrac{h(e_i(t))}{\|h(e(t))\|_1}, & \|e(t)\|_1 \neq 0, \\ 0, & \|e(t)\|_1 = 0, \end{cases} \tag{6}$$

where $i \in \mathcal{C}_\nu$, $\nu = 1, 2, \ldots, \wp$, $m \in \mathcal{W}$, $\xi_{\nu i}^m$, θ_ν and η are positive constants, $h(e_i(t)) = (h(e_{i1}(t)), h(e_{i2}(t)), \ldots, h(e_{in}(t)))^T$, $h(e(t)) = (h^T(e_1(t)), h^T(e_2(t)), \ldots, h^T(e_N(t)))^T$. $h(\cdot) : \mathbb{R} \to \Gamma$ is a quantizer, where $\Gamma = \{\pm w : w = \rho^i w_0, 0 < \rho < 1, i = 0, \pm 1, \pm 2, \ldots\} \cup \{0\}$ with sufficient large constant $w_0 > 0$. For $\forall v \in \mathbb{R}$, the quantizer $h(v)$ is defined as follows:

$$h(v) = \begin{cases} w, & \text{if } \dfrac{1}{1+\delta}w < v \leq \dfrac{1}{1-\delta}w, \\ 0, & \text{if } v = 0, \\ -q(-v), & \text{if } v < 0, \end{cases} \tag{7}$$

where $\delta = \frac{1-\rho}{1+\rho}$. According to the analysis of Xu et al. (2017), there exists a Filippov solution $\Delta \in [-\delta, \delta)$ such that $h(v) = (1 + \Delta)v$ for $\forall v \in \mathbb{R}$.

Remark 3. Note that, if $\|e(t)\|_1 = 0$ for some finite time t, then the synchronization is achieved and the controller is not needed, which is equivalent to $u_{\nu i}^m(t) = 0$. Therefore, the controllers in (6) are reasonable. In addition, the controllers in (6) are more practical than the following controllers in Yang and Lu (2016) and Zhang and Yang et al. (2017) as following:

$$u_i^m(t) = \begin{cases} -\xi_i^m e_i(t) - \eta\dfrac{e_i(t)}{\|e(t)\|_1}, & \|e(t)\|_1 \neq 0, \\ 0, & \|e(t)\|_1 = 0. \end{cases}$$

In practice, signal quantization is necessary because the arithmetic in practical control system is finite and capacity of communication channels is limited.

Remark 4. Compared with those finite-time control techniques in Abdurahman et al. (2015, 2016), Cui et al. (2014), Duan et al. (2018), Li and Yang et al. (2016), Wang (2018), Yang (2014), Yang and Cao et al. (2017), Yang et al. (2013) and Zhou et al. (2017), the controllers in (6) are designed without employing sign function, which do not introduce any chattering. Therefore, the controllers in (6) essentially improve those finite-time control techniques in Abdurahman et al. (2015, 2016), Cui et al. (2014), Duan et al. (2018), Li and Yang et al. (2016), Wang (2018), Yang (2014), Yang and Cao et al. (2017), Yang et al. (2013) and Zhou et al. (2017).

Based on the controllers in (6), our first main result is given below.

Theorem 1. *Suppose that* (H_1)*–*(H_3) *hold. Then the coupled DFCNNs (3) can be finite-timely synchronized onto DFCNNs (4) under the controllers (6) if the control gains* $\xi_{\nu i}^m, \theta_\nu$ *and* η *satisfy:*

$$\xi_{\nu i}^m > \frac{1}{(1-\delta)}(\|A_\nu\|_1\|\zeta\|_1 + k_{iim}\underline{\phi} + \sum_{j=1,j\neq i}^{N}|k_{jim}|\bar{\phi}) \triangleq \chi_{\nu i}^m, \tag{8}$$

$$\theta_\nu \geq \frac{(\|S_\nu\|_1 + \|D_\nu\|_1)\|\zeta\|_1}{q_2(1-\delta)}, \tag{9}$$

$$\eta = \sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}(\||A_\nu|\varrho\|_1 + \||S_\nu|\varrho\|_1 + \||D_\nu|\varrho\|_1) + \sigma. \tag{10}$$

Moreover, the settling time is estimated as $T = \frac{q\sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\|e_i(0)\|_1}{\underline{\rho}\sigma}$, *where* $\bar{\phi} = \max\{\phi_i, i = 1,2,\ldots,n\}$, $\underline{\phi} = \min\{\phi_i, i = 1,2,\ldots,n\}$, $\sigma > 0, q = \min\{q_1, q_2\}, \underline{\rho} = \min\{\rho_m, m \in \mathcal{W}\}, (\rho_1, \rho_2, \ldots, \rho_w)^T = \frac{1}{\upsilon}(-\Pi - \kappa)^{-1}\mathbf{1}_\omega$, $\kappa_m = \max\{\chi_{\nu i}^m - \xi_{\nu i}^m, i \in \mathcal{C}_\nu, m \in \mathcal{W}, \nu = 1,2\ldots,\wp\} < 0$, $\kappa = \mathrm{diag}(\kappa_1, \kappa_2, \ldots, \kappa_\omega)$, υ *is the maximum of the row sums of* $(-\Pi-\kappa)^{-1}$.

Proof. Due to $\chi_{\nu i}^m < \xi_{\nu i}^m$, it is easy to obtain that $\kappa_m = \max\{\chi_{\nu i}^m - \xi_{\nu i}^m, i \in \mathcal{C}_\nu, m \in \mathcal{W}, \nu = 1,2\ldots,\wp\} < 0$. Denote $\kappa = \mathrm{diag}(\kappa_1, \kappa_2, \ldots, \kappa_\omega)$. What is more, $-\Pi$ is an irreducible matrix since matrix Π is irreducible. Then, by Lemma 3, one knows that $-\Pi-\kappa$ is a nonsingular M-matrix. Furthermore, it follows from Lemma 2 that $(-\Pi-\kappa)^{-1}$ exists and all the elements of $(-\Pi-\kappa)^{-1}$ are nonnegative. Since $(-\Pi-\kappa)^{-1}$ is invertible, there is at least one positive constant in each row of $(-\Pi-\kappa)^{-1}$. Denote υ as the maximum of the row sums of $(-\Pi-\kappa)^{-1}$. Then all the elements of $(\rho_1, \rho_1, \ldots, \rho_w)^T = \frac{1}{\upsilon}(-\Pi-\kappa)^{-1}\mathbf{1}_\omega$ are positive and $\rho_m\kappa_m + \sum_{l\in\mathcal{W}}\pi_{ml}\rho_l = -\frac{1}{\upsilon} < 0, m \in \mathcal{W}, \max\{\rho_m, m \in \mathcal{W}\} = 1$ and $\underline{\rho} = \min\{\rho_m, m \in \mathcal{W}\}$.

When $r_t = m \in \mathcal{W}$, consider the following Lyapunov–Krasovskii functional:

$$V(e(t), m, t) = \sum_{i=1}^{3} V_i(e(t), m, t), \tag{11}$$

where

$$V_1(e(t), m, t) = \rho_m \sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\|e_i(t)\|_1,$$

$$V_2(e(t), m, t) = \sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\frac{(\|S_\nu\|_1 + \|D_\nu\|_1)\|\zeta\|_1}{q_2}\int_{q_2 t}^{t}\|e_i(s)\|_1 ds,$$

$$V_3(e(t), m, t) = \sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\underline{\rho}c_\nu\int_0^t\int_{q_1 s}^{s}\|e_i(u)\|_1 du ds,$$

$$c_\nu = \min\{c_j^\nu, j = 1,2\ldots,n\}.$$

Let $\mathcal{L}$ be the weak infinitesimal generator of the random process $(e(t), r_t, t)$. Then, by the chain rule in Clarkewrited (1983), differentiating $V_1(e(t), m, t)$ along the solutions of (5) and considering controller (6) produce that, for $\|e(t)\|_1 \neq 0$,

$$\begin{aligned}\mathcal{L}V_1(e(t), m, t) =& \rho_m\sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\mathbf{1}_n^T\mathrm{diag}(\mathrm{sign}(e_i(t)))\Big(-C_\nu\int_{q_1 t}^{t}e_i(s)ds \\ &+ A_\nu\gamma_i(t) + \bigvee S_\nu\circ\alpha_i(q_2 t) \\ &- \bigvee S_\nu\circ\beta_\nu(q_2 t) + \bigwedge D_\nu\circ\alpha_i(q_2 t) \\ &- \bigwedge D_\nu\circ\beta_\nu(q_2 t) + \sum_{j=1}^{N}k_{ijm}\Phi e_j(t) \\ &- (\xi_{\nu i}^m + \theta_\nu)h(e_i(t)) - \eta\frac{h(e_i(t))}{\|h(e(t))\|_1}\Big) \\ &+ \sum_{l\in\mathcal{W}}\pi_{ml}\rho_l\sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\|e_i(t)\|_1. \end{aligned}\tag{12}$$

It is obvious from (H_2) that,

$$\begin{aligned}\mathbf{1}_n^T\mathrm{diag}(\mathrm{sign}(e_i(t)))A_\nu\gamma_i((t)) \leq& \sum_{j=1}^{n}\sum_{l=1}^{n}\zeta_l|a_{jl}^\nu\|e_{il}(t)| \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n}|a_{jl}^\nu|\varrho_l \\ \leq& \|A_\nu\|_1\|\zeta\|_1\|e_i(t)\|_1 + \||A_\nu|\varrho\|_1. \end{aligned}\tag{13}$$

On the other hand, suppose there exist π and ϖ such that $\bigvee_{l=1}^{n}s_{jl}^\nu\alpha_{il}(q_2 t) = s_{j\pi}^\nu\alpha_{i\pi}(q_2 t)$ and $\bigvee_{l=1}^{n}s_{jl}^\nu\beta_{\nu l}(q_2 t) = s_{j\varpi}^\nu\beta_{\nu\varpi}(q_2 t)$. Then, it follows that $s_{j\varpi}^\nu\alpha_{i\varpi}(q_2 t) - s_{j\varpi}^\nu\beta_{\nu\varpi}(q_2 t) \leq s_{j\pi}^\nu\alpha_{i\pi}(q_2 t) - s_{j\varpi}^\nu\beta_{\nu\varpi}(q_2 t) \leq s_{j\pi}^\nu\alpha_{i\pi}(q_2 t) - s_{j\pi}^\nu\beta_{\nu\pi}(q_2 t)$ and $|s_{j\pi}^\nu\alpha_{i\pi}(q_2 t) - s_{j\varpi}^\nu\beta_{\nu\varpi}(q_2 t)| \leq \max(|s_{j\varpi}^\nu\alpha_{i\varpi}(q_2 t) - s_{j\varpi}^\nu\beta_{\nu\varpi}(q_2 t)|, |s_{j\pi}^\nu\alpha_{i\pi}(q_2 t) - s_{j\pi}^\nu\beta_{\nu\pi}(q_2 t)|) \leq \sum_{l=1}^{n}|s_{jl}^\nu\|\alpha_{il}(q_2 t) - \beta_{\nu l}(q_2 t)|$. Therefore, one obtains from (H_2) that

$$\begin{aligned}&\mathbf{1}_n^T\mathrm{diag}(\mathrm{sign}(e_i(t)))(\bigvee S_\nu\circ\alpha_i(q_2 t) - \bigvee S_\nu\circ\beta_\nu(q_2 t)) \\ &\leq \sum_{j=1}^{n}\sum_{l=1}^{n}|s_{jl}^\nu\|\alpha_{il}(q_2 t) - \beta_{\nu l}(q_2 t)| \\ &\leq \sum_{j=1}^{n}\sum_{l=1}^{n}\zeta_l|s_{jl}^\nu\|e_{il}(q_2 t)| + \sum_{j=1}^{n}\sum_{l=1}^{n}|s_{jl}^\nu|\varrho_l \\ &\leq \|S_\nu\|_1\|\zeta\|_1\|e_i(q_2 t)\|_1 + \||S_\nu|\varrho\|_1. \end{aligned}\tag{14}$$

By similar analysis methods in (14), one can obtain that

$$\begin{aligned}&\mathbf{1}_n^T\mathrm{diag}(\mathrm{sign}(e_i(t)))(\bigwedge D_\nu\circ\alpha_i(q_2 t) - \bigwedge D_\nu\circ\beta_\nu(q_2 t)) \\ &\leq \sum_{l=1}^{n}\sum_{j=1}^{n}|d_{jl}^\nu\|\alpha_{il}(q_2 t) - \beta_{\nu l}(q_2 t)| \\ &\leq \|D_\nu\|_1\|\zeta\|_1\|e_i(q_2 t)\|_1 + \||D_\nu|\varrho\|_1. \end{aligned}\tag{15}$$

Since $\delta \in (0,1)$ and $h(e_{ij}(t)) = (1+\Delta_{ij})e_{ij}(t)$, $\Delta_{ij} \in [-\delta, \delta)$, for all $i \in \mathcal{C}_\nu$, $\nu = 1,2\ldots,\wp, j = 1,2,\ldots,n$, one has $\mathrm{sign}(e_{ij}(t)) = \mathrm{sign}(h(e_{ij}(t)))$. Therefore,

$$\begin{aligned}&\mathbf{1}_n^T\mathrm{diag}(\mathrm{sign}(e_i(t)))\eta\frac{h(e_i(t))}{\|h(e(t))\|_1} = \eta\frac{\|h(e_i(t))\|_1}{\|h(e(t))\|_1} \quad \text{and} \\ &\times\sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\|h(e_i(t))\|_1 = \|h(e(t))\|_1, \end{aligned}\tag{16}$$

$$\begin{aligned}&\mathbf{1}_n^T\mathrm{diag}(\mathrm{sign}(e_i(t)))(\xi_{\nu i}^m + \theta_\nu)h(e_i(t)) = (\xi_{\nu i}^m + \theta_\nu)\|h(e_i(t))\|_1 \\ &\geq (\xi_{\nu i}^m + \theta_\nu)(1-\delta)\|e_i(t)\|_1. \end{aligned}\tag{17}$$

Combining (13)–(17) with (12), one can obtain that

$$\begin{aligned}\mathcal{L}V_1(e(t), m, t) \leq& \sum_{\nu=1}^{\wp}\sum_{i\in\mathcal{C}_\nu}\Bigg\{\sum_{l\in\mathcal{W}}\pi_{ml}\rho_l\|e_i(t)\|_1 + \rho_m\Big[\Big(\|A_\nu\|_1\|\zeta\|_1 \\ &+ k_{iim}\underline{\phi} + \sum_{j=1,j\neq i}^{N}|k_{jim}|\bar{\phi} \\ &- (\xi_{\nu i}^m + \theta_\nu)(1-\delta)\Big)\|e_i(t)\|_1 \end{aligned}$$

$$
\begin{aligned}
&+ (\|S_v\|_1 + \|D_v\|_1)\|\zeta\|_1\|e_i(q_2 t)\|_1 \\
&- C_v\int_{q_1 t}^{t}\|e_i(s)\|_1 \mathrm{d}s\Big]\Big\} + \rho_m\Big[\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}(\||A_v|\varrho\|_1 \\
&+ \||S_v|\varrho\|_1 + \||D_v|\varrho\|_1) - \eta\Big] \qquad (18)
\end{aligned}
$$

Similarly, taking the derivative of $V_2(e(t), m, t)$ and $V_3(e(t), m, t)$ derives that

$$
\begin{aligned}
\mathcal{L}V_2(e(t), m, t) = \sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\Big(&\frac{(\|S_v\|_1 + \|D_v\|_1)\|\zeta\|_1}{q_2}\|e_i(t)\|_1 - (\|S_v\|_1 \\
&+ \|D_v\|_1)\|\zeta\|_1\|e_i(q_2 t)\|_1\Big), \qquad (19)
\end{aligned}
$$

$$
\mathcal{L}V_3(e(t), m, t) = \sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\underline{\rho}c_v\int_{q_1 t}^{t}\|e_i(s)\|_1\mathrm{d}s. \qquad (20)
$$

Substituting (19)–(20) into (18) derives

$$
\begin{aligned}
\mathcal{L}V(e(t), m, t) \le \sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\Big[\Big(&\sum_{l\in\mathcal{W}}\pi_{ml}\rho_l + \rho_m(\|A_v\|_1\|\zeta\|_1 + k_{iim}\underline{\phi} \\
&+ \sum_{j=1,j\ne i}^{N}|k_{jim}|\bar{\phi} - \xi_i^m(1-\delta))\Big) \\
&+ \Big(\frac{(\|S_v\|_1 + \|D_v\|_1)\|\zeta\|_1}{q_2} - \theta_v(1-\delta)\Big)\Big] \\
&\times \|e_i(t)\|_1 \\
&+ \rho_m\Big[\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}(\||A_v|\varrho\|_1 + \||S_v|\varrho\|_1 \\
&+ \||D_v|\varrho\|_1) - \eta\Big]. \qquad (21)
\end{aligned}
$$

Since $\rho_m(\chi_i^m - \xi_i^m) + \sum_{l\in\mathcal{W}}\pi_{ml}\rho_l \le -\frac{1}{v} < 0$, for $m \in \mathcal{W}$, $i \in \mathcal{C}_v$, $v = 1, 2\ldots, \wp$, it is derived from conditions (10)–(11) and (21) that

$$
\begin{aligned}
\mathcal{L}V(e(t), m, t) \le& \sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\Big(\sum_{l\in\mathcal{W}}\pi_{ml}\rho_l + \rho_m(\chi_{vi}^m - \xi_{vi}^m)\Big)\|e_i(t)\|_1 \\
&+ \sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\Big(\frac{(\|S_v\|_1 + \|D_v\|_1)\|\zeta\|_1}{q_2} - \theta_v(1-\delta)\Big) \\
&\times \|e_i(t)\|_1 \\
&+ \rho_m\Big[\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}(\||A_v|\varrho\|_1 + \||S_v|\varrho\|_1 \\
&+ \||D_v|\varrho\|_1) - \eta\Big] \\
\le& -\underline{\rho}\sigma, \qquad (22)
\end{aligned}
$$

where $\underline{\rho} = \min\{\rho_m, m \in \mathcal{W}\}$, $\sigma > 0$ is a constant.

By the arbitrariness of $m \in \mathcal{W}$, it follows from (22) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\{V(e(t), r_t, t)\} \le -\underline{\rho}\sigma. \qquad (23)
$$

By the same discussions as those in Li and Yang et al. (2016), Yang (2014), Yang and Cao et al. (2017), Zhang and Yang et al. (2017) and Zhou et al. (2017), there exists $t_1 \in (0, +\infty)$

$$
\lim_{t\to t_1}\mathbf{E}\{V(e(t), r_t, t)\} = 0 \text{ and } \mathbf{E}\{V(e(t), r_t, t)\} \equiv 0, \quad \text{for } t \ge t_1. \qquad (24)
$$

Integrating both sides of the inequality (23) from 0 to t_1 produces

$$
t_1 \le \frac{V(e(0), r_0, 0)}{\underline{\rho}\sigma}. \qquad (25)
$$

Note that (24) also means that $\lim_{t\to T}\mathbf{E}\{\|e_i(t)\|_1\} = 0$ and $\mathbf{E}\{\|e_i(t)\|_1\} \equiv 0$, for $t \ge T$, where $T = qt_1$. As matter as fact, seeing from the inequalities (24)–(25), one can get $\mathbf{E}\{V(e(t_1), m, t_1)\} = 0$, *i.e.*, $\mathbf{E}\{\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\|e_i(t_1)\|_1\} = 0$ and $\mathbf{E}\{\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\int_{q_2 t_1}^{t_1}\|e_i(s)\|_1 \mathrm{d}s\} = 0$ and $\mathbf{E}\{\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\int_0^{t_1}\int_{q_1 s}^{s}\|e_i(u)\|_1\mathrm{d}u\mathrm{d}s\} = 0$. Furthermore, $\int_{qt_1}^{t_1}\|e_i(s)\|_1\mathrm{d}s = 0$, where $q = \min\{q_1, q_2\}$. If not, suppose that $\|e_i(qt_1)\| > 0$. Then, by the continuity of $\|e_i(s)\|_1$, there is an arbitrary small interval $[qt_1, \varsigma) \subset [qt_1, t_1]$ such that $\|e_i(s)\|_1 > 0$ for $s \in [qt_1, \varsigma)$, which imply that $\int_{qt_1}^{\varsigma}\|e_i(s)\|_1\mathrm{d}s > 0$. That is, at least one of $\mathbf{E}\{\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\int_{q_2 t_1}^{t_1}\|e_i(s)\|_1\mathrm{d}s\} > 0$ and $\mathbf{E}\{\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\int_0^{t_1}\int_{q_1 s}^{s}\|e_i(u)\|_1\mathrm{d}u\mathrm{d}s\} > 0$ is true, which contradicts with $\mathbf{E}\{V(e(t_1), m, t_1)\} = 0$. Finally, one derives that $\|e_i(qt_1)\|_1\mathrm{d}s = 0$, and the synchronization time can correct to $T = qt_1$. Hence, the coupled discontinuous neural networks (3) can synchronize onto (1) in T. This completes proof.

Remark 5. It should be emphasized that the effect of time-delays on FTS or finite-time stability is extremely difficult to overcome. However, the analysis methods developed in this paper can be applied to FTS of dynamical systems with or without delays, which are completely different from those used in Abdurahman et al. (2015, 2016), Cui et al. (2014), Duan et al. (2018), Gao et al. (2017), Jiang et al. (2015), Tang (1998), Tang et al. (2017), Wang (2018), Xu et al. (2017), and Yang et al. (2013). From the proof of Theorem 1, the key step is to obtain the inequality (23). If the 2-norm based Lyapunov functions as those in are used, Abdurahman et al. (2015, 2016), Cui et al. (2014), Duan et al. (2018), Gao et al. (2017), Jiang et al. (2015), Tang (1998), Tang et al. (2017), Wang (2018), Xu et al. (2017), and Yang et al. (2013), the inequality (23) cannot be derived. Hence, the 1-norm-based analytical technique is effective for finite-time synchronization of delayed systems.

Remark 6. Noted that, in spite of the effects of both proportional leakage and time-varying delays, the FTCS of Markovian coupled DFCNNs (3) can still be achieved. Moreover, the settling time is estimated. On the other hand, Theorem 1 is different from the results in Yang and Song et al. (2015). It is reported in Yang and Song et al. (2015) that coupled discontinuous neural networks with infinite-time distributed delay can achieve synchronization in finite time but the settling time cannot be estimated. The main reason for the estimation of the settling time for coupled discontinuous neural networks with proportional delay is that $qt \to +\infty$ when $t \to +\infty$. Therefore, $\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\frac{(\|S_v\|_1+\|D_v\|_1)\|\zeta\|_1}{q_2}\int_{q_2 t}^{t}\|e_i(s)\|_1\mathrm{d}s = 0$ and $\sum_{v=1}^{\wp}\sum_{i\in\mathcal{C}_v}\underline{\rho}c_v\int_0^t\int_{q_1 s}^{s}\|e_i(u)\|_1\mathrm{d}u\mathrm{d}s = 0$ when the synchronization has been realized. ■

Specially, when $\wp = 1$, the FTCS becomes the finite-time complete synchronization with non-chattering quantized control. In this case, the network (3) with N identical nodes (1) can be described as:

$$
\begin{cases}
\dot{x}_i(t) = -C\int_{q_1 t}^{t}x_i(s)\mathrm{d}s + Af(x_i(t)) + \bigvee S\circ f(x_i(q_2 t)) \\
\qquad + \bigwedge D\circ f(x_i(q_2 t)) + \bigvee R\circ v \\
\qquad + \bigwedge H\circ v + J + \sum_{j=1}^{N}k_{ij}(r_t)\Phi x_j(t) + u_i(r_t, t), \\
\qquad\quad i \in \mathcal{N} = \{1, 2, \ldots, N\},\ t > 0, \\
x_i(0) = \varphi_i(0),
\end{cases} \qquad (26)
$$

and the target trajectory $y(t)$ satisfies that

$$\begin{cases}\dot{y}(t)=-C\int_{q_1t}^{t}y(s)\mathrm{d}s+Af(y(t))+\bigvee S\circ f(y(q_2t))\\ \qquad +\bigwedge D\circ f(y(q_2t))\\ \qquad +\bigvee R\circ \upsilon+\bigwedge H\circ \upsilon+J,\\ y(0)=\psi(0).\end{cases} \tag{27}$$

The following result can be derived from Theorem 1.

Corollary 1. *Suppose that* (H_1)*–*(H_3) *hold. Then, under the controllers in (6), the coupled DFCNNs (26) can be finite-timely synchronized with the DFCNN (27) if the control gains* ξ_i^m, θ *and* η *satisfy:*

$$\xi_i^m > \frac{1}{(1-\delta)}(\|A\|_1\|\zeta\|_1+k_{iim}\underline{\phi}+\sum_{j=1,j\neq i}^{N}|k_{jim}|\bar{\phi}\triangleq \chi_i^m), \tag{28}$$

$$\theta \geq \frac{(\|S\|_1+\|D\|_1)\|\zeta\|_1}{q_2(1-\delta)}, \tag{29}$$

$$\eta = N(\||A|\varrho\|_1+\||S|\varrho\|_1+\||D|\varrho\|_1)+\sigma. \tag{30}$$

Moreover, the settling time is estimated as $T=\frac{q\sum_{i=1}^{N}\|e_i(0)\|_1}{\underline{\rho}\sigma}$, *where* $\bar{\phi}=\max\{\phi_i, i=1,2,\ldots,n\}$, $\underline{\phi}=\min\{\phi_i, i=1,2,\ldots,n\}$, $\sigma>0$, $q=\min\{q_1,q_2\}$, $\underline{\rho}=\min\{\rho_m, m\in\mathcal{W}\}$, $(\rho_1,\rho_1,\ldots,\rho_w)^T=\frac{1}{\upsilon}(-\Pi-\kappa)^{-1}\mathbf{1}_\omega$, $\kappa_m=\max\{\chi_i^m-\xi_i^m, i\in\mathcal{N}, m\in\mathcal{W}\}<0$, $\kappa=\mathrm{diag}(\kappa_1,\kappa_2,\ldots,\kappa_\omega)$, υ *is the maximum of the row sums of* $(-\Pi-\kappa)^{-1}$.

When $\wp=1$ and $N=1$, we can realize finite-time driver-response synchronization. The driver system can be modeled as:

$$\begin{cases}\dot{x}(t)=-C\int_{q_1t}^{t}x(s)\mathrm{d}s+Af(x(t))+\bigvee S\circ f(x(q_2t))\\ \qquad +\bigwedge D\circ f(x(q_2t))+\bigvee R\circ \upsilon\\ \qquad +\bigwedge H\circ \upsilon+J,\quad t>0,\\ x(0)=\varphi(0),\end{cases} \tag{31}$$

and the controlled response system is described as

$$\begin{cases}\dot{y}(t)=-C\int_{q_1t}^{t}y(s)\mathrm{d}s+Af(y(t))+\bigvee S\circ f(y(q_2t))\\ \qquad +\bigwedge D\circ f(y(q_2t))+\bigvee R\circ \upsilon\\ \qquad +\bigwedge H\circ \upsilon+J+u(t),\quad t>0,\\ y(0)=\psi(0).\end{cases} \tag{32}$$

Then, similar to (5), it is easy to obtain the following error system:

$$\begin{cases}\dot{z}(t)=-C\int_{q_1t}^{t}z(s)\mathrm{d}s+A\hat{\gamma}(t)+\bigvee S\circ \alpha(q_2t)-\bigvee S\circ \beta(q_2t)\\ \qquad +\bigwedge D\circ \alpha(q_2t)-\bigwedge D\circ \beta(q_2t)+u(t),\quad t>0,\\ z(0)=\psi(0)-\varphi(0),\end{cases} \tag{33}$$

where $z(t)=y(t)-x(t)=(z_1(t),z_2(t),\ldots,z_n(t))^T$, $\hat{\gamma}(t)=\hat{\alpha}(t)-\hat{\beta}(t)$, $\hat{\alpha}(t)\in F(x)$, $\hat{\beta}(t)\in F(y)$.

Theorem 2. *Suppose that* (H_1)*–*(H_2) *hold. Then the DFCNN (32) can be finite-timely synchronized with the DFCNN (31) under the following controller*

$$u(t)=\begin{cases}-\Xi h(z(t))-\widehat{\eta}\dfrac{h(z(t))}{\|h(z(t))\|_1}, & \|z(t)\|_1\neq 0,\\ 0, & \|z(t)\|_1=0,\end{cases} \tag{34}$$

where $\frac{h(z(t))}{\|h(z(t))\|_1}=(\frac{h(z_1(t))}{\|h(z(t))\|_1},\frac{h(z_2(t))}{\|h(z(t))\|_1},\ldots,\frac{h(z_n(t))}{\|h(z(t))\|_1})^T$, $h(z(t))=(h(z_1(t)),h(z_2(t)),\ldots,h(z_n(t)))^T$, $\Xi=\mathrm{diag}(\widehat{\xi}_1,\widehat{\xi}_2,\ldots,\widehat{\xi}_n)$, $\widehat{\xi}_j>\frac{1}{(1-\delta)}(\sum_{l=1}^{n}\zeta_l|a_{jl}|+\frac{1}{q_2}\sum_{l=1}^{n}\zeta_l(|s_{jl}|+|d_{jl}|))$, $j=1,2,\ldots,n$, $\widehat{\eta}=\sum_{j=1}^{n}\sum_{l=1}^{n}(|a_{jl}|+|s_{jl}|+|d_{jl}|)\varrho_j+\widehat{\sigma}$, $\widehat{\sigma}>0$. *Moreover, the settling time is estimated as:* $T=\frac{q\sum_{j=1}^{n}|z_j(0)|}{\widehat{\sigma}}$, *where* $q=\min\{q_1,q_2\}$.

Proof. Consider the following Lyapunov–Krasovskii functional:

$$\bar{V}(t)=\|z(t)\|_1+\underline{c}\int_0^t\int_{q_1s}^{s}\|z(u)\|_1\mathrm{d}u\mathrm{d}s+\frac{1}{q_2}\sum_{j=1}^{n}\sum_{l=1}^{n}|\zeta_l|(|s_{jl}|+|d_{jl}|)\int_{q_2t}^{t}|z_j(s)|\mathrm{d}s, \tag{35}$$

where $\underline{c}=\min\{c_j, j=1,2,\ldots,n\}$.

Using similar procedure as that given in Theorem 1, when $\|e(t)\|_1\neq 0$, one derives

$$\begin{aligned}\dot{\bar{V}}(t)=&\mathbf{1}_n^T\mathrm{diag}(\mathrm{sign}(z(t)))\Big[-C\int_{q_1t}^{t}z(s)\mathrm{d}s+A\hat{\gamma}(t)+\bigvee S\circ\alpha(q_2t)\\ &-\bigvee S\circ\beta(q_2t)\\ &+\bigwedge D\circ\alpha(q_2t)-\bigwedge D\circ\beta(q_2t)-\Xi h(z(t))\\ &-\eta\frac{h(z(t))}{\|h(z(t))\|_1}\Big]+\underline{c}\int_{q_1t}^{t}\|e(u)\|_1\mathrm{d}u\mathrm{d}s\\ &+\frac{1}{q_2}\sum_{j=1}^{n}\sum_{l=1}^{n}|\zeta_l|(|s_{jl}|+|d_{jl}|)|z_j(t)|-\sum_{j=1}^{n}\sum_{l=1}^{n}|\zeta_l|\\ &\times(|s_{jl}|+|d_{jl}|)|z_j(q_2t)|.\end{aligned} \tag{36}$$

By using the same analysis method in the proof of Theorem 1, it follows that

$$\begin{aligned}\dot{\bar{V}}(t)\leq&\sum_{j=1}^{n}[\sum_{l=1}^{n}\zeta_j|a_{jl}|+\frac{1}{q_2}\sum_{l=1}^{n}|\zeta_l|(|s_{jl}|+|d_{jl}|)-\xi_j(1-\delta)]|z_j(t)|\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n}(|a_{jl}|+|s_{jl}|+|d_{jl}|)\varrho_j-\eta\\ \leq&-\widehat{\sigma}.\end{aligned} \tag{37}$$

The rest part is same as that given in the proof of Theorem 1. Thus the finite-time synchronization between systems (31) and (32) can be achieved. The proof is completed.

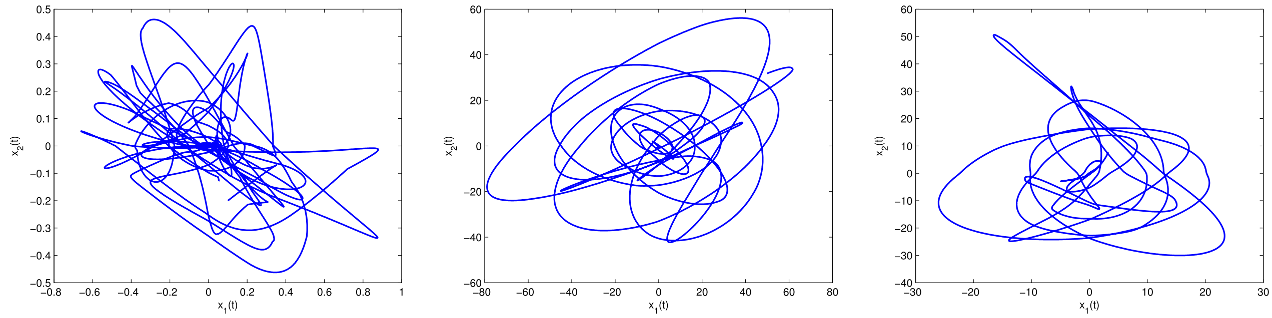
Remark 7. Since the controllers in (34) are very simple and do not need any information of the delay and sign function, they are more easier to apply in practice than those in Duan et al. (2018) and Wang (2018). Recently, FTS for DFCNNs with discrete delays in Duan et al. (2018) and FCNNs with proportional delays in Wang (2018) were studied, respectively. To overcome the effect of delays, complicated controllers with the information of delays were designed to delete the corresponding terms directly. Furthermore, the controllers in Duan et al. (2018) and Wang (2018) have to utilize the sign function to drive the coupled systems synchronization in finite time. Thus, the controller (34) essentially improves the corresponding those in Duan et al. (2018) and Wang (2018). ■

4. Numerical examples

In this section, two numerical examples are given to illustrate the effectiveness of Theorems 1 and 2. The time step size is taken as 0.01 and the quantizer density $\rho=0.8$ in the simulations.

Example 1. Consider a network consisting of 6 identical nodes with Markovian topology, which is supposed to be divided into three clusters: $\mathcal{C}_1=\{1,2,3\}$, $\mathcal{C}_2=\{4\}$, $\mathcal{C}_3=\{5,6\}$. The coupled FCNNs are described by

$$\begin{aligned}\dot{x}_i(t)=&-C_\nu\int_{q_1t}^{t}x_i(s)\mathrm{d}s+A_\nu f(x_i(t))+\bigvee S_\nu\circ f(x_i(q_2t))\\ &+\bigwedge D_\nu\circ f(x_i(q_2t))+\bigvee R_\nu\circ\upsilon_\nu\end{aligned}$$



(a) (b) (c)

Fig. 1. Trajectories of the each node of the three clusters with the same initial condition $\varphi(0) = (0.1, -0.2)^T$ under different system parameters: (a) $C_1, A_1, S_1, D_1, R_1, H_1, v_1$; (b) $C_2, A_2, S_2, D_2, R_2, H_2, v_2$; (c) $C_3, A_3, S_3, D_3, R_3, H_3, v_3$.

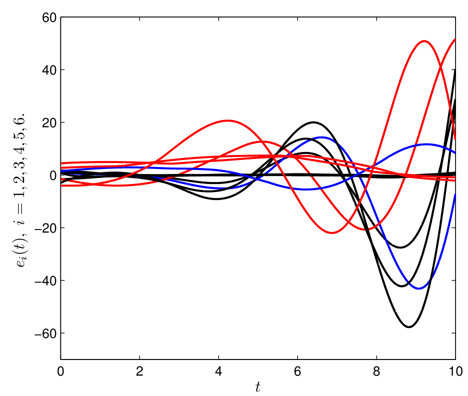


Fig. 2. Trajectories $e_i(t)$ of the coupled networks (38) without controllers.

$$+ \bigwedge H_v \circ v_v + J_v + \sum_{j=1}^{6} k_{ij}(r_t)\Phi x_j(t) + u_i(r_t, t), \;\; i \in \mathcal{C}_v,$$
$$v = 1, 2, 3, \tag{38}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$, $q_1 = 0.6$, $q_2 = 0.65$, $J_v = 0$, $C_1 = \mathrm{diag}(2.2, 2.5)$, $C_2 = \mathrm{diag}(0.9, 0.8)$, $C_3 = \mathrm{diag}(0.5, 0.3)$, $v_1 = (0.2, 0.2)^T$, $v_2 = (0, 0)^T$, $v_3 = (0.1, 0.1)^T$, $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)))^T$ with

$$f_j(x_{ij}(t)) = \begin{cases} \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|) + 0.02, & \text{if } x_j(t) \geq 0, \\ \quad j = 1, 2, & \\ \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|) - 0.08, & \text{if } x_j(t) < 0, \\ \quad j = 1, 2, & \end{cases}$$

$$A_1 = \begin{pmatrix} -0.3 & -1.2 \\ 0.3 & -0.3 \end{pmatrix}, S_1 = \begin{pmatrix} 0.95 & -0.2 \\ -1 & 0.4 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -0.08 & -0.02 \\ 0.5 & -0.7 \end{pmatrix}, A_2 = \begin{pmatrix} 0.7 & -0.5 \\ -0.5 & 0.5 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} -0.2 & 1.6 \\ 1.2 & 0.5 \end{pmatrix}, D_2 = \begin{pmatrix} 1.8 & -0.1 \\ 1.1 & 0.4 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1.5 \\ 0.9 & 0.5 \end{pmatrix}, R_1 = R_2 = R_3 = \begin{pmatrix} 2.7 & -0.12 \\ -0.7 & 0.5 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 1.7 & 0.6 \\ 0.4 & 0.5 \end{pmatrix}, D_3 = \begin{pmatrix} 1.8 & -0.1 \\ -0.8 & 1.4 \end{pmatrix},$$

$$H_1 = H_2 = H_3 = \begin{pmatrix} -1.2 & -0.3 \\ -0.1 & 2 \end{pmatrix}.$$

Let $\mathcal{W} = \{1, 2\}$, $\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and

$$K_1 = \begin{pmatrix} -1.5 & 0.5 & 1 & 0 & 0 & 0 \\ 0.4 & -1.5 & 1.1 & 1 & 0 & -1 \\ 1 & 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & -1 & -0.6 & 0.6 \\ -1 & 0 & 0 & 1 & -1 & 1 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} -1 & 0.5 & 0.5 & -0.1 & 0 & 0.1 \\ 0.3 & -1 & 0.7 & 0.4 & 0 & -0.4 \\ 0 & 0.5 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & -0.4 \\ 0.6 & 0.8 & 0.6 & -2 & -1 & 1 \\ -0.5 & 0.5 & 0 & 0 & -0.3 & 0.3 \end{pmatrix}.$$

Fig. 1 shows the chaotic-like trajectories of the each node of the three clusters under the same initial values $\varphi(0) = (0.1, -0.2)^T$. It is easy to check that the discontinuous activation functions $f_j(\cdot)$, $j = 1, 2$ satisfy (H_1). Moreover, by simple computation, one can get $\zeta_1 = \zeta_2 = 1$, $\varrho_1 = \varrho_2 = 0.1$. Hence, (H_1)–(H_3) are all satisfied.

By simple computation, one can obtain from (8)–(10) that $\chi_{11}^1 = 4.95$, $\chi_{12}^1 = 4.5$, $\chi_{13}^1 = 4.6125$, $\chi_{24}^1 = 4.725$, $\chi_{35}^1 = 4.5$, $\chi_{36}^1 = 4.725$, $\chi_{11}^2 = 3.2625$, $\chi_{12}^2 = 4.275$, $\chi_{13}^2 = 3.125$, $\chi_{24}^2 = 4.1625$, $\chi_{35}^2 = 3.0375$, $\chi_{36}^2 = 4.875$, $\theta_1 = 4.6212$, $\theta_2 = 8.6538$, $\theta_3 = 8.1346$, then, with any irreducible Π, the coupled network (38) can be finite-timely synchronized by controllers (6).

Take the generator matrix as

$$\Pi = \begin{pmatrix} -80 & 80 \\ 70 & -70 \end{pmatrix}. \tag{39}$$

In the simulations, the initial values of the coupled network are randomly chosen as $\varphi_1(t) = (0.69, -0.43)^T$, $\varphi_2(t) = (1.48, 0.18)^T$, $\varphi_3(t) = (-0.27, 0.04)^T$, $\varphi_4(t) = (-0.10, 0.35)^T$, $\varphi_5(t) = (0.75, 1.05)^T$. Choosing $\eta = 5$ (*i.e.* $\sigma = 0.285$), $\xi_{11}^1 = 5$, $\xi_{12}^1 = 4.6$, $\xi_{13}^1 = 4.7$, $\xi_{24}^1 = 4.8$, $\xi_{35}^1 = 4.6$, $\xi_{36}^1 = 4.8$, $\xi_{11}^2 = 3.3$, $\xi_{12}^2 = 4.3$, $\xi_{13}^2 = 3.2$, $\xi_{24}^2 = 4.2$, $\xi_{35}^2 = 3.1$, $\xi_{36}^2 = 4.7$, we can get $\kappa = -\mathrm{diag}(0.1, 0.0788)$. Furthermore, one can get $\rho_1 = 0.9998$, $\rho_2 = 1$. Moreover, the settling time can be explicitly obtained:

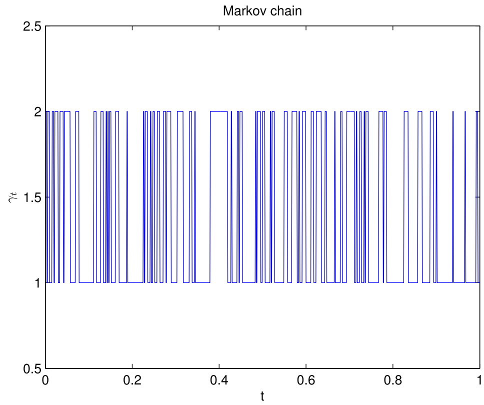


Fig. 3. Markov chain generated by probability transition matrix corresponding to the generator (39) with $\Delta = 0.01$ and $r_0 = 1$.

$T = 53.2502$. Fig. 2 shows the error trajectories of coupled networks (38) without controllers (6). Fig. 3 shows the Markovian chain generated by probability transition matrix corresponding to generator (39) with $\gamma_0 = 1$. Fig. 4 describes the error trajectories of coupled networks (38) with controllers (6). The states variables of the nodes in the network (38) are presented in Fig. 5, which shows that the cluster synchronization is realized.

Example 2. Consider the following three dimensional DFCNN as the drive system:

$$\begin{aligned}\dot{x}(t) &= -C\int_{q_1 t}^{t} x(s)\mathrm{d}s + Af(x(t)) + \bigvee S\circ f(x(q_2 t)) \\ &\quad + \bigwedge D\circ f(x(q_2 t)) + \bigvee R\circ v + \bigwedge H\circ v + J, \end{aligned} \tag{40}$$

and the response system is described as:

$$\begin{aligned}\dot{y}(t) &= -C\int_{q_1 t}^{t} y(s)\mathrm{d}s + Af(y(t)) + \bigvee S\circ f(y(q_2 t)) \\ &\quad + \bigwedge D\circ f(y(q_2 t)) + \bigvee R\circ v + \bigwedge H\circ v \\ &\quad + J + u(t), \end{aligned} \tag{41}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$, $y(t) = (y_1(t), y_2(t), y_3(t))^T$, $q_1 = 0.5$, $q_2 = 0.75$, $J = (0.9, 0.9, 0.9)^T$, $v = (1, 2, 1)^T$, $C = \mathrm{diag}(2, 1, 1.2)$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), f_3(x_3(t)))^T$ with

$$f_j(x_j(t)) = \begin{cases} \frac{1}{2}(|x_j(t)+1| - |x_j(t)-1|) + 0.05, & \text{if } x_j(t) \geq 0, \\ \quad j = 1, 2, 3, \\ \frac{1}{2}(|x_j(t)+1| - |x_j(t)-1|) - 0.05, & \text{if } x_j(t) < 0, \\ \quad j = 1, 2, 3. \end{cases}$$

$$A = \begin{pmatrix} 0.7 & -0.5 & 0 \\ -0.5 & -0.4 & 0.5 \\ 0.3 & -1.2 & 0.5 \end{pmatrix},$$

$$S = \begin{pmatrix} -0.2 & 0.7 & 1.6 \\ 1.2 & -0.5 & 0.5 \\ 1.5 & -0.1 & 0.5 \end{pmatrix}, D = \begin{pmatrix} 1.8 & -0.1 & 0.9 \\ 1.1 & -2 & 0.4 \\ 2.1 & -2 & 1.4 \end{pmatrix},$$

$$R = \begin{pmatrix} 2.7 & 0.6 & -0.12 \\ -0.7 & 0.5 & 0.5 \\ 0.8 & 1.2 & 1.3 \end{pmatrix},$$

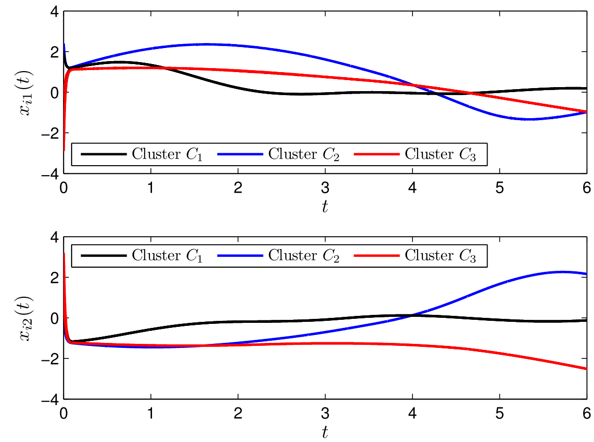
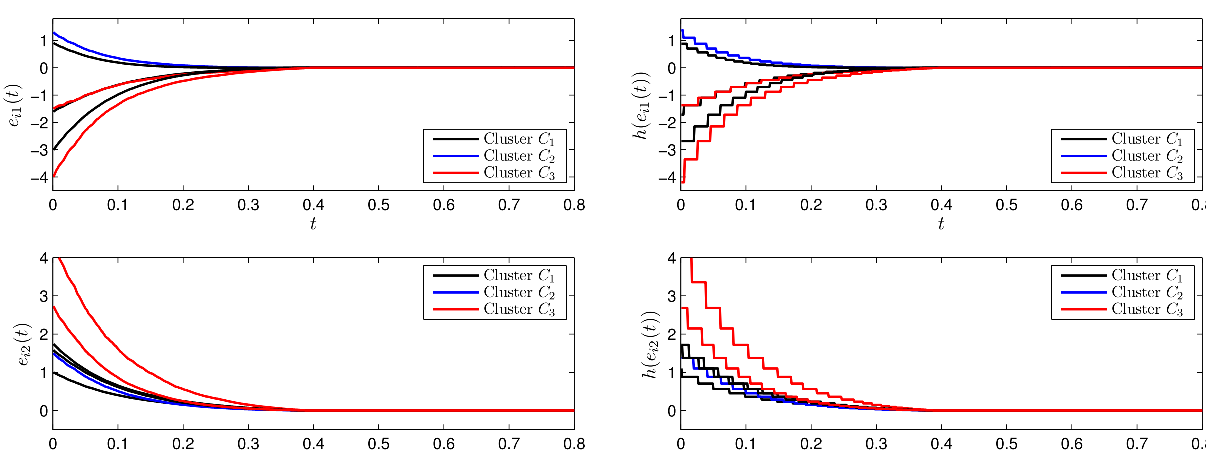
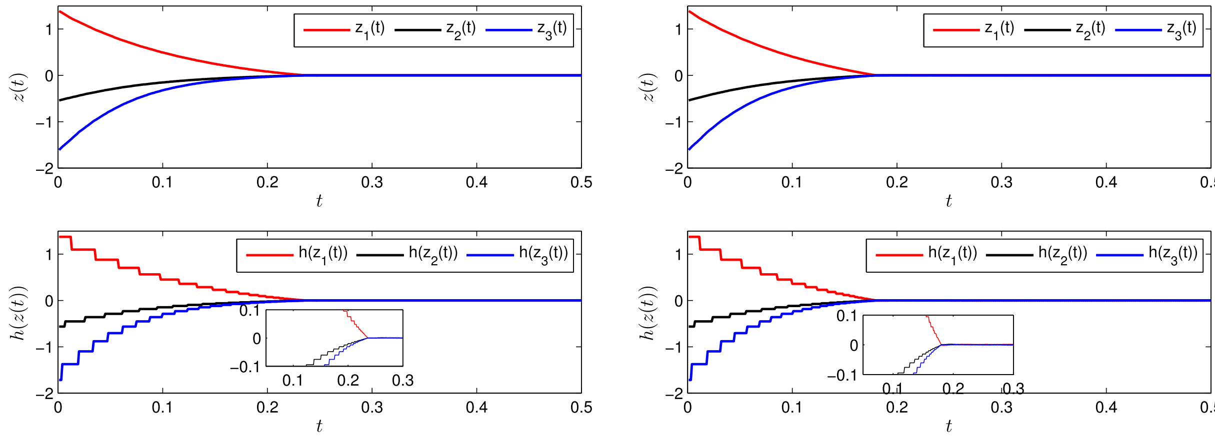


Fig. 5. Trajectories of states $x_{i1}(t)$ and $x_{i2}(t)$ of the coupled networks (38) via controllers (6) with $\eta = 5$, (*i.e.* $\sigma = 0.285$), where $i = 1, 2, \ldots, 6$.



(a) (b)

Fig. 4. Trajectories (a) $e_{i1}(t), e_{i2}(t)$ and (b) $h(e_{i1}(t)), h(e_{i1}(t))$ $(i = 1, 2, \ldots, 6)$ of the coupled networks (38) via controllers (6) with $\eta = 5$, (*i.e.* $\sigma = 0.285$).



(a) (b)

Fig. 6. Time evolutions of $z(t)$ and $h(z(t))$ under the quantized controller (34) with different $\widehat{\eta}$: (a) $\widehat{\eta}=3$ ($\widehat{\sigma}=0.78$); (b) $\widehat{\eta}=6$ ($\widehat{\sigma}=3.78$).

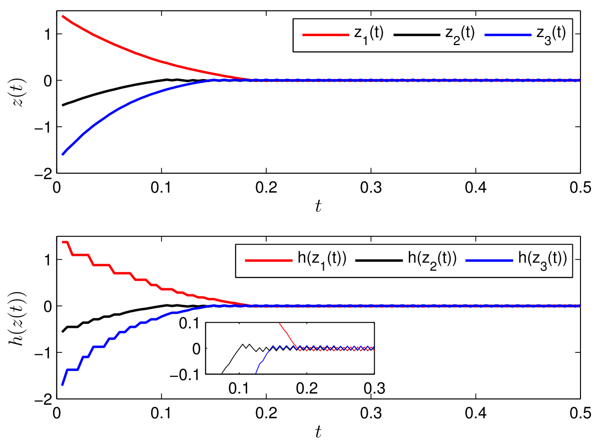


Fig. 7. Time evolutions of $z(t)$ and $h(z(t))$ under the quantized controller $u(t)=-\Xi h(z(t))-\widehat{\eta}\text{sign}(h(z(t)))$, where the control gains are the same as those used in Fig. 6.

$$H=\begin{pmatrix}-1.2 & 0.1 & -0.3\\ -0.1 & 2 & 0.4\\ -1.1 & -0.9 & 0.4\end{pmatrix}.$$

It is easy to check that the discontinuous activation functions $f_j(\cdot), j=1,2,3$ satisfy (H_1). Moreover, by simple computation, one can get $\zeta_1=\zeta_2=\zeta_3=1$, $\varrho_1=\varrho_2=0.1$. Hence, (H_1)–(H_2) are all satisfied.

By simple computation, it follows from (8)–(10) that $\widehat{\xi}_1>9.3$, $\widehat{\xi}_2>10.125$, $\widehat{\xi}_3>13.65$, and $\widehat{\eta}=2.22+\widehat{\sigma}$. In the simulations, the initial values of systems (40) and (41) are randomly chosen as $x(0)=(0.62, 0.92, 1.42)^T$, $y(0)=(2.01, 0.38, -0.19)^T$, respectively. Choosing $\widehat{\xi}_1=9.4$, $\widehat{\xi}_2=10.2$, $\widehat{\xi}_3=13.7$, and different $\widehat{\sigma}$, the driver-response systems (40) and (41) can realize finite-time synchronization via the controller (34). Especially, when $\widehat{\sigma}=0.78$, we get $\widehat{\eta}=3$ and $T\le 2.2692$. When $\widehat{\sigma}=3.78$, we get $\widehat{\eta}=6$ and $T\le 0.4683$, which can be respectively described by (a) and (b) of Fig. 6.

It should be noted that the chattering will be induced if the controller in system (41) is replaced by $u(t)=-\Xi h(z(t))-\widehat{\eta}\text{sign}(h(z(t)))$ with the same control gains as those used in Fig. 6. The time evolutions of $z(t)$ and $h(z(t))$ under the quantized controller $u(t)=-\Xi h(z(t))-\widehat{\eta}\text{sign}(h(z(t)))$ are shown in Fig. 7, where the chattering is obvious. One can find from Figs. 6 and 7 that the non-chattering controller (34) are better than $u(t)=-\Xi h(z(t))-\widehat{\eta}\text{sign}(h(z(t)))$.

Remark 8. The parameter η (or $\widehat{\eta}$) in the controller (6) (or (34)) play two roles: (1) eliminates the effects of uncertainties, and (2) realize finite-time synchronization. Particularly, σ (or $\widehat{\sigma}$) is a tunable parameter, which can adjust the length of the settling time T. Generally speaking, larger value of σ (or $\widehat{\sigma}$) can decrease the synchronization time T, as shown in Fig. 6.

5. Conclusions

In this paper, FTCS of coupled DFCNNs with Markovian topology, proportional leakage, and time-varying delays has been studied. The designed non-chattering controllers can not only save both communication channels and bandwidth, but also overcome the effects of the uncertainties caused by time-varying delays, Filippov solution, as well as Markovian jumping and further achieve FTCS. By constructing new Lyapunov–Krasovskii functionals and utilizing several effective analytical methods, sufficient synchronization criteria have been derived. Moreover, the settling time is explicitly estimated. Numerical simulations illustrated the effectiveness of theoretical results.

Note that the settling time is heavily dependent on the initial condition. If the initial condition of the considered system is unknown, the settling time cannot be estimated. Therefore, fixed-time synchronization is our next research topic, where the settling time is not dependent on the initial condition. Moreover, intermittent control scheme is a useful control technique to reduce control cost. How to achieve fixed-time synchronization via intermittent control is also challenging.

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