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Finite-time synchronization of delayed fuzzy cellular neural networks with discontinuous activations ☆

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Received 1 July 2017; received in revised form 4 February 2018; accepted 26 April 2018

Abstract

In this paper, we study the finite-time synchronization issue for delayed fuzzy cellular neural networks with discontinuous activations. Under the framework of differential inclusions, by utilizing the discontinuous state feedback control method and constructing Lyapunov functionals, new and useful finite-time synchronization criteria for the considered networks are established, which significantly generalize and improve recent works in literature. Finally, two examples with simulations are presented to show the effectiveness of the synchronization schemes.

Keywords: Fuzzy cellular neural network; Discontinuous activation; Finite-time synchronization

1. Introduction

During the last three decades, Cellular neural networks (CNNs), proposed by Chua and Yang [1,2], have attracted rapidly growing interest because of their wide range of potential applications in many fields as in combinatorial optimization, pattern recognition, signal processing and so on, see [3–5] and the reference therein. In the implementation of neural networks, such as in mathematical modelling of real world problems, the uncertainty or vagueness is inevitable, in order to take vagueness into consideration, Yang and Chua in [6,7] further introduced the so-called fuzzy cellular neural networks (FCNNs) on the basis of traditional CNNs, which have fuzzy logic between its template and input and/or output besides the "sum of product" operation. Since then, many studies have been shown that FCNNs provide a useful paradigm in image processing and pattern recognition, see [8–11] and the references therein. It was

☆ This work was supported by the National Natural Science Foundation of China (11701007, 11601007, 11771059), Natural Science Foundation of Anhui Province (1808085QA01), Key Program of University Natural Science Research Fund of Anhui Province (KJ2017A088), Key Program of Scientific Research Fund for Young Teachers of Anhui University of Science and Technology (QN201605), and the Doctoral Fund of Anhui University of Science and Technology (11668).

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https://doi.org/10.1016/j.fss.2018.04.017

also found that time-delay inevitably appears owing to the finite switching speed of the amplifiers and communication time, and it may destroy a stable network and cause sustained oscillations, bifurcation or chaos [12]. Thus, it is of great importance to study the dynamical behaviors of the FCNNs with time delays.

In most results considering the stabilization or synchronization of neural networks, the convergent mode is asymptotically stable or exponential stable of the synchronization error dynamics, which means that the response system can track the drive system over the infinite horizon. From a practical point of view, however, the convergence time often required to be faster or even finite. For example, the intercept missiles can track the aim under windage and signal interference in the finite-time, when a missile traversed the sky [13]. Hence, in order to achieve a convergence in a given time, finite-time control methods have been known to be useful and efficient. In recent years, some researchers have applied finite-time control techniques to realize synchronization for kinds of neural networks, see [14–20] and the references therein. Very recently, based on the finite-time stability theory, differential inequality techniques and the analysis approach, A. Abdurahman et al. in [21] and W. Wang in [22] studied the finite-time synchronization problem of FCNNs with time-varying delays or time-varying coefficients and proportional delays, respectively. Nevertheless, it is noteworthy that the quoted results on finite-time synchronization concern fuzzy neural networks where the neuron activations are modeled by Lipschitz or the following Lipschitz-like continuous functions:

(H) There exist a real constant F_j^1 and a positive real number F_j^2 such that

$$F_j^1 \le \frac{f_j(u)-f_j(v)}{u-v} \le F_j^2$$

for each $u, v \in \mathbb{R}, u \neq v$ and $j = 1, ..., n$.

Actually, such an assumption might not be realistic in the true environment of networks because the high-gain hypothesis is often imposed on the activation functions [23], in view of this, a neural network modeled by a differential equation with discontinuous activation functions should be considered instead of a continuous form, and then much attention has been devoted to the dynamics of neural networks with discontinuous activations in the last years, see, e.g., [24–29] and the reference therein. However, no analysis for the finite-time synchronization of delayed FCNNs with discontinuity effects of the activations has been done up to now, and there exist open room for further improvement. On the other hand, it is worth mentioning that nonlinear systems with discontinuous characteristics are often to achieve finite-time stability or convergence [30], and thus it is interesting and important to investigate the finite-time synchronization dynamics between the drive and response systems.

Inspired by the aforementioned discussions, the aim of this paper is to establish some novel finite-time synchronization criteria of delayed fuzzy neural networks with discontinuous activations, the methods used here are mainly based on the well-known finite-time convergence theory for discontinuous delay differential equations, the obtained theoretical results are more general and they effectually complement or improve the previously known results. Numerical analysis and simulations demonstrate the effectiveness of our new results.

The rest of the paper is organized as follows. In Section 2, the drive system and response system are introduced. In addition, some assumptions and definitions together with some useful lemmas needed in this paper are presented. Section 3 is devoted to investigating the finite-time synchronization between two delayed FCNNs with discontinuous activations. In Section 4, two examples with their numerical simulations are presented to illustrate the effectiveness of the main results. Some conclusions are finally drawn in Section 5.

2. Model description and preliminaries

Consider the following delayed FCNNs with discontinuous activations

$$\begin{aligned}\dot{x}_i(t) = &- c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} v_j + \bigwedge_{j=1}^n T_{ij} v_j \\ &+ \bigwedge_{j=1}^n \alpha_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^n \beta_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^n S_{ij} v_j + I_i,\ i = 1, 2, ..., n, \end{aligned} \tag{2.1}$$

Please cite this article in press as: L. Duan et al., Finite-time synchronization of delayed fuzzy cellular neural networks with discontinuous activations, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.04.017

where $x_i(t)$ denotes the state of the ith unit at time t; c_i represents the passive decay rate to the state of ith unit; a_{ij}, b_{ij} are the elements of feedback and feed-forward templates; α_{ij} and β_{ij} are elements of fuzzy feedback MIN template, fuzzy feedback MAX template, respectively; T_{ij} and S_{ij} are fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively; v_i and I_i denote input and bias of the ith neuron, respectively. f_i is the activation function, and $\tau_i(t) \geq 0$ corresponds to the transmission delay along the axon of the ith unit, $i, j = 1, 2, ..., n$.

Throughout this paper, we make the following assumptions on the activation functions.

(H1) For each $i = 1, 2, ..., n$, $f_i : \mathbb{R} \to \mathbb{R}$ is continuous except on a countable set of isolate points $\{\rho_\kappa^i\}$, where there exist finite right and left limits $f_i(\rho_\kappa^{i+})$ and $f_i(\rho_\kappa^{i-})$, respectively. Moreover, f_i has at most a finite number of jump discontinuities in every compact interval of $\mathbb{R}$.

(H2) For every $i = 1, 2, ..., n$, there exist nonnegative constants L_i and Q_i such that

$$|\gamma_i - \eta_i| \leq L_i |x_i - y_i| + Q_i, \tag{2.2}$$

$\forall x_i, y_i \in \mathbb{R}$, where $\gamma_i \in \overline{co}[f_i(x_i)]$, $\quad \eta_i \in \overline{co}[f_i(y_i)]$.

Remark 2.1. If f_i is discontinuous at x_i, then $\overline{co}[f_i(x_i)]$ is an interval with non-empty interior, that is,

$$\overline{co}[f_i(x_i)] = \Big[\min\{f_i(x_i^-), f_i(x_i^+)\}, \max\{f_i(x_i^-), f_i(x_i^+)\}\Big], \quad i = 1, 2, ..., n.$$

If f_i is continuous at x_i, then $\overline{co}[f_i(x_i)] = \{f_i(x_i)\}$ is a singleton.

Now, we introduce the concept of Filippov solutions from Filippov [31] for discontinuous system (2.1).

Definition 2.1 *(Filippov solution).* A function $x = (x_1, x_2, ..., x_n)^{\mathrm{T}}$ is said to be a solution of (2.1) on $[-\tau, T)$ $\big(\tau = \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}} \tau_i(t), T \in (0, +\infty]\big)$ if

(1) $x = (x_1, x_2, ..., x_n)^{\mathrm{T}}$ is continuous on $[-\tau, T]$ and absolutely continuous on $[0, T)$;

(2) there exists a measurable function $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)^{\mathrm{T}} : [-\tau, T) \to \mathbb{R}^n$ such that $\gamma_j(t) \in \overline{co}[f_j(x_j(t))]$ for a.a. $t \in [-\tau, T)$ and

$$\begin{aligned}\dot{x}_i(t) = &- c_i x_i(t) + \sum_{j=1}^n a_{ij}\gamma_j(t) + \sum_{j=1}^n b_{ij}v_j + \bigwedge_{j=1}^n T_{ij}v_j + \bigwedge_{j=1}^n \alpha_{ij}\gamma_j(t - \tau_j(t)) \\ &+ \bigvee_{j=1}^n \beta_{ij}\gamma_j(t - \tau_j(t)) + \bigvee_{j=1}^n S_{ij}v_j + I_i, \quad \text{for a.a.} \quad t \in [0, T), \quad i = 1, 2, ..., n.\end{aligned} \tag{2.3}$$

Remark 2.2. The validity and rationality of the existence of measurable selection function γ has been explicitly studied and described in many references, see, e.g. [28], we omit it here.

Definition 2.2 *(Initial value problem).* For any continuous function $\varphi = (\varphi_1, \varphi_2, ..., \varphi_n)^{\mathrm{T}} : [-\tau, 0] \to \mathbb{R}^n$ and any measurable selection $\psi = (\psi_1, \psi_2, ..., \psi_n)^{\mathrm{T}} : [-\tau, 0] \to \mathbb{R}^n$, such that $\psi_j(s) \in \overline{co}[f_j(\varphi_j(s))](j = 1, 2, ..., n)$ for a.a. $s \in [-\tau, 0]$ by an initial value problem associated to (2.1) with initial condition (φ, ψ) means the following problem: find functions $x(t)$, $\gamma(t)$, such that $x(t)$ is a solution of (2.1) on $[-\tau, T)$ for some $T > 0$, $\gamma(t)$ is an output solution associated to $x(t)$, and

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum\limits_{j=1}^{n} a_{ij}\gamma_j(t) + \sum\limits_{j=1}^{n} b_{ij}v_j + \bigwedge\limits_{j=1}^{n} T_{ij}v_j + \bigwedge\limits_{j=1}^{n} \alpha_{ij}\gamma_j(t-\tau_j(t)) \\ \qquad + \bigvee\limits_{j=1}^{n} \beta_{ij}\gamma_j(t-\tau_j(t)) + \bigvee\limits_{j=1}^{n} S_{ij}v_j + I_i, \quad \text{for a.a.} \quad t \in [0, T), \\ \gamma_j(t) \in \overline{co}[f_j(x_j(t))], \quad \text{for a.a.} \quad t \in [0, T), \\ x_i(s) = \varphi_i(s), \quad \forall s \in [-\tau, 0], \\ \gamma_j(s) = \psi_j(s), \quad \text{for a.a. } s \in [-\tau, 0]. \end{cases} \tag{2.4}$$

In this paper, we consider system (2.1) as the drive system, the response system is given as follows

$$\begin{aligned} \dot{y}_i(t) = &- c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}v_j + \bigwedge_{j=1}^{n} T_{ij}v_j \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij} f_j(y_j(t-\tau_j(t))) + \bigvee_{j=1}^{n} \beta_{ij} f_j(y_j(t-\tau_j(t))) + \bigvee_{j=1}^{n} S_{ij}v_j + I_i \\ &+ u_i(t), \ i = 1, 2, ..., n, \end{aligned} \tag{2.5}$$

where $y_i(t)$ corresponds to the state variable of the ith neuron of response system and $u_i(t)$ indicates the controller to be designed, the other parameters have the dynamic meanings as those in (2.1). Similar to Definition 2.2, the initial conditions associated with the response system (2.5) are given by

$$\begin{cases} \dot{y}_i(t) = -c_i y_i(t) + \sum\limits_{j=1}^{n} a_{ij}\eta_j(t) + \sum\limits_{j=1}^{n} b_{ij}v_j + \bigwedge\limits_{j=1}^{n} T_{ij}v_j + \bigwedge\limits_{j=1}^{n} \alpha_{ij}\eta_j(t-\tau_j(t)) \\ \qquad + \bigvee\limits_{j=1}^{n} \beta_{ij}\eta_j(t-\tau_j(t)) + \bigvee\limits_{j=1}^{n} S_{ij}v_j + I_i + u_i(t), \quad \text{for a.a.} \quad t \in [0, T), \\ \eta_j(t) \in \overline{co}[f_j(y_j(t))], \quad \text{for a.a.} \quad t \in [0, T), \\ y_i(s) = \phi_i(s), \quad \forall s \in [-\tau, 0], \\ \eta_j(s) = \chi_j(s), \quad \text{for a.a. } s \in [-\tau, 0]. \end{cases} \tag{2.6}$$

Definition 2.3 *(See [21]).* Drive-response systems (2.1) and (2.5) are said to be synchronized in a finite time, if for a suitable designed controller $u_i(t)$, there exists a constant $\mathcal{T} > 0$ such that

$$\lim_{t\to\mathcal{T}} |y_i(t) - x_i(t)| = 0,$$

and

$$|y_i(t) - x_i(t)| = 0,$$

for $t > \mathcal{T}, i = 1, ..., n$.

Definition 2.4 *(See [32]).* Given $g : \mathbb{R}^n \to \mathbb{R}$, the right directional derivative of g at x in the direction $v \in \mathbb{R}^n$ is defined as

$$g'(x, v) = \lim_{\rho\downarrow 0} \frac{g(x+\rho v) - g(x)}{\rho}$$

when this limit exists. The generalized directional derivative of g at x in the direction $v \in \mathbb{R}^n$ is defined as

$$g^0(x, v) = \limsup_{\substack{y\to x \\ \rho\downarrow 0}} \frac{g(y+\rho v) - g(y)}{\rho}.$$

A function $g : \mathbb{R}^n \to \mathbb{R}$, which is locally Lipschitz near $x \in \mathbb{R}^n$, is said to be regular at x if, for all $v \in \mathbb{R}^n$, the right directional derivative of g at x in the direction of v exists, and $g'(x, v) = g^0(x, v)$.

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Definition 2.5 *(See [32]).* A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular if $V(x)$ is:
(i) regular in $\mathbb{R}^n$;
(ii) positive definite, i.e., $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$;
(iii) radially unbounded, i.e., $V(x) \to +\infty$ as $\|x\|_2 \to +\infty$, where $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Remark 2.3. Let

$$V(x(t)) = \sum_{i=1}^{n} |x_i(t)|^p, \quad p \geq 1.$$

It is readily seen that $V(x(t))$ is a locally Lipschitz and convex function on $\mathbb{R}^n$, hence it is regular in $\mathbb{R}^n$. It is easy to see that $V(x(t))$ also satisfies (ii) and (iii) in Definition 2.5. Therefore, $V(x(t))$ is C-regular.

Lemma 2.1 *(See [29,32]). Suppose that $V(x(t)) : \mathbb{R}^n \to \mathbb{R}$ is C-regular, and that $x(t) : [0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact interval of $[0, +\infty)$. Let $v(t) = V(x(t))$, and suppose that there exists a continuous function $\Upsilon : (0, +\infty) \to \mathbb{R}$, with $\Upsilon(\sigma) > 0$ for $(0, +\infty)$, such that*

$$\dot{v}(t) \leq -\Upsilon(v(t))$$

and

$$\int_0^{v(0)} \frac{1}{\Upsilon(\sigma)} d\sigma = \mathbb{T} < +\infty.$$

Then we have

$$v(t) = 0, \quad t \geq \mathbb{T}$$

and

$$x(t) = 0, \quad t \geq \mathbb{T},$$

i.e., $v(t)$ converges to 0, and $x(t)$ converges to $x = 0$, in finite time $\mathbb{T}$ (namely, settling time or halting time). In particular,
(1) if $\Upsilon(\sigma) = Q_1\sigma + Q_2\sigma^{\mu}$, for any $\sigma \in (0, +\infty)$, where $\mu \in (0, 1)$ and $Q_1, Q_2 > 0$, then

$$\mathbb{T} = \frac{1}{Q_1(1-\mu)} \ln \frac{Q_1 v^{1-\mu}(0) + Q_2}{Q_2};$$

(2) if $\Upsilon(\sigma) = Q\sigma^{\mu}$, for all $\sigma \in (0, +\infty)$, where $\mu \in (0, 1)$ and $Q > 0$, then

$$\mathbb{T} = \frac{v^{1-\mu}(0)}{Q(1-\mu)}.$$

The following lemmas will be crucial to compute the time derivative along the solutions of error system of the Lyapunov functions introduced in the next section.

Lemma 2.2 *(See [24]). Let $e(t)$ be a solution of the error system, which is defined on $[0, T)$, $T \in (0, +\infty]$. Then, the function $|e(t)|$ is absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}t}|e(t)| = \nu^T(t)\dot{e}(t) = \sum_{i=1}^{n} \nu_i(t)\dot{e}_i(t) \quad \textit{for a.a. } t \in [0, T),$$

where

$$\overline{\mathrm{co}}[\mathrm{sign}(e_i(t))] \ni \nu_i(t) = \begin{cases} \mathit{sign}(e_i(t)) & \textit{if } e_i(t) \neq 0, \\ \textit{arbitrary chosen in } [-1, 1] & \textit{if } e_i(t) = 0. \end{cases}$$

Remark 2.4. Since the sign function in the following designed state feedback controller is discontinuous on the error function, and if the error function does not equal to 0, then it can be seen from Lemma 2.2 that

$$\nu_i(t) \cdot \mathrm{sign}(e_i(t)) = 1, \quad i = 1, 2, ..., n,$$

which plays an important role in the proof of the main results in Section 3.

Lemma 2.3 *(See [6]). Let $x, \tilde{x}$ be the two states of system (2.1), then we have*

$$\left| \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(\tilde{x}_j) \right| \le \sum_{j=1}^{n} |\alpha_{ij}||f_j(x_j) - f_j(\tilde{x}_j)|,$$

$$\left| \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(\tilde{x}_j) \right| \le \sum_{j=1}^{n} |\beta_{ij}||f_j(x_j) - f_j(\tilde{x}_j)|.$$

Lemma 2.4 *(Jesen inequality, see [33]). If $a_1, a_2, ..., a_n$ are positive numbers and $0 < r < p$, then*

$$\left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}}.$$

Lemma 2.5 *(See [33]). If $a_1, a_2, ..., a_n$ are positive numbers, then*

$$n a_1 a_2 \cdots a_n \le a_1^n + a_2^n + \cdots + a_n^n.$$

3. Main results

In this section, we shall establish some new criteria to guarantee the finite-time synchronization between the discontinuous drive system (2.1) and discontinuous response system (2.5).

Suppose that $x_i(t)$, $y_i(t)$ are two arbitrary solutions of system (2.1) and system (2.5), and let $e_i(t) = y_i(t) - x_i(t)$, then the error system can be written as follows:

$$\begin{aligned} \dot{e}_i(t) = &- c_i e_i(t) + \sum_{j=1}^{n} a_{ij}(\eta_j(t) - \gamma_j(t)) + \bigwedge_{j=1}^{n} \alpha_{ij} \eta_j(t - \tau_j(t)) - \bigwedge_{j=1}^{n} \alpha_{ij} \gamma_j(t - \tau_j(t)) \\ &+ \bigvee_{j=1}^{n} \beta_{ij} \eta_j(t - \tau_j(t)) - \bigvee_{j=1}^{n} \beta_{ij} \gamma_j(t - \tau_j(t)) + u_i(t). \end{aligned} \tag{3.1}$$

It is readily seen from Lemma 2.1 that the finite-time synchronization problem between response system (2.5) and drive system (2.1) can be transformed to the equivalent problem of the finite-time stability of the error system (3.1).

Now, we are ready to state and prove our main results as follows.

Theorem 3.1. *Let (H1) and (H2) hold, if the error system (3.1) is controlled with the following control law,*

$$u_i(t) = -\rho_i e_i(t) - sign(e_i(t))(\lambda_i + k|e_i(t)|^{\mu}) - \omega_i sign(e_i(t))|e_i(t - \tau_i(t))|, i = 1, 2, ..., n,$$

where the control parameters $\rho_i, \lambda_i, \omega_i$ are positive constants satisfying

$$\rho_i > \sum_{j=1}^{n} |a_{ji}| L_i - c_i, \quad \omega_i \ge \sum_{j=1}^{n} \big(|\alpha_{ji}| + |\beta_{ji}|\big) L_i, \quad \lambda_i \ge \sum_{j=1}^{n} (|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|) Q_j,$$

$k > 0$ denotes a tunable constant and $0 < \mu < 1$ is a positive constant, then the response system (2.5) can synchronize with the drive system (2.1) in a finite time

$$\mathbb{T}_1 = \frac{1}{(1-\mu)\min\limits_{1\le i\le n}\left\{c_i+\rho_i-\sum\limits_{j=1}^{n}|a_{ji}|L_i\right\}}\ln\left(1+\min_{1\le i\le n}\left\{c_i+\rho_i-\sum_{j=1}^{n}|a_{ji}|L_i\right\}k^{-1}V_1^{1-\mu}(0)\right),$$

where $V_1=\sum\limits_{i=1}^{n}|e_i(0)|$.

Proof. Consider the following Lyapunov function:

$$V_1(t)=\sum_{i=1}^{n}|e_i(t)|. \tag{3.2}$$

Calculating the derivative of $V_1(t)$ along the trajectories of system (3.1), and using Lemma 2.2, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_1(t)=&\sum_{i=1}^{n}\nu_i(t)\Bigg[-c_ie_i(t)+\sum_{j=1}^{n}a_{ij}(\eta_j(t)-\gamma_j(t))+\bigwedge_{j=1}^{n}\alpha_{ij}\eta_j(t-\tau_j(t))-\bigwedge_{j=1}^{n}\alpha_{ij}\gamma_j(t-\tau_j(t))\\&+\bigvee_{j=1}^{n}\beta_{ij}\eta_j(t-\tau_j(t))-\bigvee_{j=1}^{n}\beta_{ij}\gamma_j(t-\tau_j(t))+u_i(t)\Bigg]\\\le&-\sum_{i=1}^{n}c_i|e_i(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||\eta_j(t)-\gamma_j(t)|+\sum_{i=1}^{n}\left|\bigwedge_{j=1}^{n}\alpha_{ij}\eta_j(t-\tau_j(t))-\bigwedge_{j=1}^{n}\alpha_{ij}\gamma_j(t-\tau_j(t))\right|\\&+\sum_{i=1}^{n}\left|\bigvee_{j=1}^{n}\beta_{ij}\eta_j(t-\tau_j(t))-\bigvee_{j=1}^{n}\beta_{ij}\gamma_j(t-\tau_j(t))\right|-\sum_{i=1}^{n}\rho_i|e_i(t)|-\sum_{i=1}^{n}\lambda_i\\&-k\sum_{i=1}^{n}|e_i(t)|^{\mu}-\sum_{i=1}^{n}\omega_i|e_i(t-\tau_i(t))|,\quad\text{for a.a. } t\ge 0.\end{aligned}\tag{3.3}$$

It is shown from (2.2) that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||\eta_j(t)-\gamma_j(t)|&\le\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|(L_j|e_j(t)|+Q_j)\\&=\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|(L_i|e_i(t)|+Q_i).\end{aligned}\tag{3.4}$$

On the other hand, it is readily obtained from Lemma 2.3 and (2.2) that

$$\begin{aligned}\left|\bigwedge_{j=1}^{n}\alpha_{ij}\eta_j(t-\tau_j(t))-\bigwedge_{j=1}^{n}\alpha_{ij}\gamma_j(t-\tau_j(t))\right|&\le\sum_{j=1}^{n}|\alpha_{ij}||\eta_j(t-\tau_j(t))-\gamma_j(t-\tau_j(t))|\\&\le\sum_{j=1}^{n}|\alpha_{ij}|\big(L_j|e_j(t-\tau_j(t))|+Q_j\big),\end{aligned}\tag{3.5}$$

similarly,

$$\begin{aligned}\left|\bigvee_{j=1}^{n}\beta_{ij}\eta_j(t-\tau_j(t))-\bigvee_{j=1}^{n}\beta_{ij}\gamma_j(t-\tau_j(t))\right|&\le\sum_{j=1}^{n}|\beta_{ij}||\eta_j(t-\tau_j(t))-\gamma_j(t-\tau_j(t))|\\&\le\sum_{j=1}^{n}|\beta_{ij}|\big(L_j|e_j(t-\tau_j(t))|+Q_j\big).\end{aligned}\tag{3.6}$$

Substituting (3.4)–(3.6) into (3.3), yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) \le & -\sum_{i=1}^{n} c_i|e_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||\eta_j(t)-\gamma_j(t)| + \sum_{i=1}^{n}\left|\bigwedge_{j=1}^{n}\alpha_{ij}\eta_j(t-\tau_j(t)) - \bigwedge_{j=1}^{n}\alpha_{ij}\gamma_j(t-\tau_j(t))\right| \\ & + \sum_{i=1}^{n}\left|\bigvee_{j=1}^{n}\beta_{ij}\eta_j(t-\tau_j(t)) - \bigvee_{j=1}^{n}\beta_{ij}\gamma_j(t-\tau_j(t))\right| - \sum_{i=1}^{n}\rho_i|e_i(t)| - \sum_{i=1}^{n}\lambda_i \\ & - k\sum_{i=1}^{n}|e_i(t)|^{\mu} - \sum_{i=1}^{n}\omega_i|e_i(t-\tau_i(t))| \\ \le & -\sum_{i=1}^{n}\Big(c_i+\rho_i-\sum_{j=1}^{n}|a_{ji}|L_i\Big)|e_i(t)| - \sum_{i=1}^{n}\Big(\omega_i - \sum_{j=1}^{n}\big(|\alpha_{ji}|+|\beta_{ji}|\big)L_i\Big)|e_i(t-\tau_i(t))| \\ & - \sum_{i=1}^{n}\Big(\lambda_i - \sum_{j=1}^{n}(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)Q_j\Big) - k\sum_{i=1}^{n}|e_i(t)|^{\mu}, \quad \text{for a.a. } t \ge 0. \end{aligned} \tag{3.7}$$

Considering $0<\mu<1$, it can be found from Lemma 2.4 that

$$\sum_{i=1}^{n}|e_i(t)| \le \left(\sum_{i=1}^{n}|e_i(t)|^{\mu}\right)^{1/\mu}. \tag{3.8}$$

By virtue of

$$\rho_i > \sum_{j=1}^{n}|a_{ji}|L_i - c_i, \quad \omega_i \ge \sum_{j=1}^{n}\big(|\alpha_{ji}|+|\beta_{ji}|\big)L_i, \quad \lambda_i \ge \sum_{j=1}^{n}(|a_{ij}|+|\alpha_{ij}|+|\beta_{ij}|)Q_j,$$

and combining (3.7) and (3.8), produces

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) \le -\min_{1\le i\le n}\left\{c_i+\rho_i-\sum_{j=1}^{n}|a_{ji}|L_i\right\}V_1(t) - kV_1^{\mu}(t), \quad \text{for a.a. } t\ge 0.$$

Therefore, according to Lemma 2.1, the two drive-response fuzzy neural networks can be synchronized in a finite time, and the settling time can be explicitly expressed by

$$\mathbb{T}_1 = \frac{1}{(1-\mu)\min\limits_{1\le i\le n}\left\{c_i+\rho_i-\sum\limits_{j=1}^{n}|a_{ji}|L_i\right\}}\ln\left(1+\min_{1\le i\le n}\left\{c_i+\rho_i-\sum_{j=1}^{n}|a_{ji}|L_i\right\}k^{-1}V_1^{1-\mu}(0)\right).$$

This completes the proof.

Theorem 3.2. *Let (H1) and (H2) hold. If the error system (3.1) is controlled with the following control law*

$$u_i(t) = -\rho_i e_i(t) - sign(e_i(t))(\lambda_i + k|e_i(t)|^{\mu}) - \sum_{j=1}^{n}\omega_{ij}sign(e_i(t))|e_j(t-\tau_j(t))|, i=1,2,...,n,$$

where the control parameters $\rho_i, \lambda_i, \omega_{ij}$ *are positive constants satisfying* $\rho_i \ge -c_i + \mathfrak{a}_i^* + \mathscr{A}_i^*, \lambda_i \ge \mathcal{A}_i^* + \alpha_i^* + \beta_i^*, \omega_{ij} \ge \tilde{\alpha}_{ij} + \tilde{\beta}_{ij}, \mathfrak{a}_i^* = L_i|a_{ii}|, \mathcal{A}_i^* = \sum_{j=1}^{n} Q_j|a_{ij}|, \mathscr{A}_i^* = \sum_{j=1, j\ne i}^{n}\sum_{m=1}^{p-1}|a_{ij}|^{p\mathbb{K}_m}L_j^{p\mathbb{L}_m} + \sum_{j=1, j\ne i}^{n}|a_{ji}|^{p\mathbb{K}_p}L_i^{p\mathbb{L}_p}, \sum_{m=1}^{p}\mathbb{K}_m = \sum_{m=1}^{p}\mathbb{L}_m = 1, \tilde{\alpha}_{ij} = |\alpha_{ij}|L_j, \alpha_i^* = \sum_{j=1}^{n}|\alpha_{ij}|Q_j, \tilde{\beta}_{ij} = |\beta_{ij}|L_j, \beta_i^* = \sum_{j=1}^{n}|\beta_{ij}|Q_j$, *and* $k>0$ *denotes a tunable constant. Then the response system (2.5) can be synchronized with the drive system (2.1) in a finite time*

$$\mathbb{T}_2 = \frac{V_2^{\frac{1-\mu}{p}}(0)}{k(1-\mu)},$$

in which $V_2(0) = \sum_{i=1}^{n}|e_i(0)|^p, p\ge 2$.

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Proof. Consider the following Lyapunov function:

$$V_2(t)=\sum_{i=1}^{n}|e_i(t)|^p,\qquad p\geq 2. \tag{3.9}$$

Calculating the derivative of $V_2(t)$ along the trajectories of error system (3.1), we obtain from Lemma 2.2 that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t)=&\sum_{i=1}^{n}p|e_i(t)|^{p-1}\nu_i(t)\Big[-c_ie_i(t)+\sum_{j=1}^{n}a_{ij}(\eta_j(t)-\gamma_j(t))+\bigwedge_{j=1}^{n}\alpha_{ij}\eta_j(t-\tau_j(t))\\
&-\bigwedge_{j=1}^{n}\alpha_{ij}\gamma_j(t-\tau_j(t))+\bigvee_{j=1}^{n}\beta_{ij}\eta_j(t-\tau_j(t))-\bigvee_{j=1}^{n}\beta_{ij}\gamma_j(t-\tau_j(t))+u_i(t)\Big]\\
\leq&-\sum_{i=1}^{n}pc_i|e_i(t)|^p+\sum_{i=1}^{n}\sum_{j=1}^{n}p|a_{ij}||e_i(t)|^{p-1}|\eta_j(t)-\gamma_j(t)|\\
&+\sum_{i=1}^{n}p|e_i(t)|^{p-1}\Big|\bigwedge_{j=1}^{n}\alpha_{ij}\eta_j(t-\tau_j(t))-\bigwedge_{j=1}^{n}\alpha_{ij}\gamma_j(t-\tau_j(t))\Big|\\
&+\sum_{i=1}^{n}p|e_i(t)|^{p-1}\Big|\bigvee_{j=1}^{n}\beta_{ij}\eta_j(t-\tau_j(t))-\bigvee_{j=1}^{n}\beta_{ij}\gamma_j(t-\tau_j(t))\Big|-\sum_{i=1}^{n}p\rho_i|e_i(t)|^p\\
&-\sum_{i=1}^{n}p\lambda_i|e_i(t)|^{p-1}-\sum_{i=1}^{n}pk|e_i(t)|^{p-1+\mu}-\sum_{i=1}^{n}\sum_{j=1}^{n}p\omega_{ij}|e_i(t)|^{p-1}|e_j(t-\tau_j(t))|,\quad\text{for a.a. } t\geq 0.
\end{aligned}\tag{3.10}$$

Now we estimate the right-hand sides of (3.10) term by term, firstly, in view of Lemma 2.5, one has

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}p|a_{ij}||e_i(t)|^{p-1}|\eta_j(t)-\gamma_j(t)|\\
\leq&\sum_{i=1}^{n}\sum_{j=1}^{n}p|a_{ij}||e_i(t)|^{p-1}(L_j|e_j(t)|+Q_j)\\
=&\sum_{i=1}^{n}pL_i|a_{ii}||e_i(t)|^p+\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}pL_j|a_{ij}||e_i(t)|^{p-1}|e_j(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}pQ_j|a_{ij}||e_i(t)|^{p-1}\\
=&\sum_{i=1}^{n}p\mathbb{c}_i^*|e_i(t)|^p+\sum_{i=1}^{n}p\mathcal{A}_i^*|e_i(t)|^{p-1}+\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}p\Big(\prod_{m=1}^{p-1}|a_{ij}|^{\mathbb{K}_m}L_j^{\mathbb{L}_m}|e_i(t)|\Big)(|a_{ij}|^{\mathbb{K}_p}L_j^{\mathbb{L}_p}|e_j(t)|)\\
\leq&\sum_{i=1}^{n}p\mathbb{c}_i^*|e_i(t)|^p+\sum_{i=1}^{n}p\mathcal{A}_i^*|e_i(t)|^{p-1}+\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\Big(\sum_{m=1}^{p-1}|a_{ij}|^{p\mathbb{K}_m}L_j^{p\mathbb{L}_m}|e_i(t)|^p+|a_{ij}|^{p\mathbb{K}_p}L_j^{p\mathbb{L}_p}|e_j(t)|^p\Big)\\
=&\sum_{i=1}^{n}p\mathbb{c}_i^*|e_i(t)|^p+\sum_{i=1}^{n}p\mathcal{A}_i^*|e_i(t)|^{p-1}+\sum_{i=1}^{n}\Big(\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{m=1}^{p-1}|a_{ij}|^{p\mathbb{K}_m}L_j^{p\mathbb{L}_m}+\sum_{\substack{j=1\\j\neq i}}^{n}|a_{ji}|^{p\mathbb{K}_p}L_i^{p\mathbb{L}_p}\Big)|e_i(t)|^p\\
=&\sum_{i=1}^{n}p(\mathbb{c}_i^*+\mathscr{A}_i^*)|e_i(t)|^p+\sum_{i=1}^{n}p\mathcal{A}_i^*|e_i(t)|^{p-1}.
\end{aligned}\tag{3.11}$$

Secondly, it follows from Lemma 2.3 and (2.2) that

$$
\begin{aligned}
&\sum_{i=1}^{n} p|e_i(t)|^{p-1}\left|\bigwedge_{j=1}^{n} \alpha_{ij}\eta_j(t-\tau_j(t)) - \bigwedge_{j=1}^{n} \alpha_{ij}\gamma_j(t-\tau_j(t))\right| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n} p|e_i(t)|^{p-1}|\alpha_{ij}||\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n} p|e_i(t)|^{p-1}|\alpha_{ij}|(L_j|e_j(t-\tau_j(t))| + Q_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p|\alpha_{ij}|L_j|e_i(t)|^{p-1}|e_j(t-\tau_j(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} p|\alpha_{ij}|Q_j|e_i(t)|^{p-1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p\tilde{\alpha}_{ij}|e_i(t)|^{p-1}|e_j(t-\tau_j(t))| + \sum_{i=1}^{n} p\alpha_i^*|e_i(t)|^{p-1}.
\end{aligned}
\tag{3.12}
$$

Similarly,

$$
\begin{aligned}
&\sum_{i=1}^{n} p|e_i(t)|^{p-1}\left|\bigvee_{j=1}^{n} \beta_{ij}\eta_j(t-\tau_j(t)) - \bigvee_{j=1}^{n} \beta_{ij}\gamma_j(t-\tau_j(t))\right| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n} p|e_i(t)|^{p-1}|\beta_{ij}||\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n} p|e_i(t)|^{p-1}|\beta_{ij}|(L_j|e_j(t-\tau_j(t))| + Q_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p|\beta_{ij}|L_j|e_i(t)|^{p-1}|e_j(t-\tau_j(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} p|\beta_{ij}|Q_j|e_i(t)|^{p-1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} p\tilde{\beta}_{ij}|e_i(t)|^{p-1}|e_j(t-\tau_j(t))| + \sum_{i=1}^{n} p\beta_i^*|e_i(t)|^{p-1}.
\end{aligned}
\tag{3.13}
$$

Substituting (3.11)–(3.13) into (3.10) yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) \le &-\sum_{i=1}^{n} p(-c_i + \mathbb{a}_i^* + \mathscr{A}_i^* - \rho_i)|e_i(t)|^p + \sum_{i=1}^{n} p(\mathcal{A}_i^* - \lambda_i + \alpha_i^* + \beta_i^*)|e_i(t)|^{p-1} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} p(\tilde{\alpha}_{ij} + \tilde{\beta}_{ij} - \omega_{ij})|e_i(t)|^{p-1}|e_j(t-\tau_j(t))| - \sum_{i=1}^{n} pk|e_i(t)|^{p-1+\mu}.
\end{aligned}
\tag{3.14}
$$

In view of $\rho_i \ge -c_i + \mathbb{a}_i^* + \mathscr{A}_i^*$, $\lambda_i \ge \mathcal{A}_i^* + \alpha_i^* + \beta_i^*$, $\omega_{ij} \ge \tilde{\alpha}_{ij} + \tilde{\beta}_{ij}$ and denoting $q = p - 1 + \mu$, then it follows from (3.14) and Lemma 2.4 that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) &\le -\sum_{i=1}^{n} pk|e_i(t)|^q \\
&\le -pk\left(\sum_{i=1}^{n} |e_i(t)|^p\right)^{\frac{q}{p}} \\
&= -pkV_2^{\frac{q}{p}}(t).
\end{aligned}
$$

Therefore, we conclude from Lemma 2.1 that the drive system (2.1) and response system (2.5) are finite-timely synchronized in finite time

$$\mathbb{T}_2 = \frac{V_2^{\frac{1-\mu}{p}}(0)}{k(1-\mu)}.$$

The proof is completed.

Remark 3.1. It is found that the settling time $\mathbb{T}_1$ and $\mathbb{T}_2$ are inversely proportional to the tunable constant k, while the control input $u_i(t)$ is directly proportional to the tunable constant k, so in order to achieve an ideal settling time, and to make the control cost as low as possible, the tunable constant should be rationally chosen according to practical design requirements.

Remark 3.2. If the activation functions $f(x)$ is continuous and satisfies Lipschitz condition, then the constant Q in (2.2) will be 0. Considering this case, the state feedback controllers designed in Theorems 3.1 and 3.2 are effective as well. Meanwhile, the finite-time synchronization problem of FCNNs with continuous activations has been well studied in [21] and [22], it should be noted that boundedness, as one of the prerequisite conditions, has been imposed on the neuron activation functions, but one can observe that there is no other additional restriction on the neuron activation functions except for the discontinuity and generalized Lipschitz condition. These comparisons demonstrate that the obtained theoretical results are more general and they effectually complement or improve existing ones to some extent.

4. Numerical examples

In this section, two examples are shown to verify the effectiveness of finite-time synchronization scheme obtained in previous section.

Example 4.1. Consider the following two dimensional fuzzy neural networks as the drive system:

$$\begin{cases} \dot{x}_1(t) = -c_1x_1(t) + \sum\limits_{j=1}^{2} a_{ij}f_j(x_j(t)) + \bigwedge\limits_{j=1}^{2} \alpha_{ij}f_j(x_j(t-\tau_j(t))) \\ \qquad + \bigvee\limits_{j=1}^{2} \beta_{ij}f_j(x_j(t-\tau_j(t))) + 0.2, \\ \dot{x}_2(t) = -c_2x_2(t) + \sum\limits_{j=1}^{2} a_{ij}f_j(x_j(t)) + \bigwedge\limits_{j=1}^{2} \alpha_{ij}f_j(x_j(t-\tau_j(t))) \\ \qquad + \bigvee\limits_{j=1}^{2} \beta_{ij}f_j(x_j(t-\tau_j(t))) + 0.1, \end{cases} \tag{4.1}$$

where

$$f_i(x) = \begin{cases} \tanh(x) + 0.5, & x \geq 0, \\ \tanh(x) - 0.5, & x < 0, \end{cases} \quad i = 1, 2. \tag{4.2}$$

It is easy to see that the discontinuous activations satisfy Assumptions (H1) and (H2) with $L_i = Q_i = 1, i = 1, 2$. The parameters of system (4.1) are assumed that $c_1 = c_2 = 1, a_{11} = 1.75, a_{12} = -4.8, a_{21} = -1.2, a_{22} = 2.5, \alpha_{11} = \beta_{11} = -1.5, \alpha_{12} = \beta_{12} = -0.4, \alpha_{21} = \beta_{21} = -0.3, \alpha_{22} = \beta_{22} = -1.8, \tau_i(t) = \frac{e^t}{1+e^t}, i = 1, 2$.

Now, we study the finite-time synchronization control of discontinuous fuzzy neural networks by using the results in this paper. The corresponding response system is described by

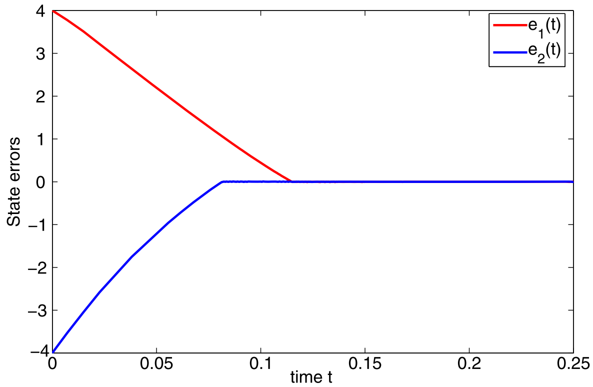

Fig. 1. The synchronization errors between the drive-response systems in Example 4.1 with initial values $\varphi(s) = (-1.5, 3.2)^{\mathrm{T}}, \phi(s) = (2.5, -1.8)^{\mathrm{T}}, s \in [-1, 0]$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\begin{cases} \dot{y}_1(t) = -c_1 y_1(t) + \sum\limits_{j=1}^{2} a_{ij} f_j(y_j(t)) + \bigwedge\limits_{j=1}^{2} \alpha_{ij} f_j(y_j(t-\tau_j(t))) \\ \qquad + \bigvee\limits_{j=1}^{2} \beta_{ij} f_j(y_j(t-\tau_j(t))) + 0.2 + u_1(t), \\ \dot{y}_2(t) = -c_2 y_2(t) + \sum\limits_{j=1}^{2} a_{ij} f_j(y_j(t)) + \bigwedge\limits_{j=1}^{2} \alpha_{ij} f_j(y_j(t-\tau_j(t))) \\ \qquad + \bigvee\limits_{j=1}^{2} \beta_{ij} f_j(y_j(t-\tau_j(t))) + 0.1 + u_2(t), \end{cases} \tag{4.3}$$

where the system parameters are the same as those in system (4.1), and the finite-time controller $u_i(t)$ is designed as follows

$$u_i(t) = -\rho_i e_i(t) - sign(e_i(t))(\lambda_i + 2|e_i(t)|^{\frac{1}{2}}) - \omega_i sign(e_i(t))|e_i(t-\tau_i(t))|,$$

where $e_i(t) = y_i(t) - x_i(t)$ for $i = 1, 2$.

Choosing $\rho_1 = 3, \rho_2 = 9.5, \omega_1 = 4, \omega_2 = 5, \lambda_1 = 9.5, \lambda_2 = 8$. By calculation, we can easily obtain that the Assumptions in Theorem 3.1 are all satisfied. Therefore, the two drive-response systems (4.1) and (4.3) can be synchronized in a finite time. The time evolution of synchronization errors between drive-response systems (4.1) and (4.3) with different initial values is showed in Fig. 1.

Example 4.2. Consider the following three dimensional fuzzy neural networks as the drive system:

$$\begin{aligned} \dot{x}_i(t) = &- c_i x_i(t) + \sum_{j=1}^{3} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{3} b_{ij} v_j + \bigwedge_{j=1}^{3} T_{ij} v_j \\ &+ \bigwedge_{j=1}^{3} \alpha_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{3} \beta_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{3} S_{ij} v_j + I_i, i = 1, 2, 3, \end{aligned} \tag{4.4}$$

where

$$f_i(x) = \begin{cases} \tanh(x) + 0.2\sin(x) + 0.8, & x \geq 0, \\ \tanh(x) + 0.2\cos(x) - 0.7, & x < 0, \end{cases} \quad i = 1, 2, 3. \tag{4.5}$$

It is readily seen that the activations (4.5) satisfy Assumptions (H1) and (H2) with $L_i = 1.2, Q_i = 0.8, i = 1, 2, 3$. The system parameters are taken as $c_1 = 1.5, c_2 = 1, c_3 = 1.2, a_{11} = 1.8, a_{12} = -3.4, a_{13} = -3.1, a_{21} = -3.2, a_{22} = 1.5, a_{23} = 4.5, a_{31} = -4, a_{32} = -4.2, a_{33} = 1.3, b_{11} = T_{11} = S_{11} = 0.1, b_{22} = T_{22} = S_{22} = 0.2, b_{33} = T_{33} = S_{33} = 0.5, b_{ij} = T_{ij} = S_{ij} = 0, i \neq j, i, j = 1, 2, 3, \alpha_{11} = \beta_{11} = -1.5, \alpha_{12} = \beta_{12} = -1.2, \alpha_{13} = \beta_{13} = -0.8, \alpha_{21} =$

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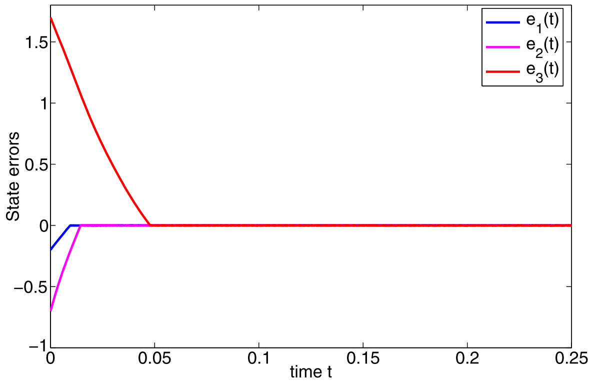

Fig. 2. The synchronization errors between the drive-response systems in Example 4.2 with initial values $\varphi(s) = (0.1, 0.2, -0.2)^{\mathrm{T}}$, $\phi(s) = (-0.1, -0.5, 1.5)^{\mathrm{T}}$, $s \in [-1, 0]$.

$\beta_{21} = -2.2, \alpha_{22} = \beta_{22} = -1.4, \alpha_{23} = \beta_{23} = -2.5, \alpha_{31} = \beta_{31} = -0.2, \alpha_{32} = \beta_{32} = -1.4, \alpha_{33} = \beta_{33} = -1.2, I_i = -0.5, \tau_i = 0.5, v_i = 1, i = 1, 2, 3.$

Based on the above system parameters, we consider the corresponding response system as follows:

$$\begin{aligned}\dot{y}_i(t) = &- c_i y_i(t) + \sum_{j=1}^{3} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{3} b_{ij} v_j + \bigwedge_{j=1}^{3} T_{ij} v_j \\ &+ \bigwedge_{j=1}^{3} \alpha_{ij} f_j(y_j(t - \tau_j(t))) + \bigvee_{j=1}^{3} \beta_{ij} f_j(y_j(t - \tau_j(t))) + \bigvee_{j=1}^{3} S_{ij} v_j + I_i \\ &+ u_i(t), i = 1, 2, 3, \end{aligned} \tag{4.6}$$

where the system parameters are the same as those used in system (4.4), and the finite-time controller $u_i(t)$ is designed as follows

$$u_i(t) = -\rho_i e_i(t) - sign(e_i(t))(\lambda_i + |e_i(t)|^{\frac{1}{2}}) - \sum_{j=1}^{3} \omega_{ij} sign(e_i(t))|e_j(t - \tau_j(t))|,$$

where $e_i(t) = y_i(t) - x_i(t)$ for $i = 1, 2, 3$.

Taking $\rho_1 = 17.5, \rho_2 = 20, \rho_3 = 20.5, \lambda_1 = 13, \lambda_2 = 18, \lambda_3 = 13, \omega_{11} = 3.8, \omega_{12} = 3.7, \omega_{13} = 3, \omega_{21} = 5.4, \omega_{22} = 3.5, \omega_{23} = 6.5, \omega_{31} = 5, \omega_{32} = 3.4, \omega_{33} = 3.1$, and $p = 2, \mathbb{K}_i = \mathbb{L}_i = 0.5$, for $i = 1, 2$. By simple calculations, one can easily check that all the algebra conditions in Theorem 3.2 are satisfied. Accordingly, the finite-time synchronization of the drive-response systems (4.4) and (4.6) can be achieved. This fact is strongly supported by numerical simulations in Fig. 2.

Remark 4.1. In this paper, we introduce finite-time control technique to synchronize delayed FCNNs with discontinuous activations. In the literature, there are many results concerning exponential or asymptotical synchronization. However, to the best of our knowledge, there is no published paper dealing with finite-time synchronization of discontinuous delayed FCNNs. On the other hand, it is easy to see from Figs. 1 and 2 that our results can effectively shorten the synchronization time greatly than the method of exponential and asymptotical synchronization. Hence, the results of this paper are new and extend existing results.

5. Conclusion

In this paper, the problem of finite-time synchronization of fuzzy cellular neural networks with discontinuous activations and time delays has been extensively discussed. Based on generalized Lyapunov method and finite-time convergence theory, by designing useful state feedback controllers, some novel finite-time synchronization criteria have been established. Finally, two numerical examples with simulations are presented to show the effective performance of the proposed finite-time control scheme. It is known that time delay, especially leakage delay or proportional delay, plays an important influence on system dynamics, to reveal the influence of time delay on the dynamics of finite time synchronization is our next research interest, which is a challenging topic.

Acknowledgements

The authors would like to express the sincere appreciation to the editor and reviewers for their helpful comments in improving the presentation and quality of the paper. The revision work was carried out when the first author was visiting Prof. Xianhua Tang at Central South University, and he would like to thank the staff in the School of Mathematics and Statistics for their help and thank the university for its excellent facilities and support during his stay.

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Please cite this article in press as: L. Duan et al., Finite-time synchronization of delayed fuzzy cellular neural networks with discontinuous activations, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.04.017