Pricing FX-TARN Under Lévy Processes Using Numerical Methods

Valentin Bandelier

April 28, 2017

Version: 0.6

Swiss Federal Institute of Technology Lausanne - EPFL

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

School of Basic Sciences - SB
Institute of Mathematics - MATH

Master Thesis

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Abstract

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Abstract (different language)

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Acknowledgement

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This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Contents

1 Introduction

> *Finance is the art of passing currency from hand to hand until it finally disappears.*
>
> — **Robert W. Sarnoff** (1918-1997)

This thesis presents different numerical methods for pricing FX-TARN under Lévy processes. In general, options with path dependents payoff, such as this product, are evaluated by Monte Carlo simulations. We will describe two other methods based on Finite Difference (FD) and Fast Fourier Transform (FFT). The initial chapter starts, in Section 1.1, with an historic of existing works that allowed this project to born. Then, in Section 1.2, the FX-TARN product is presented. In section 1.3, an example of term sheet illustrates this exotic product.

Finally, the chapter concludes with an overview of the thesis in section 1.4.

1.1 Motivation

1.2 FX-TARN Description

An FX Target Accrual Redemption Note (FX-TARN) is a financial product that allows an investor to accumulate an amount of cash until a certain *target accrual level* U over a predefined schedule. More precisely, the contract between the bank and the client imposes cash flow on scheduled dates (fixing dates). We can replicate these cash flows with a series of FX call options (resp. FX put options) with strike K, that the bank sells to a client, and at the same time a series of FX put options (resp. FX call options) with the same strike K, that the bank buys from the client. Sometimes, the client leg that the bank buys is combined with a leverage factor g called *gear factor*. The scheduling is defined by a number of fixing dates $t_1, t_2, \ldots, t_N$ that corresponds to the option expiry dates. Finally, the product knock-out if the total sum of payouts (from the bank's point of view) exceeds the given target U. There are three types of knock-out when the target U is breached that we will see in the next section:

- **No Gain :** the last payment is disallowed when the target U is breached,
- **Part Gain :** only a part of the payment is allowed such that only the target is paid,
- **Full Gain :** the last payment is allowed when the target U is breached.

Payoff Definition

Define the following notations:

- $S(t)$: FX rate at time t,
- K : strike,
- t_0 : today's date,
- $t_1, t_2, \ldots, t_M$: fixing dates,
- U : target accrual level,
- $A(t)$: accumulated gains at time t,
- N_f : notional foreign amount.

On each fixing date $t_n, n = 1, \ldots, N$, if the target level U is not breached by the accumulated amount $A(t_m)$, the gain per unit of notional foreign amount from the point of view of the investor is given by

$$\tilde{C}_n = \beta(S(t_n) - K) \times \mathbf{1}_{\{\beta S(t_n) \geq \beta K\}},$$

and the loss

$$\tilde{C}_n^* = -g \times \beta(K - S(t_n)) \times \mathbf{1}_{\{\beta S(t_n) \leq \beta K\}},$$

where β is the strategy, i.e. $\beta = 1$ the investor buys call options, $\beta = -1$ the investor buys put options.

Denote $t_{\tilde{N}}$ the first fixing date before maturity on which the target level U is breached by the total accumulated gain (without the loss part), i.e.

$$\tilde{N} = \min\{n : A(t_n) \geq U\}, \qquad n = 1, 2, \ldots, N.$$

If the target U is not breached, set $\tilde{N} = N$. For $t_n \leq t_{\tilde{N}}$ we can write the actual payment as

$$C_n(S(t_n), A(t_{n-1})) = \tilde{C}_n \times (\mathbf{1}_{A(t_{n-1}) + \tilde{C}} + W_n \times \mathbf{1}_{\{A(t_{n-1} + \tilde{C} \geq U\}}),$$

and $C_n = 0$ for $t_n > t_{\tilde{N}}$. As a loss can not occur at the same time as a gain and consequently does not depend on the knock-out condition, we can set

$$C_n^* = \tilde{C}_n^*.$$

$A(t_{n-1})$ is the accumulated gain immediately after the fixing date t_{n-1} and W_n is the weight corresponding to the type of knock-out when the target is breached. Therefore, the accumulated gain $A(t)$ is a step function such that $A(t) = A(t_{n-1})$, for $t_{n-1} \leq t < t_n$ with

$$A(t_n) = A(t_{n-1}) + C_n(S(t_n), A(t_{n-1})).$$

We can model the weights W_n for the different types of knock-out as follow:

$$W_n = \begin{cases} 0, & \text{for No Gain,} \\ \frac{U - A(t_{n-1})}{\beta \times (S(t_n) - K)}, & \text{for Part Gain,} \\ 1, & \text{for Full Gain.} \end{cases}$$

Finally, the net present value of FX-TARN in domestic currency for FX rate realization $\mathbf{S} = (S(t_1), S(t_2), \ldots, S(t_N))$ is

$$P(\mathbf{S}) = N_f \times \sum_{n=1}^{N} \frac{C_n(S(t_n), A(t_{n-1})) + C_n^*(S(t_n))}{B_d(t_0, t_n)}, \qquad A(t_0) = 0,$$

where $B_d(t_0, t_n)^{-1}$ is the domestic discounting factor from t_n to t_0.

1.3 Example of Term Sheet

1.4 Overview of the Thesis

Lévy Processes

2

> *Paul Lévy was a painter in the probabilistic world.*
>
> — **Michel Loève** (1907-1979)

The Lévy processes play a central role in mathematical finance. They can describe the reality of financial markets in a more accurate way than models based on the geometric Brownian motion used in particular in Black-Scholes model. Indeed we can observe in the real world that the asset price processes have some jumps. Moreover, the log returns of the underlying have empirical distribution with fat tails and skewness which deviates from normality supposed by Black and Scholes. We begin this chapter, in section 2.1, with the definition of a Lévy process and expose its fundamental properties. Next, in section 2.2, we presents two main results about Lévy processes which are the Lévy-Khinchine formula and the Lévy-Itô decomposition. In section 2.3, the Lévy measure and path properties of a Lévy process are exposed. Finally, the section 2.4 presents the class of exponential Lévy processes and the equivalent martingale measure used to describe the asset price in financial modeling.

2.1 Definitions and properties

The Lévy processes, which are the continuous-time case of random walks, are ingredients for building continuous-time stochastic models. The simplest Lévy process is the linear drift. The Wiener process, Poisson process, and compound Poisson process are the most famous examples of Lévy processes. We will see later in this chapter that the sum of a linear drift, a Wiener process, and a compound Poisson process is again a Lévy process. It is called a *Lévy jump-diffusion* process.

Definition 2.1 (*Wiener Process*)

*A stochastic process $W = \{W_t, t \geq 0\}$, with $W_0 = 0$, is a **Wiener process**, also called a standard Brownian motion, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if:*

1. *W has independent increments, i.e. $(W_{t+s} - W_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*

2. *W has stationary increments, i.e. the distribution of $(W_{t+s} - W_t)$ does not depend on t.*

3. *W has Gaussian increments, i.e. $(W_{t+s} - W_t) \sim \mathcal{N}(0, s)$.*

4. *W is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to t} \mathbb{P}(|W_t - W_s| < \epsilon) = 0.$$

This motion was discovered by Brown in 1827 and taken back by Bachelier (1900) to model the stock market prices. Only in 1923 the Brownian was defined and constructed rigorously by R. Wiener.

Definition 2.2 (*Poisson process*)

Let $(\tau_i)_{i \geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $N = \{N_t, t \geq 0\}$, with $N_0 = 0$, defined by

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{t \geq T_n\}}$$

*is called **Poisson process** with intensity λ.*
This process has the following properties:

1. *N has independent increments, i.e. $(N_{t+s} - N_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*

2. *N has stationary increments, i.e. the distribution of $(N_{t+s} - N_t)$ does not depend on t.*

3. *N has Poisson increments, i.e. $(N_{t+s} - N_t)$ has a Poisson distribution with parameter λs.*

4. *N is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to 0} \mathbb{P}(|N_{t+s} - N_t| < \epsilon) = 0.$$

When the process is characterized by a constant intensity parameter λ, we say that the process is homogeneous. If the intensity parameter varies with time t as $\lambda(t)$, the process is said to be non-homogeneous.

The Poisson process, which bears the name of the French physicist and mathematician Siméon Denis Poisson, defines a counting process. It counts the number of random times (T_n) which occur in $[0, t]$. Therefore, this is an increasing pure jump process. The jumps of size 1 occur at times T_n and the intervals between two jumps are exponentially distributed. If we compare definitions 2.1 and 2.2, we can see that only the fourth property differs between the two processes, only the distribution changes. The main idea of a Lévy process is to ignore the distribution of increments.

Definition 2.3 (*Lévy process*)

A cadlag stochastic process $X = \{X_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with real values is called a ***Lévy process*** *if it has the following properties:*

1. *X has independent increments, i.e. $(X_{t+s} - X_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*
2. *X has stationary increments, i.e. the distribution of $(X_{t+s} - X_t)$ does not depend on t.*
3. *X is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to 0} \mathbb{P}(|X_{t+s} - X_t| < \epsilon) = 0.$$

The third condition does not imply that the sample paths are continuous. In fact, the Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths. This condition serves to exclude jumps at non-random times. In other words, for a given t, the probability of seeing a jump at t is zero, discontinuities occur at random time. The compound Poisson process is a good example of a Lévy process.

Definition 2.4 (*Compound Poisson process*)

A ***compound Poisson process*** *with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $X = \{X_t, t \geq 0\}$ defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where jumps size Y_i are i.i.d. with the density function f and $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity λ, independent from $(Y_i)_{i \geq 1}$.

We can easily deduce the following properties from this definition:

1. The sample paths of X are cadlag piecewise constant functions.
2. The jump times $(T_i)_{i \geq 1}$ have the same law as the jump times of the Poisson process N_t. They can be expressed as partial sums of an independent exponential random variable with parameter λ.
3. The jump sizes $(Y_i)_{i \geq 1}$ are i.i.d. with law f.

We can also see that the Poisson process itself can be seen as a compound Poisson process with $Y_i \equiv 1$. This explains the origin of the name of the definition. Finally, the compound Poisson process allows us to work with jump sizes which have an arbitrary distribution.

2.2 Lévy-Khinchine formula and Lévy-Itô decomposition

We will now present in this section two main results about Lévy processes: the *Lévy-Khinchine formula* and the *Lévy-Itô decomposition*. Let's start with the relationship between infinitely divisible distributions and Lévy process.

Definition 2.5 (*Infinite divisibility*)

A probability distribution F is said to be ***infinitely divisible*** *if for any integer $n \geq 2$, there exists n i.i.d. random variables $Y_1, \ldots, Y_n$ such that $Y_1 + \cdots + Y_n$ has distribution F.*

If X is a Lévy process, for any $t > 0$ the distribution of X_t is infinitely divisible. This comes from the fact that for any $n \geq 1$,

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}), \tag{2.1}$$

and the property of stationary and independent increments. Let define now the characteristic function and characteristic exponent of X_t.

Definition 2.6 (*Characteristic function and exponent*)

The ***characteristic function*** *Φ_t of a random variable X_t with cumulative distribution F_t is given by*

$$\Phi_t(u) = \mathbb{E}\left[e^{iuX_t}\right] = \int_{-\infty}^{\infty} e^{i\theta x} dF_t(x).$$

Its ***characteristic exponent*** *is given by*

$$\Psi_t(u) = \log\left(\mathbb{E}\left[e^{iuX_t}\right]\right),$$

for $u \in \mathbb{R}$ and $t > 0$.

Then using twice equation (2.1) we obtain for any positive integers m, n that

$$m\Psi_1(u) = \Psi_m(u) = n\Psi_{m/n}(u).$$

Hence for any rational $t = \frac{m}{n} > 0$ we have

$$\Psi_t(u) = t\Psi_1(u).$$

We can generalize this relation for all $t > 0$ with the help of the almost sure continuity of X and a sequence of rational $\{t_n, n \geq 1\}$ such that $t_n \downarrow t$.

In conclusion, any Lévy process has the property that for all $t > 0$

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{t\Psi(u)},$$

where $\Psi(u) = \Psi_1(u)$ is the characteristic exponent of X_1.

Then it is clear that each Lévy process has an infinitely divisible distribution. This allows us to apply the celebrated Lévy-Khinchine formula.

Theorem 2.7 (*Lévy-Khintchine formula*)

Each Lévy process can be characterized by a triplet (γ, σ, ν) *with* $\gamma \in \mathbb{R}, \sigma \geq 0$ *and* ν *a measure satisfying* $\nu(\{0\}) = 0$ *and*

$$\int_{\mathbb{R}} \min\{1, |x|^2\}\nu(dx) < \infty.$$

In term of this triplet the characteristic function of the Lévy process equals:

$$\begin{aligned}\Phi_t(u) &= \mathbb{E}\left[\exp(iuX_t)\right] \\ &= \exp\left(t\left(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right)\nu(dx)\right)\right).\end{aligned} \tag{2.2}$$

(The proof can be find in Tankov and Cont (2003))

The triplet (γ, σ, ν) is called the *Lévy* or *characteristic triplet*. Moreover, γ is called the *drift term*, σ the *Gaussian* or *diffusion coefficient* and $\nu(dx)$ is the *Lévy measure*, being the intensity of jumps of size x. This brings us to the following great result which is the Lévy-Itô decomposition.

Theorem 2.8 (*Lévy-Itô decomposition*)

Consider a triplet (γ, σ, ν) *where* $\gamma \in \mathbb{R}$, $\sigma \geq 0$ *and* ν *is a measure satisfying* $\nu(\{0\}) = 0$ *and*

$$\int_{\mathbb{R}} \min\{1, |x|^2\}\nu(dx) < \infty.$$

Then, there exists exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *on which four independent Lévy processes exist,* $X^{(1)}, X^{(2)}, X^{(3)}$ *and* $X^{(4)}$*, where* $X^{(1)}$ *is a constant drift,* $X^{(2)}$ *is a Wiener process,* $X^{(3)}$ *is a compound Poisson process and* $X^{(4)}$ *is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time interval. Taking* $X = X^{(1)} + X^{(2)} + X^{(3)} + X^{(4)}$*, we have that there exists a probability space on which a Lévy process* $X = \{X_t, 0 \leq t \leq T\}$ *with characteristic exponent*

$$\Psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right)\nu(dx),$$

for all $u \in \mathbb{R}$, is defined.
(See Kyprianou (2006) for the proof)

The Lévy process is characterized by its triplet (γ, σ, ν). The simplest Lévy process is the linear *drift* with the triplet $(\gamma, 0, 0)$. Adding a *diffusion* component we get the triplet $(\gamma, \sigma, 0)$ which is the case of the Black-Scholes model. A *pure jump* process will be identified by the triplet $(0, 0, \nu)$ and finally a *Lévy jump-diffusion* process will have the complete triplet (γ, σ, ν). The figure 2.1 illustrates some examples of Lévy processes.

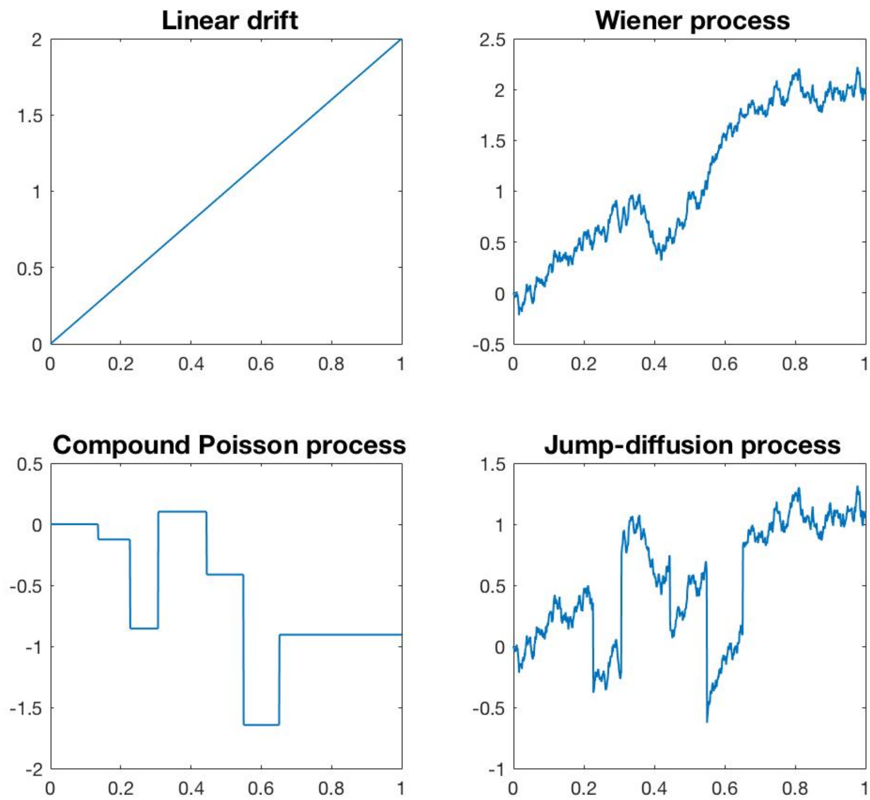

Fig. 2.1: Examples of Lévy processes: a linear drift with Lévy triplet $(2, 0, 0)$, a Wiener process with Lévy triplet $(2, 1, 0)$, a compound Poisson process with Lévy triplet $(0, 0, \lambda \cdot f_J)$, where $\lambda = 5$, and $f_J \sim \mathcal{N}(0, 1)$ and finally a jump-diffusion process with Lévy triplet $(2, 1, \lambda \cdot f_J)$.

2.3 Lévy measure and path properties

The *Lévy measure* dictates the behavior of the jumps.

Definition 2.9 (*Lévy measure*)

Let $x = \{X_t, t \geq 0\}$ be a Lévy process on $\mathbb{R}$. The measure ν on $\mathbb{R}$ defined by

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0, 1] : \Delta x \neq 0, \Delta x \in A\}\right],$$

*is called the **Lévy measure** of X: $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A.*

For example, the Lévy measure of a compound Poisson process is given by $\nu(dx) = \lambda f_J(dx)$. In other words, the expected number of jumps, in a time interval of length 1, is λ and the jump size is distributed according to f_J.

More generally, if ν is a finite measure, that is $\lambda = \nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(dx) < \infty$, then we can defined $f(dx) = \frac{\nu(dx)}{\lambda}$, which is a probability measure. Then, λ is the expected number of jumps and $f(dx)$ is the distribution of the jump size x. If $\nu(\mathbb{R}) = \infty$, an infinite number of (small) jumps is expected.

Proposition 2.10 (*Finite and infinite activity*)

Let $X = \{X_t, t \geq 0\}$ be a Lévy process with triplet (γ, σ, ν).

1. *If $\nu(\mathbb{R}) < \infty$ then almost all paths of X have a finite number of jumps on every compact interval. In that case, the Lévy process has **finite activity**.*
2. *If $\nu(\mathbb{R}) = \infty$ then almost all paths of X have a infinite number of jumps on every compact interval. In that case, the Lévy process has **infinite activity**.*

(See Theorem 21.3 in Sato (1999) for the proof)

Then the Lévy jump models can be classified into two categories from their Lévy measure: jump-diffusion or pure jump models. The jump-diffusions models are modeled by a Gaussian part (Wiener process) combined with a jump part (compound Poisson process), that has finitely many jumps in every time interval, i.e. finite activity models. The second category consists of models with an infinite number of jumps in every interval, i.e. infinite activity models. In these models, there is no need of Gaussian part because the dynamics of jumps are already rich enough to generate nontrivial small time behavior. Merton model and Variance Gamma model are respectively good examples of jump-diffusion and pure jump models. We can see in figure 2.2 that the Lévy density in Variance Gamma model allows infinite number of jumps, while the Merton model has a finite number of jumps on every time interval.

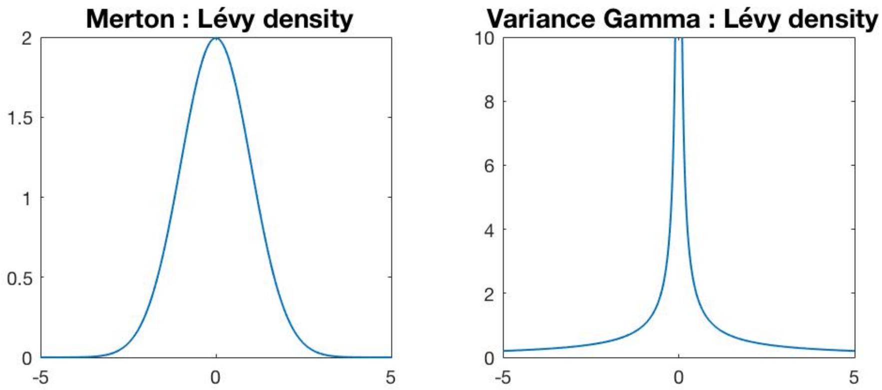

Fig. 2.2: The density of Lévy measure in the Merton model (left) and the Variance Gamma model (right).

2.4 Exponential Lévy processes and Equivalent martingale measure

In finance, it is common to model the stock price process as exponentials of a Lévy process:

$$S_t = S_0 e^{X_t}.$$

The advantage of this representation is that the stock prices process is nonnegative and the log returns $\log(S_{t+s}/S_t)$, for $s, t > 0$, follow the distribution of increments of length s dictate by the Lévy process $X = \{X_t, t \geq 0\}$. Thus they have independent and stationary increments. If we choose a process such that $X_0 = 0$, we get $e^{X_0} = 1$ and therefore $S_0 = S_0 e^{X_0} = S_0$.

In order to avoid an arbitrage opportunity, the discounted and reinvested process $\hat{S} = \{\hat{S}_t = e^{-(r-q)t} S_t, t \geq 0\}$ has to be a martingale under an *equivalent martingale measure* (EMM) $\mathbb{Q}$, called the *risk-neutral measure*. Recall that r is the *(domestic) risk-free rate* and q is the *continuous dividend yield (or foreign interest rate)* of the asset. In other words, we are looking for a measure $\mathbb{Q}$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[\hat{S}_T | \mathcal{F}_t\right] = \hat{S}_t.$$

Since the market is not complete under Lévy processes, there exists several ways to find a risk-neutral measure. We will see two different methods to determine this probability measure.

2.4.1 Esscher transform method

The first approach to find an EMM $\mathbb{Q}$ is proposed by Gerber, Shiu, et al. (1994) using the Esscher transform. Suppose that the Lévy process $X = \{X_t, t \geq 0\}$ has a density function $f(x;t)$. Now multiply this density by an exponential factor $e^{\theta t}$ to get a new density function:

$$f(x;t,\theta) = \frac{e^{\theta x} f(x;t)}{\int_{\mathbb{R}} e^{\theta y} f(y;t) dy}.$$

Note that the denominator ensures the properties of $f(x;t,\theta)$ to be a density function, i.e.

$$\int_{\mathbb{R}} f(y;t,\theta) dy = 1.$$

With this transformation we obtain a new probability function defined by

$$d\mathbb{P}_t^\theta = \frac{d\mathbb{P}_t}{\int_{\mathbb{R}} e^{\theta y} d\mathbb{P}_t} = \frac{d\mathbb{P}_t}{M(\theta;t)},$$

where $M(\theta;t)$ is the moment-generating function and $\mathbb{P}$ is the real world probability measure. The goal is to determine the parameter θ such that $\mathbb{P}^\theta$ is an EEM. Take a look on the moment-generating function of X_t under $\mathbb{P}$,

$$M(u;t) = \mathbb{E}\left[e^{uX_t}\right] = \Phi_t(-iu),$$

and the moment-generating function of X_t under $\mathbb{P}^\theta$,

$$\begin{aligned} M(u;t,\theta) &= \int_{\mathbb{R}} e^{ux} f(x;t,\theta) dx \\ &= \frac{\int_{\mathbb{R}} e^{(u+\theta)x} f(x;t) dx}{\int_{\mathbb{R}} e^{\theta y} f(y;t) dy} \\ &= \frac{M(u+\theta;t)}{M(\theta;t)} \\ &= \frac{\Phi_t(-i(u+\theta))}{\Phi_t(-i\theta)}. \end{aligned} \tag{2.3}$$

The martingale condition on $\hat{S} = \{\hat{S}_t = S_0 e^{-(r-q)t + X_t}, t \geq 0\}$ gives us the following relation:

$$S_0 = e^{-(r-q)t} \mathbb{E}^{\mathbb{P}^\theta}\left[S_t\right] = e^{-(r-q)t} S_0 \underbrace{\mathbb{E}^{\mathbb{P}^\theta}\left[e^{X_t}\right]}_{=M(u;t,\theta)} = e^{-(r-q)t} S_0 \frac{\Phi_t(-i(u+\theta))}{\Phi_t(-i\theta)}.$$

Therefore, θ is given by the explicit equation

$$e^{(r-q)t} = \frac{\Phi_t(-i(1+\theta))}{\Phi_t(-i\theta)}.$$

Thus the solution θ^* of this equation gives us the Esscher transform martingale measure and we have $\mathbb{Q} \equiv \mathbb{P}^{\theta^*}$.

Characterization of the risk-neutral Lévy process

With the help of equation (2.3) we have that

$$\Phi_t^\theta(-iu) = \frac{\Phi_t(-i(u+\theta))}{\Phi_t(-i\theta)} \iff \Phi_t^\theta(z) = \frac{\Phi_t(z-i\theta))}{\Phi_t(-i\theta)}.$$

We can also add that the new Lévy process is characterized by the triplet $(\gamma^\theta, \sigma^\theta, \nu^\theta(dx))$, and with the Lévy-Khintchine formula 2.7 combined to the definition (2.6) of the characteristic exponent, we can recover

$$\begin{aligned}
\gamma^\theta &= \gamma + \sigma^2\theta + \int_{-1}^{1} \left(e^{\theta x} - 1\right) \nu(dx), \\
\sigma^\theta &= \sigma, \\
\nu^\theta(dx) &= e^{\theta x}\nu(dx).
\end{aligned}$$

2.4.2 Mean-correction method

The second way to obtain an equivalent martingale measure $\mathbb{Q}$ is to correct the mean of the exponential Lévy process to satisfy the martingale condition of the discounted stock price process $\hat{S} = \{\hat{S}_t = e^{-(r-q)t}S_t, t \geq 0\}$. The idea is to add a drift to the Lévy process to kill the drift of the discounted asset price process. Therefore we obtain a new Lévy process $\tilde{X} = \left\{\tilde{X}_t = X_t + \omega t, t \geq 0\right\}$ and consequently

$$\begin{aligned}
S_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-(r-q)t}S_t\right] \\
&= S_0 e^{-(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{\tilde{X}_t}\right] \\
&= S_0 e^{-(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{X_t+\omega t}\right] \\
&= S_0 e^{[\omega-(r-q))+\Psi(-i)]t}
\end{aligned}$$

Hence we have that ω have to be equal to $[(r-q) - \Psi(-i)]$, where Ψ is the characteristic exponent of X_1. Moreover we have that the new risk-neutral Lévy process $\tilde{X}$ is characterized by the triplet (γ^*, σ, ν) with

$$\gamma^* = \gamma + (r-q) - \Psi(-i). \tag{2.4}$$

The mean-correction method is simpler than the Esscher transform method and this is the method we will use throughout this thesis. There are several other measures that can be found in the book of Miyahara (2011).

3 Financial Mathematic Models

> *Essentially, all models are wrong, but some are useful.*
>
> — **George E. P. Box** (1919-2013)

In this chapter, we will take a look on some popular models in financial mathematics. To begin, in section 3.1 we will describe the Black-Scholes model (1973) and compute its risk-neutral characteristic function. In section 3.2 we will talk about *jump-diffusion models*. These models evolve with a diffusion process, punctuated by jumps at random intervals. We can model this behavior with a Wiener process and a compound Poisson process to characterized the jumps with size distribution f_J. In fact, we will talk about two examples: the Merton model (1976) and the Kou model (2002). Finally, the section 3.3 is devoted to *pure jump models*. This category of models is characterized by infinite number of jumps in any time interval, called *infinite activity* models. These particular models don't need a Brownian part because the dynamic of the process is already modeled by an infinity of small jumps. However, we will see that it is possible to construct these models by a Brownian subordination, which is called a time-changed Browninan motion. At the end of this chapter we will have seen two examples which are the Normal Inverse Gaussian (NIG) model, proposed by Barndorff-Nielsen (1997) the Variance Gamma (VG) model, proposed by Madan et al. (1998).

3.1 Black-Scholes model

Samuelson (1965) was the first one to introduce Brownian motion to model asset prices. Then his work was taken over by Black and Scholes (1973) to create the most famous model, the Black-Scholes model. In this model, the stock price $S = \{S_t, t \geq 0\}$ follows a geometric Brownian motion, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are respectively the drift and the volatility of the process. This stochastic differential equation has a unique solution which is

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

In fact this model is based on an exponential Lévy process $X = \{X_t, t \geq 0\}$ defined by

$$X_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Hence his characteristic triplet is $\left(\mu - \frac{1}{2}\sigma^2, \sigma, 0\right)$.

Risk-neutral Characteristic Function

Recall that X_t in this model is described by the characteristic triplet $(\gamma, \sigma, 0)$ with $\gamma = \left(\mu - \frac{1}{2}\sigma^2\right)$. Thus the Lévy-Khintchine formula 2.7 gives us the characteristic function of X_t

$$\Phi_t(u) = \exp\left\{t\left(\left(\mu - \frac{1}{2}\sigma^2\right) iu - \frac{1}{2}\sigma^2 u^2\right)\right\}.$$

Hence the characteristic exponent of X_1 evaluated at $-i$ is

$$\Psi(-i) = \mu - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 = \mu.$$

With equation (2.4), we obtain the risk-neutral drift

$$\gamma^* = r - q - \frac{1}{2}\sigma^2,$$

and the risk-neutral characteristic function is given by

$$\Phi_t^{\mathrm{RN}}(u) = \exp\left\{t\left(i\gamma^* u - \frac{1}{2}\sigma^2 u^2\right)\right\}.$$

Finally the risk-neutral stock price process is defined by

$$\begin{aligned} S_t &= S_0 \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \\ &= S_0 \exp\left\{X_t^{\mathrm{BS}}(r, q, \sigma)\right\} \end{aligned}$$

3.2 Jump-diffusion models

Consider now the Lévy jump-diffusion process $X = \{X_t, t \geq 0\}$. It is modeled by a drifted Brownian motion and a compound Poisson process. Therefore we can write it in the form

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

with $\gamma \in \mathbb{R}, \sigma \in \mathbb{R}_+, W = \{W_t, t \geq 0\}$ is a Wiener process, $N = \{N_t, t \geq 0\}$ is a Poisson process with parameter λ and $Y = \{Y_t, t \geq 0\}$ is an i.i.d sequence of random variables with density f_J.

The characteristic function of X_t is given by

$$\begin{aligned}
\Phi_t(u) &= \mathbb{E}\left[e^{iuX_t}\right] \\
&= \mathbb{E}\left[\exp\left\{iu\left(\gamma t + \sigma W_t + \sum_{i=1}^{N_t}\right)\right\}\right] \\
&= \exp\{iu\gamma t\}\,\mathbb{E}\left[\exp\{iu\sigma W_t\}\right]\mathbb{E}\left[\exp\left\{iu\sum_{i=1}^{N_t} Y_i\right\}\right],
\end{aligned}$$

by independence of W_t and N_t. Since $W_t \sim \mathcal{N}(0, \sigma^2 t)$ and $N_t \sim \text{Poisson}(\lambda t)$, we have

$$\begin{aligned}
\mathbb{E}\left[e^{iu\sigma W_t}\right] &= e^{-\frac{1}{2}\sigma^2 u^2 t}, \\
\mathbb{E}\left[e^{iu\sum_{i=1}^{N_t} Y_i}\right] &= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{iunY}\right]\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} \Phi_Y(u)^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= e^{\lambda t(\Phi_Y(u)-1)} \\
&= e^{\lambda t \int_{\mathbb{R}} (e^{iuy}-1)F(dy)}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\Phi_t(u) &= \exp\{iu\gamma t\}\exp\left\{-\frac{1}{2}\sigma^2 u^2 t\right\}\exp\left\{\lambda t\int_{\mathbb{R}}\left(e^{iuy}-1\right)f_J(dy)\right\} \\
&= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}\left(e^{iuy}-1\right)\lambda f_J(dy)\right)\right\}. \qquad (3.1)
\end{aligned}$$

Then we have a characterization of Lévy jump-diffusion process by its characteristic triplet $(\gamma, \sigma, \lambda \cdot f_J)$.

3.2.1 Merton Model

Under the Black-Scholes model, the stock price is supposed to be continuous. Unfortunately this is not the case in reality. Merton (1976) is the first to use the notion of discontinuous price process to model asset returns. In his model, Merton uses a Normal distribution to model the jump size, i.e. $f_J \sim \mathcal{N}(\alpha, \delta^2)$. Then the Lévy processes is

$$X_t = \mu t + \sigma W_t + \sum_{i=0}^{N_t} Y_i,$$

with $Y_i \sim \mathcal{N}(\alpha, \delta^2)$. Hence, the density function of the jump size is

$$f_J(x) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x-\alpha)^2}{2\delta^2}},$$

and the Lévy density is

$$\nu(x) = \lambda f_J(x) = \frac{\lambda}{\sqrt{2\pi}\delta} e^{-\frac{(x-\alpha)^2}{2\delta^2}}.$$

Then there are four parameters in the Merton model excluding the drift parameter μ:

- σ - the diffusion volatility,
- λ - the jump intensity,
- α - the mean of jump size,
- δ - the standard deviation of jump size.

Risk-neutral Characteristic Function

With the help of equation 3.1, we obtain the characteristic function of the model under the real world measure $\mathbb{P}$:

$$\begin{aligned}
\Phi_t(u) &= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1\right) \lambda f_J(dy)\right)\right\} \\
&= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\Phi_Y(u) - 1\right)\right)\right\} \\
&= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda\left(e^{iu\alpha - \frac{1}{2}\delta^2 u^2} - 1\right)\right)\right\},
\end{aligned}$$

where Φ_Y is the characteristic function of a jump Y. Hence the model is characterized by the triplet $(\gamma, \sigma, \lambda \cdot f_J)$.

We can now compute the characteristic exponent in order to apply the mean-correction and get the risk-neutral process.

$$\Psi(-i) = \gamma + \frac{1}{2}\sigma^2 + \lambda \left(e^{\alpha + \frac{1}{2}\delta^2} - 1 \right).$$

Applying equation (2.4), we obtain the risk-neutral drift

$$\gamma^* = (r - q) - \frac{1}{2}\sigma^2 - \lambda \left(e^{\alpha + \frac{1}{2}\delta^2} - 1 \right),$$

and the risk-neutral characteristic function of the Merton jump-diffusion model is given by

$$\Phi_t^{\mathrm{RN}}(u) = \exp\left\{ t \left(i\gamma^* u - \frac{1}{2}\sigma^2 u^2 + \lambda \left(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1 \right) \right) \right\}.$$

The risk-neutral stock price process is finally

$$S_t = S_0 \exp\left\{ X_t^{\mathrm{Mer}}(r, q, \sigma, \lambda, \alpha, \delta) \right\},$$

where X^{Mer} is the Lévy jump-diffusion process characterized by the triplet $(\gamma^*, \sigma, \lambda \cdot f_J)$.

3.2.2 Kou Model

The Kou model (2002) is very similar to Merton's one. The only difference is in the distribution of the jump size, which is double-exponential. Then the Lévy process under Kou model is

$$X_t = \gamma t + \sigma W_t + \sum_{i=0}^{N_t} Y_i,$$

with $Y_i \sim \mathrm{DoubleExp}(p, \eta_1, \eta_2)$. In other words, jump size has the density

$$f_J(x) = \begin{cases} p \cdot \eta_1 e^{-\eta_1 x}, & \text{if } x \geq 0, \\ (1-p) \cdot \eta_2 e^{\eta_2 x}, & \text{if } x < 0. \end{cases}$$

The probability p represents the probability of an upward jump and $(1-p)$ the probability of a downward jump. Thus the Lévy density is given by

$$\nu(x) = \lambda \left(p \cdot \eta_1 e^{-\eta_1 x} \mathbf{1}_{x \geq 0} + (1-p) \cdot \eta_2 e^{\eta_2 x} \mathbf{1}_{x<0} \right).$$

Then there are five parameters in the Kou model excluding the drift parameter μ:

- σ - the diffusion volatility,
- λ - the jump intensity,

- p - the probability of an upward jump,
- η_1, η_2 - control the decay of the tails in the distribution.

Risk-neutral Characteristic Function

A preliminary computation of the characteristic function of a double exponential random variable Y is needed.

$$\begin{aligned}\Phi_Y(u) &= \int_{\mathbb{R}} e^{iuy} f_J(y) dy \\ &= \int_0^{\infty} e^{iuy} p \cdot \eta_1 e^{-\eta_1 y} dy + \int_{-\infty}^{0} e^{iuy}(1-p) \cdot \eta_2 e^{\eta_2 y} dy \\ &= p \cdot \eta_1 \left[\frac{e^{(iu-\eta_1)y}}{iu-\eta_1}\right]_0^{\infty} + (1-p) \cdot \eta_2 \left[\frac{e^{(iu+\eta_2)y}}{iu+\eta_2}\right]_{-\infty}^{0} \\ &= \frac{p \cdot \eta_1}{\eta_1 - iu} + \frac{(1-p) \cdot \eta_2}{\eta_2 + iu}\end{aligned}$$

Now as for Merton model, the equation (3.1) gives us the characteristic function of X_t

$$\begin{aligned}\Phi_t(u) &= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\Phi_Y(u) - 1\right)\right)\right\} \\ &= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\frac{p \cdot \eta_1}{\eta_1 - iu} + \frac{(1-p) \cdot \eta_2}{\eta_2 + iu} - 1\right)\right)\right\}.\end{aligned}$$

Hence the model is characterized by the triplet $(\gamma, \sigma, \lambda \cdot f_J)$.

The characteristic exponent of this process gives us

$$\Psi(-i) = \gamma + \frac{1}{2}\sigma^2 + \lambda\left(\frac{p \cdot \eta_1}{\eta_1 + 1} + \frac{(1-p) \cdot \eta_2}{\eta_2 + 1} - 1\right).$$

Consequently we obtain the risk-neutral drift

$$\gamma^* = (r-q) - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p \cdot \eta_1}{\eta_1 + 1} + \frac{(1-p) \cdot \eta_2}{\eta_2 + 1} - 1\right),$$

and the risk-neutral characteristic function of the Double Exponential Kou jump-diffusion model

$$\Phi_t^{\text{RN}}(u) = \exp\left\{t\left(i\gamma^* u - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\frac{p \cdot \eta_1}{\eta_1 + iu} + \frac{(1-p) \cdot \eta_2}{\eta_2 + iu} - 1\right)\right)\right\}.$$

Therefore we can model the risk-neutral stock price process by

$$S_t = S_0 \exp\left\{X_t^{\text{Kou}}(r, q, \sigma, \lambda, p, \eta_1, \eta_2)\right\},$$

where X_t^{Kou} is the Lévy jump-diffusion process characterized by the triplet $(\gamma^*, \sigma, \lambda \cdot f_J)$.

3.3 Pure jump models

To go beyond the jump-diffusion process, initially proposed by Merton in 1976, we can talk about infinite activity models. There exist a lot of paper about these kind of Lévy processes. We will see two different models which are the *Normal Inverse Gaussian* (NIG) model, proposed by Barndorff-Nielsen in 1997, and the *Variance Gamma* (VG) model, proposed by Madan et al. in 1998. They are both particular cases of the Generalized Hyperbolic model, developed by Eberlein and Prause (1998).

These two models can be described as a Brownian motion $W = \{W_t, t \geq 0\}$ with constant drift θ and volatility σ evaluated at a random time $T = \{T_t, t \geq 0\}$,

$$X_t = \theta T_t + \sigma W_{T_t}.$$

This process is called *time changed Brownian motion* with constant drift. Moreover, the process T is called the subordinator process, which is an increasing Lévy process. The subordinating processes in the NIG and VG models are respectively an *Inverse Gaussian* process and a *Gamma* process.

3.3.1 Normal Inverse Gaussian Model

First of all, we will present the subordinating Inverse Gaussian process which is used to construct the Normal Inverse Gaussian (NIG) process.

Inverse Gaussian Process

Let $T \sim \text{IG}(a, b)$ be an inverse Gaussian random variable. This is in fact the first time that a Brownian motion with drift $b > 0$ reaches the level $a > 0$ Its density function is given by

$$f_{\text{IG}}(x; a, b) = \frac{ae^{ab}}{\sqrt{2\pi}} x^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}\left(\frac{a^2}{x} + b^2 x\right)\right\}, \qquad x > 0,$$

and its characteristic function is

$$\Phi_{\text{IG}}(u; a, b) = \exp\left\{-a\left(\sqrt{-2iu + b^2} - b\right)\right\}.$$

Note that if $X_1, \ldots, X_n$ are independent IG random variables with parameters $(a/n, b)$, then $X_1 + \cdots + X_n \sim \text{IG}(a, b)$. Thus this distribution is infinitely divisible and we are able to define an IG process $X^{\text{IG}} = \{X_t^{\text{IG}}, t \geq 0\}$ as a process that starts at 0 and has independent and stationary increments such that $X_t^{\text{IG}} \sim \text{IG}(at, b)$. Hence it has the following characteristic function

$$\begin{aligned}\Phi_t^{\text{IG}}(u; at, b) &= \mathbb{E}\left[e^{iuX_t^{\text{IG}}}\right] \\ &= \exp\left\{-at\left(\sqrt{-2iu + b^2} - b\right)\right\}\end{aligned}$$

Now let's verify the non-decreasing condition for a subordinator. We have that

$$\begin{aligned}\mathbb{P}\left(X_{t+\Delta t}^{\text{IG}} < X_t^{\text{IG}}\right) &= \mathbb{P}\left(X_{t+\Delta t}^{\text{IG}} - X_t^{\text{IG}} < 0\right) \\ &= \mathbb{P}\left(X_{\Delta t}^{\text{IG}} < 0\right) = 0,\end{aligned}$$

since an IG random variable takes only positive values. Thus it is a good candidate as subordinator.

Normal Inverse Gaussian Process

As mention by Geman (2002), we can represent the NIG process by a time-changed Brownian motion with an IG process as subordinator. Let $W = \{W_t, t \geq 0\}$ be a standard Brownian motion and $T = \{T_t, t \geq 0\}$ be an IG process with parameters $a = 1$ and b. Then the NIG process is given by

$$X_t = \theta T_t + \sigma W_{T_t}.$$

Thus its characteristic function is

$$\begin{aligned}\Phi_t^{\text{NIG}}(u; \theta, \sigma) &= \mathbb{E}\left[e^{iuX_t}\right] \\ &= \mathbb{E}\left[e^{\left(iu\theta - \frac{\sigma^2 u^2}{2}\right)T_t}\right] \\ &= \Phi_t^{\text{IG}}\left(u\theta + i\frac{\sigma^2 u^2}{2}\right) \\ &= \exp\left\{-t\left(\sqrt{-2i\left(u\theta + i\frac{\sigma^2 u^2}{2}\right) + b^2} - b\right)\right\} \\ &= \exp\left\{-t\left(\sqrt{b^2 - 2iu\theta + \sigma^2 u^2} - b\right)\right\} \\ &= \exp\left\{-t\sigma\left(\sqrt{\frac{b^2}{\sigma^2} + \frac{\theta^2}{\sigma^4} - \left(\frac{\theta}{\sigma^2} + iu\right)^2} - \frac{b}{\sigma}\right)\right\}.\end{aligned}$$

To simplify the notation, we can set

$$
\begin{aligned}
\alpha^2 &= \frac{b^2}{\sigma^2} + \frac{\theta^2}{\sigma^4}, \\
\beta &= \frac{\theta}{\sigma^2}, \\
\delta &= \sigma.
\end{aligned}
$$

The NIG process becomes

$$X_t^{\mathrm{NIG}} = \beta\delta^2 T_t + \delta W_{T_t},$$

and we get the characteristic function given by Barndorff-Nielsen (1997) in the form

$$\Phi_t^{\mathrm{NIG}}(u;\alpha,\beta,\delta) = \exp\left\{t\delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right)\right\}.$$

Then the NIG model has three parameters to control the shape of the distribution:

- α - tail heaviest of steepness,
- β - symmetry,
- δ - scale.

Risk-neutral Characteristic Function

Here, since the characteristic triplet $(\gamma, 0, \nu)$ is not trivial, we will find a risk-neutral characteristic function using the following form of the stock price

$$S_t = S_0 e^{(r-q)t+\omega t+X_t^{\mathrm{NIG}}}.$$

Hence we have that

$$
\begin{aligned}
S_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-(r-q)t}S_t\right] \\
&= S_0\mathbb{E}^{\mathbb{Q}}\left[e^{\omega t+X_t^{\mathrm{NIG}}}\right] \\
&= S_0 e^{\omega t}\Phi_t^{\mathrm{NIG}}(-i).
\end{aligned}
$$

Therefore we must have $e^{\omega t}\Phi_t^{\mathrm{NIG}}(-i) = 1$ or equivalently

$$\omega = -\delta t\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+1)^2}\right).$$

This gives us the risk-neutral drift

$$\gamma^* = (r-q) - \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+1)^2}\right),$$

and the risk-neutral characteristic function is

$$\Phi_t^{RN}(u) = \exp\left\{t\left(i\gamma^* u + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right)\right)\right\}.$$

Finally, the risk-neutral stock price process is in the form

$$S_t = S_0 \exp\{\gamma^* t + X_t^{\text{NIG}}(\alpha, \beta, \delta)\},$$

where X_t^{NIG} is the Normal Inverse Gaussian process and γ^* is the risk-neutral drift.

3.3.2 Variance Gamma Model

Madan et al. in 1998 had the same approach as the previous Normal Inverse Gaussian model. The difference is that the random time in the Brownian motion is Gamma distributed. In a second time, since this process has also finite variation, it can be represent by the difference of two increasing processes. The first one models the price increases while the second one reflects the price decreases. To begin, let us introduce the subordinating Gamma process used to construct the Variance Gamma process.

Gamma process

The Gamma density function $f_\gamma(x; a, b)$ with parameters $a, b > 0$ is given by

$$f_\gamma(x; a, b) = x^{a-1}\frac{b^a e^{-bx}}{\Gamma(a)},$$

where Γ is the Euler gamma function. Then its characteristic function is

$$\Phi_\gamma(u; a, b) = \left(1 - \frac{iu}{b}\right)^{-a}.$$

This distribution is also infinitely divible because if $X_1, \ldots, X_n \sim \Gamma(a/n, b)$, we have that $X_1 + \cdots + X_n \sim \Gamma(a, b)$. Therefore, we can define a Gamma process $X^{\text{Gam}} = \left\{X_t^{\text{Gam}}, t \geq 0\right\}$, which is a stochastic process that starts at 0 and has stationary and independent increments such that $X_t^{\text{Gamma}} \sim \Gamma(at, b)$. The corresponding characteristic function is given by

$$\begin{aligned}
\Phi_t^{\text{Gam}}(u; at, b) &= \mathbb{E}\left[e^{iuX_t^{\text{Gam}}}\right] \\
&= \left(1 - \frac{iu}{b}\right)^{-at} \\
&= \left(\frac{1}{1 - \frac{iu\nu}{\mu}}\right)^{\frac{\mu^2}{\nu}t},
\end{aligned}$$

where μ and ν are respectively the mean rate and the variance rate of the process.

Variance Gamma process

As in the case of the Normal Inverse Gamma process, we can represent the Variance Gamma process as a time-changed Brownian motion

$$X_t = \theta T_t + \sigma W_{T_t},$$

with $T = \{T_t, t \geq 0\}$ a gamma process with mean rate $\mu = 1$ and variance rate ν. Therefore the characteristic function of this process is

$$\begin{aligned}
\Phi_t^{\text{VG}}(u; \theta, \sigma, \nu) &= \mathbb{E}\left[e^{iuX_t}\right] \\
&= \Phi_t^{\text{Gam}}\left(u\theta + i\frac{\sigma^2 u^2}{2}\right) \\
&= \left(\frac{1}{1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2}\right)^{\frac{t}{\nu}}.
\end{aligned} \tag{3.2}$$

Then we have that the VG model has three parameters:

- θ - drift of the Brownian motion,
- σ - volatility of the Brownian motion,
- ν - variance rate of the time change.

Madan et al. (1998) showed that the VG process has finite variation. Therefore we can represent this process by the difference of two independent and increasing gamma process with mean rate $\mu_\pm$ variance rate $\nu_\pm$, i.e.

$$X_t = \gamma_t^+(\mu_+, \nu_+) - \gamma_t^-(\mu_-, \nu_-),$$

where γ_t^+ and γ_t^- correspond respectively to the positive and negative shocks. Therefore the characteristic function of this representation is

$$\begin{aligned}\Phi_t^{\mathrm{VG}}(u) &= \mathbb{E}\left[e^{iu(\gamma_t^+ - \gamma_t^-)}\right] \\ &= \Phi_{\gamma_t^+}(u)\Phi_{-\gamma_t^-}(u) \\ &= \left(\frac{1}{1-\frac{iu\nu_+}{\mu_+}}\right)^{\frac{\mu_+^2}{\nu_+}t}\left(\frac{1}{1+\frac{iu\nu_-}{\mu_-}}\right)^{\frac{\mu_-^2}{\nu_-}t}.\end{aligned} \tag{3.3}$$

Thus, comparing the both characteristic functions (3.2) and (3.3), we get the following relations

$$\begin{aligned}\frac{\mu_+^2}{\nu_+} = \frac{\mu_-^2}{\nu_-} &= \frac{1}{\nu}, \\ \frac{\nu_+\nu_-}{\mu_+\mu_-} &= \frac{\sigma^2\nu}{2}, \\ \frac{\nu_+}{\mu_+} - \frac{\nu_-}{\mu_-} &= \theta\nu.\end{aligned}$$

Hence we have that

$$\begin{aligned}\mu_+ &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}, \\ \mu_- &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2}, \\ \nu_+ &= \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}\right)^2 \nu, \\ \nu_- &= \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2}\right)^2 \nu.\end{aligned}$$

Finally, the VG process is effectively the difference of two independent gamma processes.

Risk-neutral Characteristic Function

Just recall that the characteristic function under real world probability $\mathbb{P}$ is given by

$$\begin{aligned}\Phi_t^{\text{VG}}(u) &= \left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)^{-\frac{t}{\nu}} \\ &= \exp\left\{-\frac{t}{\nu}\ln\left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)\right\}.\end{aligned}$$

In the same way as in the NIG model, we can construct the risk-neutral drift by considering

$$S_t = S_0 e^{(r-q)t+\omega t+X_t^{\text{VG}}}.$$

Then

$$\begin{aligned}S_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-(r-q)t}S_t\right] \\ &= S_0\mathbb{E}^{\mathbb{Q}}\left[e^{\omega t+X_t^{\text{VG}}}\right] \\ &= S_0 e^{\omega t}\Phi_t^{\text{VG}}(-i),\end{aligned}$$

and we must have that $e^{\omega t}\Phi_t^{\text{VG}}(-i) = 1$ or in other words

$$\omega = \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right).$$

At the end we obtain the risk-neutral drift

$$\gamma^* = (r-q) + \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right),$$

and the risk-neutral characteristic function is given by

$$\Phi_t^{\text{RN}}(u) = \exp\left\{t\left(i\gamma^* u - \frac{1}{\nu}\ln\left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)\right)\right\}.$$

Finally, the risk-neutral stock price process is

$$S_t = S_0\exp\left\{\gamma^* t + X_t^{\text{VG}}(\theta,\sigma,\nu)\right\},$$

where X_t^{VG} is the Variance Gamma process and γ^* is the risk-neutral drift.

3.4 Summary

To summarize, we can see that in all models the risk-neutral stock price process can be written in the form:

$$S_t = S_0 \exp\left\{\gamma^* t + X_t\right\},$$

with the risk-neutral drift γ^* and a drift-less Lévy process X_t. Moreover, the risk-neutral characteristic function is in the form:

$$\Phi_t^{\mathrm{RN}}(u) = \exp\left\{t\left(i\gamma^* u + \Psi(u)\right)\right\},$$

where Ψ is the characteristic exponent of X_1. Tables 3.1, 3.2 and 3.3 illustrate respectively the drift-less Lévy process X_t, the risk-neutral drift γ^* and the risk-neutral characteristic exponent $\Psi(u)$ for all the models which we have just studied in this chapter.

Models	Lévy process X_t	Comments
Black-Scholes	σW_t	
Merton	$\sigma W_t + \sum_{i=1}^{N_t} Y_i$	$Y_i \sim \mathcal{N}(\alpha, \delta^2)$
Kou	$\sigma W_t + \sum_{i=1}^{N_t} Y_i$	$Y_i \sim \text{DoubleExp}(p, \eta_1, \eta_2)$
Normal Inverse Gaussian	$\beta\delta T_t + \delta W_{T_t}$	$T_t \sim \text{IG}\left(\delta\sqrt{\alpha^2 - \beta^2}\right)$
Variance Gamma	$\theta T_t + \sigma W_{T_t}$	$T_t \sim \text{Gamma}(1, \nu)$

Tab. 3.1: Drift-less Lévy processes X_t for several models.

Models	Risk-neutral drift γ^*
Black-Scholes	$r - q - \frac{1}{2}\sigma^2$
Merton	$r - q - \frac{1}{2}\sigma^2 - \lambda\left(e^{\alpha + \frac{1}{2}\delta^2} - 1\right)$
Kou	$r - q - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p\cdot\eta_1}{\eta_1 + 1} + \frac{(1-p)\cdot\eta_2}{\eta_2 + 1} - 1\right)$
Normal Inverse Gaussian	$r - q - \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}\right)$
Variance Gamma	$r - q + \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right)$

Tab. 3.2: Risk-neutral drifts γ^* for several models.

Models	Risk-neutral characteristic exponent $\Psi(u)$
Black-Scholes	$-\frac{1}{2}\sigma^2 u^2$
Merton	$-\frac{1}{2}\sigma^2 u^2 + \lambda \left(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1 \right)$
Kou	$-\frac{1}{2}\sigma^2 u^2 + \lambda \left(\frac{p \cdot \eta_1}{\eta_1 + iu} + \frac{(1-p) \cdot \eta_2}{\eta_2 + iu} - 1 \right)$
Normal Inverse Gaussian	$\delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2} \right)$
Variance Gamma	$-\frac{1}{\nu} \ln \left(1 - iu\theta\nu + \frac{\sigma^2 \nu}{2} u^2 \right)$

Tab. 3.3: Risk-neutral characteristic exponent $\Psi(u)$ for several models.

4 Numerical Methods

> *FFT is the most important numerical algorithm of our lifetime.*
>
> — **Gilbert Strang** (1934)

Section Introduction

4.1 Monte Carlo

4.2 Finite Difference Method

Taylor Expansion

Recall that the Taylor expansion for a function $f \in C^{\infty}$ infinitely many differentiable is given by

$$\begin{aligned} f(x) &= f(a) + (x-a)\frac{\partial f}{\partial x}(a) + \frac{(x-a)^2}{2!}\frac{\partial^2 f}{\partial x^2}(a) + \cdots + \frac{(x-a)^n}{n!}\frac{\partial^n f}{\partial x^n}(a) + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(a). \end{aligned}$$

If f is only $(n+1)$ times continuously differentiable, i.e. $f \in C^{(n+1)}$, we can write

$$f(x) = \sum_{k=0}^{n+1} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(x) + O((x-a)^{n+1}),$$

where $O((x-a)^{n+1})$ represents the remainder in Landau notation.

Forward and Backward Difference Approximation of First Derivative

In order to approximate $\frac{\partial f}{\partial t}(x)$ assume that f is twice continuously differentiable, i.e. $f \in C^2$. By a first order Taylor expansion we can write

$$f(x+h) = f(x) + h\frac{\partial f}{\partial x}(x) + O(h), \tag{4.1}$$

$$f(x-h) = f(x) - h\frac{\partial f}{\partial x}(x) + O(h). \tag{4.2}$$

The equation (4.1) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x+h) - f(x)}{h} + O(h), \tag{4.3}$$

which is known as *forward difference* approximation of the first derivative.

On the other hand, the equation (4.2) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x) - f(x-h)}{h} + O(h), \tag{4.4}$$

which is known as *backward difference* approximation of the first derivative.

Central Difference Approximation of First Derivative

Now assume that $f \in C^3$. Then with a second order Taylor expansion, we have

$$f(x+h) = f(x) + h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2), \tag{4.5}$$

$$f(x-h) = f(x) - h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2). \tag{4.6}$$

Subtracting equation (4.6) from (4.5) we get

$$f(x+h) - f(x-h) = 2h\frac{\partial f}{\partial x}(x) + O(h^2).$$

Therefore we obtain the *central difference* approximation of the first derivative

$$\frac{\partial f}{\partial x}(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2). \tag{4.7}$$

Central Difference Approximation of Second Derivative

Finally summing equations (4.5) and (4.6) we get

$$2f(x) = f(x+h) + f(x-h) + h^2 \frac{\partial^2 f}{\partial x^2}(x) + O(h^2).$$

Then the *central difference* approximation of the second derivative is given by

$$\frac{\partial^2 f}{\partial x^2}(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \tag{4.8}$$

Option Pricing under the Generalized Black-Scholes model

Consider the Generalized Black-Scholes model, which includes the *local volatility* $\sigma(S,t)$ and term structures of *interest rate* $r(t)$ and *dividend rate* $q(t)$. The price of an asset S under such model follows the *stochastic differential equation* (SDE):

$$dS_t = (r(t) - q(t))S_t dt + \sigma(S_t, t) S_t dW_t.$$

Then we know that the value of an option $v(S,t)$ on that asset S satisfies the following *partial differential equation* (PDE):

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\sigma(S,t)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(t) - q(t))S\frac{\partial v}{\partial S} = r(t)v(S,t) & \\ v(S,T) = \Psi(S) & \text{Terminal Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, t) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, t) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

Now if we use the change of variable $\tau = (T - t)$ to express *time to maturity*, we obtain the following PDE:

$$\begin{cases} -\frac{\partial v}{\partial \tau} + \frac{\sigma(S,\tau)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(\tau) - q(\tau))S\frac{\partial v}{\partial S} = r(\tau)v(S,\tau) & \\ v(S,0) = \Psi(S) & \text{Initial Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, \tau) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, \tau) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

To begin, we have to define the domain of the problem

$$D = \{S_{\min} \leq S \leq S_{\max}; 0 \leq \tau \leq T\}$$

and set it to a discrete grid

$$\bar{D} = \begin{Bmatrix} S_j = S_{\min} + (j-1)h; & h = \frac{S_{\max} - S_{\min}}{N}; & j = 1, \ldots, N+1 \\ t_k = 0 + (k-1)\Delta t; & \Delta t = \frac{T}{M}; & k = 1, \ldots, M+1 \end{Bmatrix},$$

where N is the number of subintervals in the S-direction and M is the number of subintervals in the τ-direction.

Forward Euler Approximation

The Forward Euler approximation constructs the *explicit* discretization of the Generalized Black-Scholes PDE. In other words, we approximate the theta term $\frac{\partial v}{\partial t}(S,t)$ using a *forward difference* approximation (4.3):

$$\frac{\partial v}{\partial t}(S,t) \approx \frac{v(S, t+\Delta t) - v(S,t)}{\Delta t}.$$

The *central difference* approximation of the first derivative (4.7) for the delta term $\frac{\partial v}{\partial S}(S,t)$ gives us

$$\frac{\partial v}{\partial S}(S,t) \approx \frac{v(S+h,t) - v(S-h,t)}{2h}$$

and the *central difference* approximation of the second derivative (4.8) for the gamma term $\frac{\partial^2 v}{\partial S^2}(S,t)$ gives

$$\frac{\partial^2 v}{\partial S^2}(S,t) \approx \frac{v(S+h,t) - 2v(S,t) + v(S-h,t)}{h^2}.$$

Backward Euler Approximation

θ-Method and Crank-Nicolson Approximation

4.3 The Convolution Method

5 Conclusion

Citation.

— **Author**
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Lausanne, April 28, 2017

Valentin Bandelier