Pricing TARN Using Numerical Methods

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Master Thesis

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Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Abstract (different language)

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Acknowledgement

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This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

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Monte-Carlo Method

> *Citation*

— **Author**
(Author description)

Finite Difference Method

Section Introduction

2.1 Taylor Expansion

Recall that the Taylor expansion for a function $f \in C^\infty$ infinitely many differentiable is given by

$$f(x) = f(a) + (x - a)\frac{\partial f}{\partial x}(x) + \frac{(x-a)^2}{2!}\frac{\partial^2 f}{\partial x^2}(x) + \cdots + \frac{(x-a)^n}{n!}\frac{\partial^n f}{\partial x^n}(x) + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(x).$$

If f is only $(n+1)$ times continuously differentiable, i.e. $f \in C^{(n+1)}$, we can write

$$f(x) = \sum_{k=0}^{n+1} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(x) + O((x-a)^{n+1}),$$

where $O((x-a)^{n+1})$ represents the remainder in Landau notation.

2.1.1 Forward and Backward Difference Approximation of First Derivative

In order to approximate $\frac{\partial f}{\partial t}(x)$ assume that f is twice continuously differentiable, i.e. $f \in C^2$. By a first order Taylor expansion we can write

$$f(x + h) = f(x) + h\frac{\partial f}{\partial x}(x) + O(h), \tag{2.1}$$

$$f(x - h) = f(x) - h\frac{\partial f}{\partial x}(x) + O(h). \tag{2.2}$$

The equation (2.1) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x + h) - f(x)}{h} + O(h), \tag{2.3}$$

which is known as *forward difference* approximation of the first derivative.

On the other hand, the equation (2.2) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x) - f(x - h)}{h} + O(h), \tag{2.4}$$

which is known as *backward difference* approximation of the first derivative.

2.1.2 Central Difference Approximation of First Derivative

Now assume that $f \in C^3$. Then with a second order Taylor expansion, we have

$$f(x) = f(x + h) + h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2), \tag{2.5}$$

$$f(x) = f(x - h) - h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2). \tag{2.6}$$

Subtracting equation (2.6) from (2.5) we get

$$f(x + h) - f(x - h) = 2h\frac{\partial f}{\partial x}(x) + O(h^2).$$

Therefore we obtain the *central difference* approximation of the first derivative

$$\frac{\partial f}{\partial x}(x) = \frac{f(x + h) - f(x - h)}{2h} + O(h^2). \tag{2.7}$$

2.1.3 Central Difference Approximation of Second Derivative

Finally summing equations (2.5) and (2.6) we get

$$2f(x) = f(x + h) + f(x - h) + h^2\frac{\partial^2 f}{\partial x^2}(x) + O(h^2).$$

Then the *central difference* approximation of the second derivative is given by

$$\frac{\partial^2 f}{\partial x^2}(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \qquad (2.8)$$

2.2 Option Pricing under the Generalized Black-Scholes model

Consider the Generalized Black-Scholes model, which includes the *local volatility* $\sigma(S,t)$ and term structures of *interest rate* $r(t)$ and *dividend rate* $q(t)$. The price of an asset S under such model follows the *stochastic differential equation* (SDE):

$$dS_t = (r(t) - q(t))S_t dt + \sigma(S_t, t)S_t dW_t.$$

Then we know that the value of an option $v(S,t)$ on that asset S satisfies the following *partial differential equation* (PDE):

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\sigma(S,t)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(t) - q(t))S\frac{\partial v}{\partial S} = r(t)v(S,t) \\ v(S,T) = \Psi(S) & \text{Terminal Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, t) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, t) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

Now if we use the change of variable $\tau = (T - t)$ to express *time to maturity*, we obtain the following PDE:

$$\begin{cases} -\frac{\partial v}{\partial \tau} + \frac{\sigma(S,\tau)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(\tau) - q(\tau))S\frac{\partial v}{\partial S} = r(\tau)v(S,\tau) \\ v(S,0) = \Psi(S) & \text{Initial Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, \tau) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, \tau) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

To begin, we have to define the domain of the problem

$$D = \{S_{\min} \leq S \leq S_{\max}; 0 \leq \tau \leq T\}$$

and set it to a discrete grid

$$\bar{D} = \begin{cases} S_j = S_{\min} + (j-1)h; & h = \frac{S_{\max} - S_{\min}}{N}; & j = 1, \ldots, N+1 \\ t_k = 0 + (k-1)\Delta t; & \Delta t = \frac{T}{M}; & k = 1, \ldots, M+1 \end{cases},$$

where N is the number of subintervals in the S-direction and M is the number of subintervals in the τ-direction.

2.2.1 Forward Euler Approximation

The Forward Euler approximation constructs the *explicit* discretization of the Generalized Black-Scholes PDE. In other words, we approximate the theta term $\frac{\partial v}{\partial t}(S, t)$ using a *forward difference* approximation (2.3):

$$\frac{\partial v}{\partial t}(S, t) \approx \frac{v(S, t + \Delta t) - v(S, t)}{\Delta t}.$$

The *central difference* approximation of the first derivative (2.7) for the delta term $\frac{\partial v}{\partial S}(S, t)$ gives us

$$\frac{\partial v}{\partial S}(S, t) \approx \frac{v(S + h, t) - v(S - h, t)}{2h}$$

and the *central difference* approximation of the second derivative (2.8) for the gamma term $\frac{\partial^2 v}{\partial S^2}(S, t)$ gives

$$\frac{\partial^2 v}{\partial S^2}(S, t) \approx \frac{v(S + h, t) - 2v(S, t) + v(S - h, t)}{h^2}.$$

2.2.2 Backward Euler Approximation

2.2.3 θ-Method and Crank-Nicolson Approximation

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Colophon

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Declaration

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Lausanne, March 14, 2017

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Valentin Bandelier