Pricing FX-TARN Under Lévy Processes Using Numerical Methods

Valentin Bandelier

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Swiss Federal Institute of Technology Lausanne - EPFL

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School of Basic Sciences - SB
Institute of Mathematics - MATH

Master Thesis

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Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Abstract (different language)

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Acknowledgement

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Contents

1 Introduction

> *Finance is the art of passing currency from hand to hand until it finally disappears.*
>
> — **Robert W. Sarnoff**
> (1918-1997)

This thesis presents different numerical methods for pricing FX-TARN under Lévy processes. In general, options with path dependents payoff, such as this product, are evaluated by Monte Carlo simulations. We will describe two other methods based on Finite Difference (FD) and Fast Fourier Transform (FFT). The initial chapter starts, in Section 1.1, with an historic of existing works that allowed this project to born. Then, in Section 1.2, the FX-TARN product is presented. In section 1.3, an example of term sheet illustrates this exotic product.

Finally, the chapter concludes with an overview of the thesis in section 1.4.

1.1 Motivation

1.2 FX-TARN Description

A FX Target Accrual Redemption Note (FX-TARN) is a financial product that allows an investor to accumulate an amount of cash until a certain *target accrual level* U over a predefined schedule. More precisely, a bank sells a series of FX call options (resp. FX put options) with strike K to a client and at the same time buys a series of FX put options (resp. FX call options) with the same strike K from the client. Sometimes, the client leg that the bank buys is combined to a leverage factor g called *gear factor*. The scheduling is defined by a number of fixing dates $t_1, t_2, \ldots, t_N$ that corresponds to the option expiry dates. Finally, the product knock-out if the total sum of payouts (from the bank's point of view) exceeds the given target U. There is three types of knock-out when the target U is breach that we will see in the next section:

- **No Gain :** the last payment is disallowed when the target U is breached,

- **Part Gain :** only a part of the payment is allowed such that only the target is paid,
- **Full Gain :** the last payment is allowed when the target U is breached.

Payoff Definition

Define the following notations:

- $S(t)$: FX rate at time t,
- K : strike,
- t_0 : today's date,
- $t_1, t_2, \ldots, t_M$: fixing dates,
- U : target accrual level,
- $A(t)$: accumulated gains at time t,
- N_f : notional foreign amount.

On each fixing date $t_n, n = 1, \ldots, N$, if the target level U is not breached by the accumulated amount $A(t_m)$, the gain per unit of notional foreign amount from the point of view of the investor is given by

$$\tilde{C}_n = \beta(S(t_n) - K) \times \mathbf{1}_{\{\beta S(t_n) \geq \beta K\}},$$

and the loss

$$\tilde{C}_n^* = -g \times \beta(K - S(t_n)) \times \mathbf{1}_{\{\beta S(t_n) \leq \beta K\}},$$

where β is the strategy, i.e. $\beta = 1$ the investor buys call options, $\beta = -1$ the investor buys put options.

Denote $t_{\tilde{N}}$ the first fixing date before maturity on which the target level U is breached by the total accumulated gain (without the loss part), i.e.

$$\tilde{N} = \min\{n : A(t_n) \geq U\}, \qquad n = 1, 2, \ldots, N.$$

If the target U is not breached, set $\tilde{N} = N$. For $t_n \leq t_{\tilde{N}}$ we can write the actual payment as

$$C_n(S(t_n), A(t_{n-1})) = \tilde{C}_n \times (\mathbf{1}_{A(t_{n-1})+\tilde{C}} + W_n \times \mathbf{1}_{\{A(t_{n-1}+\tilde{C} \geq U\}}),$$

and $C_n = 0$ for $t_n > t_{\tilde{N}}$. As a loss can not occur at the same time as a gain and consequently does not depend on the knock-out condition, we can set

$$C_n^* = \tilde{C}_n^*.$$

$A(t_{n-1})$ is the accumulated gain immediately after the fixing date t_{n-1} and W_n is the weight corresponding to the type of knock-out when the target is breached. Therefore, the accumulated gain $A(t)$ is a step function such that $A(t) = A(t_{n-1})$, for $t_{n-1} \leq t < t_n$ with

$$A(t_n) = A(t_{n-1}) + C_n(S(t_n), A(t_{n-1})).$$

We can model the weights W_n for the different types of knock-out as follow:

$$W_n = \begin{cases} 0, & \text{for No Gain,} \\ \frac{U - A(t_{n-1})}{\beta \times (S(t_n) - K)}, & \text{for Part Gain,} \\ 1, & \text{for Full Gain.} \end{cases}$$

Finally, the net present value of FX-TARN in domestic currency for FX rate realization $\mathbf{S} = (S(t_1), S(t_2), \ldots, S(t_N))$ is

$$P(\mathbf{S}) = N_f \times \sum_{n=1}^{N} \frac{C_n(S(t_n), A(t_{n-1})) + C_n^*(S(t_n))}{B_d(t_0, t_n)}, \qquad A(t_0) = 0,$$

where $B_d(t_0, t_n)^{-1}$ is the domestic discounting factor from t_n to t_0.

1.3 Example of Term Sheet

1.4 Overview of the Thesis

Lévy Processes

2

> *Paul Lévy was a painter in the probabilistic world.*
>
> — **Michel Loève** (1907-1979)

The Lévy processes play a central role in mathematical finance. They can describe the reality of financial markets in more accurate way than models based on the geometric Brownian motion used in particular in Black-Scholes model. Indeed we can observe in the real world that the asset price processes have jumps. Moreover, the log returns of the underlying have empirical distribution with fat tails and skewness which deviates from normality supposed by Black and Scholes. We begin this chapter, in section 2.1, with the definition of a Lévy process and expose its fundamental properties. Next, in section 2.2, we presents two main results about Lévy processes which are the Lévy-Khinchine formula and the Lévy-Itô decomposition.

2.1 Definitions and properties

The Lévy processes, which are the continuous-time case of random walks, are ingredients for building continuous-time stochastic models. The simplest Lévy process is the linear drift. The Wiener process, Poisson process and compound Poisson process are the most famous examples of Lévy processes. We will see later in this chapter that the sum of a linear drift, a Wiener process and a compound Poisson process is again a Lévy process. It is called a *Lévy jump-diffusion* process.

Definition 2.1 (*Wiener Process*)

*A stochastic process $W = \{W_t, t \geq 0\}$, with $W_0 = 0$, is a **Wiener process**, also called a standard Brownian motion, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if:*

1. *W has independent increments, i.e. $(W_{t+s} - W_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*
2. *W has stationary increments, i.e. the distribution of $(W_{t+s} - W_t)$ does not depend on t.*
3. *W has Gaussian increments, i.e. $(W_{t+s} - W_t) \sim \mathcal{N}(0, s)$.*

4. *W is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to t} \mathbb{P}(|W_t - W_s| < \epsilon) = 0.$$

This motion was discovered by Brown in 1827 and taken back by Bachelier (1900) to model the stock market prices. Only in 1923 the Brownian was defined and constructed rigorously by R. Wiener.

Definition 2.2 (*Poisson process*)

Let $(\tau_i)_{i \geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $N = \{N_t, t \geq 0\}$, with $N_0 = 0$, defined by

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{t \geq T_n\}}$$

*is called **Poisson process** with intensity λ.*
This process has the following properties:

1. *N has independent increments, i.e. $(N_{t+s} - N_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*
2. *N has stationary increments, i.e. the distribution of $(N_{t+s} - N_t)$ does not depend on t.*
3. *N has Poisson increments, i.e. $(N_{t+s} - N_t)$ has a Poisson distribution with parameter λs.*
4. *N is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to 0} \mathbb{P}(|N_{t+s} - N_t| < \epsilon) = 0.$$

When the process is characterized by a constant intensity parameter λ, we say that the process is homogeneous. If the intensity parameter varies with time t as $\lambda(t)$, the process is said to be non-homogeneous.

The Poisson process, which bears the name of the French physicist and mathematician Siméon Denis Poisson, defines a counting process. It counts the number of random times (T_n) which occur in $[0, t]$. Therefore, this is an increasing pure jump process. The jumps of size 1 occur at times T_n and the intervals between two jumps are exponentially distributed. If we compare definitions 2.1 and 2.2, we can see that only property 4 differs between the two processes, only the distribution changes. The main idea of a Lévy process is to ignore the distribution of increments.

Definition 2.3 (*Lévy process*)

*A cadlag stochastic process $X = \{X_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with real values is called a **Lévy process** if it has the following properties:*

1. *X has independent increments, i.e. $(X_{t+s} - X_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*
2. *X has stationary increments, i.e. the distribution of $(X_{t+s} - X_t)$ does not depend on t.*
3. *X is stochastically continuous, i.e. $\forall \epsilon > 0 : \lim_{s \to 0} \mathbb{P}(|X_{t+s} - X_t| < \epsilon) = 0$.*

The third condition does not imply that the sample paths are continuous. In fact the Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths. This condition serves to exclude jumps at non-random times. In other words, for a given t, the probability of seeing a jump at t is zero, discontinuities occur at random time. The compound Poisson process is a good example of Lévy process.

Definition 2.4 (*Compound Poisson process*)

A ***compound Poisson process*** *with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $X = \{X_t, t \geq 0\}$ defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where jumps size Y_i are i.i.d. with the density function f and $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity λ, independent from $(Y_i)_{i \geq 1}$.

We can easily deduce the following properties from this definition:

1. The sample paths of X are cadlag piecewise constant functions.
2. The jump times $(T_i)_{i \geq 1}$ have the same law as the jump times of the Poisson process N_t. They can be expressed as partial sums of independent exponential random variable with parameter λ.
3. The jump sizes $(Y_i)_{i \geq 1}$ are i.i.d. with law f.

We can also see that the Poisson process itself can be seen as a compound Poisson process with $Y_i \equiv 1$. This explains the origin of the name of the definition. Finally the compound Poisson process allows us to work with a process with jump sizes can have an arbitrary distribution.

2.2 Lévy-Khinchine formula and Lévy-Itô decomposition

We will now present in this section two main results about Lévy processes: the *Lévy-Khinchine formula* and the *Lévy-Itô decomposition*. Let's start with the relationship between infinitely divisible distributions and Lévy process.

Definition 2.5 (*Infinite divisibility*)

A probability distribution F is said to be ***infinitely divisible*** *if for any integer $n \geq 2$, there exists n i.i.d. random variables $Y_1, \dots, Y_n$ such that $Y_1 + \dots + Y_n$ has distribution F.*

If X is a Lévy process, for any $t > 0$ the distribution of X_t is infinitely divisible. This comes from the fact that for any $n \geq 1$,

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \dots + (X_t - X_{(n-1)t/n}), \tag{2.1}$$

and the property of stationary and independent increments. Let define now the characteristic function and characteristic exponent of X_t.

Definition 2.6 (*Characteristic function and exponent*)

The ***characteristic function*** *Φ_t of a random variable X_t with cumulative distribution F_t is given by*

$$\Phi_t(u) = \mathbb{E}\left[e^{iuX_t}\right] = \int_{-\infty}^{\infty} e^{i\theta x} dF_t(x).$$

Its ***characteristic exponent*** *is given by*

$$\Psi_t(u) = -\log\left(\mathbb{E}\left[e^{iuX_t}\right]\right),$$

for $u \in \mathbb{R}$ and $t > 0$.

Then using twice equation (2.1) we obtain for any positive integers m, n that

$$m\Psi_1(u) = \Psi_m(u) = n\Psi_{m/n}(u).$$

Hence for any rational $t = \frac{m}{n} > 0$ we have

$$\Psi_t(u) = t\Psi_1(u).$$

We can generalize this relation for all $t > 0$ with the help of the almost sure continuity of X and a sequence of rational $\{t_n, n \geq 1\}$ such that $t_n \downarrow t$.

In conclusion, any Lévy process has the property that for all $t > 0$

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{-t\Psi(u)},$$

where $\Psi(u) = \Psi_1(u)$ is the characteristic exponent of X_1.

Then it is clear that each Lévy process has an infinitely divisible distribution. This allows us to apply the celebrated Lévy-Khinchine formula.

Theorem 2.7 (*Lévy-Khintchine formula*)

Each Lévy process can be characterized by a triplet (μ, σ, ν) with $\mu \in \mathbb{R}, \sigma \geq 0$ and ν a measure satisfying $\nu(0) = 0$ and

$$\int_{\mathbb{R}} \min\{1, |x|^2\}\nu(dx) < \infty.$$

In term of this triplet the characteristic function of the Lévy process equals:

$$\begin{aligned} \Phi(u) &= \mathbb{E}\left[\exp(iuX_t)\right] \\ &= \exp\left(t\left(i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\nu(dx)\right)\right)\right). \end{aligned} \tag{2.2}$$

(The proof can be find in Tankov and Cont (2003))

The triplet (μ, σ, ν) is called the *Lévy* or *characteristic triplet*. Moreover, μ is called the *drift term*, σ the *Gaussian* or *diffusion coefficient* and ν the *Lévy measure*. This brings us to the following great result which is the Lévy-Itô decomposition.

Theorem 2.8 (*Lévy-Itô decomposition*)

Consider a triplet (μ, σ, ν) where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure satisfying $\nu(0) = 0$ and

$$\int_{\mathbb{R}} \min\{1, |x|^2\}\nu(dx) < \infty.$$

Then, there exists exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes exist, $X^{(1)}, X^{(2)}, X^{(3)}$ and $X^{(4)}$, where $X^{(1)}$ is a constant drift, $X^{(2)}$ is a Wiener process, $X^{(3)}$ is a compound Poisson process and $X^{(4)}$ is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time interval. Taking $X = X^{(1)} + X^{(2)} + X^{(3)} + X^{(4)}$, we have that there exists a probability space on which a Lévy process $X = \{X_t, 0 \leq t \leq T\}$ with characteristic exponent

$$\Psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\nu(dx)\right),$$

for all $u \in \mathbb{R}$, is defined.
(See Kyprianou (2006) for the proof)

The Lévy process is charaterized by its triplet (μ, σ, ν). The simplest Lévy process is the linear *drift* with the triplet $(\mu, 0, 0)$. Adding a *diffusion* component we get the triplet $(\mu, \sigma, 0)$ which is the case of the Black-Scholes model. A *pure jump* process will be identified by the triplet $(0, 0, \nu)$ and finally a *Lévy jump-diffusion* process will have the complete triplet (μ, σ, ν). The figure 2.1 illustrates some examples of Lévy processes.

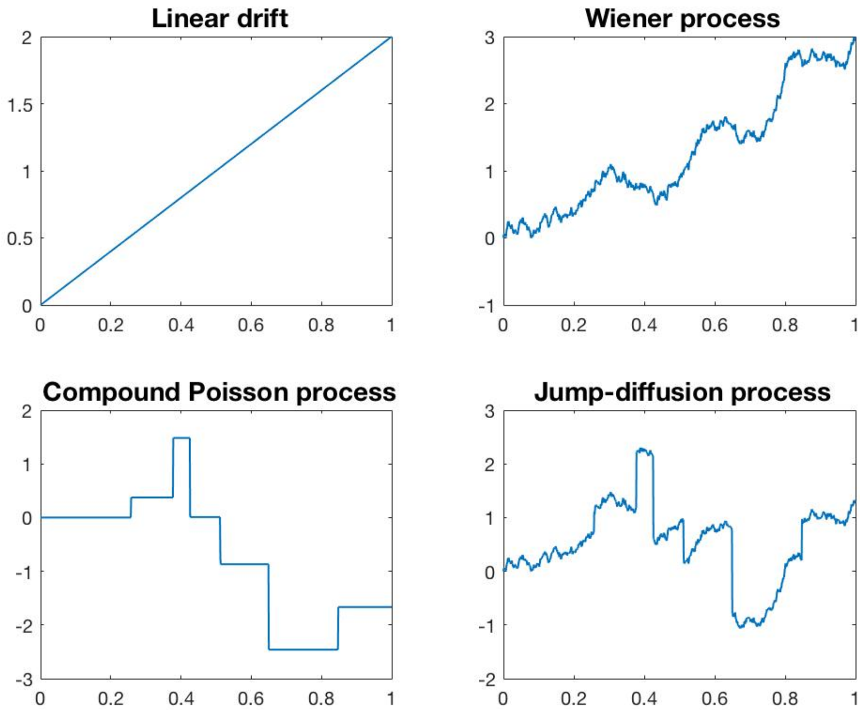

Fig. 2.1: Examples of Lévy processes: a linear drift with Lévy triplet $(2, 0, 0)$, a Wiener process with Lévy triplet $(2, 1, 0)$, a compound Poisson process with Lévy triplet $(0, 0, \lambda \times f)$, where $\lambda = 5$ and $f \sim \mathcal{N}(0, 1)$ and finally a jump-diffusion process with Lévy triplet $(2, 1, \lambda \times f)$.

3 Numerical Methods

> *FFT is the most important numerical algorithm of our lifetime.*
>
> — **Gilbert Strang** (1934)

Section Introduction

3.1 Monte Carlo

3.2 Finite Difference Method

Taylor Expansion

Recall that the Taylor expansion for a function $f \in C^{\infty}$ infinitely many differentiable is given by

$$\begin{aligned} f(x) &= f(a) + (x-a)\frac{\partial f}{\partial x}(a) + \frac{(x-a)^2}{2!}\frac{\partial^2 f}{\partial x^2}(a) + \cdots + \frac{(x-a)^n}{n!}\frac{\partial^n f}{\partial x^n}(a) + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(a). \end{aligned}$$

If f is only $(n+1)$ times continuously differentiable, i.e. $f \in C^{(n+1)}$, we can write

$$f(x) = \sum_{k=0}^{n+1} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(x) + O((x-a)^{n+1}),$$

where $O((x-a)^{n+1})$ represents the remainder in Landau notation.

Forward and Backward Difference Approximation of First Derivative

In order to approximate $\frac{\partial f}{\partial t}(x)$ assume that f is twice continuously differentiable, i.e. $f \in C^2$. By a first order Taylor expansion we can write

$$f(x+h) = f(x) + h\frac{\partial f}{\partial x}(x) + O(h), \tag{3.1}$$

$$f(x-h) = f(x) - h\frac{\partial f}{\partial x}(x) + O(h). \tag{3.2}$$

The equation (3.1) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x+h) - f(x)}{h} + O(h), \tag{3.3}$$

which is known as *forward difference* approximation of the first derivative.

On the other hand, the equation (3.2) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x) - f(x-h)}{h} + O(h), \tag{3.4}$$

which is known as *backward difference* approximation of the first derivative.

Central Difference Approximation of First Derivative

Now assume that $f \in C^3$. Then with a second order Taylor expansion, we have

$$f(x+h) = f(x) + h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2), \tag{3.5}$$

$$f(x-h) = f(x) - h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2). \tag{3.6}$$

Subtracting equation (3.6) from (3.5) we get

$$f(x+h) - f(x-h) = 2h\frac{\partial f}{\partial x}(x) + O(h^2).$$

Therefore we obtain the *central difference* approximation of the first derivative

$$\frac{\partial f}{\partial x}(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2). \tag{3.7}$$

Central Difference Approximation of Second Derivative

Finally summing equations (3.5) and (3.6) we get

$$2f(x) = f(x+h) + f(x-h) + h^2 \frac{\partial^2 f}{\partial x^2}(x) + O(h^2).$$

Then the *central difference* approximation of the second derivative is given by

$$\frac{\partial^2 f}{\partial x^2}(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \tag{3.8}$$

Option Pricing under the Generalized Black-Scholes model

Consider the Generalized Black-Scholes model, which includes the *local volatility* $\sigma(S,t)$ and term structures of *interest rate* $r(t)$ and *dividend rate* $q(t)$. The price of an asset S under such model follows the *stochastic differential equation* (SDE):

$$dS_t = (r(t) - q(t))S_t dt + \sigma(S_t, t) S_t dW_t.$$

Then we know that the value of an option $v(S,t)$ on that asset S satisfies the following *partial differential equation* (PDE):

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\sigma(S,t)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(t) - q(t))S\frac{\partial v}{\partial S} = r(t)v(S,t) & \\ v(S,T) = \Psi(S) & \text{Terminal Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, t) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, t) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

Now if we use the change of variable $\tau = (T - t)$ to express *time to maturity*, we obtain the following PDE:

$$\begin{cases} -\frac{\partial v}{\partial \tau} + \frac{\sigma(S,\tau)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(\tau) - q(\tau))S\frac{\partial v}{\partial S} = r(\tau)v(S,\tau) & \\ v(S,0) = \Psi(S) & \text{Initial Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, \tau) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, \tau) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

To begin, we have to define the domain of the problem

$$D = \{S_{\min} \le S \le S_{\max}; 0 \le \tau \le T\}$$

and set it to a discrete grid

$$\bar{D} = \left\{ \begin{array}{lll} S_j = S_{\min} + (j-1)h; & h = \frac{S_{\max} - S_{\min}}{N}; & j = 1, \ldots, N+1 \\ t_k = 0 + (k-1)\Delta t; & \Delta t = \frac{T}{M}; & k = 1, \ldots, M+1 \end{array} \right\},$$

where N is the number of subintervals in the S-direction and M is the number of subintervals in the τ-direction.

Forward Euler Approximation

The Forward Euler approximation constructs the *explicit* discretization of the Generalized Black-Scholes PDE. In other words, we approximate the theta term $\frac{\partial v}{\partial t}(S,t)$ using a *forward difference* approximation (3.3):

$$\frac{\partial v}{\partial t}(S,t) \approx \frac{v(S, t+\Delta t) - v(S,t)}{\Delta t}.$$

The *central difference* approximation of the first derivative (3.7) for the delta term $\frac{\partial v}{\partial S}(S,t)$ gives us

$$\frac{\partial v}{\partial S}(S,t) \approx \frac{v(S+h,t) - v(S-h,t)}{2h}$$

and the *central difference* approximation of the second derivative (3.8) for the gamma term $\frac{\partial^2 v}{\partial S^2}(S,t)$ gives

$$\frac{\partial^2 v}{\partial S^2}(S,t) \approx \frac{v(S+h,t) - 2v(S,t) + v(S-h,t)}{h^2}.$$

Backward Euler Approximation

θ-Method and Crank-Nicolson Approximation

3.3 The Convolution Method

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Colophon

This thesis was typeset with $\LaTeX\,2_\varepsilon$. It uses the *Clean Thesis* style developed by Ricardo Langner. The design of the *Clean Thesis* style is inspired by user guide documents from Apple Inc.

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Declaration

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Lausanne, April 15, 2017

Valentin Bandelier