Pricing FX-TARN Under Lévy Processes Using Numerical Methods

Valentin Bandelier

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Swiss Federal Institute of Technology Lausanne - EPFL

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

School of Basic Sciences - SB
Institute of Mathematics - MATH

Master Thesis

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Supervisor	Fabio Nobile EPFL Institute of Mathematics - MATH
Co-supervisor	Julien Hugonnier EPFL Swiss Finance Institute - SFI
Assistant	Francesco Statti

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Swiss Federal Institute of Technology Lausanne - EPFL
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School of Basic Sciences - SB
Route Cantonale
1015 Lausanne

Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Abstract (different language)

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Acknowledgement

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This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Contents

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Introduction

> *Finance is the art of passing currency from hand to hand until it finally disappears.*
>
> — **Robert W. Sarnoff**
> (1918-1997)

This thesis presents different numerical methods for pricing FX-TARN under Lévy processes. In general, options with path dependents payoff, such as this product, are evaluated by Monte Carlo simulations. We will describe two other methods based on Finite Difference (FD) and Fast Fourier Transform (FFT). The initial chapter starts, in Section 1.1, with an historic of existing works that allowed this project to born. Then, in Section 1.2, the FX-TARN product is presented. In section 1.3, an example of term sheet illustrates this exotic product. Finally, the chapter concludes with an overview of the thesis in section 1.4.

1.1 Motivation

1.2 FX-TARN Description

An FX Target Accrual Redemption Note (FX-TARN) is a financial product that allows an investor to accumulate an amount of cash until a certain *target accrual level* U over a predefined schedule. More precisely, the contract between the bank and the client imposes cash flow on scheduled dates (fixing dates). We can replicate these cash flows with a series of FX call options (resp. FX put options) with strike K, that the bank sells to a client, and at the same time a series of FX put options (resp. FX call options) with the same strike K, that the bank buys from the client. Sometimes, the client leg that the bank buys is combined with a leverage factor g called *gear factor*. The scheduling is defined by a number of fixing dates $t_1, t_2, \ldots, t_N$ that corresponds to the option expiry dates. Finally, the product knock-out if the total sum of payouts (from the bank's point of view) exceeds the given target U. There are three types of knock-out when the target U is breached that we will see in the next section:

- **No Gain :** the last payment is disallowed when the target U is breached,
- **Part Gain :** only a part of the payment is allowed such that only the target is paid,
- **Full Gain :** the last payment is allowed when the target U is breached.

Payoff Definition

Define the following notations:

- $S(t)$: FX rate at time t,
- K : strike(s) (could be different for each fixing dates),
- t_0 : today's date,
- $t_1, t_2, \dots, t_M$: fixing dates,
- U : target accrual level,
- $A(t)$: accumulated gains at time t,
- N_f : Accrual amount per fixing date.

On each fixing date $t_n, n = 1, \dots, N$, if the target level U is not breached by the accumulated amount $A(t_m)$, the gain per unit of notional foreign amount from the point of view of the investor is given by

$$\tilde{C}_n = \beta(S(t_n) - K) \times \mathbf{1}_{\{\beta S(t_n) \geq \beta K\}},$$

and the loss

$$\tilde{C}_n^* = -g \times \beta(K - S(t_n)) \times \mathbf{1}_{\{\beta S(t_n) \leq \beta K\}},$$

where β is the strategy, i.e. $\beta = 1$ the investor buys call options, $\beta = -1$ the investor buys put options.

Denote $t_{\tilde{N}}$ the first fixing date before maturity on which the target level U is breached by the total accumulated gain (without the loss part), i.e.

$$\tilde{N} = \min\{n : A(t_n) \geq U\}, \qquad n = 1, 2, \dots, N.$$

If the target U is not breached, set $\tilde{N} = N$. For $t_n \leq t_{\tilde{N}}$ we can write the actual payment for a gain as

$$C_n(S(t_n), A(t_{n-1})) = \tilde{C}_n \times \left(\mathbf{1}_{\{A(t_{n-1}))+\tilde{C}<U\}} + W_n \times \mathbf{1}_{\{A(t_{n-1})+\tilde{C}\geq U\}}\right), \qquad (1.1)$$

and $C_n = 0$ for $t_n > t_{\tilde{N}}$. The loss does not count in the knock-out condition but will also knock-out by the same redemption event. Therefore we have that

$$C_n^*(S(t_n), A(t_{n-1})) = \tilde{C}_n^* \times \left(\mathbf{1}_{\{A(t_{n-1})+\tilde{C}<U\}} + W_n \times \mathbf{1}_{\{A(t_{n-1})+\tilde{C}\geq U\}}\right). \tag{1.2}$$

Here, $A(t_{n-1})$ is the accumulated gain immediately after the fixing date t_{n-1} and W_n is the weight corresponding to the type of knock-out when the target is breached. We can set the weights W_n for the different types of knock-out as follow:

$$W_n = \begin{cases} 0, & \text{for No Gain,} \\ \frac{U-A(t_{n-1})}{\beta\times(S(t_n)-K)}, & \text{for Part Gain,} \\ 1, & \text{for Full Gain.} \end{cases}$$

Therefore, the accumulated gains $A(t)$ is a step function such that $A(t) = A(t_{n-1})$, for $t_{n-1} \leq t < t_n$ with

$$A(t_n) = A(t_{n-1}) + C_n(S(t_n), A(t_{n-1})).$$

In Figure 1.1, we can see the plot of the positive cash flow $C(S(t), A(t^-)) = C(S, A)$ that counts in the accumulated amount and produces jumps in the price on a fixing date.

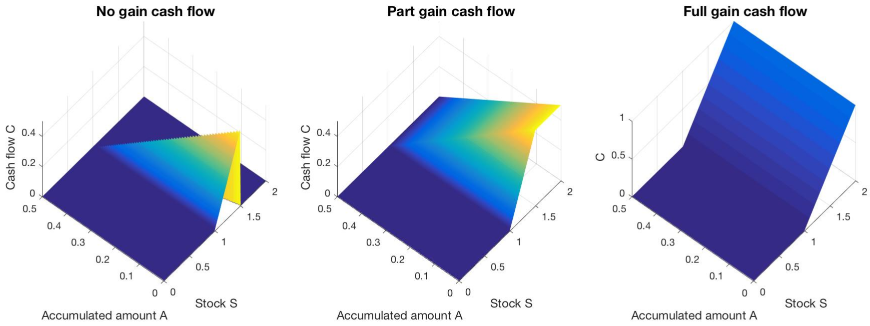


Fig. 1.1: Positive cash flow that produces jump on a fixing date for different type of knock-out.

Finally, the net present value of FX-TARN in domestic currency for FX rate realization $\mathbf{S} = (S(t_1), S(t_2), \ldots, S(t_N))$ is

$$P(\mathbf{S}) = N_f \times \sum_{n=1}^{N} \frac{\mathbf{C}_n^{\text{tot}}(S(t_n), A(t_{n-1}))}{B_d(t_0, t_n)}, \qquad A(t_0) = 0, \tag{1.3}$$

where $B_d(t_0, t_n)^{-1}$ is the domestic discounting factor from t_n to t_0 and $\mathbf{C}_n^{\text{tot}} = C_n(S(t_n), A(t_{n-1})) + C_n^*(S(t_n), A(t_{n-1}))$ is the total cash flow at time t_n.

1.3 Example of Term Sheet

In this section, we will give an example of a particular TARN called "*Leveraged Foreign Exchange Target Accrual Redemption Note with Full Settlement*", i.e. Full Gain FX-TARN with gear factor. Thus this is a Full Gain FX-TARN with gear factor. The term sheet has the following form.

USD/CHF FX-TARN	**Key Characteristics of the Transaction**
Instrument Type	Leverage FX Target Accrual Redemption Note (TARN)
Trade Date	23 May 2017
Transaction Price	0% (zero cost strategy)
Buyer of CHF	The Bank
Buyer of USD	The Client
Underlying	USD/CHF Foreign Exchange rate
Notional Amount	USD 2’080’000.00 (versus maximum notional amount USD 4’160’000.00)
Accrual Amount per Fixing Date	USD 40’000.00 (versus Leveraged Accrual Amount USD 80’000.00)
Leverage factor	2
Initial Spot Price	0.9730 CHF per USD
Strike Price	As per date schedule below
Target Redemption Level	0.4 CHF per 1 USD
Weekly Gains	On each Fixing Date, • If USD/CHF $>$ Strike Price : Weekly Gains $=$ USD/CHF - Strike Price • Otherwise : Weekly Gains $=$ 0 CHF
Cumulated Gains	In respect of any Fixing Date, the Weekly Gains on that Fixing Date plus the sum of all Weekly Gains in respect of all previous Fixing Dates.
Redemption Event	A Redemption Event is deemed to have occurred if the Cumulated Gains (including the present fixing) is greater than or equal to the Target Redemption Level on any Fixing Date.
Expiration Date	22 May 2018
Fixing Reference	Weekly (Business Days)
Fixing Dates	52 Fixings (see Schedule below)
Profile on Fixing Date	In respect of each Fixing Date : **1)** If no Redemption Event occurs and the Fixing Price is : • At or above the Strike Price, the Client will buy from the Bank the Accrual Amount at the strike price :

<table>
<tr><td></td><td>Strike Price x Accrual Amount
Or
• Below the Strike Price, the Client will buy from the Bank the Leveraged Accrual Amount at the Strike Price at the Strike Price for delivery on the relevant Settlement Date :
Leveraged Factor x Strike Price x Accrual Amount
2) If a Redemption Event occurs :
• The Client will buy from the Bank the Accrual Amount at the Strike Price for the Fixing Date that Redemption Event is deemed to have occurred :
Strike Price x Accrual Amount
• The product is then knocked out for all remaining subsequent Fixings. There will be no further obligations between the Client and the Bank.</td></tr>
<tr><td>Schedule</td><td><table>
<tr><th>Fixing</th><th>Fixing Date</th><th>Strike Level</th></tr>
<tr><td>1</td><td>30 May 2017</td><td>0.9275</td></tr>
<tr><td>2</td><td>6 June 2017</td><td>0.9275</td></tr>
<tr><td>3</td><td>13 June 2017</td><td>0.9350</td></tr>
<tr><td>4</td><td>20 June 2017</td><td>0.9350</td></tr>
<tr><td>5</td><td>27 June 2017</td><td>0.9350</td></tr>
<tr><td>6</td><td>4 July 2017</td><td>0.9420</td></tr>
<tr><td>7</td><td>18 July 2017</td><td>0.9420</td></tr>
<tr><td>8</td><td>25 July 2017</td><td>0.9420</td></tr>
<tr><td>⋮</td><td>⋮</td><td>⋮</td></tr>
<tr><td>50</td><td>8 May 2018</td><td>0.9420</td></tr>
<tr><td>51</td><td>15 May 2018</td><td>0.9420</td></tr>
<tr><td>52</td><td>22 May 2018</td><td>0.9420</td></tr>
</table>(Pay intention about the strike that is increasing in time.)</td></tr>
</table>

Tab. 1.1: FX-TARN Term Sheet example.

1.4 Overview of the Thesis

Lévy Processes

2

> *Paul Lévy was a painter in the probabilistic world.*
>
> — **Michel Loève**
> (1907-1979)

The Lévy processes play a central role in mathematical finance. They can describe the reality of financial markets in a more accurate way than models based on the geometric Brownian motion used in particular in Black-Scholes model. Indeed we can observe in the real world that the asset price processes have some jumps. Moreover, the log returns of the underlying have empirical distribution with fat tails and skewness which deviates from normality supposed by Black and Scholes. We begin this chapter, in section 2.1, with the definition of a Lévy process and expose its fundamental properties. Next, in section 2.2, we presents two main results about Lévy processes which are the Lévy-Khinchine formula and the Lévy-Itô decomposition. In section 2.3, the Lévy measure and path properties of a Lévy process are exposed. Finally, the section 2.4 presents the class of exponential Lévy processes and the equivalent martingale measure used to describe the asset price in financial modeling.

2.1 Definitions and properties

The Lévy processes, which are the continuous-time case of random walks, are ingredients for building continuous-time stochastic models. The simplest Lévy process is the linear drift. The Wiener process, Poisson process, and compound Poisson process are the most famous examples of Lévy processes. We will see later in this chapter that the sum of a linear drift, a Wiener process, and a compound Poisson process is again a Lévy process. It is called a *Lévy jump-diffusion* process.

Definition 2.1 (*Wiener Process*)

*A stochastic process $W = \{W_t, t \geq 0\}$, with $W_0 = 0$, is a **Wiener process**, also called a standard Brownian motion, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if:*

1. *W has independent increments, i.e. $(W_{t+s} - W_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*

2. *W has stationary increments, i.e. the distribution of $(W_{t+s} - W_t)$ does not depend on t.*
3. *W has Gaussian increments, i.e. $(W_{t+s} - W_t) \sim \mathcal{N}(0, s)$.*
4. *W is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to t} \mathbb{P}(|W_t - W_s| < \epsilon) = 0.$$

This motion was discovered by Brown in 1827 and taken back by Bachelier (1900) to model the stock market prices. Only in 1923 the Brownian was defined and constructed rigorously by R. Wiener.

Definition 2.2 (*Poisson process*)

Let $(\tau_i)_{i \geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $N = \{N_t, t \geq 0\}$, with $N_0 = 0$, defined by

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{t \geq T_n\}}$$

*is called **Poisson process** with intensity λ.*
This process has the following properties:

1. *N has independent increments, i.e. $(N_{t+s} - N_t)$ is independent of $\mathcal{F}_t$ for any $s > 0$.*
2. *N has stationary increments, i.e. the distribution of $(N_{t+s} - N_t)$ does not depend on t.*
3. *N has Poisson increments, i.e. $(N_{t+s} - N_t)$ has a Poisson distribution with parameter λs.*
4. *N is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to 0} \mathbb{P}(|N_{t+s} - N_t| < \epsilon) = 0.$$

When the process is characterized by a constant intensity parameter λ, we say that the process is homogeneous. If the intensity parameter varies with time t as $\lambda(t)$, the process is said to be non-homogeneous.

The Poisson process, which bears the name of the French physicist and mathematician Siméon Denis Poisson, defines a counting process. It counts the number of random times (T_n) which occur in $[0, t]$. Therefore, this is an increasing pure jump process. The jumps of size 1 occur at times T_n and the intervals between two jumps are exponentially distributed. If we compare definitions 2.1 and 2.2, we can see that only the fourth property differs between the two processes, only the distribution changes. The main idea of a Lévy process is to ignore the distribution of increments.

Definition 2.3 (*Lévy process*)

A cadlag stochastic process $X = \{X_t, t \geq 0\}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with real values is called a* ***Lévy process*** *if it has the following properties:*

1. X *has independent increments, i.e.* $(X_{t+s} - X_t)$ *is independent of* $\mathcal{F}_t$ *for any* $s > 0$*.*
2. X *has stationary increments, i.e. the distribution of* $(X_{t+s} - X_t)$ *does not depend on* t*.*
3. X *is stochastically continuous, i.e.*

$$\forall \epsilon > 0 : \lim_{s \to 0} \mathbb{P}(|X_{t+s} - X_t| < \epsilon) = 0.$$

The third condition does not imply that the sample paths are continuous. In fact, the Brownian motion is the only (non-deterministic) Lévy process with continuous sample paths. This condition serves to exclude jumps at non-random times. In other words, for a given t, the probability of seeing a jump at t is zero, discontinuities occur at random time. The compound Poisson process is a good example of a Lévy process.

Definition 2.4 (*Compound Poisson process*)

A ***compound Poisson process*** *with intensity* $\lambda > 0$ *and jump size distribution* f *is a stochastic process* $X = \{X_t, t \geq 0\}$ *defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where jumps size Y_i *are i.i.d. with the density function* f *and* $N = \{N_t, t \geq 0\}$ *is a Poisson process with intensity* λ*, independent from* $(Y_i)_{i \geq 1}$*.*

We can easily deduce the following properties from this definition:

1. The sample paths of X are cadlag piecewise constant functions.
2. The jump times $(T_i)_{i \geq 1}$ have the same law as the jump times of the Poisson process N_t. They can be expressed as partial sums of an independent exponential random variable with parameter λ.
3. The jump sizes $(Y_i)_{i \geq 1}$ are i.i.d. with law f.

We can also see that the Poisson process itself can be seen as a compound Poisson process with $Y_i \equiv 1$. This explains the origin of the name of the definition. Finally, the compound Poisson process allows us to work with jump sizes which have an arbitrary distribution.

2.2 Lévy-Khinchine formula and Lévy-Itô decomposition

We will now present in this section two main results about Lévy processes: the *Lévy-Khinchine formula* and the *Lévy-Itô decomposition*. Let's start with the relationship between infinitely divisible distributions and Lévy process.

Definition 2.5 (***Infinite divisibility***)

A probability distribution F is said to be ***infinitely divisible*** *if for any integer $n \geq 2$, there exists n i.i.d. random variables $Y_1, \dots, Y_n$ such that $Y_1 + \cdots + Y_n$ has distribution F.*

If X is a Lévy process, for any $t > 0$ the distribution of X_t is infinitely divisible. This comes from the fact that for any $n \geq 1$,

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}), \tag{2.1}$$

and the property of stationary and independent increments. Let define now the characteristic function and characteristic exponent of X_t.

Definition 2.6 (***Characteristic function and exponent***)

The ***characteristic function*** *Φ_t of a random variable X_t with cumulative distribution F_t is given by*

$$\Phi_t(u) = \mathbb{E}\left[e^{iuX_t}\right] = \int_{-\infty}^{\infty} e^{i\theta x} dF_t(x).$$

Its ***characteristic exponent*** *is given by*

$$\Psi_t(u) = \log\left(\mathbb{E}\left[e^{iuX_t}\right]\right),$$

for $u \in \mathbb{R}$ and $t > 0$.

Then using twice equation (2.1) we obtain for any positive integers m, n that

$$m\Psi_1(u) = \Psi_m(u) = n\Psi_{m/n}(u).$$

Hence for any rational $t = \frac{m}{n} > 0$ we have

$$\Psi_t(u) = t\Psi_1(u).$$

We can generalize this relation for all $t > 0$ with the help of the almost sure continuity of X and a sequence of rational $\{t_n, n \geq 1\}$ such that $t_n \downarrow t$.

In conclusion, any Lévy process has the property that for all $t > 0$

$$\mathbb{E}\left[e^{iuX_t}\right] = e^{t\Psi(u)},$$

where $\Psi(u) = \Psi_1(u)$ is the characteristic exponent of X_1.

Then it is clear that each Lévy process has an infinitely divisible distribution. This allows us to apply the celebrated Lévy-Khinchine formula.

Theorem 2.7 (*Lévy-Khintchine formula*)

Each Lévy process can be characterized by a triplet (γ, σ, ν) *with* $\gamma \in \mathbb{R}, \sigma \geq 0$ *and* ν *a measure satisfying* $\nu(\{0\}) = 0$ *and*

$$\int_{\mathbb{R}} \min\{1, |x|^2\}\nu(dx) < \infty.$$

In term of this triplet the characteristic function of the Lévy process equals:

$$\begin{aligned}\Phi_t(u) &= \mathbb{E}\left[\exp(iuX_t)\right] \\ &= \exp\left(t\left(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right)\nu(dx)\right)\right).\end{aligned} \tag{2.2}$$

(The proof can be find in Tankov and Cont (2003))

The triplet (γ, σ, ν) is called the *Lévy* or *characteristic triplet*. Moreover, γ is called the *drift term*, σ the *Gaussian* or *diffusion coefficient* and $\nu(dx)$ is the *Lévy measure*, being the intensity of jumps of size x. This brings us to the following great result which is the Lévy-Itô decomposition.

Theorem 2.8 (*Lévy-Itô decomposition*)

Consider a triplet (γ, σ, ν) *where* $\gamma \in \mathbb{R}$, $\sigma \geq 0$ *and* ν *is a measure satisfying* $\nu(\{0\}) = 0$ *and*

$$\int_{\mathbb{R}} \min\{1, |x|^2\}\nu(dx) < \infty.$$

Then, there exists exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *on which four independent Lévy processes exist,* $X^{(1)}, X^{(2)}, X^{(3)}$ *and* $X^{(4)}$*, where* $X^{(1)}$ *is a constant drift,* $X^{(2)}$ *is a Wiener process,* $X^{(3)}$ *is a compound Poisson process and* $X^{(4)}$ *is a square integrable (pure jump) martingale with an a.s. countable number of jumps of magnitude less than 1 on each finite time interval. Taking* $X = X^{(1)} + X^{(2)} + X^{(3)} + X^{(4)}$*, we have that there exists a probability space on which a Lévy process* $X = \{X_t, 0 \leq t \leq T\}$ *with characteristic exponent*

$$\Psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right)\nu(dx),$$

for all $u \in \mathbb{R}$, is defined.
(See Kyprianou (2006) for the proof)

The Lévy process is characterized by its triplet (γ, σ, ν). The simplest Lévy process is the linear *drift* with the triplet $(\gamma, 0, 0)$. Adding a *diffusion* component we get the triplet $(\gamma, \sigma, 0)$ which is the case of the Black-Scholes model. A *pure jump* process will be identified by the triplet $(0, 0, \nu)$ and finally a *Lévy jump-diffusion* process will have the complete triplet (γ, σ, ν). The figure 2.1 illustrates some examples of Lévy processes.

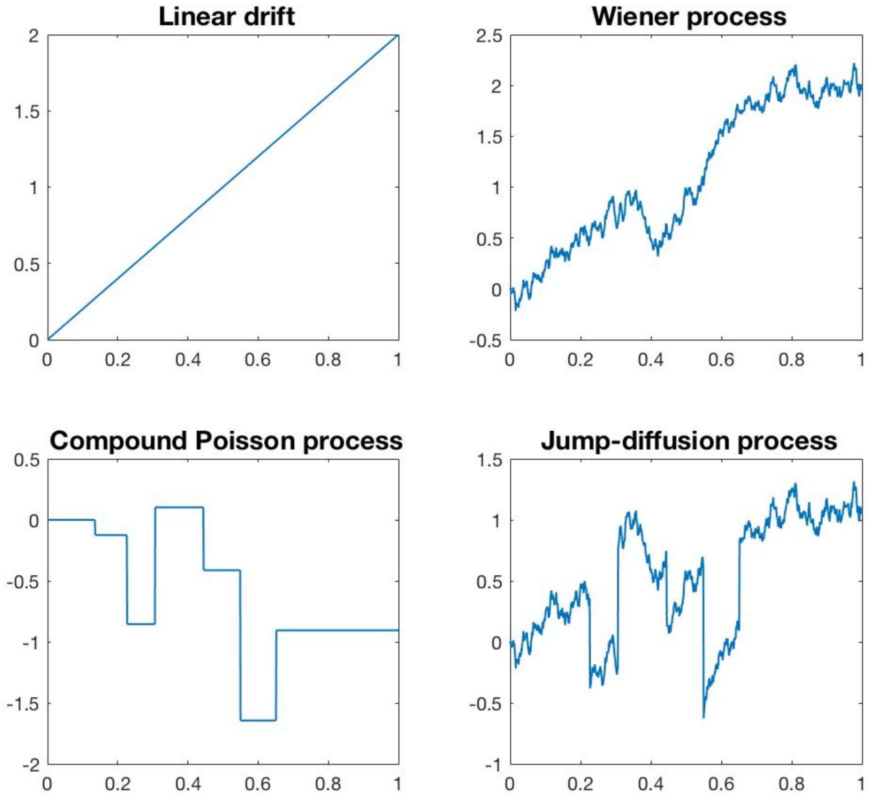


Fig. 2.1: Examples of Lévy processes: a linear drift with Lévy triplet $(2, 0, 0)$, a Wiener process with Lévy triplet $(2, 1, 0)$, a compound Poisson process with Lévy triplet $(0, 0, \lambda \cdot f_J)$, where $\lambda = 5$, and $f_J \sim \mathcal{N}(0, 1)$ and finally a jump-diffusion process with Lévy triplet $(2, 1, \lambda \cdot f_J)$.

2.3 Lévy measure and path properties

The *Lévy measure* dictates the behavior of the jumps.

Definition 2.9 (*Lévy measure*)

Let $x = \{X_t, t \geq 0\}$ be a Lévy process on $\mathbb{R}$. The measure ν on $\mathbb{R}$ defined by

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1] : \Delta x \neq 0, \Delta x \in A\}\right],$$

is called the ***Lévy measure*** *of* X*:* $\nu(A)$ *is the expected number, per unit time, of jumps whose size belongs to* A*.*

For example, the Lévy measure of a compound Poisson process is given by $\nu(dx) = \lambda f_J(dx)$. In other words, the expected number of jumps, in a time interval of length 1, is λ and the jump size is distributed according to f_J.

More generally, if ν is a finite measure, that is $\lambda = \nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(dx) < \infty$, then we can defined $f(dx) = \frac{\nu(dx)}{\lambda}$, which is a probability measure. Then, λ is the expected number of jumps and $f(dx)$ is the distribution of the jump size x. If $\nu(\mathbb{R}) = \infty$, an infinite number of (small) jumps is expected.

Proposition 2.10 (*Finite and infinite activity*)

Let $X = \{X_t, t \geq 0\}$ *be a Lévy process with triplet* (γ, σ, ν)*.*

1. *If* $\nu(\mathbb{R}) < \infty$ *then almost all paths of* X *have a finite number of jumps on every compact interval. In that case, the Lévy process has* ***finite activity****.*
2. *If* $\nu(\mathbb{R}) = \infty$ *then almost all paths of* X *have a infinite number of jumps on every compact interval. In that case, the Lévy process has* ***infinite activity****.*

(See Theorem 21.3 in Sato (1999) for the proof)

Then the Lévy jump models can be classified into two categories from their Lévy measure: jump-diffusion or pure jump models. The jump-diffusions models are modeled by a Gaussian part (Wiener process) combined with a jump part (compound Poisson process), that has finitely many jumps in every time interval, i.e. finite activity models. The second category consists of models with an infinite number of jumps in every interval, i.e. infinite activity models. In these models, there is no need of Gaussian part because the dynamics of jumps are already rich enough to generate nontrivial small time behavior. Merton model and Variance Gamma model are respectively good examples of jump-diffusion and pure jump models. We can see in figure 2.2 that the Lévy density in Variance Gamma model allows infinite number of jumps, while the Merton model has a finite number of jumps on every time interval.

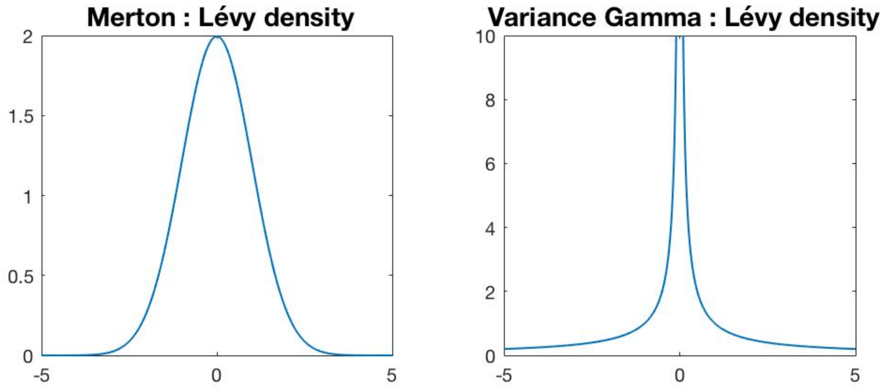


Fig. 2.2: The density of Lévy measure in the Merton model (left) and the Variance Gamma model (right).

2.4 Exponential Lévy processes and Equivalent martingale measure

In finance, it is common to model the stock price process as exponentials of a Lévy process:

$$S_t = S_0 e^{X_t}.$$

The advantage of this representation is that the stock prices process is nonnegative and the log returns $\log(S_{t+s}/S_t)$, for $s, t > 0$, follow the distribution of increments of length s dictate by the Lévy process $X = \{X_t, t \geq 0\}$. Thus they have independent and stationary increments. If we choose a process such that $X_0 = 0$, we get $e^{X_0} = 1$ and therefore $S_0 = S_0 e^{X_0} = S_0$.

In order to avoid an arbitrage opportunity, the discounted and reinvested process $\hat{S} = \{\hat{S}_t = e^{-(r-q)t} S_t, t \geq 0\}$ has to be a martingale under an *equivalent martingale measure* (EMM) $\mathbb{Q}$, called the *risk-neutral measure*. Recall that r is the *(domestic) risk-free rate* and q is the *continuous dividend yield (or foreign interest rate)* of the asset. In other words, we are looking for a measure $\mathbb{Q}$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[\hat{S}_T | \mathcal{F}_t\right] = \hat{S}_t.$$

Since the market is not complete under Lévy processes, there exists several ways to find a risk-neutral measure. We will see two different methods to determine this probability measure.

2.4.1 Esscher transform method

The first approach to find an EMM $\mathbb{Q}$ is proposed by Gerber, Shiu, et al. (1994) using the Esscher transform. Suppose that the Lévy process $X = \{X_t, t \geq 0\}$ has a density function $f(x;t)$. Now multiply this density by an exponential factor $e^{\theta t}$ to get a new density function:

$$f(x;t,\theta) = \frac{e^{\theta x} f(x;t)}{\int_{\mathbb{R}} e^{\theta y} f(y;t) dy}.$$

Note that the denominator ensures the properties of $f(x;t,\theta)$ to be a density function, i.e.

$$\int_{\mathbb{R}} f(y;t,\theta) dy = 1.$$

With this transformation we obtain a new probability function defined by

$$d\mathbb{P}_t^{\theta} = \frac{d\mathbb{P}_t}{\int_{\mathbb{R}} e^{\theta y} d\mathbb{P}_t} = \frac{d\mathbb{P}_t}{M(\theta;t)},$$

where $M(\theta;t)$ is the moment-generating function and $\mathbb{P}$ is the real world probability measure. The goal is to determine the parameter θ such that $\mathbb{P}^{\theta}$ is an EEM. Take a look on the moment-generating function of X_t under $\mathbb{P}$,

$$M(u;t) = \mathbb{E}\left[e^{uX_t}\right] = \Phi_t(-iu),$$

and the moment-generating function of X_t under $\mathbb{P}^{\theta}$,

$$\begin{aligned} M(u;t,\theta) &= \int_{\mathbb{R}} e^{ux} f(x;t,\theta) dx \\ &= \frac{\int_{\mathbb{R}} e^{(u+\theta)x} f(x;t) dx}{\int_{\mathbb{R}} e^{\theta y} f(y;t) dy} \\ &= \frac{M(u+\theta;t)}{M(\theta;t)} \\ &= \frac{\Phi_t(-i(u+\theta))}{\Phi_t(-i\theta)}. \end{aligned} \tag{2.3}$$

The martingale condition on $\hat{S} = \{\hat{S}_t = S_0 e^{-(r-q)t+X_t}, t \geq 0\}$ gives us the following relation:

$$S_0 = e^{-(r-q)t}\mathbb{E}^{\mathbb{P}^{\theta}}\left[S_t\right] = e^{-(r-q)t} S_0 \underbrace{\mathbb{E}^{\mathbb{P}^{\theta}}\left[e^{X_t}\right]}_{=M(u;t,\theta)} = e^{-(r-q)t} S_0 \frac{\Phi_t(-i(u+\theta))}{\Phi_t(-i\theta)}.$$

Therefore, θ is given by the explicit equation

$$e^{(r-q)t} = \frac{\Phi_t(-i(1+\theta))}{\Phi_t(-i\theta)}.$$

Thus the solution θ^* of this equation gives us the Esscher transform martingale measure and we have $\mathbb{Q} \equiv \mathbb{P}^{\theta^*}$.

Characterization of the risk-neutral Lévy process

With the help of equation (2.3) we have that

$$\Phi_t^\theta(-iu) = \frac{\Phi_t(-i(u+\theta))}{\Phi_t(-i\theta)} \iff \Phi_t^\theta(z) = \frac{\Phi_t(z-i\theta))}{\Phi_t(-i\theta)}.$$

We can also add that the new Lévy process is characterized by the triplet $(\gamma^\theta, \sigma^\theta, \nu^\theta(dx))$, and with the Lévy-Khintchine formula 2.7 combined to the definition (2.6) of the characteristic exponent, we can recover

$$\begin{aligned} \gamma^\theta &= \gamma + \sigma^2\theta + \int_{-1}^{1} \left(e^{\theta x} - 1\right) \nu(dx), \\ \sigma^\theta &= \sigma, \\ \nu^\theta(dx) &= e^{\theta x}\nu(dx). \end{aligned}$$

2.4.2 Mean-correction method

The second way to obtain an equivalent martingale measure $\mathbb{Q}$ is to correct the mean of the exponential Lévy process to satisfy the martingale condition of the discounted stock price process $\hat{S} = \{\hat{S}_t = e^{-(r-q)t}S_t, t \geq 0\}$. The idea is to add a drift to the Lévy process to kill the drift of the discounted asset price process. Therefore we obtain a new Lévy process $\tilde{X} = \left\{\tilde{X}_t = X_t + \omega t, t \geq 0\right\}$ and consequently

$$\begin{aligned} S_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-(r-q)t}S_t\right] \\ &= S_0 e^{-(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{\tilde{X}_t}\right] \\ &= S_0 e^{-(r-q)t}\mathbb{E}^{\mathbb{Q}}\left[e^{X_t+\omega t}\right] \\ &= S_0 e^{[\omega-(r-q))+\Psi(-i)]t} \end{aligned}$$

Hence we have that ω have to be equal to $[(r-q) - \Psi(-i)]$, where Ψ is the characteristic exponent of X_1. Moreover we have that the new risk-neutral Lévy process $\tilde{X}$ is characterized by the triplet (γ^*, σ, ν) with

$$\gamma^* = \gamma + (r-q) - \Psi(-i). \tag{2.4}$$

The mean-correction method is simpler than the Esscher transform method and this is the method we will use throughout this thesis. There are several other measures that can be found in the book of Miyahara (2011).

3

Financial Mathematic Models

> *Essentially, all models are wrong, but some are useful.*
>
> — **George E. P. Box**
> (1919-2013)

In this chapter, we will take a look on some popular models in financial mathematics. To begin, in section 3.1 we will describe the Black-Scholes model (1973) and compute its risk-neutral characteristic function. In section 3.2 we will talk about *jump-diffusion models*. These models evolve with a diffusion process, punctuated by jumps at random intervals. We can model this behavior with a Wiener process and a compound Poisson process to characterized the jumps with size distribution f_J. In fact, we will talk about two examples: the Merton model (1976) and the Kou model (2002). Finally, the section 3.3 is devoted to *pure jump models*. This category of models is characterized by infinite number of jumps in any time interval, called *infinite activity* models. These particular models don't need a Brownian part because the dynamic of the process is already modeled by an infinity of small jumps. However, we will see that it is possible to construct these models by a Brownian subordination, which is called a time-changed Browninan motion. At the end of this chapter we will have seen two examples which are the Normal Inverse Gaussian (NIG) model, proposed by Barndorff-Nielsen (1997) the Variance Gamma (VG) model, proposed by Madan et al. (1998).

3.1 Black-Scholes Model

Samuelson (1965) was the first one to introduce Brownian motion to model asset prices. Then his work was taken over by Black and Scholes (1973) to create the most famous model, the Black-Scholes model. In this model, the stock price $S = \{S_t, t \geq 0\}$ follows a geometric Brownian motion, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are respectively the drift and the volatility of the process. This stochastic differential equation has a unique solution which is

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

In fact this model is based on an exponential Lévy process $X = \{X_t, t \geq 0\}$ defined by

$$X_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Hence his characteristic triplet is $\left(\mu - \frac{1}{2}\sigma^2, \sigma, 0\right)$.

Risk-neutral Characteristic Function

Recall that X_t in this model is described by the characteristic triplet $(\gamma, \sigma, 0)$ with $\gamma = \left(\mu - \frac{1}{2}\sigma^2\right)$. Thus the Lévy-Khintchine formula 2.7 gives us the characteristic function of X_t

$$\Phi_t(u) = \exp\left\{t\left(\left(\mu - \frac{1}{2}\sigma^2\right) iu - \frac{1}{2}\sigma^2 u^2\right)\right\}.$$

Hence the characteristic exponent of X_1 evaluated at $-i$ is

$$\Psi(-i) = \mu - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 = \mu.$$

With equation (2.4), we obtain the risk-neutral drift

$$\gamma^* = r - q - \frac{1}{2}\sigma^2,$$

and the risk-neutral characteristic function is given by

$$\Phi_t^{\text{RN}}(u) = \exp\left\{t\left(i\gamma^* u - \frac{1}{2}\sigma^2 u^2\right)\right\}.$$

Finally the risk-neutral stock price process is defined by

$$\begin{aligned} S_t &= S_0 \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \\ &= S_0 \exp\left\{X_t^{\text{BS}}(r, q, \sigma)\right\} \end{aligned}$$

3.2 Jump-diffusion Models

Consider now the Lévy jump-diffusion process $X = \{X_t, t \geq 0\}$. It is modeled by a drifted Brownian motion and a compound Poisson process. Therefore we can write it in the form

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

with $\gamma \in \mathbb{R}, \sigma \in \mathbb{R}_+, W = \{W_t, t \geq 0\}$ is a Wiener process, $N = \{N_t, t \geq 0\}$ is a Poisson process with parameter λ and $Y = \{Y_t, t \geq 0\}$ is an i.i.d sequence of random variables with density f_J.

The characteristic function of X_t is given by

$$\begin{aligned}\Phi_t(u) &= \mathbb{E}\left[e^{iuX_t}\right] \\ &= \mathbb{E}\left[\exp\left\{iu\left(\gamma t + \sigma W_t + \sum_{i=1}^{N_t}\right)\right\}\right] \\ &= \exp\{iu\gamma t\}\,\mathbb{E}\left[\exp\{iu\sigma W_t\}\right]\mathbb{E}\left[\exp\left\{iu\sum_{i=1}^{N_t} Y_i\right\}\right],\end{aligned}$$

by independence of W_t and N_t. Since $W_t \sim \mathcal{N}(0, \sigma^2 t)$ and $N_t \sim \text{Poisson}(\lambda t)$, we have

$$\begin{aligned}\mathbb{E}\left[e^{iu\sigma W_t}\right] &= e^{-\frac{1}{2}\sigma^2 u^2 t}, \\ \mathbb{E}\left[e^{iu\sum_{i=1}^{N_t} Y_i}\right] &= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{iunY}\right]\mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \Phi_Y(u)^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\ &= e^{\lambda t(\Phi_Y(u) - 1)} \\ &= e^{\lambda t \int_{\mathbb{R}} (e^{iuy} - 1) F(dy)}.\end{aligned}$$

Hence we get

$$\begin{aligned}\Phi_t(u) &= \exp\{iu\gamma t\}\exp\left\{-\frac{1}{2}\sigma^2 u^2 t\right\}\exp\left\{\lambda t \int_{\mathbb{R}} \left(e^{iuy} - 1\right) f_J(dy)\right\} \\ &= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1\right)\lambda f_J(dy)\right)\right\}. \qquad (3.1)\end{aligned}$$

Then we have a characterization of Lévy jump-diffusion process by its characteristic triplet $(\gamma, \sigma, \lambda \cdot f_J)$.

3.2.1 Merton Model

Under the Black-Scholes model, the stock price is supposed to be continuous. Unfortunately this is not the case in reality. Merton (1976) is the first to use the notion of discontinuous price process to model asset returns. In his model, Merton uses a Normal distribution to model the jump size, i.e. $f_J \sim \mathcal{N}(\alpha, \delta^2)$. Then the Lévy processes is

$$X_t = \mu t + \sigma W_t + \sum_{i=0}^{N_t} Y_i,$$

with $Y_i \sim \mathcal{N}(\alpha, \delta^2)$. Hence, the density function of the jump size is

$$f_J(x) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x-\alpha)^2}{2\delta^2}},$$

and the Lévy density is

$$\nu(x) = \lambda f_J(x) = \frac{\lambda}{\sqrt{2\pi}\delta} e^{-\frac{(x-\alpha)^2}{2\delta^2}}.$$

Then there are four parameters in the Merton model excluding the drift parameter μ:

- σ - the diffusion volatility,
- λ - the jump intensity,
- α - the mean of jump size,
- δ - the standard deviation of jump size.

Risk-neutral Characteristic Function

With the help of equation 3.1, we obtain the characteristic function of the model under the real world measure $\mathbb{P}$:

$$\begin{aligned}
\Phi_t(u) &= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1\right) \lambda f_J(dy)\right)\right\} \\
&= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda\left(\Phi_Y(u) - 1\right)\right)\right\} \\
&= \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda\left(e^{iu\alpha - \frac{1}{2}\delta^2 u^2} - 1\right)\right)\right\},
\end{aligned}$$

where Φ_Y is the characteristic function of a jump Y. Hence the model is characterized by the triplet $(\gamma, \sigma, \lambda \cdot f_J)$.

We can now compute the characteristic exponent in order to apply the mean-correction and get the risk-neutral process.

$$\Psi(-i) = \gamma + \frac{1}{2}\sigma^2 + \lambda\left(e^{\alpha+\frac{1}{2}\delta^2} - 1\right).$$

Applying equation (2.4), we obtain the risk-neutral drift

$$\gamma^* = (r-q) - \frac{1}{2}\sigma^2 - \lambda\left(e^{\alpha+\frac{1}{2}\delta^2} - 1\right),$$

and the risk-neutral characteristic function of the Merton jump-diffusion model is given by

$$\Phi_t^{\mathrm{RN}}(u) = \exp\left\{t\left(i\gamma^* u - \frac{1}{2}\sigma^2 u^2 + \lambda\left(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1\right)\right)\right\}.$$

The risk-neutral stock price process is finally

$$S_t = S_0 \exp\left\{X_t^{\mathrm{Mer}}(r, q, \sigma, \lambda, \alpha, \delta)\right\},$$

where X^{Mer} is the Lévy jump-diffusion process characterized by the triplet $(\gamma^*, \sigma, \lambda \cdot f_J)$.

3.2.2 Kou Model

The Kou model (2002) is very similar to Merton's one. The only difference is in the distribution of the jump size, which is double-exponential. Then the Lévy process under Kou model is

$$X_t = \gamma t + \sigma W_t + \sum_{i=0}^{N_t} Y_i,$$

with $Y_i \sim \text{DoubleExp}(p, \eta_1, \eta_2)$. In other words, jump size has the density

$$f_J(x) = \begin{cases} p \cdot \eta_1 e^{-\eta_1 x}, & \text{if } x \geq 0, \\ (1-p) \cdot \eta_2 e^{\eta_2 x}, & \text{if } x < 0. \end{cases}$$

The probability p represents the probability of an upward jump and $(1-p)$ the probability of a downward jump. Thus the Lévy density is given by

$$\nu(x) = \lambda\left(p \cdot \eta_1 e^{-\eta_1 x}\mathbf{1}_{x\geq 0} + (1-p) \cdot \eta_2 e^{\eta_2 x}\mathbf{1}_{x<0}\right).$$

Then there are five parameters in the Kou model excluding the drift parameter μ:

- σ - the diffusion volatility,
- λ - the jump intensity,

- p - the probability of an upward jump,
- η_1, η_2 - control the decay of the tails in the distribution.

Risk-neutral Characteristic Function

A preliminary computation of the characteristic function of a double exponential random variable Y is needed.

$$\begin{aligned}
\Phi_Y(u) &= \int_{\mathbb{R}} e^{iuy} f_J(y) dy \\
&= \int_0^{\infty} e^{iuy} p \cdot \eta_1 e^{-\eta_1 y} dy + \int_{-\infty}^{0} e^{iuy} (1-p) \cdot \eta_2 e^{\eta_2 y} dy \\
&= p \cdot \eta_1 \left[\frac{e^{(iu-\eta_1)y}}{iu - \eta_1} \right]_0^{\infty} + (1-p) \cdot \eta_2 \left[\frac{e^{(iu+\eta_2)y}}{iu + \eta_2} \right]_{-\infty}^{0} \\
&= \frac{p \cdot \eta_1}{\eta_1 - iu} + \frac{(1-p) \cdot \eta_2}{\eta_2 + iu}
\end{aligned}$$

Now as for Merton model, the equation (3.1) gives us the characteristic function of X_t

$$\begin{aligned}
\Phi_t(u) &= \exp\left\{ t \left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda \left(\Phi_Y(u) - 1 \right) \right) \right\} \\
&= \exp\left\{ t \left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda \left(\frac{p \cdot \eta_1}{\eta_1 - iu} + \frac{(1-p) \cdot \eta_2}{\eta_2 + iu} - 1 \right) \right) \right\}.
\end{aligned}$$

Hence the model is characterized by the triplet $(\gamma, \sigma, \lambda \cdot f_J)$.

The characteristic exponent of this process gives us

$$\Psi(-i) = \gamma + \frac{1}{2}\sigma^2 + \lambda \left(\frac{p \cdot \eta_1}{\eta_1 - 1} + \frac{(1-p) \cdot \eta_2}{\eta_2 + 1} - 1 \right).$$

Consequently we obtain the risk-neutral drift

$$\gamma^* = (r - q) - \frac{1}{2}\sigma^2 - \lambda \left(\frac{p \cdot \eta_1}{\eta_1 - 1} + \frac{(1-p) \cdot \eta_2}{\eta_2 + 1} - 1 \right),$$

and the risk-neutral characteristic function of the Double Exponential Kou jump-diffusion model

$$\Phi_t^{\text{RN}}(u) = \exp\left\{ t \left(i\gamma^* u - \frac{1}{2}\sigma^2 u^2 + \lambda \left(\frac{p \cdot \eta_1}{\eta_1 + iu} + \frac{(1-p) \cdot \eta_2}{\eta_2 + iu} - 1 \right) \right) \right\}.$$

Therefore we can model the risk-neutral stock price process by

$$S_t = S_0 \exp\left\{ X_t^{\text{Kou}}(r, q, \sigma, \lambda, p, \eta_1, \eta_2) \right\},$$

where X_t^{Kou} is the Lévy jump-diffusion process characterized by the triplet $(\gamma^*, \sigma, \lambda \cdot f_J)$.

3.3 Pure jump Models

To go beyond the jump-diffusion process, initially proposed by Merton in 1976, we can talk about infinite activity models. There exist a lot of paper about these kind of Lévy processes. We will see two different models which are the *Normal Inverse Gaussian* (NIG) model, proposed by Barndorff-Nielsen in 1997, and the *Variance Gamma* (VG) model, proposed by Madan et al. in 1998. They are both particular cases of the Generalized Hyperbolic model, developed by Eberlein and Prause (1998).

These two models can be described as a Brownian motion $W = \{W_t, t \geq 0\}$ with constant drift θ and volatility σ evaluated at a random time $T = \{T_t, t \geq 0\}$,

$$X_t = \theta T_t + \sigma W_{T_t}.$$

This process is called *time changed Brownian motion* with constant drift. Moreover, the process T is called the *subordinator* process, which is an increasing Lévy process. The subordinating processes in the NIG and VG models are respectively an *Inverse Gaussian* process and a *Gamma* process.

3.3.1 Normal Inverse Gaussian Model

First of all, we will present the subordinating Inverse Gaussian process which is used to construct the Normal Inverse Gaussian (NIG) process.

Inverse Gaussian Process

Let $T \sim \mathrm{IG}(a, b)$ be an inverse Gaussian random variable. This is in fact the first time that a Brownian motion with drift $b > 0$ reaches the level $a > 0$ Its density function is given by

$$f_{\mathrm{IG}}(x; a, b) = \frac{a e^{ab}}{\sqrt{2\pi}} x^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}\left(\frac{a^2}{x} + b^2 x\right)\right\}, \qquad x > 0,$$

and its characteristic function is

$$\Phi_{\mathrm{IG}}(u; a, b) = \exp\left\{-a\left(\sqrt{-2iu + b^2} - b\right)\right\}.$$

Note that if $X_1, \dots, X_n$ are independent IG random variables with parameters $(a/n, b)$, then $X_1 + \dots + X_n \sim \text{IG}(a, b)$. Thus this distribution is infinitely divisible and we are able to define an IG process $X^{\text{IG}} = \{X_t^{\text{IG}}, t \geq 0\}$ as a process that starts at 0 and has independent and stationary increments such that $X_t^{\text{IG}} \sim \text{IG}(at, b)$. Hence it has the following characteristic function

$$\begin{aligned}\Phi_t^{\text{IG}}(u; at, b) &= \mathbb{E}\left[e^{iuX_t^{\text{IG}}}\right] \\ &= \exp\left\{-at\left(\sqrt{-2iu + b^2} - b\right)\right\}\end{aligned}$$

Now let's verify the non-decreasing condition for a subordinator. We have that

$$\begin{aligned}\mathbb{P}\left(X_{t+\Delta t}^{\text{IG}} < X_t^{\text{IG}}\right) &= \mathbb{P}\left(X_{t+\Delta t}^{\text{IG}} - X_t^{\text{IG}} < 0\right) \\ &= \mathbb{P}\left(X_{\Delta t}^{\text{IG}} < 0\right) = 0,\end{aligned}$$

since an IG random variable takes only positive values. Thus it is a good candidate as subordinator.

Normal Inverse Gaussian Process

As mention by Geman (2002), we can represent the NIG process by a time-changed Brownian motion with an IG process as subordinator. Let $W = \{W_t, t \geq 0\}$ be a standard Brownian motion and $T = \{T_t, t \geq 0\}$ be an IG process with parameters $a = 1$ and b. Then the NIG process is given by

$$X_t = \theta T_t + \sigma W_{T_t}.$$

Thus its characteristic function is

$$\begin{aligned}\Phi_t^{\text{NIG}}(u; \theta, \sigma) &= \mathbb{E}\left[e^{iuX_t}\right] \\ &= \mathbb{E}\left[e^{\left(iu\theta - \frac{\sigma^2 u^2}{2}\right)T_t}\right] \\ &= \Phi_t^{\text{IG}}\left(u\theta + i\frac{\sigma^2 u^2}{2}\right) \\ &= \exp\left\{-t\left(\sqrt{-2i\left(u\theta + i\frac{\sigma^2 u^2}{2}\right) + b^2} - b\right)\right\} \\ &= \exp\left\{-t\left(\sqrt{b^2 - 2iu\theta + \sigma^2 u^2} - b\right)\right\} \\ &= \exp\left\{-t\sigma\left(\sqrt{\frac{b^2}{\sigma^2} + \frac{\theta^2}{\sigma^4} - \left(\frac{\theta}{\sigma^2} + iu\right)^2} - \frac{b}{\sigma}\right)\right\}.\end{aligned}$$

To simplify the notation, we can set

$$\alpha^2 = \frac{b^2}{\sigma^2} + \frac{\theta^2}{\sigma^4},$$
$$\beta = \frac{\theta}{\sigma^2},$$
$$\delta = \sigma.$$

Then the subordinator $T_t \sim \text{IG}\left(t, \delta\sqrt{\alpha^2 - \beta^2}\right)$ and the NIG process becomes

$$X_t^{\text{NIG}} = \beta\delta^2 T_t + \delta W_{T_t},$$

and we get the characteristic function given by Barndorff-Nielsen (1997) in the form

$$\Phi_t^{\text{NIG}}(u; \alpha, \beta, \delta) = \exp\left\{t\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right)\right\}.$$

Then the NIG model has three parameters to control the shape of the distribution:

- α - tail heaviest of steepness,
- β - symmetry,
- δ - scale.

Note that the parameters have to satisfy the conditions $\alpha, \delta > 0$ and $-\alpha < \beta < \alpha$.

Risk-neutral Characteristic Function

Here, since the characteristic triplet $(\gamma, 0, \nu)$ is not trivial, we will find a risk-neutral characteristic function using the following form of the stock price

$$S_t = S_0 e^{(r-q)t + \omega t + X_t^{\text{NIG}}}.$$

Hence we have that

$$\begin{aligned} S_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-(r-q)t} S_t\right] \\ &= S_0 \mathbb{E}^{\mathbb{Q}}\left[e^{\omega t + X_t^{\text{NIG}}}\right] \\ &= S_0 e^{\omega t} \Phi_t^{\text{NIG}}(-i). \end{aligned}$$

Therefore we must have $e^{\omega t}\Phi_t^{\text{NIG}}(-i) = 1$ or equivalently

$$\omega = -\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2}\right).$$

This gives us the risk-neutral drift

$$\gamma^* = (r-q) - \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+1)^2}\right),$$

and the risk-neutral characteristic function is

$$\Phi_t^{RN}(u) = \exp\left\{t\left(i\gamma^* u + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right)\right)\right\}.$$

Finally, the risk-neutral stock price process is in the form

$$S_t = S_0 \exp\{\gamma^* t + X_t^{\mathrm{NIG}}(\alpha,\beta,\delta)\},$$

where X_t^{NIG} is the Normal Inverse Gaussian process characterized by the Lévy triplet $(\gamma^*, 0, \nu^{\mathrm{NIG}}$, with

$$\nu^{\mathrm{NIG}}(dx) = \frac{\delta\alpha}{\pi}\frac{\exp(\beta x)K_1(\alpha|x|)}{|x|}dx,$$

where K_λ is the modified Bessel of the second king with index λ.

3.3.2 Variance Gamma Model

Madan et al. in 1998 had the same approach as the previous Normal Inverse Gaussian model. The difference is that the random time in the Brownian motion is Gamma distributed. In a second time, since this process has also finite variation, it can be represent by the difference of two increasing processes. The first one models the price increases while the second one reflects the price decreases. To begin, let us introduce the subordinating Gamma process used to construct the Variance Gamma process.

Gamma process

The Gamma density function $f_\Gamma(x;a,b)$ with parameters $a, b > 0$ is given by

$$f_\Gamma(x;a,b) = x^{a-1}\frac{b^a e^{-bx}}{\Gamma(a)},$$

where Γ is the Euler gamma function. Then its characteristic function is

$$\Phi_\Gamma(u;a,b) = \left(1-\frac{iu}{b}\right)^{-a}.$$

This distribution is also infinitely divible because if $X_1, \dots, X_n \sim \mathrm{Gamma}(a/n, b)$, we have that $X_1 + \cdots + X_n \sim \mathrm{Gamma}(a, b)$. Therefore, we can define a Gamma process $X^{\text{Gam}} = \left\{X_t^{\text{Gam}}, t \geq 0\right\}$, which is a stochastic process that starts at 0 and has stationary and independent increments such that $X_t^{\text{Gamma}} \sim \Gamma(at, b)$. The corresponding characteristic function is given by

$$\begin{aligned}\Phi_t^{\text{Gam}}(u; at, b) &= \mathbb{E}\left[e^{iuX_t^{\text{Gam}}}\right] \\ &= \left(1 - \frac{iu}{b}\right)^{-at} \\ &= \left(\frac{1}{1 - \frac{iu\nu}{\mu}}\right)^{\frac{\mu^2}{\nu}t},\end{aligned}$$

where μ and ν are respectively the mean rate and the variance rate of the process.

Variance Gamma process

As in the case of the Normal Inverse Gamma process, we can represent the Variance Gamma process as a time-changed Brownian motion

$$X_t = \theta T_t + \sigma W_{T_t},$$

with $T = \{T_t, t \geq 0\}$ a gamma process with mean rate $\mu = 1$ and variance rate ν. Therefore the characteristic function of this process is

$$\begin{aligned}\Phi_t^{\text{VG}}(u; \theta, \sigma, \nu) &= \mathbb{E}\left[e^{iuX_t}\right] \\ &= \Phi_t^{\text{Gam}}\left(u\theta + i\frac{\sigma^2 u^2}{2}\right) \\ &= \left(\frac{1}{1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2}\right)^{\frac{t}{\nu}}.\end{aligned} \tag{3.2}$$

Then we have that the VG model has three parameters:

- θ - drift of the Brownian motion,
- σ - volatility of the Brownian motion,
- ν - variance rate of the time change.

Madan et al. (1998) showed that the VG process has finite variation. Therefore we can represent this process by the difference of two independent and increasing gamma process with mean rate $\mu_\pm$ variance rate $\nu_\pm$, i.e.

$$X_t = \gamma_t^+(\mu_+, \nu_+) - \gamma_t^-(\mu_-, \nu_-),$$

where γ_t^+ and γ_t^- correspond respectively to the positive and negative shocks. Therefore the characteristic function of this representation is

$$\begin{aligned}\Phi_t^{\mathrm{VG}}(u) &= \mathbb{E}\left[e^{iu(\gamma_t^+ - \gamma_t^-}\right] \\ &= \Phi_{\gamma_t^+}(u)\Phi_{-\gamma_t^-}(u) \\ &= \left(\frac{1}{1-\frac{iu\nu_+}{\mu_+}}\right)^{\frac{\mu_+^2}{\nu_+}t}\left(\frac{1}{1+\frac{iu\nu_-}{\mu_-}}\right)^{\frac{\mu_-^2}{\nu_-}t}. \qquad (3.3)\end{aligned}$$

Thus, comparing the both characteristic functions (3.2) and (3.3), we get the following relations

$$\begin{aligned}\frac{\mu_+^2}{\nu_+} = \frac{\mu_-^2}{\nu_-} &= \frac{1}{\nu}, \\ \frac{\nu_+\nu_-}{\mu_+\mu_-} &= \frac{\sigma^2\nu}{2}, \\ \frac{\nu_+}{\mu_+} - \frac{\nu_-}{\mu_-} &= \theta\nu.\end{aligned}$$

Hence we have that

$$\begin{aligned}\mu_+ &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}, \\ \mu_- &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2}, \\ \nu_+ &= \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}\right)^2 \nu, \\ \nu_- &= \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2}\right)^2 \nu.\end{aligned}$$

Finally, the VG process is effectively the difference of two independent gamma processes.

Risk-neutral Characteristic Function

Just recall that the characteristic function under real world probability $\mathbb{P}$ is given by

$$\begin{aligned}\Phi_t^{\text{VG}}(u) &= \left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)^{-\frac{t}{\nu}} \\ &= \exp\left\{-\frac{t}{\nu}\ln\left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)\right\}.\end{aligned}$$

In the same way as in the NIG model, we can construct the risk-neutral drift by considering

$$S_t = S_0 e^{(r-q)t+\omega t+X_t^{\text{VG}}}.$$

Then

$$\begin{aligned}S_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-(r-q)t}S_t\right] \\ &= S_0\mathbb{E}^{\mathbb{Q}}\left[e^{\omega t+X_t^{\text{VG}}}\right] \\ &= S_0 e^{\omega t}\Phi_t^{\text{VG}}(-i),\end{aligned}$$

and we must have that $e^{\omega t}\Phi_t^{\text{VG}}(-i) = 1$ or in other words

$$\omega = \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right).$$

At the end we obtain the risk-neutral drift

$$\gamma^* = (r-q) + \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right),$$

and the risk-neutral characteristic function is given by

$$\Phi_t^{\text{RN}}(u) = \exp\left\{t\left(i\gamma^* u - \frac{1}{\nu}\ln\left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)\right)\right\}.$$

Finally, the risk-neutral stock price process is

$$S_t = S_0\exp\left\{\gamma^* t + X_t^{\text{VG}}(\theta, \sigma, \nu)\right\},$$

where X_t^{VG} is the Variance Gamma process characterized by the Lévy triplet $(\gamma^*, 0, \nu^{\text{VG}})$, with

$$\nu^{\text{VG}}(dx) = \begin{cases}\frac{C\exp(Gx)}{|x|}dx, & x < 0,\\ \frac{C\exp(-Mx)}{x}dx, & x > 0,\end{cases}$$

where

$$\begin{aligned} C &= \frac{1}{\nu} > 0, \\ G &= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu \right)^{-1} > 0, \\ M &= \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu \right)^{-1} > 0. \end{aligned}$$

3.4 Summary

To summarize, we can see that in all models the risk-neutral stock price process can be written in the form:

$$S_t = S_0 \exp\left\{\gamma^* t + X_t\right\},$$

with the risk-neutral drift γ^* and a drift-less Lévy process X_t. Moreover, the risk-neutral characteristic function is in the form:

$$\Phi_t^{\mathrm{RN}}(u) = \exp\left\{t\left(i\gamma^* u + \Psi(u)\right)\right\},$$

where Ψ is the characteristic exponent of X_1. Tables 3.1, 3.2, 3.3 and 3.4 illustrate respectively the drift-less Lévy process X_t, the risk-neutral drift γ^*, the Lévy density $\nu(dy)$ and the risk-neutral characteristic exponent $\Psi(u)$ for all the models which we have just studied in this chapter.

Models	Lévy process X_t	Comments
Black-Scholes	σW_t	
Merton	$\sigma W_t + \sum_{i=1}^{N_t} Y_i$	$Y_i \sim \mathcal{N}(\alpha, \delta^2)$
Kou	$\sigma W_t + \sum_{i=1}^{N_t} Y_i$	$Y_i \sim$ DoubleExp(p, η_1, η_2)
Normal Inverse Gaussian	$\beta\delta^2 T_t + \delta W_{T_t}$	$T_t \sim$ IG $\left(t, \delta\sqrt{\alpha^2 - \beta^2}\right)$
Variance Gamma	$\theta T_t + \sigma W_{T_t}$	$T_t \sim$ Gamma $\left(\frac{t}{\nu}, \frac{1}{\nu}\right)$

Tab. 3.1: Drift-less Lévy processes X_t for several models.

Models	Risk-neutral drift γ^*
Black-Scholes	$r - q - \frac{1}{2}\sigma^2$
Merton	$r - q - \frac{1}{2}\sigma^2 - \lambda\left(e^{\alpha+\frac{1}{2}\delta^2} - 1\right)$
Kou	$r - q - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p \cdot \eta_1}{\eta_1 - 1} + \frac{(1-p)\cdot\eta_2}{\eta_2+1} - 1\right)$
Normal Inverse Gaussian	$r - q - \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}\right)$
Variance Gamma	$r - q + \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right)$

Tab. 3.2: Risk-neutral drifts γ^* for several models.

Models	Lévy density $\nu(dx)$
Black-Scholes	0
Merton	$\frac{\lambda}{\sqrt{2\pi}\delta} e^{-\frac{(x-\alpha)^2}{2\delta^2}} dx$
Kou	$\lambda\left(p \cdot \eta_1 e^{-\eta_1 x}\mathbf{1}_{x\geq 0} + (1-p)\cdot\eta_2 e^{\eta_2 x}\mathbf{1}_{x<0}\right) dx$
Normal Inverse Gaussian	$\frac{\delta\alpha}{\pi}\frac{\exp(\beta x)K_1(\alpha\|x\|)}{\|x\|} dx$
Variance Gamma	$\left(\frac{C\exp(-Mx)}{x}\mathbf{1}_{x\geq 0} + \frac{C\exp(Gx)}{\|x\|}\mathbf{1}_{x<0}\right) dx$

Tab. 3.3: Lévy density $\nu(dy)$ for several models.

Models	Risk-neutral characteristic exponent $\Psi(u)$
Black-Scholes	$-\frac{1}{2}\sigma^2 u^2$
Merton	$-\frac{1}{2}\sigma^2 u^2 + \lambda\left(e^{i\alpha u - \frac{1}{2}\delta^2 u^2} - 1\right)$
Kou	$-\frac{1}{2}\sigma^2 u^2 + \lambda\left(\frac{p\cdot\eta_1}{\eta_1 - iu} + \frac{(1-p)\cdot\eta_2}{\eta_2 + iu} - 1\right)$
Normal Inverse Gaussian	$\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right)$
Variance Gamma	$-\frac{1}{\nu}\ln\left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)$

Tab. 3.4: Risk-neutral characteristic exponent $\Psi(u)$ for several models.

4

Numerical Methods

> *FFT is the most important numerical algorithm of our lifetime.*
>
> — **Gilbert Strang**
> (1934)

Section Introduction

4.1 Monte Carlo Method

In this section we will briefly recall the principle of the Monte Carlo simulations and present algorithms to simulate the different processes that we have studied in the last chapter. The idea in the Monte Carlo method is to simulate M sample paths of the stock price process $\mathbf{S}_m, m = 1, \ldots, M$, under the corresponding model and for each path, compute the present value $P(\mathbf{S}_i)$ of the financial product. Then, by the law of the large numbers, we obtain the following proxy:

$$\hat{P}(\mathbf{S}) = \frac{1}{M}\sum_{m=1}^{M} P(\mathbf{S}_m) \xrightarrow{M\to\infty} P(\mathbf{S}),$$

where $\mathbf{S} = (S(t_1), \ldots, S(t_N))$ is the realization of the stock price. The standard error of the estimate is given by

$$\text{SE} = \sqrt{\frac{1}{M-1}\sum_{i=1}^{M}\left(\hat{P}(\mathbf{S}) - P(\mathbf{S}_i)\right)^2}.$$

Remark that the standard error decreases with the square root of the number of sample paths M. By the law of large numbers, we can construct the following 95% confidence interval for the real price $P(\mathbf{S})$

$$\left[\hat{P}(\mathbf{S}) - 1.96\frac{\text{SE}}{\sqrt{M}}, \hat{P}(\mathbf{S}) + 1.96\frac{\text{SE}}{\sqrt{M}}\right]$$

Recall that in our case, the present value of the FX TARN is given by equation (1.3):

$$P(\mathbf{S}) = N_f \times \sum_{n=1}^{N} \frac{C_n(S(t_n), A(t_{n-1})) + C_n^*(S(t_n))}{B_d(t_0, t_n)}, \qquad A(t_0) = 0,$$

where C_n and C_n^* are respectively the gain and the loss on the n^{th} fixing date given by equations (1.1) and (1.2). The variable $A(t_n)$ models the accumulated gains until the date t_n and $B_d(t_0, t_n)^{-1} = e^{-r_d(t_n - t_0)}$ is the domestic discounting factor from t_n to t_0. In the rest of the thesis, we will consider the present value per unit of notional $(N_f = 1)$.

4.1.1 Simulations under Black-Scholes model

We have seen that the Lévy process in the Black-Scholes model is given by

$$X_t^{\text{BS}} = \left(r - q - \frac{1}{2}\right) t + \sigma W_t,$$

where W_t is a Wiener process. Therefore, by discretization of time, we get

$$\Delta X_t^{\text{BS}} = \left(r - q - \frac{1}{2}\sigma^2\right) \Delta t + \sigma\sqrt{\Delta t} Z,$$

with $Z \sim \mathcal{N}(0,1)$. Finally we easily have

$$\begin{aligned} S_{t+\Delta_t} &= \exp\left\{X_t^{\text{BS}} + \left(r - q - \frac{1}{2}\sigma^2\right) \Delta t + \sigma\sqrt{\Delta t} Z\right\} \\ &= S_t \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right) \Delta t + \sigma\sqrt{\Delta t} Z\right\}. \end{aligned}$$

We can use the command `random('norm',0,1)` in MATLAB to generate random normal variable.

4.1.2 Simulations under Jump-diffusion models

A jump-diffusion process is nothing else than a Brownian motion with drift to which is added by a jump process modeled by a compound Poisson process. In other words, we have

$$X_t^{\text{JD}} = \gamma^* t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

where $N_t \sim \text{Poisson}(\lambda t)$ and the jump size Y_i has density function f_J. We have seen the two special case where the distribution f_J is normal $\mathcal{N}(\alpha, \delta^2)$ in the Merton model and double exponential $\text{DoubleExp}(p, \eta_1, \eta_2)$ in the Kou model.

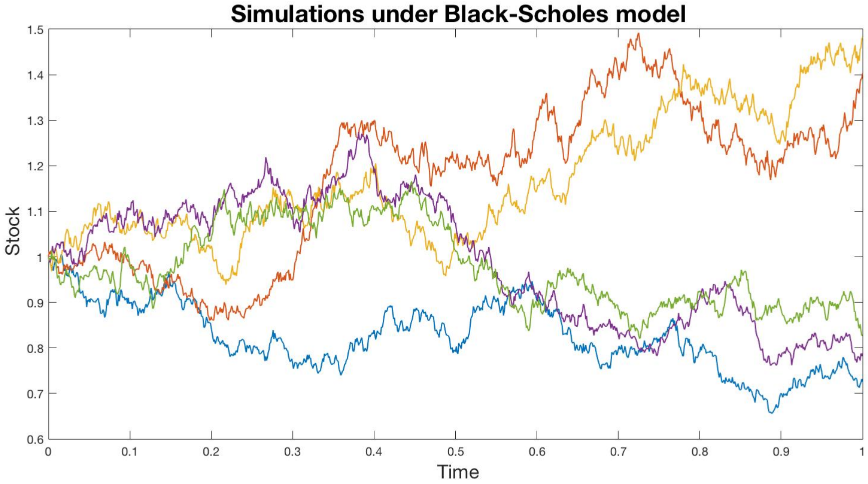


Fig. 4.1: Simulations of stock price process under Black-Scholes model. $S_0 = 1, r = 0.01, q = 0.02, \sigma = 0.3, T = 1, dt = 0.001, M = 5.$

Therefore, we have that

$$\Delta X_t^{\text{JD}} = \gamma^* \Delta t + \sigma \sqrt{\Delta t} Z + J(\Delta t),$$

where $J(\Delta t)$ is the sum of all jumps between t and $t + \Delta t$, i.e.

$$J(\Delta t) = \sum_{i=1}^{N_{\Delta t}} Y_i.$$

We can use the command MATLAB `random('poiss',`$\lambda\Delta t$`)` to simulate the variable $N_{\Delta t}$.

Merton model

In his model, Merton supposed that the jump size is normally distributed with mean α and standard deviation δ, i.e. $Y_i \sim \mathcal{N}(\alpha, \delta)$. Then, recall that the risk-neutral drift is given by

$$\gamma^* = r - q - \frac{1}{2}\sigma^2 - \lambda \left(e^{\alpha + \frac{1}{2}\delta^2} - 1 \right).$$

Thus we have

$$S_{t+\Delta t} = S_t \exp \left\{ \gamma^* \Delta t + \sigma \sqrt{\Delta t} Z + J(\Delta t) \right\},$$

where $J(\Delta t) \sim \mathcal{N}(N_{\Delta_t} \alpha, N_{\Delta t} \delta)$ and $N_{\Delta t} \sim \text{Poisson}(\lambda \Delta t)$.

Finally, we obtain the results of five simulated sample paths in figure 4.2.

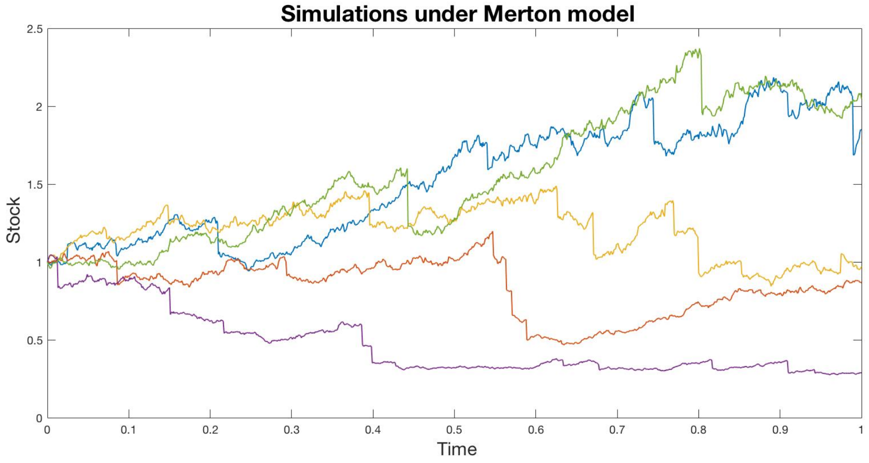


Fig. 4.2: Simulations of stock price process under Merton model.
$S_0 = 1, r = 0.01, q = 0.02, \lambda = 10, \alpha = -0.1, \delta = 0.1, \sigma = 0.3,$
$T = 1, dt = 0.001, M = 5.$

Kou model

This model is very similar to the Merton's one, but Kou proposed to use a double exponential distribution for the jump size, i.e. $Y_i \sim \text{DoubleExp}(p, \eta_1, \eta_2)$. Thus the difficulty is to simulate double exponential random variables. Note that the sum of K independent exponential random variables of parameter η has a gamma distribution with parameters K and η. In other words, if $X_1, \ldots, X_K \sim \text{Exp}(\eta)$, then $Y = \sum_{i=1}^K X_i \sim \Gamma(K, \eta)$ and

$$f_Y(y) = y^{K-1} \frac{\eta^K e^{-\eta y}}{K-1}.$$

Hence, to simulate the jumps $J(\Delta t)$, we begin by simulating a binomial random variable K that counts the number of positive jump in $[t, t + \Delta t]$,

$$K \sim \text{Binomial}(N_{\Delta t}, p), \qquad \text{with } N_{\Delta t} \sim \text{Poisson}(\lambda \Delta t).$$

Then, we simulate the positive and negative jumps

$$J^+ \sim \text{Gamma}(K, \eta_1),$$
$$J^- \sim \text{Gamma}(N_{\Delta t}, \eta_2).$$

Be careful using the MATLAB command `random('gam',`$K, 1/\eta_i$`)` because the convention of the parameters (shape/scale versus shape/rate).

Therefore the sum of jumps in the time interval $[t, t + \Delta t]$ is given by

$$J(\Delta t) = J^+ - J^-.$$

At the end, we have the same representation of the stock price as before

$$S_{t+\Delta t} = S_t \exp\left\{\gamma^* \Delta t + \sigma\sqrt{\Delta t} Z + J(\Delta t)\right\},$$

with

$$\gamma^* = r - q - \frac{1}{2}\sigma^2 - \lambda\left(\frac{p \cdot \eta_1}{\eta_1 + 1} + \frac{(1-p)\cdot \eta_2}{\eta_2 + 1} - 1\right).$$

The simulation of sample paths under Kou model are illustrated in figure 4.3

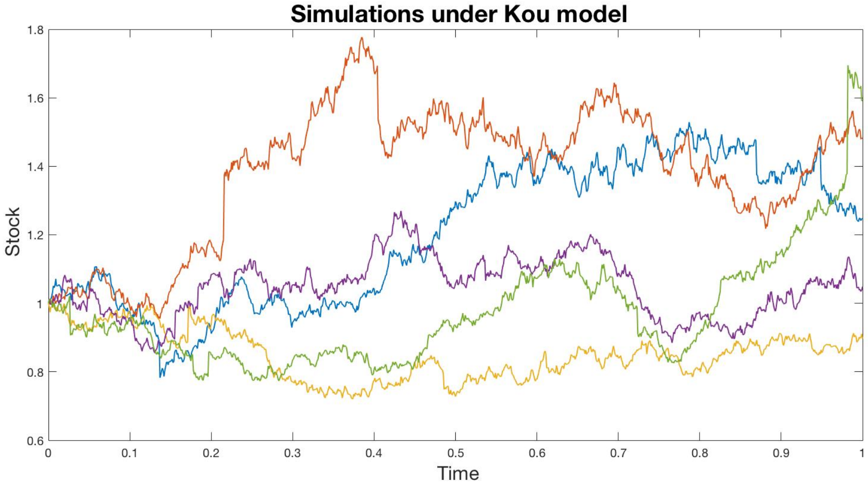


Fig. 4.3: Simulations of stock price process under Kou model. $S_0 = 1, r = 0.01, q = 0.02, \lambda = 10, p = 0.55, \eta_1 = \eta_2 = 25, \sigma = 0.3,$ $T = 1, dt = 0.001, M = 5.$

4.1.3 Simulations under Pure jump models

Recall that a pure jump process can be seen as a Brownian subordination

$$X_t^{\mathrm{PJ}} = \theta T_t + \sigma W_{T_t},$$

where $T = \{T_t, t \geq 0\}$ is a random time process, called the *subordinator*. The goal is then to simulate this subordinator and substitute it to the time into the Brownian motion with drift. In the Normal Inverse Gaussian model, this time subordinator will be a Inverse Gaussian process, and in the Variance Gamma model, it will be a Gamma process.

Normal Inverse Gaussian model

First of all, recall that the Lévy process in the Normal Inverse Gaussian model is given by

$$X_t^{\mathrm{PJ}} = \beta\delta^2 T_t + \delta W_{T_t},$$

with $T_t \sim \mathrm{IG}\left(t, \delta\sqrt{\alpha^2 - \beta^2}\right)$. Hence we have to construct a Normal Inverse Gaussian (NIG) process. To do that, we simulate an Inverse Gaussian (IG) process and set it as time parameter of the Brownian motion. In fact, we have that

$$\Delta X_t^{\mathrm{PJ}} = \beta\delta^2 \Delta T_t + \delta\sqrt{\Delta T_t} Z,$$

where $\Delta T_t \sim \mathrm{IG}\left(\Delta t, \delta\sqrt{\alpha^2 - \beta^2}\right)$ and $Z \sim \mathcal{N}(0,1)$.

Finally, we have the stock price sample path

$$S_{t+\Delta t} = S_t \exp\left\{\gamma^* \Delta t + \Delta X_t^{\mathrm{PJ}}\right\},$$

with $\gamma^* = r - q - \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}\right)$. Be careful with the convention of the MATLAB command `random('inversegaussian',`μ, λ`)`, where $\mu = \frac{a}{b}$ is the mean and $\lambda = a^2$ is the shape parameter. We can see the result of five sample path in figure 4.4.

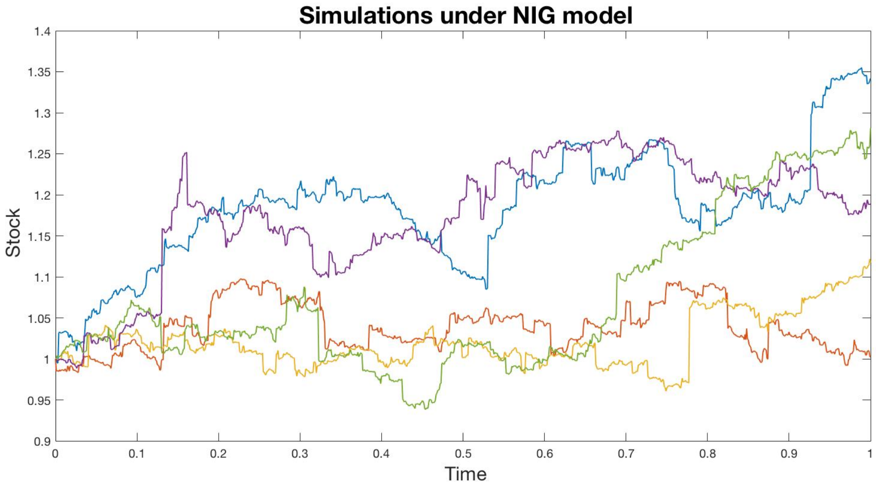


Fig. 4.4: Simulations of stock price process under NIG model. $S_0 = 1, r = 0.01, q = 0.02, \alpha = 50, \beta = 3, \delta = 1,$ $T = 1, dt = 0.001, M = 5.$

Variance Gamma model

Following the same procedure as before, we just have to change the time subordinator process by taking a Gamma process. Then we have the time-changed Brownian motion

$$X_t^{\mathrm{PJ}} = \theta T_t + \sigma W_{T_t},$$

with $T_t \sim \mathrm{Gamma}\left(\frac{t}{\nu}, \frac{1}{\nu}\right)$. Therefore, we get

$$\Delta X_t^{\mathrm{PJ}} = \theta \Delta T_t + \sigma \sqrt{\Delta T_t}\, Z,$$

where $\Delta T_t \sim \mathrm{Gamma}\left(\frac{\Delta t}{\nu}, \frac{1}{\nu}\right)$ and $Z \sim \mathcal{N}(0,1)$. Thus we get

$$S_{t+\Delta t} = S_t \exp\left\{\gamma^* \Delta t + \Delta X_t^{\mathrm{PJ}}\right\},$$

with $\gamma^* = r - q + \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{\sigma^2 \nu}{2}\right)$. The figure 4.5 illustrates the result of five simulations under this last model.

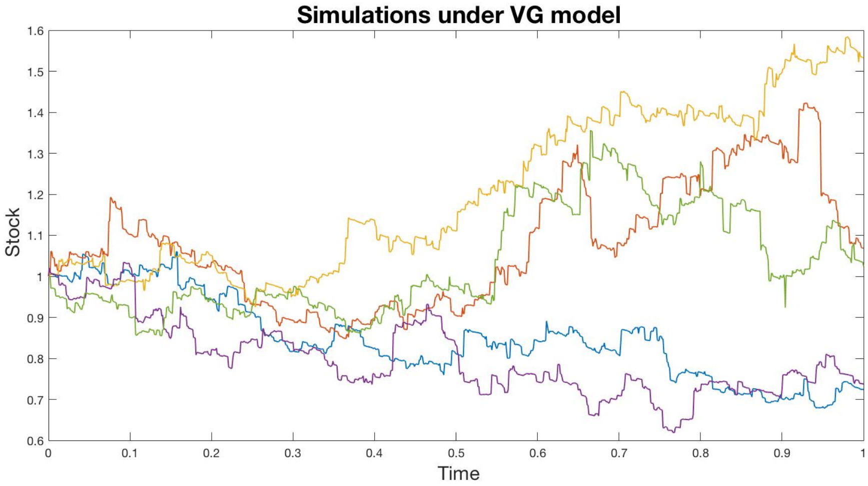


Fig. 4.5: Simulations of stock price process under VG model. $S_0 = 1, r = 0.01, q = 0.02, \theta = 0.5, \sigma = 0.3, \nu = 0.01,$ $T = 1, dt = 0.001, M = 5.$

4.1.4 Pricing FX TARN with Monte Carlo

Note that for the pricing of the FX TARN it suffices to take Δt equal to the difference of two consecutive fixing dates, i.e. the length of a period (e.g. Daily/Weekly/Monthly). This is not necessary to simulate the points between these dates because the cash flows depend only on the observations on the fixing dates.

From the simulations above, and equations (1.1) and (1.2), this is easy to compute the payoff on each fixing dates $t_n, (n = 1, \ldots, N)$, with respect to the realization S_{t_n}. This is also necessary to update the variable $A(t_n)$ that take into account the accumulated gain.

Finally, we have for each simulated scenario $\mathbf{S}_m, (m = 1, \ldots, M)$, the present value from equation (1.3)

$$P(\mathbf{S}_m) = \sum_{n=1}^{N} e^{-r_d t_n} \mathbf{C}_n^{\text{tot}}(S(t_n), A(t_{n-1})),$$

where we have taken $B_d(t_0, t_n)^{-1} = e^{-r_d(t_n - t_0)}$, with r_d is the domestic risk-free rate. Hence, the value of the FX TARN obtained with Monte-Carlo method is given by

$$\hat{P}(\mathbf{S}) = \frac{1}{M} \sum_{m=1}^{M} P(\mathbf{S}_m).$$

4.2 Finite Difference Method

This section is devoted to the Finite Difference (FD) method for different models. We will see how to solve numerically the PDE in Black-Scholes model and the PIDE in the jump-diffusion and pure jump models. The central tool of Lévy processes in FD method is the Lévy measure ν used characterized the jumps in the integral part of the PIDE. In the case of Black-Scholes, we have seen that this measure is null.

4.2.1 Black-Scholes world

This method to price Target Accrual Redemption Note (TARN) under (generalized) Black-Scholes model was proposed by Luo and Shevchenko (2015). To resume their approach, consider the value of the FX-TARN $V(S, t, A)$, where S is the spot rate and A is the accumulated amount until time t. Since the accumulation A has no effect in the diffusion between two fixing date, the option pricing PDE is still valid

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2 \frac{\partial^2 V}{\partial S^2} + (r_d(t) - r_f(t))\, S \frac{\partial V}{\partial S} - r_d(t)V = 0, \tag{4.1}$$

with r_d and r_f respectively the domestic and foreign risk free rates.

The idea is to consider the terminal condition to be

$$V(S, T, A) = 0.$$

In fact, the value of the product at the end of its life is zero because there is no more coming cash-flow. However, just before last fixing date, at $T^{-} = t_N$, the value is equal to the last payment

$$V(S, t_N^{-}, A(t_N^{-})) = C_N(S, A(t_{N-1})) + C_N^{*}(S, A(t_{N-1}) = \mathbf{C}_N^{\text{tot}}(S, A(t_{N-1})),$$

where C_N and C_N^{*} are respectively the gain and the loss at time t_N and $\mathbf{C}_N^{\text{tot}}$ is the total cash-flow. Note that $A(t_N^{-}) = A(t_{N-1})$.

We can backward this reflexion for each fixing date, which gives us for the n-th fixing date t_n,

$$\begin{aligned} V(S, t_n^{-}, A(t_n^{-})) &= V(S, t_n, A(t_n)) + \mathbf{C}_n^{\text{tot}}(S, A(t_{n-1})) \\ &= V(S, t_n, A(t_{n-1}) + C_n(S, A(t_{n-1}))) + \mathbf{C}_n^{\text{tot}}(S, A(t_{n-1})) \\ &= V(S, t_n, A^{+}(t_{n-1})) + \mathbf{C}_n^{\text{tot}}(S, A(t_{n-1})), \qquad (4.2) \end{aligned}$$

with $A^{+}(t_{n-1}) = A(t_{n-1}) + C_n(S, A(t_{n-1}))$ is the accumulated amount after a backward jump on a fixing date.

Finally, the today's FX-TARN price is given by $V(S(t_0), t_0, 0)$.

Finite difference scheme and Cubic spline interpolation

In order to get this final solution, the idea of Luo and Shevchenko is to introduce an auxiliary finite grid $0 = A_1 < A_2 < \cdots < A_J = U$ to track the accumulated amount, where U denotes the target. Then the goal is to perform J finite difference solutions between all fixing dates, for each node in the accumulated amount which correspond to different "scenarios".

Let us now introduce a finite difference grid points in the spot price variable S, $0 = S_0 < S_1 < \cdots < S_M = S_{\max}$. For a fixed time t_n, if we associate the node A_j to the accumulated amount $A(t_{n-1})$, the equation (4.2) can be written as

$$V(S_m, t_n^{-}, A_j) = V(S_m, t_n, A_j^{+}) + \mathbf{C}_n^{\text{tot}}(S_m, A_j),$$

with $A_j^{+} = A_j + C_n(S_m, A_j)$, for $m = 1, \ldots, M$. This quantity is used as initial condition at the beginning of each finite different scheme.

Since only the value $V(S_m, t_n, A_j)$ is known after a diffusion, we have to evaluate $V(S_m, t_n, A_j^{+})$ by cubic spline interpolation. Hence for a fixed j, we can extract J values $V(S_m, t_n, A_j^{+})$ from $V(S_m, t_n, A_j)$ by interpolating with respect to A_j.

To perform the finite difference it is common to write equation (4.1) in terms of log asset price $x = \log(S)$ and time to maturity $\tau = T - t$. This gives us the following PDE

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - \left(r_d - r_f - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial x} + r_d V = 0.$$

The advantage of this representation is that in a Black-Scholes world with constant parameters, all the coefficient are constant. Therefore the finite difference grid in x is given by $\log(S_{\min}) = x_0 < x_1 < \cdots < x_M = \log(S_{\max})$.

Let denote the option price at time $\tau_n = T - t_n$ and grid point x_i as V_i^n, $n = 0, 1, \ldots, N$. Hence, for an uniform grid, $\Delta x_i = x_i - X_{i-1} = \Delta x$, we get the finite difference approximation with second order accuracy

$$\begin{aligned}\frac{\partial V}{\partial x}(x_i, \tau_n) &= \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2),\\ \frac{\partial^2 V}{\partial x^2}(x_i, \tau_n) &= \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta x^2} + \mathcal{O}(\Delta x^2).\end{aligned}$$

Then with the finite difference operator F_i^n defined as

$$F_i^n \equiv -\frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - \left(r_d - r_f - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial x} + r_d V,$$

the θ-scheme can be expressed as

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} + \theta F_i^{n+1} + (1-\theta) F_i^n = 0,$$

with $\theta \in [0, 1]$. The special cases $\theta = 0, \theta = 0.5$ and $\theta = 1$ correspond respectively to the fully explicit, Crank-Nicholson and fully implicit scheme.

Boundary Conditions

Here we have used Dirichlet boundary conditions between two consecutive fixing dates τ_n and τ_{n+1} and defining the length of the period as $\tau^* = \tau_n - \tau_{n+1}$, we get

$$\begin{aligned}V(x_{\min}, \tau, A_j) &= \Psi(x_{\min}, \tau, A_j),\\ V(x_{\max}, \tau, A_j) &= \Psi(x_{\max}, \tau, A_j),\end{aligned}$$

where

$$\Psi(x, \tau, A_j) = \exp\{-r_d(\tau^* - \tau)\}\left(A_j + \mathbf{C}^{\text{tot}}(\exp\{r_d(\tau^* - \tau) + x\}, A_j)\right)$$

for $t \in [\tau_n, \tau_{n+1}]$. In fact, the quantity $(\tau^* - t)$ is the left time in the bucket time until the next fixing date. We just have to discount the value of the option between fixing date because the interpolation initialize a new value for the next step.

4.2.2 Jump-diffusion models

To go beyond the Black-Scholes world, we can generalize this method to the jump-diffusion models. As proposed by Cont and Voltchkova (2005), we can solve numerical the Partial Integro Differential equation (PIDE) evolving under Lévy processes. The PIDE is given by

$$\begin{aligned}\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{x^2} - \left(r_d - r_f - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial x} + r_d V \\ - \int_{-\infty}^{\infty} \left[V(x+y,\tau) - V(x,\tau) - (e^y - 1)\frac{\partial V}{\partial x}(x,\tau)\right]\nu(dy) = 0,\end{aligned}$$

where $\nu(\cdot)$ is the Lévy measure of the process $X = \{X_t, t \geq 0\}$ that characterized the log asset price. Recall that in the Merton and Kou models, the Lévy density are respectively given by

$$\begin{aligned}\nu^{\text{Mer}}(dy) &= \frac{\lambda}{\sqrt{2\pi}\delta} e^{-\frac{(y-\alpha)^2}{2\delta^2}} dy, \\ \nu^{\text{Kou}}(dy) &= \lambda\left(p \cdot \eta_1 e^{-\eta_1 y}\mathbf{1}_{y \geq 0} + (1-p) \cdot \eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}\right) dy.\end{aligned}$$

Therefore, some integral terms can be analytically computed as

$$\begin{aligned}\int_{-\infty}^{\infty} \nu^*(dy) &= \lambda, \\ \int_{-\infty}^{\infty} (e^y - 1)\nu^{\text{Mer}}(dy) &= \lambda\left(e^{\alpha + \frac{1}{2}\delta^2} - 1\right) \equiv c, \\ \int_{-\infty}^{\infty} (e^y - 1)\nu^{\text{Kou}}(dy) &= \lambda\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right) \equiv c.\end{aligned}$$

Hence, we can rewrite the PIDE in the form

$$\begin{aligned}\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{x^2} - \left(r_d - r_f - \frac{1}{2}\sigma^2 - c\right)\frac{\partial V}{\partial x} + (r_d + \lambda)V \\ - \int_{-\infty}^{\infty} V(x+y,\tau)\nu(dy) = 0,\end{aligned}$$

Note that in the log asset price form, the last integral term $\int_{-\infty}^{\infty} V(x+y)\nu(dy)$ has a convolution structure that allows us to compute it by Fast Fourier Transform (FFT). This is not the case in the asset price PIDE.

Localization to a bounded domain

To solve this numerical problem, we first truncate the space domain to a bounded interval $x \in (x_{\min}, x_{\max})$ as usually. The boundary conditions are the same as the Black-Scholes case. Hence the truncated Cauchy problem reads

$$\begin{cases} \dfrac{\partial V}{\partial \tau} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V}{x^2} - \left(r_d - r_f - \dfrac{1}{2}\sigma^2 - c \right) \dfrac{\partial V}{\partial x} + (r_d + \lambda) V \\ \qquad\qquad\qquad - \displaystyle\int_{-\infty}^{\infty} V(x+y,\tau)\nu(dy) = 0, & x \in (x_{\min}, x_{\max}), \\ V(x,\tau) = \Psi(x,\tau,A_j), & x \notin (x_{\min}, x_{\max}), \end{cases}$$

where

$$\Psi(x,\tau,A_j) = \exp\{-r_d(\tau^* - \tau)\} \left(A_j + \mathbf{C}^{\text{tot}}(\exp\{r_d(\tau^* - t) + x\}, A_j) \right).$$

Observe that the integral term $\int_{-\infty}^{\infty} V(x+y,\tau)\nu(dy)$ still involves the values of $V(x,t)$ outside of the truncated domain. Hence the boundary conditions have to be imposed for all $x \notin (x_{\min}, x_{\max})$ and not only at the two boundary points $x_{\min}$ and $x_{\max}$.

Truncation of the integral

For computational purpose, we need to truncate the integral term,

$$J = \int_{-\infty}^{\infty} V(x+y,\tau)\nu(dy),$$

as

$$J_{B_l,B_r} = \int_{-B_l}^{B_r} V(x+y,\tau)\nu(dy).$$

In practice, it is more convenient to use a variable truncation on

$$[-(x - x_{\min} - B_l, (x_{\max} - x + B_r] \supset [-B_l, B_r].$$

Therefore, we get

$$J \approx \int_{-(x-x_{\min})-B_l}^{(x_{\max}-x)+B_r} V(x+y,\tau) - V(x,\tau)\nu(dy).$$

If the Lévy measure ν has density $\nu(dy) = k(y)dy$, which is the case in Merton or Kou model, we have

$$J \approx \int_{x_{\min}-B_l}^{x_{\max}+B_r} V(z,\tau) - V(x,\tau)k(z-x)(dz).$$

Hence, the fully truncated problem can be written as

$$\begin{cases} \dfrac{\partial V}{\partial \tau} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V}{x^2} - \left(r_d - r_f - \dfrac{1}{2}\sigma^2 - c\right)\dfrac{\partial V}{\partial x} + (r_d + \lambda)V \\ \qquad\qquad - \displaystyle\int_{x_{\min}-B_l}^{x_{\max}+B_r} V(z,\tau)k(z-x)dz = 0, \qquad x \in (x_{\min}, x_{\max}), \\ V(x,\tau) = \Psi(x,\tau,A_j), \qquad\qquad x \in [x_{\min} - B_l, x_{\min}] \cup [x_{\max}, x_{\max} + B_r]. \end{cases}$$

Finite difference approximation

The major change from the simple case of a PDE is to extend the spatial grid to be able to compute the integral part outer of the original grid. In other words, consider the following original and extended spatial grid

$$\begin{aligned} x_i &= x_{\min} + i\Delta x, \qquad \text{for } i \in I = \{1, M-1\}, \\ x_i &= x_{\min} + i\Delta x, \qquad \text{for } i \in \tilde{I} = \{-K_l, \dots, M + K_r\}. \end{aligned}$$

One way to compute the integral term is to use the trapezoidal rule using the extended computational grid $j \in \tilde{I}$. This means that for a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an internal grid point x_i, denoting $k_i = k(x_i)$, we have

$$\int_{x_{\min}-B_l}^{x_{\max}+B_r} f(z)k(z-x_i) \approx \sum_{l=-K_l}^{M+K_r} \Delta x f(x_l) k(\underbrace{x_l - x_i}_{x_{l-i}}) = \sum_{-K_l}^{M+K_r} \Delta x f(x_l) k_{l-i}.$$

Hence our integral can be approximated by

$$\int_{x_{\min}-B_l}^{x_{\max}+B_r} V(z,\tau_n)k(z-x)dz \approx \Delta x \sum_{l=-K_l}^{M+K_r} V_l^n k_{l-i}.$$

This development brings us to the *Forward Euler* finite difference approximation

$$\begin{cases} \dfrac{V_i^{n+1} - V_i^n}{\Delta t} - \dfrac{1}{2}\sigma^2 \dfrac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta x^2} - \left(r_d - r_f - \dfrac{1}{2}\sigma^2 - c\right)\dfrac{V_{i+1}^n - V_{i-1}^n}{2\Delta x} \\ \qquad\qquad + (r_d + \lambda)V_i^n - \Delta x \displaystyle\sum_{l=-K_l}^{M+K_r} V_l^n k_{l-i} = 0, \qquad i \in I, \\ V_i^n = \Psi(x_i, \tau_n, A_j), \qquad\qquad i \in \tilde{I} \backslash I. \end{cases}$$

With the following vectorial notation

$\mathbf{V}^n = \{V_i^n, i \in I\}$: vector of unknowns on internal nodes,

$\tilde{\mathbf{V}}^n = \{V_i^n, i \in \tilde{I}\}$: vector of nodal values on the extended grid, we can write the Forward Euler finite difference in the algebraic form as

$$\begin{cases} \dfrac{\mathbf{V}^{n+1} - \mathbf{V}^{n}}{\Delta t} + A\mathbf{V}^{n} - J\tilde{\mathbf{V}}^{n} = \mathbf{F}^{n}, & n = 0, \dots, N-1, \\ \tilde{\mathbf{V}}_{I}^{n+1} = \mathbf{V}^{n+1}, \qquad \tilde{\mathbf{V}}_{\tilde{I}\setminus I}^{n+1} = \mathbf{\Psi}_{\tilde{I}\setminus I}^{n+1}, & n = 0, \dots, N-1, \end{cases}$$

with the matrix $A \in \mathbb{R}^{(M-1)\times(M-1)}$ and right hand side vector $\mathbf{F}^{n} \in \mathbb{R}^{M-1}$

$$A = \begin{bmatrix} \alpha & \gamma & & & \\ \delta & \alpha & \gamma & & \\ & \delta & \ddots & \ddots & \\ & & \ddots & & \end{bmatrix} \qquad \text{with} \begin{cases} \alpha = \frac{\sigma^2}{\Delta x^2} + r_d + \lambda, \\ \gamma = -\frac{\sigma^2}{2\Delta x^2} - \frac{1}{2\Delta x}\left(r_d - r_f - \frac{\sigma^2}{2} - c\right), \\ \delta = -\frac{\sigma^2}{2\Delta x^2} + \frac{1}{2\Delta x}\left(r_d - r_f - \frac{\sigma^2}{2} - c\right), \end{cases}$$

and

$$\mathbf{F}^{n} = \begin{bmatrix} -\delta\Psi(x_0, \tau_n) \\ 0 \\ \vdots \\ 0 \\ -\gamma\Psi(X_M, \tau_n) \end{bmatrix}.$$

The matrix $J = \{J_{il}\} = \{k_{l-i}\}$ is a rectangular, Toepliz matrix, $J \in \mathbb{R}^{(M-1)\times(\tilde{M}+1}$, with $\tilde{M} = M + K_l + K_r$ the number of intervals in the extended grid.

$$J = \Delta x \begin{bmatrix} k_{-K_l-1} & k_{-K_l} & \cdots & \cdots & \cdots & \cdots & k_{M+K_r-1} \\ k_{-K_l-2} & k_{-K_l-1} & k_{-K_l} & \cdots & \cdots & \cdots & k_{M+K_r-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ k_{-K_l-M+1} & \cdots & \cdots & k_{-K_l-1} & k_{-K_l} & \cdots & k_{K_r+1} \end{bmatrix}.$$

Finally it is common to use an Explicit-Implicit scheme to reduce instability. The idea is to treat implicitly the convection-diffusion term and explicitly the integral term.

$$\begin{cases} \dfrac{\mathbf{V}^{n+1} - \mathbf{V}^{n}}{\Delta t} + A\mathbf{V}^{n+1} - J\tilde{\mathbf{V}}^{n} = \mathbf{F}^{n}, & n = 0, \dots, N-1, \\ \tilde{\mathbf{V}}_{I}^{n+1} = \mathbf{V}^{n+1}, \qquad \tilde{\mathbf{V}}_{\tilde{I}\setminus I}^{n+1} = \mathbf{\Psi}_{\tilde{I}\setminus I}^{n+1}, & n = 0, \dots, N-1, \end{cases}$$

At each iteration of this explicit-implicit scheme, we have to compute the integral term $\mathbf{F}_J^n = J\tilde{\mathbf{V}}^n$, in $\mathcal{O}(M \times \tilde{M})$ operations, and then solve the tridiagonal linear system

$$(I + \Delta t A)\mathbf{U}^{n+1} = \mathbf{U}^{n} + \Delta t\left(\mathbf{F}_J^m + \mathbf{F}^{n+1}\right),$$

in $\mathcal{O}(M)$ operations. However we can reduce the number of operation to $\mathcal{O}(M\log(M))$ by computing the integral term by FFT.

Computation of the integral term by FFT

The idea is to enlarge the matrix J to make it circulant. Let us first define a circulant matrix. Consider an arbitrary vector $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$. Then a circulant matrix $J_c(\alpha)$ is

$$J_c(\alpha) = \begin{bmatrix} \alpha_1 & \cdots & \alpha_{N-1} & \alpha_N \\ \alpha_N & \alpha_1 & \cdots & \alpha_{N-1} \\ \vdots & \ddots & \ddots & \vdots \\ \alpha_2 & \cdots & \alpha_N & \alpha_1 \end{bmatrix}.$$

Then the matrix vector product $w = J_c(\alpha)v$ can be computed in $\mathcal{O}(N \log(N))$ operation using the following algorithm

$$\begin{aligned} \hat{\alpha} &= \text{fft}(\alpha), \\ \hat{v} &= \text{fft}(v), \\ \hat{w} &= \text{conj}(\hat{\alpha}). * \hat{v}, \\ w &= \text{ifft}(\hat{w}). \end{aligned}$$

In our case, let $M_T = 2M + K_l + K_r - 1$ and define the vector

$$\mathbf{k} = (k_{-K_l-M+1}, \dots, k_{-1}, k_0, k_1, \dots, k_{M+K_r-1}) \in \mathbb{R}^{M_T},$$

and the circulant matrix $J_c(\mathbf{k}) \in \mathbb{R}^{M_T \times M_T}$.

Moreover, given $\tilde{\mathbf{V}}^n \in \tilde{\mathbb{M}}$, extend it by zero as

$$\tilde{\mathbf{V}}^n_{\text{ext}} = (\underbrace{0, \dots, 0}_{M-2}, \tilde{\mathbf{U}}^n) \in \mathbb{R}^{M_T}.$$

At the end, we have that

$$J\tilde{\mathbf{V}}^n = \left(J_c(\mathbf{k})\tilde{\mathbf{V}}^n_{\text{ext}}\right)_{1:M-1},$$

and we have computed the integral term in $\mathcal{O}(M \log(M))$.

Fixed point iterations for implicit scheme

In order to avoid all stability constrains, we have to solve the numerical problem with an implicit scheme. However, the resolution of the linear system of equations, which can be written as

$$B\mathbf{V}^{n+1} = \tilde{\mathbf{F}}^{n+1},$$

is very expensive ($\mathcal{O}(M^3)$) by $\mathcal{LU}$ factorization.

Then a possible remedy is to use fixed-point iterations. In other words, at time step τ_{n+1}, denote $\mathbf{V}_{(k)}^{n+1}$ the k-th fixed point iteration and compute $\mathbf{V}_{k+1}^{n+1}$ as

$$(I + \Delta t A)\mathbf{V}_{(k+1)}^{n+1} = -\Delta t J \tilde{\mathbf{V}}_{(}^{n+1} k) + \mathbf{V}^n + \Delta t \mathbf{F}^{n+1}.$$

The fixed point iterations can be started from $\mathbf{V}_{(0)}^{n+1} = \mathbf{V}^n$ and stopped as soon as $\parallel \mathbf{V}_{(k+1)}^{n+1} - \mathbf{V}_{(k)}^{n+1} \parallel tol$ for a prescribed tolerance.

Therefore, at each iteration, we have to solve a tridiagonal system, which costs $\mathcal{O}(M)$, and compute $J\tilde{\mathbf{V}}_{(k)}^{n+1}$, which costs $\mathcal{O}(M_T \log(M_T))$ if we use FFT.

4.2.3 Pure jump models

In the case of pure jump models, the Lévy measure as the property that $\nu(\mathbb{R}) = \infty$, which means that the process has infinite activity, i.e. there are infinitely many small jumps. In other words, the Lévy density $\nu(dy) = k(y)dy$ explodes at the origin, as we can see on the Figure 4.6. Unfortunately, it does not allow us to apply the preceding procedure by splitting the integral term in three parts

$$\int_{-\infty}^{\infty} \left[V(x+y,\tau) - V(x,\tau) - (e^y - 1)\frac{\partial V}{\partial x}(x,\tau) \right] \nu(dy) = \\ \int_{-\infty}^{\infty} V(x+y,\tau)k(y)dy - \int_{-\infty}^{\infty} V(x,\tau)k(y)dy - \int_{-\infty}^{\infty} (e^y - 1)\frac{\partial V}{\partial x}(x,\tau)k(y)dy,$$

as each of them is unbounded.

The idea is thus to come down to a non-singular case by approximating the process $X = \{X_t, t \geq 0\}$, by an appropriate finite activity process $X^\epsilon = \{X_t^\epsilon, t \geq 0\}$.

Cont and Voltchkova propose a possible strategy consisting in cutting out the small jumps, i.e. the singular part of $k(y)$ and approximating them with a Wiener process. Let us now define the truncated density

$$k_\epsilon(y) = \begin{cases} k(y), & \text{for } |y| \geq \epsilon \\ \frac{k(\epsilon)-k(-\epsilon)}{2} y + \frac{k(\epsilon)+k(-\epsilon}{2}, & |y| < \epsilon, \end{cases}$$

so that $k_\epsilon(y)$ corresponds to a finite activity Lévy process with intensity $\lambda(\epsilon) = \int k_\epsilon(y)dy < \infty$. This truncated density is illustrated on the Figure 4.6.

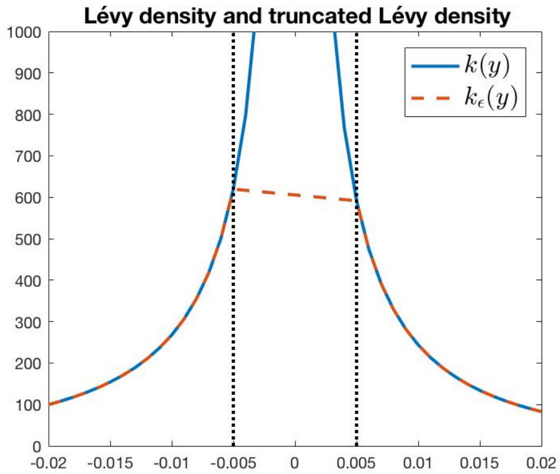


Fig. 4.6: Pure jump Lévy density (in blue) and its truncated density (in red) with $\epsilon = 0.005$.

We set now $\tilde{k}(y) = k(y) - k_\epsilon(y)$, which is non zero only for $|y| \leq \epsilon$, and split the integral terms as

$$\int_{-\infty}^{\infty} \left[V(x+y,\tau) - V(x,\tau) - (e^y - 1)\frac{\partial V}{\partial x}(x,\tau) \right] k(y)dy = I_1 + I_2,$$

with

$$I_1 = \int_{-\infty}^{\infty} \left[V(x+y,\tau) - V(x,\tau) - (e^y - 1)\frac{\partial V}{\partial x}(x,\tau) \right] k_\epsilon(y)dy$$
$$I_2 = \int_{-\epsilon}^{\epsilon} \left[V(x+y,\tau) - V(x,\tau) - (e^y - 1)\frac{\partial V}{\partial x}(x,\tau) \right] \tilde{k}(y)dy.$$

The first term corresponds to the finite activity Lévy process X^ϵ and can be treated as the previous case.

The second integral requires a bit more intention. By summing and subtracting the term $y\frac{\partial V}{\partial x}$, we have

$$I_2 = \int_{-\epsilon}^{\epsilon} \left[\underbrace{(x+y,\tau) - V(x,\tau) - y\frac{\partial V}{\partial x}(x,\tau)}_{\frac{y^2}{2}\frac{\partial^2 y}{\partial x^2} + \mathcal{O}(y^3)} - \underbrace{(e^y - 1 - y)}_{\frac{y^2}{2} + \mathcal{O}(y^3)} \frac{\partial V}{\partial x}(x,\tau) \right] \tilde{k}(y)dy$$
$$= \left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \right) \frac{\sigma^2(\epsilon)}{2} + \int_{-\epsilon}^{\epsilon} \mathcal{O}(y^3)\tilde{k}(y)dy,$$

with

$$\sigma^2(\epsilon) = \int_{-\epsilon}^{\epsilon} y^2 \tilde{k}(y)dy.$$

Therefore, we can treat the infinite activity case with a finite activity process X^ϵ characterized by the Lévy triplet $(\gamma^*, \sigma(\epsilon), k_\epsilon(y))$. Hence, up to $\mathcal{O}(y^3)$ terms, the option price $V(x,t)$ satisfies the modified PIDE

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2(\epsilon)\frac{\partial^2 V}{x^2} - \left(r_d - r_f - \frac{1}{2}\sigma^2(\epsilon) - c(\epsilon) \right) \frac{\partial V}{\partial x} + (r_d + \lambda(\epsilon))V$$
$$- \int_{-\infty}^{\infty} V(x+y,\tau)\nu(dy) = 0,$$

with

$$\lambda(\epsilon) \equiv \int_{-\infty}^{\infty} k_\epsilon(y)dy,$$
$$c(\epsilon) \equiv \int_{-\infty}^{\infty} \left(y^2 - 1 \right) k_\epsilon(y)dy,$$
$$\sigma^2(\epsilon) \equiv \int_{-\infty}^{\infty} y^2 k_\epsilon(y)dy.$$

For more details about the consistency, stability and convergence of finite difference scheme for exponential Lévy models, please see in the main reference of this section, [CV05].

4.3 The Convolution Method

4.4 Summary

Calibration

5

> *If you want to know the value of a security, use the price of another security that is as similar to it as possible. All the rest is modelling.*
>
> — **Emanuel Derman**
> (1946)

In order to be able to price a FX-TARN, we have to calibrate our models to the market option prices. The aim of the calibration is to estimate the unknown parameters of the model which reproduce the same prices as in the market. This is analogous than the implied volatility in the Black-Scholes framework. For this purpose, we will use European vanilla option prices calculated from the implied volatility given by BLOOMBERG.

Firstly, in section 5.1, we will expose the calibration inputs and how we can compute the European vanilla option price. Secondly, in section 5.2, we will explain the procedure that we used to calibration our models. Finally, in section 5.3, we will discuss about the results of the calibrations of our models.

5.1 Calibration inputs

All the data we have used in this chapter come from the provider BLOOMBERG. We have recovered the implicit volatility quotes with respect to the delta Call or delta Put and tenor for the exchange rate USD/CHF on the 23 May 2017. The delta notations are listed in the Table 5.1 with the corresponding theoretical delta.

Notation	Delta	Notation	Delta
5D Put	-0.05	5D Call	0.05
10D Put	-0.10	10D Call	0.10
15D Put	-0.15	15D Call	0.15
25D Put	-0.25	25D Call	0.25
35D Put	-0.35	35D Call	0.35

Tab. 5.1: Notation convention of delta Call and delta Put.

For the calibration, we have only used the volatility quotes for maturities 1W, 2W, 3W, 1M, 2M, 3M, 4M, 6M, 9M and 1Y. At the end, that represents 110 implied volatilities including ATM ($\Delta = \pm 0.50$) quote.

The Figure 5.1 shows us the CHF and USD risk-free rate curves used in our calibration.

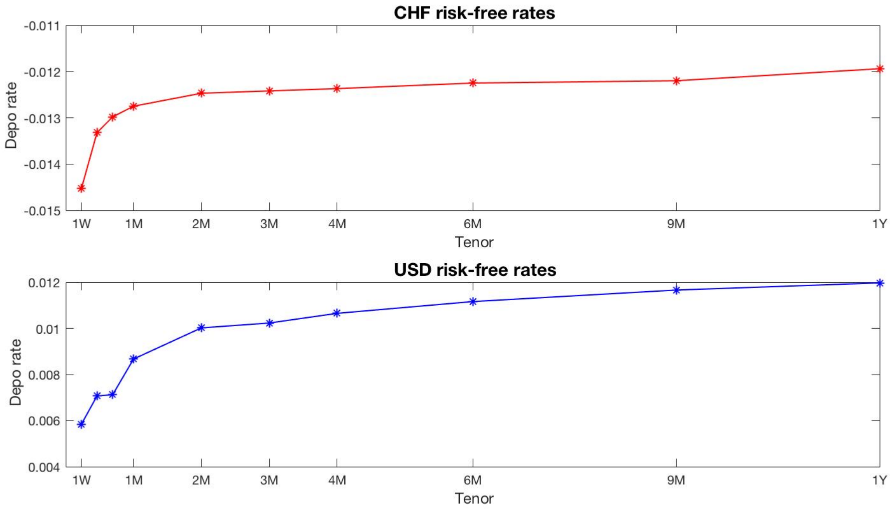


Fig. 5.1: Risk-free rates from BLOOMBERG used for the calibration.

Since the implied volatilities are quoted with respect to the delta, to compute the European vanilla option price, we have to recover the corresponding strike. In the Black-Scholes framework, we have the following relationship between the delta Δ and the strike K:

$$K = S_0 e^{(r_d - r_f)T} \exp\left(-\beta\sigma\sqrt{T}\Phi^{-1}\left(e^{-r_f T}|\Delta|\right) + \frac{1}{2}\sigma^2 T\right),$$

where $\beta = 1$ for a Call option and $\beta = -1$ for a Put option and Φ is the standard Gaussian cumulative probability function. The initial spot price $S_0 = 0.9730$ is naturally given by the term sheet of the FX-TARN.

From here, we can use the Black-Scholes formula to compute the European vanilla option prices:

$$V(K,T) = \beta\left(S_0 e^{-r_f T}\Phi(\beta d_+) - K e^{-r_d T}\Phi(\beta d_-)\right),$$

with

$$d_\pm = \frac{\log\left(\frac{S_0 e^{(r_d - r_f)T}}{K}\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

To avoid having prices closed to zero, we have chosen to compute only in-the-money (ITM) option prices. Finally, we get the European option prices illustrated in Figure 5.2

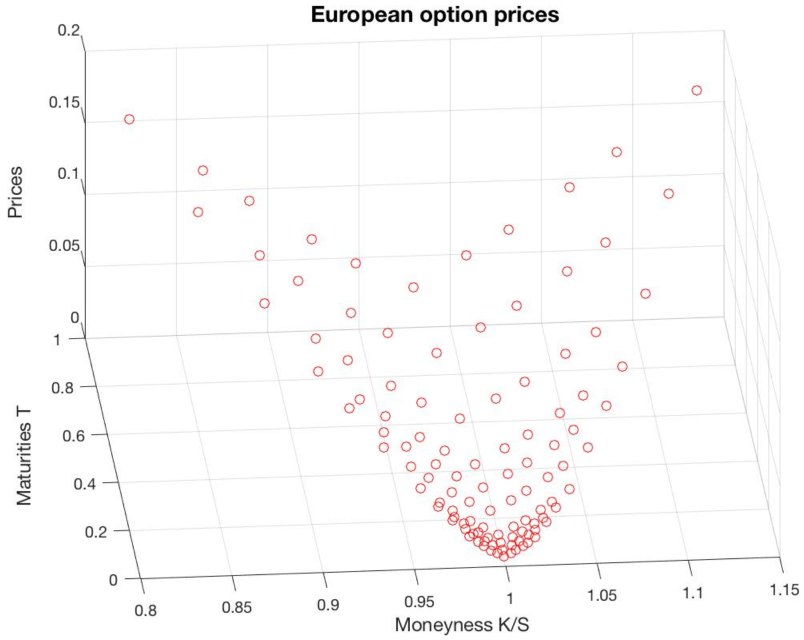


Fig. 5.2: European option prices; on the left hand side, the ITM Call options, and on the right hand side, the ITM Put options.

5.2 Calibration method

To calibrate our models, we search to minimize the root-mean-square error (RMSE) of the differences between the model prices and the observed prices:

$$\mathrm{RMSE}(\Theta) = \sqrt{\sum_{i=1}^{N} \frac{1}{N} \left|V(\Theta; K_i, T_i) - V^{\mathrm{obs}}(K_i, T_i)\right|^2},$$

where Θ is the parameter vector of the model.

For example, the RMSE of the Black-Scholes model with 1Y ATM volatility, $\sigma = 0.07908$, is equal to $8.3 \cdot 10^{-4}$, which is quite good but we will see that the models with jumps fit better with market prices. We can see in the Figure 5.3 the prices computed with Black-Scholes model. Observe that the model prices don't fit perfectly with market prices deep-in-the-money.

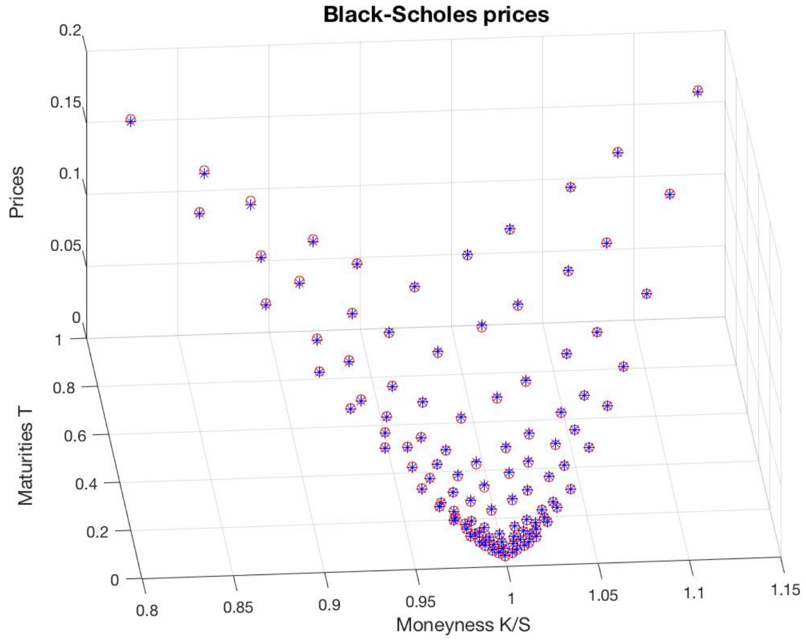


Fig. 5.3: Black-Scholes prices (in blue) in comparison with the market prices (in red).

The calibrated parameters $\hat{\Theta}$ of the model with jumps satisfy

$$\hat{\Theta} = \arg\min_{\Theta} \text{RMSE}(\Theta).$$

We can use the MATLAB command `fmincon` which allows us to find the minimum of constrained nonlinear multi-variable function. Note that the method used to price the European options under jump models is the convolution method with $N_x = 1000$ and damping factor $\alpha = -\beta$, where $\beta = 1$ for a Call option and $\beta = -1$ for a Put option. This method is in fact the fastest method to apply the minimizing function.

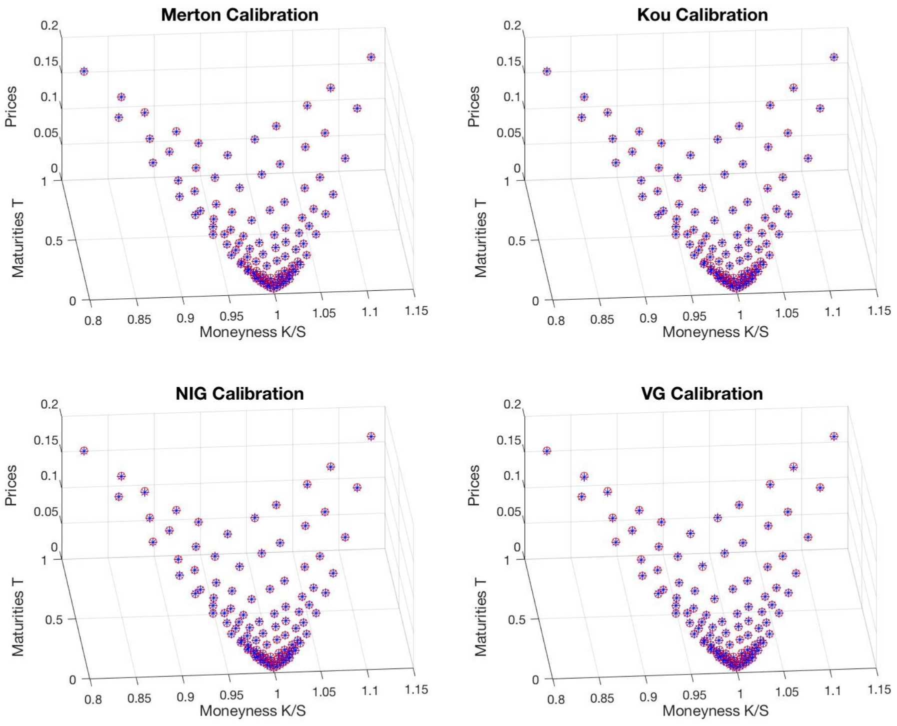
5.3 Results of calibrations

The resulting calibrated parameters and RMSE for each jumps models are summarized in the the Table 5.2 in comparison with the Black-Scholes model. We can see that all models perform better that the Black-Scholes model. The models that fit the best the market prices are the jump-diffusion models because they have more parameter and give more freedom in the calibration.

Models	Calibrated parameters $\hat{\Theta}$	RMSE
Black-Scholes(σ)	$[0.07908]$	$8.3 \cdot 10^{-4}$
Merton($\sigma, \lambda, \alpha, \delta$)	$[0.0653, 0.1201, -0.0608, 0.1545]$	$2.6 \cdot 10^{-4}$
Kou($\sigma, \lambda, p, \eta_1, \eta_2$)	$[0.0656, 0.1444, 0.0751, 4.9643, 9.6208]$	$2.4 \cdot 10^{-4}$
NIG(α, β, δ)	$[22.3418, -4.7619, 0.1402]$	$4.5 \cdot 10^{-4}$
VG(θ, σ, ν)	$[-0.0383, 0.0790, 0.1827]$	$5.3 \cdot 10^{-4}$

Tab. 5.2: Calibrated parameters $\hat{\Theta}$ and RMSE for each models.

We can see in Figure 5.4 that all the model prices are more closed to the market prices than the results obtained with Black-Scholes. This means that the FX-TARN price computed with jump model will be more relevant with respect to the market than the FX-TARN price under Black-Scholes.



Merton Calibration

Kou Calibration

NIG Calibration

VG Calibration

Fig. 5.4: Jump models calibration; model prices in blue and market prices in red.

6

Numerical Results

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7

Conclusion

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Lausanne, June 5, 2017

Valentin Bandelier