Pricing TARN Using Numerical Methods

Valentin Bandelier

March 16, 2017

Version: 0.1

Swiss Federal Institute of Technology Lausanne - EPFL

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

School of Basic Sciences - SB
Institute of Mathematics - MATH

Master Thesis

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Valentin Bandelier

Supervisor Fabio Nobile
EPFL
Institute of Mathematics - MATH

Co-supervisor Julien Hugonnier
EPFL
Swiss Finance Institute - SFI

Assistant Francesco Statti

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Swiss Federal Institute of Technology Lausanne - EPFL
Institute of Mathematics - MATH
School of Basic Sciences - SB
Route Cantonale
1015 Lausanne

Abstract

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Abstract (different language)

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Acknowledgement

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1 Monte-Carlo Method

> *Citation*
>
> — **Author**
> (Author description)

Finite Difference Method

2

Citation

— **Author**
(Author description)

Section Introduction

2.1 Taylor Expansion

Recall that the Taylor expansion for a function $f \in C^{\infty}$ infinitely many differentiable is given by

$$
\begin{aligned}
f(x) &= f(a) + (x-a)\frac{\partial f}{\partial x}(x) + \frac{(x-a)^2}{2!}\frac{\partial^2 f}{\partial x^2}(x) + \cdots + \frac{(x-a)^n}{n!}\frac{\partial^n f}{\partial x^n}(x) + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(x).
\end{aligned}
$$

If f is only $(n+1)$ times continuously differentiable, i.e. $f \in C^{(n+1)}$, we can write

$$f(x) = \sum_{k=0}^{n+1} \frac{(x-a)^k}{k!}\frac{\partial^k f}{\partial x^k}(x) + O((x-a)^{n+1}),$$

where $O((x-a)^{n+1})$ represents the remainder in Landau notation.

2.1.1 Forward and Backward Difference Approximation of First Derivative

In order to approximate $\frac{\partial f}{\partial t}(x)$ assume that f is twice continuously differentiable, i.e. $f \in C^2$. By a first order Taylor expansion we can write

$$f(x+h) = f(x) + h\frac{\partial f}{\partial x}(x) + O(h), \tag{2.1}$$

$$f(x-h) = f(x) - h\frac{\partial f}{\partial x}(x) + O(h). \tag{2.2}$$

The equation (2.1) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x+h) - f(x)}{h} + O(h), \tag{2.3}$$

which is known as *forward difference* approximation of the first derivative.

On the other hand, the equation (2.2) gives us

$$\frac{\partial f}{\partial x}(x) = \frac{f(x) - f(x-h)}{h} + O(h), \tag{2.4}$$

which is known as *backward difference* approximation of the first derivative.

2.1.2 Central Difference Approximation of First Derivative

Now assume that $f \in C^3$. Then with a second order Taylor expansion, we have

$$f(x) = f(x+h) + h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2), \tag{2.5}$$

$$f(x) = f(x-h) - h\frac{\partial f}{\partial x}(x) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x) + O(h^2). \tag{2.6}$$

Subtracting equation (2.6) from (2.5) we get

$$f(x+h) - f(x-h) = 2h\frac{\partial f}{\partial x}(x) + O(h^2).$$

Therefore we obtain the *central difference* approximation of the first derivative

$$\frac{\partial f}{\partial x}(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2). \tag{2.7}$$

2.1.3 Central Difference Approximation of Second Derivative

Finally summing equations (2.5) and (2.6) we get

$$2f(x) = f(x+h) + f(x-h) + h^2\frac{\partial^2 f}{\partial x^2}(x) + O(h^2).$$

Then the *central difference* approximation of the second derivative is given by

$$\frac{\partial^2 f}{\partial x^2}(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2). \tag{2.8}$$

2.2 Option Pricing under the Generalized Black-Scholes model

Consider the Generalized Black-Scholes model, which includes the *local volatility* $\sigma(S,t)$ and term structures of *interest rate* $r(t)$ and *dividend rate* $q(t)$. The price of an asset S under such model follows the *stochastic differential equation* (SDE):

$$dS_t = (r(t) - q(t))S_t dt + \sigma(S_t, t) S_t dW_t.$$

Then we know that the value of an option $v(S,t)$ on that asset S satisfies the following *partial differential equation* (PDE):

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\sigma(S,t)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(t) - q(t))S\frac{\partial v}{\partial S} = r(t)v(S,t) & \\ v(S,T) = \Psi(S) & \text{Terminal Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, t) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, t) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

Now if we use the change of variable $\tau = (T - t)$ to express *time to maturity*, we obtain the following PDE:

$$\begin{cases} -\frac{\partial v}{\partial \tau} + \frac{\sigma(S,\tau)S^2}{2}\frac{\partial^2 v}{\partial S^2} + (r(\tau) - q(\tau))S\frac{\partial v}{\partial S} = r(\tau)v(S,\tau) & \\ v(S,0) = \Psi(S) & \text{Initial Condition (Payoff function)} \\ \frac{\partial^2 v}{\partial S^2}(S_{\max}, \tau) = \frac{\partial^2 v}{\partial S^2}(S_{\min}, \tau) = 0 & \text{Neumann Boundary Conditions} \end{cases}$$

To begin, we have to define the domain of the problem

$$D = \{S_{\min} \leq S \leq S_{\max}; 0 \leq \tau \leq T\}$$

and set it to a discrete grid

$$\bar{D} = \begin{Bmatrix} S_j = S_{\min} + (j-1)h; & h = \frac{S_{\max} - S_{\min}}{N}; & j = 1, \ldots, N+1 \\ t_k = 0 + (k-1)\Delta t; & \Delta t = \frac{T}{M}; & k = 1, \ldots, M+1 \end{Bmatrix},$$

where N is the number of subintervals in the S-direction and M is the number of subintervals in the τ-direction.

2.2.1 Forward Euler Approximation

The Forward Euler approximation constructs the *explicit* discretization of the Generalized Black-Scholes PDE. In other words, we approximate the theta term $\frac{\partial v}{\partial t}(S,t)$ using a *forward difference* approximation (2.3):

$$\frac{\partial v}{\partial t}(S,t) \approx \frac{v(S,t+\Delta t)-v(S,t)}{\Delta t}.$$

The *central difference* approximation of the first derivative (2.7) for the delta term $\frac{\partial v}{\partial S}(S,t)$ gives us

$$\frac{\partial v}{\partial S}(S,t) \approx \frac{v(S+h,t)-v(S-h,t)}{2h}$$

and the *central difference* approximation of the second derivative (2.8) for the gamma term $\frac{\partial^2 v}{\partial S^2}(S,t)$ gives

$$\frac{\partial^2 v}{\partial S^2}(S,t) \approx \frac{v(S+h,t)-2v(S,t)+v(S-h,t)}{h^2}.$$

2.2.2 Backward Euler Approximation

2.2.3 θ-Method and Crank-Nicolson Approximation

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Colophon

This thesis was typeset with $\LaTeX\,2_\varepsilon$. It uses the *Clean Thesis* style developed by Ricardo Langner. The design of the *Clean Thesis* style is inspired by user guide documents from Apple Inc.

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Declaration

You can put your declaration here, to declare that you have completed your work solely and only with the help of the references you mentioned.

Lausanne, March 16, 2017

Valentin Bandelier