Homework-2

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October 21, 2022

1. Point charge in the presence of a charged conducting sphere. A point charge q is at $(0,0,d)$. A conductor sphere with total charge Q and with radius a is placed at the origin. Q and q have the same sign, and $d > a$. Please answer the following questions:

   (a) Calculate the electric potential distribution outside the sphere using method of images.

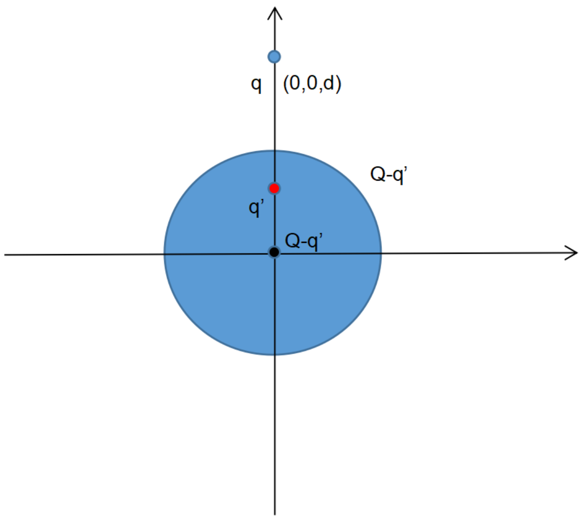

Figure 1: Image charge

The image charge q' is shown in Figure 1. Then, the outer surface will induce a charge distribution of $Q - q'$, which can be viewed as a point charge at the origin. Hence the electric potential distribution outside the sphere is:

$$q : (0,0,d), \qquad q' = -\frac{a}{d}q : (0,0,\frac{a^2}{d}), \qquad Q - q' = Q + \frac{a}{d}q : (0,0,0)$$

$$\phi(\boldsymbol{r}) = \phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{\frac{a}{d}q}{\sqrt{\frac{a^4}{d^2}+r^2-2\frac{a^2}{d}r\cos\theta}} + \frac{Q+\frac{a}{d}q}{r}\right] \tag{1}$$

   (b) Calculate the electric field distribution outside the sphere.

$$\begin{aligned}\boldsymbol{E} = -\nabla\phi \quad &= \quad -\frac{1}{4\pi\epsilon_0}\left[\frac{(d\cos\theta - r)q}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} - \frac{(\frac{a^2}{d}\cos\theta - r)\frac{a}{d}q}{(r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta)^{\frac{3}{2}}} - \frac{Q+\frac{a}{d}q}{r^2}\right]\hat{r} \\ &\quad -\frac{1}{4\pi\epsilon_0}\left[\frac{-qd\sin\theta}{(r^2+d^2-2rd\cos\theta)^{\frac{3}{2}}} + \frac{q\frac{a^3}{d^2}\sin\theta}{(r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta)^{\frac{3}{2}}}\right]\hat{\theta}\end{aligned} \tag{2}$$

   (c) Calculate the surface charge density distribution σ at the conducting sphere, and calculate the

Coulomb force per unit area exerted on the surface charge.

$$
\begin{aligned}
\sigma &= -\epsilon_0 \left.\frac{\partial \phi}{\partial r}\right|_{r=a} \\
&= -\frac{1}{4\pi}\left[\frac{(d\cos\theta - a)q}{(a^2+d^2-2ad\cos\theta)^{\frac{3}{2}}} - \frac{(\frac{a^2}{d}\cos\theta - a)\frac{a}{d}q}{(a^2+\frac{a^4}{d^2}-2a\frac{a^2}{d}\cos\theta)^{\frac{3}{2}}} - \frac{Q+\frac{a}{d}q}{a^2}\right] \\
&= -\frac{q}{4\pi a^2}\frac{a}{d}\left[\frac{1-\frac{a^2}{d^2}}{(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta)^{\frac{3}{2}}} - \frac{d}{a}\frac{Q}{q} - 1\right]
\end{aligned} \tag{3}
$$

The force per unit area is $dF = \dfrac{\sigma^2}{2\epsilon_0}dS$

$$
dF = \frac{q^2}{32\pi^2 a^2 d^2 \epsilon_0}\left[\frac{1-\frac{a^2}{d^2}}{(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta)^{\frac{3}{2}}} - \frac{d}{a}\frac{Q}{q} - 1\right]^2 \tag{4}
$$

(d) Calculate the surface Coulomb force F exerted on the point charge q from the charged sphere. Let charge $q = 1, Q = 6q, a = 1$, numerically calculate this force, and plot F as a function of d.

Using method of the image, the force on the point charge q is:

$$
F = \frac{1}{4\pi\epsilon_0}\left[\frac{q(Q+\frac{a}{d}q)}{d^2} - \frac{q(\frac{a}{d}q)}{(d-\frac{a^2}{d})^2}\right] \tag{5}
$$

With the numerical settings:

$$
F = \frac{1}{4\pi\epsilon_0}\left[\frac{6}{d^2} + \frac{1}{d^3} - \frac{d}{(d^2-1)^2}\right] \tag{6}
$$

Here, the sign " + " represents repulsive.

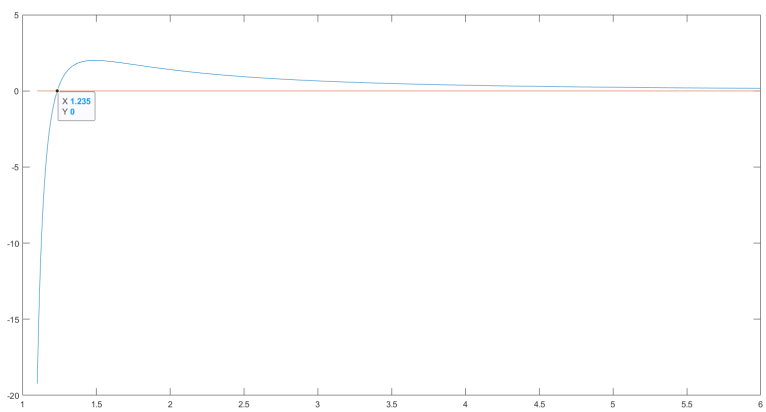

Figure 2: the *force* of d

(e) Consider the limit of $Q \gg q$, is the Coulomb force always repulsive? At which point $d = d_c$, the force becomes attractive? Derive an analytic expression for d_c(in the limit of $Q \gg q$) to the leading order of $\frac{q}{Q}$. Compare this analytic expression with the numerical value of d_c for $Q = 6q$ obtained in (d).

No, when d is small enough, the force will become attractive. Set $q/Q = x$, then the force is:

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\left[\frac{xQ^2 + \frac{a}{d}x^2Q^2}{d^2} - \frac{q^2\frac{a}{d}}{(d-\frac{a^2}{d})^2}\right] \\ &\approx \frac{1}{4\pi\epsilon_0}\left[\frac{xQ^2}{d^2} - \frac{q^2\frac{a}{d}}{(d-\frac{a^2}{d})^2}\right] \end{aligned} \tag{7}$$

Set $F = 0$, the equation is:

$$\begin{aligned} adx &= (d - \frac{a^2}{d})^2 \\ d_c^4 - axd_c^3 - 2a^2d_c^2 + a^4 &= 0 \end{aligned} \tag{8}$$

Let $a = 1, x = \frac{1}{6}$, we can numerically solve the equation:

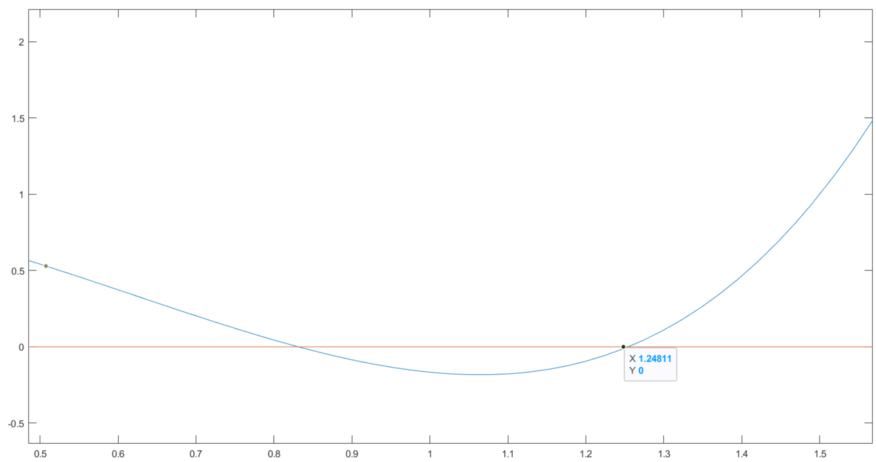

Figure 3: numerical solution

2. A point charge q is brought to a position a distribution d away from an infinite plane condutor held at zero potential. Using the method of images, find:

(a) the surface-charge density σ induced on the plane, and plot it:

The illustration is shown in Figure 4, and by the symmetry, we use $\phi(\rho, z)$ to describe the potential. Since the on the plane, the potential is zero, hence the potential distribution is:

$$\phi(\rho, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(d-z)^2+\rho^2}} - \frac{1}{\sqrt{(d+z)^2+\rho^2}}\right] \tag{9}$$

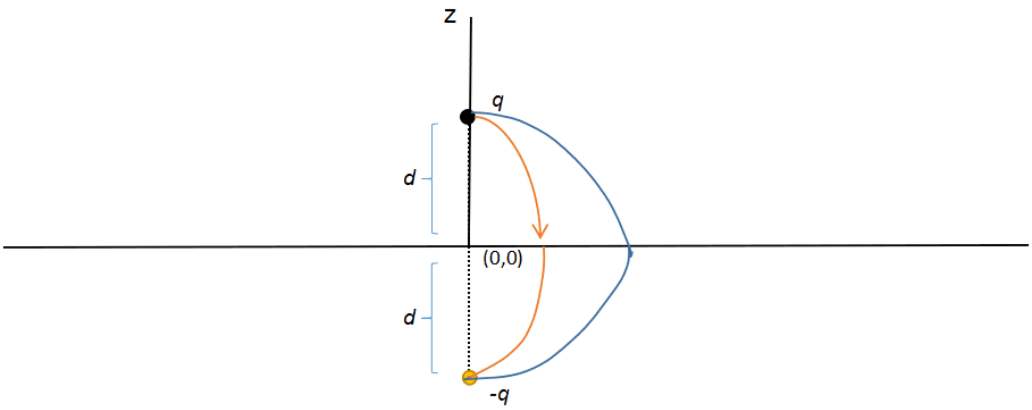

Figure 4: charge potential distribution

Then the surface charge density is:

$$\begin{aligned}\sigma = -\epsilon_0 \frac{\partial \phi}{\partial z}\bigg|_{z=0} &= -\frac{q}{4\pi}\left[\frac{d-z}{((d-z)^2+\rho^2)^{\frac{3}{2}}} - \frac{d+z}{((d+z)^2+\rho^2)^{\frac{3}{2}}}\right]\Bigg|_{z=0} \\ &= -\frac{qd}{2\pi(d^2+\rho^2)^{\frac{3}{2}}} \end{aligned} \tag{10}$$

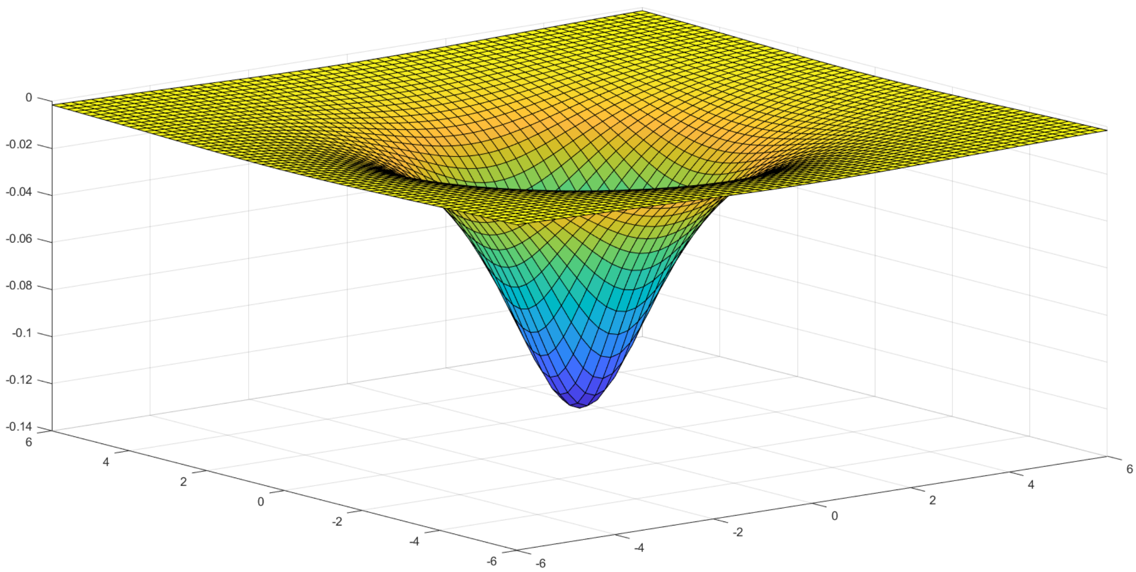

Figure 5: surface charge density distribution

(b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image:

$$|F| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2} = \frac{q^2}{16\pi\epsilon_0 d^2} \tag{11}$$

(c) the total force acting on the plane by integrating $\frac{\sigma^2}{2\epsilon_0}$ over the whole plane:

$$\begin{aligned}
|F| &= \int_0^{2\pi} d\theta \int_0^{\infty} \rho d\rho \frac{\sigma^2}{2\epsilon_0} \\
&= \frac{q^2}{4\pi\epsilon_0}\int_0^{\infty} d\rho \frac{d^2\rho}{(d^2+\rho^2)^3} \qquad (12)\\
&= \frac{q^2}{4\pi\epsilon_0 d^2}\int_0^{\infty} dx \frac{x}{(1+x^2)^3} \qquad (13)\\
&= \frac{q^2}{4\pi\epsilon_0 d^2}\frac{1}{2}(-\frac{1}{2})\frac{1}{(1+x^2)^2}\Big|_0^{\infty} \qquad (14)\\
&= \frac{q^2}{16\pi\epsilon_0 d^2} \qquad (15)
\end{aligned}$$

(d) the work necessary to remove the charge q from its position to infinity:

$$W = q\times(\phi(z=\infty,\rho=0)-\phi(z=d,\rho=0)) = \frac{q^2}{8\pi\epsilon_0 d} \qquad (16)$$

3. About the Legendre functions $\{P_l(x)\}$

(a) Derive Rodrigues'formula: $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

The general Legendre functions is:

$$P_l(x) = \sum_{k=0}^{\lceil\frac{l-1}{2}\rceil}(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k} \qquad (17)$$

Using the binomial therorem:

$$\begin{aligned}
\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l &= \frac{1}{2^l l!}\frac{d^l}{dx^l}\sum_{k=0}^{l} C_l^k(-1)^k x^{2l-2k} \\
&= \frac{1}{2^l l!}\sum_{k=0}^{\lceil\frac{l-1}{2}\rceil} C_l^k(-1)^k(2l-2k)(2l-2k-1)\cdots(l-2k+1)x^{l-2k} \\
&= \sum_{k=0}^{\lceil\frac{l-1}{2}\rceil}(-1)^k\frac{l!(2l-2k)!}{2^l l! k!(l-k)!(l-2k)!}x^{l-2k} \\
&= \sum_{k=0}^{\lceil\frac{l-1}{2}\rceil}(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k} = P_l(x) \qquad (18)
\end{aligned}$$

(b) Derive $\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0$

Proof:

$$\begin{aligned}
\frac{dP_{l+1}}{dx} &= \frac{d}{dx}\sum_{k=0}^{\lceil\frac{l}{2}\rceil}(-1)^k\frac{(2l+2-2k)!}{2^{l+1}k!(l+1-k)!(l+1-2k)!}x^{l+1-2k} \\
&= \sum_{k=0}^{\lceil\frac{l}{2}\rceil}(-1)^k\frac{(2l+2-2k)!}{2^{l+1}k!(l+1-k)!(l-2k)!}x^{l-2k} \qquad (19)
\end{aligned}$$

$$\begin{aligned}\frac{dP_{l-1}}{dx} &= \frac{d}{dx}\sum_{k=0}^{\lceil\frac{l-2}{2}\rceil}(-1)^k\frac{(2l-2-2k)!}{2^{l-1}k!(l-1-k)!(l-1-2k)!}x^{l-1-2k} \\ &= \sum_{k=0}^{\lceil\frac{l-2}{2}\rceil}(-1)^k\frac{(2l-2-2k)!}{2^{l-1}k!(l-1-k)!(l-2-2k)!}x^{l-2-2k} \end{aligned}\tag{20}$$

$$\begin{aligned}\frac{dP_{l+1}}{dx}-\frac{dP_{l-1}}{dx} &= \frac{(2l+2)!}{2^{l+1}(l+1)!l!}x^l+\sum_{k=1}^{\lceil\frac{l}{2}\rceil}\Big[(-1)^k\frac{(2l+2-2k)!}{2^{l+1}k!(l+1-k)!(l-2k)!} \\ &\quad -(-1)^{k-1}\frac{(2l-2k)!}{2^{l-1}(k-1)!(l-k)!(l-2k)!}\Big]x^{l-2k} \\ &= (2l+1)\frac{(2l)!}{2^l l!l!}x^l+\sum_{k=1}^{\lceil\frac{l}{2}\rceil}\Big[(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!} \\ &\quad (\frac{(2l+2-2k)(2l+1-2k)}{2(l+1-k)}+2k)\Big]x^{l-2k} \\ &= (2l+1)\sum_{k=0}^{\lceil\frac{l-1}{2}\rceil}(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k} \\ &= (2l+1)P_l \end{aligned}\tag{21}$$

Hence, $\dfrac{dP_{l+1}}{dx}-\dfrac{dP_{l-1}}{dx}-(2l+1)P_l=0$

(c) Using the identity $(l+1)P_{l+1}-(2l+1)xP_l+lP_{l-1}=0$ to calculate the following integral:

$$I_2=\int_{-1}^{1}dxx^2P_l(x)P_{l'}(x)$$

$$\begin{aligned}I_2 &= \int_{-1}^{1}dx(xP_l)(xP_{l'}) \\ &= \int_{-1}^{1}dx(\frac{(l+1)P_{l+1}+lP_{l-1}}{2l+1})(\frac{(l'+1)P_{l'+1}+l'P_{l'-1}}{2l'+1}) \\ &= \int_{-1}^{1}dx\frac{(l+1)(l'+1)}{(2l+1)(2l'+1)}P_{l+1}P_{l'+1}+\frac{l(l'+1)}{(2l+1)(2l'+1)}P_{l-1}P_{l'+1}+\dots \\ &= \frac{2(l+1)^2}{(2l+1)^2(2l+3)}\delta_{l,l'}+\frac{2l(l-1)}{(2l+1)(2l-1)(2l-3)}\delta_{l-1,l'+1}+\dots \\ &= \begin{cases}\dfrac{2(l+1)^2}{(2l+1)^2(2l+3)}+\dfrac{2l^2}{(2l+1)^2(2l-1)} & l=l' \\ \dfrac{2l(l-1)}{(2l+1)(2l-1)(2l-3)} & l=l'+2 \\ \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} & l=l'-2\end{cases}\end{aligned}\tag{22}$$

4. Numerically solve Poisson equation in two dimension:

$$\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial y^2}=-\frac{\rho(x,y)}{\epsilon_0}\tag{23}$$

Solve the potential inside the rectangle of length L and width W. The rectangle is divided into $N_x \times N_y$ discrete sites($L = 2W = 2m$):

$$r_{i,j} = (x_i, y_j) = (\frac{iL}{N_x}, \frac{jW}{N_y}) = (ih_x, jh_y) \qquad 0 \leqslant i \leqslant N_x, 0 \leqslant j \leqslant N_y \tag{24}$$

with boundary condition $\phi_0 = 1V$:

$$\phi(x=0,y) = 0, \phi(x=L,y) = \phi_0 \qquad \phi(x, y=0) = \frac{\phi_0 x}{L}, \phi(x, y=W) = \frac{\phi_0 x}{L} \tag{25}$$

Then the Poisson equation in matrix form:

$$\boldsymbol{A} \cdot \boldsymbol{\phi} = -\frac{\boldsymbol{\rho}}{\epsilon_0} + \boldsymbol{b} \tag{26}$$

(a) Charge density is zero inside the rectangle, $\rho(\boldsymbol{r}) = 0$

The second difference equation is:

$$\begin{cases} \left.\dfrac{\partial^2 \phi}{\partial x^2}\right|_{x=x_i, y=y_j} = \dfrac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h_x^2} \\ \left.\dfrac{\partial^2 \phi}{\partial y^2}\right|_{x=x_i, y=y_j} = \dfrac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{h_y^2} \end{cases} \tag{27}$$

Then we can translate the Eq (23) into the linear equations:

$$\begin{cases} \dfrac{1}{h_x^2}\phi_{i+1,j} + \dfrac{1}{h_y^2}\phi_{i,j+1} - 2(\dfrac{1}{h_x^2} + \dfrac{1}{h_y^2})\phi_{i,j} + \dfrac{1}{h_x^2}\phi_{i-1,j} + \dfrac{1}{h_y^2}\phi_{i,j-1} = b_{i,j} \\ \phi_{0,j} = 0, \phi_{N_x,j} = \phi_0, \phi_{i,0} = \dfrac{i}{N_x}\phi_0, \phi_{i,N_y} = \dfrac{i}{N_x}\phi_0 \end{cases} \tag{28}$$

Reset the vector:

$$\phi_j^h = \begin{vmatrix} \phi_{1,j} \\ \phi_{2,j} \\ \vdots \\ \phi_{N_x-1,j} \end{vmatrix} \quad b_j^h = \begin{vmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{N_x-1,j} \end{vmatrix} \quad \phi^h = \begin{vmatrix} \phi_1^h \\ \phi_2^h \\ \vdots \\ \phi_{N_y-1}^h \end{vmatrix} \quad b^h = \begin{vmatrix} b_1^h \\ b_2^h \\ \vdots \\ b_{N_y-1}^h \end{vmatrix}$$

$$C = \begin{bmatrix} -2(\frac{1}{h_x^2} + \frac{1}{h_y^2}) & \frac{1}{h_x^2} & \cdots & & \\ \frac{1}{h_x^2} & -2(\frac{1}{h_x^2} + \frac{1}{h_y^2}) & \frac{1}{h_x^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{h_x^2} & -2(\frac{1}{h_x^2} + \frac{1}{h_y^2}) & \frac{1}{h_x^2} \\ & & & \frac{1}{h_x^2} & -2(\frac{1}{h_x^2} + \frac{1}{h_y^2}) \end{bmatrix}_{(N_x-1)\times(N_x-1)} \tag{29}$$

$$A = \begin{bmatrix} C & \frac{1}{h_y^2}I & \cdots & & \\ \frac{1}{h_y^2}I & C & \frac{1}{h_y^2}I & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{h_y^2}I & C & \frac{1}{h_y^2}I \\ & & & \frac{1}{h_y^2}I & C \end{bmatrix} \tag{30}$$

Hence the equation can be written as:

$$A\phi^h = b^h \tag{31}$$

with b^h

$$-b_1^h = \begin{vmatrix} \frac{1}{h_y^2 N_x}\phi_0 \\ \frac{2}{h_y^2 N_x}\phi_0 \\ \vdots \\ (\frac{N_x-1}{h_y^2 N_x} + \frac{1}{h_x^2})\phi_0 \end{vmatrix} \quad -b_{N_y-1}^h = \begin{vmatrix} \frac{1}{h_y^2 N_x}\phi_0 \\ \frac{2}{h_y^2 N_x}\phi_0 \\ \vdots \\ (\frac{N_x-1}{h_y^2 N_x} + \frac{1}{h_x^2})\phi_0 \end{vmatrix} \quad \begin{cases} -b_{N_x-1,j} = \frac{1}{h_x^2}\phi_0 \quad (j \neq 1, N_y-1) \\ -b_{i,j} = 0 \quad (i \neq 1, N_x-1; j \neq 1, N_y-1) \end{cases}$$

The potential distribution is:

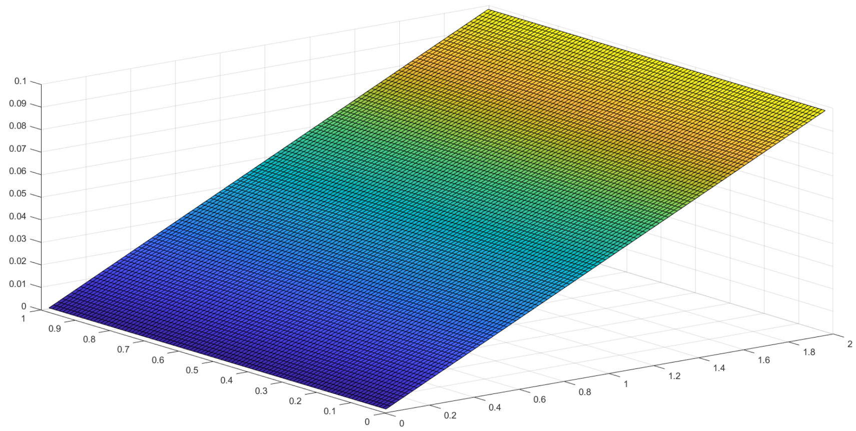

Figure 6: The electrostatic potentials for $\rho(\boldsymbol{r}) = 0$

(b) Charge density obeys a Gaussian distribution, $\rho(\boldsymbol{r}) = \rho_0 e^{-20\frac{|\boldsymbol{r}-\boldsymbol{r}_c|^2}{W^2}}, \boldsymbol{r}_c = (\frac{L}{2}, \frac{W}{2}), \rho_0 = 1 \quad C/m^2$

The equation becomes:

$$A\phi^h = b^h - \frac{\rho^h}{\epsilon_0} \tag{32}$$

After numerical calculation, the potential distribution is as the following:

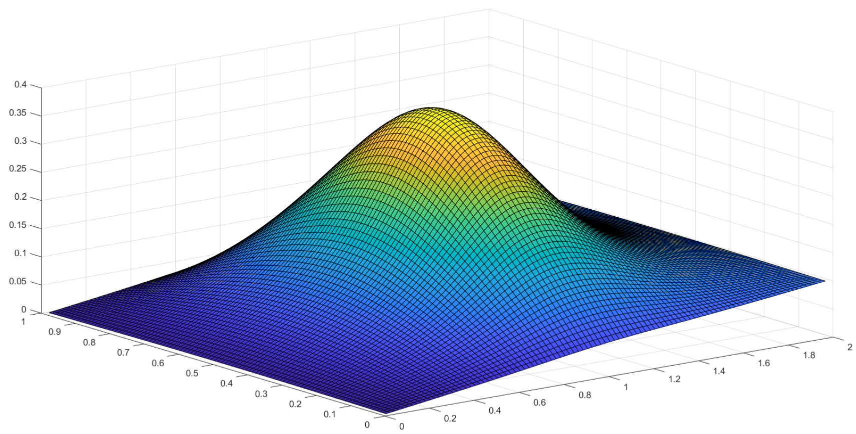

Figure 7: The electrostatic potential for $\rho(\boldsymbol{r}) = \rho_0 e^{-20|\boldsymbol{r}-\boldsymbol{r}_c|^2/W^2}$

Here is my code:

```
%----------------------------------initial parameter------------------%
L = 2;
W = 1;
Nx = input('Enter the Nx:');
Ny = input('Enter the Ny:');

h_x = L/Nx;
h_y = W/Ny;

%----------------------------------initial Matrix---------------------%
%initial C%
C = zeros(Nx-1);
for i = 1:Nx-1
    C(i,i)=-2*(1/(h_x)^2+1/(h_y)^2);
    for j = 1:Nx-1
        if abs(i-j)==1
            C(i,j)=1/(h_x)^2;
        end
    end
end

%initial A%
I = eye(Nx-1);

B = zeros((Nx-1)*(Ny-1));
row = (Nx-1)*ones(1,Ny-1);
B_1 = mat2cell(B,row,row);

for i = 1:Ny-1
    B_1{i,i} = C;
end

for i = 1:Ny-1
    for j = 1:Ny-1
        if abs(i-j)==1
            B_1{i,j}=1/(h_y)^2*I;
        end
    end
end

A = cell2mat(B_1);

%--------------------initial x,y,b------------------%
x = zeros(1,Nx-1);
y = zeros(1,Ny-1);
phi_0=0.1;

for i = 1:Nx-1
    x(i)=i*h_x;
end

for j = 1:Ny-1
    y(j)=j*h_y;
end

%initial b%
b = zeros((Nx-1)*(Ny-1),1);
for i = 1:Nx-1
    for j = 1:Ny-1
        if i == Nx-1
            b((j-1)*(Nx-1)+Nx-1) = b((j-1)*(Nx-1)+Nx-1)+1/(h_x)^2*phi_0;
        end
```

```
            if j == Ny-1
                b((j-1)*(Nx-1)+i) = b((j-1)*(Nx-1)+i)+i/(h_y^2*Nx)*phi_0;
            end
            if j == 1
                b((j-1)*(Nx-1)+i) = b((j-1)*(Nx-1)+i)+i/(h_y^2*Nx)*phi_0;
            end
        end
end
b=-b;
%------------------solve the equation--------------%
%------------------------\rho=0----------------------%
phi = A\b;

%-------------------------\rho------------------------%
%----------------set e_0 =1,rho_0=10-----------------%
rho = zeros((Nx-1)*(Ny-1),1);
for i = 1:Nx-1
    for j = 1:Ny-1
        rho((j-1)*(Nx-1)+i) = 10*exp(-20*((i*h_x-L/2)^2+(j*h_y-W/2)^2)/W^2);
    end
end

phi_1 = A\(b-rho);

%-------------------------plot------------------------%
map1 = reshape(phi',[Nx-1,Ny-1])';
map2 = reshape(phi_1',[Nx-1,Ny-1])';

[X, Y] = meshgrid(x, y);
surf(X,Y,map1);
surf(X,Y,map2);
```