上海科技大学
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Midterm Exam

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1. **Colulomb's law and Gauss theorem**

   (a) What is the electric field and electrostatic potential generated by a single point charge, and multiple point charges? How to define continuous charge density? What are the electric field and electrostatic potential generated by a continuous charge density distribution?

   (b) Suppose a charge q is in an electric field $\boldsymbol{E}(\boldsymbol{r})$, how much work is needed to move the charge from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$? What is the electrostatic potential energy for two point charges q_1 and q_2 separated by $\boldsymbol{r}$? Given a collection of point charges $\{q_i, i = 1, 2, 3, \ldots, N\}$, what is the electrostatic potential energy? Given continuous charge density distribution $\rho(\boldsymbol{r})$, what is the total potential energy? What is the energy density for the electric field $\boldsymbol{E}(\boldsymbol{r})$ generated by $\rho(\boldsymbol{r})$?

   (c) Try to prove that a collection of charged particles by themselves is unstable. The charge particles cannot stay in equilibrium without other external forces. (Hint: a particle stays in equilibrium position when its potential energy $U(x)$ satisfies $\frac{\partial U}{\partial x} = 0$ and $\frac{\partial^2 U}{\partial x^2} > 0$)

(a) The electric field and potential generated by a single point charge q located at $\boldsymbol{r}_0$ is

$$\boldsymbol{E}(\boldsymbol{r};\boldsymbol{r}_0) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}, \qquad \phi(\boldsymbol{r};\boldsymbol{r}_0) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{1}$$

The electric field and potential generated by multiple point charges q_i located at $\boldsymbol{r}_i$ $(i = 1, \ldots, N)$ is

$$\boldsymbol{E}(\boldsymbol{r};\boldsymbol{r}_1,\ldots) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} q_i \frac{\boldsymbol{r}-\boldsymbol{r}_i}{|\boldsymbol{r}-\boldsymbol{r}_i|^3}, \qquad \phi(\boldsymbol{r};\boldsymbol{r}_1,\ldots) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|} \tag{2}$$

If the charges are so small and exist everywhere in space, they can descibed by a charge density $\rho(\boldsymbol{r})$. If there is a Δq in a small volume ΔV, then $\Delta q \approx \rho(\boldsymbol{r})\Delta V$. Hence we can define the continuous charge density:

$$\rho(\boldsymbol{r}) = \lim_{\Delta V \to 0}\frac{\Delta q(\boldsymbol{r})}{\Delta V} \tag{3}$$

Converting the sum over $\boldsymbol{r}$ into an integral over $\boldsymbol{r}$ through $\sum_{r_i}(\cdots) \to \int d\boldsymbol{r}'(\cdots)$, we have the continuous form:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int d\boldsymbol{r}'\rho(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \qquad \phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int d\boldsymbol{r}'\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{4}$$

(b) The force acting on the charge q at any point is

$$\boldsymbol{F} = q\boldsymbol{E} \tag{5}$$

Hence the work done in moving the charge from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ is

$$W = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2}\boldsymbol{F}\cdot d\boldsymbol{l} = -q\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2}\boldsymbol{E}\cdot d\boldsymbol{l} = q\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2}\nabla\phi\cdot d\boldsymbol{l} = q\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2}d\phi = q(\phi(\boldsymbol{r}_2)-\phi(\boldsymbol{r}_1)) \tag{6}$$

The potential energy can be viewed that q_2 is brought from infinity to a point $\boldsymbol{r}_2$ where $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ in a region of electric field produced by q_1

$$W = \frac{q_2}{4\pi\epsilon_0}\frac{q_1}{|\boldsymbol{r}_2-\boldsymbol{r}_1|} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\boldsymbol{r}|} \tag{7}$$

For a collection of point charges, the potential energy of the charge q_i is

$$W_i = \frac{q_i}{4\pi\epsilon_0}\sum_{j\neq i}^{N}\frac{q_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|} \tag{8}$$

The total potential energy of all the charges is

$$W = \frac{1}{2}\times\frac{1}{4\pi\epsilon_0}\sum_{i,j,i\neq j}^{N}\frac{q_iq_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|} \tag{9}$$

If we set the $i=j$ term as zero, then Eq (9) can be written as

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i,j=1}^{N}\frac{q_iq_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|} \tag{10}$$

Then the total continuous potential energy is

$$W = \frac{1}{8\pi\epsilon_0}\iint d\boldsymbol{r}d\boldsymbol{r}'\frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{11}$$

For a electric field $\boldsymbol{E}(\boldsymbol{r})$ generated by $\rho(\boldsymbol{r})$, we make use of Possion equation $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$ to eliminate the charge density:

$$W = \frac{1}{2}\int d\boldsymbol{r}\rho(\boldsymbol{r})\phi(\boldsymbol{r}) = -\frac{\epsilon_0}{2}\int d\boldsymbol{r}\phi\nabla^2\phi = \frac{\epsilon_0}{2}\int d\boldsymbol{r}|\nabla\phi|^2 = \frac{\epsilon_0}{2}\int d\boldsymbol{r}|\boldsymbol{E}(\boldsymbol{r}|^2 \tag{12}$$

Hence the energy density is

$$w(\boldsymbol{r}) = \frac{\epsilon_0}{2}|\boldsymbol{E}(\boldsymbol{r})|^2 \tag{13}$$

(c) We denote m_i as the mass of the point charge, and consider the Lagrangian of the system:

$$\mathcal{L} = \sum_i\frac{1}{2}m_i\dot{\boldsymbol{r}}_i^2 - \frac{1}{8\pi\epsilon_0}\sum_{i,j}\frac{q_iq_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|} \tag{14}$$

The action is given by

$$S = \int dt\mathcal{L}(\boldsymbol{r},\dot{\boldsymbol{r}}) \tag{15}$$

If the system is stable, it reads $\delta S \equiv 0$ with any perturbation $\delta\boldsymbol{r}_k$

$$\begin{aligned}\delta S &= \int dt\left(m_k\dot{\boldsymbol{r}}_k\delta\dot{\boldsymbol{r}}_k + \frac{1}{4\pi\epsilon_0}\sum_{j\neq k}\frac{q_kq_j}{|\boldsymbol{r}_k-\boldsymbol{r}_j|^2}\delta\boldsymbol{r}_k\right)\\ &= \int dt\left(-m_k\ddot{\boldsymbol{r}}_k + \frac{1}{4\pi\epsilon_0}\sum_{j\neq k}\frac{q_kq_j}{|\boldsymbol{r}_k-\boldsymbol{r}_j|^2}\right)\delta\boldsymbol{r}_k\end{aligned} \tag{16}$$

Hence we have

$$-m_k\ddot{\boldsymbol{r}}_k + \frac{1}{4\pi\epsilon_0}\sum_{j\neq k}\frac{q_kq_j}{|\boldsymbol{r}_k-\boldsymbol{r}_j|^2} = 0 \tag{17}$$

The particle is in equilibrium, then $\ddot{\boldsymbol{r}}_k = 0$, for any point charge q_k:

$$\sum_{j\neq k} \frac{q_j}{|\boldsymbol{r}_k - \boldsymbol{r}_j|^2} = 0 \tag{18}$$

If S is in maximum, then $\Delta S = \delta S + \frac{1}{2}\delta^2 S < 0$:

$$\begin{aligned} \delta^2 S &= \int dt \left(-\frac{1}{2\pi\epsilon_0} \sum_{j\neq k} \frac{q_k q_j}{|\boldsymbol{r}_k - \boldsymbol{r}_j|^3} \right) (\delta \boldsymbol{r}_k)^2 < 0 \\ &\Longrightarrow \sum_{j\neq k} \frac{q_k q_j}{|\boldsymbol{r}_k - \boldsymbol{r}_j|^3} > 0 \end{aligned} \tag{19}$$

On the other hand, the energy of the system must be in minimum:

$$H = T + V = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \frac{1}{8\pi\epsilon_0} \sum_{i,j} \frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}, \qquad \frac{\partial H}{\partial \lambda} = 0, \qquad \frac{\partial^2 H}{\partial \lambda^2} > 0 \tag{20}$$

Set the parameter λ as $\boldsymbol{r}_k$, then we have:

$$\sum_{j\neq k} \frac{q_k q_j}{|\boldsymbol{r}_k - \boldsymbol{r}_j|^3} > 0 \tag{21}$$

Differentiate the Eq (18) by $\boldsymbol{r}_k$, it satisfies:

$$\sum_{j\neq k} \frac{-q_j}{|\boldsymbol{r}_k - \boldsymbol{r}_j|^3} = 0 \tag{22}$$

which is a contradiction to Eq (21).

2. **Method of images and Green's functions for typpe-I boundary conditions**

   (a) Consider an infinite parallel conductor plate, with a point charge q at $z = b$ above the plate. The electrostatic potential on the conductor plate is zero. What is the potential above the conductor plate?

   (b) Write down the Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ for such a geometry.

   (c) The potential on the conductor plane $z = 0$ is specified as:

$$\begin{cases} \phi(\rho, \varphi, z = 0) = V_0 & 0 \leqslant \rho \leqslant a \\ \phi(\rho, \varphi, z = 0) = 0 & \rho > a \end{cases} \tag{23}$$

   Please find an integral expression in cylindrical coordinates for the potential at a point $\boldsymbol{r}$ above the conductor plate.

   (a) The image charge is under the plate at $z = -b$ with total charge $-q$

   Hence the potential distribution can be viewed as the production of the two point charge:

$$\phi(\rho, \varphi, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{(z-b)^2 + \rho^2}} - \frac{1}{\sqrt{(z+b)^2 + \rho^2}} \right] \tag{24}$$

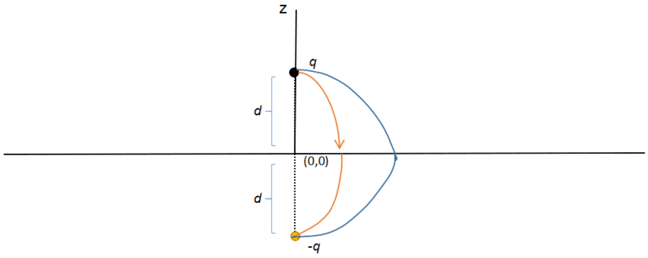

Figure 1: Electrostatic potential distribution

(b) Set $\boldsymbol{r}_+ = (\rho', \varphi', z'), \boldsymbol{r}_- = (\rho', \varphi', -z')$, then the Green's function $G(\boldsymbol{r}, \boldsymbol{r}_+)$ is written as

$$\begin{aligned} G(\boldsymbol{r}, \boldsymbol{r}_+) &= \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_+|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_-|} \\ &= \frac{1}{\sqrt{(\rho\cos\varphi - \rho'\cos\varphi')^2 + (\rho\sin\varphi - \rho'\sin\varphi')^2 + (z-z')^2}} \\ &\quad - \frac{1}{\sqrt{(\rho\cos\varphi - \rho'\cos\varphi')^2 + (\rho\sin\varphi - \rho'\sin\varphi')^2 + (z+z')^2}} \end{aligned} \tag{25}$$

(c) The Green's function satisfies with the *Dirichlet boundary conditions*, hence the formal solution is (here we use Ω to denote charge density ρ in cylindrical coordinates, and $\boldsymbol{r}'$ is along z-axis):

$$\begin{aligned} \phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\int d\boldsymbol{r}_+ \Omega(\boldsymbol{r}_+) G(\boldsymbol{r}, \boldsymbol{r}_+) + \frac{1}{4\pi}\oint_S \phi(\boldsymbol{r}_+)\frac{\partial G(\boldsymbol{r}, \boldsymbol{r}_+)}{\partial z'} da' \\ &= \frac{1}{4\pi\epsilon_0}\int_0^\infty dz' \int_0^{2\pi} d\varphi' \int_0^\infty \rho' d\rho' \Omega(\boldsymbol{r}_+) G(\boldsymbol{r}, \boldsymbol{r}_+) + \frac{1}{4\pi}\oint_{\rho'\leqslant a} V_0 \left.\frac{\partial G}{\partial z'}\right|_{z'=0} da' \\ &= \frac{1}{4\pi\epsilon_0}\int_0^\infty dz' \int_0^{2\pi} d\varphi' \int_0^\infty \rho' d\rho' \Omega(\boldsymbol{r}_+)\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_+|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_-|}\right) \\ &\quad + \frac{V_0}{2\pi}\int_0^{2\pi} d\varphi' \int_0^a \rho' d\rho' \left(\frac{z}{[(\rho\cos\varphi - \rho'\cos\varphi')^2 + (\rho\sin\varphi - \rho'\sin\varphi')^2 + z^2]^{\frac{3}{2}}}\right) \end{aligned} \tag{26}$$

3. **Laplace equation in Cartesian coordinates**

Consider a hollow cuboid, $0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, 0 \leqslant z \leqslant c$, with electrostatic potential fixed on the walls of the cube:

$$\begin{cases} \phi(0,y,z) = \phi(a,y,z) = \phi(x,0,z) = \phi(x,b,z) = 0 \\ \phi(x,y,0) = \phi(x,y,c) = V_0 \end{cases}$$

(a) Write down Laplace equation in Cartesian coordinates, then decompose the equation into three partial differential equations with respect to three different variables x, y, z respectively.

(b) Find the electrostatic potential inside the cube.

(c) What is the electric field inside the cube? What is the surface charge density on the surface $z = c$?

(a) The Laplace equation in Cartesian coordinates is formed as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0 \tag{27}$$

Introduce the separation of variables $\phi = X(x)Y(y)Z(z)$,

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}X+\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y+\frac{1}{Z}\frac{\partial^2}{\partial z^2}Z=0 \tag{28}$$

Then decompose the equation as

$$\begin{cases}\dfrac{1}{X}\dfrac{\partial^2}{\partial x^2}X=-\alpha^2\\[2ex] \dfrac{1}{Y}\dfrac{\partial^2}{\partial y^2}Y=-\beta^2\\[2ex] \dfrac{1}{Z}\dfrac{\partial^2}{\partial z^2}Z=\gamma^2\end{cases} \tag{29}$$

where $\alpha^2+\beta^2=\gamma^2, \alpha, \beta, \gamma > 0$

(b) Integrate the Eq (29), we obtain the general solution:

$$X(x)=a_1e^{i\alpha x}+a_2e^{-i\alpha x},\qquad Y(y)=b_1e^{i\beta y}+b_2e^{-i\beta y},\qquad Z(z)=c_1e^{\gamma z}+c_2e^{-\gamma z} \tag{30}$$

Apply the boundary conditions:

$$\begin{aligned}X(x)&=\sin(\alpha_n x)\\ Y(y)&=\sin(\beta_m y)\\ Z(z)&=c_1\sinh(\sqrt{\alpha_n^2+\beta_m^2}z)+c_2\cosh(\sqrt{\alpha_n^2+\beta_m^2}z)\end{aligned} \tag{31}$$

where

$$\begin{cases}\alpha_n=\dfrac{n\pi}{a}\\[2ex] \beta_m=\dfrac{m\pi}{b}\\[2ex] \gamma_{nm}=\pi\sqrt{\dfrac{n^2}{a^2}+\dfrac{m^2}{b^2}}\end{cases} \tag{32}$$

The general solution to ϕ:

$$\begin{aligned}\phi(x,y,z)=&\sum_{n,m}A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}z)\\ &+\sum_{n,m}B_{nm}\sin(\alpha_n x)\sin(\beta_m y)\cosh(\gamma_{nm}z)\end{aligned} \tag{33}$$

Make use of $\phi(x,y,0)=\phi(x,y,c)=V_0$ and orthogonality of the basis function:

$$\begin{aligned}
B_{nm} &= \frac{4V_0}{ab}\int_0^a dx\int_0^b dy\,\sin(\alpha_n x)\sin(\beta_m y)\\
&= \begin{cases}\dfrac{16V_0}{nm\pi^2} & n,m \text{ are both odd}\\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{34}$$

$$\begin{aligned}
A_{nm} &= \frac{4V_0(1-\cosh(\gamma_{nm}c))}{ab\sinh(\gamma_{nm}c)}\int_0^a dx\int_0^b dy\,\sin(\alpha_n x)\sin(\beta_m y)\\
&= \begin{cases}\dfrac{16V_0(1-\cosh(\gamma_{nm}c))}{nm\pi^2\sinh(\gamma_{nm}c)} & n,m \text{ are both odd}\\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{35}$$

Hence the potential inside the cube is:

$$\phi(x,y,z)=\frac{16V_0}{\pi^2}\sum_n^{odd}\sum_m^{odd}\frac{1}{nm}\sin(\alpha_n x)\sin(\beta_m y)\left(\frac{1-\cosh(\gamma_{nm}c)}{\sinh(\gamma_{nm}c)}\sinh(\gamma_{nm}z)+\cosh(\gamma_{nm}z)\right) \tag{36}$$

(c) Applying $\boldsymbol{E}=-\nabla\phi$ and $\sigma=-\epsilon_0\left.\dfrac{\partial\phi}{\partial z}\right|_{z=c}$, we have

$$\begin{aligned}
\boldsymbol{E}(x,y,z) &= -\frac{16V_0}{\pi^2}\sum_n^{odd}\sum_m^{odd}\frac{\alpha_n}{nm}\cos(\alpha_n x)\sin(\beta_m y)\left(\frac{1-\cosh(\gamma_{nm}c)}{\sinh(\gamma_{nm}c)}\sinh(\gamma_{nm}z)+\cosh(\gamma_{nm}z)\right)\hat{x}\\
&\quad -\frac{16V_0}{\pi^2}\sum_n^{odd}\sum_m^{odd}\frac{\beta_m}{nm}\sin(\alpha_n x)\cos(\beta_m y)\left(\frac{1-\cosh(\gamma_{nm}c)}{\sinh(\gamma_{nm}c)}\sinh(\gamma_{nm}z)+\cosh(\gamma_{nm}z)\right)\hat{y}\\
&\quad -\frac{16V_0}{\pi^2}\sum_n^{odd}\sum_m^{odd}\frac{\gamma_{nm}}{nm}\sin(\alpha_n x)\sin(\beta_m y)\left(\frac{1-\cosh(\gamma_{nm}c)}{\sinh(\gamma_{nm}c)}\cosh(\gamma_{nm}z)+\sinh(\gamma_{nm}z)\right)\hat{z}
\end{aligned} \tag{37}$$

$$\sigma=-\frac{16\epsilon_0 V_0}{\pi^2}\sum_n^{odd}\sum_m^{odd}\frac{\gamma_{nm}}{nm}\sin(\alpha_n x)\sin(\beta_m y)\left(\frac{1-\cosh(\gamma_{nm}c)}{\sinh(\gamma_{nm}c)}\cosh(\gamma_{nm}c)+\sinh(\gamma_{nm}c)\right) \tag{38}$$

4. **Boundary-value problem in spherical coordinates**

(a) A conductor sphere with radius a is centered at the origin. The potential on the surface of the conductor is fixed as:

$$\phi(r=a,\theta,\varphi)=V_1\cos\theta$$

Suppose there is no charge outside the sphere, what is the electrostatic potential outside the sphere?

(b) There are two point charges $+q$ at $(0,0,b)$ and $-q$ at $(0,0,-b)$, both outside the sphere. The electrostatic potential on the surface of the sphere is the same as that in (a). Please find the expression for the electrostatic potential outside the sphere (Hint:take use of Green's function method)

(a) The potential satisfies with *Laplace equation*, and possesses azimuthal symmetry $m = 0$, the general solution is:

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta) \tag{39}$$

Since there is no charge outside the sphere, the electrostatic potential at the infty is finite. Hence, all A_l is zero. The coefficients B_l are found by evaluating the Eq (39) on the surface of the sphere:

$$V_1 \cos\theta = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta) \tag{40}$$

Using orthogonality of the Legendre series, the B_l are:

$$\begin{aligned} B_l &= \frac{(2l+1)a^{l+1}V_1}{2} \int_0^{\pi} \cos\theta P_l(\cos\theta)\sin\theta d\theta \\ &= \frac{(2l+1)a^{l+1}V_1}{2} \int_{-1}^{1} x P_l(x) dx \\ &= \frac{(2l+1)a^{l+1}V_1}{2} \int_{-1}^{1} P_1(x) P_l(x) dx \\ &= \frac{(2l+1)a^{l+1}V_1}{2} \frac{2}{3}\delta_{1,l} \\ &= \begin{cases} a^2 V_1 & l = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{41}$$

Hence, the potential outside the sphere is

$$\phi_1(r,\theta,\varphi) = V_1 \frac{a^2}{r^2}\cos\theta \qquad (r \geqslant a) \tag{42}$$

(b) According to the linear superposition principle, this problem can be decomposed as two parts:

$$\begin{cases} \nabla^2\phi = 0 \\ \phi|_{r=a} = V_1\cos\theta \end{cases}, \qquad \begin{cases} \nabla^2\phi = -\dfrac{q(\delta(\boldsymbol{r}-\boldsymbol{r}_+) - \delta(\boldsymbol{r}-\boldsymbol{r}_-))}{\epsilon_0} \\ \phi|_{r=a} = 0 \end{cases} \tag{43}$$

where $\boldsymbol{r}_+ = (0,0,b), \boldsymbol{r}_- = (0,0,-b)$.

The first one is already solved. For the second one, we apply the method of images:

$$\begin{aligned} \phi_2(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\Bigg[& \frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{\frac{a}{b}q}{\sqrt{r^2+\frac{a^4}{b^2}-2\frac{ra^2}{b}\cos\theta}} \\ & + \frac{-q}{\sqrt{r^2+b^2+2rb\cos\theta}} + \frac{\frac{a}{b}q}{\sqrt{r^2+\frac{a^4}{b^2}+2\frac{ra^2}{b}\cos\theta}} \Bigg] \end{aligned} \tag{44}$$

Hence the entire solution is:

$$\phi = \phi_1 + \phi_2 \tag{45}$$

$$\begin{aligned} &= \frac{q}{4\pi\epsilon_0}\Bigg[\frac{1}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{1}{\sqrt{r^2\frac{b^2}{a^2}+a^2-2rb\cos\theta}} \\ & - \frac{1}{\sqrt{r^2+b^2+2rb\cos\theta}} + \frac{1}{\sqrt{r^2\frac{b^2}{a^2}+a^2+2rb\cos\theta}} \Bigg] + V_1\frac{a^2}{r^2}\cos\theta \qquad (r \geqslant a) \end{aligned} \tag{46}$$

5. **Multipole expansion**

(a) Given some localized charge density $\rho(\boldsymbol{r}')$, the electrostatic potential at $\boldsymbol{r}$ is:

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int d\boldsymbol{r}'\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{47}$$

For $r >> r'$, please derive the expressions for the contributions to $\phi(\boldsymbol{r})$ from charge monopole, dipole, and quadrupole moments in Cartesian coordinates. Write down the explicit expressions of charge monopole, dipole, and quadrupole moments in Cartesian coordinates.

(b) For $r >> r'$, please derive the expressions for the contributions to $\phi(\boldsymbol{r})$ from charge monopole, dipole, and quadrupole moments in spherical coordinates. Write down the explicit expressions of charge monopole, dipole, and quadrupole moments in spherical coordinates.

(c) Derive the relationships between the mulitipole moments in Cartesian coordinates and in spherical coordinates (for monopole, dipole, and quadrupole moments)

(a) Set $\boldsymbol{r} = (x, y, z), \boldsymbol{r}' = (x', y', z')$, and the Taylor expansion:

$$f(\boldsymbol{r}-\boldsymbol{r}') = f(\boldsymbol{r}) - \boldsymbol{r}'\cdot\nabla f(\boldsymbol{r}) + \frac{1}{2!}(\boldsymbol{r}'\cdot\nabla)^2 f(\boldsymbol{r}) + \cdots \tag{48}$$

For $r >> r'$, $|\boldsymbol{r}-\boldsymbol{r}'| \approx r$, the potential can be expanded as

$$\begin{aligned}\phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\int d\boldsymbol{r}'\rho(\boldsymbol{r}')\left(\frac{1}{r} - \boldsymbol{r}'\cdot\nabla\frac{1}{r} + \frac{1}{2!}(\boldsymbol{r}'\cdot\nabla)^2\frac{1}{r} + \cdots\right) \\ &= \frac{1}{4\pi\epsilon_0}\int d\boldsymbol{r}'\rho(\boldsymbol{r}')\left(\frac{1}{r} + \boldsymbol{r}'\cdot\frac{\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{i,j}x_i'x_j'\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{r} + \cdots\right)\end{aligned} \tag{49}$$

and for $\dfrac{\partial^2}{\partial x_i\partial y_j}\dfrac{1}{r}$,

$$\frac{\partial^2}{\partial x_i\partial y_j}\frac{1}{r} = \frac{3x_ix_j - \delta_{ij}r^2}{r^5} \tag{50}$$

hence,

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int d\boldsymbol{r}'\rho(\boldsymbol{r}')\left(\frac{1}{r} + \boldsymbol{r}'\cdot\frac{\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{i,j}(3x_i'x_j' - r'^2\delta_{ij})\frac{x_ix_j}{r^5} + \cdots\right) \tag{51}$$

The expressions for charge monopole, dipole, and quadrupole moments are

$$q = \int d\boldsymbol{r}'\rho(\boldsymbol{r}') \tag{52}$$

$$\boldsymbol{p} = \int d\boldsymbol{r}'\boldsymbol{r}'\rho(\boldsymbol{r}') \tag{53}$$

$$Q_{ij} = \int d\boldsymbol{r}'(3x_i'x_j' - r'^2\delta_{ij})\rho(\boldsymbol{r}') \tag{54}$$

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{i,j}Q_{ij}\frac{x_ix_j}{r^5} + \cdots\right] \tag{55}$$

Obviously, $Q_{ij} = Q_{ji}$

(b) Make use of $\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l,m}^*(\theta',\varphi')Y_{l,m}(\theta,\varphi)$, and here $r_> = r, r_< = r'$, the potential in a spherical coordinates is written as

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{l,m}\frac{Y_{l,m}(\theta,\varphi)}{r^{l+1}} \tag{56}$$

where

$$q_{l,m} = \int d\boldsymbol{r}' Y_{l,m}^*(\theta',\varphi')r'^l\rho(\boldsymbol{r}') \tag{57}$$

These coefficients are called multipole moments. The monopole, dipole, quadrupole correspond with $l=0, l=1, l=2$ respectively:

$$\begin{aligned}
&monopole: && 4\pi q_{0,0}Y_{0,0} && (58)\\
&dipole: && \sum_{m=-1}^{1}\frac{4\pi}{3}q_{1,m}Y_{1,m} && (59)\\
&quadrupole: && \sum_{m=-2}^{2}\frac{4\pi}{5}q_{2,m}Y_{2,m} && (60)
\end{aligned}$$

(c) By calculation,

$$\begin{aligned}
q_{0,0} &= \frac{1}{\sqrt{4\pi}}\int d\boldsymbol{r}'\rho(\boldsymbol{r}') = \frac{1}{\sqrt{4\pi}}q && (61)\\
q_{1,1} &= -\sqrt{\frac{3}{8\pi}}\int d\boldsymbol{r}'(x'-iy')\rho(\boldsymbol{r}') = -\sqrt{\frac{3}{8\pi}}(p_x-ip_y)\\
q_{1,0} &= \sqrt{\frac{3}{4\pi}}p_z && (62)\\
q_{1,-1} &= \sqrt{\frac{3}{8\pi}}(p_x+ip_y)\\
q_{2,2} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int d\boldsymbol{r}'(x'-iy')^2\rho(\boldsymbol{r}') = \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11}-2iQ_{12}-Q_{22})\\
q_{2,1} &= -\frac{1}{6}\sqrt{\frac{15}{2\pi}}(Q_{13}-iQ_{23})\\
q_{2,0} &= \frac{1}{4}\sqrt{\frac{5}{\pi}}Q_{33} && (63)\\
q_{2,-1} &= \frac{1}{6}\sqrt{\frac{15}{2\pi}}(Q_{31}+iQ_{32})\\
q_{2,-2} &= \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11}+2iQ_{21}-Q_{22})
\end{aligned}$$

(64)