上海科技大学
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Homework-3

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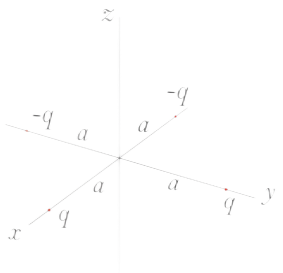
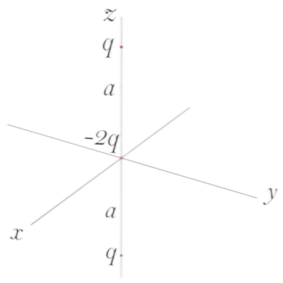
1. **Multipoles**

   (a) Please calculate the total charge, dipole moment and quadrupole moment for the charge distributions shown in two figure (a) and (b).

   (b) Please expand the electrostatic potential in fig.(b) using multipole expansion, clarify the different contributions from total charge, dipole moments. Plot the potential (in terms of multipole expansion) in the x-y plane as a function of distance r for $r > a$. Compare this with the exact result calculated using Coulomb's law.



(a)



(b)

(a) The total charge in (a):

$$Q = \int d\boldsymbol{r}' \rho(\boldsymbol{r}') = \int d\boldsymbol{r}' q \left[\delta(y' - a) + \delta(x' - a) - \delta(y' + a) - \delta(x' + a)\right] = 0 \tag{1}$$

The dipole moment in (a):

$$\boldsymbol{p} = \int d\boldsymbol{r}' \rho(\boldsymbol{r}')\boldsymbol{r}' = \int d\boldsymbol{r}' q \left[\delta(y' - a) + \delta(x' - a) - \delta(y' + a) - \delta(x' + a)\right] \boldsymbol{r}' = 2qa\hat{\boldsymbol{x}} + 2qa\hat{\boldsymbol{y}} \tag{2}$$

The quadrupole moment in (a):

$$\mathcal{Q}_{ij} = \int d\boldsymbol{r}' (3x_i'x_j' - r'^2\delta_{ij})\rho(\boldsymbol{r}') = 0 \tag{3}$$

Similarly, for (b):

$$\begin{aligned} Q &= 0 && (4)\\ \boldsymbol{p} &= 0 && (5)\\ \mathcal{Q}_{33} = -2\mathcal{Q}_{22} = -2\mathcal{Q}_{11} = 4a^2q \quad &, \quad \mathcal{Q}_{12} = \mathcal{Q}_{21} = \mathcal{Q}_{13} = \mathcal{Q}_{31} = \mathcal{Q}_{32} = \mathcal{Q}_{23} = 0 && (6) \end{aligned}$$

(b) The electrostatic potential is

$$\begin{aligned} \varphi(\boldsymbol{r}) &= \int d\boldsymbol{r}' \frac{\rho(\boldsymbol{r}')}{4\pi\epsilon_0|\boldsymbol{r} - \boldsymbol{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \int d\boldsymbol{r}' \rho(\boldsymbol{r}')(\frac{1}{r} + \boldsymbol{r}' \cdot \frac{\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{i,j} x_i'x_j' \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{r} + \cdots) \\ &= \frac{1}{4\pi\epsilon_0}(0 + 0 + a^2 q \frac{\partial^2}{\partial z^2}\frac{1}{r} + \cdots) \\ &= \frac{1}{4\pi\epsilon_0}(\frac{3a^2qz^2}{r^5} - \frac{a^2q}{r^3} + \cdots) \end{aligned} \tag{7}$$

The total charge and dipole have no contributions for the potential. The potential calculated by Coulomb's law:

$$\varphi_C(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}(\frac{q}{\sqrt{x^2 + y^2 + (z - a)^2}} + \frac{q}{\sqrt{x^2 + y^2 + (z + a)^2}} - \frac{2q}{\sqrt{x^2 + y^2 + z^2}}) \tag{8}$$

In the x-y plane,

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(-\frac{a^2 q}{(x^2+y^2)^{\frac{3}{2}}}\right) \tag{9}$$

$$\varphi_C(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{\sqrt{x^2+y^2+a^2}} - \frac{2q}{\sqrt{x^2+y^2}}\right) \tag{10}$$

The potential is plotted in the below:

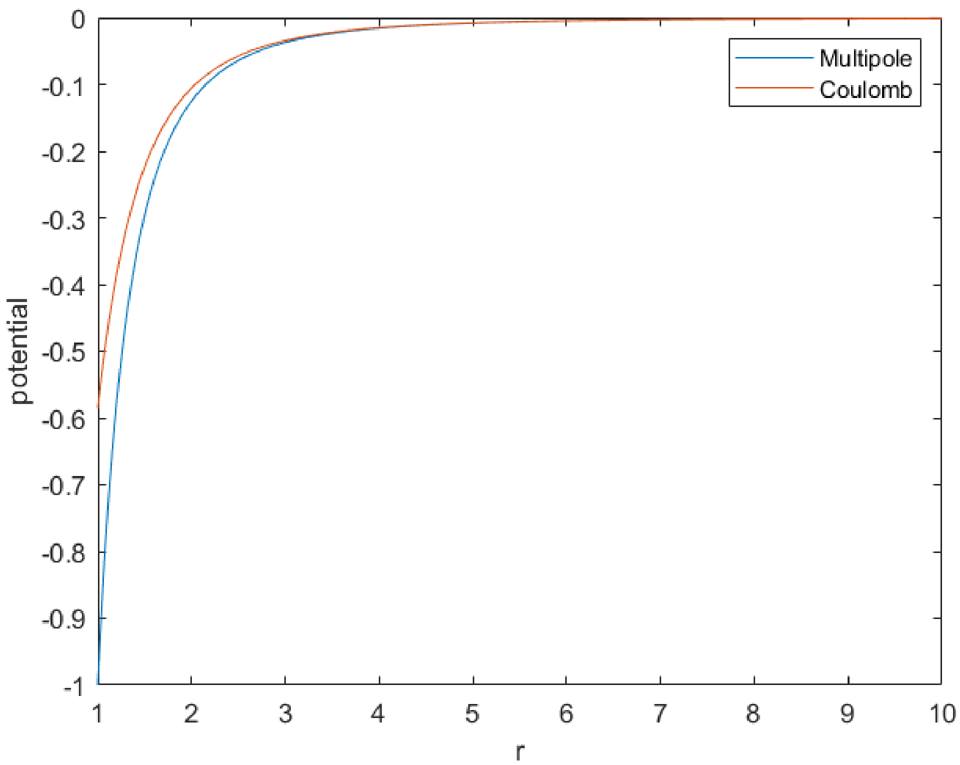


Figure 1: The potential in the $z = 0$ plane

2. **Dielectrics**: A point charge q is located outside a dielectric sphere with dielectric constant ϵ as shown in the figure.

   (a) Calculate the electrostatic potential everywhere in space.

   (b) Calculate the electric fields everywhere in space.

   (c) Calculate the electric polarization in the dielectric sphere, and calculate the bound charge density at the surface induced by the polarization.

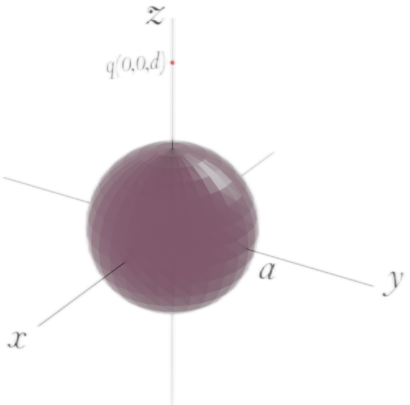
   (d) Verify that, when ϵ is infinitely large, the system is equivalent to a conductor.

(a) From the expansion of the Green's function for a spherical shell bounded by $r = a$ and $r = b$

$$G(\boldsymbol{r}, \boldsymbol{r}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)}{(2l+1)\left[1 - (\frac{a}{b})^{2l+1}\right]} \left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right) \tag{11}$$

Set $b \to \infty$, we obtain the Green's function appropriate for the "exterior" problem with a spherical boundary at $r = a$. Considering the symmetry, we hence obtain,

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \sum_{l=0}^{\infty} \left[\frac{r^l_<}{r^{l+1}_>} - \frac{a^{2l+1}}{(r_< r_>)^{l+1}}\right] \tag{12}$$



Since there's no point charge in the sphere, the potential satisfies the *Poisson equation*. Thus we can set the general solution:

$$\phi_1(\boldsymbol{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\sum_n(\dfrac{r^n}{d^{n+1}} - \dfrac{a^{2n+1}}{r^{n+1}d^{n+1}})P_n(\cos\theta) + \sum_n(a_n r^n + \dfrac{b_n}{r^{n+1}})P_n(\cos\theta) & a < r < d \\ \dfrac{q}{4\pi\epsilon_0}\sum_n(\dfrac{d^n}{r^{n+1}} - \dfrac{a^{2n+1}}{r^{n+1}d^{n+1}})P_n(\cos\theta) + \sum_n(a_n r^n + \dfrac{b_n}{r^{n+1}})P_n(\cos\theta) & r > d \end{cases} \tag{13}$$

$$\phi_2(\boldsymbol{r}) = \sum_n(c_n r^n + \frac{d_n}{r^{n+1}})P_n(\cos\theta) \qquad r \leqslant a \tag{14}$$

With the boundary condition:

$$\phi_1|_{r\to\infty} = const, \qquad \phi_2|_{r=0} = const \tag{15}$$

$$\phi_1|_{r=a} = \phi_2|_{r=a}\,, \qquad \epsilon_0\left.\frac{\partial\phi_1}{\partial r}\right|_{r=a} = \epsilon\left.\frac{\partial\phi_2}{\partial r}\right|_{r=a} \tag{16}$$

Then,

$$0 = d_n = a_n \tag{17}$$

$$\sum_n \frac{b_n}{a^{n+1}}P_n(\cos\theta) = \sum_n c_n a^n P_n(\cos\theta) \tag{18}$$

$$\frac{q}{4\pi\epsilon_0}\sum_n(\frac{na^{n-1}}{d^{n+1}} + \frac{(n+1)a^{n-1}}{d^{n+1}})P_n(\cos\theta) - \sum_n\frac{(n+1)b_n}{a^{n+2}}P_n(\cos\theta) = \frac{\epsilon}{\epsilon_0}\sum_n nc_n a^{n-1}P_n(\cos\theta) \tag{19}$$

The coefficients c_n and b_n,

$$c_n = \frac{q(2n+1)}{4\pi\left[\epsilon n + \epsilon_0(n+1)\right]d^{n+1}}, \qquad b_n = \frac{q(2n+1)a^{2n+1}}{4\pi\left[\epsilon n + \epsilon_0(n+1)\right]d^{n+1}} \tag{20}$$

Hence, the potential is formed as

$$\phi_1(\boldsymbol{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\sum_n(\dfrac{r^n}{d^{n+1}} - \dfrac{a^{2n+1}}{r^{n+1}d^{n+1}})P_n(\cos\theta) + \sum_n\dfrac{q(2n+1)a^{2n+1}}{4\pi\left[\epsilon n + \epsilon_0(n+1)\right]d^{n+1}}\dfrac{1}{r^{n+1}}P_n(\cos\theta) & a < r < d \\ \dfrac{q}{4\pi\epsilon_0}\sum_n(\dfrac{d^n}{r^{n+1}} - \dfrac{a^{2n+1}}{r^{n+1}d^{n+1}})P_n(\cos\theta) + \sum_n\dfrac{q(2n+1)a^{2n+1}}{4\pi\left[\epsilon n + \epsilon_0(n+1)\right]d^{n+1}}\dfrac{1}{r^{n+1}}P_n(\cos\theta) & r > d \end{cases} \tag{21}$$

$$\phi_2(\boldsymbol{r}) = \sum_n\frac{q(2n+1)}{4\pi\left[\epsilon n + \epsilon_0(n+1)\right]d^{n+1}}r^n P_n(\cos\theta) \qquad r \leqslant a \tag{22}$$

(b) $\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi$ gives

$$\boldsymbol{E}_1(\boldsymbol{r}) = \begin{cases} -\Big[\frac{q}{4\pi\epsilon_0}\sum_n(\frac{nr^{n-1}}{d^{n+1}} + \frac{(n+1)a^{2n+1}}{r^{n+2}d^{n+1}})P_n(\cos\theta) - \sum_n \frac{q(2n+1)a^{2n+1}}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}\frac{n+1}{r^{n+2}}P_n(\cos\theta)\Big]\hat{\boldsymbol{r}} \\ +\Big[\frac{q}{4\pi\epsilon_0}\sum_n(\frac{r^n}{d^{n+1}} - \frac{a^{2n+1}}{r^{n+1}d^{n+1}})\frac{dP_n(\cos\theta)}{d\cos\theta} + \sum_n \frac{q(2n+1)a^{2n+1}}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}\frac{1}{r^{n+1}}\frac{dP_n(\cos\theta)}{d\cos\theta}\Big](\frac{\sin\theta}{r})\hat{\boldsymbol{\theta}} \\ \qquad a<r<d \\ -\Big[\frac{q}{4\pi\epsilon_0}\sum_n(-\frac{(n+1)d^n}{r^{n+2}} + \frac{(n+1)a^{2n+1}}{r^{n+2}d^{n+1}})P_n(\cos\theta) - \sum_n \frac{q(2n+1)a^{2n+1}}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}\frac{n+1}{r^{n+2}}P_n(\cos\theta)\Big]\hat{\boldsymbol{r}} \\ +\Big[\frac{q}{4\pi\epsilon_0}\sum_n(\frac{d^n}{r^{n+1}} - \frac{a^{2n+1}}{r^{n+1}d^{n+1}})\frac{dP_n(\cos\theta)}{d\cos\theta} + \sum_n \frac{q(2n+1)a^{2n+1}}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}\frac{1}{r^{n+1}}\frac{dP_n(\cos\theta)}{d\cos\theta}\Big](\frac{\sin\theta}{r})\hat{\boldsymbol{\theta}} \\ \qquad r>d \end{cases}$$

$$\boldsymbol{E}_2(\boldsymbol{r}) = -\Big[\sum_n \frac{q(2n+1)n}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}r^{n-1}P_n(\cos\theta)\Big]\hat{\boldsymbol{r}} + \Big[\sum_n \frac{q(2n+1)}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}r^n\frac{dP_n(\cos\theta)}{d\cos\theta}\Big]\frac{\sin\theta}{r}\hat{\boldsymbol{\theta}} \qquad r\leqslant a \tag{23}$$

(c)

$$\begin{aligned}\boldsymbol{P} &= (\epsilon-\epsilon_0)\boldsymbol{E}_2 \\ &= -\Big[\sum_n \frac{(\epsilon-\epsilon_0)q(2n+1)n}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}r^{n-1}P_n(\cos\theta)\Big]\hat{\boldsymbol{r}} + \Big[\sum_n \frac{(\epsilon-\epsilon_0)q(2n+1)}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}r^n\frac{dP_n(\cos\theta)}{d\cos\theta}\Big]\frac{\sin\theta}{r}\hat{\boldsymbol{\theta}}\end{aligned} \tag{24}$$

$$\begin{aligned}\sigma_{\boldsymbol{P}} &= \boldsymbol{P}\cdot\hat{\boldsymbol{r}}|_{r=a} \\ &= -\Big[\sum_n \frac{(\epsilon-\epsilon_0)q(2n+1)n}{4\pi\left[\epsilon n+\epsilon_0(n+1)\right]d^{n+1}}a^{n-1}P_n(\cos\theta)\Big]\end{aligned} \tag{25}$$

(d) As $\epsilon\to\infty$

$$\phi_2(\boldsymbol{r})\to 0, \qquad \boldsymbol{E}_2\to 0, \qquad \phi_1(\boldsymbol{r})\to \begin{cases} \frac{q}{4\pi\epsilon_0}\sum_n(\frac{r^n}{d^{n+1}} - \frac{a^{2n+1}}{r^{n+1}d^{n+1}})P_n(\cos\theta) & a<r<d \\ \frac{q}{4\pi\epsilon_0}\sum_n(\frac{d^n}{r^{n+1}} - \frac{a^{2n+1}}{r^{n+1}d^{n+1}})P_n(\cos\theta) & r>d \end{cases} \tag{26}$$

That is, the system degenerates as a conducted sphere which the potential is zero everywhere inside the sphere. Thus it becomes a perfect conductor, and the potential on the surface is bounded by zero. Hence the system is equivalent to a conductor.