Advanced Communication Theory

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Contents

1 Introductory Concepts

1.1 SISO System

1. A digital modulator includes analogue subunits (e.g., AM/FM/PM); adding specific digital blocks before these can turn an analogue modulator into a digital one (e.g., FM + digital blocks = FSK).
2. The receiver is the inverse of the transmitter; only the ADC's quantiser has no inverse in the DAC.
3. Even digital transmitters contain analogue subsystems like up-converters; both transmission and reception always deal with analogue signals.
4. Two spectral efficiency metrics:
   - **EUE** (Energy Utilisation Efficiency) = E_b/N_0
   - **BUE** (Bandwidth Utilisation Efficiency) = B/r_b

1.2 Antenna Array (SISO → MISO, SIMO, MIMO)

1. **Modelling Approaches**: **Non-parametric** treats each antenna as an independent SISO system with per-path β, while **parametric** treats the array as a coordinated system, accounting for geometry and wave propagation.
   Limitations of non-parametric modelling:
   (a) Ignores **Cartesian coordinates** and antenna orientation.
   (b) Ignores **signal directions**.
   (c) Ignores **propagation models** (e.g., plane or spherical waves).
2. **An array system** employs $N > 1$ antennas in 3D space, referenced to a common origin.

   The antenna positions are described by: $\underline{\underline{r}} = [\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N] = \begin{bmatrix}\underline{r}_x & \underline{r}_y & \underline{r}_z\end{bmatrix}^T \in \mathbb{R}^{3\times N}$, where $\underline{r}_k \in \mathbb{R}^{3\times 1}$ denotes the position of the k-th antenna, and $\underline{r}_x$, $\underline{r}_y$, $\underline{r}_z \in \mathbb{R}^{N\times 1}$ are vectors of x, y, and z coordinates respectively.

   The array aperture is defined as the maximum distance between any 2 elements, i.e.: $\max_{\forall i,j} \left\|\underline{r}_i - \underline{r}_j\right\|$

1.3 From EM-Waves to Array Manifold Vector

EM Fields Let $\underline{r} = [x, y, z]^T$ be the spatial coordinate in Cartesian form, with $\|\underline{r}\| = R$. Its spherical counterpart is: $\underline{r} = [R, \theta, \phi]^T$, where $(\theta, \phi) \triangleq$ (azimuth, elevation).

The direction of propagation is given by the unit vector: $\underline{u}(\theta, \phi) = [\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi]^T$. Hence, $\underline{r} = R \cdot \underline{u}$ and $\underline{u}^T\underline{r} = R$.

Assuming Tx at the origin, the electric field at point $\underline{r}$ is: $\underline{E}(\underline{r}, t) = \underline{E}_0 \cdot \exp\left(j2\pi F_c t - j\frac{2\pi}{\lambda}\underline{u}^T\underline{r}\right)$, which reflects both temporal oscillation and spatial phase shift.

Array Manifold Vector Derivation In a receiving array, assume the reference point $\underline{0}_3$ is located at the Rx. Each antenna element is positioned at $\underline{r}_k$. According to the plane wave model, the received electric field at the k-th antenna is:

$$\underline{E}(\underline{r}_k, t) = \underline{E}_0 \cdot \exp\left(j2\pi F_c t - j\frac{2\pi}{\lambda}\underline{u}^T\underline{r}_k\right) = \underline{E}(\underline{0}_3, t) \cdot \exp\left(-j\frac{2\pi}{\lambda}\underline{u}^T\underline{r}_k\right)$$

This shows that all antennas receive the same waveform $E(\underline{0}_3, t)$ with phase delays determined by their position along the direction $\underline{u}$.

If we assume $\underline{E}_0 = 1$ (unit amplitude), the relative response of the k-th antenna becomes:

$$\text{Response}_k = \exp\left(-j\frac{2\pi}{\lambda}\,\underline{u}^T\underline{r}_k\right) = \exp\left(-j\,\underline{r}_k^T\,\underline{k}(\theta, \phi)\right)$$

where $\underline{k}(\theta, \phi) = \dfrac{2\pi}{\lambda}\,\underline{u}(\theta, \phi) = \dfrac{2\pi}{\lambda}[\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi]^T$ is the wavenumber vector.

The array manifold vector (array response vector, source position vector, SPV) is given by:

$$\underline{S}(\theta,\phi) = \exp\left(-j\begin{bmatrix}\underline{r}_1 & \underline{r}_2 & \cdots & \underline{r}_N\end{bmatrix}^T \underline{k}(\theta,\phi)\right) = \exp\left(-j[r_x, r_y, r_z]\underline{k}(\theta,\phi)\right) = \exp\left(-j\,\underline{\underline{r}}^T\,\underline{k}(\theta,\phi)\right)$$

This vector $\underline{S}(\theta,\phi) \in \mathbb{C}^{N\times 1}$ encodes the spatial signature of the arriving wave at all N antennas. It depends only on the geometry of the array $\underline{\underline{r}}$ and the direction of arrival (DoA) (θ,ϕ).

- If signals lie in the x–y plane ($\phi = 0°$), then: $\underline{S}(\theta) = \exp\left(-j\frac{2\pi}{\lambda}(\underline{r}_x\cos\theta + \underline{r}_y\sin\theta)\right)$
- For a linear array along the x-axis: $\underline{S}(\theta) = \exp\left(-j\frac{2\pi}{\lambda}\underline{r}_x\cos\theta\right)$

System Type	Array Manifold Vector $\underline{S}$
SIMO	$\underline{S}^{(Rx)}$ or $\underline{S}$
MISO	$\underline{S}^{(Tx)}$ or $\overline{\underline{S}}$
MIMO	$\underline{S}^{(virtual)} = \underline{S}^{(Tx)} \otimes \underline{S}^{(Rx)}$

1.4 Fourier Transform Conventions

Fourier Transform We use the proper frequency f in Hz (not angular frequency ω in rad/sec). A full Fourier transform table will be provided in the exam.

$$\mathrm{FT}\{g(t)\} = G(f) \triangleq \int_{-\infty}^{\infty} g(t)\cdot e^{-j2\pi ft}\,dt$$
$$\mathrm{FT}^{-1}\{G(f)\} = g(t) \triangleq \int_{-\infty}^{\infty} G(f)\cdot e^{j2\pi ft}\,df$$

Woodward's Notation

- $\mathrm{rect}\{t\}$: width 1, centred at $t=0$; $\mathrm{rect}\{t\} \triangleq \begin{cases}1, & |t| < \frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$
- $\mathrm{sinc}\{t\}$: central peak at $t=0$ and zeros at $\pm1, \pm2, \ldots$; $\mathrm{sinc}\{t\} \triangleq \dfrac{\sin(\pi t)}{\pi t}$
- $\Lambda\{t\}$: triangular function supported on $[-1,1]$ with peak at $t=0$; $\Lambda\{t\} \triangleq \begin{cases}1-t, & 0\le t\le 1\\ 1+t, & -1\le t\le 0\end{cases}$
- $\mathrm{rep}_T\{g(t)\}$: periodic extension of $g(t)$ with period T; $\mathrm{rep}_T\{g(t)\} \triangleq \sum_{n=-\infty}^{\infty} g(t-nT)$
- $\mathrm{comb}_T\{g(t)\}$: sampling function; $\mathrm{comb}_T\{g(t)\} \triangleq \sum_{n=-\infty}^{\infty} g(nT)\cdot\delta(t-nT)$

Complex Power Convention In all signal processing operations, we never multiply two complex quantities directly. Instead, we always multiply one by the complex conjugate of the other.

For a complex scalar β, its squared magnitude is given by $\beta^2 = \beta\beta^* = |\beta|^2$. Similarly, for a complex signal $s(t)$, its power is defined as

$$P_s = \mathbb{E}\{s(t)^2\} = \mathbb{E}\{s(t)\cdot s^*(t)\}$$

2 Diversity Theory

2.1 Diversity Definition

Diversity refers to receiving multiple copies of the same signal, each with a different distortion level, and combining them using a weighted estimator based on a chosen combining rule.

Diversity techniques are classified by how the signal copies are obtained, including multi-path, time, frequency, polarisation, code, and space diversity.

Input-Output Relationship The combined output signal is: $s_{\text{div}} = w_1^* x_1 + w_2^* x_2 + \cdots + w_N^* x_N = \underline{w}^H \underline{x}$, and satisfies $\text{SNR}_{\text{div}} > \text{SNR}_i$ for every i.

If s_d represents the desired signal (scalar), then each received copy:

$$x_i = \beta_i s_d + n_i \quad \Rightarrow \quad \underline{x} = \underline{\beta} s_d + \underline{n}$$

The combined signal:

$$s_{\text{div}} = \underline{w}^H \underline{x} = \underbrace{\underline{w}^H \underline{\beta} s_d}_{\text{desired}} + \underbrace{\underline{w}^H \underline{n}}_{\text{noise}}$$

Output SNR:

$$\text{SNR}_{\text{out,div}} = \frac{\mathbb{E}\left\{\left(\underline{w}^H \underline{\beta} s_d\right)^2\right\}}{\mathbb{E}\left\{\left(\underline{w}^H \underline{n}\right)^2\right\}} = \frac{P_d\, \underline{w}^H \mathbf{R}_{\beta\beta} \underline{w}}{\underline{w}^H \mathbf{R}_{nn} \underline{w}} = \frac{P_d}{\sigma_n^2} \cdot \frac{\underline{w}^H \mathbf{R}_{\beta\beta} \underline{w}}{\underline{w}^H \underline{w}}$$

where P_d is the signal power, $\mathbf{R}_{\beta\beta}$ and $\mathbf{R}_{nn}$ are the covariance matrices of $\underline{\beta}$ and noise $\underline{n}$ respectively. In the case of spatially white noise, $\mathbf{R}_{nn} = \sigma_n^2 \mathbf{I}$, where σ_n^2 is the noise power and $\mathbf{I}$ is the identity matrix.

Diversity Combining Rules

- **Max Ratio Combining (MRC)**: chooses weights $\underline{w}_{\text{MRC}}$ to maximise output SNR; equivalent to Wiener-Hopf equation

$$\underline{w}_{\text{MRC}} = \arg\max_{\underline{w}} \left(\text{SNR}_{\text{out,div}}\right)$$

- **Selection Combining (SC)**: selects the branch with the highest individual SNR

$$w_k = \begin{cases} 1, & \text{if } \text{SNR}_k > \text{SNR}_i,\ \forall i \\ 0, & \text{otherwise} \end{cases}$$

- **Equal Gain Combining (EGC)**: all weights have equal magnitude

$$w_1 = w_2 = \cdots = w_N$$

- **Scanning Combining (SCC)**: uses the first branch whose SNR exceeds a threshold; switches when it drops below

$$\text{if } \text{SNR}_k > \text{threshold} \Rightarrow \begin{cases} w_k = 1 \\ w_j = 0,\ \forall j \neq k \end{cases} \qquad \text{else } k \leftarrow k+1$$

2.2 Wireless Channel

Parameters The channel capacity C (in bits/sec) defines the maximum achievable information rate. In addition, wireless channels are characterised by three key bandwidth parameters and their corresponding time-domain quantities, linked via reciprocal relations:

Bandwidth Parameter	Time Parameter	Reciprocal Relation
B: Channel bandwidth	T_{cs}: Symbol duration	$B = 1/T_{cs}$
B_{coh}: Coherence bandwidth	T_{spread}: Delay spread	$B_{\text{coh}} = 1/T_{\text{spread}}$
B_{Dop}: Doppler spread	T_{coh}: Coherence time	$B_{\text{Dop}} = 1/T_{\text{coh}}$

Delay spread reflects multipath richness; Doppler spread reflects time variation. Coherence bandwidth and coherence time indicate how stable the channel is across frequency and time, respectively.

- *Frequency characteristics:*
  - If $T_{cs} < T_{\text{spread}}$ (i.e. $B > B_{\text{coh}}$), the channel is *frequency-selective*.
  - If $T_{cs} > T_{\text{spread}}$ (i.e. $B < B_{\text{coh}}$), the channel is *flat*.

- *Time characteristics:*
  - If $T_{cs} < T_{\text{coh}}$ (i.e. $B > B_{\text{Dop}}$), the channel exhibits *slow fading.*
  - If $T_{cs} > T_{\text{coh}}$ (i.e. $B < B_{\text{Dop}}$), the channel exhibits *fast fading.*

Multipath In modern wireless systems, multipaths are no longer treated as unwanted interference but rather as opportunities for improvement through *multipath diversity.* The goal is to *resolve*, *estimate* and finally *utilise* the multipaths to enhance performance. That is, to identify how many independent paths can be separated and exploited.

The number of resolvable multipath components depends on the pulse duration T_{cs}. A smaller T_{cs} (i.e. larger bandwidth) allows for finer resolution.

When $T_{cs} < T_{\text{spread}}$ (i.e. frequency-selective channel), the number of resolvable paths is:

$$L = \left\lfloor \frac{T_{\text{spread}}}{T_{cs}} \right\rfloor + 1$$

2.3 RAKE Receiver

Main Assumptions

1. The channel is frequency-selective: $T_{\text{spread}} > T_{cs}$
2. The channel is slow fading: approximately time-invariant over T_{cs}

Definition Under these conditions, the received signal $r(t)$ includes L resolvable paths. A RAKE receiver is the optimum coherent receiver for multipath channels. Each path requires a dedicated correlation receiver (matched filter), meaning L correlators for L paths. These correlation branches are known as **RAKE fingers**, and their outputs $s_1, s_2, \ldots, s_L$ must be combined.

Because the multipath components arrive at different times and with different strengths, a **channel estimator** is required. This module, referred to as the **finger manager**, provides both the delay profile and the associated weights to the RAKE receiver.

There are four types of RAKE receivers depending on the combining rule (MRC, SC, EGC, SCC), as previously summarised in 2.1.

Real World Example To isolate multipaths in a wireless channel, the receiver first uses **pilot signals** to estimate path delays τ_{jk} and fading coefficients β_{jk}. A **PN code generator** then applies these delays to align correlation templates with the desired path(s).

Unlike traditional Rx that track only one path, RAKE Rx assign one correlator (finger) per path and combine their outputs using weights predicted from τ_{jk} and β_{jk}. This ensures robustness even when individual paths are weak. The combined signal is finally passed to a decision block to recover the transmitted symbol.

3 SIMO, MISO, MIMO: Modelling

3.1 Spatial Characteristics of a Wireless Channel

- *Space characteristics:*
  - A channel is *space-selective* if its transfer function varies as the receiver moves in space.
  - A channel is *spatially coherent* if its transfer function remains approximately constant over a distance D_{coh}, called the coherence distance.
- *Scale characteristics:*
  - If displacement $> D_{\text{coh}}$, the channel exhibits *small-scale fading*, caused by constructive/destructive interference of multipaths.

- If displacement $\gg \lambda$ (many wavelengths), the channel exhibits *large-scale fading*, caused by path loss and shadowing; typically frequency-independent.

Wavenumber Spectrum and Angle Spectrum

- **Wavenumber spectrum** $S_H(\underline{k})$: multipath energy distribution over spatial frequencies; $\underline{k}$ is the wavenumber vector; obtained via FT of the spatial autocorrelation function $\Phi_H(\Delta\underline{r})$.
- **Angle spectrum** $p(\theta, \phi)$: average received power vs. direction-of-arrival; θ = azimuth angle, ϕ = elevation angle.

The two are related through a ring mapping: $S_H(\underline{k}) \leftrightarrow p(\theta, \phi)$, which maps spatial frequency content to angular content for array processing.

Local Area The local area is the spatial region around a reference point $\underline{r}_o$ where the wireless channel can be assumed approximately constant. Within this area, the channel can be modelled as a sum of plane waves with negligible variation. Its radius is approximately $d \lesssim \dfrac{c}{B} = c \cdot T_{cs}$, where B is the bandwidth.

3.2 Wireless SIMO Channels

A SIMO channel (Single-Input Multiple-Output) models the impulse response received by an N-element antenna array. Each antenna receives a delayed and phase-shifted version of the transmitted signal.

Impulse Response (Single Path) For a single path with delay τ and DoA (θ, ϕ), the impulse response vector observed at the N antennas is:

$$\underline{h}(t) = \beta \cdot \underline{S}(\theta, \phi) \cdot \delta(t - \tau)$$

All entries share the same β and τ and are aligned with the same direction vector $\underline{u}(\theta, \phi)$, hence $\underline{h}(t)$ and $\underline{S}$ are colinear. The Rx array manifold vector is given by:

$$\underline{S}(\theta, \phi) = \exp\left(-j \cdot \underline{\underline{r}}^T \cdot \underline{k}(\theta, \phi)\right) \quad \in \mathbb{C}^{N \times 1}$$

The impulse response vector can be expressed as:

$$\underline{h} = \begin{bmatrix} \beta_{Tx,1} \\ \beta_{Tx,2} \\ \vdots \\ \beta_{Tx,N} \end{bmatrix} \quad \text{(non-parametric)} \qquad \text{vs} \qquad \underline{h} = \beta \cdot \underline{S}(\theta, \phi) \quad \text{(parametric)}$$

- In the **non-parametric model**, each entry is unknown and must be estimated individually. The number of unknowns grows with the number of antennas (N).
- In the **parametric model**, the unknowns are only β and DoA (θ, ϕ), independent of N.

Impulse Response (Multipath) If there are L resolvable paths, each with angle (θ_ℓ, ϕ_ℓ), delay τ_ℓ and complex gain β_ℓ, the total impulse response vector becomes:

$$\underline{h}(t) = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{S}(\theta_\ell, \phi_\ell) \cdot \delta(t - \tau_\ell)$$

In this case, we also have:

$$\underline{h} = \begin{bmatrix} \beta_{Tx,1} \\ \beta_{Tx,2} \\ \vdots \\ \beta_{Tx,N} \end{bmatrix} \quad \text{(non-parametric)} \qquad \text{vs} \qquad \underline{h} = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{S}(\theta_\ell, \phi_\ell) \quad \text{(parametric)}$$

The parametric formulation requires estimating L sets of parameters $(\beta_\ell, \theta_\ell, \phi_\ell)$ regardless of N, making it scalable and more efficient for large antenna arrays.

Modelling the Received Signal For a single transmitter sending baseband signal $m(t)$ over an L-path SIMO channel, the received signal at an N-element antenna array is:

$$\underline{x}(t) = \underline{h}(t) * m(t) + \underline{n}(t) = \sum_{\ell=1}^{L} \underline{S}(\theta_\ell, \phi_\ell) \cdot \beta_\ell \cdot m(t - \tau_\ell) + \underline{n}(t) = \underbrace{\mathbf{S}}_{N\times L} \cdot \underbrace{\underline{m}(t)}_{L\times 1} + \underbrace{\underline{n}(t)}_{N\times 1}$$

Here, $\underline{m}(t) = [\beta_1 m(t-\tau_1), \ldots, \beta_L m(t-\tau_L)]^T$ is the vector of delayed and weighted signal components. It should not be confused with the scalar baseband signal $m(t)$.

Extension to Multi-User SIMO Now consider M co-channel transmitters, each contributing L multipaths. The received signal becomes:

$$\underline{x}(t) = \sum_{i=1}^{M} \sum_{\ell=1}^{L} \underline{S}_{i\ell} \cdot \beta_{i\ell} \cdot m_i(t - \tau_{i\ell}) + \underline{n}(t) = \underbrace{\mathbf{S}}_{N\times(M\cdot L)} \cdot \underbrace{\underline{m}(t)}_{(M\cdot L)\times 1} + \underbrace{\underline{n}(t)}_{N\times 1}$$

3.3 Wireless MISO Channels

Impulse Response (Single Path) In the MISO (Multiple-Input Single-Output) case, the transmitter uses an array of $\overline{N}$ antennas located at positions

$$\underline{\underline{\overline{r}}} = \begin{bmatrix} \underline{\overline{r}}_1 & \underline{\overline{r}}_2 & \cdots & \underline{\overline{r}}_{\overline{N}} \end{bmatrix} = \begin{bmatrix} \underline{\overline{r}}_x & \underline{\overline{r}}_y & \underline{\overline{r}}_z \end{bmatrix}^T$$

where $\underline{\underline{\overline{r}}} \in \mathbb{C}^{3\times\overline{N}}$ and the overbar denotes a Tx-side parameter.

For a single path, the scalar impulse response observed at the Rx is:

$$h(t) = \beta \cdot \underline{\overline{S}}^H \cdot \underline{\delta}\left(t - \frac{d}{c}\right)$$

The Tx array manifold vector is given by:

$$\underline{\overline{S}}(\overline{\theta}, \overline{\phi}) = \exp\left(+j \cdot \underline{\underline{\overline{r}}}^T \cdot \underline{k}(\overline{\theta}, \overline{\phi})\right) \quad \in \mathbb{C}^{\overline{N}\times 1}$$

where the wavenumber vector is:

$$\underline{k}(\overline{\theta}, \overline{\phi}) = \frac{2\pi}{\lambda_c} \cdot \underline{u}(\overline{\theta}, \overline{\phi}) \quad \text{with} \quad \underline{u}(\overline{\theta}, \overline{\phi}) = \begin{bmatrix} \cos\overline{\theta}\cos\overline{\phi} & \sin\overline{\theta}\cos\overline{\phi} & \sin\overline{\phi} \end{bmatrix}^T$$

- The exponential term uses a **positive** sign, indicating energy radiating *out* from the Tx array.
- The structure of $\underline{\overline{S}}(\overline{\theta}, \overline{\phi})$ is analogous to that of the Rx-side array manifold vector $\underline{S}(\theta, \phi)$, but conjugated when used in MISO channel expression.

Impulse Response (Multipath) For a MISO multipath channel with L resolvable paths, the channel impulse response is:

$$h(t) = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{\overline{S}}_\ell^H \cdot \underline{\delta}(t - \tau_\ell)$$

where $\underline{\overline{S}}_\ell = \underline{\overline{S}}(\overline{\theta}_\ell, \overline{\phi}_\ell) \in \mathbb{C}^{\overline{N}\times 1}$ is the Tx array manifold vector for the ℓ-th path.

Modelling the Received Signal We consider two cases for the transmitted baseband signal $m(t)$:

- **Case 1:** $m(t)$ is demultiplexed into $\overline{N}$ different signals, forming the vector $\underline{m}(t) \in \mathbb{C}^{\overline{N}\times 1}$. Then:

$$x(t) = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{\overline{S}}_\ell^H \cdot \underbrace{(\underline{\overline{w}} \circ \underline{m}(t-\tau_\ell))}_{\overline{N}\times 1} + n(t)$$

- **Case 2:** All Tx antennas transmit the same signal $m(t)$, and the model becomes:

$$x(t) = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{\overline{S}}_\ell^H \cdot \underbrace{(\underline{\overline{w}} \cdot m(t-\tau_\ell))}_{\overline{N}\times 1} + n(t)$$

In both cases, $\underline{\overline{w}} \in \mathbb{C}^{\overline{N}\times 1}$ is a transmit weight vector applied across the Tx array elements.

The symbol $\circ$ denotes the Hadamard (element-wise) product, meaning the multiplication is performed element-by-element between vectors of the same dimension.

Transmit Diversity Transmit diversity exploits multiple Tx antennas to provide robustness against fading by sending redundant information across different spatial paths. It is typically categorised as:

- **Closed-loop transmit diversity**: Requires CSI feedback from the Rx. The Tx initially transmits omnidirectionally. After receiving the signal, the Rx estimates the optimal direction and sends it back to the Tx via an uplink feedback channel. The Tx then updates its weights to direct energy toward the Rx. This improves performance but introduces additional overhead due to the feedback link.
- **Open-loop transmit diversity**: No feedback is required. Schemes like STBC (Space-Time Block Coding) are used, which work well for frequency-flat channels. However, the number of transmit antennas must be a power of two (e.g. 2, 4), which limits antenna design flexibility.

★ STBC (Open-Loop MISO Example) ★ For two Tx antennas over flat fading, the transmitter encodes m_1, m_2 into an orthogonal block, which is sent over two time slots and two antennas:

$$\begin{bmatrix} m_1 & m_2 \\ -m_2^* & m_1^* \end{bmatrix}$$

Let the flat-fading coefficients after summing over L unresolvable paths be:

$$\beta_{1,\mathrm{Rx}} = \sum_{j=1}^{L} \beta_{1j,\mathrm{Rx}}, \quad \beta_{2,\mathrm{Rx}} = \sum_{j=1}^{L} \beta_{2j,\mathrm{Rx}}$$

The received signals over two time slots are:

$$\begin{cases} x_1 = \beta_{1,\mathrm{Rx}} m_1 - \beta_{2,\mathrm{Rx}} m_2^* + n_1 \\ x_2 = \beta_{1,\mathrm{Rx}} m_2 + \beta_{2,\mathrm{Rx}} m_1^* + n_2 \end{cases}$$

These can be written in two equivalent forms: **(★ Exam focus ★)**

- **Message-formulation:**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{bmatrix} \cdot \begin{bmatrix} \beta_{1,\mathrm{Rx}} \\ \beta_{2,\mathrm{Rx}} \end{bmatrix} + \underline{n}$$

- **Channel-formulation:**

$$\underline{x} = \begin{bmatrix} \beta_{1,\mathrm{Rx}} & -\beta_{2,\mathrm{Rx}} \\ \beta_{2,\mathrm{Rx}}^* & \beta_{1,\mathrm{Rx}}^* \end{bmatrix} \cdot \begin{bmatrix} m_1 \\ m_2^* \end{bmatrix} + \underline{n}$$

i.e. $\underline{x} = \mathbf{H}\underline{m} + \underline{n}$

This channel matrix is orthogonal:

$$\mathbf{H}^H\mathbf{H} = \left(|\beta_{1,\mathrm{Rx}}|^2 + |\beta_{2,\mathrm{Rx}}|^2\right) \cdot \mathbf{I}_2 = \|\underline{\beta}_{\mathrm{Rx}}\|^2 \cdot \mathbf{I}_2$$

Thus, decoding can be done by matched filtering:

$$\begin{aligned} \underline{G} &= \mathbf{H}^H \underline{x} = \mathbf{H}^H(\mathbf{H}\underline{m} + \underline{n}) \\ &= \|\underline{\beta}_{\mathrm{Rx}}\|^2 \underline{m} + \underline{\tilde{n}} = \begin{bmatrix} \|\underline{\beta}_{\mathrm{Rx}}\|^2 m_1 + \tilde{n}_1 \\ \|\underline{\beta}_{\mathrm{Rx}}\|^2 m_2^* + \tilde{n}_2 \end{bmatrix} \end{aligned}$$

Thus, symbols m_1, m_2 can then be recovered independently. $\beta_{1,\mathrm{Rx}}$ and $\beta_{2,\mathrm{Rx}}$ can be obtained by sending pilot symbols as m_1, m_2. The receiver only needs to estimate them locally as no CSI feedback is needed.

3.4 Wireless MIMO Channels

Impulse Response (Single Path) In the MIMO (Multiple-Input Multiple-Output) case, both Tx and Rx use antenna arrays. Denote their element locations as:

$$\overline{\underline{r}} = \begin{bmatrix} \overline{r}_1 & \cdots & \overline{r}_{\overline{N}} \end{bmatrix} \in \mathbb{R}^{3\times\overline{N}}, \quad \underline{\underline{r}} = \begin{bmatrix} \underline{r}_1 & \cdots & \underline{r}_N \end{bmatrix} \in \mathbb{R}^{3\times N}$$

where overbars indicate Tx-side parameters, and underscores denote Rx-side ones.

For a single path with gain β, direction-of-departure $(\overline{\theta}, \overline{\phi})$, and direction-of-arrival (θ, ϕ), the matrix-valued impulse response is:

$$\underline{h}(t) = \beta \cdot \underline{S}(\theta, \phi) \cdot \overline{\underline{S}}^H(\overline{\theta}, \overline{\phi}) \cdot \underline{\delta}\left(t - \frac{\rho}{c}\right)$$

where:

$$\begin{cases} \bigstar \text{ Tx:} & \overline{\underline{S}} = \overline{\underline{S}}(\overline{\theta}, \overline{\phi}) = \exp\left(+j \cdot \overline{\underline{r}}^T \cdot \underline{k}(\overline{\theta}, \overline{\phi})\right) \in \mathbb{C}^{\overline{N}\times 1} \\ \bigstar \text{ Rx:} & \underline{S} = \underline{S}(\theta, \phi) = \exp\left(-j \cdot \underline{\underline{r}}^T \cdot \underline{k}(\theta, \phi)\right) \in \mathbb{C}^{N\times 1} \end{cases}$$

Impulse Response (Multipath) For a MIMO channel with L resolvable paths, the total impulse response is:

$$\underline{h}(t) = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{S}_\ell \cdot \overline{\underline{S}}_\ell^H \cdot \underline{\delta}(t - \tau_\ell)$$

with $\underline{S}_\ell = \underline{S}(\theta_\ell, \phi_\ell) \in \mathbb{C}^{N\times 1}$ and $\overline{\underline{S}}_\ell = \overline{\underline{S}}(\overline{\theta}_\ell, \overline{\phi}_\ell) \in \mathbb{C}^{\overline{N}\times 1}$.

Modelling the Received Signal We consider two Tx strategies for baseband signal $m(t)$:

- **Case 1:** The signal is demultiplexed into $\overline{N}$ streams forming $\underline{m}(t) \in \mathbb{C}^{\overline{N}\times 1}$. Then:

$$\underline{x}(t) = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{S}_\ell \cdot \overline{\underline{S}}_\ell^H \cdot \underbrace{(\overline{\underline{w}} \circ \underline{m}(t - \tau_\ell))}_{\overline{N}\times 1} + \underline{n}(t)$$

- **Case 2:** All Tx elements send the same $m(t)$, yielding:

$$\underline{x}(t) = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{S}_\ell \cdot \overline{\underline{S}}_\ell^H \cdot \underbrace{(\overline{\underline{w}} \cdot m(t - \tau_\ell))}_{\overline{N}\times 1} + \underline{n}(t)$$

In both cases, $\underline{x}(t) \in \mathbb{C}^{N\times 1}$ is the received vector, $\overline{\underline{w}} \in \mathbb{C}^{\overline{N}\times 1}$ is the transmit weight vector, and $\underline{n}(t)$ is additive noise at the Rx. The operator $\circ$ denotes the Hadamard (element-wise) product.

3.5 Multipath Clustering

In practice, multiple rays often arrive in groups (clusters) with similar angles and delays. Each cluster contains L_{scat} rays and can be approximated as a single path with effective gain:

$$\beta_\ell = \sum_{k=1}^{L_{\mathrm{scat}}} \beta_{\ell k}$$

- **SIMO:** $h_\ell(t) = \beta_\ell \cdot \underline{S}_\ell \cdot \delta(t - \tau_\ell)$
- **MISO:** $h_\ell(t) = \beta_\ell \cdot \overline{\underline{S}}_\ell^H \cdot \underline{\delta}(t - \tau_\ell)$
- **MIMO:** $h_\ell(t) = \beta_\ell \cdot \underline{S}_\ell \cdot \overline{\underline{S}}_\ell^H \cdot \underline{\delta}(t - \tau_\ell)$

3.6 MIMO Modelling Without Geometric Information

In some scenarios, we do not know the spatial structure of the Tx/Rx arrays (e.g. their geometry or steering directions). In this case, the MIMO channel is modelled directly by a complex gain matrix.

Let $\underline{\beta}_{j,\text{Tx}} \in \mathbb{C}^{N\times 1}$ denote the gain vector from Tx antenna j to all N Rx antennas. Stack these into a full channel matrix $\mathbf{H}$, where columns represent Tx antennas while rows represent Rx:

$$\underline{x}[n] = \mathbf{H} \cdot \underline{m}[n] + \underline{n}[n], \quad \text{with } \mathbf{H} = \begin{bmatrix} \underline{\beta}_{1,\text{Tx}} & \cdots & \underline{\beta}_{\overline{N},\text{Tx}} \end{bmatrix}$$

where $\underline{m}[n] \in \mathbb{C}^{\overline{N}\times 1}$ is the transmitted symbol vector and $\underline{x}[n] \in \mathbb{C}^{N\times 1}$ is the received vector.

The second-order statistics are captured by the covariance matrix:

$$\mathbf{R}_{xx} = \mathbf{H}\mathbf{R}_{mm}\mathbf{H}^H + \sigma_n^2 \mathbf{I}_N$$

This model allows signal processing without requiring knowledge of array geometry.

If the array geometry is known, the channel matrix $\mathbf{H}$ can be expressed using array manifolds:

$$\mathbf{H} = \sum_{\ell=1}^{L} \beta_\ell \cdot \underline{S}_\ell \cdot \overline{\underline{S}}_\ell^H \quad \text{or} \quad \mathbf{H} = \mathbf{S} \cdot \mathbf{B} \cdot \overline{\mathbf{S}}^H$$

where:

- $\underline{S}_\ell$, $\overline{\underline{S}}_\ell$ are the Rx and Tx array manifold vectors for path ℓ
- $\mathbf{S} \in \mathbb{C}^{N\times L}$, $\overline{\mathbf{S}} \in \mathbb{C}^{\overline{N}\times L}$: stacked manifold matrices
- $\mathbf{B} = \text{diag}(\beta_1, \ldots, \beta_L)$: path gain matrix

This links the statistical and geometric views of the channel.

3.7 MIMO Channel Capacity

The capacity of a SISO channel (in bit/s) is given by the Shannon formula:

$$C_{\text{SISO}} = B \cdot \log_2\left(1 + \frac{P}{\sigma_n^2}\right) \quad [\text{bit/s}]$$

where P is the signal power, σ_n^2 is the noise power, and B is the bandwidth.

The capacity of a MIMO channel (in bit/s) is defined as:

$$C = B \cdot \log_2\left(\frac{\det(\mathbf{R}_{xx})}{\det(\mathbf{R}_{nn})}\right) \quad [\text{bit/s}]$$

where $\mathbf{R}_{xx}$ is the covariance matrix of received signal, $\mathbf{R}_{nn}$ is the noise covariance matrix, and B is bandwidth (Hz).

Assuming spatially white noise $\mathbf{R}_{nn} = \sigma_n^2\mathbf{I}$, the formula simplifies to:

$$C = B \cdot \log_2\left(\det\left(\mathbf{I} + \frac{1}{\sigma_n^2}\mathbf{H}\mathbf{R}_{mm}\mathbf{H}^H\right)\right)$$

If the transmitted streams are uncorrelated with power allocation $\mathbf{R}_{mm} = \text{diag}(P_1, \ldots, P_{\overline{N}})$, then:

$$C = B\sum_{j=1}^{\overline{N}} \log_2\left(1 + \frac{\|\underline{\beta}_{j,\text{Tx}}\|^2 P_j}{\sigma_n^2}\right)$$

3.8 MIMO Equivalence and Spatial Convolution

A MIMO system can be equivalently interpreted as a "virtual SIMO" or "virtual MISO" by spatially convolving the Tx and Rx arrays into a single virtual array. This allows the use of classical SIMO tools on MIMO systems.

Let the Tx and Rx array coordinates be denoted as:

$$\text{Tx-array: } \underline{\underline{\overline{r}}} \in \mathbb{R}^{3\times\overline{N}}, \quad \text{Rx-array: } \underline{\underline{r}} \in \mathbb{R}^{3\times N}$$

Then, the virtual array formed by spatial convolution is:

$$\underline{\underline{r}}_{\text{virtual}} = \underline{\underline{\overline{r}}} \otimes \mathbf{1}_N^T + \mathbf{1}_{\overline{N}}^T \otimes \underline{\underline{r}} \quad \in \mathbb{R}^{3\times(\overline{N}\cdot N)}$$

The corresponding array manifold vector becomes:

$$\underline{S}_{\text{virtual}} = \overline{\underline{S}}^* \otimes \underline{S}$$

This virtual array of size $\overline{N} \cdot N$ enables MIMO signal processing to be recast as an extended SIMO problem with enhanced spatial resolution.

4 SIMO, MIMO: Array Receivers

4.1 General Problem Formulation ($M < N$)

Problem Statement We observe a complex vector signal $\underline{x}(t) \in \mathbb{C}^{N\times 1}$ as:

$$\underline{x}(t) = \underbrace{\mathbf{S}(\underline{p})}_{N\times M} \cdot \underbrace{\underline{m}(t)}_{M\times 1} + \underbrace{\underline{n}(t)}_{N\times 1}$$

Here:

- $\underline{x}(t)$ is the received signal, $\underline{m}(t)$ is the unknown signal vector, $\underline{n}(t)$ is additive noise.
- N is the number of sensors (Rx geometry known), M is the number of sources (unknown), $M < N$.
- $\mathbf{S}$ is the array manifold matrix parameterised by $\underline{p}$ (e.g. DOAs).

Tasks include:

- **Detection**: Detect the number of sources M,
- **Estimation**: Estimate parameters of $\underline{m}(t)$ and $\underline{n}(t)$ (e.g. power, DOA),
- **Reception**: Recover a desired component $m_i(t)$.

Theoretical Covariance Matrix The covariance of the observed signal is:

$$\mathbf{R}_{xx} \triangleq \mathbb{E}\left\{\underline{x}(t)\underline{x}^H(t)\right\} = \mathbf{S}\mathbf{R}_{mm}\mathbf{S}^H + \mathbf{R}_{nn}$$

Where:

- $\mathbf{R}_{mm} = \mathbb{E}\left\{\underline{m}(t)\underline{m}^H(t)\right\} \quad (M \times M)$:
  signal covariance matrix, Hermitian, diagonal entries = powers P_i,
- $\mathbf{R}_{nn} = \mathbb{E}\left\{\underline{n}(t)\underline{n}^H(t)\right\} \quad (N \times N)$:
  noise covariance matrix. Assuming isotropic AWGN noise, we have $\mathbf{R}_{nn} = \sigma_n^2 \cdot \mathbf{I}_N$.

Practical Covariance Matrix Assume we collect L snapshots over time $t_1, t_2, \ldots, t_L$, forming:

$$\begin{aligned}
\mathbf{X} &\triangleq [\underline{x}(t_1),\ \underline{x}(t_2),\ \ldots,\ \underline{x}(t_L)] \qquad (N \times L) \\
&= [\mathbf{S}\cdot\underline{m}(t_1) + \underline{n}(t_1), \quad \mathbf{S}\cdot\underline{m}(t_2) + \underline{n}(t_2), \quad \ldots, \quad \mathbf{S}\cdot\underline{m}(t_L) + \underline{n}(t_L)] \\
&= \mathbf{S}\cdot\mathbf{M} + \mathbf{N}
\end{aligned}$$

$$\text{with} \quad \begin{cases} \mathbf{S} = [\underline{S}_1,\ \underline{S}_2,\ \ldots,\ \underline{S}_M] & (N \times M) \\ \mathbf{M} = [\underline{m}(t_1),\ \underline{m}(t_2),\ \ldots,\ \underline{m}(t_L)] & (M \times L) \\ \mathbf{N} = [\underline{n}(t_1),\ \underline{n}(t_2),\ \ldots,\ \underline{n}(t_L)] & (N \times L) \end{cases}$$

where the array manifold matrix $\mathbf{S}$ is the same as in the theoretical model (because it's time-invariant), while $\mathbf{M}$ and $\mathbf{N}$ consist of signal and noise samples collected at each time instant $t_1, t_2, \ldots, t_L$, respectively. Each column corresponds to one temporal snapshot.

The sample covariance matrix is:

$$\begin{aligned} \hat{\mathbf{R}}_{xx} &= \frac{1}{L}\sum_{l=1}^{L} \underline{x}(t_l)\, \underline{x}(t_l)^H = \frac{1}{L}\,\mathbf{X}\mathbf{X}^H && \text{(practical)} \\ &= \mathbb{E}\left\{\underline{x}(t)\, \underline{x}(t)^H\right\} = \mathbf{S}\,\hat{\mathbf{R}}_{mm}\,\mathbf{S}^H + \hat{\mathbf{R}}_{nn} && \text{(theoretical)} \end{aligned}$$

This connects the **theoretical** model to the **empirical** data estimate. In an array system, the covariance matrix $\mathbf{R}_{xx}$ (whether theoretical or practical) encapsulates all the **geometrical** and **statistical** information of the sources with respect to the array. Therefore, all parameter estimation and source characterisation must ultimately rely on this matrix.

Generating Snapshots with a Given Covariance Matrix To generate L synthetic snapshots of the signal vector $\underline{x}(t)$ with a predefined covariance matrix $\mathbf{R}_{xx}$, each snapshot $\underline{x}(t_\ell)$ can be formed as:

$$\underline{x}(t_\ell) = \mathbf{E}\mathbf{D}^{1/2}\underline{z}(t_\ell)$$

where $\mathbf{E}$ and $\mathbf{D}$ are the eigenvector and eigenvalue matrices of $\mathbf{R}_{xx}$, respectively. The vector $\underline{z}(t_\ell) \in \mathbb{C}^N$ is a complex Gaussian vector with zero mean and unit variance, satisfying:

$$\mathbb{E}\left\{\underline{z}(t_\ell)\underline{z}(t_\ell)^H\right\} = \mathbf{I}_N$$

Collecting all L snapshots, the data matrix becomes:

$$\mathbf{X} = [\underline{x}(t_1),\ \underline{x}(t_2),\ \ldots,\ \underline{x}(t_L)] = \left[\mathbf{E}\mathbf{D}^{1/2}\underline{z}(t_1),\ \mathbf{E}\mathbf{D}^{1/2}\underline{z}(t_2),\ \ldots,\ \mathbf{E}\mathbf{D}^{1/2}\underline{z}(t_L)\right]$$

This approach ensures that the generated $\mathbf{X}$ satisfies the desired second-order statistics defined by $\mathbf{R}_{xx}$. In the similar fashion we can generate any collection of snapshots with a given cov. matrix.

4.2 Detection (Number of Sources)

Theoretical Detection via Eigen-Decomposition We use the Maximum Likelihood (ML) rule to select the hypothesis with the maximum likelihood value. This is equivalent to finding the number of sources M based on the statistics of the observed signal $\underline{x}(t)$, particularly its covariance matrix $\mathbf{R}_{xx}$. Under the model:

$$\mathbf{R}_{xx} = \underbrace{\mathbf{S}\mathbf{R}_{mm}\mathbf{S}^H}_{\mathbf{R}_{\text{signals}}} + \mathbf{R}_{nn}$$

we assume $\mathbf{R}_{nn} = \sigma_n^2\mathbf{I}_N$ (AWGN). When $M < N$, the rank of $\mathbf{R}_{\text{signals}}$ is M. Thus, we perform eigen-decomposition:

$$\mathbf{R}_{xx} = \mathbf{E}\mathbf{D}\mathbf{E}^H$$

where $\mathbf{D} = \Lambda + \sigma_n^2 \mathbf{I}_N$, and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_M, 0, \ldots, 0)$ is the diagonal matrix of signal eigenvalues. The eigenvalue matrix $\mathbf{D}$ becomes:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 + \sigma_1^2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 + \sigma_2^2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_M + \sigma_M^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \sigma_{M+1}^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix}$$

- **Theoretical:** All noise variances are equal, i.e. $\sigma_1^2 = \cdots = \sigma_N^2 = \sigma_n^2$, so we can count the multiplicity of the minimum eigenvalue of $\mathbf{R}_{xx}$ to determine the number of sources:

$$M = N - (\text{multiplicity of minimum eigenvalue of } \mathbf{R}_{xx})$$

- **Practical:** The eigenvalues are no longer exactly equal but approximately clustered, i.e. $\sigma_1^2 \neq \sigma_2^2 \neq \cdots \neq \sigma_N^2$, although $\sigma_1^2 \approx \cdots \approx \sigma_M^2 \approx \sigma_n^2$. In this case, we cannot directly count repeated eigenvalues, and must instead apply **information-theoretic criteria** such as AIC or MDL to estimate M.

Information-Theoretic Criteria (for awareness only) In practice, we use model selection criteria to choose the best hypothesis from a set of models $\{H_k\}$, based on the L observations. Two widely used approaches are:

- **Akaike Information Criterion (AIC)**:
  - Select the model that minimises:

$$AIC(k) = -2 \ln \left(\max_k \text{LF}^{(k)} \right) + 2k$$

  - The first term is the negative log-likelihood (maximum likelihood estimator), and the second term is a bias correction.
- **Minimum Description Length (MDL)**:
  - Minimises a slightly stronger cost function:

$$MDL(k) = -\ln \left(\max_k \text{LF}^{(k)} \right) + \frac{1}{2} k \ln L$$

  - Similar to AIC, but penalises model complexity more strictly using the additional $\frac{1}{2} k \ln L$ term.

For each k, we compute the likelihood function:

$$\text{LF}^{(k)} = \ln \left(\frac{\left(\prod_{\ell=k+1}^{N} d_\ell \right)^{1/(N-k)}}{\frac{1}{N-k} \sum_{\ell=k+1}^{N} d_\ell} \right) = \ln \left(\frac{\textbf{geometric mean} \text{ of smallest } N-k \text{ eigenvalues of } \widehat{\mathbf{R}}_{xx}}{\textbf{arithmetic mean} \text{ of smallest } N-k \text{ eigenvalues of } \widehat{\mathbf{R}}_{xx}} \right)$$

Both AIC and MDL are used to determine the number of sources M by selecting the value of k that best explains the observed data under the Gaussian noise model. Once M is determined, the noise power $\hat{\sigma}_n^2$ can be estimated by averaging the smallest $N - M$ eigenvalues:

$$\hat{\sigma}_n^2 = \frac{1}{N-M} \sum_{i=M+1}^{N} \sigma_i^2$$

4.3 Estimation (Signal / Channel Parameters)

Observation Space and Subspace We work in an N-dimensional complex observation space $\mathcal{H}$, where each observation vector $\underline{x} \in \mathcal{H}$ has N entries.

- **Subspace structure:**
  - A vector $\underline{a} \in \mathbb{C}^{N\times 1}$ spans a 1D subspace $\mathcal{L}[\underline{a}] = \{\lambda \underline{a} \mid \lambda \in \mathbb{C}\}$.
  - A matrix $\mathbf{S} \in \mathbb{C}^{N\times M}$ spans an M-dim subspace $\mathcal{L}[\mathbf{S}] = \{\mathbf{S}\underline{a} \mid \underline{a} \in \mathbb{C}^{M\times 1}\}$.
- **Orthogonal projection:**
  - Any $\underline{x} \in \mathcal{H}$ can be projected onto $\mathcal{L}[\mathbf{S}]$ using $\mathbb{P}_{\mathbf{S}} = \mathbf{S}(\mathbf{S}^H\mathbf{S})^{-1}\mathbf{S}^H$, and $\mathbb{P}_{\mathbf{S}}\underline{x} \in \mathcal{L}[\mathbf{S}]$.
  - The projection matrix satisfies $\mathbb{P}_{\mathbf{S}}^2 = \mathbb{P}_{\mathbf{S}}$ and $\mathbb{P}_{\mathbf{S}}^H = \mathbb{P}_{\mathbf{S}}$ (Hermitian and idempotent).
- **Orthogonal complement:**
  - The orthogonal complement is $\mathcal{L}[\mathbf{S}]^\perp$ with $\dim(\mathcal{L}[\mathbf{S}]^\perp) = N - M$.
  - The projection matrix is $\mathbb{P}_{\mathbf{S}}^\perp = \mathbf{I}_N - \mathbb{P}_{\mathbf{S}}$, and $\mathbb{P}_{\mathbf{S}}^\perp\underline{x} \in \mathcal{L}[\mathbf{S}]^\perp$.
- **Geometric interpretation:**
  - Any $\underline{x} \in \mathcal{H}$ can be decomposed as $\underline{x} = \mathbb{P}_{\mathbf{S}}\underline{x} + \mathbb{P}_{\mathbf{S}}^\perp\underline{x}$.
  - If $\underline{x} \in \mathcal{L}[\mathbf{S}]$, then $\mathbb{P}_{\mathbf{S}}\underline{x} = \underline{x}$ and $\mathbb{P}_{\mathbf{S}}^\perp\underline{x} = \mathbf{0}$.

Problem Description We adopt the powerful parameter estimation technique known as the **signal-subspace approach**, which partitions the observation space into the **signal subspace** and its orthogonal complement, the **noise subspace**.

The received signal is modelled as:

$$\underline{x}(t) = \sum_{i=1}^{M} \beta_i m_i(t)\underline{S}_i + \underline{n}(t) = \mathbf{S}\cdot \underline{m}(t) + \underline{n}(t)$$

where $\mathbf{S} \in \mathbb{C}^{N\times M}$ and $\underline{x}(t) \in \mathbb{C}^{N\times 1}$. This leads to the following estimation scenario:

1. **Observation Space:** A complex N-dimensional space (the ambient space for array measurements).
2. **Signal Subspace:** A linear subspace of dimension M, determined by the received signal.
   - The matrix $\mathbf{S} \in \mathbb{C}^{N\times M}$ consists of M column vectors (steering vectors), each associated with a source $m_i(t)$.
   - These columns span the M-dimensional subspace $\mathcal{L}[\mathbf{S}]$, and the signal vector $\mathbf{S}\underline{m}(t)$ resides within this subspace.
   - The observed data $\underline{x}(t)$ is a noisy version of this point and fluctuates around it.
   - The **noise subspace** is the orthogonal complement $\mathcal{L}[\mathbf{S}]^\perp$, of dimension $N - M$.
3. **Manifold:** A highly non-linear subspace representing the system's structural properties, determined solely by the sensor geometry and orientation.
   - The manifold corresponds to the set of all possible array responses $\underline{S}(\underline{p})$, where $\underline{p}$ is the source parameters (e.g. DOA).
   - For each value group of $\underline{p}$, the manifold returns a single steering vector $\underline{S}(\underline{p}) \in \mathbb{C}^{N\times 1}$.
   - Each column of $\mathbf{S}$ is a point lying on this manifold, corresponding to $\underline{S}(\underline{p}_i)$ for some source i.

Goal: To determine the M intersection points between the signal subspace and the manifold.

In other words, we have a **subspace** that reflects the structure of the measured Rx data and a **manifold** that encodes the physical model of the array system. Estimating the signal parameters involves finding where the manifold intersects with the signal subspace.

To extract the signal and noise subspaces, we perform eigen-decomposition of $\widehat{\mathbf{R}}_{xx}$:

$$\widehat{\mathbf{R}}_{xx} = \mathbf{E}\mathbf{D}\mathbf{E}^H = [\mathbf{E}_s \ \mathbf{E}_n]\begin{bmatrix}\mathbf{D}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_n\end{bmatrix}[\mathbf{E}_s \ \mathbf{E}_n]^H = \mathbf{E}_s\mathbf{D}_s\mathbf{E}_s^H + \mathbf{E}_n\mathbf{D}_n\mathbf{E}_n^H$$

Then:

$$\begin{aligned} \text{Signal subspace}: &\quad \mathcal{L}[\mathbf{S}] = \mathcal{L}[\mathbf{E}_s] = \mathcal{L}[\mathbf{E}_n]^\perp \\ \text{Noise subspace}: &\quad \mathcal{L}[\mathbf{S}]^\perp = \mathcal{L}[\mathbf{E}_n] = \mathcal{L}[\mathbf{E}_s]^\perp \\ \text{Observation space}: &\quad \mathcal{L}[\widehat{\mathbf{R}}_{xx}] = \mathcal{L}[\underline{x}(t)] = \mathcal{L}[\mathbf{X}] \end{aligned}$$

Note: $\mathbf{S} \neq \mathbf{E}_s$, but they span the same subspace.

Manifold In antenna array processing, the **manifold** refers to the geometrical set of array responses traced by a parameterised steering vector $\mathbf{a}(p) \in \mathbb{C}^N$, where p denotes signal-related parameters such as DOA, frequency, etc. The manifold encapsulates how the physical array structure maps signal parameters to measurement vectors in the N-dimensional observation space.

- A **single-parameter manifold** $\mathcal{A} = \{\mathbf{a}(p) \mid p \in \Omega\}$ forms a 1D curve; a **two-parameter manifold** (e.g., $\mathbf{a}(p, q)$) forms a surface.
- The curve's geometry is determined solely by sensor positions and orientations. Changing the array geometry modifies the manifold shape.
- Key parameters include: arc length s, velocity $\dot{\mathbf{a}}(p)$, length of manifold l_m and curvature κ.
- A "good" manifold has high curvature and distinguishable structure; "bad" manifolds (e.g., with flat regions or self-intersections) lead to ambiguous parameter estimates.
- In linear arrays, the manifold is a **hyperhelix** (curvatures are constant for every arc-length s or parameter p), and is characterised by the Cartan matrix $\mathbf{C}$. The Frobenius norm of $\mathbf{C}$, denoted $\|\mathbf{C}\|_F = \sqrt{\sum_{i,j} |\mathbf{C}_{ij}|^2}$, reflects the symmetry of the array geometry:
  - $\|\mathbf{C}\|_F = 1$: the array is **symmetric** — sensors are symmetrically placed around the origin.
  - $\|\mathbf{C}\|_F = \sqrt{2}$: the array is **fully asymmetric** — sensors are completely off-balance with no symmetry.
  - $1 < \|\mathbf{C}\|_F < \sqrt{2}$: the array is **partially symmetric** — there is some structural symmetry, but not complete.
- There are many parameterisations of a hypersurface. We should always try to find a parameterization that makes solving the problem easier.

MUSIC Algorithm The **Multiple Signal Classification (MUSIC)** algorithm belongs to the family of **Signal-Subspace type techniques**. It estimates the intersection between the **signal subspace** and the **array manifold**, in order to estimate signal parameters p such as the direction-of-arrival (DOA).

Assumptions: the number of sources M is known. The array geometry is known.

Steps:

1. Receive the array signal $\underline{x}(t) \in \mathbb{C}^{N\times 1}$ over L snapshots.
2. Estimate the covariance matrix: $\mathbf{R}_{xx} = \frac{1}{L}\sum_{l=1}^{L} \underline{x}(t_l)\, \underline{x}(t_l)^H$
3. Perform eigen-decomposition of $\widehat{\mathbf{R}}_{xx}$.
4. Find the noise subspace $\mathcal{L}[\mathbf{E}_n]$, spanned by the eigenvectors corresponding to the smallest $N - M$ eigenvalues.
5. For each candidate parameter p, compute the steering vector $\underline{S}(p)$, and evaluate the projection of $\underline{S}(p)$ onto the noise subspace, obtaining the cost function $\xi(p)$:
$$\xi(p) = \underline{S}(p)^H\, \mathbf{P}_{\mathbf{E}_n}\, \underline{S}(p), \quad \text{where} \quad \mathbf{P}_{\mathbf{E}_n} = \mathbf{E}_n \left(\mathbf{E}_n^H \mathbf{E}_n\right)^{-1} \mathbf{E}_n^H = \mathbf{E}_n \mathbf{E}_n^H$$
6. The estimated parameters $\hat{p}_1, \ldots, \hat{p}_M$ correspond to the **minima** of cost function $\xi(p)$, i.e.
$$\hat{p}_1, \ldots, \hat{p}_M = \arg\min_p \, \xi(p)$$

Limitations:

- MUSIC fails if signals are **coherent** (e.g. due to multipath or jamming).
- In this case, $\underline{S}(p) \notin \mathcal{L}[\mathbf{E}_n]^\perp$, and the subspace separation breaks down.
- **Solution:** Apply *spatial smoothing* to decorrelate signals before using MUSIC.

Spatial Smoothing (Preprocessing of MUSIC) When some source signals are coherent (i.e. fully correlated), the signal subspace dimension is less than M and MUSIC breaks down. In this case, spatial smoothing helps restore full-rank structure.

1. Partition the uniform linear array into overlapping subarrays.

2. For each subarray, compute the covariance matrix $\mathbf{R}_{xx,i}$.
3. Average the subarray covariance matrices:

$$\mathbf{R}_{xx,\text{smooth}} = \frac{1}{K}\sum_{i=1}^{K}\mathbf{R}_{xx,i}$$

4. Apply MUSIC using $\mathbf{R}_{xx,\text{smooth}}$ instead of $\mathbf{R}_{xx}$.

Estimation of Signal Powers After obtaining the DOA estimates and the noise power $\hat{\sigma}_n^2$, we can estimate the source power correlation matrix $\mathbf{R}_{mm}$ based on the theoretical signal model:

$$\mathbf{R}_{xx} = \mathbf{S}\mathbf{R}_{mm}\mathbf{S}^H + \sigma_n^2\mathbf{I}_N$$

To solve for $\mathbf{R}_{mm}$, we apply the Moore-Penrose pseudo-inverse $\mathbf{S}^{\#} = (\mathbf{S}^H\mathbf{S})^{-1}\mathbf{S}^H$:

$$\mathbf{R}_{mm} = \mathbf{S}^{\#}(\mathbf{R}_{xx} - \sigma_n^2\mathbf{I}_N)\mathbf{S}^{\#H}$$

This approach enables post-MUSIC estimation of signal powers and cross-correlations among the sources.

4.4 Reception (Signal Recovery)

Main Categories of Beamformers Two main categories based on the sensor architecture:

- **Category 1: Single Sensor**, uses a single physical antenna with a directional beam pattern.
- **Category 2: Array of Sensors**, employs multiple sensors arranged in specific geometries (e.g. line, planar arrays). Variants include:
  - **Switched Beamformer:** Selects one beam (from a fixed set) that maximises reception for the user or source. Switches between beams as the target moves.
  - **Adaptive Beamformer:** Dynamically adjusts the beam pattern based on signal statistics to track user direction or suppress interference.

Reception Problem and Beamformer Definition A **beamformer** is an array system designed to **enhance** a desired signal and **suppress** interference and noise by shaping the array pattern. It steers a high-gain beam toward the direction of arrival (DOA) of the desired signal, while placing **nulls** toward unwanted sources (interferers), typically via adaptive control of weights.

The array pattern is a function of the DOA θ, the manifold vector $\underline{S}(\theta)$, and the weight vector $\underline{w}$. The gain is defined as:

$$g(\theta) = \underline{w}^H\underline{S}(\theta)$$

This characterises the array's response to a signal arriving from angle θ. If no weighting is applied, i.e., $\underline{w} = \mathbf{1}_N$, the array responds equally in all directions as defined by $g(\theta) = \mathbf{1}^H\underline{S}(\theta)$.

Common visualisation:

- **Cartesian plot:** Gain versus azimuth angle
- **Polar plot:** Spatial distribution of gain (e.g. beam shape)

Beam structure:

- **Mainlobe:** The primary direction of maximum gain, whose angular width is called **Beamwidth:**
  - If d is inter-sensor spacing, N is number of sensors, and λ is the wavelength

$$\text{beamwidth}^\circ = 2\sin^{-1}\left(\frac{\lambda}{Nd}\right)\times\frac{180}{\pi}$$

  - If $d = \frac{\lambda}{2}$, then it can be simplified as:

$$\text{beamwidth}^\circ = 2\sin^{-1}\left(\frac{2}{N}\right)\times\frac{180}{\pi}$$

- **Key trend:** Increasing N (sensor number), increasing d (aperture), or decreasing λ leads to narrower beams (higher spatial resolution).

- **Sidelobes:** Secondary peaks (ideally suppressed).

Steering and Weight Design: To steer the mainlobe toward a desired direction θ_0, we align the weight vector with the manifold vector at that angle:

$$\underline{w}_{\text{mainlobe}} = \exp\left(-j \cdot \underline{\underline{r}}^T \cdot \underline{k}(\theta_0)\right) = \underline{S}(\theta_0)$$

This ensures constructive interference in that direction, while allowing for destructive interference (nulls) elsewhere—i.e., the array pattern is shaped by phasing.

Some Popular Beamformers

- **Wiener-Hopf Beamformer**:

$$\underline{w} = c\,\mathbf{R}_{xx}^{-1}\,\underline{S}_{\text{desired}}$$

  - c is a constant scalar
  - Maximises output SINR (Signal-to-Interference-plus-Noise Ratio)
  - Conventional beamformer: resolution depends on SNR (which affects the quality of R_{xx})
  - Does not require interferer DOAs

- **Modified Wiener-Hopf Beamformer**:

$$\underline{w} = c\,\mathbf{R}_{n+J}^{-1}\,\underline{S}_{\text{desired}}$$

  - $\mathbf{R}_{n+J}$: covariance matrix where desired signal is removed, only noise and interference remain
  - Robust to pointing errors (errors in desired DOA)

- **Minimum Variance (Capon) Beamformer**:

$$\min_{\underline{w}} \quad \underline{w}^H \mathbf{R}_{xx} \underline{w} \quad \text{s.t.} \quad \underline{w}^H \underline{S}(\theta_0) = 1$$

$$\Rightarrow \quad \text{Solution:} \quad \underline{w} = c\,\mathbf{R}_{xx}^{-1}\underline{S}_{\text{desired}}$$

  - Minimises output power while maintaining unit gain in desired direction
  - Optimises resolution under variance constraint
  - Unlike Wiener-Hopf beamformer, constant c is chosen under constraint

- **Supersolution Beamformer (with known interferer DOAs)**:

$$\underline{w} = \mathbb{P}_{S_J}^{\perp}\,\underline{S}_{\text{desired}}, \quad \mathbf{S} = [\underline{S}_{\text{desired}}, \mathbf{S}_j]$$

  - Projects desired signal onto the orthogonal complement of interference subspace
  - Asymptotically cancels all interferers
  - Requires estimation of all DOAs

- **Supersolution Beamformer (without interferer DOAs)**:

$$\underline{w} = \mathbb{P}_{\mathbf{E}_{n_j}}\,\underline{S}_{\text{desired}}, \quad \text{where } \mathbf{E}_{n_j} : \text{noise subspace of } \mathbf{R}_{n+J}$$

  - Uses noise subspace (from covariance with desired signal removed)
  - Cancels interferers indirectly without needing their DOAs
  - Practical when DOAs of interference sources are unknown

- **Maximum Likelihood (ML) Beamformer**:

$$\underline{w} = \text{col}_{\text{des}}(\mathbf{S}^{\#}), \quad \text{with} \quad \mathbf{S}^{\#} = \mathbf{S}(\mathbf{S}^H\mathbf{S})^{-1}$$

  - Based on pseudo-inverse (least-squares solution)
  - col_{des}: column of $\mathbf{S}^{\#}$ corresponding to desired source

Beamformers in Mobile Communications Beamforming supports multiple schemes in mobile networks: analogue access schemes (e.g. FDMA in AMPS, TACS, NMT), digital methods (e.g. TDMA in GSM, IS136; CDMA), and duplexing approaches like FDD and TDD.

Key advantages:

- **Signal gain** — better coverage
- **Interference rejection** — higher capacity
- **Spatial diversity** — multipath robustness
- **Power efficiency** — lower energy cost

4.5 Performance Criteria and Theoretical Bounds

Two Popular Performance Evaluation Criteria

- **Output SNIR** (SNIR_{out}): Signal-to-Noise-plus-Interference Ratio at the beamformer output.

$$\begin{aligned} y(t) &= \underline{w}^H \underline{x}(t) = \underline{w}^H \left(\mathbf{S}\underline{m}(t) + \underline{n}(t)\right) = \underline{w}^H \left(\underline{S}_1 m_1(t) + \mathbf{S}_J \underline{m}_J(t) + \underline{n}(t)\right) \\ P_y &= \underline{w}^H \mathbf{R}_{xx}\, \underline{w} \qquad \text{(assuming desired, interfs \& noise are uncorrelated)} \\ &= \underline{w}^H \left(\mathbf{R}_{dd} + \mathbf{R}_{JJ} + \mathbf{R}_{nn}\right) \underline{w} \end{aligned}$$

$$\text{SNIR}_{\text{out}} = \frac{P_{d,out}}{P_{J,out} + P_{n,out}} = \frac{P_1 (\underline{w}^H \underline{S}_1)^2}{\sum_{i=2}^{M} \sum_{j=2}^{M} \rho_{ij}\, \underline{w}^H \underline{S}_i \underline{S}_j^H\, \underline{w} + \sigma_n^2 \underline{w}^H \underline{w}} \qquad (\times \text{ not examinable})$$

- **Outage Probability** (OP): The probability that the output SNIR / SIR falls below a predefined threshold, the lower the better.

$$\text{OP} = \Pr\left(\text{SNIR}_{\text{out}} < \text{SNIR}_{\text{pr}}\right) \text{ or } \Pr\left(\text{SIR}_{\text{out}} < \text{SIR}_{\text{pr}}\right)$$

Three Theoretical Bounds

- **Detection and Resolution Bounds:**
  - The uncertainty σ_e (angular estimation error) decreases with higher SNR and more snapshots:

$$\sigma_e \propto \frac{1}{\sqrt{SNR \cdot L}}$$

  - **Square-root Law (Detection)**: Determines the minimum angular separation for reliable detection. Affected by manifold slope and estimation uncertainty.

$$\Delta\theta_p = f\left(\underline{\dot{S}}, \sigma_e\right)$$

  - **Fourth-root Law (Resolution)**: Determines the minimum angular separation for resolving two closely spaced sources. Affected by manifold slope, curvature, and estimation uncertainty.

$$\Delta\theta_{\text{res}} = f\left(\underline{\dot{S}}, \sqrt[4]{\kappa_1}, \sqrt{\sigma_e}\right)$$

  - In frequency-selective channels, two multipath components cannot be resolved if the delay difference is smaller than the symbol period.
- **Cramér–Rao Bound, CRB (Estimation Accuracy):**
  - The lower bound on estimation error variance, expressed via uncertainty hypersphere radius:

$$\sigma_e = \sqrt{\text{CRB}} = \sqrt{\frac{1}{2(SNR \times L)C}}$$

  - L is the number of snapshots.
  - $C = 1$ for ideal algorithms; $C < 1$ reflects additional uncertainty from imperfect estimators.

5 Advanced Multi-Antenna Wireless Comms

5.1 Increasing Degrees-of-Freedom (DoF)

To address 5G challenges such as high path loss, dense co-channel interference ($M > N$), and multipath fading, two key strategies are proposed to increase the Degrees-of-Freedom (DoF, i.e. N) in beamformer-based communication systems:

- **Solution 1: Massive MIMO**
  - Increase the number of antennas N such that $M < N$ (number of signals < number of sensors).
  - More computational power and hardware are needed.
  - This expands the observation space, enhancing resolution and interference suppression.
  - Two implementations:
    - *Non-parametric massive MIMO*: Antennas operate independently. Problematic as increasing N also increases the number of unknowns.
    - *Parametric massive MIMO*: Antennas are processed jointly. This is effective as $N \uparrow$ but unknowns do not increase proportionally.
- **Solution 2: Spatiotemporal Array Extensions (STAR Manifolds)**
  - Keep N fixed, but use time or structure diversity to virtually extend the array to size $N \cdot N_{\text{ext}}$, so that $M < N \cdot N_{\text{ext}}$.
  - Only computational complexity increases (not hardware).
  - Examples: virtual arrays, spatiotemporal receivers.

5.2 Massive MIMO (MaMI)

Introduction Massive MIMO (maMI) employs a **very large number of antennas** N (no clear definition, e.g., hundreds or thousands) operated coherently and adaptively. This increases spatial resolution and improves throughput and energy efficiency by focusing signal energy more precisely.

- **Use case:** TDD, later extended to FDD. The bigger N, the more signal paths can be detected and resolved, providing more channel parameters, thus increasing the system performance.
- **Benefits:**
  - Higher data rates and reduced latency
  - Simpler MAC layer
  - Improved energy efficiency
  - Better interference suppression and jamming robustness
- **Trade-offs:**
  - Increased hardware complexity (many RF chains needed)
  - High computational complexity for baseband processing
  - Calibration and synchronisation across many RF chains
  - Elevated power consumption and stricter internal efficiency demands
  - Challenging channel estimation using traditional models

5.3 Spatiotemporal Wireless Communications

Introduction Spatiotemporal processing increases the DoF by jointly exploiting both spatial and temporal dimensions. It provides an alternative by extending the array manifold in time.

- **Use case:**
  - **TDMA/FDMA (2G):** few users ($M < N$), array can be used to null co-channel interferences.

  - **CDMA (3G):** many weak interferers ($M > N$), array must handle stronger MAI; spatiotemporal methods can increase DoF.
- **Classification:**
  - **Decoupled Space-Time Receivers:**
    - **Space then Time:** spatial beamforming followed by matched filtering (e.g., OFDM/CDMA). Each path is isolated individually (i.e. no diversity).
    - **Time then Space:** filter bank followed by beamforming. All multipaths are separated and combined (i.e. multipath diversity).
  - **STAR-Rx:** Spatiotemporal Array Receivers integrate space and time into a joint manifold.

Spatiotemporal Manifolds (Extended Manifolds) Spatiotemporal manifolds extend the classical array manifold vector $\underline{S}$, which is only related to **spatial** parameters, to include additional **temporal** parameters. This leads to a high-dimensional manifold $\underline{h}$ constructed from $\underline{S}$ via mappings like:

$$\underline{h} = \underline{S} \otimes f(\text{extension parameters})$$

yielding an $N \cdot N_{\text{ext}}$ dimensional observation space.

Some parameters and examples of extended manifolds are as follows.

Extended parameters	**Examples of extended manifolds: (Not examinable)**
• Time delay ℓ • Polarisation • Subcarriers • Bandwidth • Doppler frequency	• Basic STAR: $\underline{h}^{\text{STAR}} = \underline{S} \otimes \mathbf{J}^{\ell}\underline{c}$ • Polar-STAR: $\underline{h}^{\text{POL-STAR}} = \underline{S} \otimes \underline{q} \otimes \mathbf{J}^{\ell}\underline{c}$ • Doppler-STAR: $\underline{h}^{\text{DOP-STAR}} = \underline{S} \otimes \left(\mathbf{J}^{\ell}\underline{c} \circ \mathcal{F}_D\right)$ • Multicarrier-STAR: $\underline{h}^{\text{MC-STAR}} = \underline{S} \otimes \mathbf{J}^{\ell}\underline{\alpha}[\ell, F_k]$ • Virtual-STAR: $\underline{h}^{\text{virtual-STAR}} = \underline{S} \otimes \overline{\underline{S}}^{*} \otimes \mathbf{J}^{\ell}\underline{c}$

Shifting Matrix J The shifting matrix $\mathbf{J} \in \mathbb{C}^{N_{\text{ext}} \times N_{\text{ext}}}$ is defined as:

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \underline{0}_{N_{\text{ext}}-1}^{T} & 0 \\ \mathbf{I}_{N_{\text{ext}}-1} & \underline{0}_{N_{\text{ext}}-1} \end{bmatrix}$$

Property: $\mathbf{J}^{\ell}\,\underline{x}$ shifts vector $\underline{x}$ down by ℓ positions; its transpose $(\mathbf{J}^{T})^{\ell}\,\underline{x}$ shifts it up by ℓ. In both cases, the newly introduced entries are zero-padded.

The shifting matrix can be used to model delay ℓ in temporal parameters.

Generic Spatiotemporal Architecture and Data Cube The spatiotemporal receiver architecture starts by discretising the received signal $x(t)$ into a high-dimensional vector $x[n] \in \mathbb{C}^{NN_{\text{ext}} \times 1}$ via a manifold extender. This 3D observation (space × delay × symbol) is vectorised for subsequent beamforming. The manifold $\underline{h}$ characterises spatial and temporal channel properties and is estimated through training or blind methods.

Example of STAR Receiver To begin with, we define the extended manifold $\underline{h}_{ij} \in \mathbb{C}^{NN_{\text{ext}} \times 1}$ corresponding to the j-th multipath of the i-th user. It is constructed by combining both spatial and temporal characteristics of the channel. Specifically:

$$\underline{h}_{ij} = \underline{S}_{ij} \otimes \underline{w}_{\xi_{ij}} \in \mathbb{C}^{NN_{\text{ext}} \times 1}$$

where:

- $\underline{S}_{ij} = \underline{S}(\theta_{ij}, \phi_{ij}) \in \mathbb{C}^{N \times 1}$ is the spatial array manifold vector, defined by the direction-of-arrival angles (θ, ϕ).
- $\underline{w}_{\xi_{ij}} \in \mathbb{C}^{N_{\text{ext}} \times 1}$ is the delay manifold vector, determined by the pseudo-random code shift ξ_{ij} for that path.

This STAR formulation allows us to model arbitrary numbers of delayed multipaths from multiple users, with each path being uniquely characterised by its direction and temporal delay. The spatiotemporal manifold $\underline{h}_{ij}$ thus serves as a foundational building block for forming the composite channel matrix $\mathbf{H}_1$ used in the signal model below.

To achieve spatiotemporal extension, the received signal vector $x(t) \in \mathbb{C}^{N \times 1}$ is discretised into $\underline{x}[n] \in \mathbb{C}^{NN_{\text{ext}} \times 1}$, where:

$$N_{\text{ext}} = 2 \times q \times N_c$$

with N_c **being the code period** and q the oversampling factor (e.g. $q = 1$). The extended vector is generated by applying a tapped-delay line to each antenna element.

- **Sampling period** T_s: Defined as $T_s = \frac{T_c}{q}$, where T_c is the chip duration (i.e. the time width of one chip in a spread-spectrum signal) and q is the oversampling factor. When $q = 1$, the received signal $x(t)$ is sampled at the chip rate, so $T_s = T_c$.
- **Discrete delay** ℓ: The delay τ is discretised into an integer index using $\ell = \left\lfloor \frac{\tau}{T_s} \right\rfloor \bmod N_c$. This ensures that $\ell \in [0, N_c)$, enabling proper alignment with the code symbol and maintaining synchronisation.

Assuming $q = 1$, each element $\underline{x}_i[n] \in \mathbb{C}^{2N_c}$ captures the multipath reflections across time within the interval $2N_c$, ensuring the whole data symbol (and ISI paths) are covered.

Assembling $\underline{x}_i[n]$ together, the resulting space-time vector is:

$$\underline{x}[n] = \left[\underline{x}_1[n]^T,\ \underline{x}_2[n]^T,\ \ldots,\ \underline{x}_N[n]^T\right]^T = \text{vec}\left(\mathbf{X}[n]^T\right)$$

where $\mathbf{X}[n] \in \mathbb{C}^{N \times 2N_c}$ stacks the outputs of tapped delays from all N antennas, and $\underline{x}[n] \in \mathbb{C}^{2N_c N \times 1}$.

The overall signal model for M users (assuming user 1 is the desired one) and a frequency-selective channel of K_i multipath components for user i is (the following not examinable):

$$\underline{x}[n] = \mathbf{H}_1\, \underline{m}_1[n] + \underline{I}_{\text{ISI}}[n] + \underline{I}_{\text{MAI}}[n] + \underline{n}[n]$$

where:

- $\mathbf{H}_1 = [\underline{h}_{11}, \underline{h}_{12}, \ldots, \underline{h}_{1K_1}] \in \mathbb{C}^{NN_{\text{ext}} \times K_1}$ contains the STAR manifold vectors for user 1.
- $\underline{m}_1[n] \in \mathbb{C}^{K_1 \times 1}$ is the symbol vector, encoding multipath amplitudes.
- For slow fading: $\underline{m}_1[n] = a[n] \cdot \underline{\beta}_1$, constant across time.
  For fast fading: $\underline{m}_1[n] = a[n] \cdot \underline{\beta}_1[n]$, varies with n.

The ISI term is formed by delayed copies of past symbols:

$$\underline{I}_{\text{ISI}}[n] = \left(\mathbf{I}_N \otimes (\mathbf{J}^T)^{N_c}\right) \mathbf{H}_1\, \underline{m}_1[n-1] + (\mathbf{I}_N \otimes \mathbf{J}_{N_c})\, \mathbf{H}_1\, \underline{m}_1[n+1]$$

where $\mathbf{J}$ is the shifting matrix.

This construction enables a unified modelling of spatial and temporal multipath propagation, and can be used with standard array processing techniques (e.g. beamforming, MUSIC, etc.) on the constructed $\mathbb{R}_{xx} = \mathbb{E}\{\underline{x}[n]\, \underline{x}[n]^H\} \in \mathbb{C}^{NN_{\text{ext}} \times NN_{\text{ext}}}$.

Spatiotemporal Channel Estimation (M > N) Similar to the MUSIC algorithm in conventional arrays ($M < N$), the STAR framework allows blind estimation of manifold parameters when $M > N$. A 2D subspace-based cost function can be used to estimate delay ℓ and angle θ, defined as:

$$\zeta(\theta, \ell) = \frac{1}{\underline{h}(\theta, \ell)^H\, \mathbb{P}_n\, \underline{h}(\theta, \ell)} \quad \text{(not examinable)} \tag{21}$$

where $\mathbb{P}_n$ projects onto the noise subspace of the second-order statistics:

$$\mathbb{R}_{xx} = \mathbb{E}\left\{\underline{x}[n]\, \underline{x}[n]^H\right\}$$

Estimation: STAR CDMA Rx Properties To resolve $M > N$ scenarios (more sources than sensors), subspace methods such as STAR-CDMA receivers use:

- Blind estimation of channel responses (pilot free)
- Separation of desired paths from multi-access interference (MAI)
- High-resolution delay and DOA discrimination via 2D manifold patterns
- Robustness to near–far effect

STAR Reception Pattern The array pattern for spatiotemporal reception is defined as:

$$g(\theta, \ell) = \underline{w}^H \, \underline{h}(\theta, \ell), \quad \text{with } \ell = \left\lfloor \frac{\tau}{T_s} \right\rfloor \bmod N_c$$

where $\underline{w}$ is the beamforming vector and $\underline{h}(\theta, \ell)$ encodes the array response at DOA θ and TOA ℓ. Here delay $\ell \in [0, N_c)$ is the discrete delay index corresponding to integer multiples of the sampling period T_s.

Default pattern:

- For space-only reception: $g(\theta) = \frac{1}{N}\underline{1}_{2NN_c}^H \, \underline{h}(\theta, \ell)$, i.e. $\underline{w} = \underline{1}_{2NN_c}$
- For spatiotemporal reception: $g(\theta, \ell) = (1_N \otimes \underline{c})^T \, \underline{h}(\theta, \ell)$, i.e. $\underline{w} = \underline{h}\,(90^\circ, 0T_c)$

Spatiotemporal Beamformers In STAR beamforming, weights are constructed from the extended manifold matrix

$$\mathbf{H}_i = [\underline{h}_{i1}, \underline{h}_{i2}, \ldots, \underline{h}_{iL_p}] \in \mathbb{C}^{2N_cN \times L_p}$$

where L_p is the number of multipaths.

For single-user (SU-Rx) or multi-user (MU-Rx) cases, the following methods apply:

- **RAKE-type**: $\underline{w}_i = \mathbf{H}_i \, \underline{\beta}_i$, a linear combination of manifold vectors.
- **Subspace-type** (not examinable): $\underline{w}_i = c \cdot \mathbf{P}_n \mathbf{H}_i \left(\mathbf{H}_i^H \mathbf{P}_n \mathbf{H}_i\right)^{-1} \underline{\beta}_i$, where $\mathbf{P}_n$ projects onto the noise subspace. This generalises subspace-based beamforming to the spatiotemporal case.

Capacity Expressions in Spatiotemporal Systems The capacity depends on the type of receiver:

- **SISO:** $C = B \log_2(1 + \mathrm{SNIR}_{\mathrm{out}})$ [bits/sec]
- **MIMO/ST** (not examinable):
  $C = B \log_2 \left(\det(\mathbf{R}_{xx}) / \det(\mathbf{R}_{nn})\right)$ or $C = B \log_2 \det \left(\mathbf{I}_N + \frac{1}{\sigma_n^2} \mathbf{S}_{\mathrm{IR}} \mathbf{m}_m \mathbf{S}_{\mathrm{IR}}^H\right)$
- **In the wideband limit** ($B \to \infty$):
  $C = \mathrm{DoF} \cdot 1.44 \cdot \frac{P_s}{N_0 + N_J}$, where degrees of freedom (DoF) are:
  - SISO: 1;
  - SIMO: N;
  - virtual-SIMO: $\bar{N}N$
  - spatiotemporal-SIMO: NN_{ext};
  - virtual-ST-SIMO: $\bar{N}NN_{\mathrm{ext}}$.

5.4 mmWave Communications

Introduction Millimetre wave (mmWave) communications exploit the underutilised spectrum between 30–300 GHz, with both unlicensed (e.g. 60 GHz, used by IEEE 802.11ad) and licensed bands (e.g. 71–76, 81–86, 92–95 GHz) allocated for high-speed point-to-point links. Compared to sub-6 GHz systems, mmWave offers dramatically wider bandwidths (up to 2 GHz) and supports large-scale antenna arrays (32–256 elements), enabling very high data rates and precise directional beamforming.

- **Advantages:**
  - Achieves high data throughput and low air latency.

  - Strong spatial multiplexing gain via highly directional beams.
  - Helps alleviate spectrum congestion in traditional frequency bands.
- **Challenges:**
  - Severe atmospheric attenuation, especially from rain and molecular absorption at specific frequencies.
  - High free-space path loss and sensitivity to blockage from buildings, people, etc.
  - Limited propagation range, requiring powerful beamforming algorithms.
  - High power consumption and complexity, especially for digital beamformers operating at multi-GHz sampling rates.
  - Need for low-cost RF chains and improved channel estimation under rapid fading and large bandwidths.
- **Solution:**
  - Leverage compact antenna elements to deploy massive MIMO arrays.
  - Implement powerful hybrid or digital beamforming schemes to counteract attenuation and blockage.

Software Defined Radio (SDR) and Digital Beamforming Modern mmWave communication systems increasingly adopt SDR architectures, where traditionally hardware-implemented components (e.g. mixers, filters, modulators, demodulators) are instead realised via software. This provides flexibility and scalability.

A typical system includes **a digital signal processor (DSP) for baseband tasks** and **SDR hardware to handle RF interfacing**. Beamforming operations—especially in massive MIMO systems—can be digitally executed per antenna branch.

A single antenna branch of a digital beamformer consists of signal modulation and downconversion stages. The received signal $m(t)\cos(2\pi F_{RF}t - \psi_k)$ is mixed with a local oscillator, resulting in intermediate frequency $F_{IF} = F_{RF} - F_{LO}$. After filtering and digitisation, the signal is separated into in-phase (I) and quadrature (Q) components and recombined into a baseband complex signal:

$$x_k(t_\ell) = I_2 + jQ_2 = m(t_\ell)e^{j\psi_k}$$

This forms the k-th element of the array manifold vector.

Digital vs Analogue vs Hybrid Beamforming Three beamforming architectures are compared:

- **Digital Beamforming:**
  - Weights applied at baseband; highest flexibility.
  - One transceiver per antenna: high power and cost.
  - Best suited for frequency-selective channels.
- **Analogue Beamforming:**
  - One baseband port with RF-domain phase shifters.
  - Simpler, but limited pattern shaping and flexibility.
  - Lower cost, best for flat channels and wide coverage.
- **Hybrid Beamforming:**
  - A compromise: digital beamforming on limited streams, followed by analogue beamforming.
  - Reduces cost while retaining partial flexibility.
  - Suited for practical mmWave deployments.

	Digital BF	Hybrid BF	Analogue BF
Weights location	Baseband	RF + baseband	RF
Transceivers	N	$< N$	1
Best for	Flexibility, capacity	Trade-off (cost + performance)	Coverage
Channel type	Selective	Both flat and selective	Flat

Digital beamforming achieves high spatial selectivity but requires one complete RF chain per antenna, which increases power consumption and complexity. In contrast, analogue beamformers apply weights early (at RF), reducing the dynamic range requirements of later blocks. Hybrid beamforming offers a power-efficient and scalable solution by balancing digital flexibility with analogue simplicity.

5.5 Distributed Antenna Arrays (LAA)

Introduction Distributed arrays (also referred to as large aperture arrays, LAA) consist of antenna elements placed over a wide physical area. This configuration enables improved spatial diversity and resolution, particularly useful for localisation and tracking.

Wideband vs Narrowband Assumption

- **Narrowband assumption (NB):** Sensors receive the same part of the waveform (i.e. baseband waveform does not change across sensors).

$$\underline{x}(t) = \sum_{i=1}^{M} \underline{S}_i \, m_i(t) + \underline{n}(t)$$
$$\mathbf{R}_{xx} = \sum_{i=1}^{M} P_i \, \underline{S}_i \, \underline{S}_i^H + \mathbf{R}_{nn}$$

- **Wideband assumption (WB):** Sensors receive different parts of the waveform due to large inter-element spacing. This is typical in distributed arrays and leads to the need for a wideband signal model.

$$\underline{x}(t) = \sum_{i=1}^{M} \underline{S}_i \circ \underline{m}_i(t) + \underline{n}(t)$$
$$\mathbf{R}_{xx} = \sum_{i=1}^{M} \underline{S}_i \, \underline{S}_i^H \circ \mathbf{R}_{\underline{m}_i \underline{m}_i} + \mathbf{R}_{nn}$$

- WB applicability depends on *array geometry, source location and signal bandwidth.*
- The spherical wave manifold is given by:

$$\underline{S}_i = \underline{S}(\theta_i, \rho_i, \text{geometry}, f_c)$$

5.6 5G

Introduction

- 5G is the 5th generation of mobile networks, viewed either as a continuation of 4G LTE or a technological revolution.
- Designed to support massive data and connectivity demands (e.g. IoT, smart cities).
- Initially coexists with 4G networks, evolving to standalone deployments.
- Ecosystem includes: D2D communication, M2M, vehicle-to-vehicle, indoor femto, and cloud-controlled architectures.

Revolutionary Changes

- **Latency:** 5G achieves ultra-low latency, enabling real-time wireless interaction.

$$\text{Total latency} \approx \text{UL: } 0.5\,\text{ms} + \text{DL: } 0.5\,\text{ms} = 1\,\text{ms}$$

  - Compared to 3G/4G, this is significantly faster.
  - Enables real-time applications such as tactile internet and autonomous driving.
- **Communication families (ITU-defined):**

  - **eMBB** (enhanced Mobile Broadband): high-speed services like UHD and VR.
  - **mMTC** (massive Machine Type Communications): dense IoT, smart cities.
  - **URLLC / mCC** (Ultra-Reliable Low-Latency Communications): mission-critical links, e.g. robotics, aerospace.

5G Operating Principles

- Operates in high-frequency bands (cmWave/mmWave) enabling more bandwidth.
- Small cells and dense deployment improve coverage and capacity.
- Massive MIMO and beamforming technologies track users and adapt to their locations.
  - Massive MIMO arrays (10s–100s of antennas) for improved gain and spatial multiplexing.
  - Beamforming: ensures high directional gain and efficient spectrum reuse.
  - Tx/Rx implemented via either digital, analogue, or hybrid beamforming architectures.
- Flexibility in numerology: variable subcarrier spacing, slot and subframe lengths.

Key Capacity Enablers

1. **Spectrum Efficiency**: through large arrays and spatiotemporal MIMO.
2. **Spectrum Extension**: up to 100 GHz (mmWave focus between 10–100 GHz).
3. **Network Density**: many small cells + D2D links.
4. **Overall Impact**: up to $10^4\times$ performance over legacy systems.

Standardisation and Capabilities

- 5G = "New Radio" (NR), OFDM-based, with configurable subcarrier spacings:

$$\Delta f = 15 \times 2^m \text{ kHz}, \quad m \in \{0, 1, 2, 3\}$$

- Frame structure: 10 ms frame, 1 ms subframe, variable slot durations.
- Peak data rate: 20 Gbps; latency: 1 ms; mobility: 500 km/h; connection density: $10^6/\text{km}^2$.