Supplementary Material to: Parallax Bundle Adjustment on Manifold with Convexified Initialization

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This report provides theorem proofs, additional derivations and experimental results to support the paper [1]. Therefore, it should not be considered a self-contained document, but rather regarded as an appendix of [1], and cited as:

"L. Liu, T. Zhang, Y. Liu, B. Leighton, L. Zhao, S. Huang, and G. Dissanayake, Parallax bundle adjustment on manifold with convexified initialization, (supplementary material)"

1 The Proof of Theorem 1

Theorem 1. *Under the formulation [1](9),* $\mathbf{H_{FF}}$ *is consistently non-singular for any* $\mathcal{X}$ *and* $\mathbf{H_{FF}} \geq \mathbf{I}$.

Consider the feature j and the corresponding sub-block matrix $\mathbf{H_{FF}}_j$ in $\mathbf{H_{FF}} = blkdiag(\mathbf{H_{FF}}_1, \cdots, \mathbf{H_{FF}}_n)$. Denote $\mathbf{J}_{i,j} = \frac{\partial \mathbf{e}_{i,j}}{\partial \mathbf{F}_j}$ for $(i \in \mathbb{T}_j)$, $\mathbf{J}_{m_j,j}$ is the jacobian for the feature's main anchor observation, $\mathbf{J}_{a_j,j}$ is the jacobian for its associate anchor observation. We have

$$\mathbf{H_{FF}}_j \geq \mathbf{J}^\intercal_{m_j,j}\mathbf{J}_{m_j,j} + \mathbf{J}^\intercal_{a_j,j}\mathbf{J}_{a_j,j}. \tag{1}$$

On the one hand, using (27) in Appendix B , we have

$$\mathbf{J}^\intercal_{m_j,j}\mathbf{J}_{m_j,j} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & (S(\mathbf{n}_j)\mathbf{A}_{\mathbf{n}_j})^\intercal (S(\mathbf{n}_j)\mathbf{A}_{\mathbf{n}_j}) \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix}. \tag{2}$$

On the other hand, we have

$$\mathbf{J}^\intercal_{a_j,j}\mathbf{J}_{a_j,j} \geq \begin{bmatrix} \frac{\partial \mathbf{e}_{i,j}}{\partial \theta_j} & \mathbf{0} \end{bmatrix}^\intercal \begin{bmatrix} \frac{\partial \mathbf{e}_{i,j}}{\partial \theta_j} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} (\partial\xi \mathbf{J}^N_\theta)^\intercal (\partial\xi \mathbf{J}^N_\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{3}$$

where $\partial\xi$ is the jacobian of normalize function with respect to ray $\mathbf{N}_{a_j,j}$, and $\mathbf{J}^N_{\theta_j}$ is the jacobian of $\mathbf{N}_{a_j,j}$ with respect to parallax angle θ_j.

Using equation (28) from Appendix B

$$\mathbf{N}_{a_j,j} = \cos\theta_j \|\mathbf{a} \times \mathbf{n}_m\| \mathbf{n}_m + \sin\theta_j (\mathbf{a} - (\mathbf{a} \cdot \mathbf{n}_m)\mathbf{n}_m). \tag{4}$$

$$\mathbf{J}^N_{\theta_j} = -\sin(\theta)\|\mathbf{a} \times \mathbf{n}_m\| \mathbf{n}_m + \cos(\theta_j)(\mathbf{a} - (\mathbf{a}^\intercal \mathbf{n}_m)\mathbf{n}_m) \tag{5}$$

where

$$\mathbf{a} = \mathbf{p}_{m_j} - \mathbf{p}_{a_j}, \quad \mathbf{b} = \mathbf{p}_{m_j} - \mathbf{p}_i, \quad \mathbf{n}_m = \mathbf{R}_{m_j}\mathbf{n}_{m_j}.$$

Observe that

$$\begin{aligned} &(\mathbf{a} - (\mathbf{a}\cdot\mathbf{n}_m)\mathbf{n}_m) \perp \mathbf{n}_m, \\ &\|(\mathbf{a} - (\mathbf{a}\cdot\mathbf{n}_m)\mathbf{n}_m)\| = \|\mathbf{a}\times\mathbf{n}_m\| = \sin(\pi - \alpha_j), \end{aligned}$$

we have

$$\begin{aligned} \mathbf{N}_{a,j}\cdot\mathbf{J}_\theta^N &= 0 \\ \|\mathbf{N}_{a,j}\| = \|\mathbf{J}_\theta^N\| &= \lambda. \end{aligned}$$

Thus rewrite (4) as

$$\begin{aligned} \mathbf{N}_{a,j} &= \lambda\mathbf{u}, \qquad \mathbf{u}^\intercal\mathbf{u} = 1 \\ \mathbf{J}_\theta^N &= \lambda\mathbf{v}, \qquad \mathbf{v}^\intercal\mathbf{v} = 1 \\ \mathbf{u}\cdot\mathbf{v} &= 0. \end{aligned} \tag{6}$$

Substitute into (3) with the expression for normalize function's jacobian $\partial\xi$ from Appendix B (17), we get

$$\partial\xi\mathbf{J}_\theta^N = (\mathbf{I}_3 - \mathbf{u}\mathbf{u}^\intercal)\mathbf{v} = \mathbf{v}. \tag{7}$$

Now we have

$$\begin{bmatrix} (\partial\xi\mathbf{J}_\theta^N)^\intercal(\partial\xi\mathbf{J}_\theta^N) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{8}$$

Therefore, $\mathbf{H}_{\mathbf{FF}_j} \geq \mathbf{I}_3$ and $\mathbf{H}_{\mathbf{FF}} \geq \mathbf{I}$.

2 Proof of Theorem 2

Theorem 2. *With accurate initial estimate for orientation, the formulation [1](13) can always converge to a near-optimal solution for the BA problem [1](9). Furthermore, the problem [1](13) is convex when EG pairs are noise-free.*

Consider the following function

$$h_{\mathbf{V}}(\mathbf{x}) := \|\check{\mathbf{x}} - \mathbf{V}\|^2, \tag{9}$$

where $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{V} \in \mathbb{R}^3$ ($\|\mathbf{V}\| = 1$). It is a fact that

$$h_{\mathbf{V}}(\mathbf{x} + \lambda\Delta\mathbf{x}) \leq \max\{h_{\mathbf{V}}(\mathbf{x}), h_{\mathbf{V}}(\mathbf{x} + \Delta\mathbf{x})\} \tag{10}$$

for any $\Delta\mathbf{x} \in \mathbb{R}^3$ and any $\lambda \in (0, 1)$.

Considering the problem [1](13) and the linearity of $\bar{\mathbf{N}}_{j,i}$ w.r.t. $\mathbf{p}$, [1](13) can be rewritten as

$$\begin{aligned} \min_{\{\mathbf{p}\}} h(\mathbf{p}, \bar{\mathbf{R}}, \bar{\mathbf{F}}) &= \min_{\{\mathbf{p}\}} \sum_{i\in\mathbb{T}_j, j} \|\frac{\bar{\mathbf{A}}_i\mathbf{p}}{\|\bar{\mathbf{A}}_i\mathbf{p}\|} - \bar{\mathbf{V}}_{ij}\|^2 \\ &= \min_{\{\mathbf{p}\}} \sum_{i\in\mathbb{T}_j, j} h_{\mathbf{V}_i}(\bar{\mathbf{A}}_i\mathbf{p}), \end{aligned} \tag{11}$$

where $\bar{\mathbf{V}}_{ij} := \bar{\mathbf{R}}_i \mathbf{v}_{ij}^{(l)}$ is a directional vector.

Denoting the global minimum of the convex problem in [1](13) as $\bar{\mathbf{p}}$, we have

$$
\begin{aligned}
& h(\bar{\mathbf{p}} + \lambda \Delta\mathbf{p}, \bar{\mathbf{R}}, \bar{\mathbf{F}}) \\
&= \sum_{i \in \mathbb{T}_j, j} h_{\mathbf{V}_i}(\bar{\mathbf{A}}_i(\bar{\mathbf{p}} + \lambda \Delta\mathbf{p})) \\
(using(10)) &\leq \sum_{i \in \mathbb{T}_j, j} \max\{h_{\mathbf{V}_i}(\bar{\mathbf{A}}_i \bar{\mathbf{p}}), h_{\mathbf{V}_i}(\bar{\mathbf{A}}_i(\bar{\mathbf{p}} + \Delta\mathbf{p}))\} \\
&\leq h(\bar{\mathbf{p}}, \bar{\mathbf{R}}, \bar{\mathbf{F}}) + h(\bar{\mathbf{p}} + \Delta\mathbf{p}, \bar{\mathbf{R}}, \bar{\mathbf{F}})
\end{aligned}
\tag{12}
$$

for any $\Delta\mathbf{p} \in \mathbb{R}^{3M}$ and $\lambda \in (0, 1)$. The inequality above indicates a fact: if we perform optimization for the problem [1](13), i.e., solve $\min_{\mathbf{p}} h(\mathbf{p}, \bar{\mathbf{R}}, \bar{\mathbf{F}})$ from any initial guess $\mathbf{p} \in \mathbb{R}^{3M}$, the converged value $\mathbf{p}_{op}$ after optimization will be a near-optimal solution, i.e., $h(\mathbf{p}, \bar{\mathbf{R}}, \bar{\mathbf{F}}) \leq 2h(\bar{\mathbf{p}}, \bar{\mathbf{R}}, \bar{\mathbf{F}})$. When $\bar{\mathbf{R}}$ is close to the optimal estimate, $\mathbf{p}_{op}$ will be also a near-optimal solution of the original PMBA optimization problem [1](9) clearly. Under noise-free condition, $\mathbf{p}_{op}$ is an exact solution due to $0 \leq h(\mathbf{p}, \bar{\mathbf{R}}, \bar{\mathbf{F}}) \leq 2h(\bar{\mathbf{p}}, \bar{\mathbf{R}}, \bar{\mathbf{F}}) \leq 0$ thus the problem [1](13) is convex.

3 More supporting data

We show PMBA test results on our own datasets. Images for the Usyd-mainquad and Victoria-cottage are taken at Sydney University campus using our self-developed portable mapping platform. The fake-pile images are taken from the Google Tango device [2]. All datasets can be downloaded from OpenSLAM[1].

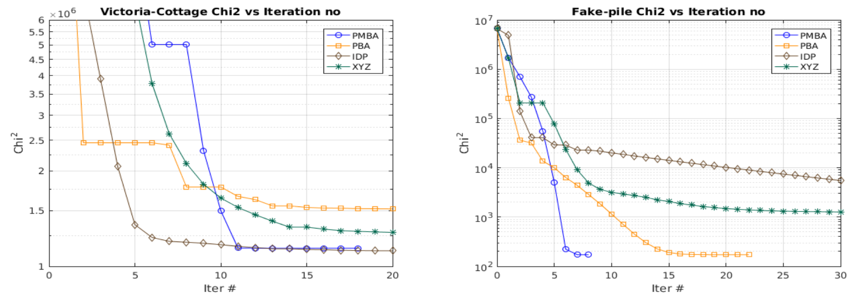
3 .1 PMBA convergence results

In this section we present more convergence plots illustrating the efficiency and accuracy of our PMBA formulation. See Figure 1. The plots show Mean Square Error (labeled as Chi^2) at each stage of iteration. As explained in [1], although PMBA uses a ray direction based cost function. For fairness of comparison, we compute its corresponding UV pixel error. Despite this treatment, PMBA is able to reach good global minimum better than or close to the best results other BA's can achieve.

3 .2 Reconstruction results from PMBA

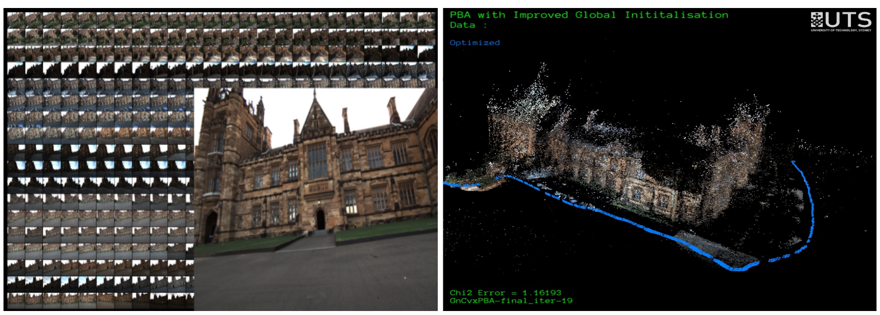
We present reconstruction results from our collected outdoor data sets. See Fig 2 and 3.

[1] https://svn.openslam.org/data/svn/ParallaxBA/



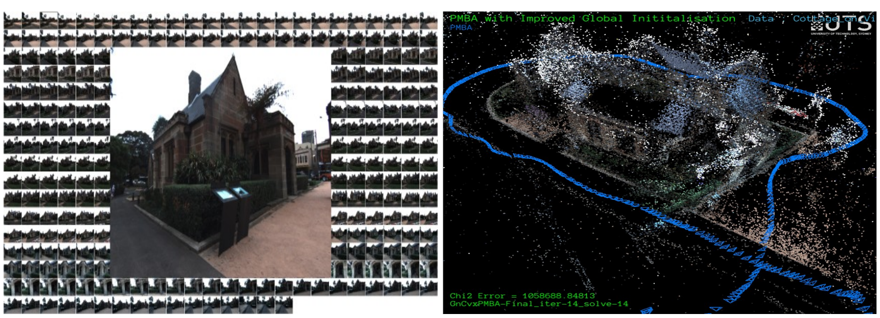
(c) Victoria-cottage (d) Fake-pile

Fig. 1: Convergence plots for PMBA, PBA, IDP and XYZ



(a) Images (b) PMBA output pose and map

Fig. 2: Usyd-mainquad dataset



(a) Images (b) PMBA output pose and map

Fig. 3: Victoria-cottage dataset

3 .3 Results of full PMBA pipeline at various stages

We use publicly available datasets to test our full PMBA pipeline. We use Ladybug-1370, Trafalgar-126 and Venice-427 datasets from "Bundle adjustment in the large" (BAL) database [2], and the College dataset from OpenSLAM. We present screenshots at various stages of our pipeline in in Fig. 4, 5, 6, and 7. In the snapshots, the PMBA pipeline data is in blue, reference data is in red. BAL is the reference data in tests that take BAL inputs. PBA generated results is the reference data in the College dataset. In all figures, column 1 shows optimzation results at QPLC stage, column 2 shows selected iteration results in Convex initialization, column 3 is a typical iteration result in full-PMBA stage and column 4 shows the final map.

The figures shows that reasonable initial values were formed at QPLC stage. The convex pose-graph stage has a very large convergence region such that imperfect outputs from the QPLC stage can gradually converge to motion trajectory with a topology similar to that of Ground Truth. This is especially obvious in "Ladybug-1370" and "Venice-427". . We notice that in "Ladybug-1370", BAL's optimal trajectory (in red) contains an erroneous camera pose, shown as red dot at top right corner of the red trajectory, our method did not encounter this stray pose at all.

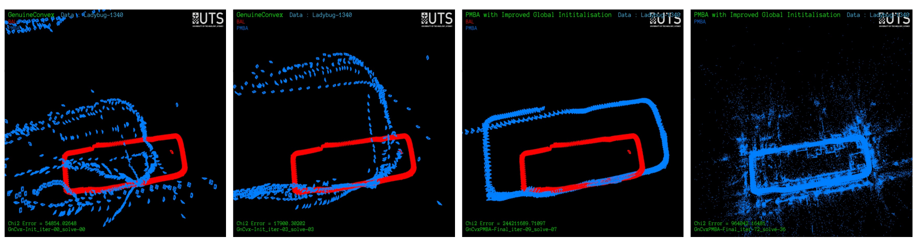


Fig. 4: Ladybug-1370 results

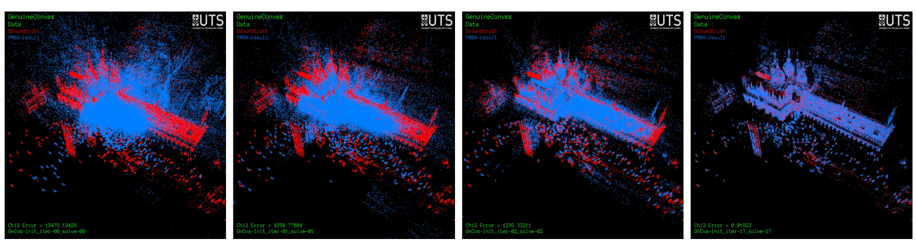


Fig. 5: Venice-427 results

[2] http://grail.cs.washington.edu/projects/bal/

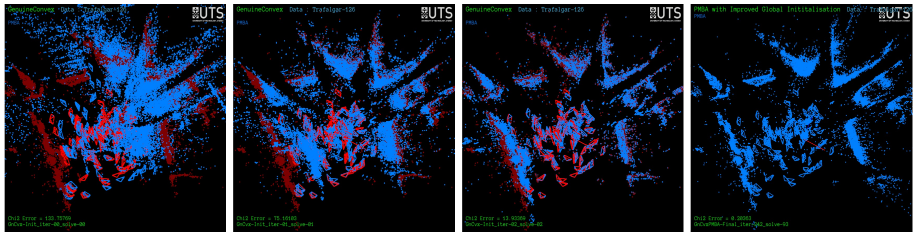


Fig. 6: Trafalgar-126

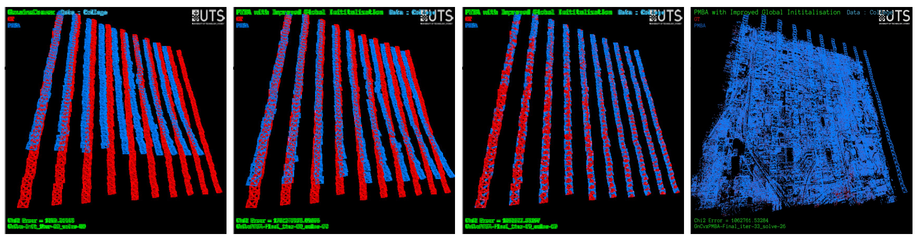


Fig. 7: Ladybug-1370 results

A Appendix – Global ray re-examined

For ease of mathematical manipulation, define

$$\mathbf{a} = \mathbf{p}_{m_j} - \mathbf{p}_{a_j}, \quad \mathbf{b} = \mathbf{p}_{m_j} - \mathbf{p}_i, \quad \mathbf{n}_m = \mathbf{R}_{m_j}\mathbf{n}_{m_j} \tag{13}$$

From Figure [1] 2, we can see that α_j is the angle between $\mathbf{a}$ and $\mathbf{n}_m$, so compute its sine and cosine as

$$\sin(\alpha_j) = \frac{\|\mathbf{a} \times \mathbf{n}_m\|}{\|\mathbf{a}\|\|\mathbf{n}_m\|} = \frac{\|\mathbf{a} \times \mathbf{n}_m\|}{\|\mathbf{a}\|}, \quad \cos(\alpha_j) = \frac{\mathbf{a} \cdot \mathbf{n}_m}{\|\mathbf{a}\|\|\mathbf{n}_m\|} = \frac{\mathbf{a}^\intercal \mathbf{n}_m}{\|\mathbf{a}\|}$$

For easy of subsequent Jacobian derivation. we re-write the expression of observation ray vector from [1](7) as

$$\begin{aligned}
\mathbf{N}_{j,i} &= \sin(\alpha_j - \theta_j)\|\mathbf{p}_{m_j} - \mathbf{p}_{a_j}\|\mathbf{R}_{m_j}\mathbf{n}_j + \sin(\theta_j)(\mathbf{p}_{m_j} - \mathbf{p}_i) \\
&= \sin(\alpha_j - \theta_j)\|\mathbf{a}\|\mathbf{n}_m + \sin(\theta_j)\mathbf{b} \\
&= \cos(\theta_j)\frac{\|\mathbf{a} \times \mathbf{n}_m\|}{\|\mathbf{a}\|}\|\mathbf{a}\|\mathbf{n}_m - \sin(\theta_j)\frac{\mathbf{a}^\intercal \mathbf{n}_m}{\|\mathbf{a}\|}\|\mathbf{a}\|\mathbf{n}_m + \sin(\theta_j)\mathbf{b} \\
&= \cos(\theta_j)\|\mathbf{a} \times \mathbf{n}_m\|\mathbf{n}_m - \sin(\theta_j)(\mathbf{b} - (\mathbf{a}^\intercal \mathbf{n}_m)\mathbf{n}_m)
\end{aligned} \tag{14}$$

B Appendix – Jacobians

From [1](9), an observation's jacobian can be calculated as

$$\frac{\partial \mathbf{e}_{i,j}}{\partial \mathcal{X}_{j,i}} = \frac{\partial \breve{\mathbf{N}}_{j,i}}{\mathcal{X}_{j,i}} \tag{15}$$

B .1 Jacobian of observation ray

The global ray direction in [1](9) $\breve{\mathbf{N}}_{j,i}$ can be computed as

$$\breve{\mathbf{N}}_{j,i} = \xi(\mathbf{N}), \quad \xi(\mathbf{x}) \quad = \frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{16}$$

where ξ is the normalization function. And its derivative is

$$\partial\xi(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}\mathbf{I}_3 - \frac{1}{\|\mathbf{x}\|^3}\mathbf{x}\mathbf{x}^\intercal \tag{17}$$

Therefore (15) becomes

$$\frac{\partial \mathbf{e}_{i,j}}{\partial \mathcal{X}_{j,i}} = \partial\xi(\mathbf{N}_{j,i})\frac{\partial \mathbf{N}_{j,i}}{\mathcal{X}_{j,i}} \tag{18}$$

Define Jacobians of ray $\mathbf{N}$ with respect to $\mathbf{n}_m$, $\mathbf{a}$, $\mathbf{b}$ and parallax angle θ_j as

$$\begin{aligned}
\mathbf{J}^N_{n_m} &= \frac{\partial \mathbf{N}}{\partial \mathbf{n}_m} = \cos(\theta_j)(\|S(\mathbf{a})\mathbf{n}_m\|\mathbf{I}_3 + \mathbf{n}_m\frac{(S(\mathbf{a})\mathbf{n}_m)^\intercal}{\|S(\mathbf{a})\mathbf{n}_m\|}S(\mathbf{a})) - \sin(\theta_j)(\mathbf{n}_m\mathbf{a}^\intercal + (\mathbf{a}^\intercal\mathbf{n}_m)\mathbf{I}_3)\\
\mathbf{J}^N_{a} &= \frac{\partial \mathbf{N}}{\partial \mathbf{a}} = -\cos(\theta)\mathbf{n}_m\frac{(S(\mathbf{a})\mathbf{n}_m)^\intercal}{\|S(\mathbf{a})\mathbf{n}_m\|}S(\mathbf{n}_m) - \sin(\theta_j)\mathbf{n}_m\mathbf{n}_m^\intercal\\
\mathbf{J}^N_{b} &= \frac{\partial \mathbf{N}}{\partial \mathbf{b}} = \sin\theta_j\\
\mathbf{J}^N_{\theta} &= \frac{\partial \mathbf{N}}{\partial \theta_j} = -\sin(\theta)\|\mathbf{a}\times\mathbf{n}_m\|\mathbf{n}_m + \cos(\theta_j)(\mathbf{b} - (\mathbf{a}^\intercal\mathbf{n}_m)\mathbf{n}_m)
\end{aligned} \tag{19}$$

The Jacobian of $\mathbf{N}$ w.r.t. the main anchor is

$$\partial_m\mathbf{N} = \mathbf{J}^N_{n_m}\partial_m\mathbf{n}_m + \mathbf{J}^N_{a}\partial_m\mathbf{a} + \sin(\theta)_j)\partial_m\mathbf{a} \tag{20}$$

The Jacobian of $\mathbf{N}$ w.r.t. the associated anchor is

$$\partial_a\mathbf{N} = \mathbf{J}^N_{a}\partial_a\mathbf{a} \tag{21}$$

The Jacobian of $\mathbf{N}$ w.r.t. the current pose is

$$\partial_i\mathbf{N} = -\sin(\theta_j)\mathbf{I}_3 \tag{22}$$

The Jacobian of $\mathbf{N}$ w.r.t. the feature is

$$\partial_f\mathbf{N} = \mathbf{J}^N_{n_m}(\partial_f\mathbf{n}_m) + \mathbf{J}^N_{\theta} \tag{23}$$

The Jacobians of $\mathbf{n}_m$

$$\begin{aligned}
\partial_m\mathbf{n}_m &= \begin{bmatrix}-\mathbf{R}_{m_j}S(\mathbf{n}_j) & \mathbf{0}_{3,3}\end{bmatrix},\\
\partial_f\mathbf{n}_m &= \begin{bmatrix}-\mathbf{R}_{m_j}S(\mathbf{n}_j)\mathbf{C}\end{bmatrix},
\end{aligned} \tag{24}$$

The Jacobians of $\mathbf{a}$

$$\begin{aligned}
\partial_m\mathbf{a} &= \begin{bmatrix}\mathbf{0}_{3,3} & \mathbf{R}_m\end{bmatrix}\\
\partial_a\mathbf{a} &= \begin{bmatrix}\mathbf{0}_{3,3} & -\mathbf{R}_a\end{bmatrix}
\end{aligned} \tag{25}$$

B .2 Jacobians of observations to state variables

The jacobian for an observation (factor) with respect to each involved state variable (vertex) is given in Table 1.

Table 1: Jacobian for a factor node

Involved vertices	Jacobian calculation
Main anchor	$\partial\xi\ \partial_m\mathbf{N}$
Associated Achor	$\partial\xi\ \partial_a\mathbf{N}$
Jacobian For Pose i	$\partial\xi\ \partial_i\mathbf{N}$
Jacobian For feature $(\mathbf{n},\theta)$	$\partial\xi\ \partial_f\mathbf{N}$

B .3 Special factor m_j: current pose is main anchor

Involved vertexes : Obviously, this factor only involves one vertex – feature $(\mathbf{n},\theta)$.

Error function

$$\begin{aligned}
&\mathbf{e}_{i,j} = \xi(\mathbf{N}_{m_j,j}) - \mathbf{z},\ \mathbf{N}_{m_j,j} = \mathbf{R}_{m_j}\mathbf{n},\ \|\mathbf{N}_{m_j,j}\| = 1 \\
&\partial\xi(\mathbf{N}_{m_j,i}) = 1
\end{aligned} \tag{26}$$

Jacobian For $(\mathbf{n},\boldsymbol{\theta})$

$$\begin{bmatrix} 0 & S(\mathbf{n})\mathbf{A}_n \end{bmatrix} \tag{27}$$

where n is the feature's local direction vector and $\mathbf{A}_n$ is its the null space.

B .4 Special factor a_j: current pose is associate anchor

This factor involves three vertices: $\mathbf{X}_m$, $\mathbf{X}_a$ and $(\mathbf{n},\theta)$.
Since $\mathbf{p}_{a_j}$ and $\mathbf{p}_i$ are the same positions, we have $\mathbf{a} = \mathbf{b}$, so the Jacobians for ray $\mathbf{N}_{j,a_j}$ can be simplified as:

$$\begin{aligned}
\mathbf{N}_{j,a_j} &= \cos(\theta_j)\|\mathbf{a}\times\mathbf{n}_m\|\mathbf{n}_m + \sin(\theta_j)(\mathbf{a} - (\mathbf{a}^\intercal\mathbf{n}_m)\mathbf{n}_m) \\
\mathbf{J}^N_{\theta_j} &= -\sin(\theta)\|\mathbf{a}\times\mathbf{n}_m\|\mathbf{n}_m + \cos(\theta_j)(\mathbf{a} - (\mathbf{a}^\intercal\mathbf{n}_m)\mathbf{n}_m) \\
\mathbf{J}^N_a &= -\cos(\theta_j)\mathbf{n}_m\frac{(S(\mathbf{a})\mathbf{n}_m)^\intercal}{\|S(\mathbf{a})\mathbf{n}_m\|}S(\mathbf{n}_m) - \sin(\theta_j)\mathbf{n}_m\mathbf{n}_m^\intercal + \sin(\theta_j)\mathbf{I}_3
\end{aligned} \tag{28}$$

Note that $\mathbf{J}^N_{n_m}$ is same as (19).

References

1. L. Liu, T. Zhang, Y. Liu, B. Leighton, L. Zhao, Huang, S., Dissanayake, G.: Parallax bundle adjustment on manifold with convexified initialization. Submitted to WAFR 2018
2. Liu, L., Wang, Y., Zhao, L., Huang, S.: Evaluation of Different SLAM Algorithms using Google Tangle Data. In: 2017 IEEE Conference on Industrial Electronics and Applications. 10.1109/ICIEA.2017.8283158