



STOR 320 Modeling I

Lecture 15

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Introduction

- Read Vigorously
 - Part IV in R4DS
 - Chapters 6 and 7 in ModernDive
- Goal: Understand the Relationship Between Variables
- Purpose:
 - Explanation
 - Prediction
- Classic Model:
 - Single Outcome Variable (Y)
 - Multiple Predictor Variables (X_1, X_2, \dots, X_P)
 - Multiple Regression Model:
$$Y = \beta_0 + \beta_1 X_1 + \dots + \beta_P X_P + \varepsilon$$



Linear Regression

- Model Deconstruction:

$$Y = \underbrace{\beta_0 + \beta_1 X_1 + \cdots + \beta_P X_P}_{\text{Signal}} + \underbrace{\varepsilon}_{\text{Noise}}$$

- Noise: Unexplainable Error
 - $E(\varepsilon) = 0$
 - $Var(\varepsilon) = \sigma^2$
- Signal: Helps Us Understand in the Variation in Y
 - $E(Y|X_1, \dots, X_P)$ = Expected Value of Y Given Information about X_1, \dots, X_P
 - Used For Prediction/Explanation



Linear Regression

- Once We Have Data
 - Estimate the Parameters

$$(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_P)$$

- Calculate the Fitted Values

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \dots + \hat{\beta}_P X_P$$

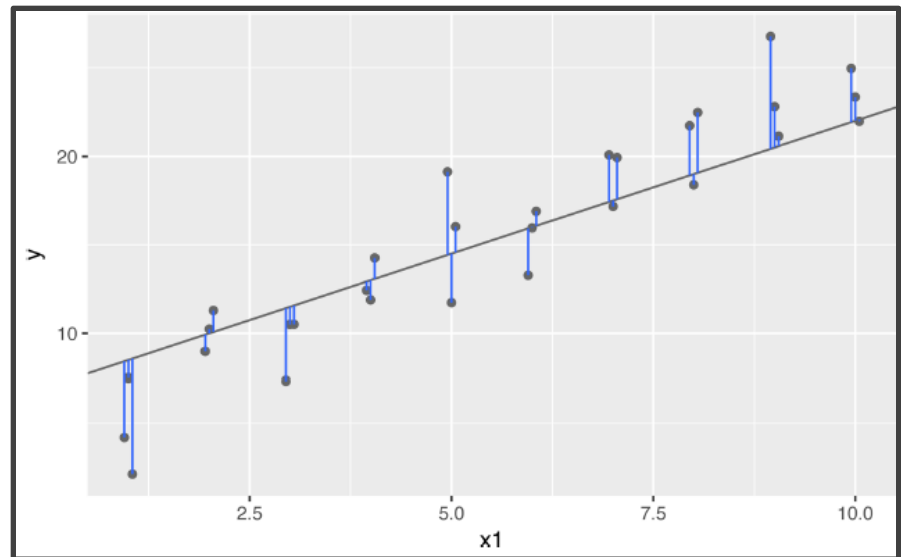
- Obtain the Residuals

$$\hat{\varepsilon} = Y - \hat{Y}$$

- Evaluate the Noise ($\hat{\sigma}^2$)
 - Key: Pick Estimates $(\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_P)$ where $\hat{\varepsilon} \approx 0$ and $\hat{\sigma}^2$ is Small

Optimization

- Optimization Problem:
 - One Outcome Variable (Y)
 $y_1, y_2, y_3 \dots, y_n$
 - One Predictor Variable (X)
 $x_1, x_2, x_3 \dots, x_n$
 - Choice of $\hat{\beta}_0$ and $\hat{\beta}_1$
$$\hat{\varepsilon}_k = y_k - \hat{y}_k$$
$$= y_k - (\hat{\beta}_0 + \hat{\beta}_1 x_k)$$





Optimization

- Optimization Problem (Cont.):
 - Loss Functions:

- Sum of Squared Errors $SSE = \sum \hat{\epsilon}_k^2$
- Mean Squared Error $MSE = \frac{1}{N} \sum \hat{\epsilon}_k^2$
- Root MSE $RMSE = \sqrt{\frac{1}{N} \sum \hat{\epsilon}_k^2}$

- Mean Absolute Error $MAE = \frac{1}{N} \sum |\hat{\epsilon}_k|$



Family of Models

- Family of Models:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

- Empty Model: $Y = \beta_0 + \varepsilon$
- 1 Coefficient:
 - $Y = \beta_0 + \beta_1 X_1 + \varepsilon$
 - $Y = \beta_0 + \beta_2 X_2 + \varepsilon$
- 2 Coefficients:
 - $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$
- Fact: Adding More Predictor Variables Will Always Cause the Loss Function to Decrease



Data Partition

- Good Practice:
 - Randomly Split Full Dataset Into Two Datasets
 - Training Data
 - 80%-90% of Original Data
 - Used for Model Fitting
 - Testing Data
 - 20%-10% of Original Data
 - Used for Model Selection



Example

- Modeling Real Experimental Data
 - Question: What Factors Improve Hourly Salary?

- Hypothesis 1: Experience



- Hypothesis 2: Education





Example

- Modeling Real Experimental Data
 - Data From 10,000 Individuals
 - X_1 = Experience (# of Years)
 - X_2 = Education (# of Years)
 - Y = Salary (dollars/hour)
 - Preview of Data:

```
## # A tibble: 6 x 3
##   salary experience education
##   <dbl>         <int>      <int>
## 1   47.9           27         9
## 2   37.8           24         2
## 3   35.6           19         7
## 4   34.0           17         8
## 5   39.7           25         4
## 6   37.4           23         5
```



Example

```
set.seed(216)
DATA$SPLIT=sample(x=c("TRAIN", "TEST"), size=10000,
                  replace=T, prob=c(0.85, 0.15))
TRAIN=DATA %>% filter(SPLIT=="TRAIN")
TEST=DATA %>% filter(SPLIT=="TEST")
glimpse(TRAIN)
```

```
## Rows: 8,525
## Columns: 4
## $ salary      <dbl> 37.78150, 39.69892, 37.43090, 43.21785, 25.81015, 30.99309...
## $ experience  <int> 24, 25, 23, 27, 8, 17, 23, 15, 28, 20, 13, 29, 19, 24, 29,...
## $ education   <int> 2, 4, 5, 4, 5, 5, 3, 7, 0, 3, 4, 7, 6, 7, 4, 3, 6, 7, 5, 7...
## $ SPLIT       <chr> "TRAIN", "TRAIN", "TRAIN", "TRAIN", "TRAIN", "TRAIN", "TRAIN", "TRA...
```

```
glimpse(TEST)
```

```
## Rows: 1,475
## Columns: 4
## $ salary      <dbl> 47.88340, 35.60634, 34.00961, 39.45598, 38.36103, 50.13343...
## $ experience  <int> 27, 19, 17, 22, 21, 30, 21, 24, 22, 16, 13, 20, 31, 21, 23...
## $ education   <int> 9, 7, 8, 8, 8, 9, 15, 8, 9, 8, 11, 9, 7, 10, 13, 8, 10, 8,...
## $ SPLIT       <chr> "TEST", "TEST", "TEST", "TEST", "TEST", "TEST", "TEST", "TEST", "T...
```



Empty Model (MODEL 0)

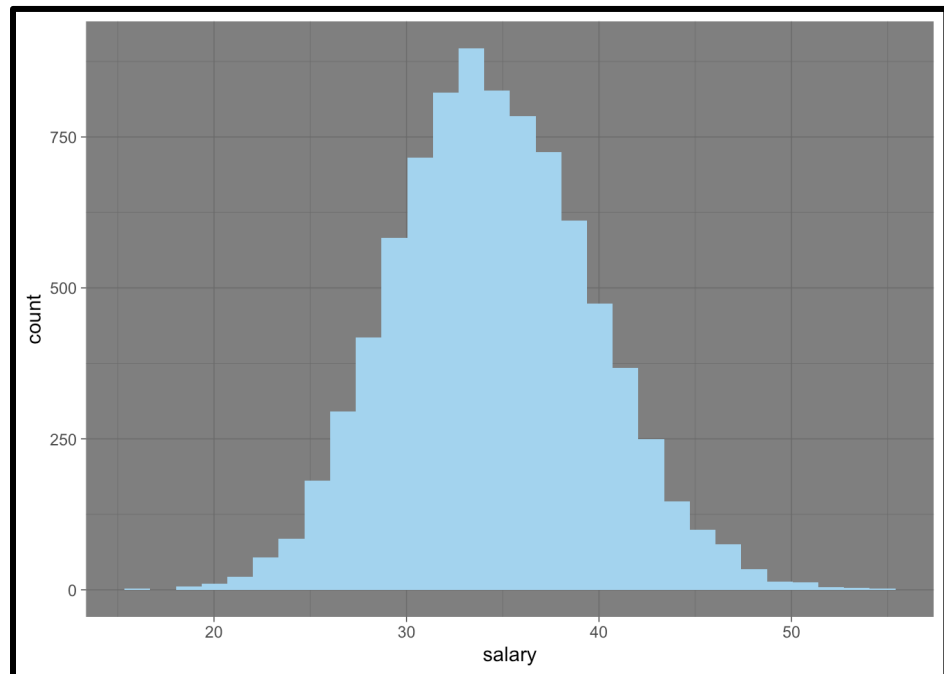
- MODEL 0

$$Y = \beta_0 + \varepsilon$$

$$E(Y) = \beta_0$$

- Summary of Salary

```
## # A tibble: 1 x 4
##   mean    sd   min   max
##   <dbl> <dbl> <dbl> <dbl>
## 1  34.5   5.15  16.1  54.8
```





Empty Model (MODEL 0)

- Function to Get Fitted Values:

```
MODEL0 = function(DATA, COEF) {  
  FIT=COEF[1]  
}
```

- Functions to Evaluate Model:

```
MSE0=function(DATA, COEF) {  
  ERROR=DATA$salary-MODEL0(DATA, COEF)  
  LOSS=mean(ERROR^2)  
  return(LOSS)  
}  
MAE0=function(DATA, COEF) {  
  ERROR=DATA$salary-MODEL0(DATA, COEF)  
  LOSS=mean(abs(ERROR))  
  return(LOSS)  
}
```



Empty Model: Optimization

- Optimization
 - Specify Possible Values of $\hat{\beta}_0$

```
COEF0=tibble(  
  beta0=seq(16,55,length=100)  
)
```

- All Possible Models





Empty Model: Optimization

- Optimization
 - We Desire to Find the $\hat{\beta}_0$ that Minimizes MSE and MAE
 - `map()`: purr Package

```
COEF0 %>%  
  mutate(MSE=purrr::map_dbl(beta0,MSE0,DATA=TRAIN),  
         MAE=purrr::map_dbl(beta0,MAE0,DATA=TRAIN),  
         rankMSE=rank(MSE),rankMAE=rank(MAE)) %>%  
  filter(rankMSE<5,rankMAE<5)
```

```
## # A tibble: 3 x 5  
##   beta0    MSE    MAE rankMSE rankMAE  
##   <dbl> <dbl> <dbl>   <dbl>   <dbl>  
## 1  34.1  26.7  4.13     3       2  
## 2  34.5  26.5  4.13     1       1  
## 3  34.9  26.7  4.15     2       3
```



Empty Model: Optimization

- Optimization
 - `optim()`: Base R

```
BESTMSE0=optim(par=16,fn=MSE0,DATA=TRAIN)
BESTMSE0$par

## [1] 34.53125

BESTMAE0=optim(par=16,fn=MAE0,DATA=TRAIN)
BESTMAE0$par

## [1] 34.34375
```

Starting Values

- `lm()`: Base R (Linear Reg)

```
LM0=lm(salary~1,data=TRAIN)
coef(LM0)

## (Intercept)
## 34.53428
```



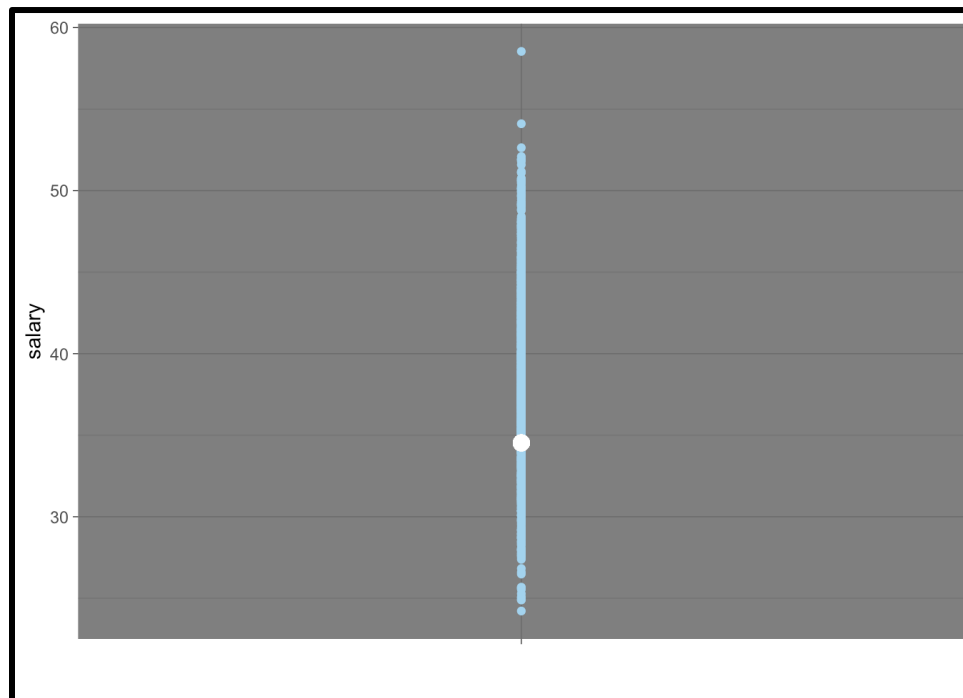

Empty Model: Final Model

- Final MODEL 0

$$Y = 34.53 + \varepsilon$$

$$E(Y) = 34.53$$

- Prediction on Test Data:





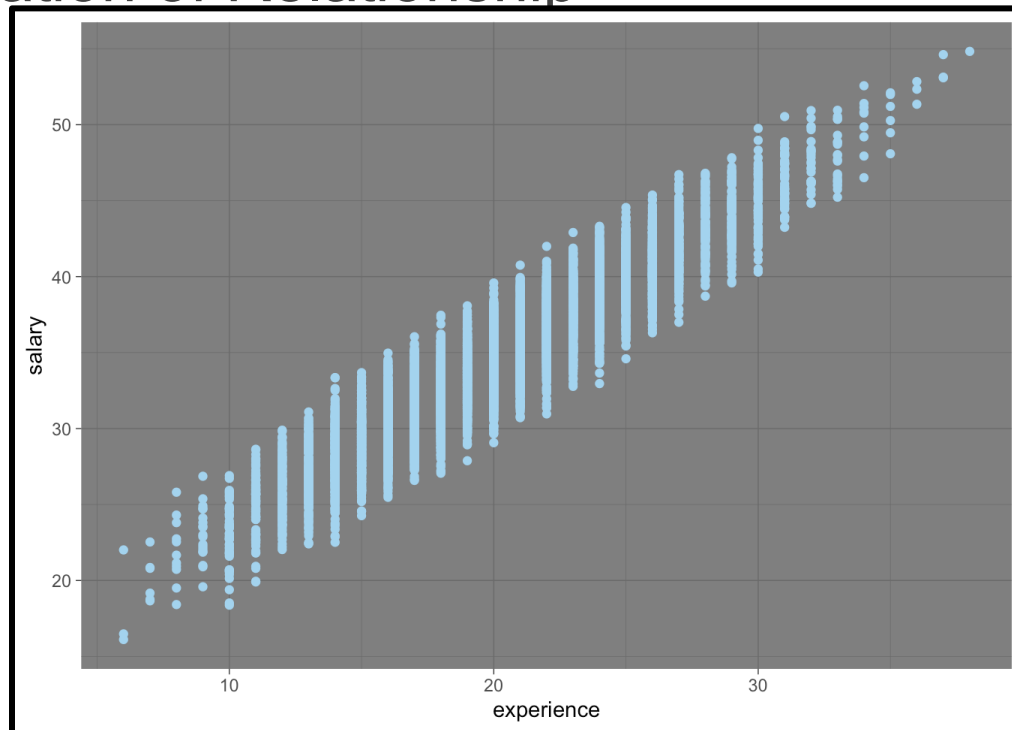
Model 1A

- MODEL 1A

$$Y = \beta_0 + \beta_1 X_1 + \varepsilon$$

$$E(Y) = \beta_0 + \beta_1 X_1$$

- Visualization of Relationship





Model 1A

- Function to Get Fitted Values

```
MODEL1A = function(DATA, COEF) {  
  FIT=COEF[1]+COEF[2]*DATA$experience  
}
```

- Functions to Evaluate Model

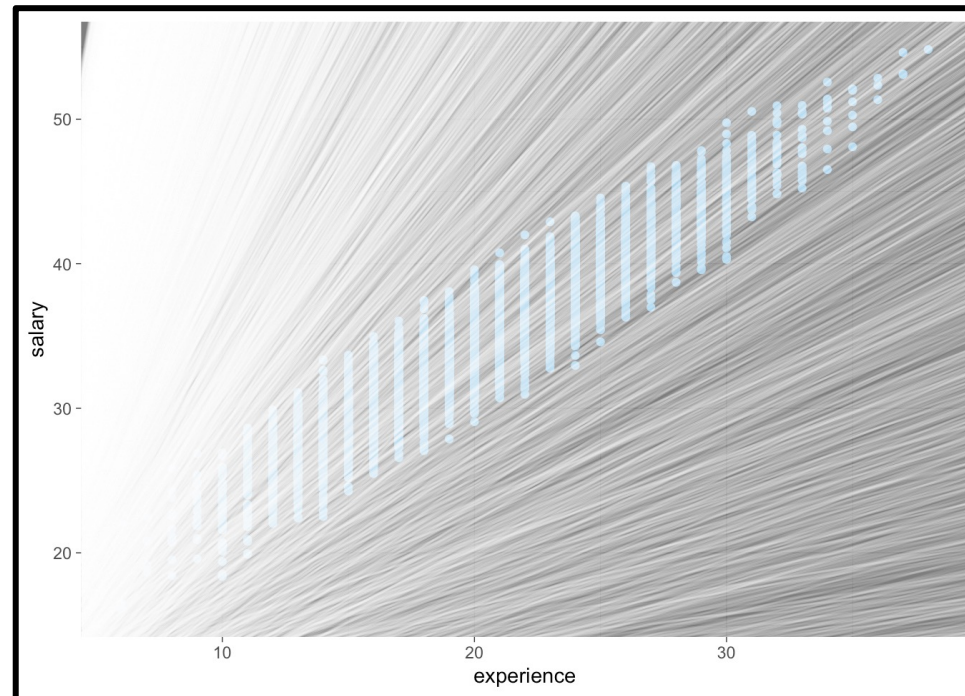
```
MSE1A=function(DATA, COEF) {  
  ERROR=DATA$salary-MODEL1A(DATA, COEF)  
  LOSS=mean(ERROR^2)  
  return(LOSS)  
}  
MAE1A=function(DATA, COEF) {  
  ERROR=DATA$salary-MODEL1A(DATA, COEF)  
  LOSS=mean(abs(ERROR))  
  return(LOSS)  
}
```



Model 1A: Optimization

- Optimization
 - Possible Values of $\hat{\beta}_0$ and $\hat{\beta}_1$
 - All Possible Models

```
set.seed(216)
COEF1A=tibble(
  beta0=runif(10000,0,10),
  beta1=runif(10000,0,10)
)
```





Model 1A: Optimization

- Optimization
 - Use of apply() Function

```
COEF1A %>%  
  mutate(MSE=apply(COEF1A,1,MSE1A,DATA=TRAIN),  
         MAE=apply(COEF1A,1,MAE1A,DATA=TRAIN),  
         rankMSE=rank(MSE),rankMAE=rank(MAE)) %>%  
  filter(rankMSE<5,rankMAE<5)
```

```
## # A tibble: 4 x 6  
##   beta0 beta1  MSE  MAE rankMSE rankMAE  
##   <dbl> <dbl> <dbl> <dbl>   <dbl>   <dbl>  
## 1  9.48  1.26  4.09  1.63     3     3  
## 2  9.46  1.23  4.00  1.62     2     2  
## 3  9.84  1.20  4.12  1.65     4     4  
## 4  9.40  1.24  3.96  1.61     1     1
```



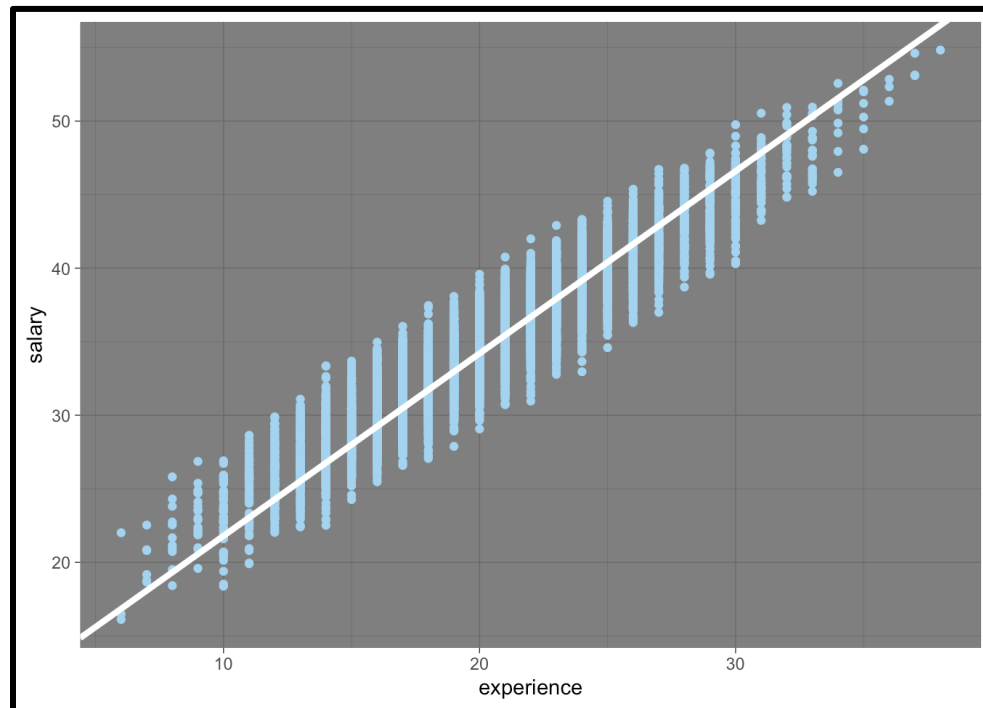
Model 1A: Final Model

- Final MODEL 1A

$$Y = 9.4 + 1.24X_1 + \varepsilon$$

$$E(Y) = 9.4 + 1.24X_1$$

- Fitted on Train Data





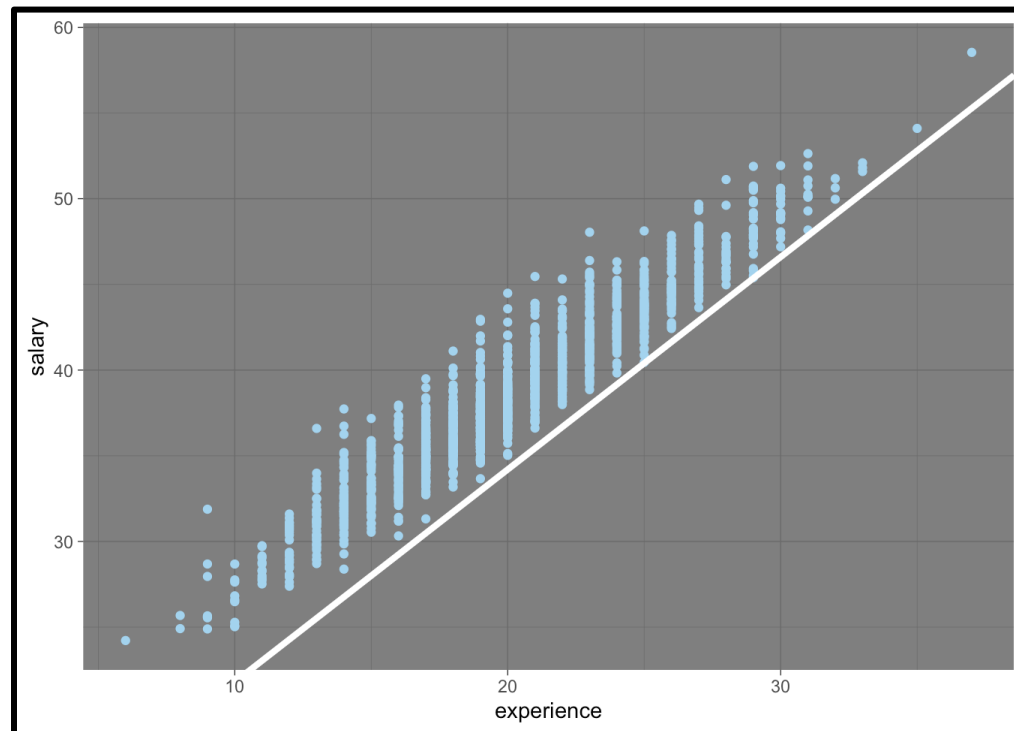
Model 1A: Final Model

- Final MODEL 1A

$$Y = 9.4 + 1.24X_1 + \varepsilon$$

$$E(Y) = 9.4 + 1.24X_1$$

- Prediction on Test Data





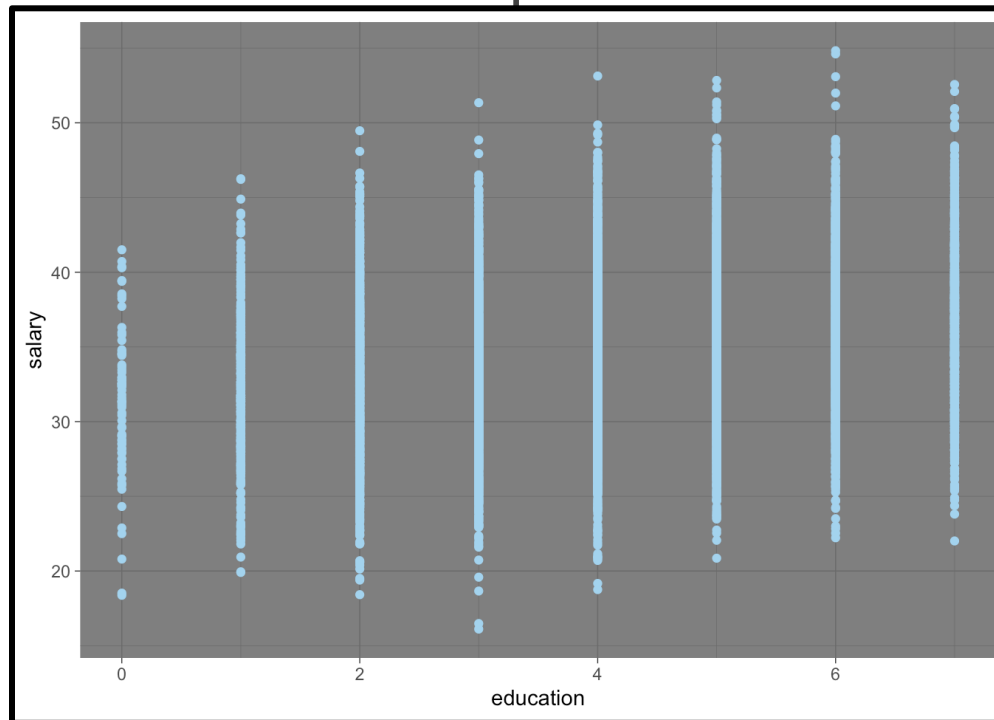
Model 1B

- MODEL 1B

$$Y = \beta_0 + \beta_1 X_2 + \varepsilon$$

$$E(Y) = \beta_0 + \beta_1 X_2$$

- Visualization of Relationship





Model 1B

- Function to Get Fitted Values

```
MODEL1B = function(DATA, COEF) {  
  FIT = COEF[1] + COEF[2] * DATA$education  
}
```

- Functions to Evaluate Model

```
MSE1B = function(DATA, COEF) {  
  ERROR = DATA$salary - MODEL1B(DATA, COEF)  
  LOSS = mean(ERROR^2)  
  return(LOSS)  
}  
  
MAE1B = function(DATA, COEF) {  
  ERROR = DATA$salary - MODEL1B(DATA, COEF)  
  LOSS = mean(abs(ERROR))  
  return(LOSS)  
}
```



Model 1B: Optimization

- Optimization
 - Use of optim() Function

```
BESTMSE1B=optim(par=c(0,0),fn=MSE1B,DATA=TRAIN)  
BESTMSE1B$par
```

```
## [1] 30.8323639 0.8543225
```

```
BESTMAE1B=optim(par=c(0,0),fn=MAE1B,DATA=TRAIN)  
BESTMAE1B$par
```

```
## [1] 30.6619753 0.8512186
```



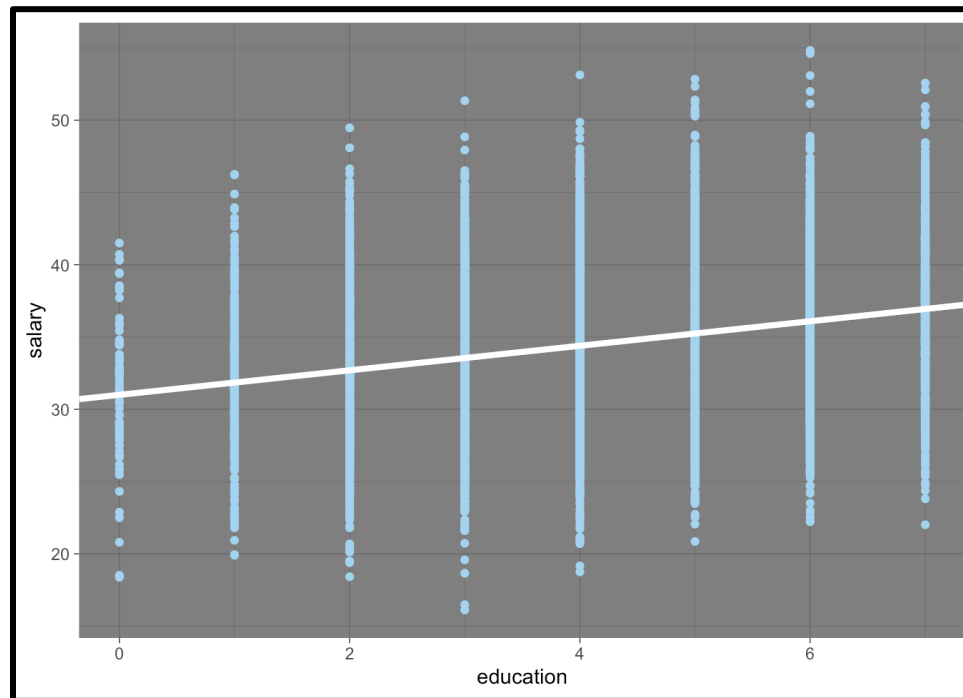
Model 1B: Final Model

- Final MODEL 1B

$$Y = 31 + 0.85X_2 + \varepsilon$$

$$E(Y) = 31 + 0.85X_2$$

- Fitted on Train Data





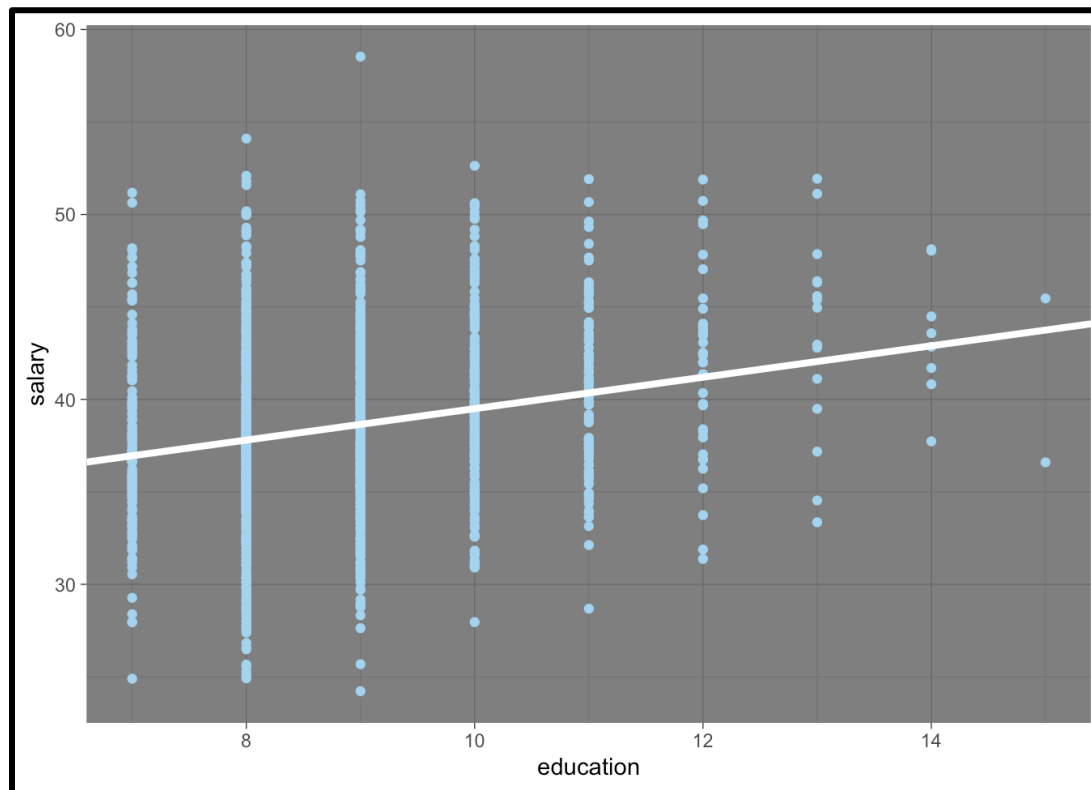
Model 1B: Final Model

- Final MODEL 1B

$$Y = 31 + 0.85X_2 + \varepsilon$$

$$E(Y) = 31 + 0.85X_2$$

- Prediction on Test Data





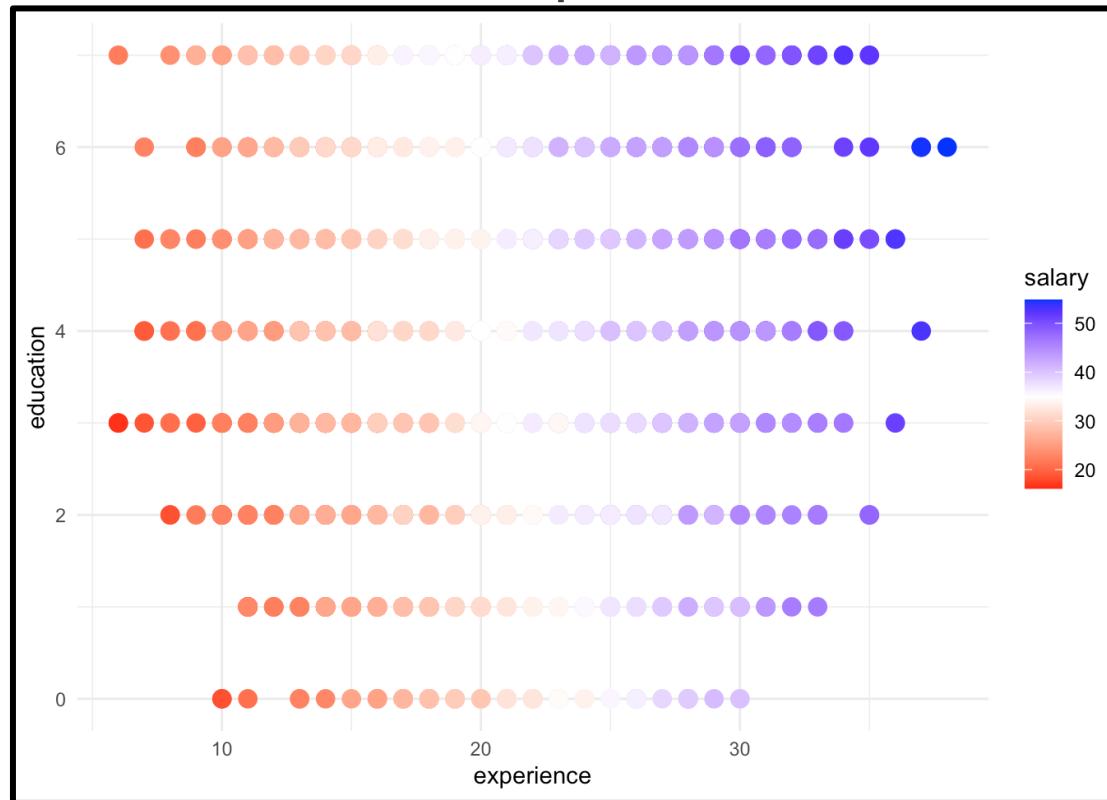
MODEL 2

- MODEL 2

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

$$E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

- Visualization of Relationship





MODEL 2

- Function to Get Fitted Values

```
MODEL2 = function(DATA, COEF) {  
  FIT = COEF[1] + COEF[2] * DATA$experience + COEF[3] * DATA$education  
}
```

- Functions to Evaluate Model

```
MSE2 = function(DATA, COEF) {  
  ERROR = DATA$salary - MODEL2(DATA, COEF)  
  LOSS = mean(ERROR^2)  
  return(LOSS)  
}  
MAE2 = function(DATA, COEF) {  
  ERROR = DATA$salary - MODEL2(DATA, COEF)  
  LOSS = mean(abs(ERROR))  
  return(LOSS)  
}
```



Multiple Regression

- Use `lm()` with `summary()`
- Final MODEL 2

$$Y = 9 + 1.08X_1 + 0.9X_2 + \varepsilon$$

$$E(Y) = 9 + 1.08X_1 + 0.9X_2$$

```
LM2=lm(salary~experience+education,data=TRAIN)
summary(LM2)

##
## Call:
## lm(formula = salary ~ experience + education, data = TRAIN)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -3.6426 -0.6776 -0.0138  0.6838  3.7675
##
## Coefficients:
##              Estimate Std. Error t value      Pr(>|t|)
## (Intercept)  8.996672   0.058760   153.1 <0.0000000000000002 ***
## experience   1.079243   0.002474   436.3 <0.0000000000000002 ***
## education    0.902851   0.006635   136.1 <0.0000000000000002 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.025 on 8522 degrees of freedom
## Multiple R-squared:  0.9605, Adjusted R-squared:  0.9604
## F-statistic: 1.035e+05 on 2 and 8522 DF, p-value: < 0.00000000000000022
```

Model Summary



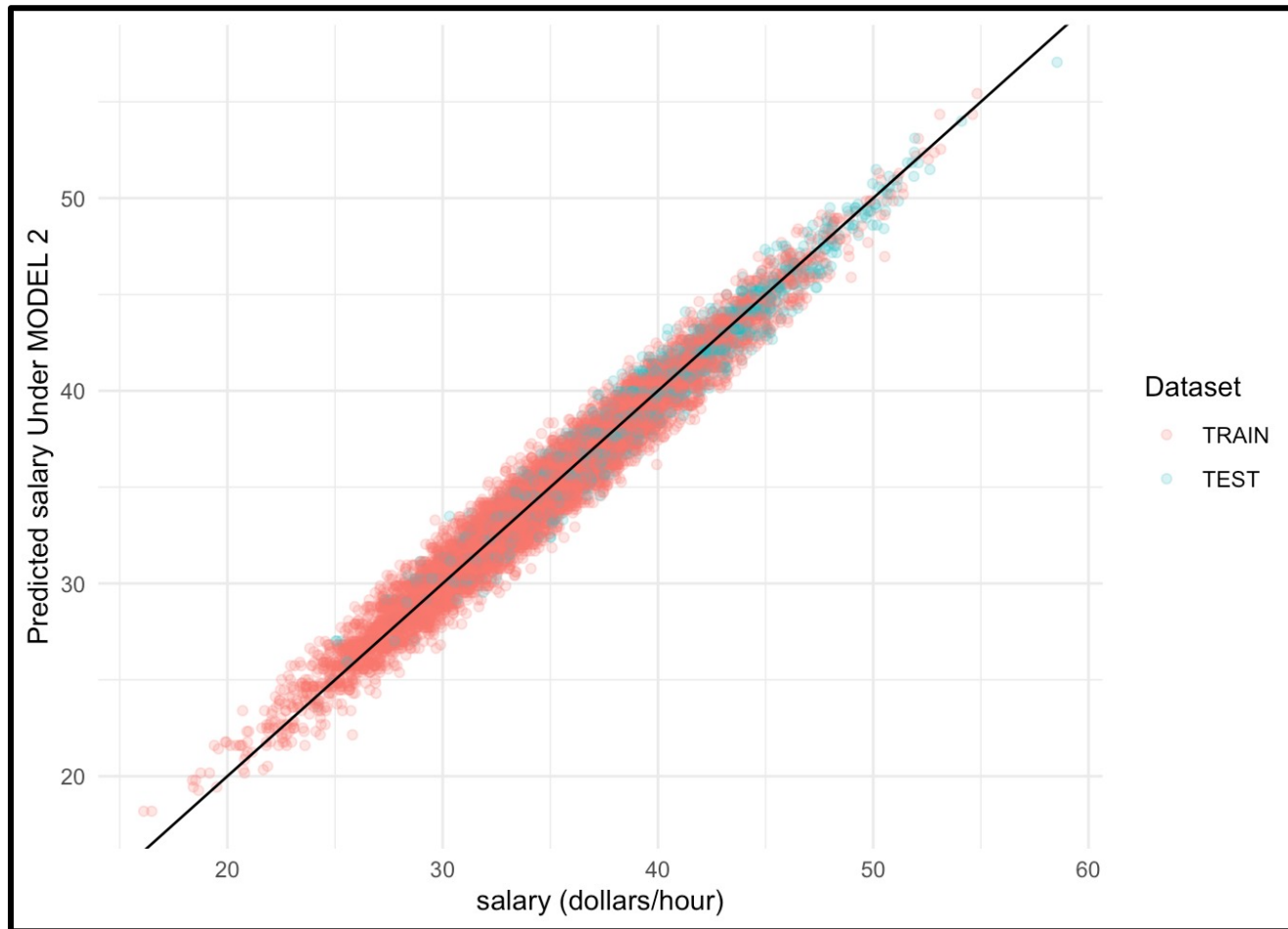
```
LM2=lm(salary~experience+education,data=TRAIN)
summary(LM2)
```

```
##
## Call:
## lm(formula = salary ~ experience + education, data = TRAIN)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -3.6426 -0.6776 -0.0138  0.6838  3.7675
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  8.996672   0.058760   153.1 <0.0000000000000002 ***
## experience   1.079243   0.002474   436.3 <0.0000000000000002 ***
## education    0.902851   0.006635   136.1 <0.0000000000000002 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.025 on 8522 degrees of freedom
## Multiple R-squared:  0.9605, Adjusted R-squared:  0.9604
## F-statistic: 1.035e+05 on 2 and 8522 DF, p-value: < 0.00000000000000022
```




Visualization

- Comparing Predicted Values to Actual Values for MODEL 2





Model Evaluation

- Out-of-Sample Evaluation

```
MODELS=c("MODEL 0", "MODEL 1A", "MODEL 1B", "MODEL 2")
MSE=c(MSE0(TEST, c(34.53)),
      MSE1A(TEST, c(9.4, 1.24)),
      MSE1B(TEST, c(31, 0.85)),
      MSE2(TEST, c(9, 1.07, 0.9)))
MAE=c(MAE0(TEST, c(34.53)),
      MAE1A(TEST, c(9.4, 1.24)),
      MAE1B(TEST, c(31, 0.85)),
      MAE2(TEST, c(9, 1.07, 0.9)))
COMPARE=tibble(MODELS=MODELS, MSE=MSE, MAE=MAE)
print(COMPARE)
```

```
## # A tibble: 4 x 3
##   MODELS      MSE    MAE
##   <chr>    <dbl> <dbl>
## 1 MODEL 0  42.0   5.17
## 2 MODEL 1A 21.5   4.31
## 3 MODEL 1B 24.5   3.94
## 4 MODEL 2  0.965 0.786
```