Optimal Transport Based Filtering with Nonlinear State Equality Constraints

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Introduction

We consider the problem of Optimal Transport based Bayesian filtering (OTF) in presence of state dependent nonlinear equality-constraints.

Contributions

Our main contribution is in proposing a framework to extend the OTF, where nonlinear state equality constraints are present. To the best of our knowledge, there is no prior work on OTF with state constraints.

Filter model

The recursion starts from the initial distribution $p(\boldsymbol{x}_0)$. In the prediction step, we evaluate the prior distribution of $\boldsymbol{x}_k$, given $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1}$ as

$$p(\boldsymbol{x}_k|\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1}) = \int p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})p(\boldsymbol{x}_{k-1}|\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1})d\boldsymbol{x}_{k-1} \quad (1)$$

where $p(\boldsymbol{x}_k|\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1})$ is the posterior distribution at k^{th} time step, $p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1})$ encapsulates the state transition model from $k-1$ to k, and $p(\boldsymbol{x}_{k-1}|\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1})$ is the posterior state distribution at $(k-1)^{\text{th}}$ time step.

In the next update step, given the new measurement $\boldsymbol{y}_k$ at time step k, the updated posterior distribution of the state $\boldsymbol{x}_k$ is computed using the Bayes rule as

$$\boxed{p(\boldsymbol{x}_k|\boldsymbol{y}_1, \cdots, \boldsymbol{y}_k) = \frac{p(\boldsymbol{y}_k|\boldsymbol{x}_k)p(\boldsymbol{x}_k|\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1})}{\int p(\boldsymbol{y}_k|\boldsymbol{x}_k)p(\boldsymbol{x}_k|\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{k-1})d\boldsymbol{x}_k}. \quad (2)}$$

- Equally weighted prior ensemble at time step k is denoted by $\boldsymbol{X}_k^- \in [\boldsymbol{x}_{1,k}^-, \boldsymbol{x}_{2,k}^-, ..., \boldsymbol{x}_{N,k}^-]$ and the equally weighted posterior ensemble by $\boldsymbol{X}_k^+ \in [\boldsymbol{x}_{1,k}^+, \boldsymbol{x}_{2,k}^+, ..., \boldsymbol{x}_{N,k}^+]$.
- The matrix $\boldsymbol{T} := [t_{ij}]$ is a coupling between $\boldsymbol{X}_k^-$ and $\boldsymbol{X}_k^+$,

$$\sum_{i=1}^{N} t_{ij} = 1/N, \ \sum_{j=1}^{N} t_{ij} = w_i, \ \text{and} \ \ t_{ij} \geq 0; \tag{3}$$

where $w_i \propto$ likelihood-function$(\boldsymbol{y}_i, \boldsymbol{x}_{i,k}^-)$.

Optimal matrix $\boldsymbol{T}$ is solved by optimizing the following linear programming problem,

$$\boldsymbol{T}^* = \underset{\boldsymbol{T}}{\operatorname{argmin}} \sum_{i=1}^{N} \sum_{j=1}^{N} t_{ij} D(\boldsymbol{x}_{i,k}^{-}, \hat{\boldsymbol{x}}_{j,k}^{+}) \tag{4}$$

subjected to conditions in (3), where $D(\boldsymbol{x}_{i,k}^{-}, \boldsymbol{x}_{j,k}^{+})$ is a distance metric between $\boldsymbol{x}_{i,k}^{-}$ and $\hat{\boldsymbol{x}}_{j,k}^{+}$. The $\hat{\boldsymbol{x}}_{j,k}^{+}$ denotes posterior samples with sample weights equal to w_i. The equally weighted prior and unequally weighted (weighted with w_i) posterior have the same sample locations.

Posterior samples:

$$\boldsymbol{X}_k^{+} = \boldsymbol{X}_k^{-} N \boldsymbol{T} \tag{5}$$

(interpreted as a re-sampling).

Based upon the OT filtering step: $\boldsymbol{X}_k^+ = \boldsymbol{X}_k^- N \boldsymbol{T}$ we present an alternative sampling technique.

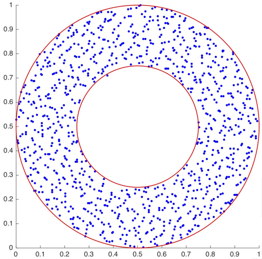

Figure 1: Uniform samples generated from an annulus

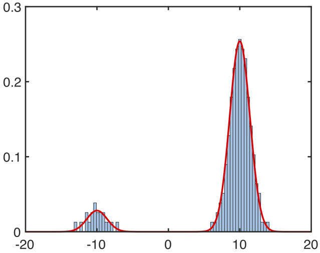

Figure 2: Histogram generated from Bi-modal distribution

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Problem formulation

Assume that for all $k \geq 1$, the state vectors $\boldsymbol{x}_k$ satisfies the equality constraint

$$\boxed{\boldsymbol{g}(\boldsymbol{x}_k) = \boldsymbol{d}_k \qquad (6)}$$

where $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^s$ and $\boldsymbol{d}_k \in \mathbb{R}^s$ are known. The objective of a nonlinear equality constrained Bayesian filtering is to evaluate (5), such that it also satisfies (6).

Common techniques[1]:

- Nonlinear equality-constrained filtering (NLeq)
- Projected filtering (Proj)
- Measurement-Augmented filtering (MA)

We adapt these techniques for OTF and name them: OTNLeq, OTProj, and OTMA respectively.

We reinforce the nonlinear equality constraints using the concepts of measurement-augmentation and projection feedback together. This is essentially OTNLeq combined with OTMA, which is named as OTNLeqMA.

[1] Bruno O. Soares Teixeira et al. "Unscented filtering for equality-constrained nonlinear systems". In: *2008 American Control Conference*. IEEE, 2008. DOI: 10.1109/acc.2008.4586463. URL: https://doi.org/10.1109%2Facc.2008.4586463.

We consider a simple pendulum with a point mass at the end.

- We use a non-minimal co-ordinates $\boldsymbol{x} = [x, y]$ to denote the location of the point mass on a 2-dimensional plane.
- The positive x-direction point towards the right, while the positive y-direction points downwards.

The kinematics of this pendulum is modeled as:

$$\ddot{x} = \frac{1}{L^2}(-gxy - x(\dot{x}^2 + \dot{y}^2)) \tag{7a}$$

$$\ddot{y} = \frac{1}{L^2}(gx^2 - y(\dot{x}^2 + \dot{y}^2)) \tag{7b}$$

Experimental Setup I

- L is the length of the pendulum taken to be 1 meter, and g is the gravity term, which is taken to be 9.8 m/sec^2.
- Since the length of the pendulum is fixed we have $x(t)^2 + y(t)^2 = L^2$ for $t \geq 0$ as our state dependent equality constraint.
- We set real initial location of the pendulum point mass $\boldsymbol{x}_0 = [x(0), y(0)]$ as $\boldsymbol{x}_0 = [L\cos(30^o), L\sin(30^o)]$.
- The samples are generated by sampling uniformly from $\pm 5^o$ about the mean position.
- We assume no process noise in this numerical study.

Experimental Setup II

We assume that we can measure the location of the point mass using visual measurement with measurement noise and that we do not have access to the angular measurements. The measurement model is:

$$\boldsymbol{y}_k = \boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{v}_k, \tag{8}$$

where $\boldsymbol{h}(\boldsymbol{x}_k) := H\boldsymbol{x}_k$ with $H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. The stochastic noise variable , $\boldsymbol{v}_k \sim \mathbb{N}(\boldsymbol{0}, \boldsymbol{R})$, where $\boldsymbol{R} = \text{diag}([0.01\ 0.01])\text{m}^2$. The measurements are available at every discrete time step of $\Delta t = 0.05$ secs.

Constraint — $\boldsymbol{g}(\hat{\boldsymbol{x}}_k) = \hat{x}_k^2 + \hat{y}_k^2 = 1$, where $\hat{x}_k$ and $\hat{y}_k$ denotes the position estimates of the point mass.

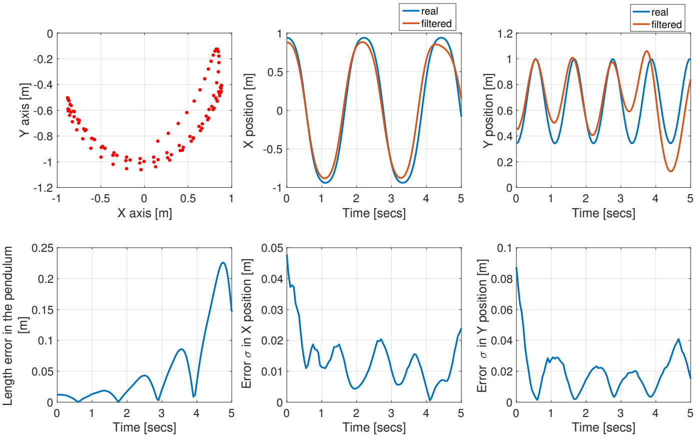

Figure 3: OT filter responses for a simple pendulum without compensating for the length constraint, with 10 samples

Results II

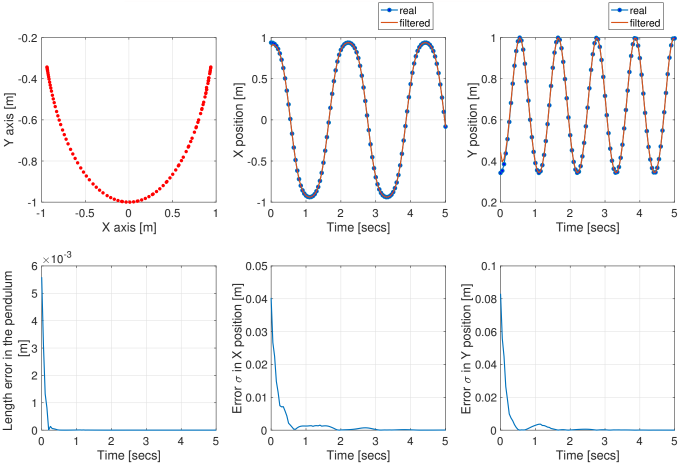

Figure 4: OT filter responses for a simple pendulum after compensating for the length constraint using OTNLeqMA

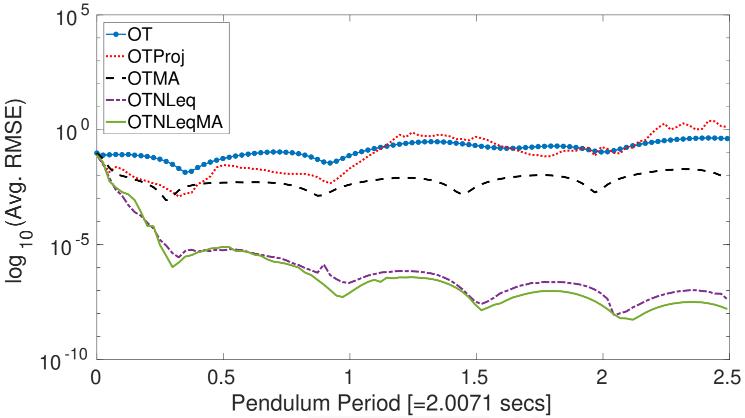

Figure 5: Average RMS constraint error over 100 Monte Carlo runs

Conclusion

- We addressed the nonlinear equality-constrained filtering problem for nonlinear systems when OT filtering is used for its state estimation.
- Three existing methodologies for equality-constrained filtering problems using KF and UKF, were coupled with OT filtering and their performances were evaluated using a numerical example.
- We further showed numerically, that the proposed OTNLeqMA filter provided the least constraint error compared to others.

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Thank You