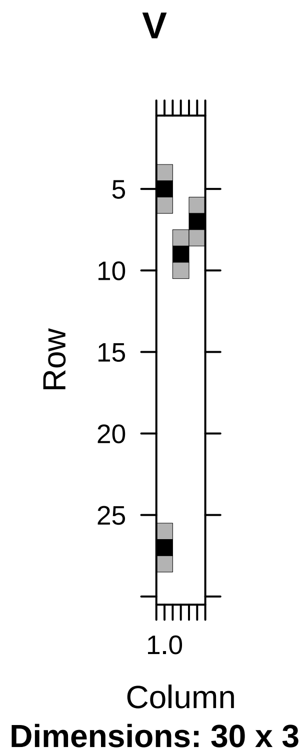
Covariance Model for Teleconnection

1. Covariance Model

1.1 nearest neighbour and teleconnection

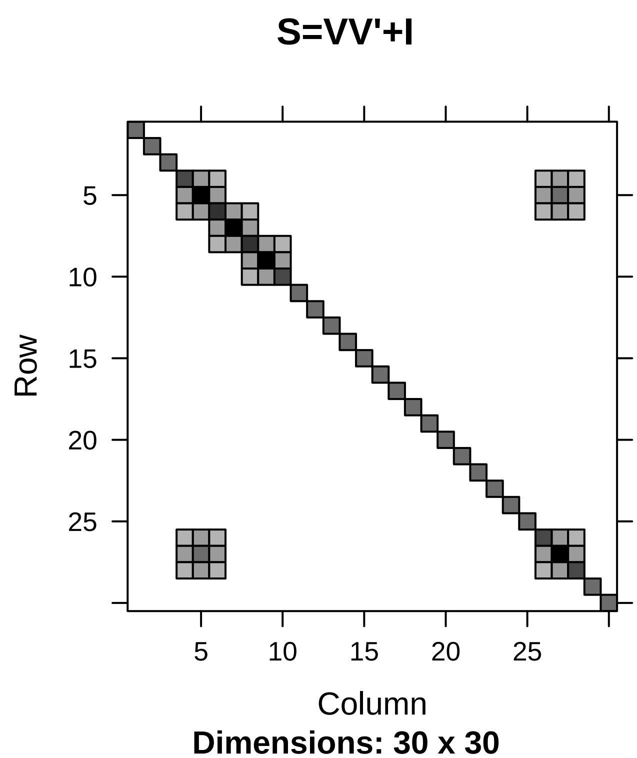
Consider the matrix V:

```
##       [,1] [,2] [,3]
##  [1,]  0.0  0.0  0.0
##  [2,]  0.0  0.0  0.0
##  [3,]  0.0  0.0  0.0
##  [4,]  0.5  0.0  0.0
##  [5,]  1.0  0.0  0.0
##  [6,]  0.5  0.0  0.5
##  [7,]  0.0  0.0  1.0
##  [8,]  0.0  0.5  0.5
##  [9,]  0.0  1.0  0.0
## [10,]  0.0  0.5  0.0
## [11,]  0.0  0.0  0.0
## [12,]  0.0  0.0  0.0
## [13,]  0.0  0.0  0.0
## [14,]  0.0  0.0  0.0
## [15,]  0.0  0.0  0.0
## [16,]  0.0  0.0  0.0
## [17,]  0.0  0.0  0.0
## [18,]  0.0  0.0  0.0
## [19,]  0.0  0.0  0.0
## [20,]  0.0  0.0  0.0
## [21,]  0.0  0.0  0.0
## [22,]  0.0  0.0  0.0
## [23,]  0.0  0.0  0.0
## [24,]  0.0  0.0  0.0
## [25,]  0.0  0.0  0.0
## [26,]  0.5  0.0  0.0
## [27,]  1.0  0.0  0.0
## [28,]  0.5  0.0  0.0
## [29,]  0.0  0.0  0.0
## [30,]  0.0  0.0  0.0
```

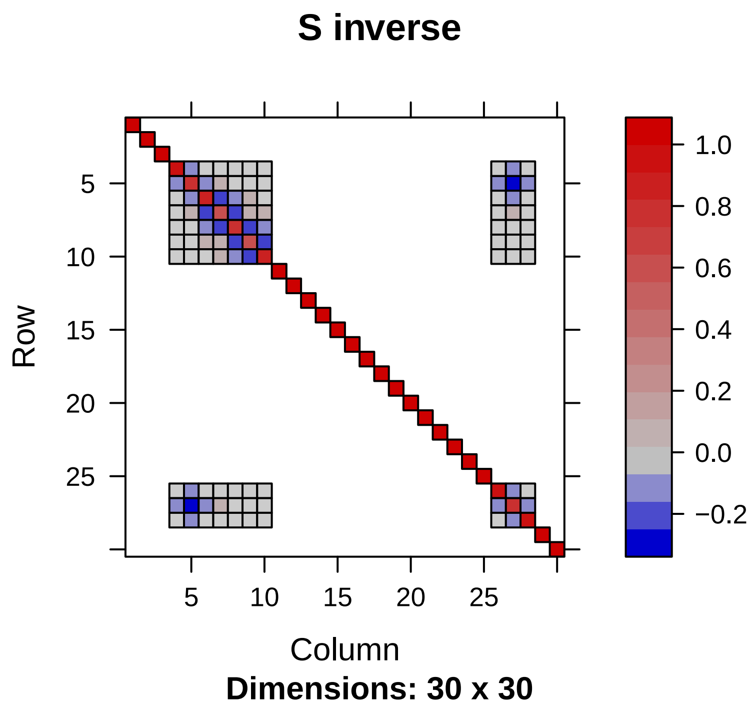


Let the covariance matrix be

$$S = VV^\top + I$$



Meanwhile the inverse of S is



1.2 link with factor model

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}, F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

$$X = \mu + AF + \epsilon$$

$$\Sigma = cov(X) = AA^\top + I$$

A is not unique. Change the model a little bit, let

$$V = [v_1, \cdots, v_r]$$

, and

$$\Lambda = diag(\lambda_1, \cdots, \lambda_r)$$

. Then we have

$$\Sigma = V\Lambda V^\top + I = \sum_{i=1}^{r} \lambda_i v_i v_i^\top + I$$

.

Note that each column of V: coefficients for certain factor for different features X

We have:

- Suppose: features X_i is indexed by i, associated with the distance
- neighbourhood effect: row vector similarity, i.e. the more similar, the more correlated,

- teleconnection effect: remote i and j are driven by the same factor, i.e. the column of V (the common factor) has two separate (remote) peaks

Possible model assumption:

- If column of V, i.e. v_i is **single peaked**, then it can be used to model neighbourhood effect
- If v_i is **double peaked**, then it can be used to model teleconnection
- We can use kernel function to model the peak.
- We can assume the support of the kernel (or the peak) is sparse, since not all the features can be related to the same factor.

For example,

- Let $K(\cdot)$ be a kernel function
- v_i has peak center p_i
- $v_i = K(\frac{i-p_i}{h})$, where h is the bandwidth
- We can assume one teleconnection, for example, $v_1 = K(\frac{i-p_i^{(1)}}{h}) + K(\frac{i-p_i^{(2)}}{h})$
- $\Sigma = V\Lambda V^\top + I = \sum_{i=1}^r \lambda_i v_i v_i^\top + I$

In this way,Σ is under a spiked covariance model with different sparsity structure of the spikes (v_i): **the covariance is the sum of many single peaked and one double peaked vectors**.

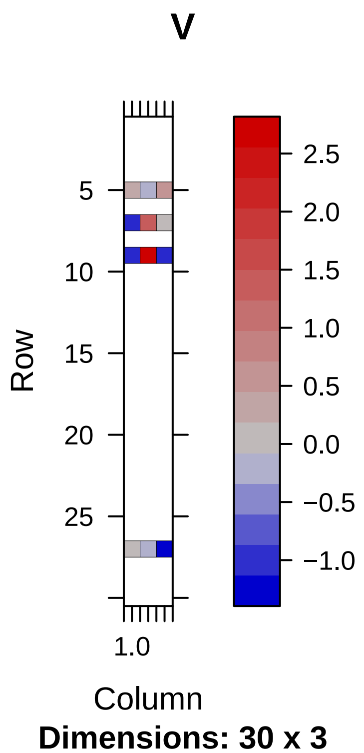
2. Different Sparsity

2.1 group sparsity

Joint sparse for each row of V:

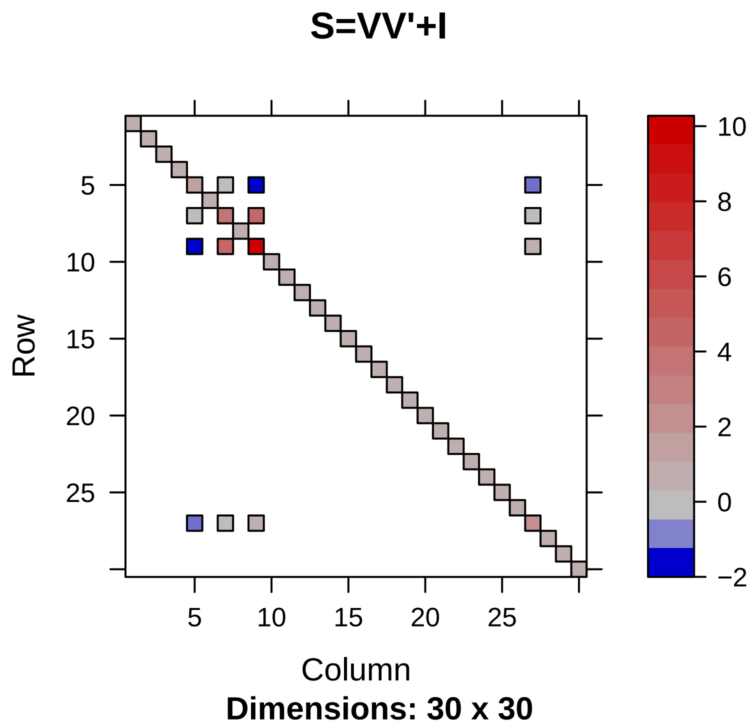
```
##              [,1]        [,2]        [,3]
##  [1,]  0.00000000  0.00000000  0.00000000
##  [2,]  0.00000000  0.00000000  0.00000000
##  [3,]  0.00000000  0.00000000  0.00000000
##  [4,]  0.00000000  0.00000000  0.00000000
##  [5,]  0.21786217 -0.14753578  0.69851857
##  [6,]  0.00000000  0.00000000  0.00000000
##  [7,] -0.95436614  1.34256617 -0.02431167
##  [8,]  0.00000000  0.00000000  0.00000000
##  [9,] -1.06753303  2.55876278 -0.91427162
## [10,]  0.00000000  0.00000000  0.00000000
## [11,]  0.00000000  0.00000000  0.00000000
## [12,]  0.00000000  0.00000000  0.00000000
## [13,]  0.00000000  0.00000000  0.00000000
## [14,]  0.00000000  0.00000000  0.00000000
## [15,]  0.00000000  0.00000000  0.00000000
## [16,]  0.00000000  0.00000000  0.00000000
## [17,]  0.00000000  0.00000000  0.00000000
## [18,]  0.00000000  0.00000000  0.00000000
## [19,]  0.00000000  0.00000000  0.00000000
## [20,]  0.00000000  0.00000000  0.00000000
## [21,]  0.00000000  0.00000000  0.00000000
## [22,]  0.00000000  0.00000000  0.00000000
## [23,]  0.00000000  0.00000000  0.00000000
## [24,]  0.00000000  0.00000000  0.00000000
## [25,]  0.00000000  0.00000000  0.00000000
## [26,]  0.00000000  0.00000000  0.00000000
```

```
## [27,]  0.05423719 -0.08871028 -1.13665536
## [28,]  0.00000000  0.00000000  0.00000000
## [29,]  0.00000000  0.00000000  0.00000000
## [30,]  0.00000000  0.00000000  0.00000000
```

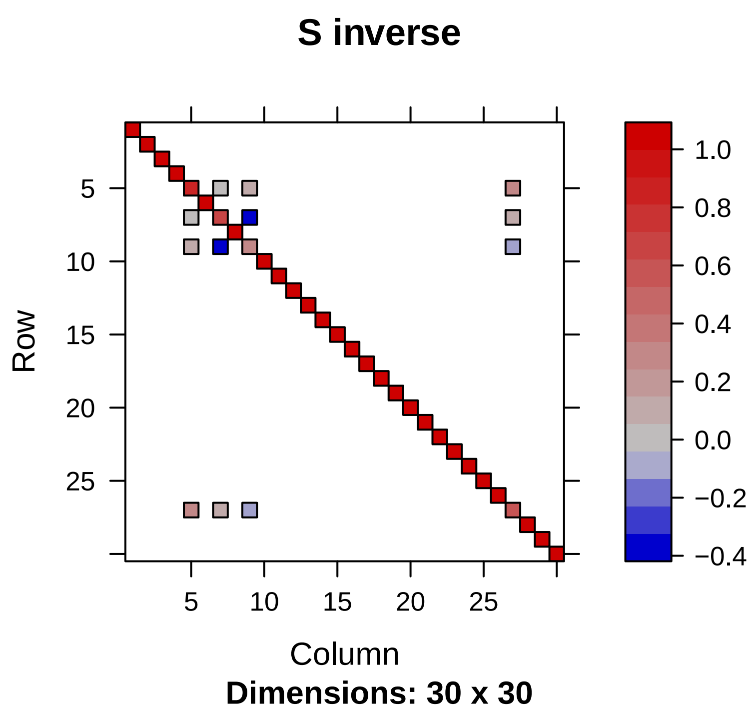


Let the covariance matrix be

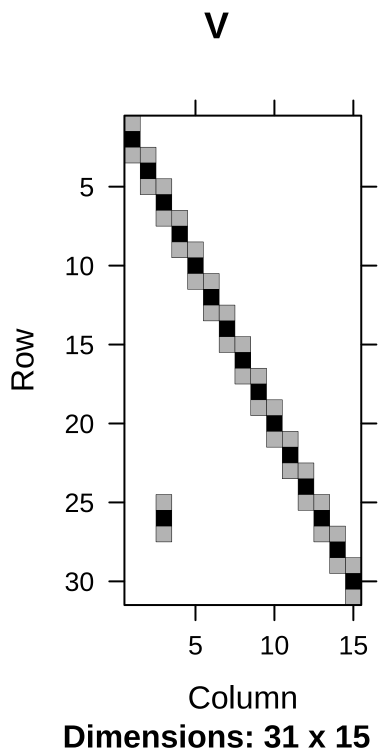
$$S = VV^{\top} + I$$

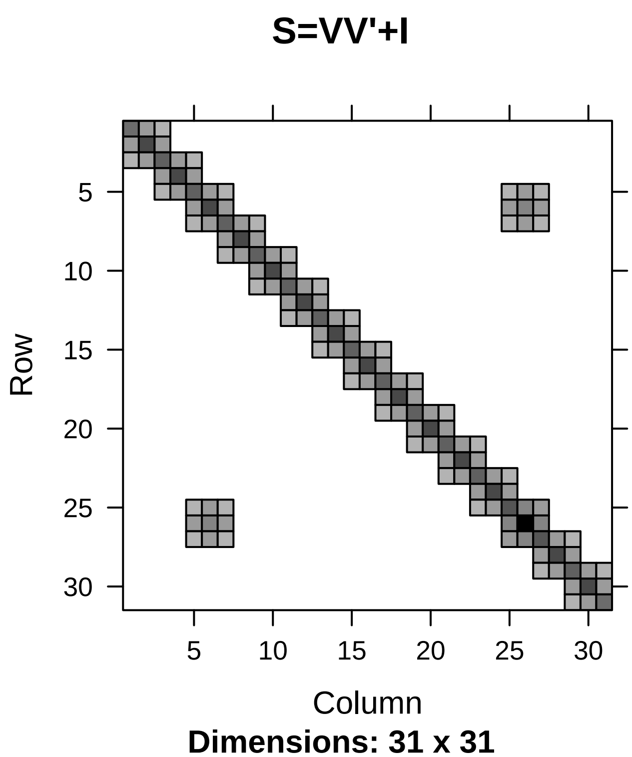


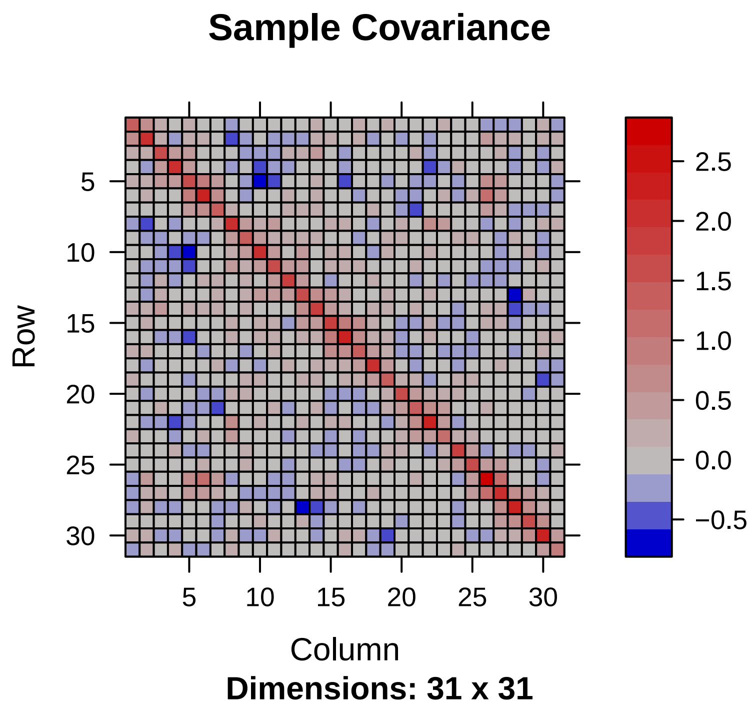
Meanwhile the inverse of S is

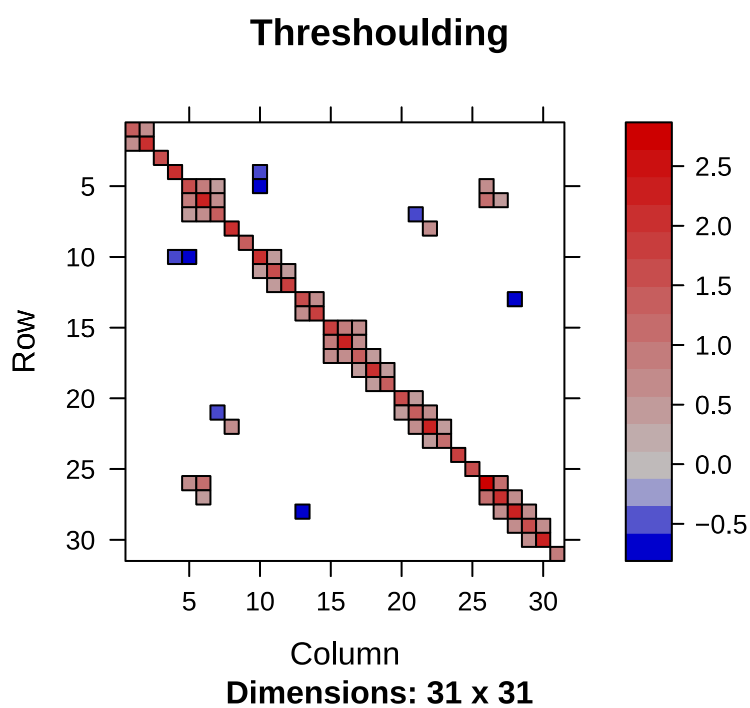


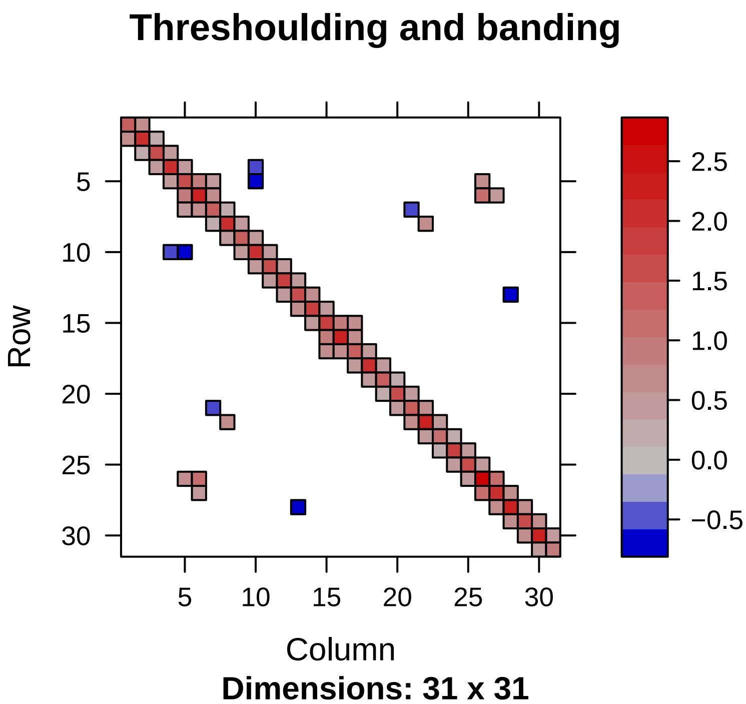
3. Estimation

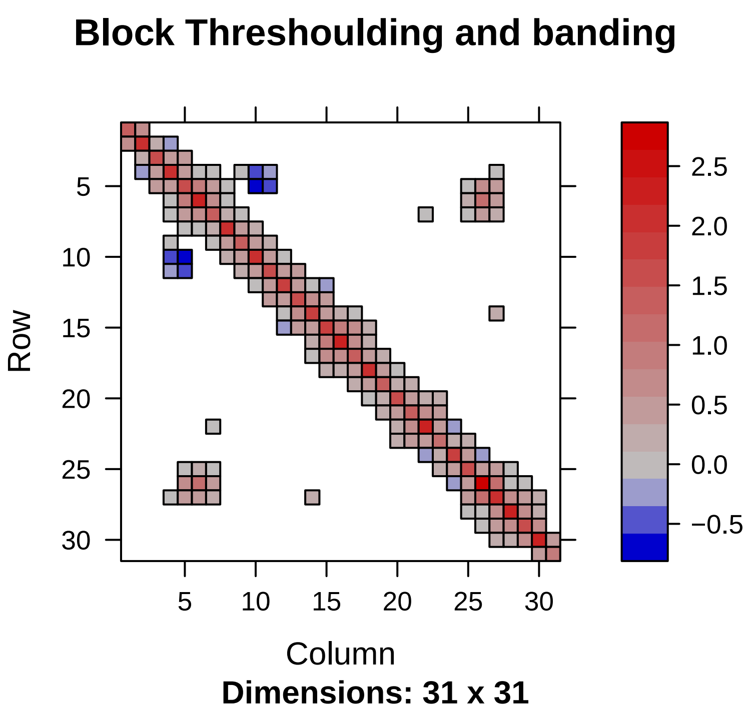


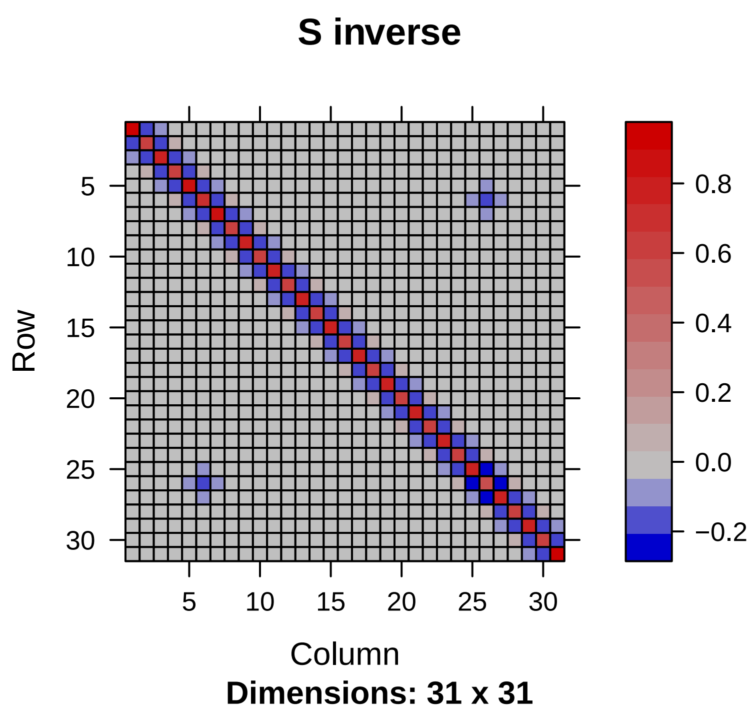



Sample Covariance
Row
5
10
15
20
25
30
Column
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
Dimensions: 31 x 31


Threshoulding
Row
5
10
15
20
25
30
Column
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
Dimensions: 31 x 31


Threshoulding and banding
Row
Column
Dimensions: 31 x 31


Block Threshoulding and banding
Row
Column
Dimensions: 31 x 31




```
base::norm(S-Shat,"F")
```

```
## [1] 5.446327
```

```
base::norm(S-Stap,"F")
```

```
## [1] 4.215985
```

```
base::norm(S-Stb,"F")
```

```
## [1] 3.309273
```

```
base::norm(S-Str,"F")
```

```
## [1] 2.969419
```