

Matrix Norm

June 28, 2016

Here are some simulation results on the norm compression inequality for the Schatten p norm.

1. function for calculating the Schatten p norm

It is calculated based on the definition using singular values(use SVD to get singular values).

$$\|A\| = \left(\sum_{i=1}^r \sigma_i^p(A) \right)^{\frac{1}{p}}$$

```
# Calculate the Schatten p norm
schatten_norm=function(B,p){
  res=rep(0,length(p))
  for(i in 1:length(p)){
    res[i]=(sum((svd(B)$d)^p[i]))^(1/p[i])
  }
  return(res)
}
```

2. function to compress matrix A

A must be a square matrix, and the row and column partition is the same(but each can be unevenly partitioned), with Schatten p norm.

```
# Compress the matrix A according to partition d, with Schatten p norm
norm_compress=function(A,d,p){
  n=length(d)
  A1=matrix(0,n,n)
  D=c(0,cumsum(d))
  for(i in 1:n){
    for(j in 1:n){
      idx=(D[i]+1):D[i+1]
      idy=(D[j]+1):D[j+1]
      A1[i,j]=schatten_norm(A[idx,idy],p)
    }
  }
  return(A1)
}
```

3. function to do simulation

- input: size of the matrix (n), and partition (d, sum is n)
- output:
 - plot of Schatten p norm for matrix A, and its compressed version N(A), for $p = 1, 1.1, 1.2, \dots, 4.8, 4.9, 5$.
 - Eigenvalues of A

- difference between norm of A , and norm of $N(A)$, i.e. $\text{norm}(A) - \text{norm}(N(A))$

Note that the matrix A is simply generated with n^2 uniform $[0, 1]$ random numbers.

```
# Do simulation: for matrix size n by n, partition d
simu=function(n,d){
  A=matrix(runif(n*n),n,n)
  #A=matrix(rnorm(n*n),n,n)
  p=(10:50)/10
  r0=r1=rep(0,length(p))
  for(i in 1:length(p)){
    A1=norm_compress(A,d,p[i])
    r0[i]=schatten_norm(A,p[i]) # norm of A
    r1[i]=schatten_norm(A1,p[i]) # norm of compressed A, which is A1
  }
  plot(p,r0,col="red",type="o",xlab="p",ylab="Norm",
       main=paste0("Schatten p norm: ", "Partition:",
                   paste0(d,sep=","),collapse = ""), " Size:",n))
  points(p,r1,col="blue",type="o")
  legend("topright",c("Norm of A","Norm of Compressed N(A)"),col=c("red","blue"),
        lty =1,pch=1)
  print("Eigenvalues of A:")
  print(eigen(A)$values)
  print("norm(A)-norm(N(A)):")
  print(round(r0-r1,2))
}
```

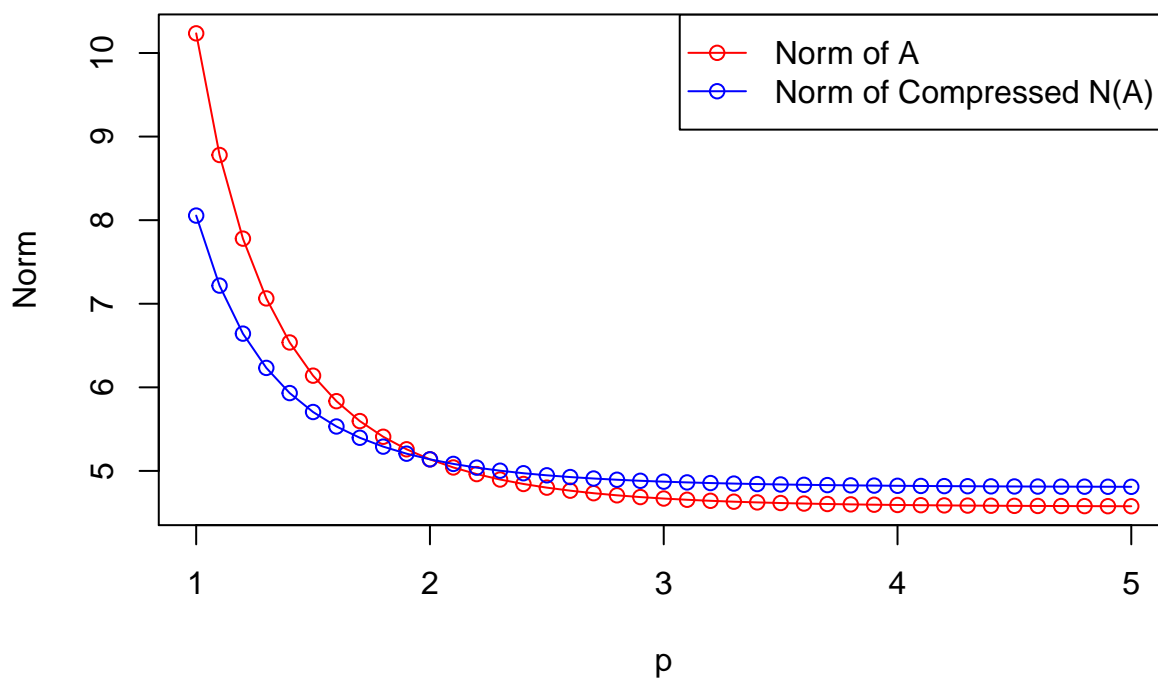
4. some results

From the following results,

- When $p = 2$ (the 11th difference), the two norms are equal, which confirms your finding that for the Frobenius norm, the equal sign holds.
- It seems that when $1 \leq p < 2$, the norm is decreased by compression, while when $p > 2$, the norm is increased by compression.

```
# 9 by 9, partition with 3+3+3
simu(9,c(3,3,3))
```

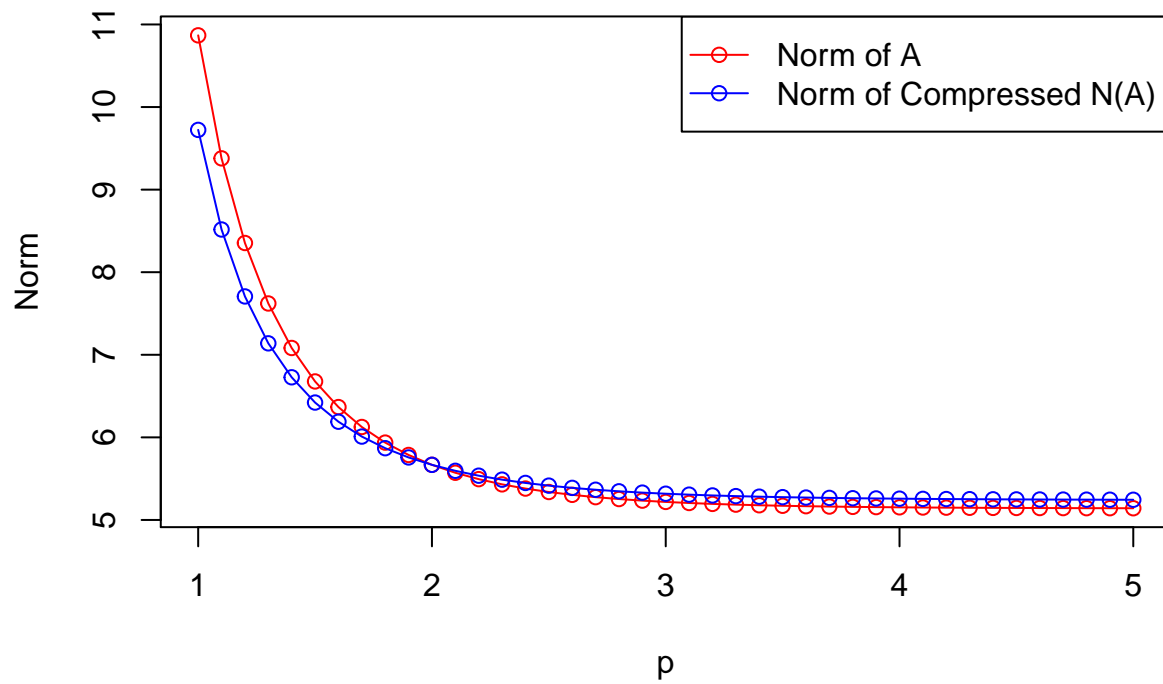
Schatten p norm: Partition:3,3,3, Size:9



```
## [1] "Eigenvalues of A:"
## [1] 4.4072927+0.0000000i -0.8704632+0.1720976i -0.8704632-0.1720976i
## [4] 0.3878734+0.3783216i 0.3878734-0.3783216i -0.3107763+0.3410219i
## [7] -0.3107763-0.3410219i 0.0317276+0.4397760i 0.0317276-0.4397760i
## [1] "norm(A)-norm(N(A)):"
## [1] 2.18 1.56 1.14 0.83 0.61 0.43 0.30 0.20 0.12 0.05 0.00
## [12] -0.04 -0.08 -0.11 -0.13 -0.15 -0.16 -0.18 -0.19 -0.19 -0.20 -0.21
## [23] -0.21 -0.22 -0.22 -0.22 -0.22 -0.23 -0.23 -0.23 -0.23 -0.23 -0.23
## [34] -0.23 -0.23 -0.23 -0.23 -0.23 -0.23 -0.23 -0.23
```

```
# 9 by 9, partition with 2+7
simu(9,c(2,7))
```

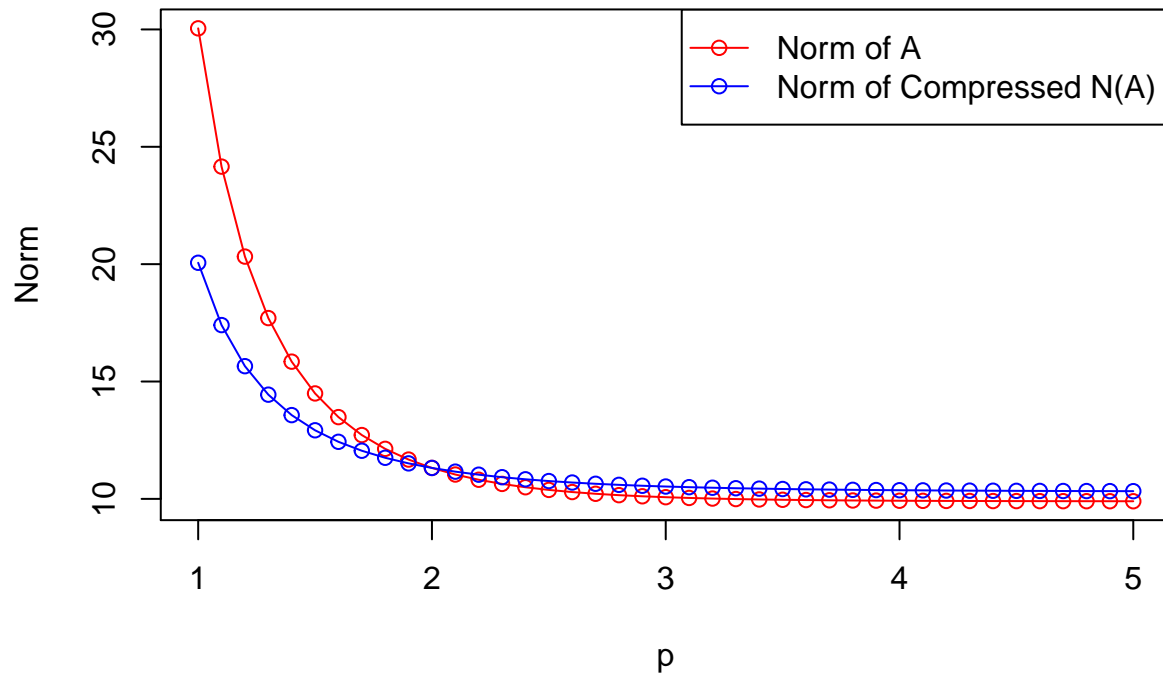
Schatten p norm: Partition:2,7, Size:9



```
## [1] "Eigenvalues of A:"
## [1] 5.0434417+0.0000000i 0.9093430+0.0000000i -0.7394352+0.0000000i
## [4] -0.1003789+0.5841030i -0.1003789-0.5841030i 0.4994334+0.0266024i
## [7] 0.4994334-0.0266024i -0.4293387+0.0000000i -0.1438740+0.0000000i
## [1] "norm(A)-norm(N(A)):"
## [1] 1.14 0.86 0.65 0.48 0.35 0.26 0.18 0.12 0.07 0.03 0.00
## [12] -0.02 -0.04 -0.06 -0.07 -0.08 -0.08 -0.09 -0.09 -0.10 -0.10 -0.10
## [23] -0.10 -0.10 -0.10 -0.10 -0.10 -0.10 -0.10 -0.10 -0.10 -0.10 -0.10
## [34] -0.10 -0.10 -0.10 -0.10 -0.10 -0.10 -0.10 -0.10
```

```
# 20 by 20, partition with 5+5+5+5
simu(20,c(5,5,5,5))
```

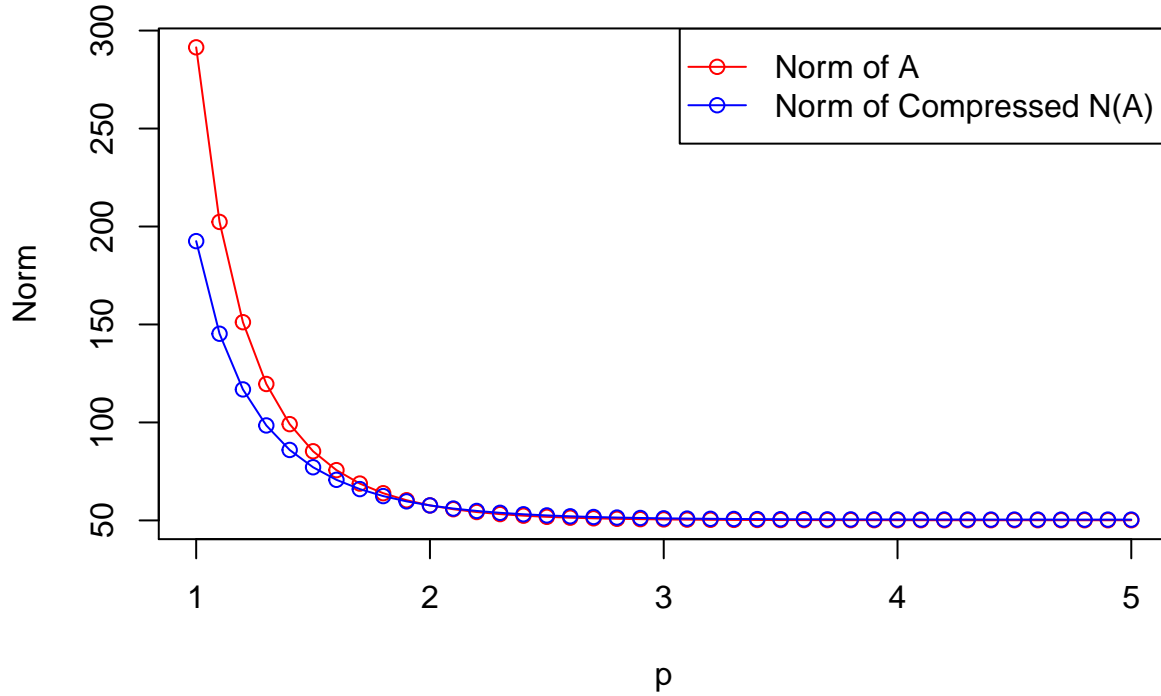
Schatten p norm: Partition:5,5,5,5, Size:20



```
## [1] "Eigenvalues of A:"
## [1] 9.79318829+0.0000000i -0.62654712+0.9528665i -0.62654712-0.9528665i
## [4] -0.95951679+0.4937081i -0.95951679-0.4937081i 1.02182567+0.1660219i
## [7] 1.02182567-0.1660219i 0.68631571+0.7286214i 0.68631571-0.7286214i
## [10] 0.25008915+0.9422016i 0.25008915-0.9422016i -0.89939100+0.0000000i
## [13] -0.54472761+0.5191661i -0.54472761-0.5191661i -0.08642149+0.6776904i
## [16] -0.08642149-0.6776904i 0.02649700+0.5223696i 0.02649700-0.5223696i
## [19] 0.49579308+0.0000000i 0.02601844+0.0000000i
## [1] "norm(A)-norm(N(A)):"
## [1] 9.99 6.75 4.67 3.26 2.28 1.57 1.05 0.67 0.38 0.16 0.00
## [12] -0.12 -0.22 -0.29 -0.34 -0.38 -0.41 -0.43 -0.44 -0.45 -0.46 -0.46
## [23] -0.46 -0.46 -0.46 -0.46 -0.46 -0.46 -0.45 -0.45 -0.45 -0.45 -0.44
## [34] -0.44 -0.44 -0.44 -0.44 -0.43 -0.43 -0.43 -0.43
```

```
# 100 by 100, partition with 20+30+50
simu(100,c(20,30,50))
```

Schatten p norm: Partition:20,30,50, Size:100



```
## [1] "Eigenvalues of A:"
## [1] 49.8848774+0.0000000i 2.2879848+1.8827344i 2.2879848-1.8827344i
## [4] -2.8186798+0.5478591i -2.8186798-0.5478591i 0.6913392+2.7666969i
## [7] 0.6913392-2.7666969i -2.7935987+0.0000000i 2.4062303+1.3329653i
## [10] 2.4062303-1.3329653i -1.6040937+2.2283969i -1.6040937-2.2283969i
## [13] -2.1641539+1.6638292i -2.1641539-1.6638292i 1.5838820+2.1944107i
## [16] 1.5838820-2.1944107i -0.3532959+2.6195501i -0.3532959-2.6195501i
## [19] 2.5893061+0.4466512i 2.5893061-0.4466512i -0.2197868+2.5458653i
## [22] -0.2197868-2.5458653i -2.3395344+0.9963556i -2.3395344-0.9963556i
## [25] 1.1839387+2.1934616i 1.1839387-2.1934616i 1.9067979+1.5521791i
## [28] 1.9067979-1.5521791i -1.6656124+1.7353764i -1.6656124-1.7353764i
## [31] 0.6336171+2.2984474i 0.6336171-2.2984474i -2.3496885+0.0000000i
## [34] 2.3262704+0.2375525i 2.3262704-0.2375525i -1.1350354+1.9369760i
## [37] -1.1350354-1.9369760i -1.5176816+1.6443942i -1.5176816-1.6443942i
## [40] 2.0213477+0.8464816i 2.0213477-0.8464816i 2.1473169+0.0000000i
## [43] -0.0943813+2.1241497i -0.0943813-2.1241497i -1.8722472+0.9986954i
## [46] -1.8722472-0.9986954i -1.6788142+1.2896998i -1.6788142-1.2896998i
## [49] -0.6030688+1.9914310i -0.6030688-1.9914310i 1.4841291+1.4181129i
## [52] 1.4841291-1.4181129i 0.1975092+2.0062363i 0.1975092-2.0062363i
## [55] 1.2106955+1.5477435i 1.2106955-1.5477435i -1.8527051+0.6344401i
## [58] -1.8527051-0.6344401i 0.7469945+1.7561518i 0.7469945-1.7561518i
## [61] -1.7544315+0.0773162i -1.7544315-0.0773162i 1.7226710+0.0000000i
## [64] -0.2181912+1.6464069i -0.2181912-1.6464069i -0.9778680+1.3336124i
## [67] -0.9778680-1.3336124i -1.6185446+0.1911581i -1.6185446-0.1911581i
## [70] 1.2140767+1.0856621i 1.2140767-1.0856621i 0.1410172+1.5775469i
## [73] 0.1410172-1.5775469i 0.7172424+1.2700818i 0.7172424-1.2700818i
## [76] -0.3546477+1.3125444i -0.3546477-1.3125444i 1.3216301+0.2577053i
## [79] 1.3216301-0.2577053i 0.1737668+1.3201104i 0.1737668-1.3201104i
## [82] -1.1476644+0.2885374i -1.1476644-0.2885374i -0.8854563+0.5839652i
```

```
## [85] -0.8854563-0.5839652i 0.7364845+0.6423440i 0.7364845-0.6423440i
## [88] 0.9229788+0.2096345i 0.9229788-0.2096345i 0.0320662+0.8789885i
## [91] 0.0320662-0.8789885i 0.6084332+0.3449096i 0.6084332-0.3449096i
## [94] -0.5940588+0.0000000i 0.5698348+0.0000000i -0.4427627+0.0000000i
## [97] 0.2161907+0.3676756i 0.2161907-0.3676756i -0.1003528+0.4042499i
## [100] -0.1003528-0.4042499i
## [1] "norm(A)-norm(N(A)):"
## [1] 98.89 57.06 34.31 21.16 13.18 8.16 4.94 2.83 1.46 0.57 0.00
## [12] -0.35 -0.56 -0.67 -0.72 -0.72 -0.71 -0.68 -0.64 -0.60 -0.57 -0.53
## [23] -0.50 -0.47 -0.45 -0.43 -0.41 -0.39 -0.38 -0.37 -0.36 -0.35 -0.34
## [34] -0.34 -0.33 -0.33 -0.33 -0.32 -0.32 -0.32 -0.32
```

5. Other Matrices

However, the difference between the norms is not always monotone decreasing with p . I tried Toeplitz matrix from n to 1 (1 to n still monotone), something like:

```
toeplitz(6:1)
```

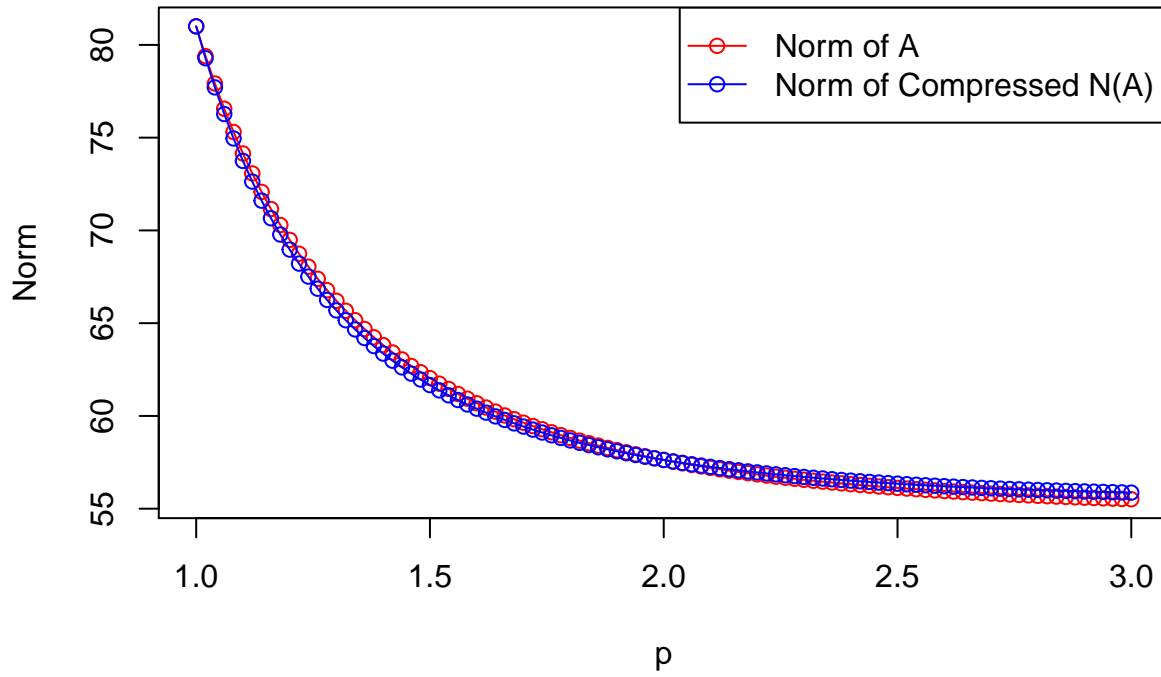
```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    6    5    4    3    2    1
## [2,]    5    6    5    4    3    2
## [3,]    4    5    6    5    4    3
## [4,]    3    4    5    6    5    4
## [5,]    2    3    4    5    6    5
## [6,]    1    2    3    4    5    6
```

Even though the inequality still holds, $p = 1$ becomes another equality, and the difference increase first and then decrease (and finally negative after $p = 2$).

```
simu2=function(n,d){
  A=toeplitz(n:1)
  #A=t(A)%*%A
  p=(50:150)/50
  r0=r1=rep(0,length(p))
  for(i in 1:length(p)){
    A1=norm_compress(A,d,p[i])
    r0[i]=schatten_norm(A,p[i]) # norm of A
    r1[i]=schatten_norm(A1,p[i]) # norm of compressed A, which is A1
  }
  par(mfrow=c(1,1))
  plot(p,r0,col="red",type="o",xlab="p",ylab="Norm",main=paste0("Schatten p norm: ", "Partition:",paste0(
  points(p,r1,col="blue",type="o")
  legend("topright",c("Norm of A", "Norm of Compressed N(A)"),col=c("red","blue"),lty=1,pch=1)
  print("norm(A)-norm(N(A)):")
  print(round(r0-r1,2))
  plot(p,r0-r1,xlab="p",ylab="Norm Difference",col="blue",type="o",main=paste0("Norm(A)-Norm(N(A)): ", "l
  abline(0,0)
}

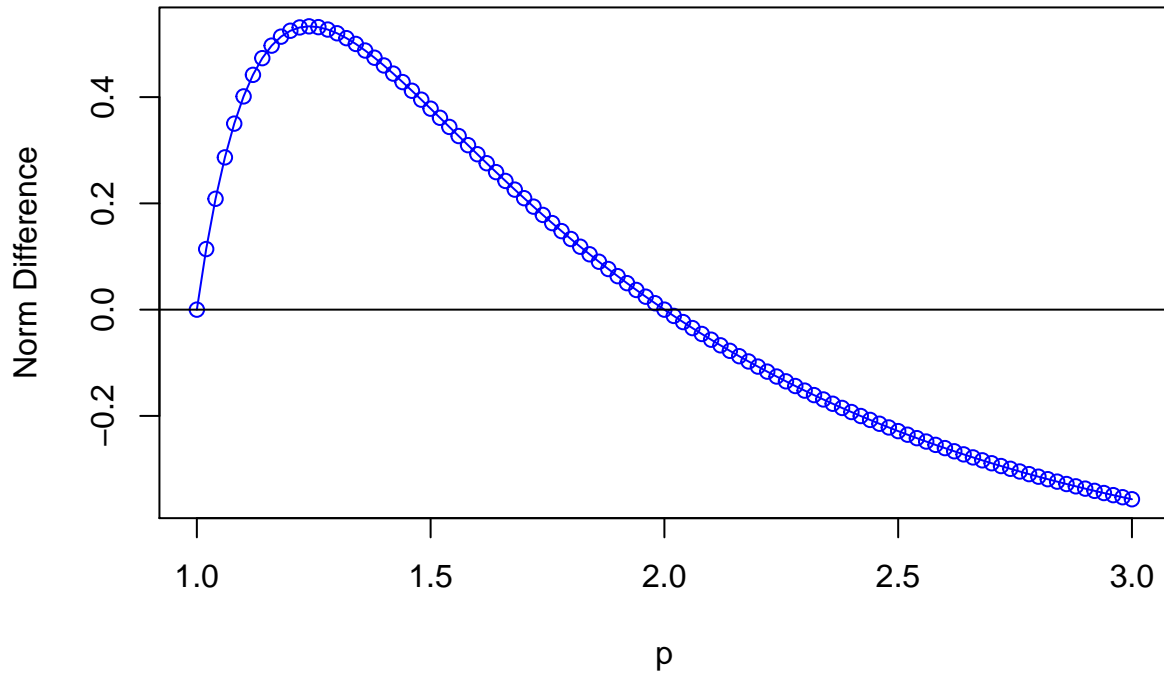
simu2(9,c(3,3,3))
```

Schatten p norm: Partition:3,3,3, Size:9



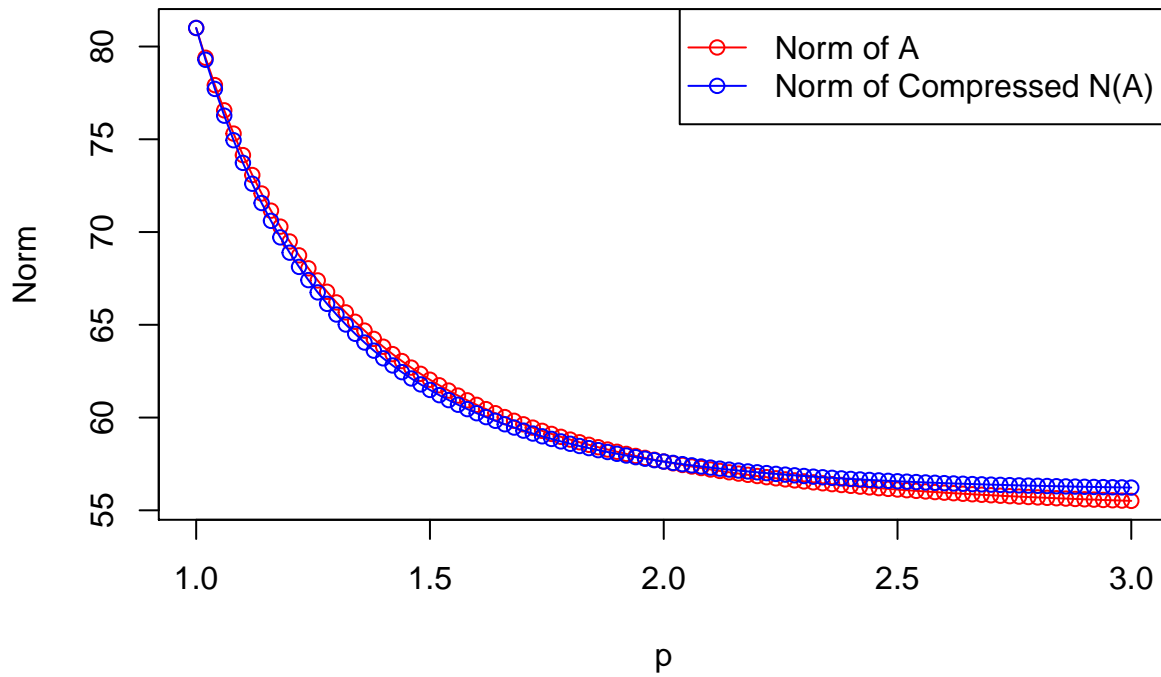
```
## [1] "norm(A)-norm(N(A)):"
## [1] 0.00 0.11 0.21 0.29 0.35 0.40 0.44 0.47 0.50 0.51 0.52
## [12] 0.53 0.53 0.53 0.53 0.52 0.51 0.50 0.49 0.47 0.46 0.44
## [23] 0.43 0.41 0.40 0.38 0.36 0.34 0.33 0.31 0.29 0.28 0.26
## [34] 0.24 0.23 0.21 0.19 0.18 0.16 0.15 0.13 0.12 0.10 0.09
## [45] 0.08 0.06 0.05 0.04 0.02 0.01 0.00 -0.01 -0.02 -0.03 -0.05
## [56] -0.06 -0.07 -0.08 -0.09 -0.10 -0.11 -0.12 -0.13 -0.13 -0.14 -0.15
## [67] -0.16 -0.17 -0.18 -0.18 -0.19 -0.20 -0.21 -0.21 -0.22 -0.23 -0.24
## [78] -0.24 -0.25 -0.25 -0.26 -0.27 -0.27 -0.28 -0.28 -0.29 -0.29 -0.30
## [89] -0.30 -0.31 -0.31 -0.32 -0.32 -0.33 -0.33 -0.34 -0.34 -0.35 -0.35
## [100] -0.35 -0.36
```


Norm(A)–Norm(N(A)): Partition:3,3,3, Size:9



```
simu2(9,c(2,7))
```

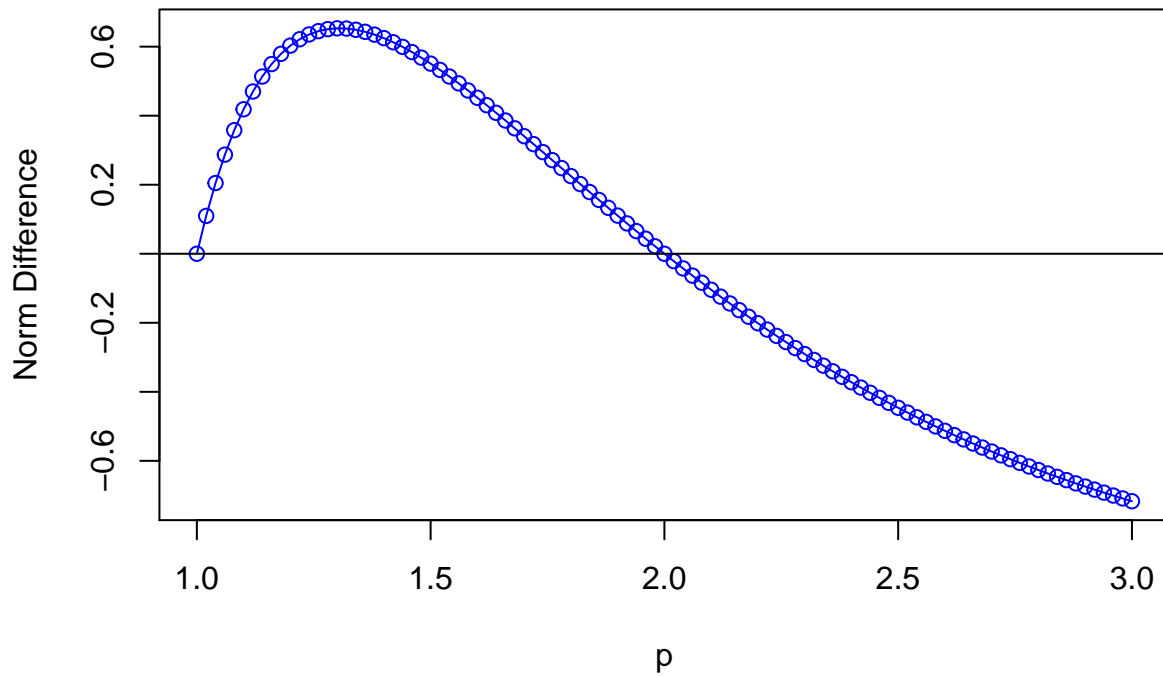
Schatten p norm: Partition:2,7, Size:9



```
## [1] "norm(A)-norm(N(A)):"
## [1] 0.00 0.11 0.20 0.29 0.36 0.42 0.47 0.51 0.55 0.58 0.60
```

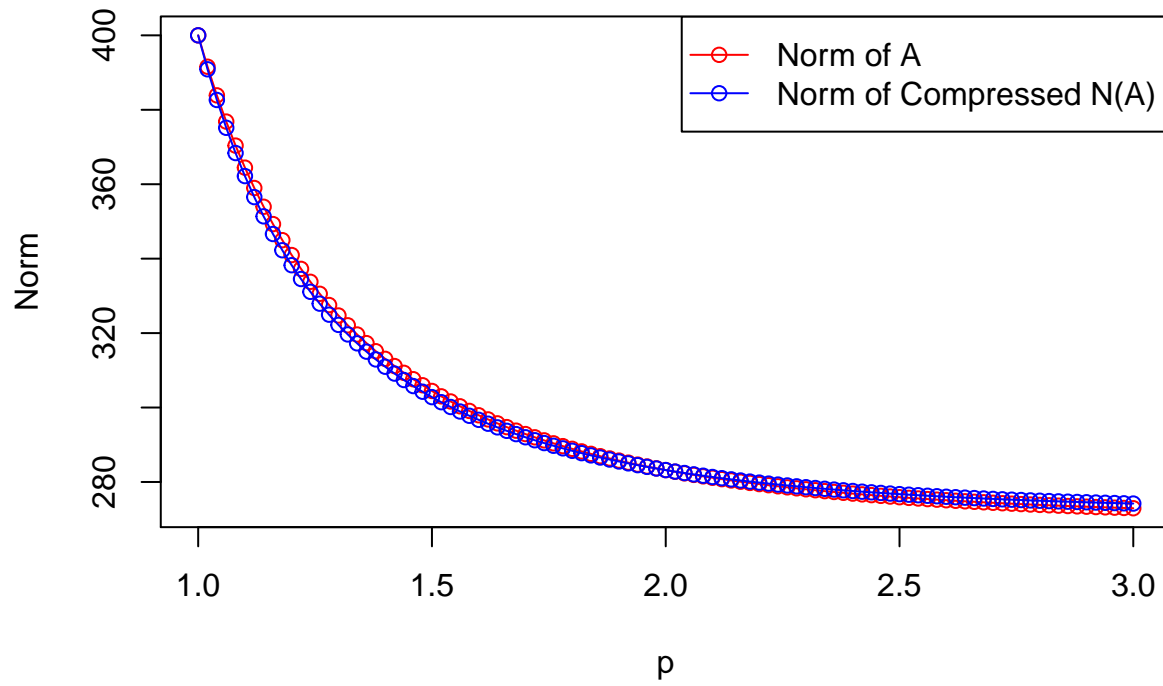
```
## [12] 0.62 0.64 0.64 0.65 0.65 0.65 0.65 0.64 0.63 0.62 0.61
## [23] 0.60 0.58 0.57 0.55 0.53 0.51 0.49 0.47 0.45 0.43 0.41
## [34] 0.39 0.36 0.34 0.32 0.29 0.27 0.25 0.23 0.20 0.18 0.16
## [45] 0.13 0.11 0.09 0.07 0.04 0.02 0.00 -0.02 -0.04 -0.06 -0.08
## [56] -0.10 -0.12 -0.14 -0.16 -0.18 -0.20 -0.22 -0.24 -0.26 -0.27 -0.29
## [67] -0.31 -0.32 -0.34 -0.36 -0.37 -0.39 -0.40 -0.42 -0.43 -0.45 -0.46
## [78] -0.47 -0.49 -0.50 -0.51 -0.53 -0.54 -0.55 -0.56 -0.57 -0.58 -0.60
## [89] -0.61 -0.62 -0.63 -0.64 -0.65 -0.66 -0.67 -0.67 -0.68 -0.69 -0.70
## [100] -0.71 -0.72
```

Norm(A)–Norm(N(A)): Partition:2,7, Size:9



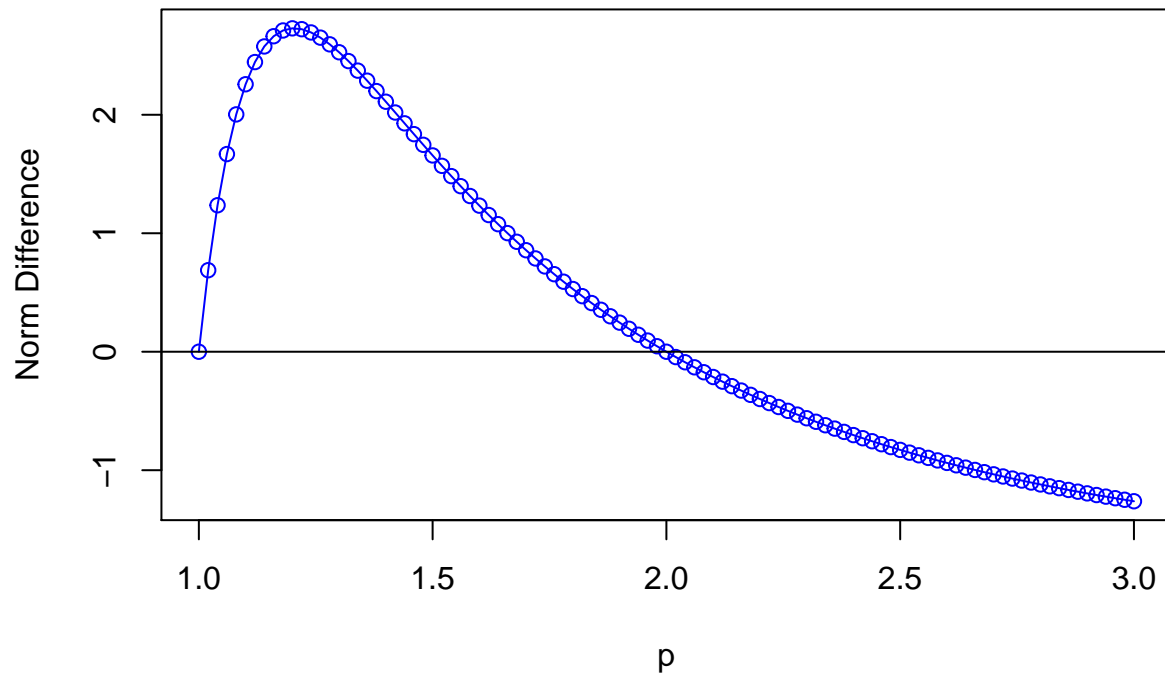
```
simu2(20,c(5,5,5,5))
```

Schatten p norm: Partition:5,5,5,5, Size:20



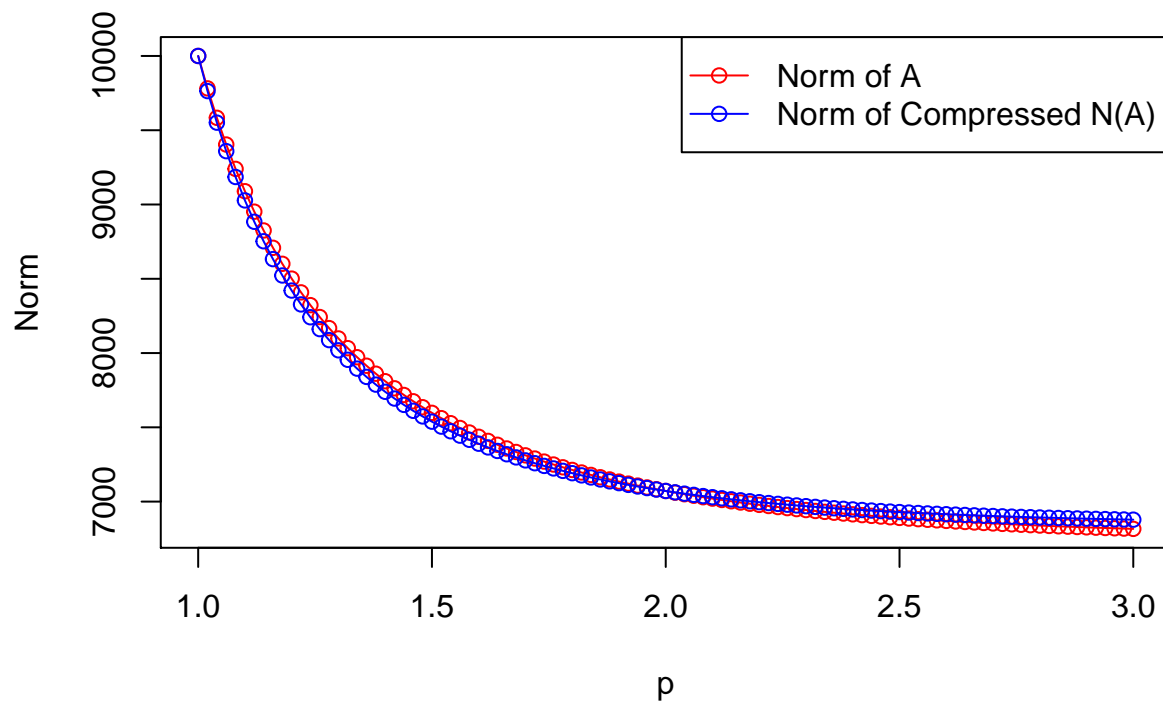
```
## [1] "norm(A)-norm(N(A)):"
## [1] 0.00 0.69 1.24 1.67 2.00 2.26 2.44 2.58 2.66 2.71 2.73
## [12] 2.72 2.69 2.65 2.59 2.53 2.45 2.37 2.29 2.20 2.11 2.02
## [23] 1.93 1.84 1.75 1.66 1.57 1.48 1.40 1.31 1.23 1.15 1.08
## [34] 1.00 0.93 0.86 0.79 0.72 0.65 0.59 0.53 0.47 0.41 0.35
## [45] 0.30 0.25 0.19 0.14 0.09 0.05 0.00 -0.05 -0.09 -0.13 -0.17
## [56] -0.21 -0.25 -0.29 -0.33 -0.36 -0.40 -0.43 -0.47 -0.50 -0.53 -0.56
## [67] -0.59 -0.62 -0.65 -0.68 -0.70 -0.73 -0.76 -0.78 -0.80 -0.83 -0.85
## [78] -0.87 -0.90 -0.92 -0.94 -0.96 -0.98 -1.00 -1.02 -1.03 -1.05 -1.07
## [89] -1.09 -1.10 -1.12 -1.14 -1.15 -1.17 -1.18 -1.20 -1.21 -1.22 -1.24
## [100] -1.25 -1.26
```

Norm(A)–Norm(N(A)): Partition:5,5,5,5, Size:20



```
simu2(100,c(20,30,50))
```

Schatten p norm: Partition:20,30,50, Size:100



```
## [1] "norm(A)-norm(N(A)):"
## [1] 0.00 18.00 32.73 44.69 54.34 62.04 68.08 72.73 76.21 78.69
```

```

## [11] 80.32 81.25 81.57 81.39 80.77 79.79 78.51 76.97 75.22 73.29
## [21] 71.22 69.02 66.72 64.34 61.90 59.41 56.89 54.34 51.78 49.20
## [31] 46.63 44.06 41.50 38.95 36.42 33.91 31.43 28.97 26.54 24.14
## [41] 21.77 19.44 17.14 14.87 12.64 10.44 8.28 6.15 4.07 2.01
## [51] 0.00 -1.98 -3.92 -5.83 -7.70 -9.53 -11.33 -13.09 -14.82 -16.51
## [61] -18.17 -19.80 -21.39 -22.95 -24.48 -25.97 -27.44 -28.87 -30.27 -31.64
## [71] -32.99 -34.30 -35.59 -36.84 -38.07 -39.28 -40.45 -41.60 -42.72 -43.82
## [81] -44.90 -45.95 -46.97 -47.98 -48.96 -49.92 -50.85 -51.77 -52.66 -53.53
## [91] -54.38 -55.22 -56.03 -56.83 -57.60 -58.36 -59.10 -59.82 -60.53 -61.22
## [101] -61.89

```

Norm(A)–Norm(N(A)): Partition:20,30,50, Size:100

