Lecture 1

Review of Probability

Introduction to Probability by Grimstead & Snell

Book: *Grimstead & Snell*

Solve exercises in this book for practice.

Key Topics

- Random variables
- Probability distributions
- Moments
- Sums of random variables
- Central Limit Theorem

Random Variables

A **random variable** is a function whose values are random, but distribution is known through its *probability distribution function*.

Toy Examples

Fair Coin (Discrete):

$$X_{\text{coin}} = \begin{cases} 1 & \text{head} \\ -1 & \text{tails} \end{cases}$$

$$P_{\text{coin}}(1) = P_{\text{coin}}(-1) = \frac{1}{2}$$

Sum of 2 Dice:

$$X_{\text{dice}} = \{2, 3, 4, \dots, 12\}$$

X_{dice}	2	3	4	5	6	7	8	9	10	11	12
P_{dice}	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Continuous Uniform Distribution

Let X_{uniform} be uniformly distributed on the open interval $(0, 1)$.

Discrete Case: Sum of probabilities equals 1:

$$\sum P = 1$$

Continuous Case: Total probability is the integral of the density:

$$\int_{-\infty}^{\infty} P_{\text{uniform}}(x)\, dx = \int_0^1 1\, dx = 1$$

Normal Distribution

The **Normal Distribution** is the most fundamental example of a continuous distribution.

Moments

Let X be a random variable.

1st Moment (Mean / Average / Expectation):

$$\mathbb{E}[X]$$

$$\mathbb{E}[X_{\text{coin}}] = P_{\text{coin}}(1) \cdot 1 + P_{\text{coin}}(-1) \cdot (-1) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0$$

$$\mathbb{E}[X_{\text{dice}}] = \frac{1}{36}(2) + \frac{2}{36}(3) + \cdots + \frac{1}{36}(12) = 7$$

$$\mathbb{E}[X_{\text{unif}}] = \int_a^b p_{\text{unif}}(x)\, x\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

2nd Moment:

$$\mathbb{E}[X^2]$$

$$\mathbb{E}[X_{\text{coin}}^2] = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$$

ℓ^{th} Moment:

$$\mathbb{E}[X^\ell]$$

Variance / Standard Deviation

The variance of X is defined as:

$$\mathbb{E}[(X - \mathbb{E}[X])^2]$$

Critical Property of Expectation

Let X_1, X_2 be random variables, and $\alpha \in \mathbb{R}$. Then:

$$\mathbb{E}[\alpha X_1 + X_2] = \alpha \mathbb{E}[X_1] + \mathbb{E}[X_2]$$

Variance Expansion Formula

$$\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2\mathbb{E}[X]X + (\mathbb{E}[X])^2]$$
$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Thus,

$$\text{Var}(X) = \sigma^2 = \sigma_X^2$$

Standard Deviation

The standard deviation of X is:

$$\sqrt{\mathrm{Var}(X)} = \sigma$$

Interpretation:

- Large standard deviation $\Rightarrow$ distribution is more spread out.
- Small standard deviation $\Rightarrow$ distribution is less spread out.

Remark: The standard **Normal Distribution** has mean 0 and standard deviation 1.

Skewness

$$\mathrm{Skew}(X) = \mathbb{E}\left[\left(\frac{X - \mathbb{E}[X]}{\sigma}\right)^3\right]$$

Examples:

- Skewness of the standard normal distribution is 0.
- Skewness of the sum of two dice rolls is also 0.

Interpretation: Skewness = 0 reflects a distribution that is balanced around its mean.

Unfair Coin Example

Let:

$$X_{\text{coin}} = \begin{cases} 1 & \text{Heads} \\ -1 & \text{Tails} \end{cases}$$

with probabilities:

$$P(1) = \frac{3}{4}, \quad P(-1) = \frac{1}{4}$$

Then:

$$\mathrm{Skew}(X_{\text{coin}}) = \frac{-\frac{3}{4}}{\left(\frac{\sqrt{3}}{2}\right)^3} < 0$$

Also:

$$\mathbb{E}[X_{\text{coin}}] = \frac{3}{4}(1) + \frac{1}{4}(-1) = \frac{1}{2}$$

Negative skewness: Distribution is skewed *above* the mean. Opposite for **positive** skewness.

Kurtosis

$$\mathrm{Kurt}(X) = \mathbb{E}\left[\left(\frac{X - \mathbb{E}[X]}{\sigma}\right)^4\right]$$

Kurtosis of the normal distribution is 3.

Excess Kurtosis:

$$\text{Excess Kurt}(X) = \text{Kurt}(X) - 3$$

- Excess **positive** kurtosis: Indicates more extreme behavior in the distribution (heavier tails).
- Excess **negative** kurtosis: Indicates less weight in the tails — extremes are less likely.

Relevant Example: Financial Interpretation

Let X be the random variable of daily return on a portfolio.

- **Mean:** A positive mean indicates the portfolio has been profitable on average.
- **Variance:** Small variance indicates consistent returns.
- **Skewness:** Negative skewness means more days are profitable.
- **Excess Kurtosis:** Small (or negative) excess kurtosis $\Rightarrow$ general outcome is consistent on a day-to-day basis.

Correlation

Let X, Y be random variables.

Covariance:

$$\begin{aligned}\text{Cov}(X,Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \text{Var}(X) \quad \text{if } X = Y \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\end{aligned}$$

Correlation:

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Note: If $X = Y$, then $\sigma_X = \sigma_Y$ and

$$\text{Corr}(X,Y) = \frac{\text{Var}(X)}{\text{Var}(X)} = 1$$

Properties:

$$-1 \leq \text{Corr}(X,Y) \leq 1$$

- $\text{Corr}(X,Y) \approx 1$: X and Y are ßimilar".
- $\text{Corr}(X,Y) \approx -1$: X and Y are öpposite".
- $\text{Corr}(X,Y) \approx 0$: distributions do not have much in common.

Relevant Example: A diversified portfolio has less variance in returns.

The Central Limit Theorem

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent and identically distributed (i.i.d.) random variables.

Let

$$\tilde{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

Then

$$\sqrt{n}\left(\tilde{X}_n - \mathbb{E}[X_1]\right)$$

converges in distribution to a normal distribution with mean 0 and variance

$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \dots$$