Lecture 2

Correlation and Covariance Matrices

Toy Example

An investor has \$10,000 to invest in two stocks, S_1 and S_2.

Assume the initial price of both stocks at time $t = 0$ is:

$$S_1(0) = S_2(0) = 100$$

The price of S_1 and S_2 at time $t = 1$ has the following probability distributions:

$$S_1(1) = \begin{cases} 120 & \text{with probability } \frac{3}{5} \\ 90 & \text{with probability } \frac{2}{5} \end{cases}$$

$$S_2(1) = \begin{cases} 110 & \text{with probability } \frac{4}{5} \\ 90 & \text{with probability } \frac{1}{5} \end{cases}$$

Return on Investment (ROI)

Define X_1 as the ROI for full investment in S_1:

$$X_1 = \begin{cases} \frac{120-100}{100} = \frac{1}{5} & \text{with probability } \frac{3}{5} \\ \frac{90-100}{100} = -\frac{1}{10} & \text{with probability } \frac{2}{5} \end{cases}$$

Expected ROI for X_1

$$\mathbb{E}[X_1] = \frac{3}{5}\left(\frac{1}{5}\right) + \frac{2}{5}\left(-\frac{1}{10}\right) = \frac{4}{50} = 0.08 \quad \text{or } 8\% \text{ ROI.}$$

ROI for X_2

$$\mathbb{E}[X_2] = \frac{4}{5}\left(\frac{1}{10}\right) + \frac{1}{5}\left(-\frac{1}{10}\right) = \frac{3}{50} = 0.06 \quad \text{or } 6\% \text{ ROI.}$$

Standard Deviations

For X_1:

$$\sigma_{X_1} = \sqrt{\mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2}$$

$$= \sqrt{\frac{5}{500} - \left(\frac{4}{50}\right)^2} \approx 0.147 \quad \text{or } 14.7\%.$$

For X_2:

$$\sigma_{X_2} = 0.08 \quad \text{or } 8\%.$$

ROI for Diversified Portfolio X_3

Assume 50% allocation to each stock:

$$X_3 = \text{ROI with 50\% in } S_1 \text{ and 50\% in } S_2.$$

Possible outcomes for X_3:

$$X_3 = \begin{cases} \frac{1}{2}\frac{20}{100} + \frac{1}{2}\frac{10}{100} & \text{prob. } \frac{3}{5}\cdot\frac{4}{5} \\ \frac{1}{2}\frac{20}{100} + \frac{1}{2}\frac{-10}{100} & \text{prob. } \frac{3}{5}\cdot\frac{1}{5} \\ \frac{1}{2}\frac{-10}{100} + \frac{1}{2}\frac{10}{100} & \text{prob. } \frac{2}{5}\cdot\frac{4}{5} \\ \frac{1}{2}\frac{-10}{100} + \frac{1}{2}\frac{-10}{100} & \text{prob. } \frac{2}{5}\cdot\frac{1}{5} \end{cases}$$

Expected ROI of X_3:

$$\mathbb{E}[X_3] = \frac{1}{2}\mathbb{E}[X_1] + \frac{1}{2}\mathbb{E}[X_2] = 0.07 \quad \text{or } 7\%.$$

Variance and Standard Deviation of X_3:

$$\sigma_{X_3}^2 = \mathbb{E}[X_3^2] - (\mathbb{E}[X_3])^2$$

$$\sigma_{X_3} \approx 0.0837 \quad \text{or } 8.37\%.$$

General Case with Weights w_1 and w_2

Assume $w_1, w_2 \geq 0$ and $w_1 + w_2 = 1$ are the weights invested in S_1 and S_2.

Portfolio value:

$$P(0) = 10{,}000, \quad P(1) = \text{value at } t = 1$$

$$P(1) = \left(1 + w_1\frac{S_1(1) - S_1(0)}{S_1(0)} + w_2\frac{S_2(1) - S_2(0)}{S_2(0)}\right)P(0).$$

General ROI formula with weights w_1, w_2

The ROI of the portfolio is:

$$\frac{P(1) - P(0)}{P(0)} = w_1\frac{S_1(1) - S_1(0)}{S_1(0)} + w_2\frac{S_2(1) - S_2(0)}{S_2(0)}$$

Thus,

$$X_3 = w_1X_1 + w_2X_2$$

Variance of X_3

$$\sigma_{X_3}^2 = \mathbb{E}[X_3^2] - (\mathbb{E}[X_3])^2$$

Expanding:

$$= \mathbb{E}\left[(w_1X_1 + w_2X_2)^2\right] - (w_1\mathbb{E}[X_1] + w_2\mathbb{E}[X_2])^2$$

Which gives:

$$\sigma_{X_3}^2 = w_1^2 \mathrm{Var}(X_1) + w_2^2 \mathrm{Var}(X_2) + 2w_1 w_2 \mathrm{Cov}(X_1, X_2)$$

where

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$$

Matrix Formulation

$$\sigma_{X_3}^2 = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_1, X_2) & \mathrm{Var}(X_2) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

More General Picture

Let $X_1, X_2, \ldots, X_k$ be the random variables representing returns of stocks $S_1, S_2, \ldots, S_k$.

Let Y be the investment account with weights $w_1, w_2, \ldots, w_k$ where:

$$\sum w_i = 1, \quad w_i \geq 0$$

Portfolio Variance

The variance of portfolio Y is:

$$\sigma_Y^2 = \mathrm{Var}(Y) = \vec{w}^{\,T} \, \mathrm{Cov}(X_1, \ldots, X_k) \, \vec{w}$$

where

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}$$

and the covariance matrix is:

$$\mathrm{Cov}(X_1, \ldots, X_k) = (\mathrm{Cov}(X_i, X_j))_{k \times k}$$

Takeaway

Quadratic optimization allows one to find weights that minimize the volatility of a portfolio subject to predetermined constraints and volatility models.

Efficient Frontier

Let $S_1, \ldots, S_k$ be stocks with returns $X_1, \ldots, X_k$ as random variables (R.O.I.).

Consider all possible portfolios of investments in $S_1, \ldots, S_k$.

A portfolio Y (distribution of return) **dominates** a portfolio Z if:

$$\mathbb{E}[Y] \geq \mathbb{E}[Z] \quad \text{and} \quad \sigma_Y \leq \sigma_Z.$$

Definition of Efficient Frontier

The efficient frontier is the collection of all portfolios not dominated by any other.

Key Fact

If $\vec{w}_1$ and $\vec{w}_2$ are distinct weight vectors such that the portfolios with weights $\vec{w}_1$ and $\vec{w}_2$ have minimum possible variance, then any other weight $\vec{w}_3$ that also minimizes variance is of the form:

$$\vec{w}_3 = c\,\vec{w}_2 + (1-c)\,\vec{w}_1$$

for some $c \in \mathbb{R}$.