33. (2 points) Alice and Bob have agreed to meet between 1 PM and 2 PM. Both arrive independently at some random time, uniformly distributed. Let X_A be the arrival time of Alice, in minutes after 1PM, and let X_B be the corresponding arrival time of Bob. Then the waiting time for Alice is $W_A = \max(X_B - X_A, 0)$. Use a suitable simulation to generate 1000 random samples of W_A, find its mean, and make an empirical cumulative distribution function.

34. (2 points) Suppose X has an exponential distribution with parameter λ and $Y|X = x$ has a Poisson distribution with parameter x. For $\lambda = 1$, generate at least 1000 random samples from the conditional distribution of $X|Y = 2$ and make a histogram.

35. (2 points) Suppose X and Y have standard normal distributions. Make at least 1,000 random samples from $Z = Y|(X + Y \geq 1)$. Do you think that Z has a normal distribution? What are its approximate mean and standard deviation?

36. (2 points) Suppose X is a real valued random variable (discrete or continuous) and $Y = aX + b + Z$, where a, b are real numbers and Z is an "error" term that is independent of X and that satisfies $\mathcal{E}(Z) = 0$. Show that $\mathcal{E}(Y|X) = aX + b$.

37. (5 points) Rejection Sampling (J. v. Neumann 1951). We want to generate random samples from some discrete distribution over a finite range $\mathcal{R}$, with pmf $p(x)$ for $x \in \mathcal{R}$. However, we don't know $p(x)$ exactly. Instead we only know some function $\ell(x)$ that is a constant multiple of $p(x)$, i.e. $\ell(x) = cp(x)$, with unknown c. *Of course* $c = \sum_{x\in\mathcal{R}} \ell(x)$, *but if the range* $\mathcal{R}$ *is very large and/or the function* $\ell(x)$ *is expensive to evaluate, this may be hard to compute.* We must also know some number $M \geq \max_{cx\in\mathcal{R}} \ell(x)$ and N, the number of elements in $\mathcal{R}$.
In **rejection sampling**, one repeats the following steps until enough samples have been generated.

a) Generate a random sample $y \in \mathcal{R}$ from the uniform distribution Y on $\mathcal{R}$, i.e. $\mathcal{P}(Y = y) = \frac{1}{N}$ for all $y \in \mathcal{R}$.
b) Generate a random sample $U \sim U(0, 1)$ from the uniform distribution on $(0, 1)$, independent of Y.
c) If $U < \frac{\ell(y)}{M}$, **accept** the point y and set $X = y$. Otherwise **reject** the point and try again. Let $\mathcal{A}$ be the event that a point Y that is generated in step a) is accepted.

Show that X has the pmf $p(x)$, i.e. $\mathcal{P}(X = x) = p(x)$ for all x, using the following steps. Let $x \in \mathcal{R}$ be arbitrary.

a) Explain in one sentence why $\mathcal{P}(X = x) = \mathcal{P}(Y = x|\mathcal{A})$.
b) Determine $\mathcal{P}(\mathcal{A}|Y = x)$ (*easy!*).
c) Use the law of total probability formula to show that $\mathcal{P}(\mathcal{A}) = \frac{c}{MN}$.
d) Compute $P(X = x) = P(Y = x|\mathcal{A})$ by reversing the conditioning in b). *Use a) and c) for this step.*

38. (5 points) Suppose that X has a Poisson distribution with parameter $\lambda = 50$, U has a uniform $U(0, 1)$ distribution, and $Y|X = x, U = p$ has a $B(x, p)$ distribution.
a) Use a simulation to make a histogram of the distribution of Y.

b) Use a simulation to make a histogram of the conditional distribution of $X|Y = 25$.

39. (5 points) Mixtures. Let Y_1 and Y_2 be two random variables which have the same range $\mathcal{R}$, and let w_1, w_2 probabilities with $w_1 + w_2 = 1$. Then the mixture Y of Y_1 and Y_2 is defined as follows:

a) Select $X \in \{1, 2\}$ at random, with $\mathcal{P}(X = 1) = w_1$, $\mathcal{P}(X = 2) = w_2$.
b) If $X = 1$, draw a sample Y_1 and set $Y = Y_1$. Otherwise, draw a sample Y_2 and set $Y = Y_2$.

1. Suppose $E(Y_1) = \mu_1$ and $E(Y_2) = \mu_2$. What is $E(Y|X = 1)$? What is $E(Y|X = 2)$? What is $E(Y)$?
2. Suppose $var(Y_1) = \sigma_1^2$ and $var(Y_2) = \sigma_2^2$. What is $E(Y^2|X = 1)$? What is $E(Y^2|X = 2)$? What is $var(Y)$? *Careful!*
3. Generate a sample of size 10,000 from $Y_1 \sim N(-2, 1)$, $Y_2 \sim N(1, 2)$, $w_1 = \frac{1}{5}$, $w_2 = \frac{4}{5}$ and make a probability histogram. *Clearly this is not a normal distribution, and a mixture is not a sum!*

40. (2 points) Problem 4.4 #6 in Chihara/Hesterberg. *First use the Central Limit Theorem to find the approximate distribution of the mean height of random samples of 30 boys, then use* `dnorm()`.