ANLY-511 Homework Problems
Version of 9/24/17

Explain your work and give concise reasoning. Attach `R` *code with comments if applicable. Using Markdown is the best way to d this. Do not print out any data or any detailed results of simulations. Your solutions for each homework set should fit on six to eight pages.*

25. (2 points) Consider the symmetric random walk that was discussed in the video of 9/21, where the states are integers $x \in \mathbb{Z}$ and the transition probabilities are

$$\mathcal{P}(X_{i+1} = x+1|X_i = x) = \mathcal{P}(X_{i+1} = x-1|X_i = x) = \frac{1}{2}$$

for all i and all x and $X_0 = 0$. Let T be the random time when $X_T = 15$ for the first time. Use a simulation to generate a few hundred values of T and then make a box plot of T. *Your answer should consist of commented simulation code and either the box plot or a description (max, min, quartiles, median). Also state the observed probability of not hitting $X = 15$ at all.*

26. (2 points) In American roulette it is possible to bet on a block of six numbers, consisting e.g. of the numbers $\{1, 2, 3, \ldots, 6\}$ or $\{4, 5, 6, \ldots, 9\}$. The casino will pay you six times your bet if one of these numbers comes up, and you lose your bet otherwise. Propose a modification of the St. Petersburg system for somebody who only uses this bet, and explain it. *Note that if you win in every sixth spin, you are close to breaking even in such a bet. Therefore you don't have to increase your bet each time you lose. It is not necessary to give a simulation.*

27. (2 points) Bob's preferred bet in American roulette consists in betting \$1 on black numbers and simultaneously \$2 on even numbers (see the roulette board in the course slides). Find all possible outcomes of a single game and their probabilities, that is, find the probability distribution of the outcome of a single bet. Then compute its expected value.

28. (2 points) Consider a room that is paved with $2 \times n$ square tiles which are labeled from 1 to $2n$. A frog performs a random walk by hopping from one tile to a randomly chosen adjacent tile in each time step. All adjacent tiles are chosen with the same probability. The frog can never hop into a wall of the room.
True or not true: the transition matrix for this random walk is symmetric, that is, it satisfies $\mathcal{P}(X_{i+1} = k|X_i = j) = \mathcal{P}(X_{i+1} = j|X_i = k)$ for all i and all possible staes $1 \le j,\ k \le 2n$. Explain your answer.

29. (2 points) Consider the following game: you get to roll an n-sided fair die once. If the outcome is the number k, then you get to throw k darts at a target. The probability of hitting the target is p. Dart throws are independent of one another. Let X be the result of the roll of the die, and let Y be the number of hits when you throw the darts. Set up the joint probability mass function (pmf) for X and Y.

30. (5 points) Consider the random walk performed by the caveman in the class slides of 9/21.
a) Using the transition matrix that was dervied in class, compute $\mathcal{P}(X_3 = 3|X_0 = 1)$. Then do the same computation directly. *Do not print out the entire transition matrix.*
b) Find the first time T such that the chance of the caveman's survival for more

than T steps is less 50% no matter where he starts, i.e. find T such that $P(X_T = 9|X_0 = k) \geq .5$ for all k and $P(X_{T-1} = 9|X_0 = j) < 0.5$ for at least one j, using `R` .

31. (5 points) Let N be a random variable with a Poisson distribution with parameter $\lambda > 0$. Given that $N = n$, let X be a binomial $B(n, p)$ distribution where $0 < p < 1$.
a) Set up the joint probability mass function for N and X, in terms of the parameters λ and p.
b) Write an `R` function with input λ, p, k that simulates k values of X.
c) Pick some values of λ and p and simulate sufficiently many instances in each case to obtain an estimate of $\mathcal{E}(X)$. Use `sapply` or `replicate`, do not use `for` loops. Then guess a formula for $\mathcal{E}(X)$ and explain why it makes sense to you. *To document this, only state your choice of λ and p, the number of simulations, and your estimate for the expected value in each case.*

32. (5 points) Let $X \sim B(80, .2)$ and $Y \sim B(100, .7)$ be independent binomial random variables. Let $Z = X + Y$. Find the following conditional quantities, using `R` simulations or exact computations:
a) $P(X < 12|X < 18)$ and $E(X|X < 18)$ (exact computation)
b) the cumulative distribution function of $X|(12 \leq X \leq 20)$ (plot of the exact cdf)
c) the cumulative distribution function of $X|Z = 90$ (simulation and plot of the ecdf)
d) $E(Z|X = k)$ for $k = 110$, 15, 20 (simulation)
e) $E(X|Z = k)$ for $k = 80$, 90, 100 (simulation).