HW8

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Problem 1 (6.4-3)

$f(x_1=5;\theta)=\frac{\theta}{2\sqrt{x_1}}e^{-\theta\sqrt{x_1}}=\frac{\theta}{2\sqrt{5}}e^{-\sqrt{5}\theta}$
$f(x_2=9;\theta)=\frac{\theta}{2\sqrt{x_2}}e^{-\theta\sqrt{x_2}}=\frac{\theta}{6}e^{-3\theta}$
$f(x_3=9;\theta)=\frac{\theta}{2\sqrt{x_3}}e^{-\theta\sqrt{x_3}}=\frac{\theta}{6}e^{-3\theta}$
$f(x_4=10;\theta)=\frac{\theta}{2\sqrt{x_4}}e^{-\theta\sqrt{x_4}}=\frac{\theta}{2\sqrt{10}}e^{-\sqrt{10}\theta}$
$\mathcal{L}(\theta;x_1=5,x_2=9,x_3=9,x_4=10)=f(x_1=5,x_2=9,x_3=9,x_4=10;\theta)=\frac{\theta^4}{720\sqrt{2}}e^{(-\sqrt{5}-3-3-\sqrt{10})\theta}$
$\ell(\theta;x_1=5,x_2=9,x_3=9,x_4=10)=\frac{\theta^3}{180\sqrt{2}}e^{(-\sqrt{5}-3-3-\sqrt{10})\theta}+\frac{\theta^4}{720\sqrt{2}}e^{(-\sqrt{5}-3-3-\sqrt{10})\theta}(-\sqrt{5}-3-3-\sqrt{10})$
Set

$$\ell=0$$
$$\frac{\hat{\theta}^3}{180\sqrt{2}}e^{(-\sqrt{5}-3-3-\sqrt{10})\hat{\theta}}=\frac{\hat{\theta}^4}{720\sqrt{2}}e^{(-\sqrt{5}-3-3-\sqrt{10})\hat{\theta}}(\sqrt{5}+3+3+\sqrt{10})$$
$$1=(\sqrt{5}+3+3+\sqrt{10})\frac{\hat{\theta}}{4}$$
$$\hat{\theta}=\frac{4}{\sqrt{5}+3+3+\sqrt{10}}$$

Problem 2 (6.4-7)

$\mathcal{L}(\theta;x_1,\dots,x_n)=f(x_1,\dots,x_n;\theta)=\frac{e^{-\sum_{i=1}^n x_i}}{(1-e^{-\theta})^n}$
$log\mathcal{L}(\theta;x_1,\dots,x_n)=-\sum_{i=1}^n x_i-nlog(1-e^{-\theta})$
$(log\mathcal{L})'=-\frac{ne^{-\theta}}{1-e^{-\theta}}=-\frac{n}{e^{\theta}-1}$
Since $(log\mathcal{L})'$ can never be equal to 0, the MLE of θ does not exist.

Problem 3 (6.4-10)

$f(x=95;\mu)=\frac{1}{\sqrt{2\pi}15}e^{-\frac{(x-\mu)^2}{2\times15^2}}=\frac{1}{\sqrt{2\pi}15}e^{-\frac{(95-\mu)^2}{2\times15^2}}$
$f(y=130;\mu)=\frac{1}{\sqrt{2\pi}20}e^{-\frac{(y-\mu)^2}{2\times20^2}}=\frac{1}{\sqrt{2\pi}20}e^{-\frac{(130-\mu)^2}{2\times20^2}}$
$\mathcal{L}(\mu;x=95,y=130)=f(x=95,y=130;\mu)=\frac{1}{2\pi\times15\times20}e^{-\frac{(95-\mu)^2}{2\times15^2}-\frac{(130-\mu)^2}{2\times20^2}}$
$log\mathcal{L}=log\frac{1}{2\pi\times15\times20}-\frac{(95-\mu)^2}{2\times15^2}-\frac{(130-\mu)^2}{2\times20^2}$
$(log\mathcal{L})'=\frac{95-\mu}{15^2}+1.3\frac{130-1.3\mu}{20^2}$
Set

$$(log\mathcal{L})'=0$$
$$\frac{95-\hat{\mu}}{15^2}+1.3\frac{130-1.3\hat{\mu}}{20^2}=0$$
$$\hat{\mu}\approx 97.4367$$

Problem 4 (6.4-25)

$E[X] = a_1 E[X_1] + \cdots + a_n E[X_n] = a_1\mu + \cdots + a_n\mu = (a_1 + \cdots + a_n)\mu = \mu$
So, for constants $a_1, \ldots, a_n$, under the condition $a_1 + \cdots + a_n = 1$, X is an unbiased estimator.

Problem 5 (6.4-28)

2 unbiased estimators of θ are $\frac{10}{9}\hat{\theta}_1$ and $\frac{5}{6}\hat{\theta}_2$. $E[\frac{10}{9}\hat{\theta}_1] = E[\frac{5}{6}\hat{\theta}_2] = \theta$
$Var(\frac{10}{9}\hat{\theta}_1) = \frac{100}{81}Var(\hat{\theta}_1) = \frac{100}{27}$
$Var(\frac{5}{6}\hat{\theta}_2) = \frac{25}{36}Var(\hat{\theta}_2) = \frac{25}{18}$
$Relative\ Efficiency = \frac{Var(\frac{5}{6}\hat{\theta}_2)}{Var(\frac{10}{9}\hat{\theta}_1)} = \frac{8}{3}$
Since $Relative\ Efficiency > 1$, $\frac{5}{6}\hat{\theta}_2$ is more efficient.

Problem 6 (6.4-8)

$\mathcal{L}(\theta; x_1, \ldots, x_n) = f(x_1, \ldots, x_n; \theta) = (\sqrt{\frac{2}{\pi}})^n \prod_{i=1}^n x_i^2 e^{-\frac{\sum_{i=1}^n x_i^2}{2\theta^2}} \frac{1}{\theta^{3n}}$
$log\mathcal{L}(\theta; x_1, \ldots, x_n) = nlog\sqrt{\frac{2}{\pi}} + 2\sum_{i=1}^n logx_i - \frac{\sum_{i=1}^n x_i^2}{2\theta^2} - 3nlog\theta$
$(log\mathcal{L})'(\theta; x_1, \ldots, x_n) = \frac{\sum_{i=1}^n x_i^2}{\theta^3} - \frac{3n}{\theta}$
Set

$$(log\mathcal{L})' = 0$$
$$\frac{\sum_{i=1}^n x_i^2}{\hat{\theta}^3} - \frac{3n}{\hat{\theta}} = 0$$
$$\hat{\theta} = \sqrt{\frac{\sum_{i=1}^n x_i^2}{3n}}$$

Problem 7 (6.4-29)

From the pdf, we know that X_1, X_2, X_3 follows exponential distribution with mean θ.
$E[\hat{\theta}_1] = E[X_1] = \theta$
$E[\hat{\theta}_2] = E[\frac{X_1+X_2}{2}] = \frac{1}{2}(E[X_1] + E[X_2]) = \theta$
$E[\hat{\theta}_3] = E[\frac{X_1+2X_2}{3}] = \frac{1}{3}(E[X_1] + 2E[X_2]) = \theta$
Therefore, all θ_i are unbiased.
$Var(\hat{\theta}_1) = Var(X_1) = \theta^2$
$Var(\hat{\theta}_2) = Var(\frac{X_1+X_2}{2}) = \frac{1}{4}(Var(X_1) + Var(X_2)) = \frac{1}{2}\theta^2$
$Var(\hat{\theta}_3) = Var(\frac{X_1+2X_2}{3}) = \frac{1}{9}(Var(X_1) + 4Var(X_2)) = \frac{5}{9}\theta^2$
Since $Var(\hat{\theta}_1) > Var(\hat{\theta}_3) > Var(\hat{\theta}_2)$, $\hat{\theta}_2$ is the most efficient estimator.

Problem 8 (6.4-36)

(a)

$$\begin{aligned}
E[W] &= E[a\bar{X} + (1-a)\bar{Y}] \\
&= E[\frac{a}{n}\sum_{i=1}^{n} X_i + \frac{1-a}{n}\sum_{i=1}^{n} Y_i] \\
&= \frac{a}{n}\sum_{i=1}^{n} E[X_i] + \frac{1-a}{n}\sum_{i=1}^{n} E[Y_i] \\
&= \frac{a}{n}n\mu + \frac{1-a}{n}n\mu \\
&= \mu
\end{aligned}$$

(b)

$$\begin{aligned}
Var(W) &= Var(a\bar{X} + (1-a)\bar{Y}) \\
&= Var(\frac{a}{n}\sum_{i=1}^{n} X_i + \frac{1-a}{n}\sum_{i=1}^{n} Y_i) \\
&= \frac{a^2}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{(1-a)^2}{n^2}\sum_{i=1}^{n} Var(Y_i) \\
&= \frac{a^2\sigma_1^2}{n} + \frac{(1-a)^2\sigma_2^2}{n} \\
(Var(W))' &= \frac{2\sigma_1^2}{n}a - \frac{2\sigma_2^2}{n}(1-a) \\
\text{Set } (Var(W))' &= 0 \\
\frac{2\sigma_1^2}{n}a &= \frac{2\sigma_2^2}{n}(1-a) \\
a &= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}
\end{aligned}$$

Since $Var(W)$ is a convex function, $a = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$ is the solution to minimize variance of W.

Bonus 1 (6.4-14)

```
nums <- c(2,3,5,9,10)
mean(nums)
```

```
## [1] 5.8
```

```
var(nums)
```

```
## [1] 12.7
```

Mean is 5.8 and Variance is 12.7.

$$\begin{cases} \frac{1}{2}(\hat{\alpha} + \hat{\beta}) = 5.8 \\ \frac{1}{12}(\hat{\beta} - \hat{\alpha})^2 = 12.7 \end{cases} \implies \begin{cases} \hat{\alpha} = 5.8 - \sqrt{38.1} \approx -0.3725 \\ \hat{\beta} = 5.8 + \sqrt{38.1} \approx 11.9725 \end{cases}$$

Bonus 2 (6.4-19)

(a) Method of Moments

$$E[X|\theta] = \int_2^\infty x\frac{\theta 2^\theta}{x^{\theta+1}}dx$$
$$= \int_2^\infty \frac{\theta 2^\theta}{x^\theta}dx$$
$$= -\frac{1}{-\theta+1}\theta 2^\theta x^{-\theta+1}\Big|_2^\infty$$
$$= \frac{2\theta}{\theta-1}$$
$$\bar{x} = \frac{2\hat{\theta}}{\hat{\theta}-1}$$
$$\hat{\theta} = \frac{\bar{x}}{\bar{x}-2} \text{ where } \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$$

(b) Maximum Likelihood

$\mathcal{L}(\theta; x_1, \ldots, x_n) = f(x_1, \ldots, x_n; \theta) = \frac{\theta^n 2^{n\theta}}{\prod_{i=1}^n x_i^{\theta+1}}$

$log\mathcal{L}(\theta; x_1, \ldots, x_n) = nlog\theta + n\theta log2 - \sum_{i=1}^n (\theta+1)logx_i$

$(log\mathcal{L})'(\theta; x_1, \ldots, x_n) = \frac{n}{\theta} + nlog2 - \sum_{i=1}^n logx_i$

Set

$$(log\mathcal{L})' = 0$$
$$\frac{n}{\hat{\theta}} + nlog2 - \sum_{i=1}^n logx_i = 0$$
$$\hat{\theta} = \frac{n}{\sum_{i=1}^n logx_i - nlog2}$$
$$= \frac{n}{log\frac{\prod_{i=1}^n x_i}{2^n}}$$
$$= \frac{n}{log\prod_{i=1}^n \frac{x_i}{2}}$$
$$= \frac{n}{\sum_{i=1}^n log\frac{x_i}{2}}$$