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Отчёт по курсовой работе
«Разработка программы для моделирования стационарного
двумерного распределения температуры»
по дисциплине
«Математические модели систем с распределёнными параметрами»

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Постановка задачи

Вариант N4.

Постановка задачи. Используя интегро-интерполяционный метод, разработать подпрограмму для моделирования распределения температуры в цилиндре, описываемого математической моделью

$$-\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r k_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2(r, z) \frac{\partial u}{\partial z} \right) \right] = f(r, z),$$

$$0 < c_{11} \leq k_1(r, z) \leq c_{12}, \quad 0 < c_{21} \leq k_2(r, z) \leq c_{22}$$

$$0 \leq r \leq R, \quad 0 \leq z \leq L$$

с граничными условиями, определяемыми вариантом задания. Для решения системы алгебраических уравнений использовать метод сопряжённых градиентов с предобуславливанием. Матрица алгебраической системы должна храниться в упакованной форме

Форма (4)

Форма (4) отличается от формы (3) тем, что индексы главных диагональных элементов не хранятся и элементы главной диагонали располагаются в отдельном массиве Diag. В массиве A хранятся ненулевые элементы строго верхней треугольной части матрицы. Так как матрица хранится построчно, то в массиве IC хранятся номера столбцов ненулевых элементов верхнего треугольника матрицы. В массиве IR хранятся указатели на начало каждой строки в массивах A и IC. IR(N+1) содержит количество ненулевых элементов в строго верхнем треугольнике матрицы A плюс один.

	1	2	3	4	5	6	7	8	9
DIAG	13	14	15	16	17	18	19	20	21

	1	2	3	4	5	6	7	8	9	10	11	12
A	7	1	8	2	3	9	4	10	5	6	11	12
IC	2	4	3	5	6	5	7	6	8	9	8	9

	1	2	3	4	5	6	7	8	9	10
IR	1	3	5	6	8	10	11	12	13	13

$$u|_{r=0} - \text{ограничено},$$

$$u|_{z=0} = \varphi_3(r)$$

$$-k_1(r) \frac{\partial u}{\partial r} \Big|_{r=R} = \chi_2 u|_{r=R} - \varphi_2(z), \quad \chi_2 \geq 0$$

$$-k_2 \frac{\partial u}{\partial z} \Big|_{z=L} = \chi_4 u|_{z=L} - \varphi_4(r), \quad \chi_4 \geq 0$$

Дискретная модель

Введем в прямоугольнике $[0, R] \times [0, L]$ равномерную основную сетку

$$r_i = ih_r$$

$$h_r = \frac{R}{N_r}$$

$$z_j = jh_z$$

$$h_z = \frac{L}{N_z}$$

и вспомогательную сетку

$$r_{i-\frac{1}{2}} = \frac{r_i + r_{i-1}}{2}, \quad i = 1, 2, \dots, N_r$$

$$z_{j-\frac{1}{2}} = \frac{z_j + z_{j-1}}{2}, \quad j = 1, 2, \dots, N_z$$

Так как используются равномерные сетки, то шаги вспомогательной сетки определяются как

$$h_i = \begin{cases} h_r, & i = 1, \dots, N_r - 1 \\ \frac{h_r}{2}, & i = 0, N_r \end{cases} \quad h_j = \begin{cases} h_z, & j = 1, \dots, N_z \\ \frac{h_z}{2}, & j = 0, N_z + 1 \end{cases}$$

Умножим исходное уравнение на r , проинтегрируем по вспомогательной сетке:

$$\begin{aligned} & - \left[\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \frac{\partial}{\partial r} \left(rk(r) \frac{\partial u}{\partial r} \right) dr dz + \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r \frac{\partial^2 u}{\partial z^2} dr dz \right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r f dr dz \\ & - \left[\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \left(r_{i+\frac{1}{2}} k \left(r_{i+\frac{1}{2}} \right) \frac{\partial u}{\partial r} \Big|_{r_{i+\frac{1}{2}}} \right) dz - \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \left(r_{i-\frac{1}{2}} k \left(r_{i-\frac{1}{2}} \right) \frac{\partial u}{\partial r} \Big|_{r_{i-\frac{1}{2}}} \right) dz \right. \\ & \quad \left. + \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \left(r \frac{\partial u}{\partial z} \Big|_{z_{j+\frac{1}{2}}} \right) dr - \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \left(r \frac{\partial u}{\partial z} \Big|_{z_{j-\frac{1}{2}}} \right) dr \right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r f dr dz \end{aligned}$$

Воспользуемся формулой средних прямоугольников для вычисления значений интегралов:

$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \phi(r, z) dr \approx h_r \phi(r_i, z) = h_r \phi_i$$

$$\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \phi(r, z) dz \approx h_z \phi(r, z_j) = h_r \phi_j$$

$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r_i \phi dr dz \approx r_i h_r h_z \phi_{i,j}$$

Также аппроксимируем производные по формуле центральных разностей:

$$k \left(r_{i+\frac{1}{2}} \right) \frac{\partial u}{\partial r} \Big|_{r=r_{i+\frac{1}{2}}, z=z_j} = k \left(r_{i+\frac{1}{2}} \right) \frac{u_{i+1,j} - u_{i,j}}{h_r}$$

$$k \left(r_{i-\frac{1}{2}} \right) \frac{\partial u}{\partial r} \Big|_{r=r_{i-\frac{1}{2}}, z=z_j} = k \left(r_{i-\frac{1}{2}} \right) \frac{u_{i,j} - u_{i-1,j}}{h_r}$$

$$\frac{\partial u}{\partial z} \Big|_{z=z_{j+\frac{1}{2}}, r=r_j} = \frac{u_{i,j+1} - u_{i,j}}{h_z}$$

$$\frac{\partial u}{\partial z} \Big|_{z=z_{j-\frac{1}{2}}, r=r_j} = \frac{u_{i,j} - u_{i,j-1}}{h_z}$$

Получим:

$$\begin{aligned} & - \left[h_z r_{i+\frac{1}{2}} k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z r_{i-\frac{1}{2}} k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r} \right. \\ & \quad \left. + h_r r_i k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r r_i k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j} - u_{i,j-1}}{h_z} \right] \\ & = r_i h_r h_z f_{i,j} \end{aligned}$$

..... (1)

$$i=1,2,\dots,N_r - 1 ; j = 1,2,\dots,N_z - 1$$

Аппроксимация граничных условий:

$$\begin{aligned}
 & i = 0, \quad j = 1, \dots, N_z - 1, \quad u|_{r=0} - \text{ограниченно}, \\
 & - \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r k_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2(r, z) \frac{\partial u}{\partial z} \right) \right] = f(r, z) \\
 & - \left[\frac{\partial}{\partial r} \left(k_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{1}{r} k_1(r, z) \frac{\partial u}{\partial r} + \frac{\partial}{\partial z} \left(k_2(r, z) \frac{\partial u}{\partial z} \right) \right] \Big|_{r=0} = f(r, z)|_{r=0} \\
 & \lim_{r \rightarrow 0} \frac{1}{r} k_1(r, z) \frac{\partial u}{\partial r}, \quad \frac{\partial u}{\partial r} \Big|_{r=0} = 0, \\
 & \lim_{r \rightarrow 0} \frac{1}{r} k_1(r, z) \frac{\partial u}{\partial r} = \frac{\partial}{\partial r} \left(k_1(r, z) \frac{\partial u}{\partial r} \right) \Big|_{r=0} \\
 & - \left[2 \frac{\partial}{\partial r} \left(k_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2(r, z) \frac{\partial u}{\partial z} \right) \right] \Big|_{r=0} = f(r, z)|_{r=0} \\
 & u|_{r=0} - \text{ограниченно} \Rightarrow \frac{\partial u}{\partial r} \Big|_{r=0} = 0 \\
 & \int_{r_i}^{r_{i+1/2}} r \phi(r, z) dr \approx \phi(r_i, z) \int_{r_i}^{r_{i+1/2}} r dr = \phi(r_i, z) \frac{r_{i+1/2}^2}{2} = h_i \frac{r_{i+1/2}}{2} \phi(r_i, z), i = 0, r_i = 0, r_{i+1/2} = h_i \\
 & - \left[h_j r_{i+1/2} k_1(r_{i+1/2}, z_j) \frac{v_{i+1,j} - v_{i,j}}{h_{i+1}} - 0 \right. \\
 & \quad \left. + h_i \frac{r_{i+1/2}}{2} k_2(r_i, z_{j+1/2}) \frac{v_{i,j+1} - v_{i,j}}{h_{j+1}} - h_i \frac{r_{i+1/2}}{2} k_2(r_i, z_{j-1/2}) \frac{v_{i,j} - v_{i,j-1}}{h_j} \right] \\
 & = h_i h_j \frac{r_{i+1/2}}{2} f_{i,j} \\
 & - \left[h_z r_{i+1/2} k_1(r_{i+1/2}, z_j) \frac{v_{i+1,j} - v_{i,j}}{h_r} - 0 \right. \\
 & \quad \left. + \frac{h_r}{2} \frac{r_{i+1/2}}{2} k_2(r_i, z_{j+1/2}) \frac{v_{i,j+1} - v_{i,j}}{h_z} - \frac{h_r}{2} \frac{r_{i+1/2}}{2} k_2(r_i, z_{j-1/2}) \frac{v_{i,j} - v_{i,j-1}}{h_z} \right] \\
 & = \frac{h_r}{2} h_z \frac{r_{i+1/2}}{2} f_{i,j}
 \end{aligned}$$

Аналогично воспользуемся интегро-интерполяционным методом, получим:

$$\begin{aligned}
 & - \left[h_z r_{i+\frac{1}{2}} k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z r_{i-\frac{1}{2}} k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r} \right. \\
 & \quad \left. + h_r r_i k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r r_i k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j} - u_{i,j-1}}{h_z} \right] \\
 & = r_i h_r h_z f_{i,j} \quad \text{при } i = 1, 2, \dots, N_r - 1; j = 1, 2, \dots, N_z - 1 \quad \dots (1) \\
 & - \left[-h_z R \left(\chi_2 u_{N,j} - \varphi_2(z_j) \right) - h_z r_{N-\frac{1}{2}} k_1 \left(r_{N-\frac{1}{2}}, z_j \right) \frac{u_{N,j} - u_{N-1,j}}{h_r} \right. \\
 & \quad \left. + \frac{h_r}{2} R k_2 \left(R, z_{j+\frac{1}{2}} \right) \frac{u_{N,j+1} - u_{N,j}}{h_z} - \frac{h_r}{2} R k_2 \left(R, z_{j-\frac{1}{2}} \right) \frac{u_{N,j} - u_{N,j-1}}{h_z} \right] \\
 & = \frac{h_r}{2} R h_z f_{N,j} \quad \text{при } i = N_r; j = 1, 2, \dots, N_z - 1 \quad \dots (2)
 \end{aligned}$$

$$u_{i,0} = \varphi_3(0) \text{ при } i = 0, \dots, N_r; j = 0 \dots (3)$$

$$\begin{aligned} & - \left[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1 \left(r_{i+\frac{1}{2}}, L \right) \frac{u_{i+1,N} - u_{i,N}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1 \left(r_{i-\frac{1}{2}}, L \right) \frac{u_{i,N} - u_{i-1,N}}{h_r} \right. \\ & \quad \left. - h_r r_i \left(\chi_4 u_{i,N} - \varphi_4(r_i) \right) - h_r r_i k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,N} - u_{i,N-1}}{h_z} \right] \\ & = \frac{r_i h_r h_z f_{i,N}}{2} \text{ при } i = 1, 2, \dots, N_r; j = N_z \dots (4) \end{aligned}$$

$$\begin{aligned} & - \left[h_z r_{i+\frac{1}{2}} k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - 0 + h_r \frac{r_{i+\frac{1}{2}}}{4} k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r \frac{r_{i+\frac{1}{2}}}{4} k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j} - u_{i,j-1}}{h_z} \right] \\ & = h_r h_z \frac{r_{i+\frac{1}{2}}}{4} f_{i,j} \text{ при } i = 0; j = 1, 2, \dots, N_z - 1 \dots (5) \end{aligned}$$

В результате получается система линейных алгебраических уравнений вида $Au=b$ размерности $N=(N_z-1)(N_r+1)$ Рассмотрим более подробно структуру этой системы. Для дальнейшей работы необходимо перенумеровать компоненты векторов u и b . Для этого используем приведенный индекс. Сперва для фиксированного r движемся по оси z , потом переходим к следующему значению r .

$$u_{i,j} = v_k$$

$$u_{i,j-1} = v_{k-1}$$

$$u_{i,j+1} = v_{k+1}$$

$$u_{i+1,j} = v_{k+N_z+1}$$

$$u_{i-1,j} = v_{k-N_z+1}$$

При таком обозначении новый индекс k можно рассчитать так: $k=i*(N_z-1)+j$

Матрица A квадратная, симметричная, пятидиагональная.

Хранить будем только 3 диагонали.

$$A = \begin{pmatrix} a & b & & c & & & & & & & & & \\ b & a & b & & c & & & & & & & & \\ & b & a & 0 & & c & & & & & & & \\ c & & 0 & a & b & & c & & & & & & \\ & c & & b & a & b & & c & & & & & \\ & & c & & b & a & 0 & & c & & & & \\ & & & c & & 0 & a & b & & c & & & \\ & & & & c & & b & a & b & & c & & \\ & & & & & c & & b & a & 0 & & c & \\ & & & & & & c & & 0 & a & b & & \\ & & & & & & & c & & b & a & b & \\ & & & & & & & & c & & b & a & \end{pmatrix}$$

Анализ порядка аппроксимации уравнения и граничных условий, выражение для главного члена погрешности аппроксимации

Невязка и порядок погрешность аппроксимации уравнения

Преобразование:

$$\begin{aligned} - \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r k_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2(r, z) \frac{\partial u}{\partial z} \right) \right] &= f(r, z) \\ - \left[\frac{\partial}{\partial r} \left(r k_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(r k_2(r, z) \frac{\partial u}{\partial z} \right) \right] &= r f(r, z) \\ \tilde{k}_1(r, z) &= r k_1(r, z), \quad \tilde{k}_2(r, z) = r k_2(r, z), \quad \tilde{q}(r, z) = r q(r, z) \\ \tilde{f}(r, z) &= r f(r, z) \end{aligned}$$

$$- \left[\frac{\partial}{\partial r} \left(\tilde{k}_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(\tilde{k}_2(r, z) \frac{\partial u}{\partial z} \right) \right] = \tilde{f}(r, z)$$

При анализе порядка аппроксимации, для простого, будем писать просто k_1, k_2, f вместо $\tilde{k}_1, \tilde{k}_2, \tilde{f}$

Невязка определяется как разность между правой и левой частью уравнения при условии, что вместо приближенного решения мы подставляем туда точное:

$$\begin{aligned} \xi_{i,j} &= h_r h_z f_{i,j} + h_z k_1(x_{i+1/2}, y_j) \frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1(x_{i-1/2}, y_j) \frac{u_{i,j} - u_{i-1,j}}{h_r} \\ &\quad + h_r k_2(x_i, y_{j+1/2}) \frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r k_2(x_i, y_{j-1/2}) \frac{u_{i,j} - u_{i,j-1}}{h_z} \end{aligned}$$

Раскладываем по степеням h точное решение в узлах и коэффициент k

$$u_{i+1,j} = u(x_i + h_r, y_j)$$

$$= u_{i,j} + h_r \frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2} \frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^3}{6} \frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24} \frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i+1,j} - u_{i,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} + \frac{h_r}{2} \frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6} \frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^3}{24} \frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i+\frac{1}{2},j} = k_1\left(r_i + \frac{h_r}{2}, z_j\right)$$

$$= k_{1,i,j} + \frac{h_r}{2} \frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8} \frac{\partial^2 k_{1,i,j}}{\partial r^2} + \frac{h_r^3}{48} \frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$\begin{aligned} k_{1,i+\frac{1}{2},j} \frac{u_{i+1,j} - u_{i,j}}{h_r} &= \left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} + h_r \left[\frac{1}{2} k_1 \frac{\partial^2 u}{\partial r^2} + \frac{1}{2} \frac{\partial k_1}{\partial r} \frac{\partial u}{\partial r} \right]_{i,j} + h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \right. \\ &\quad \left. \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} + h_r^3 \left[\frac{1}{24} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{16} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \right. \\ &\quad \left. \frac{1}{48} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^4) \end{aligned}$$

$$u_{i-1,j} = u(r_i - h_r, z_j) = u_{i,j} - h_r \frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2} \frac{\partial^2 u_{i,j}}{\partial r^2} - \frac{h_r^3}{6} \frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24} \frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i,j} - u_{i-1,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} - \frac{h_r}{2} \frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6} \frac{\partial^3 u_{i,j}}{\partial r^3} - \frac{h_r^3}{24} \frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i-\frac{1}{2},j} = k_1\left(r_i - \frac{h_r}{2}, z_j\right) = k_{1,i,j} - \frac{h_r}{2} \frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8} \frac{\partial^2 k_{1,i,j}}{\partial r^2} - \frac{h_r^3}{48} \frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$\begin{aligned} k_{1,i-\frac{1}{2},j} \frac{u_{i,j} - u_{i-1,j}}{h_r} &= \left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} - h_r \left[\frac{1}{2} k_1 \frac{\partial^2 u}{\partial r^2} + \frac{1}{2} \frac{\partial k_1}{\partial r} \frac{\partial u}{\partial r} \right]_{i,j} + h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \right. \\ &\quad \left. \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} - h_r^3 \left[\frac{1}{24} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{16} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \right. \\ &\quad \left. \frac{1}{48} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^4) \end{aligned}$$

$$\begin{aligned}
& h_z k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r} \\
& \left[\left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} + h_r \left[\frac{1}{2} k_1 \frac{\partial^2 u}{\partial r^2} + \frac{1}{2} \frac{\partial k_1}{\partial r} \frac{\partial u}{\partial r} \right]_{i,j} + h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} + \right. \\
& \left. + h_r^3 \left[\frac{1}{24} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{16} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{48} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right]_{i,j} - \right. \\
& = h_z \left[- \left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} + h_r \left[\frac{1}{2} k_1 \frac{\partial^2 u}{\partial r^2} + \frac{1}{2} \frac{\partial k_1}{\partial r} \frac{\partial u}{\partial r} \right]_{i,j} - h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} \right. \\
& \left. + h_r^3 \left[\frac{1}{24} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{16} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{48} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^4) \right]
\end{aligned}$$

Сокращаются четные степени

$$h_z k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r} = h_z \left[h_r \left(k_1 \frac{\partial^2 u}{\partial r^2} + \frac{\partial k_1}{\partial r} \frac{\partial u}{\partial r} \right)_{i,j} + h_r^3 \left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right)_{i,j} + O(h_r^4) \right]$$

т.к. $k_1 \frac{\partial^2 u}{\partial r^2} + \frac{\partial k_1}{\partial r} \frac{\partial u}{\partial r} = \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right)$, получаем, что

$$\begin{aligned}
& h_z k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r} = h_z \left[h_r \left(\frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) \right)_{i,j} + \right. \\
& \left. h_r^3 \left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right)_{i,j} + O(h_r^4) \right]
\end{aligned}$$

$$u_{i,j+1} = u(r_i, z_j + h_z) = u_{i,j} + h_z \frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2} \frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^3}{6} \frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24} \frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\frac{u_{i,j+1} - u_{i,j}}{h_z} = \frac{\partial u_{i,j}}{\partial z} + \frac{h_z}{2} \frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^2}{6} \frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^3}{24} \frac{\partial^4 u_{i,j}}{\partial z^4} + O(h^4)$$

$$k_{2,i,j+\frac{1}{2}} = k_2 \left(r_i, z_j + \frac{h_z}{2} \right) = k_{2,i,j} + \frac{h_z}{2} \frac{\partial k_{2,i,j}}{\partial z} + \frac{h_z^2}{8} \frac{\partial^2 k_{2,i,j}}{\partial z^2} + \frac{h_z^3}{48} \frac{\partial^3 k_{2,i,j}}{\partial z^3} + O(h_z^4)$$

$$\begin{aligned}
& k_{2,i,j+\frac{1}{2}} \frac{u_{i+1,j} - u_{i,j}}{h_z} = \left[k_2 \frac{\partial u}{\partial z} \right]_{i,j} + h_z \left[\frac{1}{2} k_2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{2} \frac{\partial k_2}{\partial z} \frac{\partial u}{\partial z} \right]_{i,j} + h_z^2 \left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \right. \\
& \left. \frac{1}{4} \frac{\partial k_2}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial u}{\partial z} \right]_{i,j} + h_z^3 \left[\frac{1}{24} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{12} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{16} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \right. \\
& \left. \frac{1}{48} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right]_{i,j} + O(h_z^4)
\end{aligned}$$

$$u_{i,j-1} = u(r_i, z_j - h_z) = u_{i,j} - h_z \frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2} \frac{\partial^2 u_{i,j}}{\partial z^2} - \frac{h_z^3}{6} \frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24} \frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\begin{aligned}
\frac{u_{i,j}-u_{i,j-1}}{h_z} &= \frac{\partial u_{i,j}}{\partial z} - \frac{h_z}{2} \frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^2}{6} \frac{\partial^3 u_{i,j}}{\partial z^3} - \frac{h_z^3}{24} \frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^4) \\
k_{2,i,j-\frac{1}{2}} &= k_2 \left(r_i, z_j - \frac{h_z}{2} \right) = k_{2,i,j} - \frac{h_z}{2} \frac{\partial k_{2,i,j}}{\partial z} + \frac{h_z^2}{8} \frac{\partial^2 k_{2,i,j}}{\partial z^2} - \frac{h_z^3}{48} \frac{\partial^3 k_{2,i,j}}{\partial z^3} + O(h_z^4) \\
k_{2,i,j-\frac{1}{2}} \frac{u_{i,j}-u_{i,j-1}}{h_z} &= \left[k_2 \frac{\partial u}{\partial z} \right]_{i,j} - h_z \left[\frac{1}{2} k_2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{2} \frac{\partial k_2}{\partial z} \frac{\partial u}{\partial z} \right]_{i,j} + h_z^2 \left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \right. \\
&\quad \left. \frac{1}{4} \frac{\partial k_2}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial u}{\partial z} \right]_{i,j} - h_z^3 \left[\frac{1}{24} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{12} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{16} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \right. \\
&\quad \left. \frac{1}{48} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right]_{i,j} + O(h_z^4)
\end{aligned}$$

$$\begin{aligned}
h_r k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j}-u_{i,j-1}}{h_z} = \\
h_r \left[\left[k_2 \frac{\partial u}{\partial r} \right]_{i,j} + h_z \left[\frac{1}{2} k_2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{2} \frac{\partial k_2}{\partial z} \frac{\partial u}{\partial z} \right]_{i,j} + h_z^2 \left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4} \frac{\partial k_2}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial u}{\partial z} \right]_{i,j} + \right. \\
\left. + h_z^3 \left[\frac{1}{24} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{12} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{16} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{48} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right]_{i,j} - \right. \\
\left. - \left[k_2 \frac{\partial u}{\partial z} \right]_{i,j} + h_z \left[\frac{1}{2} k_2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{2} \frac{\partial k_2}{\partial z} \frac{\partial u}{\partial z} \right]_{i,j} - h_z^2 \left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4} \frac{\partial k_2}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial u}{\partial z} \right]_{i,j} + \right. \\
\left. + h_z^3 \left[\frac{1}{24} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{12} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{16} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{48} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right]_{i,j} + O(h_z^4) \right]
\end{aligned}$$

Четные степени сокращаются

$$\begin{aligned}
h_r k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j}-u_{i,j-1}}{h_z} &= h_r \left[h_z \left(k_2 \frac{\partial^2 u}{\partial z^2} + \right. \right. \\
&\quad \left. \left. \frac{\partial k_2}{\partial z} \frac{\partial u}{\partial z} \right)_{i,j} + h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right]
\end{aligned}$$

Так как $k_2 \frac{\partial^2 u}{\partial z^2} + \frac{\partial k_2}{\partial z} \frac{\partial u}{\partial z} = \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right)$, получаем, что

$$\begin{aligned}
h_r k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j}-u_{i,j-1}}{h_z} &= h_r \left[h_z \left(\frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right)_{i,j} + \right. \\
&\quad \left. h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right]
\end{aligned}$$

Подставляем в невязку получившиеся разложения

$$\begin{aligned}
\xi_{i,j} &= h_r h_z f_{i,j} + h_z k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j}-u_{i-1,j}}{h_r} + \\
&\quad h_r k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r h_z f_{i,j} +
\end{aligned}$$

$$h_z \left[h_r \left(\frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) \right)_{i,j} + h_r^3 \left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right)_{i,j} + \right. \\ \left. O(h_r^4) \right] + h_r \left[h_z \left(\frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right)_{i,j} + h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \right. \right. \\ \left. \left. \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right]$$

Группируем по степени h_r и h_z

$$\xi_{i,j} = h_r h_z f_{i,j} + h_z k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r} + \\ h_r k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j} - u_{i,j-1}}{h_z} = h_r h_z \left[f + \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \right. \\ \left. \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right]_{i,j} + h_z \left[h_r^3 \left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right)_{i,j} + \right. \\ \left. O(h_r^4) \right] + h_r \left[h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right]$$

Чтобы вычислить порядок аппроксимации, нормируем невязку

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{h_r h_z}$$

$$\tilde{\xi}_{i,j} = f_{i,j} + k_1 \left(r_{i+\frac{1}{2}}, z_j \right) \frac{u_{i+1,j} - u_{i,j}}{h_r^2} - k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r^2} + \\ k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1} - u_{i,j}}{h_z^2} - k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j} - u_{i,j-1}}{h_z^2} = \left[f + \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \right. \\ \left. \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right]_{i,j} + h_r^2 \left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right)_{i,j} + O(h_r^3) + \\ h_z^2 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^3)$$

Выполним обратную замену.

$$\begin{aligned}
\xi_{i,j} = & \left[rf + \frac{\partial}{\partial r} \left(rk_1 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(rk_2 \frac{\partial u}{\partial z} \right) \right]_{i,j} \\
& + h_r^2 \left(\frac{1}{12} rk_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial rk_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 rk_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 rk_1}{\partial r^3} \frac{\partial u}{\partial r} \right)_{i,j} \\
& + O(h_r^3) \\
& + rh_z^2 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} \\
& + O(h_z^3)
\end{aligned}$$

$$\left[rf + \frac{\partial}{\partial r} \left(rk_1 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(rk_2 \frac{\partial u}{\partial z} \right) \right]_{i,j} = 0$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2$$

$$p_z = 2 - 0 = 2$$

Главный член погрешности по r

$$\Phi_r = \frac{1}{12} rk_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial rk_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 rk_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 rk_1}{\partial r^3} \frac{\partial u}{\partial r}$$

Главный член погрешности по z

$$\Phi_z = r \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)$$

Невязка и порядок погрешности аппроксимации граничного условия

1) $u|_{r=0} - \text{ограниченно} \Rightarrow \frac{\partial u}{\partial r} \Big|_{r=0} = 0, i = 0, j = 1, 2, \dots, N_z - 1$ (Лекция 10. p27)

$$\xi_{i,j} = \frac{h_r}{2} h_z f_{i,j} + \left[2h_z k_1(r_{i+1/2}, z_j) \frac{u_{i+1,j} - u_{i,j}}{h_r} - 0 + \right.$$

$$+ \frac{h_r}{2} k_2(r_i, z_{j+1/2}) \frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2} k_2(r_i, z_{j-1/2}) \frac{u_{i,j} - u_{i,j-1}}{h_z} \Big]$$

Подставляем полученные ранее произведения:

$$\begin{aligned} \xi_{i,j} = & \frac{h_r}{2} h_z f_{i,j} + \left[2h_z \left(\left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} + \frac{h_r}{2} \left[\frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) \right]_{i,j} \right. \right. \\ & + h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^3) \Big) - 0 \\ & + \frac{h_r}{2} \left[h_z \left(\frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right)_{i,j} \right. \\ & \left. \left. + h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right] \right] \end{aligned}$$

Группируем по степеням h_r и h_z

$$\begin{aligned} \xi_{i,j} = & \frac{h_r}{2} h_z \left[f + 2 \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) + qu \right]_{i,j} + h_z 2 \left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} \\ & + 2h_z \left(h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^3) \right) \\ & + \frac{h_r}{2} \left[h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right] \end{aligned}$$

Для вычисления порядка аппроксимации нормируем невязку

$$\begin{aligned} \tilde{\xi}_{i,j} = & \frac{\xi_{i,j}}{2h_z} \\ \tilde{\xi}_{i,j} = & \frac{h_r}{4} \left[f + 2 \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) + qu \right]_{i,j} + \left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} \\ & + h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^3) \\ & + \frac{h_r}{4} \left[h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right] \end{aligned}$$

$$\left. \frac{\partial u}{\partial r} \right|_{r=0} = 0,$$

$$- \left[2 \frac{\partial}{\partial r} \left(k_1(r, z) \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2(r, z) \frac{\partial u}{\partial z} \right) + q(r, z)u \right] \Big|_{r=0} = f(r, z)|_{r=0}$$

Аналогично выполним обратную замену, получим:

Порядок аппроксимации уравнения по r и z:

$$p_r = 2$$

$$p_z = 2$$

Главный член погрешности по x

$$\Omega_r = \left[\frac{1}{6} r k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial r k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 r k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]$$

Главный член погрешности по y

$$\Omega_z = r \left[\frac{1}{48} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{24} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{32} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{96} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right]$$

$$2) -k_1(r) \frac{\partial u}{\partial r} \Big|_{r=R} = \chi_2 u|_{r=R} - \phi_2(z), (\text{Лекция 10. p19})$$

$$\begin{aligned} \xi_{i,j} = & \frac{h_r}{2} h_z f_{i,j} - h_z \left(\chi_2 u_{i,j} - \phi_2(z_j) \right) - h_z k_1 \left(r_{i-\frac{1}{2}}, z_j \right) \frac{u_{i,j} - u_{i-1,j}}{h_r} \\ & + \frac{h_r}{2} k_2 \left(r_i, z_{j+\frac{1}{2}} \right) \frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2} k_2 \left(r_i, z_{j-\frac{1}{2}} \right) \frac{u_{i,j} - u_{i,j-1}}{h_z} \end{aligned}$$

Подставляем полученные ранее произведения:

$$\begin{aligned} & \xi_{i,j} \\ = & \frac{h_r}{2} h_z f_{i,j} - h_z \left(\chi_2 u_{i,j} - \phi_2(z_j) \right) \\ & - h_z \left[\left[k_1 \frac{\partial u}{\partial r} \right]_{i,j} - \frac{h_r}{2} \left[\frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) \right]_{i,j} + h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} - \right. \\ & \left. - h_r^3 \left[\frac{1}{24} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{16} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{48} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^4) \right] \\ & + \frac{h_r}{2} \left[h_z \left(\frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right)_{i,j} + h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} \right. \\ & \left. + O(h_z^4) \right] \end{aligned}$$

Группируем по степеням hr и hz

$$\begin{aligned}\xi_{i,j} = & \frac{h_r}{2} h_z \left[f + \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \left(\frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right) \right]_{i,j} - h_z \left[k_1 \frac{\partial u}{\partial r} + (\chi_2 u - \phi_2(z)) \right]_{i,j} \\ & - h_z \left[h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} + O(h_r^3) \right] \\ & + \frac{h_r}{2} \left[h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} + O(h_z^4) \right]\end{aligned}$$

Для вычисления порядка аппроксимации нормируем невязку

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{2h_z}$$

$$\begin{aligned}\tilde{\xi}_{i,j} = & \frac{h_r}{2} \left[f + \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \left(\frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right) \right]_{i,j} - \left[k_1 \frac{\partial u}{\partial r} + (\chi_2 u - \phi_2(z)) \right]_{i,j} \\ & - h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial u}{\partial r} \right]_{i,j} \\ & + O(h_r^3) \frac{h_r}{2} \left[h_z^2 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{24} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)_{i,j} \right. \\ & \left. + O(h_z^3) \right]\end{aligned}$$

$$\begin{aligned}\left[k_1 \frac{\partial u}{\partial r} + \chi_2 u - \phi_2(z) \right]_{r=b} &= 0 \\ f + \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) &= 0\end{aligned}$$

Аналогично выполним обратную замену, получим:

Порядок аппроксимации уравнения по r и z:

$$\begin{aligned}p_r &= 2 - 0 = 2, \\ p_z &= 2 - 0 = 2\end{aligned}$$

Главные члены погрешности

$$\begin{aligned}\Omega_r &= - \left[\frac{1}{6} r k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4} \frac{\partial r k_1}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{8} \frac{\partial^2 r k_1}{\partial r^2} \frac{\partial u}{\partial r} \right] \\ \Omega_z &= r \left(\frac{1}{24} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{12} \frac{\partial k_2}{\partial z} \frac{\partial^3 u}{\partial z^3} + \frac{1}{16} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{48} \frac{\partial^3 k_2}{\partial z^3} \frac{\partial u}{\partial z} \right)\end{aligned}$$

$$3) \quad -k_2 \frac{\partial u}{\partial z} \Big|_{z=L} = \chi_4 u|_{z=L} - \phi_4(r), \chi_4 \geq 0 \quad i = 1, 2, \dots, N_r - 1, j = N_z \quad (\text{Лекция 10, стр. 24})$$

$$\begin{aligned}
\xi_{i,j} = & \left[k_2 \frac{\partial u}{\partial z} + \chi_4 u - \phi_4(r) \right]_{i,j} + \frac{h_z}{2} \left[f + \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) \right]_{i,j} \\
& - h_z^2 \left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4} \frac{\partial k_2}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial u}{\partial z} \right]_{i,j} + O(h_z^3) \\
& + \frac{h_z}{2} \left[h_r^2 \left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6} \frac{\partial k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{8} \frac{\partial^2 k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{24} \frac{\partial^3 k_1}{\partial r^3} \frac{\partial u}{\partial r} \right) + O(h_r^3) \right] \\
& \left[k_2 \frac{\partial u}{\partial z} + \chi_4 u - \phi_4(r) \right]_{z=d} = 0 \\
& f + \frac{\partial}{\partial r} \left(k_1 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(k_2 \frac{\partial u}{\partial z} \right) = 0
\end{aligned}$$

Порядок аппроксимации уравнения по r и z:

$$\begin{aligned}
p_r &= 2 - 0 = 2, \\
p_z &= 2 - 0 = 2
\end{aligned}$$

Главные члены погрешности

$$\begin{aligned}
\Omega_r &= \frac{1}{24} r k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12} \frac{\partial r k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{16} \frac{\partial^2 r k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{48} \frac{\partial^3 r k_1}{\partial r^3} \frac{\partial u}{\partial r} \\
\Omega_z &= -r \left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4} \frac{\partial k_2}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial u}{\partial z} \right]
\end{aligned}$$

Решение системы методом сопряженных градиентов

Пусть $w^{(0)}$ - произвольное начальное приближение, тогда $Aw - Aw^{(0)} = g - Aw^{(0)}$, что даст нам невязку $r^{(0)} = A(w - w^{(0)})$, предполагается, что у нас есть система из $s^{(i)}$, где $i=1,2,\dots,n$, линейно-независимых векторов, тогда можем разложить по базису этих векторов с соответствующими коэффициентами $w - w^{(0)} = \sum_{i=1}^n a_i s^{(i)}$, найти коэффициенты можем с помощью СЛАУ $\sum_{i=1}^n a_i A s^{(i)} = r^{(0)}$, решение системы сильно упростится, если $(A s^{(i)}, s^{(i)}) = 0$ при $i \neq j$, а при $i = j$, скалярное произведение равнялось не 0 значению, в таком случае мы говорим об ортогональности. Из этого мы можем выразить коэффициенты $a_i = \frac{(r^{(0)}, s^{(i)})}{(A s^{(i)}, s^{(i)})}$, и выразить решение $w = w^{(0)} + \sum_{i=1}^n a_i s^{(i)}$.

Рассмотрим частичную сумму $w^{(n)} = w$, $w^{(n)} = w^{(0)} + \sum_{i=1}^n a_i s^{(i)}$, $w^{(k)} = w^{(0)} + \sum_{i=1}^k a_i s^{(i)}$, $w^{(k)} = w^{(k-1)} + a_k A s^{(k)}$, для невязки получим рекуррентное соотношение $r^{(k)} = r^{(k-1)} - a_k A s^{(k)}$.

$$w^{(0)}, \quad r^{(0)} = g - A w^{(0)}, \quad s^{(1)} = ?$$

$$k = 1, 2, \dots, n, \quad a_k = \frac{(r^{(0)}, s^{(k)})}{(A s^{(k)}, s^{(k)})}$$

$$w^{(k)} = w^{(k-1)} + a_k s^{(k)}, \quad r^{(k)} = r^{(k-1)} - a_k A s^{(k)}$$

$$s^{(k+1)} = ?$$

При явном методе сопряженных градиентов $s^{(1)}$ берут равным $r^{(0)}$, $s^{(k+1)} = r^{(k)} + \beta_k s^{(k)}$, с вводом дополнительного коэффициента $\beta_k = \frac{(r^{(k)}, r^{(k)})}{(r^{(k-1)}, r^{(k-1)})}$ при $\sqrt{(r^{(k)}, r^{(k)})} < \gamma \varepsilon$, явный метод обладает тем свойством что при отсутствии ошибок округления мы можем получить точное решение не позднее чем на n-ом шаге, но возникает двойственность, из-за ошибок округления происходит разрушение ортогональности последовательности s и в результате к неточности, и метод становится итерационным.

Неявный метод

$$Aw = b, \quad A = A^T, \quad (Ay, y) > 0, \quad y \neq 0$$

$x^{(0)}$ – произвольное начальное приближение

$$r^{(0)} = b - Ax^{(0)}, \quad Bw^{(0)} = r^{(0)}, \quad s^{(1)} = w^{(0)}, \quad Bg = b, \quad \gamma = \sqrt{(g, b)}$$

$$k = 1, 2, \dots, K_{max}$$

$$a_k = \frac{(w^{(k-1)}, r^{(k-1)})}{(As^{(k-1)}, s^{(k-1)})}$$

$$x^{(k)} = x^{(k-1)} + a_k s^{(k)}, \quad r^{(k)} = r^{(k-1)} + a_k As^{(k-1)}$$

$$Bw^{(k)} = r^{(k)}, \quad \sqrt{(w^{(k)}, r^{(k)})} < \gamma \varepsilon$$

$$\beta_k = \frac{(w^{(k)}, r^{(k)})}{(w^{(k-1)}, r^{(k-1)})}, \quad s^{(k+1)} = w^{(k)} + \beta_k s^{(k)}$$

О выборе матрицы предобуславливания

$$Aw = b, \quad A = A^T, \quad (Ay, y) > 0, \quad y \neq 0$$

$$B = B^T, \quad (By, y) > 0, \quad y \neq 0$$

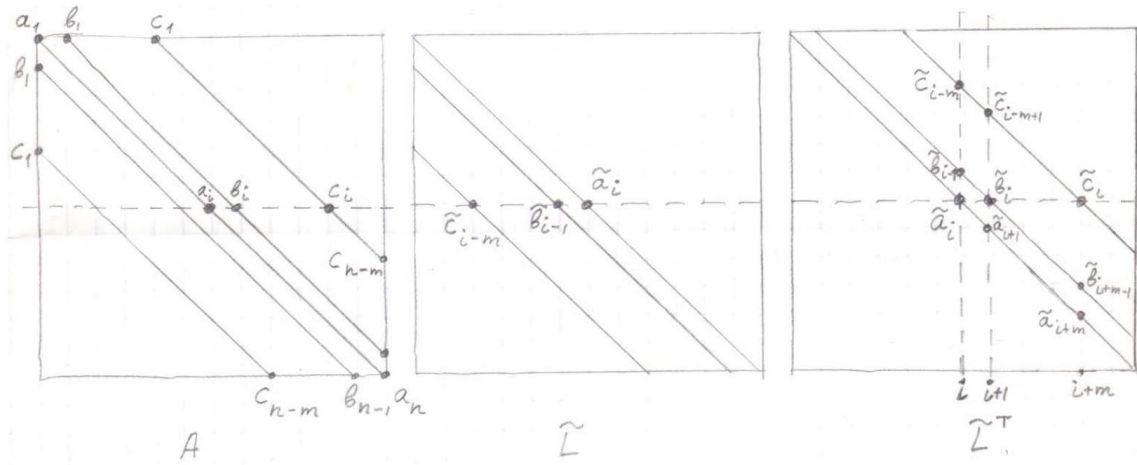
$$B = D, \quad D = \begin{bmatrix} a_{11} & - & - \\ - & \dots & - \\ - & - & a_{nn} \end{bmatrix}, \quad B = \tilde{L} \tilde{L}^T$$

$$\tilde{l}_{ij} = 0, \quad i < j$$

$$Bw^{(0)} = r^{(0)}, \quad \tilde{L}y_0 = r_0, \quad \tilde{L}^T w_0 = y_0,$$

$$Bw^{(k)} = r^{(k)}, \quad \tilde{L}y_k = r_k, \quad \tilde{L}^T w_k = y_k$$

Неполное разложение Холевского



$$a_i = \tilde{a}_i^2 + \tilde{b}_{i-1}^2 + \tilde{c}_{i-m}^2, \quad b_i = \tilde{a}_i \tilde{b}_i, \quad c_i = \tilde{a}_i \tilde{c}_i,$$

$$\tilde{a}_i = \sqrt{a_i - \tilde{b}_{i-1}^2 - \tilde{c}_{i-m}^2}, \quad i = 1, 2, \dots, n, \quad \tilde{b}_0 = 0, \quad \tilde{c}_{i-m} = 0, \quad i = 1, 2, \dots, m$$

$$\tilde{b}_i = \frac{b_i}{\tilde{a}_i}, \quad \tilde{c}_i = \frac{c_i}{\tilde{a}_i},$$

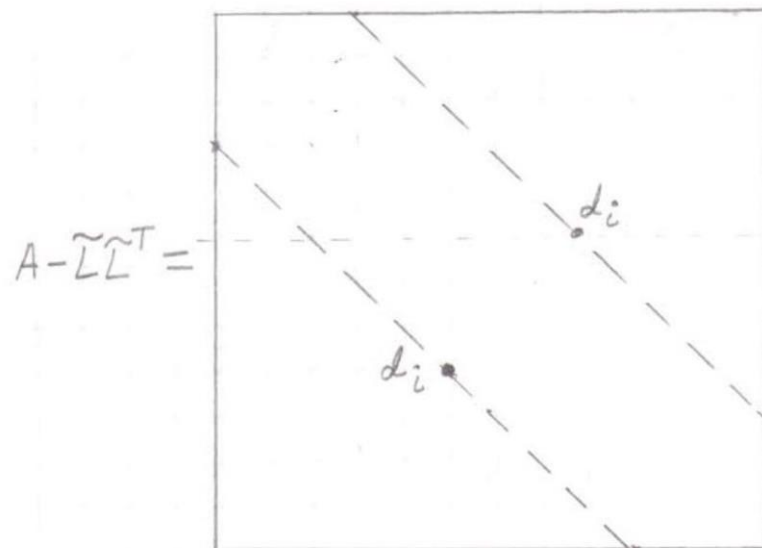
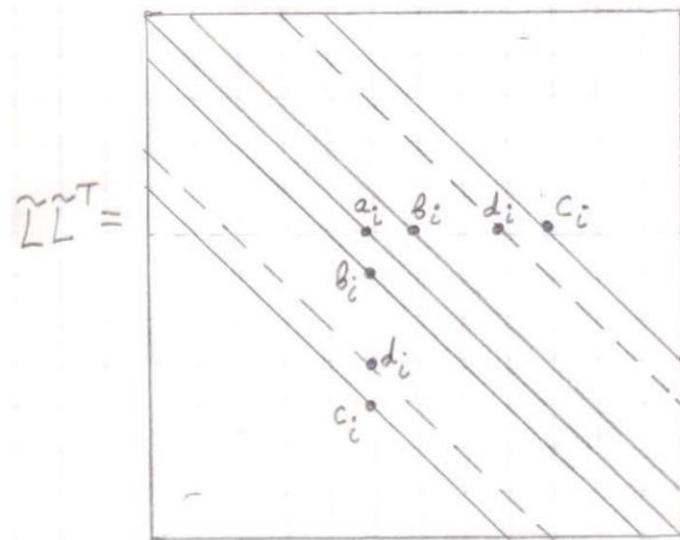
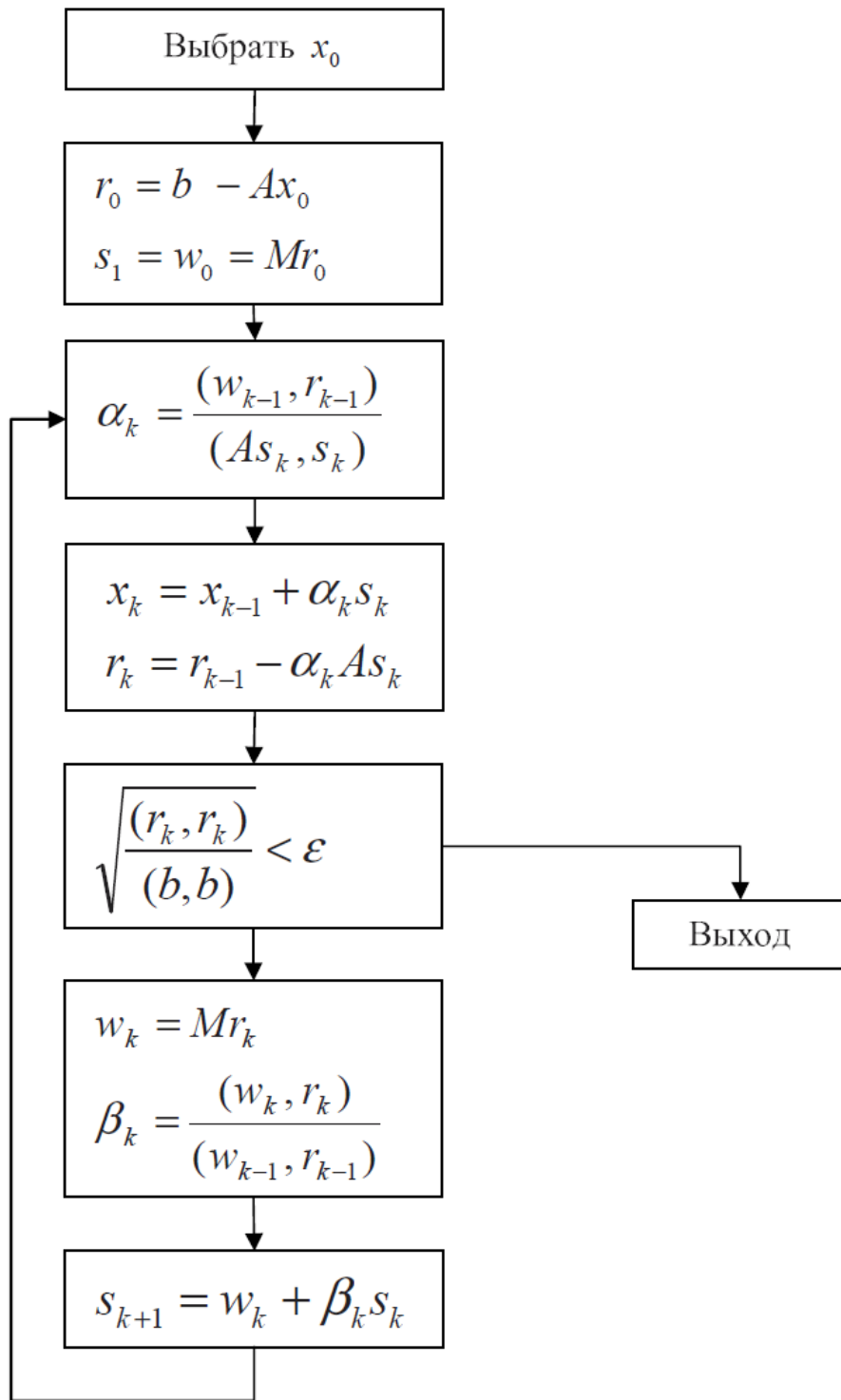


Схема применения метода выглядит следующим образом:



Здесь я использовал параметр $\varepsilon = 10^{-6}$.

Тесты

Для всех тектов:

$$\begin{aligned} R &= 1, & L &= 1 \\ \chi_2 &= 1, & \chi_4 &= 1 \end{aligned}$$

Константный тест

$$\begin{aligned} k_1 &= k_2 = 1 \\ u &= 1 \\ f &= 0 \\ \varphi_2(z) &= 1, & \varphi_3(r) &= 1, & \varphi_4(r) &= 1 \end{aligned}$$

Линейный тест

$$\begin{aligned} k_1 &= r + 1, & k_2 &= z + 1 \\ u &= r^2 \\ f &= -6r - 4 \\ \varphi_2(z) &= 5, & \varphi_3(r) &= r^2, & \varphi_4(r) &= r^2 \end{aligned}$$

Нелинейный тест

$$\begin{aligned} k_1 &= r^2 + 1, & k_2 &= z^2 + 1 \\ u &= r^4 + r^2 z^2 \\ f &= -24r^4 - 14r^2 z^2 - 18r^2 - 4z^2 \\ \varphi_2(z) &= 5z^2 + 9, & \varphi_3(r) &= r^4, & \varphi_4(r) &= r^4 + 5r^2 \end{aligned}$$

Результаты

Константный случай

Число разбиений Nr, Nz	Максимальная погрешность	Отношение погрешностей	Число итераций метода
4	8.88178419700E-16	0	11
8	3.82518982489E-07	2.32E-09	27
16	6.30674762192E-07	0.60652337056	16
32	2.62734831580E-06	0.24004231125	107
64	5.99528580780E-06	0.43823570719	208
128	6.36365821094E-06	0.94211310681	413

Линейный случай

Число разбиений Nr, Nz	Максимальная погрешность	Отношение погрешностей	Число итераций метода
4	1.95399252334E-14	0	16
8	1.14468746837E-06	1.71E-08	32
16	2.26973880935E-06	0.504325636	65
32	5.50640104335E-06	0.412200054	130
64	1.24175795404E-05	0.443435939	252
128	2.36257855251E-05	0.525594357	486

Нелинейный случай

Число разбиений Nr, Nz	Максимальная погрешность	Отношение погрешностей	Число итераций метода
4	3.38167202120E-02	0	16
8	8.92482305903E-03	3.789063378	42
16	2.25182768122E-03	3.963368571	87
32	5.80594786091E-04	3.87848416	179
64	1.70294493403E-04	3.40935737	359
128	4.16449411629E-05	4.0892	717

Вывод

Погрешность решения дифференциального уравнения складывается из двух: погрешности аппроксимации (появляется при переходе от непрерывного уравнения к системе разностных) и погрешности решения алгебраической системы.

В линейном и константном случаях погрешность аппроксимации отсутствует, ее небольшой рост с увеличением количества разбиений связан с накоплением ошибки округления.

А в нелинейном случае наблюдается уменьшение ошибки в 4 раза при увеличении в 2 раза разбиений по оси r и z . Погрешность решения алгебраической системы мала по сравнению с погрешностью аппроксимации, она возрастает незаметно. Погрешность аппроксимации, в свою очередь, уменьшается, т.к. мы увеличиваем количество разбиений. Причем, согласно теории, при одновременном удвоении числа разбиений погрешность аппроксимации должна уменьшаться в 4 раза, т.к. порядок аппроксимации метода равен 2. Как видим, наблюдаемые результаты очень близок к теоретическому.

Приложение

```
import java.util.Arrays;
import java.util.HashMap;
import java.util.function.Function;

public class N4 {
    private final static double EPS = 1e-6;
    private static int N = 5;
    private static final double R0 = 0;
    private static final double R1 = 1;
    private static final double L = 1;
    private static final double Chi2 = 1;
    private static final double Chi4 = 1;

    private enum SystemParameters {
        DIAGONAL_A, DIAGONAL_B, DIAGONAL_C, VECTOR_G
    }

    @FunctionalInterface
    public interface FunctionTwoArgs<A, B, R> {
        R apply(A a, B b);
    }

    public static void main(String[] args) {

        System.out.println("N4");
        System.out.println("---->>> Константный случай");
        test(    (r, z) -> 1.0,
                (r, z) -> 1.0,
                (r, z) -> 0.0,
                (z) -> 1.0,
                (r) -> 1.0,
                (r) -> 1.0,
                (r, z) -> 1.0);

        System.out.println("\n\n---->>> Линейный случай");
        test(    (r, z) -> r + 1.0,
                (r, z) -> z + 1.0,
```

```

        (r, z) -> -6 * r - 4,
        (z) -> 5.0,
        (r) -> r * r,
        (r) -> r * r,
        (r, z) -> r * r);

System.out.println("\n\n---->>> Нелинейный случай");
test(    (r, z) -> r * r + 1,
        (r, z) -> 1 + z * z,
        (r, z) -> -24 * r*r*r*r - 14 * r*r * z*z - 18*r*r - 4*z*z,
        (z) -> 5*z*z + 9,
        (r) -> r * r * r * r,
        (r) -> r * r * r * r + 5 * r * r,
        (r, z) -> r * r * r * r + z * z * r * r);
}

private static void test(FunctionTwoArgs<Double, Double, Double> k1,
                        FunctionTwoArgs<Double, Double, Double> k2,
                        FunctionTwoArgs<Double, Double, Double> f,
                        Function<Double, Double> phi2,
                        Function<Double, Double> phi3,
                        Function<Double, Double> phi4,
                        FunctionTwoArgs<Double, Double, Double> u)
{
    HashMap<SystemParameters, double[]> system;
    N = 5;
    double hR = (R1 - R0) / (N - 1);
    double hZ = L / (N - 1);
    double r;
    double z = 0;
    double[] result = new double[N * N];
    system = getSystem(k1, k2, f, phi2, phi3, phi4);
    for (int i = 0; i < N; ++i) {
        r = R0;
        for (int j = 0; j < N; ++j) {
            result[i * N + j] = u.apply(r, z);
            r += hR;
        }
        z += hZ;
    }
    System.out.println("Отклонения от точного решения\n"
        + Arrays.toString( sub(multiply(system, result),

```

```

        system.get(SystemParameters.VECTOR_G))));

System.out.println("Ошибка");
double prevError = 0;
double nowError;
N = 5;
System.out.println("\t\tN\tError\tRatio\t");
for (int i = 2; i <= 8; ++i) {
    N = (int) Math.round(Math.pow(2, i)) + 1;
    system = getSystem(k1, k2, f, phi2, phi3, phi4);
    result = ConjugateGradientMethod(system,
system.get(SystemParameters.VECTOR_G), getEMatrix());
    nowError = getMaxError(result, u);
    System.out.println("\t\t " + (N - 1) + "\t " + nowError + " \t "
+ prevError / nowError);
    prevError = nowError;
}
}

private static double[]
ConjugateGradientMethod(HashMap<SystemParameters, double[]> system,
double[] first,
HashMap<SystemParameters,
double[]> bMatrix) {
    double[] result = Arrays.copyOf(first, first.length);
    double[] r = sub(system.get(SystemParameters.VECTOR_G),
multiply(system, first));
    double[] p = solveB(bMatrix, r);
    double[] b = solveB(bMatrix, system.get(SystemParameters.VECTOR_G));
    double[] s = Arrays.copyOf(p, p.length);
    double alpha; double beta; double[] newR; double[] newP; int k;
    for (k = 1; k <= 10000; k++) {
        alpha = multiply(p, r) / multiply(multiply(system, s), s);
        result = addition(result, multiply(alpha, s));
        newR = sub(r, multiply(alpha, multiply(system, s)));
        newP = solveB(bMatrix, newR);
        double check = Math.sqrt(multiply(newP, newR) / multiply(b,
system.get(SystemParameters.VECTOR_G)));
        if (check < EPS) {
            ++k;
            break;

```

```

        }
        beta = multiply(newP, newR) / multiply(p, r);
        s = addition(newP, multiply(beta, s));
        r = newR; p = newP;
    }
    System.out.println("(Число итераций:\t" + k + ")");
    return result;
}

private static double[] getADiag(FunctionTwoArgs<Double, Double, Double>
k2) {
    double hR = (R1 - R0) / (N - 1);
    double hZ = L / (N - 1);
    double scale = hR / hZ;
    double[] result = new double[N * N];
    double z = hZ;
    double r;

    for (int j = 1; j < N - 1; j++) {
        r = R0;
        result[j * N] = -(scale / 4) * r * k2.apply(r, z - hZ / 2); //
#(2)
        r += hR;
        for (int i = 1; i < N - 1; i++) {
            result[j * N + i] = -(scale) * r * k2.apply(r, z - hZ / 2);
// #(1)
            r += hR;
        }
        result[j * N + N - 1] = -(scale / 2) * r * k2.apply(r, z - hZ /
2); // #(3)
        z += hZ;
    }

    r = R0;
    for (int i = 0; i < N; i++) {
        result[N * (N - 1) + i] = -scale * r * k2.apply(r, z - hZ / 2);
// # (5)
        r += hR;
    }

    return result;
}

```

```

    }

    private static double[] getCDiag(FunctionTwoArgs<Double, Double, Double>
k1,
                                FunctionTwoArgs<Double, Double, Double>
k2) {

        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double scale = hZ / hR; double z = hZ;
        double r;
        double[] result = new double[N * N];

        for (int i = 0; i < N; i++) { // #(4)
            result[i] = 1;
        }

        for (int j = 1; j < N - 1; j++) {
            r = R0;
            result[j * N] = scale * (r + hR / 2) * k1.apply(r + hR / 2, z) //
#(2)
                + (1 / scale) * r * k2.apply(r, z + hZ / 2)
                + (1 / scale) * r * k2.apply(r, z - hZ / 2);
            r += hR;

            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = scale * (r + hR / 2) * k1.apply(r + hR
/2, z) // #(1)
                    + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                    + (1 / scale) * r * k2.apply(r, z + hZ / 2)
                    + (1 / scale) * r * k2.apply(r, z - hZ / 2);

                r += hR;
            }

            result[j * N + N - 1] = hZ * r * Chi2 // #(3)
                + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                + (1 / scale / 2) * r * k2.apply(r, z + hZ / 2)
                + (1 / scale / 2) * r * k2.apply(r, z - hZ / 2);
            z += hZ;
        }
    }

```

```

        r = R0;
        for (int i = 0; i < N; i++) { // #(5)
            result[N * (N - 1) + i] = scale * (r + hR / 2) * k1.apply(r + hR
/2, z)
                + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                + hR * r * Chi4
                + (1 / scale) * r * k2.apply(r, z - hZ / 2);
            r += hR;
        }
        return result;
    }

    private static double[] getDDiag(FunctionTwoArgs<Double, Double, Double>
k1) {

        double hR = (R1 - R0) / (N - 1);
        double hZ = L / (N - 1);
        double scale = hZ / hR;
        double z = hZ;
        double r;
        double[] result = new double[N * N];

        for (int j = 1; j < N - 1; j++) {
            r = R0;
            for (int i = 0; i < N - 1; i++) {
                result[j * N + i] = -scale * (r + hR / 2) * k1.apply(r + hR
/2, z); // #(2) & (1)
                r += hR;
            }
            z += hZ;
        }
        return result;
    }

    private static double[] getEDiag(FunctionTwoArgs<Double, Double, Double>
k2) {
        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double scale = hR / hZ;
        double[] result = new double[N * N]; double z = hZ;
        double r;
        for (int j = 1; j < N - 1; j++) {
            r = R0;

```

```

        result[j * N] = -scale * r * k2.apply(r, z + hZ / 2) / 2; r +=
hR; // #(2)
        for (int i = 1; i < N - 1; i++) {
            result[j * N + i] = -scale * r * k2.apply(r, z + hZ / 2); r
+= hR; // #(1)
        }
        result[j * N + N - 1] = -scale * r * k2.apply(r, z + hZ / 2) / 2;
z += hZ; // #(3)
    }
    return result;
}

private static double[] getVectorG(FunctionTwoArgs<Double, Double,
Double> f,
                                   Function<Double, Double> phi2,
                                   Function<Double, Double> phi3,
                                   Function<Double, Double> phi4) {

    double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
    double[] result = new double[N * N]; double z = hZ;

    double r = R0;
    for (int i = 0; i < N; i++) { // # (4)
        result[i] = phi3.apply(r);
        r += hR;
    }

    for (int j = 1; j < N - 1; j++) {
        r = R0;
        result[j * N] = hR * hZ * r * f.apply(r, z) / 4; // # (2)
        r += hR;
        for (int i = 1; i < N - 1; i++) {
            result[j * N + i] = hR * hZ * r * f.apply(r, z); // # (1)
            r += hR;
        }
        result[j * N + N - 1] = hR * hZ * r * f.apply(r, z) / 2 + hZ * r
* phi2.apply(z); // # (3)
        z += hZ;
    }

    r = R0;
    for (int i = 0; i < N; i++) {

```



```

        result[N * (N - 1) + i] = hR * r * phi4.apply(r) + hR * hZ * r *
f.apply(r, z) / 2; // # (5)
        r += hR;
    }
    return result;
}

private static HashMap<SystemParameters, double[]>
getSystem(FunctionTwoArgs<Double, Double, Double> k1,

FunctionTwoArgs<Double, Double, Double> k2,

FunctionTwoArgs<Double, Double, Double> f,

Function<Double, Double> phi2,

Function<Double, Double> phi3,

Function<Double, Double> phi4)
{
    double[] a = getADiag(k2);
    double[] c = getCDiag(k1, k2);
    double[] d = getDDiag(k1);
    double[] e = getEDiag(k2);
    double[] g = getVectorG(f, phi2, phi3, phi4);

    for (int i = 0; i < N; i++) {
        g[N + i] -= g[i] * a[N + i];
        a[N + i] = 0;
        g[N * (N - 2) + i] -= g[N * (N - 1) + i] * e[N * (N - 2) + i];
        e[N * (N - 2) + i] = 0;
    }

    HashMap<SystemParameters, double[]> system = new HashMap<>();

    system.put(SystemParameters.DIAGONAL_A, c);
    system.put(SystemParameters.DIAGONAL_B, d);
    system.put(SystemParameters.DIAGONAL_C, e);
    system.put(SystemParameters.VECTOR_G, g);
    return system;
}

```

```

        private      static      double      getMaxError(double[]      solve,
FunctionTwoArgs<Double, Double, Double> u) {
    double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
    double z = 0; double r;
    double maxError = 0; double nowError;
    for (int j = 0; j < N; j++) { r = R0;
        for (int i = 0; i < N; i++) {
            nowError = Math.abs(u.apply(r, z) - solve[j * N + i]); if
(nowError > maxError) {
                maxError = nowError;
            }
            r += hR;
        }
        z += hZ;
    }
    return maxError;
}

        private      static      HashMap<SystemParameters,      double[]>
getBMatrix(HashMap<SystemParameters, double[]> system)
    {
        HashMap<SystemParameters, double[]> result = new HashMap<>(); int
squareN = N * N;
        double[] a = new double[squareN];
        double[] b = new double[squareN];
        double[] c = new double[squareN];
        result.put(SystemParameters.DIAGONAL_A, a);
        result.put(SystemParameters.DIAGONAL_B, b);
        result.put(SystemParameters.DIAGONAL_C, c);
        a[0] = Math.sqrt(system.get(SystemParameters.DIAGONAL_A)[0]);
        for (int i = 1; i < N; i++) {
            b[i - 1] = system.get(SystemParameters.DIAGONAL_B)[i - 1] / a[i
- 1];
            a[i] = Math.sqrt(system.get(SystemParameters.DIAGONAL_A)[i] -
Math.pow(b[i - 1], 2));
        }
        for (int i = N; i < squareN; i++) {
            c[i - N] = system.get(SystemParameters.DIAGONAL_C)[i - N];
            b[i - 1] = system.get(SystemParameters.DIAGONAL_B)[i - 1] / a[i
- 1];
            a[i] = Math.sqrt(system.get(SystemParameters.DIAGONAL_A)[i] -
Math.pow(b[i - 1], 2) - Math.pow(c[i - N], 2));

```

```

    }
    return result;
}

private static double[] solveB(HashMap<SystemParameters, double[]>
bMatrix, double[] g) {
    int squareN = N * N;

    double[] y = new double[squareN];
    double[] a = bMatrix.get(SystemParameters.DIAGONAL_A);
    double[] b = bMatrix.get(SystemParameters.DIAGONAL_B);
    double[] c = bMatrix.get(SystemParameters.DIAGONAL_C);

    y[0] = g[0] / a[0];
    for (int i = 1; i < N; i++) {
        y[i] = (g[i] - b[i - 1] * y[i - 1]) / a[i];
    }

    for (int i = N; i < squareN; i++) {
        y[i] = (g[i] - b[i - 1] * y[i - 1] - c[i - N] * y[i - N]) / a[i];
    }

    double[] result = new double[squareN];
    result[squareN - 1] = y[squareN - 1] / a[squareN - 1];

    for (int i = squareN - 2; i >= N * (N - 1); i--) {
        result[i] = (y[i] - b[i] * result[i + 1]) / a[i];
    }
    for (int i = N * (N - 1) - 1; i >= 0; i--) {
        result[i] = (y[i] - b[i] * result[i + 1] - c[i] * result[i + N])/
a[i];
    }
    return result;
}

private static HashMap<SystemParameters, double[]> getEMatrix() {
    HashMap<SystemParameters, double[]> e = new HashMap<>();
    int squareN = N * N;
    double[] a = new double[squareN];
    for (int j = 0; j < squareN; j++) {
        a[j] = 1;
    }
}

```

```

        e.put(SystemParameters.DIAGONAL_A, a);
e.put(SystemParameters.DIAGONAL_B, new double[squareN]);
e.put(SystemParameters.DIAGONAL_C, new double[squareN]); return e;
    }

    private static double multiply(double[] leftVector, double[]
rightVector)
    {
        double result = 0;
        for (int i = 0; i < leftVector.length; i++) {
            result += leftVector[i] * rightVector[i];
        }
        return result;
    }

    private static double[] multiply(HashMap<SystemParameters, double[]>
system, double[] vector) {
        double[] result = new double[vector.length];
        double[] diagA = system.get(SystemParameters.DIAGONAL_A); double[]
diagB = system.get(SystemParameters.DIAGONAL_B); double[] diagC =
system.get(SystemParameters.DIAGONAL_C); for (int i = 0; i < vector.length;
i++) {
            result[i] = diagA[i] * vector[i];
        }
        for (int i = 0; i < vector.length - 1; i++) { result[i] += diagB[i]
* vector[i + 1];
        }
        for (int i = 0; i < vector.length - N; i++) { result[i] += diagC[i]
* vector[i + N];
        }
        for (int i = 1; i < vector.length; i++) { result[i] += diagB[i - 1]
* vector[i - 1];
        }
        for (int i = N; i < vector.length; i++) { result[i] += diagC[i - N]
* vector[i - N];
        }
        return result;
    }

    private static double[] multiply(double number, double[] vector)
{ double[] result = new double[vector.length];

```

```

        for (int i = 0; i < vector.length; i++)
        {
            result[i] = vector[i] * number;
        }
        return result;
    }

    private static double[] addition(double[] leftVector, double[]
rightVector) {
        double[] result = new double[leftVector.length];
        for (int i = 0; i < leftVector.length; i++) {
            result[i] = leftVector[i] + rightVector[i];
        }
        return result;
    }

    private static double[] sub(double[] leftVector, double[] rightVector)
{ double[] result = new double[leftVector.length];
        for (int i = 0; i < leftVector.length; i++) {
            result[i] = leftVector[i] - rightVector[i];
        }
        return result;
    }
}

```