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ОТЧЕТ ПО КУРСОВОЙ РАБОТЕ

по дисциплине
«Математические модели систем с распределёнными параметрами»
Вариант Q12

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Оглавление

Постановка задания

Вариант Q. Постановка задачи. Используя интегро-интерполяционный метод, разработать подпрограмму для моделирования распределения температуры в полом цилиндре, описываемого математической моделью

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right]=f(r,z),$$
$$0<\mathrm{c}_{11}\le k_1(r,z)\le c_{12},\quad 0<\mathrm{c}_{21}\le \mathrm{k}_2(\mathrm{r},\mathrm{z})\le \mathrm{c}_{22},$$
$$0<R_0\le r\le R_1,\ 0\le z\le L$$

с граничными условиями, определяемыми вариантом задания. Для решения системы алгебраических уравнений использовать метод сопряжённых градиентов с предобусловливанием. Матрица алгебраической системы должна храниться в упакованной форме

Форма (4)

Форма (4) отличается от формы (3) тем, что индексы главных диагональных элементов не хранятся и элементы главной диагонали располагаются в отдельном массиве Diag. В массиве A хранятся ненулевые элементы строго верхней треугольной части матрицы. Так как матрица хранится построчно, то в массиве IC хранятся номера столбцов ненулевых элементов верхнего треугольника матрицы. В массиве IR хранятся указатели на начало каждой строки в массивах A и IC. IR(N+1) содержит количество ненулевых элементов в строго верхнем треугольнике матрицы A плюс один.

	1	2	3	4	5	6	7	8	9
DIAG	13	14	15	16	17	18	19	20	21

	1	2	3	4	5	6	7	8	9	10	11	12
A	7	1	8	2	3	9	4	10	5	6	11	12
IC	2	4	3	5	6	5	7	6	8	9	8	9

	1	2	3	4	5	6	7	8	9	10
IR	1	3	5	6	8	10	11	12	13	13

12.
$$u\big|_{r=R_0}=g_1(z),\qquad -k_1\frac{\partial u}{\partial r}\bigg|_{r=R_1}=\chi_2 u\big|_{r=R_1}-g_2(z),\ \chi_2\ge 0,$$
$$k_2\frac{\partial u}{\partial z}\bigg|_{z=0}=\chi_3 u\big|_{z=0}-g_3(r),\ \chi_3\ge 0,\qquad -k_2\frac{\partial u}{\partial z}\bigg|_{z=L}=\chi_4 u\big|_{z=L}-g_4(r),\ \chi_4\ge 0$$

Дискретная модель

Введем обозначения:

$$h_r = \frac{r_{R1} + r_{R0}}{N_r}$$
$$h_z = \frac{z_L + z_0}{N_Z}$$

Основная сетка:

$$r_i = R_0 + ih_r, \quad i = 0,1,\dots,N_r$$
$$z_j = jh_z, \quad j = 0,1,\dots,N_z$$

Введем вспомогательную сетку:

$$r_{i-\frac{1}{2}} = \frac{r_i + r_{i-1}}{2}$$
$$z_{j-1/2} = \frac{z + z_{j-1}}{2}$$
$$\hbar_i = \begin{cases} \frac{h_r}{2} & i = 0 \\ h_r & i = 1,\dots,N-1 \\ \frac{h_r}{2} & i = N \end{cases}$$

Аналогично произведем разбиения для переменной z

$$\hbar_j = \begin{cases} \frac{h_z}{2} & j = 0 \\ h_z & j = 1,\dots,N-1 \\ \frac{h_z}{2} & j = N \end{cases}$$

Умножим исходное уравнение на r, проинтегрируем по вспомогательной сетке:

$$-\left[\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\frac{\partial}{\partial r}\left(rk(r)\frac{\partial u}{\partial r}\right)drdz + \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r\frac{\partial^2 u}{\partial z^2}drdz\right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} rf\,drdz$$

$$-\left[\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\left(r_{i+\frac{1}{2}}k\left(r_{i+\frac{1}{2}}\right)\frac{\partial u}{\partial r}|_{r_{i+\frac{1}{2}}}\right)dz - \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\left(r_{i-\frac{1}{2}}k\left(r_{i-\frac{1}{2}}\right)\frac{\partial u}{\partial r}|_{r_{i-\frac{1}{2}}}\right)dz\right.$$
$$\left. + \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\left(r\frac{\partial u}{\partial z}|_{z_{j+\frac{1}{2}}}\right)dr - \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\left(r\frac{\partial u}{\partial z}|_{z_{j-\frac{1}{2}}}\right)dr\right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} rf\,drdz$$

Воспользуемся формулой средних прямоугольников для вычисления значений интегралов:

$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\phi(r,z)dr \approx h_r\phi(r_i,z) = h_r\phi_i$$
$$\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\phi(r,z)dz \approx h_z\phi\left(r,z_j\right) = h_r\phi_j$$
$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r_i\phi\,drdz \approx r_ih_rh_z\phi_{i,j}$$

Также аппроксимируем производные по формуле центральных разностей:

$$k\left(r_{i+\frac{1}{2}}\right)\frac{\partial u}{\partial r}\bigg|_{r=r_{i+\frac{1}{2}},z=z_j} = k\left(r_{i+\frac{1}{2}}\right)\frac{u_{i+1,j}-u_{i,j}}{h_r}$$

$$k\left(r_{i-\frac{1}{2}}\right)\frac{\partial u}{\partial r}\bigg|_{r=r_{i-\frac{1}{2}},z=z_j} = k\left(r_{i-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i-1,j}}{h_r}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=z_{j+\frac{1}{2}},r=r_j} = \frac{u_{i,j+1}-u_{i,j}}{h_z}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=z_{j-\frac{1}{2}},r=r_j} = \frac{u_{i,j}-u_{i,j-1}}{h_z}$$

Получим:

$$-\left[h_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} + h_r r_i k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}\right] = r_i h_r h_z f_{i,j}$$

…… (1)

i=1,2,..,$N_r - 1$; j = 1,2,..,$N_z - 1$

Аппроксимация граничных условий:

1. При $i = 0, j = 1, \ldots, N_z - 1$

$$u_{i,j} = g_1(z_j)$$

2. При $i = N_r, j = 1, \ldots, N_z - 1$

$$-\left[-h_z R_1\left(\chi_2 u_{N,j} - g_2(z_j)\right) - h_z r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, z_j\right)\frac{u_{N,j}-u_{N-1,j}}{h_r} + R_1\frac{h_r}{2}k_2\left(R_1, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - R_1\frac{h_r}{2}k_2\left(R_1, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}\right] = R_1\frac{h_r}{2}h_z f_{i,j}$$

3. При $i = 1, \ldots, N_r - 1, j = 0$

$$-[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, 0\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, 0\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} + r_i h_r k_2(r_i, z_{\frac{1}{2}})\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r r_i(\chi_3 u_{i,j} - g_3(r_i)] = \frac{r_i h_r h_z f_{ij}}{2}$$

4. При $i = 1, \ldots, N_r - 1, j = N_z$

$$-\left[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, L\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, L\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} - h_r r_i(\chi_4 u_{i,N} - g_4(r_i) - h_r r_i k_2\left(r_i, z_{N-\frac{1}{2}}\right)\frac{u_{i,N}-u_{i,N-1}}{h_z}\right] = \frac{r_i h_r h_z f_{i,j}}{2}$$

5. При $i = 0, j = 0$

$$u_{i,j} = g_1(z_j)$$

6. При $i = N_r, j = 0$

$$-\left[-\frac{h_z}{2}R_1\left(\chi_2 u_{N,0} - g_2(0)\right) - \frac{h_z}{2} r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, 0\right)\frac{u_{N,0} - u_{N-1,0}}{h_r} + \frac{h_r}{2}R_1 k_2\left(R_1, z_{\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2}R_1\left(\chi_3 u_{N,0} - g_3(R_1)\right)\right] = \frac{R_1 h_r h_z f_{N,0}}{4}$$

7. При $i = 0, j = N_z$

$$u_{i,j} = g_1(L)$$

8. При $i = N_r, j = N_z$

$$-\left[-\frac{h_z}{2}R_1\left(\chi_2 u_{N,N} - g_2(L)\right) - \frac{h_z}{2} r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, L\right)\frac{u_{N,N} - u_{N-1,N}}{h_r} - \frac{h_r}{2}R_1\left(\chi_4 u_{N,N} - g_4(R_1)\right) - \frac{h_r}{2}R_1 k_2\left(R_1, z_{N-\frac{1}{2}}\right)\frac{u_{N,N} - u_{N,N-1}}{h_z}\right] = \frac{R_1 h_r h_z f_{N,N}}{4}$$

Анализ порядка аппроксимации уравнения и граничных условий, выражение для главного члена погрешности аппроксимации

Невязка и порядок погрешность аппроксимации уравнения

Преобразование:

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r k_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right] = f(r,z)$$

$$-\left[\frac{\partial}{\partial r}\left(r k_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(r k_2(r,z)\frac{\partial u}{\partial z}\right)\right] = r f(r,z)$$

$$\tilde{k}_1(r,z) = r k_1(r,z), \quad \tilde{k}_2(r,z) = r k_2(r,z), \quad \tilde{q}(r,z) = r q(r,z)$$

$$\tilde{f}(r,z) = r f(r,z)$$

$$-\left[\frac{\partial}{\partial r}\left(\tilde{k}_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(\tilde{k}_2(r,z)\frac{\partial u}{\partial z}\right)\right] = \tilde{f}(r,z)$$

При анализе порядка аппроксимации, для простого, будем писать просто k_1, k_2, f вместо $\tilde{k}_1, \tilde{k}_2, \tilde{f}$

Невязка определяется как разность между правой и левой частью уравнения при условии, что вместо приближенного решения мы подставляем туда точное:

$$\xi_{i,j} = h_r h_z f_{i,j} + h_z k_1(x_{i+1/2}, y_j)\frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1(x_{i-1/2}, y_j)\frac{u_{i,j} - u_{i-1,j}}{h_r} + h_r k_2(x_i, y_{j+1/2})\frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r k_2(x_i, y_{j-1/2})\frac{u_{i,j} - u_{i,j-1}}{h_z}$$

Раскладываем по степеням h точное решение в узлах и коэффициент k

$$u_{i+1,j} = u(x_i + h_r, y_j) = u_{i,j} + h_r\frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^3}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i+1,j} - u_{i,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} + \frac{h_r}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^3}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i+\frac{1}{2},j} = k_1\left(r_i + \frac{h_r}{2}, z_j\right) = k_{1,i,j} + \frac{h_r}{2}\frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8}\frac{\partial^2 k_{1,i,j}}{\partial r^2} + \frac{h_r^3}{48}\frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$k_{1,i+\frac{1}{2},j}\frac{u_{i+1,j}-u_{i,j}}{h_r} = \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)$$

$$u_{i-1,j} = u(r_i - h_r, z_j) = u_{i,j} - h_r\frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} - \frac{h_r^3}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i,j}-u_{i-1,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} - \frac{h_r}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} - \frac{h_r^3}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i-\frac{1}{2},j} = k_1\left(r_i - \frac{h_r}{2}, z_j\right) = k_{1,i,j} - \frac{h_r}{2}\frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8}\frac{\partial^2 k_{1,i,j}}{\partial r^2} - \frac{h_r^3}{48}\frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$k_{1,i-\frac{1}{2},j}\frac{u_{i,j}-u_{i-1,j}}{h_r} = \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} - h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} - h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)$$

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j} - u_{i-1,j}}{h_r}$$
$$= h_z\left[\begin{array}{l}\left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + \\ +h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} - \\ -\left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} - h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + \\ +h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)\end{array}\right]$$

Сокрашаются четные степени

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} = h_z\left[h_r\left(k_1\frac{\partial^2 u}{\partial r^2}+\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right]$$

т.к. $k_1\frac{\partial^2 u}{\partial r^2}+\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)$, получаем, что

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} = h_z\left[h_r\left(\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right]$$

$$u_{i,j+1} = u\left(r_i, z_j + h_z\right) = u_{i,j} + h_z\frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^3}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\frac{u_{i,j+1}-u_{i,j}}{h_z} = \frac{\partial u_{i,j}}{\partial z} + \frac{h_z}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^2}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^3}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O\left(h^4\ \right)$$

$$k_{2,i,j+\frac{1}{2}} = k_2\left(r_i, z_j + \frac{h_z}{2}\right) = k_{2,i,j} + \frac{h_z}{2}\frac{\partial k_{2,i,j}}{\partial z} + \frac{h_z^2}{8}\frac{\partial^2 k_{2,i,j}}{\partial z^2} + \frac{h_z^3}{48}\frac{\partial^3 k_{2,i,j}}{\partial z^3} + O(h_z^4)$$

$$k_{2,i,j+\frac{1}{2}}\frac{u_{i+1,j}-u_{i,j}}{h_z} = \left[k_2\frac{\partial u}{\partial z}\right]_{i,j} + h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^4)$$

$$u_{i,j-1} = u\left(r_i, z_j - h_z\right) = u_{i,j} - h_z\frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} - \frac{h_z^3}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\frac{u_{i,j}-u_{i,j-1}}{h_z} = \frac{\partial u_{i,j}}{\partial z} - \frac{h_z}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^2}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} - \frac{h_z^3}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^4)$$

$$k_{2,i,j-\frac{1}{2}} = k_2\left(r_i, z_j - \frac{h_z}{2}\right) = k_{2,i,j} - \frac{h_z}{2}\frac{\partial k_{2,i,j}}{\partial z} + \frac{h_z^2}{8}\frac{\partial^2 k_{2,i,j}}{\partial z^2} - \frac{h_z^3}{48}\frac{\partial^3 k_{2,i,j}}{\partial z^3} + O(h_z^4)$$

$$k_{2,i,j-\frac{1}{2}}\frac{u_{i,j}-u_{i,j-1}}{h_z} = \left[k_2\frac{\partial u}{\partial z}\right]_{i,j} - h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} - h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^4)$$

$$h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} =$$

$$h_r\left[\begin{array}{l}\left[k_2\frac{\partial u}{\partial r}\right]_{i,j} + h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3}+\frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + \\ +h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} - \\ -\left[k_2\frac{\partial u}{\partial z}\right]_{i,j} + h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} - h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3}+\frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + \\ +h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^4)\end{array}\right]$$

Четные степени сокрааются

$$h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r\left[h_z\left(k_2\frac{\partial^2 u}{\partial z^2}+\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Так как$k_2\frac{\partial^2 u}{\partial z^2}+\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)$, получаем, что

$$h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Подсталяем в невязку получившиеся разложения

$$\xi_{i,j} = h_r h_z f_{i,j} + h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} + h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r h_z f_{i,j} + h_z\left[h_r\left(\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right] + h_r\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Группируем по степени hr и hz

$$\xi_{i,j} = h_r h_z f_{i,j} + h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} + h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r h_z\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j} + h_z\left[h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right] + h_r\left[h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Чтобы вычислить порядок аппроксимации, нормируем невязку

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{h_r h_z}$$

$$\tilde{\xi}_{i,j} = f_{i,j} + k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r^2} - k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r^2} +$$
$$k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z^2} - k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z^2} = \left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j} +$$
$$h_r^2\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^3) + h_z^2\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} +\right.$$
$$\left.\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^3)$$

$$\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j} = 0$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2$$
$$p_z = 2 - 0 = 2$$

Главный член погрешности по r

$$\Phi_r = \frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}$$

Главный член погрешности по z

$$\Phi_z = \frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}$$

Невязка и порядок погрешности аппроксимации граничного условия

1) $-k_1(r)\frac{\partial u}{\partial r}\Big|_{r=R_1} = \chi_2 u|_{r=R_1} - g_2(z)$

$$\xi_{i,j} = \frac{h_r}{2}h_z f_{i,j} - h_z\left(\chi_2 u_{i,j} - g_2(z_j)\right) - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j} - u_{i-1,j}}{h_r}$$
$$+ \frac{h_r}{2}k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2}k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j} - u_{i,j-1}}{h_z}$$

Подставляем полученные ранее произведения:

$$
\begin{aligned}
&\xi_{i,j} \\
&= \frac{h_r}{2} h_z f_{i,j} - h_z\left(\chi_2 u_{i,j} - g_2(z_j)\right) \\
&- h_z \begin{bmatrix} \left[k_1 \frac{\partial u}{\partial r}\right]_{i,j} - \frac{h_r}{2}\left[\frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right)\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} - \\ -h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4) \end{bmatrix} \\
&+ \frac{h_r}{2}\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}\right. \\
&\left. + O(h_z^4)\right]
\end{aligned}
$$

Группируем по степеням hr и hz

$$
\begin{aligned}
\xi_{i,j} = \frac{h_r}{2} h_z &\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)\right]_{i,j} - h_z\left[k_1\frac{\partial u}{\partial r} + \left(\chi_2 u - g_2(z)\right)\right]_{i,j} \\
&- h_z\left[h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^3)\right] \\
&+ \frac{h_r}{2}\left[h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]
\end{aligned}
$$

Для вычисления порядка аппроксимации нормируем невязку

$$
\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{2h_z}
$$

$$
\begin{aligned}
\tilde{\xi}_{i,j} = \frac{h_r}{2}&\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)\right]_{i,j} - \left[k_1\frac{\partial u}{\partial r} + \left(\chi_2 u - g_2(z)\right)\right]_{i,j} \\
&- h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} \\
&+ O(h_r^3)\frac{h_r}{2}\left[h_z^2\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}\right. \\
&\left. + O(h_z^3)\right]
\end{aligned}
$$

$$
\left[k_1\frac{\partial u}{\partial r} + \chi_2 u - g_2(z)\right]_{r=b} = 0
$$

$$
f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right) = 0
$$

Порядок аппроксимации уравнения по r и z:

$$
p_r = 2 - 0 = 2,
$$

$$
p_z = 2 - 0 = 2
$$

Главные члены погрешности

$$
\Omega_r = -\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]
$$

$$\Omega_z = \frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}$$

2) $k_2\frac{\partial u}{\partial z}\Big|_{z=0} = \chi_3 u|_{z=0} - g_3(r), i = 1,2,\dots,N_r-1, j = N_z$

$$\tilde{\xi}_{i,j} = \left[k_2\frac{\partial u}{\partial z} - \chi_3 u - g_3(r)\right]_{i,j} + \frac{h_z}{2}\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j}$$
$$+ h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^3)$$
$$+ \frac{h_z}{2}\left[h_r^2\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^3)\right]$$
$$\left[k_2\frac{\partial u}{\partial z} - \chi_3 u - g_3(r)\right]_{z=0} = 0$$
$$f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right) = 0$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2,$$
$$p_z = 2 - 0 = 2,$$

Главные члены погрешности:

$$\Omega_r = \frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}$$
$$\Omega_z = \frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}$$

3) $-k_2\frac{\partial u}{\partial z}\Big|_{z=L} = \chi_4 u|_{z=L} - g_4(r), \chi_4 \geq 0\ \ i = 1,2,\dots,N_r-1, j = N_z$

$$\tilde{\xi}_{i,j} = \left[k_2\frac{\partial u}{\partial z} + \chi_4 u - g_4(r)\right]_{i,j} + \frac{h_z}{2}\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j}$$
$$- h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^3)$$
$$+ \frac{h_z}{2}\left[h_r^2\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^3)\right]$$
$$\left[k_2\frac{\partial u}{\partial z} + \chi_4 u - g_4(r)\right]_{z=d} = 0$$
$$f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right) = 0$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2,$$
$$p_z = 2 - 0 = 2$$

Главные члены погрешности

$$\Omega_r = \frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}$$

$$\Omega_z = -\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]$$

Форма Хранения Матриц

Индексы главных диагональных элементов не хранятся, элементы главной диагонали располагаются в отдельном массиве Diag. В массиве A хранятся ненулевые элементы строго верхней треугольной части матрицы. Так как матрица хранится построчно, то в массиве IC хранятся номера столбцов ненулевых элементов верхнего треугольника матрицы. В массиве IR хранятся указатели на начало каждой строки в массивах A и IC. IR(N+1) содержит количество ненулевых элементов в строго верхнем треугольнике матрицы A плюс один.

Формулы и алгоритмы решения

Для решения используется метод сопряженных градиентов системы симметричной положительно определенной матрице A. Суть метода сводится к следующему. На основе последовательно вычисляемых невязок $r_0, r_1, r_2, \ldots$

$r_n = Ax_n - b$ одновременно с их получением с использованием процедуры Грамма-Шмидта строится система ортогональных векторов s_k

$s_1 = r_0$;

$$s_{n+1} = r_n - \sum_{j=1}^{n} b_{nj}s_j;\; b_{nj} = \frac{(As_j, r_n)}{(As_j, s_j)}$$

В этих обозначениях итерационный метод записывается в виде:

$$x_n = x_0 - \sum_{j=1}^{n} a_j s_j = x_{n-1} - a_n s_n;\; a_n = \frac{(r_0, s_n)}{(As_n, s_n)}$$

Улучшая свойства сходимости метода, преобразуем формулы:

$$b_{nm} = \frac{(r_n, r_n)}{(r_{n-1}, r_{n-1})};\; a_n = \frac{(r_{n-1}, r_{n-1})}{(As_n, s_n)};$$

Предварительные вычисления состоят в нахождении вектора невязки $r_0 = Ax_0 - b$, З по выбранному вектору xo и принятии $s_1 = r_0$ Далее по рекуррентным формулам на каждом шаге последовательно вычисляются:

$$a_n = \frac{(r_{n-1}, r_{n-1})}{(As_n, s_n)}$$

$$x_n = x_{n-1} - a_n s_n$$

$$r_n = r_{n-1} - a_n A s_n$$

$$b_n = \frac{(r_n, r_n)}{(r_{n-1}, r_{n-1})};$$

$$s_{n+1} = r_n - b_n s_n$$

Решение системы неявным методом сопряженных градиентов с предобусловливанием

Полученную систему Au=b будем решать неявным методом сопряженных градиентов с предобусловливанием. Для улучшения сходимости метода сопряженных градиентов правую и левую часть системы умножают на матрицу M . Данный процесс называется предобуславливанием. Матрицу M выбирают таким образом, чтобы $cond_2(MA)$ было как можно ближе к единице, M должна хорошо аппроксимировать A^{-1}.

Алгоритм(метода сопряженных градиентов):

1) Задаются начальным приближением и погрешностью:$\overrightarrow{x_0}, \varepsilon, k = 0$

2) Рассчитывают начальное направление:$j = 0, \vec{S}_k^j = -\nabla f(\overrightarrow{x_k}), \overrightarrow{x_k^j} = \overrightarrow{x_k}$

3)

$$\vec{x}_k^{j+1} = \vec{x}_k^j + \lambda \vec{S}_k^j, \quad \lambda = \arg\min_\lambda f(\vec{x}_k^j + \lambda \vec{S}_k^j), \quad \vec{S}_k^{j+1} = -\nabla f(\vec{x}_k^{j+1}) + \omega \vec{S}_k^j, \quad \omega = \frac{||\nabla f(\vec{x}_k^{j+1})||^2}{||\nabla f(\vec{x}_k^j)||^2}$$

Если $||\vec{S}_k^{j+1}|| < \varepsilon$ или $||\vec{x}_k^{j+1} - \vec{x}_k^j|| < \varepsilon$, то $\vec{x} = \vec{x}_k^{j+1}$ и останов.
Иначе
Если (j+1)<n, то j=j+1 и переход к 3;
иначе $\vec{x}_{k+1} = \vec{x}_k^{j+1}, \quad k = k + 1$ и k=k+1 переход к 2.

Здесь ε – задаваемая точность. Использовал $\varepsilon = 10^{-6}$.
Начальное приближение x_0 брал нулевым.
Использовалось диагональное предобусловливание: $\mathrm{M} = diag\{\frac{1}{a_{11}}, \frac{1}{a_{22}}, \dots, \frac{1}{a_{kk}}\}$

Тесты

Для всех тектов:

$$R_0 = 1, R_1 = 2, L = 1$$
$$\chi_2 = 1, \qquad \chi_3 = 1, \qquad \chi_4 = 1$$

Константный тест

$$k_1 = k_2 = 1$$
$$u = 1$$
$$f = 0$$
$$g_1 = 1, \qquad g_2 = 2, \qquad g_3 = 1, \qquad g_4 = 1$$

Линейный тест

$$k_1 = r + 1, \qquad k_2 = z + 1$$
$$u = 3r + 2z$$
$$f = -8 - \frac{3}{r}$$
$$g_1(z) = 3 + 2z, \qquad g_2(z) = 2z + 15, \qquad g_3(r) = 3r - 2, \qquad g_4(r) = 3r + 6$$

Нелинейный тест

$$k_1 = r + z, \qquad k_2 = r + z$$
$$u = r^2 + z^2$$
$$f = -8z - 8r$$
$$g_1(z) = z^2 + 1, \qquad g_2(z) = z^2 + 4z + 12, \qquad g_3(r) = r^2, \qquad g_4(r) = r^2 + 2r + 3$$