Санкт-Петербургский политехнический университет Петра Великого
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Отчёт по курсовой работе

«Разработка программы для моделирования стационарного двумерного распределения температуры»

по дисциплине

«Математические модели систем с распределёнными параметрами»

Выполнил студент гр. 3530904/90102 Мэн Цзянин

Руководитель Воскобойников С.П.

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Оглавление

Постановка задачи

Вариант N4.

Постановка задачи. Используя интегро-интерполяционный метод, разработать подпрограмму для моделирования распределения температуры в цилиндре, описываемого математической моделью

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right]=f(r,z),$$

$$0<\mathrm{c}_{11}\le k_1(r,z)\le c_{12},\quad 0<\mathrm{c}_{21}\le \mathrm{k}_2(\mathrm{r,z})\le \mathrm{c}_{22}$$

$$0\le r\le R,\ 0\le z\le L$$

с граничными условиями, определяемыми вариантом задания. Для решения системы алгебраических уравнений использовать метод сопряжённых градиентов с предобусловливанием. Матрица алгебраической системы должна храниться в упакованной форме

Форма (4)

Форма (4) отличается от формы (3) тем, что индексы главных диагональных элементов не хранятся и элементы главной диагонали располагаются в отдельном массиве Diag. В массиве A хранятся ненулевые элементы строго верхней треугольной части матрицы. Так как матрица хранится построчно, то в массиве IC хранятся номера столбцов ненулевых элементов верхнего треугольника матрицы. В массиве IR хранятся указатели на начало каждой строки в массивах A и IC. IR(N+1) содержит количество ненулевых элементов в строго верхнем треугольнике матрицы A плюс один.

	1	2	3	4	5	6	7	8	9
DIAG	13	14	15	16	17	18	19	20	21

	1	2	3	4	5	6	7	8	9	10	11	12
A	7	1	8	2	3	9	4	10	5	6	11	12
IC	2	4	3	5	6	5	7	6	8	9	8	9

	1	2	3	4	5	6	7	8	9	10
IR	1	3	5	6	8	10	11	12	13	13

$$u\big|_{r=0}-\textit{ограничено},$$

$$u\big|_{z=0}=\varphi_3(r)$$

$$-k_1(r)\frac{\partial u}{\partial r}\bigg|_{r=R}=\chi_2 u\big|_{r=R}-\varphi_2(z),\ \chi_2\ge 0$$

$$-k_2\frac{\partial u}{\partial z}\bigg|_{z=L}=\chi_4 u\big|_{z=L}-\varphi_4(r),\ \chi_4\ge 0$$

Дискретная модель

Введем в прямоугольнике [0, R]×[0, L] равномерную основную сетку

$$r_i = ih_r$$

$$h_r = \frac{R}{N_r}$$

$$z_j = jh_z$$

$$h_z = \frac{L}{N_z}$$

и вспомогательную сетку

$$r_{i-\frac{1}{2}} = \frac{r_i + r_{i-1}}{2}, \qquad i = 1,2,\ldots,N_r$$

$$z_{j-\frac{1}{2}} = \frac{z_j + z_{j-1}}{2}, \qquad j = 1,2,\ldots,N_z$$

Так как используются равномерные сетки, то шаги вспомогательной сетки определяются как

$$\hbar_i = \begin{cases} h_r, & i = 1,..,N_r - 1 \\ \frac{h_r}{2}, & i = 0, N_r \end{cases} \qquad \hbar_j = \begin{cases} h_z, & j = 1,..,N_z \\ \frac{h_z}{2}, & j = 0, N_z + 1 \end{cases}$$

Умножим исходное уравнение на r, проинтегрируем по вспомогательной сетке:

$$-\left[\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \frac{\partial}{\partial r}\left(rk(r)\frac{\partial u}{\partial r}\right) drdz + \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r\frac{\partial^2 u}{\partial z^2} drdz\right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} rf\, drdz$$

$$-\left[\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\left(r_{i+\frac{1}{2}}\, k\left(r_{i+\frac{1}{2}}\right)\frac{\partial u}{\partial r}|_{r_{i+\frac{1}{2}}}\right) dz - \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\left(r_{i-\frac{1}{2}}\, k\left(r_{i-\frac{1}{2}}\right)\frac{\partial u}{\partial r}|_{r_{i-\frac{1}{2}}}\right) dz \right.$$
$$\left. + \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\left(r\frac{\partial u}{\partial z}|_{z_{j+\frac{1}{2}}}\right) dr - \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\left(r\frac{\partial u}{\partial z}|_{z_{j-\frac{1}{2}}}\right) dr\right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} rf\, drdz$$

Воспользуемся формулой средних прямоугольников для вычисления значений интегралов:

$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \phi(r,z)dr \approx h_r\phi(r_i,z) = h_r\phi_i$$

$$\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \phi(r,z)dz \approx h_z\phi\big(r,z_j\big) = h_r\phi_j$$

$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r_i\phi\, drdz \approx r_i h_r h_z \phi_{i,j}$$

Также аппроксимируем производные по формуле центральных разностей:

$$k\left(r_{i+\frac{1}{2}}\right)\frac{\partial u}{\partial r}\bigg|_{r=r_{i+\frac{1}{2}},z=z_j} = k\left(r_{i+\frac{1}{2}}\right)\frac{u_{i+1,j}-u_{i,j}}{h_r}$$

$$k\left(r_{i-\frac{1}{2}}\right)\frac{\partial u}{\partial r}\bigg|_{r=r_{i-\frac{1}{2}},z=z_j} = k\left(r_{i-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i-1,j}}{h_r}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=z_{j+\frac{1}{2}},r=r_j} = \frac{u_{i,j+1}-u_{i,j}}{h_z}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=z_{j-\frac{1}{2}},r=r_j} = \frac{u_{i,j}-u_{i,j-1}}{h_z}$$

Получим:

$$-\Bigg[h_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} + h_r r_i k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}\Bigg] = r_i h_r h_z f_{i,j}$$

$$\cdots\cdots (1)$$

i=1,2,..,$N_r - 1$; j = 1,2,..,$N_z - 1$

Аппроксимация граничных условий:

$$i = 0, \qquad j = 1, \ldots, N_z - 1, \qquad u|_{r=0} - \textit{ограниченно},$$

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right] = f(r,z)$$

$$-\left[\frac{\partial}{\partial r}\left(k_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{1}{r}k_1(r,z)\frac{\partial u}{\partial r} + \frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right]\Bigg|_{r=0} = f(r,z)|_{r=0}$$

$$\lim_{r\to 0}\frac{1}{r}k_1(r,z)\frac{\partial u}{\partial r}, \quad \frac{\partial u}{\partial r}\bigg|_{r=0} = 0,$$

$$\lim_{r\to 0}\frac{1}{r}k_1(r,z)\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(k_1(r,z)\frac{\partial u}{\partial r}\right)\bigg|_{r=0}$$

$$-\left[2\frac{\partial}{\partial r}\left(k_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right]\Bigg|_{r=0} = f(r,z)|_{r=0}$$

$$u|_{r=0} - \textit{ограниченно} \Rightarrow \frac{\partial u}{\partial r}\bigg|_{r=0} = 0$$

$$\int_{r_i}^{r_{i+1/2}} r\phi(r,z)dr \approx \phi(r_i,z)\int_{r_i}^{r_{i+1/2}} rdr = \phi(r_i,z)\frac{r_{i+1/2}^2}{2} = \hbar_i\frac{r_{i+1/2}}{2}\phi(r_i,z), i = 0, r_i = 0, r_{i+1/2} = \hbar_i$$

$$-\left[\hbar_j r_{i+1/2}k_1(r_{i+1/2},z_j)\frac{v_{i+1,j}-v_{i,j}}{h_{i+1}} - 0\right.$$
$$\left. + \hbar_i\frac{r_{i+1/2}}{2}k_2(r_i,z_{j+1/2})\frac{v_{i,j+1}-v_{i,j}}{h_{j+1}} - \hbar_i\frac{r_{i+1/2}}{2}k_2(r_i,z_{j-1/2})\frac{v_{i,j}-v_{i,j-1}}{h_j}\right]$$
$$= \hbar_i\hbar_j\frac{r_{i+1/2}}{2}f_{i,j}$$

$$-\left[h_z r_{i+1/2}k_1(r_{i+1/2},z_j)\frac{v_{i+1,j}-v_{i,j}}{h_r} - 0\right.$$
$$\left. + \frac{h_r}{2}\frac{r_{i+1/2}}{2}k_2(r_i,z_{j+1/2})\frac{v_{i,j+1}-v_{i,j}}{h_z} - \frac{h_r}{2}\frac{r_{i+1/2}}{2}k_2(r_i,z_{j-1/2})\frac{v_{i,j}-v_{i,j-1}}{h_z}\right]$$
$$= \frac{h_r}{2}h_z\frac{r_{i+1/2}}{2}f_{i,j}$$

Аналогично воспользуемся интегро-интерполяционным методом, получим:

$$-\left[h_z r_{i+\frac{1}{2}}k_1\left(r_{i+\frac{1}{2}},z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z r_{i-\frac{1}{2}}k_1\left(r_{i-\frac{1}{2}},z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r}\right.$$
$$\left. + h_r r_i k_2\left(r_i,z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r r_i k_2\left(r_i,z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}\right]$$
$$= r_i h_r h_z f_{i,j} \quad \text{при i} = 1,2,..,N_r - 1\,;\ \text{j} = 1,2,..,N_z - 1 \quad \cdots (1)$$

$$-\left[-h_z R\left(\chi_2 u_{N,j} - \varphi_2(z_j)\right) - h_z r_{N-\frac{1}{2}}k_1\left(r_{N-\frac{1}{2}},z_j\right)\frac{u_{N,j}-u_{N-1,j}}{h_r}\right.$$
$$\left. + \frac{h_r}{2}Rk_2\left(R,z_{j+\frac{1}{2}}\right)\frac{u_{N,j+1}-u_{N,j}}{h_z} - \frac{h_r}{2}Rk_2\left(R,z_{j-\frac{1}{2}}\right)\frac{u_{N,j}-u_{N,j-1}}{h_z}\right]$$
$$= \frac{h_r}{2}Rh_z f_{N,j} \quad \text{при i} = N_r\,;\ \text{j} = 1,2,..,N_z - 1 \quad \cdots (2)$$

$$u_{i,0} = \varphi_3(0) \text{ при } i = 0, \dots, N_r\,;\; j = 0 \quad \cdots (3)$$

$$-\left[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, L\right)\frac{u_{i+1,N} - u_{i,N}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, L\right)\frac{u_{i,N} - u_{i-1,N}}{h_r} - h_r r_i\left(\chi_4 u_{i,N} - \varphi_4(r_i)\right) - h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,N} - u_{i,N-1}}{h_z}\right] = \frac{r_i h_r h_z f_{i,N}}{2} \quad \text{при } i = 1,2,\dots, N_r;\; j = \mathrm{N_z} \quad \cdots (4)$$

$$-\left[\hbar_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j} - u_{i,j}}{h_r} - 0 + \hbar_r \frac{r_{i+\frac{1}{2}}}{4} k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - \hbar_r \frac{r_{i+\frac{1}{2}}}{4} k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j} - u_{i,j-1}}{h_z}\right] = \hbar_r \hbar_z \frac{r_{i+\frac{1}{2}}}{4} f_{i,j} \quad \text{при } i = 0\,;\, j = 1,2,\dots, N_z - 1 \quad \cdots (5)$$

В результате получается система линейных алгебраических уравнений вида ***A*u=b** размерности N=(N_z-1)(N_r+1)Рассмотрим более подробно структуру этой системы. Для дальнейшей работы необходимо перенумеровать компоненты векторов u и b. Для этого используем приведенный индекс. Сперва для фиксированного *r* движемся по оси *z*, потом переходим к следующему значению *r*.

$$u_{i,j} = v_k$$

$$u_{i,j-1} = v_{k-1}$$

$$u_{i,j+1} = v_{k+1}$$

$$u_{i+1,j} = v_{k+N_z+1}$$

$$u_{i-1,j} = v_{k-N_z+1}$$

При таком обозначении новый индекс k можно рассчитать так: k=i*(N_z-1)+j

Матрица *A* квадратная, симметричная, пятидиагональная.

Хранить будем только 3 диагонали.

$$A = \begin{pmatrix}
a & b & & c & & & & & & & & \\
b & a & b & & c & & & & & & & \\
& b & a & 0 & & c & & & & & & \\
c & & 0 & a & b & & c & & & & & \\
& c & & b & a & b & & c & & & & \\
& & c & & b & a & 0 & & c & & & \\
& & & c & & 0 & a & b & & c & & \\
& & & & c & & b & a & b & & c & \\
& & & & & c & & b & a & 0 & & c \\
& & & & & & c & & 0 & a & b & \\
& & & & & & & c & & b & a & b \\
& & & & & & & & c & & b & a
\end{pmatrix}$$

Анализ порядка аппроксимации уравнения и граничных условий, выражение для главного члена погрешности аппроксимации

Невязка и порядок погрешность аппроксимации уравнения

Преобразование:

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right]=f(r,z)$$

$$-\left[\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(rk_2(r,z)\frac{\partial u}{\partial z}\right)\right]=rf(r,z)$$

$$\tilde{k}_1(r,z)=rk_1(r,z),\quad \tilde{k}_2(r,z)=rk_2(r,z),\quad \tilde{q}(r,z)=rq(r,z)$$

$$\tilde{f}(r,z)=rf(r,z)$$

$$-\left[\frac{\partial}{\partial r}\left(\tilde{k}_1(r,z)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(\tilde{k}_2(r,z)\frac{\partial u}{\partial z}\right)\right]=\tilde{f}(r,z)$$

При анализе порядка аппроксимации, для простого, будем писать просто k_1, k_2, f вместо $\tilde{k}_1, \tilde{k}_2, \tilde{f}$

Невязка определяется как разность между правой и левой частью уравнения при условии, что вместо приближенного решения мы подставляем туда точное:

$$\xi_{i,j}=h_r h_z f_{i,j}+h_z k_1(x_{i+1/2},y_j)\frac{u_{i+1,j}-u_{i,j}}{h_r}-h_z k_1(x_{i-1/2},y_j)\frac{u_{i,j}-u_{i-1,j}}{h_r}$$
$$+h_r k_2(x_i,y_{j+1/2})\frac{u_{i,j+1}-u_{i,j}}{h_z}-h_r k_2(x_i,y_{j-1/2})\frac{u_{i,j}-u_{i,j-1}}{h_z}$$

Раскладываем по степеням h точное решение в узлах и коэффициент k

$$u_{i+1,j} = u\left(x_i + h_r, y_j\right)$$

$$= u_{i,j} + h_r \frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^3}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i+1,j} - u_{i,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} + \frac{h_r}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^3}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i+\frac{1}{2},j} = k_1\left(r_i + \frac{h_r}{2}, z_j\right)$$

$$= k_{1,i,j} + \frac{h_r}{2}\frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8}\frac{\partial^2 k_{1,i,j}}{\partial r^2} + \frac{h_r^3}{48}\frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$k_{1,i+\frac{1}{2},j}\frac{u_{i+1,j}-u_{i,j}}{h_r} = \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)$$

$$u_{i-1,j} = u\left(r_i - h_r, z_j\right) = u_{i,j} - h_r\frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} - \frac{h_r^3}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i,j}-u_{i-1,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} - \frac{h_r}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} - \frac{h_r^3}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i-\frac{1}{2},j} = k_1\left(r_i - \frac{h_r}{2}, z_j\right) = k_{1,i,j} - \frac{h_r}{2}\frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8}\frac{\partial^2 k_{1,i,j}}{\partial r^2} - \frac{h_r^3}{48}\frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$k_{1,i-\frac{1}{2},j}\frac{u_{i,j}-u_{i-1,j}}{h_r} = \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} - h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} - h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)$$

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r}$$

$$= h_z \left[\begin{array}{l} \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + \\ +h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} - \\ -\left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} - h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} \\ +h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4) \end{array}\right.$$

Сокращаются четные степени

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} = h_z\left[h_r\left(k_1\frac{\partial^2 u}{\partial r^2} + \frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right]$$

т.к. $k_1\frac{\partial^2 u}{\partial r^2} + \frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)$, получаем, что

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} = h_z\left[h_r\left(\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right]$$

$$u_{i,j+1} = u\left(r_i, z_j + h_z\right) = u_{i,j} + h_z\frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^3}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\frac{u_{i,j+1}-u_{i,j}}{h_z} = \frac{\partial u_{i,j}}{\partial z} + \frac{h_z}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^2}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^3}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O\left(h^4\right)$$

$$k_{2,i,j+\frac{1}{2}} = k_2\left(r_i, z_j + \frac{h_z}{2}\right) = k_{2,i,j} + \frac{h_z}{2}\frac{\partial k_{2,i,j}}{\partial z} + \frac{h_z^2}{8}\frac{\partial^2 k_{2,i,j}}{\partial z^2} + \frac{h_z^3}{48}\frac{\partial^3 k_{2,i,j}}{\partial z^3} + O(h_z^4)$$

$$k_{2,i,j+\frac{1}{2}}\frac{u_{i+1,j}-u_{i,j}}{h_z} = \left[k_2\frac{\partial u}{\partial z}\right]_{i,j} + h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2} + \frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^4)$$

$$u_{i,j-1} = u\left(r_i, z_j - h_z\right) = u_{i,j} - h_z\frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} - \frac{h_z^3}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\frac{u_{i,j}-u_{i,j-1}}{h_z}=\frac{\partial u_{i,j}}{\partial z}-\frac{h_z}{2}\frac{\partial^2 u_{i,j}}{\partial z^2}+\frac{h_z^2}{6}\frac{\partial^3 u_{i,j}}{\partial z^3}-\frac{h_z^3}{24}\frac{\partial^4 u_{i,j}}{\partial z^4}+O(h_z^4)$$

$$k_{2,i,j-\frac{1}{2}}=k_2\left(r_i,z_j-\frac{h_z}{2}\right)=k_{2,i,j}-\frac{h_z}{2}\frac{\partial k_{2,i,j}}{\partial z}+\frac{h_z^2}{8}\frac{\partial^2 k_{2,i,j}}{\partial z^2}-\frac{h_z^3}{48}\frac{\partial^3 k_{2,i,j}}{\partial z^3}+O(h_z^4)$$

$$k_{2,i,j-\frac{1}{2}}\frac{u_{i,j}-u_{i,j-1}}{h_z}=\left[k_2\frac{\partial u}{\partial z}\right]_{i,j}-h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j}+h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3}+\right.$$
$$\left.\frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j}-h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\right.$$
$$\left.\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j}+O(h_z^4)$$

$$h_r k_2\left(r_i,z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z}-h_r k_2\left(r_i,z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}=$$
$$h_r\left[\begin{array}{l}\left[k_2\frac{\partial u}{\partial r}\right]_{i,j}+h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j}+h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3}+\frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j}+\\ +h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j}-\\ -\left[k_2\frac{\partial u}{\partial z}\right]_{i,j}+h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j}-h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3}+\frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j}+\\ +h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j}+O(h_z^4)\end{array}\right]$$

Четные степени сокращаются

$$h_r k_2\left(r_i,z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z}-h_r k_2\left(r_i,z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}=h_r\left[h_z\left(k_2\frac{\partial^2 u}{\partial z^2}+\right.\right.$$
$$\left.\left.\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right)_{i,j}+h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}+O(h_z^4)\right]$$

Так как$k_2\frac{\partial^2 u}{\partial z^2}+\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}=\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)$, получаем, что

$$h_r k_2\left(r_i,z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z}-h_r k_2\left(r_i,z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}=h_r\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j}+\right.$$
$$\left.h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}+O(h_z^4)\right]$$

Подставляем в невязку получившиеся разложения

$$\xi_{i,j}=h_r h_z f_{i,j}+h_z k_1\left(r_{i+\frac{1}{2}},z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r}-h_z k_1\left(r_{i-\frac{1}{2}},z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r}+$$
$$h_r k_2\left(r_i,z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z}-h_r k_2\left(r_i,z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}=h_r h_z f_{i,j}+$$

$h_z\left[h_r\left(\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right)_{i,j}+h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j}+\right.$

$\left.O(h_r^4)\right]+h_r\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j}+h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\right.\right.$

$\left.\left.\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}+O(h_z^4)\right]$

Группируем по степени hr и hz

$\xi_{i,j}=h_rh_zf_{i,j}+h_zk_1\left(r_{i+\frac{1}{2}},z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r}-h_zk_1\left(r_{i-\frac{1}{2}},z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r}+$
$h_rk_2\left(r_i,z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z}-h_rk_2\left(r_i,z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}=h_rh_z\left[f+\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)+\right.$
$\left.\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j}+h_z\left[h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j}+\right.$
$\left.O(h_r^4)\right]+h_r\left[h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}+O(h_z^4)\right]$

Чтобы вычислить порядок аппроксимации, нормируем невязку

$\tilde{\xi}_{i,j}=\frac{\xi_{i,j}}{h_rh_z}$

$\tilde{\xi}_{i,j}=f_{i,j}+k_1\left(r_{i+\frac{1}{2}},z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r^2}-k_1\left(r_{i-\frac{1}{2}},z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r^2}+$
$k_2\left(r_i,z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z^2}-k_2\left(r_i,z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z^2}=\left[f+\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)+\right.$
$\left.\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j}+h_r^2\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j}+O(h_r^3)+$
$h_z^2\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}+O(h_z^3)$

$\left[f+\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j}=0$

Порядок аппроксимации уравнения по r и z:

$$p_r=2-0=2$$

$$p_z=2-0=2$$

Главный член погрешности по r

$$\Phi_r = \frac{1}{12}\widetilde{k_1}\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial\widetilde{k_1}}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2\widetilde{k_1}}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3\widetilde{k_1}}{\partial r^3}\frac{\partial u}{\partial r}$$

Главный член погрешности по z

$$\Phi_z = \frac{1}{12}\widetilde{k_2}\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial\widetilde{k_2}}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2\widetilde{k_2}}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3\widetilde{k_2}}{\partial z^3}\frac{\partial u}{\partial z}$$

где

$$\widetilde{k}_1(r,z)=rk_1(r,z),\quad \widetilde{k}_2(r,z)=rk_2(r,z),\quad \widetilde{q}(r,z)=rq(r,z)$$

$$\widetilde{f}(r,z)=rf(r,z)$$

Невязка и порядок погрешности аппроксимации граничного условия

1) $u|_{r=0} - ограниченно \Rightarrow \left.\frac{\partial u}{\partial r}\right|_{r=0} = 0,\ i = 0, j = 1,2,\ldots,N_z - 1$ (Лекция10. р27)

$$\xi_{i,j} = \frac{h_r}{2}h_z f_{i,j} + \left[2h_z k_1(r_{i+1/2}, z_j)\frac{u_{i+1,j} - u_{i,j}}{h_r} - 0 + \right.$$
$$\left. + \frac{h_r}{2}k_2(r_i, z_{j+1/2})\frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2}k_2(r_i, z_{j-1/2})\frac{u_{i,j} - u_{i,j-1}}{h_z}\right]$$

Подставляем полученные ранее произведения:

$$\xi_{i,j} = \frac{h_r}{2}h_z f_{i,j} + \left[2h_z\left(\left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + \frac{h_r}{2}\left[\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right]_{i,j}\right.\right.$$
$$\left. + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^3)\right) - 0$$
$$+ \frac{h_r}{2}\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j}\right.$$
$$\left.\left. + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]\right]$$

Группируем по степеням hr и hz

$$\xi_{i,j} = \frac{h_r}{2} h_z \left[f + 2\frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right) + qu \right]_{i,j} + h_z 2 \left[k_1 \frac{\partial u}{\partial r} \right]_{i,j}$$
$$+ 2h_z \left(h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r} \right]_{i,j} + O\left(h_r^3\right) \right)$$
$$+ \frac{h_r}{2} \left[h_z^3 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z} \right)_{i,j} + O\left(h_z^4\right) \right]$$

Для вычисления порядка аппроксимации нормируем невязку

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{2h_z}$$

$$\tilde{\xi}_{i,j} = \frac{h_r}{4} \left[f + 2\frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right) + qu \right]_{i,j} + \left[k_1 \frac{\partial u}{\partial r} \right]_{i,j}$$
$$+ h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r} \right]_{i,j} + O\left(h_r^3\right)$$
$$+ \frac{h_r}{4} \left[h_z^2 \left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z} \right)_{i,j} + O\left(h_z^3\right) \right]$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=0} = 0,$$
$$-\left.\left[2\frac{\partial}{\partial r}\left(k_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right) + q(r,z)u \right]\right|_{r=0} = f(r,z)|_{r=0}$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2$$
$$p_z = 2$$

Главный член погрешности по x

$$\Omega_r = \left[\frac{1}{6}\widetilde{k_1}\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial \widetilde{k_1}}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 \widetilde{k_1}}{\partial r^2}\frac{\partial u}{\partial r} \right]$$

Главный член погрешности по y

$$\Omega_z = r \left[\frac{1}{48}\widetilde{k_2}\frac{\partial^4 u}{\partial z^4} + \frac{1}{24}\frac{\partial \widetilde{k_2}}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{32}\frac{\partial^2 \widetilde{k_2}}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{96}\frac{\partial^3 \widetilde{k_2}}{\partial z^3}\frac{\partial u}{\partial z} \right]$$

где

$$\widetilde{k}_1(r,z) = rk_1(r,z), \quad \widetilde{k}_2(r,z) = rk_2(r,z), \quad \widetilde{q}(r,z) = rq(r,z)$$
$$\widetilde{f}(r,z) = rf(r,z)$$

2) $-k_1(r)\frac{\partial u}{\partial r}\Big|_{r=R} = \chi_2 u|_{r=R} - \phi_2(z)$, (Лекция10. p19)

$$\xi_{i,j} = \frac{h_r}{2} h_z f_{i,j} - h_z\left(\chi_2 u_{i,j} - \phi_2(z_j)\right) - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j} - u_{i-1,j}}{h_r} + \frac{h_r}{2} k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2} k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j} - u_{i,j-1}}{h_z}$$

Подставляем полученные ранее произведения:

$$\xi_{i,j} = \frac{h_r}{2} h_z f_{i,j} - h_z\left(\chi_2 u_{i,j} - \phi_2(z_j)\right) - h_z\begin{bmatrix}\left[k_1\frac{\partial u}{\partial r}\right]_{i,j} - \frac{h_r}{2}\left[\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} - \\ -h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)\end{bmatrix} + \frac{h_r}{2}\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Группируем по степениям hr и hz

$$\xi_{i,j} = \frac{h_r}{2} h_z\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)\right]_{i,j} - h_z\left[k_1\frac{\partial u}{\partial r} + \left(\chi_2 u - \phi_2(z)\right)\right]_{i,j} - h_z\left[h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^3)\right] + \frac{h_r}{2}\left[h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Для вычисления порядка аппроксимации нормируем невязку

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{2h_z}$$

$$\tilde{\xi}_{i,j} = \frac{h_r}{2}\left[f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \left(\frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right)\right)\right]_{i,j} - \left[k_1 \frac{\partial u}{\partial r} + \left(\chi_2 u - \phi_2(z)\right)\right]_{i,j}$$

$$- h_r^2 \left[\frac{1}{6} k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j}$$

$$+ O\left(h_r^3\right)\frac{h_r}{2}\left[h_z^2\left(\frac{1}{12} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j}\right.$$

$$\left. + O\left(h_z^3\right)\right]$$

$$\left[k_1 \frac{\partial u}{\partial r} + \chi_2 u - \phi_2(z)\right]_{r=b} = 0$$

$$f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right) = 0$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2,$$
$$p_z = 2 - 0 = 2$$

Главные члены погрешности

$$\Omega_r = -\left[\frac{1}{6}\widetilde{k_1}\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial \widetilde{k_1}}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 \widetilde{k_1}}{\partial r^2}\frac{\partial u}{\partial r}\right]$$

$$\Omega_z = \frac{1}{24}\widetilde{k_2}\frac{\partial^4 u}{\partial z^4} + \frac{1}{12}\frac{\partial \widetilde{k_2}}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{16}\frac{\partial^2 \widetilde{k_2}}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{48}\frac{\partial^3 \widetilde{k_2}}{\partial z^3}\frac{\partial u}{\partial z}$$

где

$$\widetilde{k}_1(r,z) = rk_1(r,z), \quad \widetilde{k}_2(r,z) = rk_2(r,z), \quad \widetilde{q}(r,z) = rq(r,z)$$

$$\widetilde{f}(r,z) = rf(r,z)$$

3) $-k_2 \frac{\partial u}{\partial z}\Big|_{z=L} = \chi_4 u|_{z=L} - \phi_4(r), \chi_4 \geq 0 \;\; i = 1,2,\dots,N_r - 1, j = N_z$ $(Лекция 10, стр. 24)$

$$\tilde{\xi}_{i,j} = \left[k_2 \frac{\partial u}{\partial z} + \chi_4 u - \phi_4(r)\right]_{i,j} + \frac{h_z}{2}\left[f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right)\right]_{i,j}$$

$$- h_z^2 \left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + O\left(h_z^3\right)$$

$$+ \frac{h_z}{2}\left[h_r^2\left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O\left(h_r^3\right)\right]$$

$$\left[k_2 \frac{\partial u}{\partial z} + \chi_4 u - \phi_4(r)\right]_{z=d} = 0$$

$$f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right) = 0$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2,$$
$$p_z = 2 - 0 = 2$$

Главные члены погрешности

$$\Omega_r = \frac{1}{24}\widetilde{k_1}\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial \widetilde{k_1}}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 \widetilde{k_1}}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 \widetilde{k_1}}{\partial r^3}\frac{\partial u}{\partial r}$$

$$\Omega_z = -\left[\frac{1}{6}\widetilde{k_2}\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial \widetilde{k_2}}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 \widetilde{k_2}}{\partial z^2}\frac{\partial u}{\partial z}\right]$$

где

$$\widetilde{k}_1(r,z) = rk_1(r,z), \quad \widetilde{k}_2(r,z) = rk_2(r,z), \quad \widetilde{q}(r,z) = rq(r,z)$$

$$\widetilde{f}(r,z) = rf(r,z)$$

Решение системы методом сопряженных градиентов

Пусть $w^{(0)}$- произвольное начальное приближение, тогда $Aw - Aw^{(0)} = g - Aw^{(0)}$, что даст нам невязку $r^{(0)} = A(w - w^{(0)})$, предполагается, что у нас есть система из $s^{(i)}$, где i=1,2,…,n, линейно-независимых векторов, тогда можем разложит по базису этих векторов с соответствующими коэффициентами $w - w^{(0)} = \sum_{i=1}^{n} a_i s^{(i)}$, найти коэффициенты можем с помощью СЛАУ $\sum_{i=1}^{n} a_i A s^{(i)} = r^{(0)}$, решение системы сильно упростится, если $\left(As^{(i)}, s^{(i)}\right) = 0$ при $i \neq j$, а при $i = j$, скалярное произведение равнялось не 0 значению, в таком случае мы говорим об артогональности. Из этого мы можем выразить коэффициенты $a_i = \frac{(r^{(0)}, s^{(i)})}{(As^{(i)}, s^{(i)})}$, и выразить решение $w = w^{(0)} + \sum_{i=1}^{n} a_i s^{(i)}$.

Рассмотрим частичную сумму $w^{(n)} = w,\ w^{(n)} = w^{(0)} + \sum_{i=1}^{n} a_i s^{(i)}, w^{(k)} = w^{(0)} + \sum_{i=1}^{k} a_i s^{(i)}$, $w^{(k)} = w^{(k-1)} + a_k A s^{(k)}$, для невязки получим рекуррентное соотношение $r^{(k)} = r^{(k-1)} - a_k A s^{(k)}$.

$$w^{(0)}, \qquad r^{(0)} = \mathrm{g} - Aw^{(0)}, \qquad s^{(1)} = ?$$

$$k = 1, 2, \dots, n, \qquad a_k = \frac{\left(r^{(0)}, s^{(k)}\right)}{\left(As^{(k)}, s^{(k)}\right)}$$

$$w^{(k)} = w^{(k-1)} + a_k s^{(k)}, \qquad r^{(k)} = r^{(k-1)} - a_k A s^{(k)}$$

$$s^{(k+1)} = ?$$

При явном методе сопряженных градиентов $s^{(1)}$ берут равным $r^{(0)}, s^{(k+1)} = r^{(k)} + \beta_k s^{(k)}$, с вводом дополнительного коэффициента $\beta_k = \frac{(r^{(k)}, r^{(k)})}{(r^{(k-1)}, r^{(k-1)})}$ при $\sqrt{(r^{(k)}, r^{(k)})} < \gamma\varepsilon$, явный метод обладает тем свойством что при отсутствии ошибок округления мы можем получить точное решение не позднее чем на n-ом шаге, но возникает двойственность, из-за ошибок округления происходит разрушение артогональности последовательности s и в результате к неточности, и метод становится итерационным.

Неявный метод

$$\mathrm{A}w = b, \qquad A = A^T, \qquad (Ay, y) > 0, \qquad y \neq 0$$

$$x^{(0)} - \text{произвольное начальное приблидение}$$

$$r^{(0)} = b - Ax^{(0)}, \qquad Bw^{(0)} = r^{(0)}, \qquad s^{(1)} = w^{(0)}, \qquad Bg = b, \qquad \gamma = \sqrt{(g, b)}$$

$$k = 1,2, \dots, K_{max}$$

$$a_k = \frac{\left(w^{(k-1)}, r^{(k-1)}\right)}{\left(As^{(k-1)}, s^{(k-1)}\right)}$$

$$x^{(k)} = x^{(k-1)} + a_k s^{(k)}, \qquad r^{(k)} = r^{(k-1)} + a_k A s^{(k-1)}$$

$$Bw^{(k)} = r^{(k)}, \sqrt{\left(w^{(k)}, r^{(k)}\right)} < \gamma\varepsilon$$

$$\beta_k = \frac{\left(w^{(k)}, r^{(k)}\right)}{\left(w^{(k-1)}, r^{(k-1)}\right)}, \qquad s^{(k+1)} = w^{(k)} + \beta_k s^{(k)}$$

О выборе матрицы предобусловливания

$$\mathrm{A}w = b, \qquad A = A^T, \qquad (Ay, y) > 0, \qquad y \neq 0$$

$$B = B^T, \qquad (By, y) > 0, \qquad y \neq 0$$

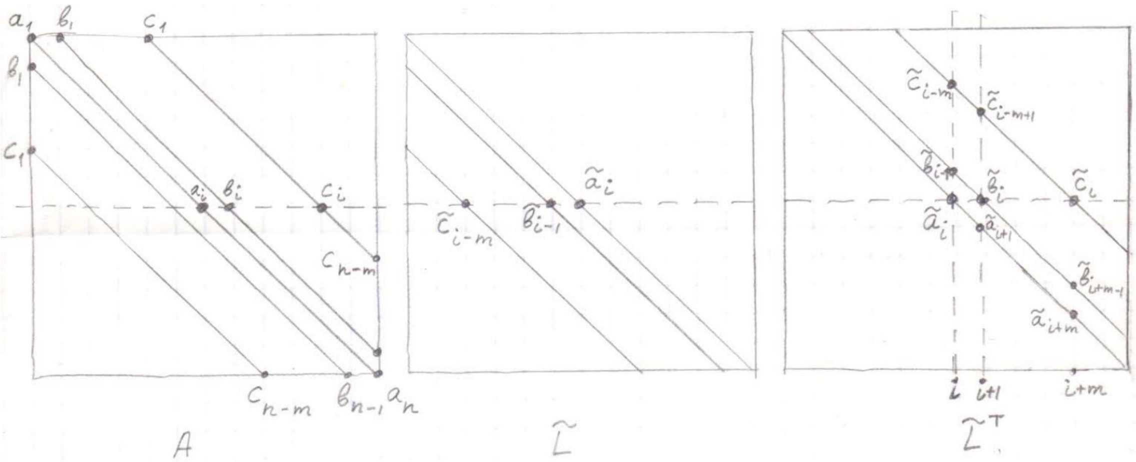
$$B = D, \qquad D = \begin{bmatrix} a_{11} & \_ & \_ \\ \_ & \cdots & \_ \\ \_ & \_ & a_{nn} \end{bmatrix}, \qquad B = \tilde{L}\,\tilde{L}^T$$

$$\tilde{l}_{ij} = 0, \quad i < j$$

$$Bw^{(0)} = r^{(0)}, \qquad \tilde{L}y_0 = r_0, \qquad \tilde{L}^T w_0 = y_0,$$

$$Bw^{(k)} = r^{(k)}, \qquad \tilde{L}y_k = r_k, \qquad \tilde{L}^T w_k = y_k$$

Неполное разложение Холевского

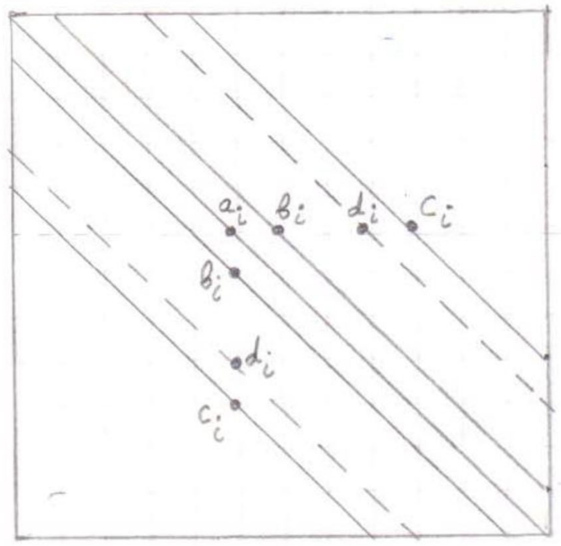


$$a_i = \tilde{a}_i^2 + \tilde{b}_{i-1}^2 + \tilde{c}_{i-m}^2, \qquad b_i = \tilde{a}_i \tilde{b}_i, \qquad c_i = \tilde{a}_i \tilde{c}_i,$$

$$\tilde{a}_i = \sqrt{a_i - \tilde{b}_{i-1}^2 - \tilde{c}_{i-m}^2}, \quad i = 1,2,\ldots,n, \quad \tilde{b}_0 = 0, \quad \tilde{c}_{i-m} = 0, \quad i = 1,2,\ldots,m$$

$$\tilde{b}_i = \frac{b_i}{\tilde{a}_i}, \qquad \tilde{c}_i = \frac{c_i}{\tilde{a}_i},$$

$\tilde{L}\tilde{L}^T =$



$A - \tilde{L}\tilde{L}^T =$

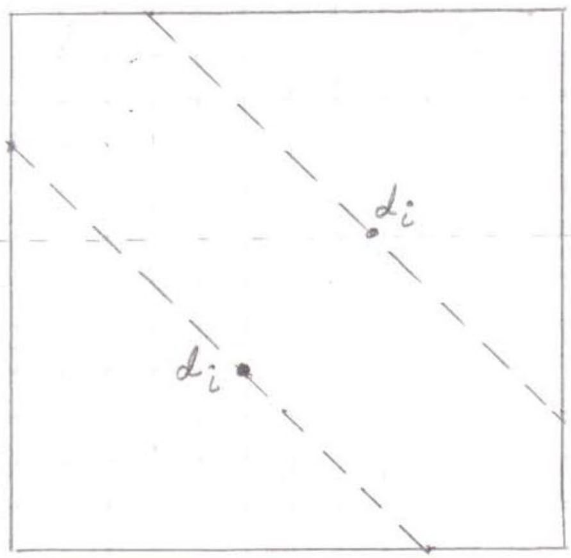
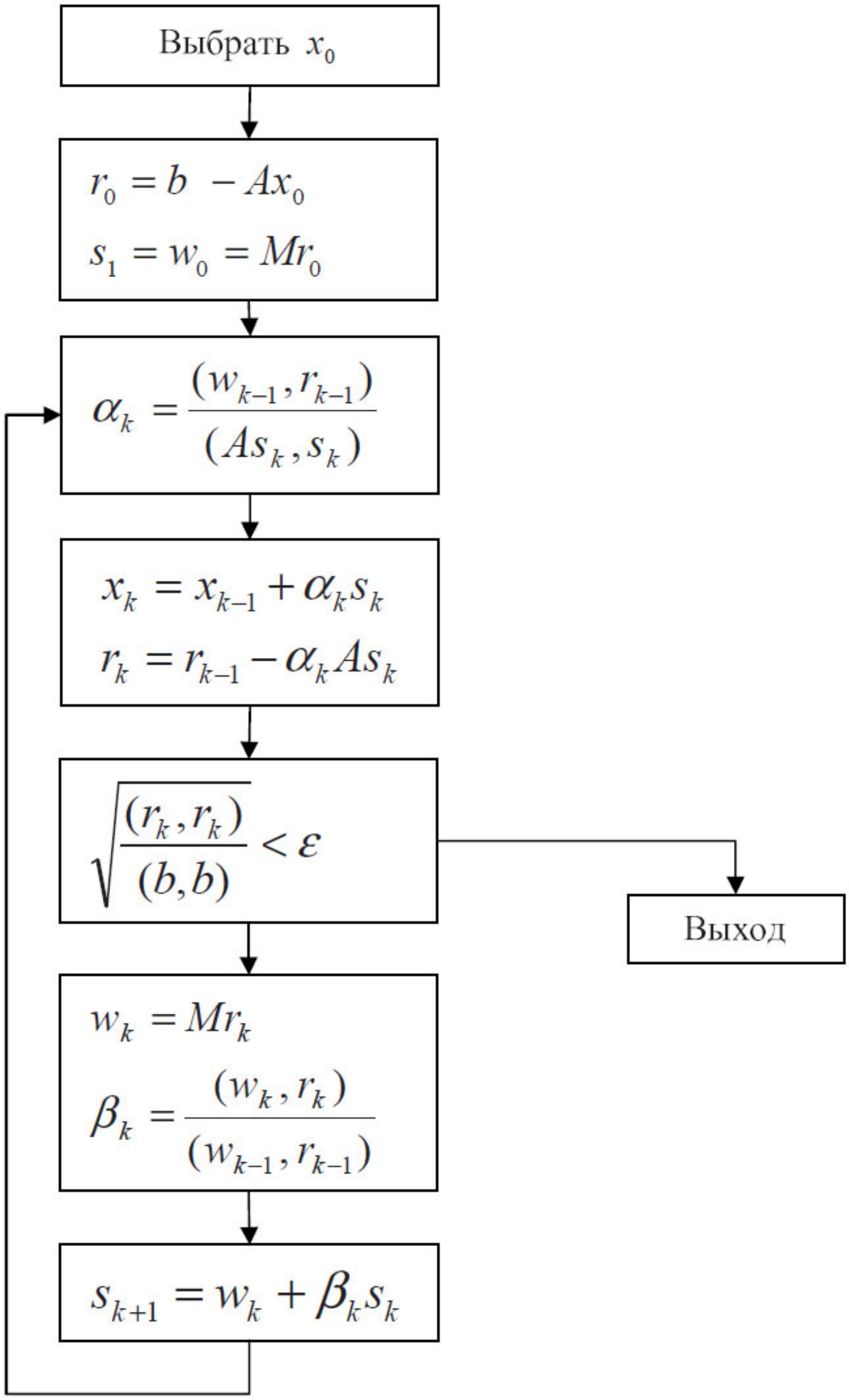


Схема применения метода выглядит следующим образом:



Здесь я использовал параметр $\varepsilon = 10^{-6}$.

Тесты

Для всех тектов:

$$R = 1, \qquad L = 1$$
$$\chi_2 = 1, \qquad \chi_3 = 1, \qquad \chi_4 = 1$$

Константный тест

$$k_1 = k_2 = 1$$
$$u = 1$$
$$f = 0$$
$$\varphi_2(z) = 1, \qquad \varphi_3(r) = 1, \qquad \varphi_4(r) = 1$$

Линейный тест

$$k_1 = r + 1, \qquad k_2 = z + 1$$
$$u = r^2$$
$$f = -6r - 4$$
$$\varphi_2(z) = 5, \qquad \varphi_3(r) = r^2, \qquad \varphi_4(r) = r^2$$

Нелинейный тест

$$k_1 = r^2 + 1, \qquad k_2 = z^2 + 1$$
$$u = r^4 + r^2 z^2$$
$$f = -24r^4 - 14r^2 z^2 - 18r^2 - 4z^2$$
$$\varphi_2(z) = 5z^2 + 9, \qquad \varphi_3(r) = r^4, \qquad \varphi_4(r) = r^4 + 5r^2$$

Результаты

Константный случай

Число разбиений Nr, Nz	Максимальная погрешность	Отношение погрешностей	Число итераций метода
4	8.88178419700E-16	0	11
8	3.82518982489E-07	2.32E-09	27
16	6.30674762192E-07	0.60652337056	16
32	2.62734831580E-06	0.24004231125	107
64	5.99528580780E-06	0.43823570719	208
128	6.36365821094E-06	0.94211310681	413

Линейный случай

Число разбиений Nr, Nz	Максимальная погрешность	Отношение погрешностей	Число итераций метода
4	1.95399252334E-14	0	16
8	1.14468746837E-06	1.71E-08	32
16	2.26973880935E-06	0.504325636	65
32	5.50640104335E-06	0.412200054	130
64	1.24175795404E-05	0.443435939	252
128	2.36257855251E-05	0.525594357	486

Нелинейный случай

Число разбиений Nr, Nz	Максимальная погрешность	Отношение погрешностей	Число итераций метода
4	3.38167202120E-02	0	16
8	8.92482305903E-03	3.789063378	42
16	2.25182768122E-03	3.963368571	87
32	5.80594786091E-04	3.87848416	179
64	1.70294493403E-04	3.40935737	359
128	4.16449411629E-05	4.0892	717

Вывод

Погрешность решения дифференциального уравнения складывается из двух: погрешности аппроксимации (появляется при переходе от непрерывного уравнения к системе разностных) и погрешности решения алгебраической системы.

В линейном и константном случаях погрешность аппроксимации отсутствует, ее небольшой рост с увеличением количества разбиений связано с накоплением ошибки округления.

А в нелинейном случае наблюдается уменьшение ошибки в 4 раза при увеличении в 2 раза разбиений по оси r и z. Погрешность решения алгебраической системы мала по сравнению с погрешностью аппроксимации, она возрастает незаметно. Погрешность аппроксимации, в свою очередь, уменьшается, т.к. мы увеличиваем количество разбиений. Причем, согласно теории, при одновременном удвоении числа разбиений погрешность аппроксимации должна уменьшаться в 4 раза, т.к. порядок аппроксимации метода равен 2. Как видим, наблюдаемые результаты очень близок к теоретическому.

Приложение

```
import java.util.Arrays;
import java.util.HashMap;
import java.util.function.Function;


public class N4 {
    private final static double EPS = 1e-6;
    private static int N = 5;
    private static final double R0 = 0;
    private static final double R1 = 1;
    private static final double L = 1;
    private static final double Chi2 = 1;
    private static final double Chi4 = 1;


    private enum SystemParameters {
        DIAGONAL_A, DIAGONAL_B, DIAGONAL_C, VECTOR_G
    }

    @FunctionalInterface
    public interface FunctionTwoArgs<A, B, R> {
        R apply(A a, B b);
    }

    public static void main(String[] args) {

        System.out.println("N4");
        System.out.println("---->>> Константый случай");
        test(   (r, z) -> 1.0,
               (r, z) -> 1.0,
               (r, z) -> 0.0,
               (z) -> 1.0,
               (r) -> 1.0,
               (r) -> 1.0,
               (r, z) -> 1.0);


        System.out.println("\n\n---->>> Линейный случай");
        test(   (r, z) -> r + 1.0,
               (r, z) -> z + 1.0,
```

```
            (r, z) -> -6 * r - 4,
            (z) -> 5.0,
            (r) -> r * r,
            (r) -> r * r,
            (r, z) -> r * r);

    System.out.println("\n\n---->>> Нелинейный случай");
    test(   (r, z) -> r * r + 1,
            (r, z) -> 1 + z * z,
            (r, z) -> -24 * r*r*r*r - 14 * r*r * z*z - 18*r*r - 4*z*z,
            (z) -> 5*z*z + 9,
            (r) -> r * r * r * r,
            (r) -> r * r * r * r + 5 * r * r,
            (r, z) -> r * r * r * r + z * z * r * r);
}

private static void test(FunctionTwoArgs<Double, Double, Double> k1,
                         FunctionTwoArgs<Double, Double, Double> k2,
                         FunctionTwoArgs<Double, Double, Double> f,
                         Function<Double, Double> phi2,
                         Function<Double, Double> phi3,
                         Function<Double, Double> phi4,
                         FunctionTwoArgs<Double, Double, Double> u)
{
    HashMap<SystemParameters, double[]> system;
    N = 5;
    double hR = (R1 - R0) / (N - 1);
    double hZ = L / (N - 1);
    double r;
    double z = 0;
    double[] result = new double[N * N];
    system = getSystem(k1, k2, f, phi2, phi3, phi4);
    for (int i = 0; i < N; ++i) {
        r = R0;
        for (int j = 0; j < N; ++j) {
            result[i * N + j] = u.apply(r, z);
            r += hR;
        }
        z += hZ;
    }
    System.out.println("Отклонения от точного решения\n"
            + Arrays.toString( sub(multiply(system, result),
```

```
                system.get(SystemParameters.VECTOR_G))));

        System.out.println("Ошибка");
        double prevError = 0;
        double nowError;
        N = 5;
        System.out.println("\t\tN\tError\tRatio\t");
        for (int i = 2; i <= 8; ++i) {
            N = (int) Math.round(Math.pow(2, i)) + 1;
            system = getSystem(k1, k2, f, phi2, phi3, phi4);
            result              =              ConjugateGradientMethod(system,
system.get(SystemParameters.VECTOR_G), getEMatrix());
            nowError = getMaxError(result, u);
            System.out.println("\t\t " + (N - 1) + "\t " + nowError + " \t "
+ prevError / nowError);
            prevError = nowError;
        }
    }


    private                         static                         double[]
ConjugateGradientMethod(HashMap<SystemParameters, double[]> system,
                                            double[] first,
                                            HashMap<SystemParameters,
double[]> bMatrix) {
        double[] result = Arrays.copyOf(first, first.length);
        double[]     r     =     sub(system.get(SystemParameters.VECTOR_G),
multiply(system, first));
        double[] p = solveB(bMatrix, r);
        double[] b = solveB(bMatrix, system.get(SystemParameters.VECTOR_G));
        double[] s = Arrays.copyOf(p, p.length);
        double alpha; double beta; double[] newR; double[] newP; int k;
        for (k = 1; k <= 10000; k++) {
            alpha = multiply(p, r) / multiply(multiply(system, s), s);
            result = addition(result, multiply(alpha, s));
            newR = sub(r, multiply(alpha, multiply(system, s)));
            newP = solveB(bMatrix, newR);
            double  check  =  Math.sqrt(multiply(newP,  newR)  /  multiply(b,
system.get(SystemParameters.VECTOR_G)));
            if (check < EPS) {
                ++k;
                break;
```

```
            }
            beta = multiply(newP, newR) / multiply(p, r);
            s = addition(newP, multiply(beta, s));
            r = newR; p = newP;
        }
        System.out.println("(Число итераций:\t" + k +")");
        return result;
    }


    private static double[] getADiag(FunctionTwoArgs<Double, Double, Double>
k2) {
        double hR = (R1 - R0) / (N - 1);
        double hZ = L / (N - 1);
        double scale = hR / hZ;
        double[] result = new double[N * N];
        double z = hZ;
        double r;

        for (int j = 1; j < N - 1; j++) {
            r = R0;
            result[j * N] = -(scale / 4) * r * k2.apply(r, z - hZ / 2); //
#(2)
            r += hR;
            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = -(scale) * r * k2.apply(r, z - hZ / 2);
// #(1)
                r += hR;
            }
            result[j * N + N - 1] = -(scale / 2) * r * k2.apply(r, z - hZ /
2); // #(3)
            z += hZ;
        }

        r = R0;
        for (int i = 0; i < N; i++) {
            result[N * (N - 1) + i] = -scale * r * k2.apply(r, z - hZ / 2);
// #（5）
            r += hR;
        }

        return result;
```

```
    }

    private static double[] getCDiag(FunctionTwoArgs<Double, Double, Double>
k1,
                                      FunctionTwoArgs<Double,  Double,  Double>
k2) {

        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double scale = hZ / hR; double z = hZ;
        double r;
        double[] result = new double[N * N];

        for (int i = 0; i < N; i++) { // #(4)
            result[i] = 1;
        }

        for (int j = 1; j < N - 1; j++) {
            r = R0;
            result[j * N] = scale * (r + hR / 2) * k1.apply(r + hR / 2, z) //
#(2)
                    + (1 / scale) * r * k2.apply(r, z + hZ / 2)
                    + (1 / scale) * r * k2.apply(r, z - hZ / 2);
            r += hR;


            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = scale * (r + hR / 2) * k1.apply(r + hR
/2, z) // #(1)
                        + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                        + (1 / scale) * r * k2.apply(r, z + hZ / 2)
                        + (1 / scale) * r * k2.apply(r, z - hZ / 2);

                r += hR;
            }

            result[j * N + N - 1] = hZ * r * Chi2 // #(3)
                    + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                    + (1 / scale / 2) * r * k2.apply(r, z + hZ / 2)
                    + (1 / scale / 2) * r * k2.apply(r, z - hZ / 2);
            z += hZ;
        }
```

```
        r = R0;
        for (int i = 0; i < N; i++) {  // #(5)
            result[N * (N - 1) + i] =  scale * (r + hR / 2) * k1.apply(r + hR
/2, z)
                    + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                    + hR * r * Chi4
                    + (1 / scale) * r * k2.apply(r, z - hZ / 2);
            r += hR;
        }
        return result;
    }

    private static double[] getDDiag(FunctionTwoArgs<Double, Double, Double>
k1) {

        double hR = (R1 - R0) / (N - 1);
        double hZ = L / (N - 1);
        double scale = hZ / hR;
        double z = hZ;
        double r;
        double[] result = new double[N * N];

        for (int j = 1; j < N - 1; j++) {
            r = R0;
            for (int i = 0; i < N - 1; i++) {
                result[j * N + i] = -scale * (r + hR / 2) * k1.apply(r + hR
/2, z); // #(2) & (1)
                r += hR;
            }
            z += hZ;
        }
        return result;
    }

    private static double[] getEDiag(FunctionTwoArgs<Double, Double, Double>
k2) {
        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double scale = hR / hZ;
        double[] result = new double[N * N]; double z = hZ;
        double r;
        for (int j = 1; j < N - 1; j++) {
            r = R0;
```

```
            result[j * N] = -scale * r * k2.apply(r, z + hZ / 2) / 2; r +=
hR; // #(2)
            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = -scale * r * k2.apply(r, z + hZ / 2); r
+= hR; // #(1)
            }
            result[j * N + N - 1] = -scale * r * k2.apply(r, z + hZ / 2) / 2;
z += hZ; // #(3)
        }
        return result;
    }

    private static double[] getVectorG(FunctionTwoArgs<Double, Double,
Double> f,
                                       Function<Double, Double> phi2,
                                       Function<Double, Double> phi3,
                                       Function<Double, Double> phi4) {

        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double[] result = new double[N * N]; double z = hZ;

        double r = R0;
        for (int i = 0; i < N; i++) { // #（4）
            result[i] = phi3.apply(r);
            r += hR;
        }

        for (int j = 1; j < N - 1; j++) {
            r = R0;
            result[j * N] = hR * hZ * r * f.apply(r, z) / 4;  // #（2）
            r += hR;
            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = hR * hZ * r * f.apply(r, z);  // #（1）
                r += hR;
            }
            result[j * N + N - 1] = hR * hZ * r * f.apply(r, z) / 2 + hZ * r
* phi2.apply(z); // #（3）
            z += hZ;
        }

        r = R0;
        for (int i = 0; i < N; i++) {
```

```
            result[N * (N - 1) + i] = hR * r * phi4.apply(r) + hR * hZ * r *
f.apply(r, z) / 2; // #（5）
            r += hR;
        }
        return result;
    }

    private        static        HashMap<SystemParameters,        double[]>
getSystem(FunctionTwoArgs<Double, Double, Double> k1,

FunctionTwoArgs<Double, Double, Double> k2,

FunctionTwoArgs<Double, Double, Double> f,

Function<Double, Double> phi2,

Function<Double, Double> phi3,

Function<Double, Double> phi4)
    {
        double[] a = getADiag(k2);
        double[] c = getCDiag(k1, k2);
        double[] d = getDDiag(k1);
        double[] e = getEDiag(k2);
        double[] g = getVectorG(f, phi2, phi3, phi4);

        for (int i = 0; i < N; i++) {
            g[N + i] -= g[i] * a[N + i];
            a[N + i] = 0;
            g[N * (N - 2) + i] -= g[N * (N - 1) + i] * e[N * (N - 2) + i];
            e[N * (N - 2) + i] = 0;
        }

        HashMap<SystemParameters, double[]> system = new HashMap<>();

        system.put(SystemParameters.DIAGONAL_A, c);
        system.put(SystemParameters.DIAGONAL_B, d);
        system.put(SystemParameters.DIAGONAL_C, e);
        system.put(SystemParameters.VECTOR_G, g);
        return system;
    }
```

```
    private      static      double      getMaxError(double[]      solve,
FunctionTwoArgs<Double, Double, Double> u) {
        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double z = 0; double r;
        double maxError = 0; double nowError;
        for (int j = 0; j < N; j++) { r = R0;
            for (int i = 0; i < N; i++) {
                nowError = Math.abs(u.apply(r, z) - solve[j * N + i]); if
(nowError > maxError) {
                    maxError = nowError;
                }
                r += hR;
            }
            z += hZ;
        }
        return maxError;
    }

    private       static       HashMap<SystemParameters,       double[]>
getBMatrix(HashMap<SystemParameters, double[]> system)
    {
        HashMap<SystemParameters, double[]> result = new HashMap<>(); int
squareN = N * N;
        double[] a = new double[squareN];
        double[] b = new double[squareN];
        double[] c = new double[squareN];
        result.put(SystemParameters.DIAGONAL_A, a);
        result.put(SystemParameters.DIAGONAL_B, b);
        result.put(SystemParameters.DIAGONAL_C, c);
        a[0] = Math.sqrt(system.get(SystemParameters.DIAGONAL_A)[0]);
        for (int i = 1; i < N; i++) {
            b[i - 1] = system.get(SystemParameters.DIAGONAL_B)[i - 1] / a[i
- 1];
            a[i]  =  Math.sqrt(system.get(SystemParameters.DIAGONAL_A)[i]  -
Math.pow(b[i - 1], 2));
        }
        for (int i = N; i < squareN; i++) {
            c[i - N] = system.get(SystemParameters.DIAGONAL_C)[i - N];
            b[i - 1] = system.get(SystemParameters.DIAGONAL_B)[i - 1] / a[i
- 1];
            a[i]  =  Math.sqrt(system.get(SystemParameters.DIAGONAL_A)[i]  -
Math.pow(b[i - 1], 2) - Math.pow(c[i - N], 2));
```

```
        }
        return result;
    }

    private  static  double[]  solveB(HashMap<SystemParameters,  double[]>
bMatrix, double[] g) {
        int squareN = N * N;

        double[] y = new double[squareN];
        double[] a = bMatrix.get(SystemParameters.DIAGONAL_A);
        double[] b = bMatrix.get(SystemParameters.DIAGONAL_B);
        double[] c = bMatrix.get(SystemParameters.DIAGONAL_C);

        y[0] = g[0] / a[0];
        for (int i = 1; i < N; i++) {
            y[i] = (g[i] - b[i - 1] * y[i - 1]) / a[i];
        }

        for (int i = N; i < squareN; i++) {
            y[i] = (g[i] - b[i - 1] * y[i - 1] - c[i - N] * y[i - N]) / a[i];
        }

        double[] result = new double[squareN];
        result[squareN - 1] = y[squareN - 1] / a[squareN - 1];

        for (int i = squareN - 2; i >= N * (N - 1); i--) {
            result[i] = (y[i] - b[i] * result[i + 1]) / a[i];
        }
        for (int i = N * (N - 1) - 1; i >= 0; i--) {
            result[i] = (y[i] - b[i] * result[i + 1] - c[i] * result[i + N])/
a[i];
        }
        return result;
    }

    private static HashMap<SystemParameters, double[]> getEMatrix() {
        HashMap<SystemParameters, double[]> e = new HashMap<>();
        int squareN = N * N;
        double[] a = new double[squareN];
        for (int j = 0; j < squareN; j++) {
            a[j] = 1;
        }
```

```
        e.put(SystemParameters.DIAGONAL_A,                              a);
e.put(SystemParameters.DIAGONAL_B,          new          double[squareN]);
e.put(SystemParameters.DIAGONAL_C, new double[squareN]); return e;
    }

    private   static   double   multiply(double[]   leftVector,   double[]
rightVector)
    {
        double result = 0;
        for (int i = 0; i < leftVector.length; i++) {
            result += leftVector[i] * rightVector[i];
        }
        return result;
    }

    private  static  double[]  multiply(HashMap<SystemParameters,  double[]>
system, double[] vector) {
        double[] result = new double[vector.length];
        double[]  diagA  =  system.get(SystemParameters.DIAGONAL_A);  double[]
diagB   =   system.get(SystemParameters.DIAGONAL_B);   double[]   diagC   =
system.get(SystemParameters.DIAGONAL_C); for (int i = 0; i < vector.length;
i++) {
            result[i] = diagA[i] * vector[i];
        }
        for (int i = 0; i < vector.length - 1; i++) { result[i] += diagB[i]
* vector[i + 1];
        }
        for (int i = 0; i < vector.length - N; i++) { result[i] += diagC[i]
* vector[i + N];
        }
        for (int i = 1; i < vector.length; i++) { result[i] += diagB[i - 1]
* vector[i - 1];
        }
        for (int i = N; i < vector.length; i++) { result[i] += diagC[i - N]
* vector[i - N];

        }
        return result;
    }

    private   static   double[]   multiply(double   number,   double[]   vector)
{ double[] result = new double[vector.length];
```

```
        for (int i = 0; i < vector.length; i++)
        {
            result[i] = vector[i] * number;
        }
        return result;
    }

    private   static   double[]   addition(double[]   leftVector,   double[]
rightVector) {
        double[] result = new double[leftVector.length];
        for (int i = 0; i < leftVector.length; i++) {
            result[i] = leftVector[i] + rightVector[i];
        }
        return result;
    }

    private static double[] sub(double[] leftVector, double[] rightVector)
{ double[] result = new double[leftVector.length];
        for (int i = 0; i < leftVector.length; i++) {
            result[i] = leftVector[i] - rightVector[i];
        }
        return result;
    }
}
```