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ОТЧЕТ ПО КУРСОВОЙ РАБОТЕ

по дисциплине
«Математические модели систем с распределёнными параметрами»
Вариант Q12

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Постановка задания

Вариант Q. Постановка задачи. Используя интегро-интерполяционный метод, разработать подпрограмму для моделирования распределения температуры в полом цилиндре, описываемого математической моделью

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right]=f(r,z),$$

$$0<c_{11}\le k_1(r,z)\le c_{12},\qquad 0<c_{21}\le k_2(r,z)\le c_{22},$$

$$0<R_0\le r\le R_1,\ 0\le z\le L$$

с граничными условиями, определяемыми вариантом задания. Для решения системы алгебраических уравнений использовать метод сопряжённых градиентов с предобусловливанием. Матрица алгебраической системы должна храниться в упакованной форме

Форма (4)

Форма (4) отличается от формы (3) тем, что индексы главных диагональных элементов не хранятся и элементы главной диагонали располагаются в отдельном массиве Diag. В массиве A хранятся ненулевые элементы строго верхней треугольной части матрицы. Так как матрица хранится построчно, то в массиве IC хранятся номера столбцов ненулевых элементов верхнего треугольника матрицы. В массиве IR хранятся указатели на начало каждой строки в массивах A и IC. IR(N+1) содержит количество ненулевых элементов в строго верхнем треугольнике матрицы A плюс один.

	1	2	3	4	5	6	7	8	9
DIAG	13	14	15	16	17	18	19	20	21

	1	2	3	4	5	6	7	8	9	10	11	12
A	7	1	8	2	3	9	4	10	5	6	11	12
IC	2	4	3	5	6	5	7	6	8	9	8	9

	1	2	3	4	5	6	7	8	9	10
IR	1	3	5	6	8	10	11	12	13	13

12. $u\big|_{r=R_0}=g_1(z),$ $\quad -k_1\dfrac{\partial u}{\partial r}\Big|_{r=R_1}=\chi_2 u\big|_{r=R_1}-g_2(z),\ \chi_2\ge 0,$

$k_2\dfrac{\partial u}{\partial z}\Big|_{z=0}=\chi_3 u\big|_{z=0}-g_3(r),\ \chi_3\ge 0,$ $\quad -k_2\dfrac{\partial u}{\partial z}\Big|_{z=L}=\chi_4 u\big|_{z=L}-g_4(r),\ \chi_4\ge 0$

Дискретная модель

Введем обозначения:

$$h_r = \frac{r_{R1} + r_{R0}}{N_r}$$
$$h_z = \frac{z_L + z_0}{N_Z}$$

Основная сетка:

$$r_i = R_0 + ih_r, \quad i = 0,1,\dots,N_r$$
$$z_j = jh_z, \quad j = 0,1,\dots,N_z$$

Введем вспомогательную сетку:

$$r_{i-\frac{1}{2}} = \frac{r_i + r_{i-1}}{2}$$
$$z_{j-1/2} = \frac{z + z_{j-1}}{2}$$
$$\hbar_i = \begin{cases} \frac{h_r}{2} & i = 0 \\ h_r & i = 1,\dots,N-1 \\ \frac{h_r}{2} & i = N \end{cases}$$

Аналогично произведем разбиения для переменной z

$$\hbar_j = \begin{cases} \frac{h_z}{2} & j = 0 \\ h_z & j = 1,\dots,N-1 \\ \frac{h_z}{2} & j = N \end{cases}$$

Умножим исходное уравнение на r, проинтегрируем по вспомогательной сетке:

$$-\left[\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\frac{\partial}{\partial r}\left(rk(r)\frac{\partial u}{\partial r}\right)drdz + \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r\frac{\partial^2 u}{\partial z^2}drdz\right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} rf\,drdz$$

$$-\left[\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\left(r_{i+\frac{1}{2}}k\left(r_{i+\frac{1}{2}}\right)\frac{\partial u}{\partial r}|_{r_{i+\frac{1}{2}}}\right)dz - \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\left(r_{i-\frac{1}{2}}k\left(r_{i-\frac{1}{2}}\right)\frac{\partial u}{\partial r}|_{r_{i-\frac{1}{2}}}\right)dz\right.$$
$$\left.+\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\left(r\frac{\partial u}{\partial z}|_{z_{j+\frac{1}{2}}}\right)dr - \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\left(r\frac{\partial u}{\partial z}|_{z_{j-\frac{1}{2}}}\right)dr\right] = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} rf\,drdz$$

Воспользуемся формулой средних прямоугольников для вычисления значений интегралов:

$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\phi(r,z)dr \approx h_r\phi(r_i,z) = h_r\phi_i$$
$$\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}}\phi(r,z)dz \approx h_z\phi\left(r,z_j\right) = h_r\phi_j$$
$$\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} r_i\phi\,drdz \approx r_ih_rh_z\phi_{i,j}$$

Также аппроксимируем производные по формуле центральных разностей:

$$k\left(r_{i+\frac{1}{2}}\right)\frac{\partial u}{\partial r}\bigg|_{r=r_{i+\frac{1}{2}},z=z_j} = k\left(r_{i+\frac{1}{2}}\right)\frac{u_{i+1,j}-u_{i,j}}{h_r}$$

$$k\left(r_{i-\frac{1}{2}}\right)\frac{\partial u}{\partial r}\bigg|_{r=r_{i-\frac{1}{2}},z=z_j} = k\left(r_{i-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i-1,j}}{h_r}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=z_{j+\frac{1}{2}},r=r_j} = \frac{u_{i,j+1}-u_{i,j}}{h_z}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=z_{j-\frac{1}{2}},r=r_j} = \frac{u_{i,j}-u_{i,j-1}}{h_z}$$

Получим:

$$-\left[h_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j} - u_{i-1,j}}{h_r} + h_r r_i k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}\right] = r_i h_r h_z f_{i,j}$$

…… (1)

i=1,2,..,$N_r - 1$; j = 1,2,..,$N_z - 1$

Аппроксимация граничных условий:

1. При $i = 0, j = 1, \dots, N_z - 1$

$$u_{i,j} = g_1(z_j)$$

2. При $i = N_r, j = 1, \dots, N_z - 1$

$$-\left[-h_z R_1\left(\chi_2 u_{N,j} - g_2(z_j)\right) - h_z r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, z_j\right)\frac{u_{N,j} - u_{N-1,j}}{h_r} + R_1\frac{h_r}{2}k_2\left(R_1, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - R_1\frac{h_r}{2}k_2\left(R_1, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z}\right] = R_1\frac{h_r}{2}h_z f_{i,j}$$

3. При $i = 1, \dots, N_r - 1, j = 0$

$$-[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, 0\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, 0\right)\frac{u_{i,j} - u_{i-1,j}}{h_r} + r_i h_r k_2(r_i, z_{\frac{1}{2}})\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r r_i(\chi_3 u_{i,j} - g_3(r_i)] = \frac{r_i h_r h_z f_{ij}}{2}$$

4. При $i = 1, \dots, N_r - 1, j = N_z$

$$-\left[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, L\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, L\right)\frac{u_{i,j} - u_{i-1,j}}{h_r} - h_r r_i(\chi_4 u_{i,N} - g_4(r_i) - h_r r_i k_2\left(r_i, z_{N-\frac{1}{2}}\right)\frac{u_{i,N}-u_{i,N-1}}{h_z}\right] = \frac{r_i h_r h_z f_{i,j}}{2}$$

5. При $i = 0, j = 0$

$$u_{i,j} = g_1(z_j)$$

6. При $i = N_r, j = 0$

$$-\left[-\frac{h_z}{2}R_1\left(\chi_2 u_{N,0} - g_2(0)\right) - \frac{h_z}{2}r_{N-\frac{1}{2}}k_1\left(r_{N-\frac{1}{2}},0\right)\frac{u_{N,0} - u_{N-1,0}}{h_r} + \frac{h_r}{2}R_1 k_2\left(R_1, z_{\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2}R_1\left(\chi_3 u_{N,0} - g_3(R_1)\right)\right] = \frac{R_1 h_r h_z f_{N,0}}{4}$$

7. При $i = 0, j = N_z$

$$u_{i,j} = g_1(L)$$

8. При $i = N_r, j = N_z$

$$-\left[-\frac{h_z}{2}R_1\left(\chi_2 u_{N,N} - g_2(L)\right) - \frac{h_z}{2}r_{N-\frac{1}{2}}k_1\left(r_{N-\frac{1}{2}},L\right)\frac{u_{N,N} - u_{N-1,N}}{h_r} - \frac{h_r}{2}R_1\left(\chi_4 u_{N,N} - g_4(R_1)\right) - \frac{h_r}{2}R_1 k_2\left(R_1, z_{N-\frac{1}{2}}\right)\frac{u_{N,N} - u_{N,N-1}}{h_z}\right] = \frac{R_1 h_r h_z f_{N,N}}{4}$$

Анализ порядка аппроксимации уравнения и граничных условий, выражение для главного члена погрешности аппроксимации

Невязка и порядок погрешность аппроксимации уравнения

Преобразование:

$$-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2(r,z)\frac{\partial u}{\partial z}\right)\right] = f(r,z)$$

$$-\left[\frac{\partial}{\partial r}\left(rk_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(rk_2(r,z)\frac{\partial u}{\partial z}\right)\right] = rf(r,z)$$

$$\tilde{k}_1(r,z) = rk_1(r,z), \quad \tilde{k}_2(r,z) = rk_2(r,z), \quad \tilde{q}(r,z) = rq(r,z)$$

$$\tilde{f}(r,z) = rf(r,z)$$

$$-\left[\frac{\partial}{\partial r}\left(\tilde{k}_1(r,z)\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(\tilde{k}_2(r,z)\frac{\partial u}{\partial z}\right)\right] = \tilde{f}(r,z)$$

При анализе порядка аппроксимации, для простого, будем писать просто k_1, k_2, f вместо $\tilde{k}_1, \tilde{k}_2, \tilde{f}$

Невязка определяется как разность между правой и левой частью уравнения при условии, что вместо приближенного решения мы подставляем туда точное:

$$\xi_{i,j} = h_r h_z f_{i,j} + h_z k_1(x_{i+1/2}, y_j)\frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1(x_{i-1/2}, y_j)\frac{u_{i,j} - u_{i-1,j}}{h_r} + h_r k_2(x_i, y_{j+1/2})\frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r k_2(x_i, y_{j-1/2})\frac{u_{i,j} - u_{i,j-1}}{h_z}$$

Раскладываем по степеням h точное решение в узлах и коэффициент k

$$u_{i+1,j} = u(x_i + h_r, y_j) = u_{i,j} + h_r\frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^3}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i+1,j} - u_{i,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} + \frac{h_r}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^3}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i+\frac{1}{2},j} = k_1\left(r_i + \frac{h_r}{2}, z_j\right) = k_{1,i,j} + \frac{h_r}{2}\frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8}\frac{\partial^2 k_{1,i,j}}{\partial r^2} + \frac{h_r^3}{48}\frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$k_{1,i+\frac{1}{2},j}\frac{u_{i+1,j}-u_{i,j}}{h_r} = \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)$$

$$u_{i-1,j} = u(r_i - h_r, z_j) = u_{i,j} - h_r\frac{\partial u_{i,j}}{\partial r} + \frac{h_r^2}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} - \frac{h_r^3}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} + \frac{h_r^4}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^5)$$

$$\frac{u_{i,j}-u_{i-1,j}}{h_r} = \frac{\partial u_{i,j}}{\partial r} - \frac{h_r}{2}\frac{\partial^2 u_{i,j}}{\partial r^2} + \frac{h_r^2}{6}\frac{\partial^3 u_{i,j}}{\partial r^3} - \frac{h_r^3}{24}\frac{\partial^4 u_{i,j}}{\partial r^4} + O(h_r^4)$$

$$k_{1,i-\frac{1}{2},j} = k_1\left(r_i - \frac{h_r}{2}, z_j\right) = k_{1,i,j} - \frac{h_r}{2}\frac{\partial k_{1,i,j}}{\partial r} + \frac{h_r^2}{8}\frac{\partial^2 k_{1,i,j}}{\partial r^2} - \frac{h_r^3}{48}\frac{\partial^3 k_{1,i,j}}{\partial r^3} + O(h_r^4)$$

$$k_{1,i-\frac{1}{2},j}\frac{u_{i,j}-u_{i-1,j}}{h_r} = \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} - h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} - h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)$$

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j} - u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j} - u_{i-1,j}}{h_r}$$
$$= h_z\left[\begin{array}{l} \left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + \\ +h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} - \\ -\left[k_1\frac{\partial u}{\partial r}\right]_{i,j} + h_r\left[\frac{1}{2}k_1\frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right]_{i,j} - h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + \\ +h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4) \end{array}\right]$$

Сокращаются четные степени

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} = h_z\Bigg[h_r\left(k_1\frac{\partial^2 u}{\partial r^2}+\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r}\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\Bigg]$$

т.к. $k_1\frac{\partial^2 u}{\partial r^2}+\frac{\partial k_1}{\partial r}\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)$, получаем, что

$$h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} = h_z\Bigg[h_r\left(\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\Bigg]$$

$$u_{i,j+1} = u(r_i, z_j + h_z) = u_{i,j} + h_z\frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^3}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\frac{u_{i,j+1}-u_{i,j}}{h_z} = \frac{\partial u_{i,j}}{\partial z} + \frac{h_z}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^2}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^3}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h^4)$$

$$k_{2,i,j+\frac{1}{2}} = k_2\left(r_i, z_j + \frac{h_z}{2}\right) = k_{2,i,j} + \frac{h_z}{2}\frac{\partial k_{2,i,j}}{\partial z} + \frac{h_z^2}{8}\frac{\partial^2 k_{2,i,j}}{\partial z^2} + \frac{h_z^3}{48}\frac{\partial^3 k_{2,i,j}}{\partial z^3} + O(h_z^4)$$

$$k_{2,i,j+\frac{1}{2}}\frac{u_{i+1,j}-u_{i,j}}{h_z} = \left[k_2\frac{\partial u}{\partial z}\right]_{i,j} + h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^4)$$

$$u_{i,j-1} = u(r_i, z_j - h_z) = u_{i,j} - h_z\frac{\partial u_{i,j}}{\partial z} + \frac{h_z^2}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} - \frac{h_z^3}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} + \frac{h_z^4}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^5)$$

$$\frac{u_{i,j}-u_{i,j-1}}{h_z} = \frac{\partial u_{i,j}}{\partial z} - \frac{h_z}{2}\frac{\partial^2 u_{i,j}}{\partial z^2} + \frac{h_z^2}{6}\frac{\partial^3 u_{i,j}}{\partial z^3} - \frac{h_z^3}{24}\frac{\partial^4 u_{i,j}}{\partial z^4} + O(h_z^4)$$

$$k_{2,i,j-\frac{1}{2}} = k_2\left(r_i, z_j - \frac{h_z}{2}\right) = k_{2,i,j} - \frac{h_z}{2}\frac{\partial k_{2,i,j}}{\partial z} + \frac{h_z^2}{8}\frac{\partial^2 k_{2,i,j}}{\partial z^2} - \frac{h_z^3}{48}\frac{\partial^3 k_{2,i,j}}{\partial z^3} + O(h_z^4)$$

$$k_{2,i,j-\frac{1}{2}}\frac{u_{i,j}-u_{i,j-1}}{h_z} = \left[k_2\frac{\partial u}{\partial z}\right]_{i,j} - h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} - h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^4)$$

$$h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} =$$

$$h_r\left[\begin{array}{l}\left[k_2\frac{\partial u}{\partial r}\right]_{i,j} + h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} + h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3}+\frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + \\ +h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} - \\ -\left[k_2\frac{\partial u}{\partial z}\right]_{i,j} + h_z\left[\frac{1}{2}k_2\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right]_{i,j} - h_z^2\left[\frac{1}{6}k_2\frac{\partial^3 u}{\partial z^3}+\frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + \\ +h_z^3\left[\frac{1}{24}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^4)\end{array}\right]$$

Четные степени сокрааются

$$h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r\left[h_z\left(k_2\frac{\partial^2 u}{\partial z^2}+\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z}\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Так как$k_2\frac{\partial^2 u}{\partial z^2}+\frac{\partial k_2}{\partial z}\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)$, получаем, что

$$h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Подсталяем в невязку получившиеся разложения

$$\xi_{i,j} = h_r h_z f_{i,j} + h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} + h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r h_z f_{i,j} + h_z\left[h_r\left(\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right)_{i,j} + h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right] + h_r\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Группируем по степени hr и hz

$$\xi_{i,j} = h_r h_z f_{i,j} + h_z k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r} - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r} + h_r k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z} - h_r k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z} = h_r h_z\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j} + h_z\left[h_r^3\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4}+\frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3}+\frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2}+\frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^4)\right] + h_r\left[h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4}+\frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3}+\frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2}+\frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Чтобы вычислить порядок аппроксимации, нормируем невязку

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{h_r h_z}$$

$$\tilde{\xi}_{i,j} = f_{i,j} + k_1\left(r_{i+\frac{1}{2}}, z_j\right)\frac{u_{i+1,j}-u_{i,j}}{h_r^2} - k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j}-u_{i-1,j}}{h_r^2} + k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1}-u_{i,j}}{h_z^2} - k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j}-u_{i,j-1}}{h_z^2} = \left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right]_{i,j} + h_r^2\left(\frac{1}{12}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^3) + h_z^2\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^3)$$

Выполним обратную замену:

$$\tilde{\xi}_{i,j} = \left[rf + \frac{\partial}{\partial r}\left(rk_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(rk_2\frac{\partial u}{\partial z}\right)\right]_{i,j} + h_r^2\left(\frac{1}{12}rk_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial rk_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 rk_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 rk_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^3) + rh_z^2\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^3)$$

$$\left[rf + \frac{\partial}{\partial r}\left(rk_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(rk_2\frac{\partial u}{\partial z}\right)\right]_{i,j} = 0$$

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2$$
$$p_z = 2 - 0 = 2$$

Главный член погрешности по r

$$\Phi_r = \frac{1}{12}rk_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial rk_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 rk_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 rk_1}{\partial r^3}\frac{\partial u}{\partial r}$$

Главный член погрешности по z

$$\Phi_z = r(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z})$$

Невязка и порядок погрешности аппроксимации граничного условия

1) $-k_1(r)\frac{\partial u}{\partial r}\Big|_{r=R_1} = \chi_2 u|_{r=R_1} - g_2(z)$

$$\xi_{i,j} = \frac{h_r}{2} h_z f_{i,j} - h_z\left(\chi_2 u_{i,j} - g_2(z_j)\right) - h_z k_1\left(r_{i-\frac{1}{2}}, z_j\right)\frac{u_{i,j} - u_{i-1,j}}{h_r} + \frac{h_r}{2} k_2\left(r_i, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2} k_2\left(r_i, z_{j-\frac{1}{2}}\right)\frac{u_{i,j} - u_{i,j-1}}{h_z}$$

Подставляем полученные ранее произведения:

$$\xi_{i,j} = \frac{h_r}{2} h_z f_{i,j} - h_z\left(\chi_2 u_{i,j} - g_2(z_j)\right) - h_z\left[\begin{array}{l}\left[k_1\frac{\partial u}{\partial r}\right]_{i,j} - \frac{h_r}{2}\left[\frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right)\right]_{i,j} + h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} - \\ -h_r^3\left[\frac{1}{24}k_1\frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^4)\end{array}\right] + \frac{h_r}{2}\left[h_z\left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)_{i,j} + h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Группируем по степеням hr и hz

$$\xi_{i,j} = \frac{h_r}{2} h_z\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)\right]_{i,j} - h_z\left[k_1\frac{\partial u}{\partial r} + \left(\chi_2 u - g_2(z)\right)\right]_{i,j} - h_z\left[h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^3)\right] + \frac{h_r}{2}\left[h_z^3\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^4)\right]$$

Для вычисления порядка аппроксимации нормируем невязку

$$\tilde{\xi}_{i,j} = \frac{\xi_{i,j}}{2h_z}$$

$$\tilde{\xi}_{i,j} = \frac{h_r}{2}\left[f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \left(\frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right)\right)\right]_{i,j} - \left[k_1\frac{\partial u}{\partial r} + \left(\chi_2 u - g_2(z)\right)\right]_{i,j} - h_r^2\left[\frac{1}{6}k_1\frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]_{i,j} + O(h_r^3)\frac{h_r}{2}\left[h_z^2\left(\frac{1}{12}k_2\frac{\partial^4 u}{\partial z^4} + \frac{1}{6}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{24}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z}\right)_{i,j} + O(h_z^3)\right]$$

$$\left[k_1\frac{\partial u}{\partial r} + \chi_2 u - g_2(z)\right]_{r=b} = 0$$

$$f + \frac{\partial}{\partial r}\left(k_1\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right) = 0$$

Аналогично выполним обратную замену, получим:

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2,$$
$$p_z = 2 - 0 = 2$$

Главные члены погрешности

$$\Omega_r = -\left[\frac{1}{6} r k_1 \frac{\partial^3 u}{\partial r^3} + \frac{1}{4}\frac{\partial r k_1}{\partial r}\frac{\partial^2 u}{\partial r^2} + \frac{1}{8}\frac{\partial^2 r k_1}{\partial r^2}\frac{\partial u}{\partial r}\right]$$
$$\Omega_z = r(\frac{1}{24} k_2 \frac{\partial^4 u}{\partial z^4} + \frac{1}{12}\frac{\partial k_2}{\partial z}\frac{\partial^3 u}{\partial z^3} + \frac{1}{16}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + \frac{1}{48}\frac{\partial^3 k_2}{\partial z^3}\frac{\partial u}{\partial z})$$

2) $k_2 \frac{\partial u}{\partial z}\Big|_{z=0} = \chi_3 u|_{z=0} - g_3(r), i = 1,2,\dots,N_r - 1, j = N_z$

$$\tilde{\xi}_{i,j} = \left[k_2 \frac{\partial u}{\partial z} - \chi_3 u - g_3(r)\right]_{i,j} + \frac{h_z}{2}\left[f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right)\right]_{i,j}$$
$$+ h_z^2\left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^3)$$
$$+ \frac{h_z}{2}\left[h_r^2\left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^3)\right]$$
$$\left[k_2 \frac{\partial u}{\partial z} - \chi_3 u - g_3(r)\right]_{z=0} = 0$$
$$f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right) = 0$$

Аналогично выполним обратную замену, получим:

Порядок аппроксимации уравнения по r и z:

$$p_r = 2 - 0 = 2,$$
$$p_z = 2 - 0 = 2,$$

Главные члены погрешности:

$$\Omega_r = \frac{1}{24} r k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12}\frac{\partial r k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{16}\frac{\partial^2 r k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{48}\frac{\partial^3 r k_1}{\partial r^3}\frac{\partial u}{\partial r}$$
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3) $-k_2 \frac{\partial u}{\partial z}\Big|_{z=L} = \chi_4 u|_{z=L} - g_4(r), \chi_4 \geq 0 \;\; i = 1,2,\dots,N_r - 1, j = N_z$

$$\tilde{\xi}_{i,j} = \left[k_2 \frac{\partial u}{\partial z} + \chi_4 u - g_4(r)\right]_{i,j} + \frac{h_z}{2}\left[f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right)\right]_{i,j}$$
$$- h_z^2\left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4}\frac{\partial k_2}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{1}{8}\frac{\partial^2 k_2}{\partial z^2}\frac{\partial u}{\partial z}\right]_{i,j} + O(h_z^3)$$
$$+ \frac{h_z}{2}\left[h_r^2\left(\frac{1}{12} k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{6}\frac{\partial k_1}{\partial r}\frac{\partial^3 u}{\partial r^3} + \frac{1}{8}\frac{\partial^2 k_1}{\partial r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{24}\frac{\partial^3 k_1}{\partial r^3}\frac{\partial u}{\partial r}\right)_{i,j} + O(h_r^3)\right]$$

$$\left[k_2 \frac{\partial u}{\partial z} + \chi_4 u - g_4(r)\right]_{z=d} = 0$$
$$f + \frac{\partial}{\partial r}\left(k_1 \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_2 \frac{\partial u}{\partial z}\right) = 0$$

Аналогично выполним обратную замену, получим:

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$$p_r = 2 - 0 = 2,$$
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Главные члены погрешности

$$\Omega_r = \frac{1}{24} r k_1 \frac{\partial^4 u}{\partial r^4} + \frac{1}{12} \frac{\partial r k_1}{\partial r} \frac{\partial^3 u}{\partial r^3} + \frac{1}{16} \frac{\partial^2 r k_1}{\partial r^2} \frac{\partial^2 u}{\partial r^2} + \frac{1}{48} \frac{\partial^3 r k_1}{\partial r^3} \frac{\partial u}{\partial r}$$
$$\Omega_z = -r\left[\frac{1}{6} k_2 \frac{\partial^3 u}{\partial z^3} + \frac{1}{4} \frac{\partial k_2}{\partial z} \frac{\partial^2 u}{\partial z^2} + \frac{1}{8} \frac{\partial^2 k_2}{\partial z^2} \frac{\partial u}{\partial z}\right]$$

Преобразования разностной схемы для применения метода сопряженных градиентов

Разностная схема с приведенными подобными членами:

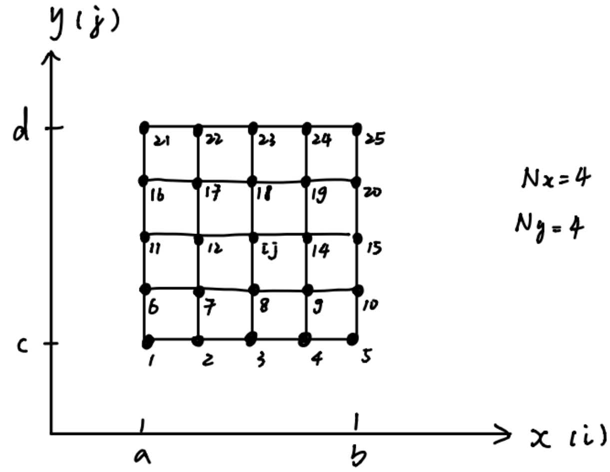
Основная сетка для $i = 1, \dots, N_r - 1,\ j = 1, \dots, N_z - 1$

$$-\frac{h_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)}{h_r} v_{i+1,j} + \left[\frac{h_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)}{h_r} + \frac{h_z r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, z_j\right)}{h_r} + \frac{h_r r_i k_2\left(r_i, z_{j+\frac{1}{2}}\right)}{h_z} + \frac{h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)}{h_z}\right] v_{i,j}$$
$$-\frac{h_z r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, z_j\right)}{h_r} v_{i-1,j} - \frac{h_r r_i k_2\left(r_i, z_{j+\frac{1}{2}}\right)}{h_z} v_{i,j+1} - \frac{h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)}{h_z} v_{i,j-1} = r_i h_r h_z f_{i,j}$$

Понижение размерности матрицы методом исключения неизвестных

Пронумеруем узлы матрицы следующим образом

Будим принимать, что $N_x = N_r = 4, N_y = N_z = 4$



Перейдем к одному индексу $i = 0, \dots, N_r,\ j = 0, \dots, N_z$

$$m = jL + i + 1, \qquad L = N_r + 1$$
$$v_{i,j-1} \to w_{m-L},$$

$$v_{i-1,j} \to w_{m-1},$$
$$v_{i,j} \to w_m,$$
$$v_{i+1,j} \to w_{m+1},$$
$$v_{i,j+1} \to w_{m+L}$$

Основная сетка

$$i = 1, \dots, N_r - 1, \;\; j = 1, \dots, N_z - 1, \;\; m = jL + i + 1$$
$$a_m = -\frac{h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)}{h_z},$$
$$b_m = -\frac{h_z r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, z_j\right)}{h_r},$$
$$c_m = \frac{h_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)}{h_r} + \frac{h_z r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, z_j\right)}{h_r} + \frac{h_r r_i k_2\left(r_i, z_{j+\frac{1}{2}}\right)}{h_z} + \frac{h_r r_i k_2\left(r_i, z_{j-\frac{1}{2}}\right)}{h_z}$$
$$d_m = -\frac{h_z r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, z_j\right)}{h_r}$$
$$e_m = -\frac{h_r r_i k_2\left(r_i, z_{j+\frac{1}{2}}\right)}{h_z}$$
$$g_m = r_i h_r h_z f_{i,j}$$
$$a_m w_{m-L} + b_m w_{m-1} + c_m w_m + d_m w_{m+1_{e_m w_{m-L}}} = g_m$$

Остальные:

1. При $i = 0, j = 1, \dots, N_z - 1$

$$u_{i,j} = g_1(z_j)$$

$$a_m = 0$$
$$b_m = 0$$
$$c_m = 1$$
$$d_m = 0$$
$$e_m = 0$$
$$g_m = g_1(z_j)$$

2. При $i = N_r, j = 1, \dots, N_z - 1$

$$-\Big[-h_z R_1\left(\chi_2 u_{N,j} - g_2(z_j)\right) - h_z r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, z_j\right)\frac{u_{N,j} - u_{N-1,j}}{h_r} + R_1 \frac{h_r}{2} k_2\left(R_1, z_{j+\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - R_1 \frac{h_r}{2} k_2\left(R_1, z_{j-\frac{1}{2}}\right)\frac{u_{i,j} - u_{i,j-1}}{h_z}\Big] = R_1 \frac{h_r}{2} h_z f_{i,j}$$

$$a_m = \frac{R_1 \dfrac{h_r}{2} k_2\left(R_1, z_{j-\frac{1}{2}}\right)}{h_z}$$

$$b_m = \frac{h_z r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, z_j\right)}{h_r}$$

$$c_m = h_z R_1 \chi_2 + \frac{h_z r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, z_j\right)}{h_r} + \frac{R_1 \dfrac{h_r}{2} k_2\left(R_1, z_{j+\frac{1}{2}}\right)}{h_z} + \frac{R_1 \dfrac{h_r}{2} k_2\left(R_1, z_{j-\frac{1}{2}}\right)}{h_z}$$

$$d_m = 0$$

$$g_m = R_1 \frac{h_r}{2} h_z f_{i,j} - h_z R_1 g_2(z_j)$$

3. При $i = 1, \dots, N_r - 1, j = 0$

$$-[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, 0\right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, 0\right) \frac{u_{i,j} - u_{i-1,j}}{h_r} + r_i h_r k_2(r_i, z_{\frac{1}{2}}) \frac{u_{i,j+1} - u_{i,j}}{h_z} - h_r r_i (\chi_3 u_{i,j} - g_3(r_i)] = \frac{r_i h_r h_z f_{ij}}{2}$$

$$a_m = 0$$

$$b_m = \frac{\dfrac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, 0\right)}{h_r}$$

$$c_m = \frac{\dfrac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, 0\right)}{h_r} + \frac{\dfrac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, 0\right)}{h_r} + \frac{r_i h_r k_2\left(r_i, z_{\frac{1}{2}}\right)}{h_z} + h_r r_i \chi_3$$

$$d_m = \frac{\dfrac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, 0\right)}{h_r}$$

$$e_m = \frac{r_i h_r k_2\left(r_i, z_{\frac{1}{2}}\right)}{h_z}$$

$$g_m = \frac{r_i h_r h_z f_{ij}}{2} + h_r r_i g_3(r_i)$$

4. При $i = 1, \dots, N_r - 1, j = N_z$

$$-\left[\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, L\right) \frac{u_{i+1,j} - u_{i,j}}{h_r} - \frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, L\right) \frac{u_{i,j} - u_{i-1,j}}{h_r} - h_r r_i (\chi_4 u_{i,N} - g_4(r_i) - h_r r_i k_2\left(r_i, z_{N-\frac{1}{2}}\right) \frac{u_{i,N} - u_{i,N-1}}{h_z}\right] = \frac{r_i h_r h_z f_{i,j}}{2}$$

$$a_m = \frac{h_r r_i k_2\left(r_i, z_{N-\frac{1}{2}}\right)}{h_z}$$

$$b_m = \frac{\dfrac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, L\right)}{h_r}$$

$$c_m = \frac{\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, L\right)}{h_r} + \frac{\frac{h_z}{2} r_{i-\frac{1}{2}} k_1\left(r_{i-\frac{1}{2}}, L\right)}{h_r} + h_r r_i \chi_4 + \frac{h_r r_i k_2\left(r_i, z_{N-\frac{1}{2}}\right)}{h_z}$$

$$d_m = \frac{\frac{h_z}{2} r_{i+\frac{1}{2}} k_1\left(r_{i+\frac{1}{2}}, L\right)}{h_r}$$

$$e_m = 0$$

$$g_m = \frac{r_i h_r h_z f_{i,j}}{2} + h_r r_i g_4(r_i)$$

5. При $i = 0, j = 0$

$$u_{i,j} = g_1(z_j)$$

$$a_m = 0$$
$$b_m = 0$$
$$c_m = 1$$
$$d_m = 0$$
$$e_m = 0$$
$$g_m = g_1(z_j)$$

6. При $i = N_r, j = 0$

$$-\left[-\frac{h_z}{2} R_1\left(\chi_2 u_{N,0} - g_2(0)\right) - \frac{h_z}{2} r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, 0\right)\frac{u_{N,0} - u_{N-1,0}}{h_r} + \frac{h_r}{2} R_1 k_2\left(R_1, z_{\frac{1}{2}}\right)\frac{u_{i,j+1} - u_{i,j}}{h_z} - \frac{h_r}{2} R_1\left(\chi_3 u_{N,0} - g_3(R_1)\right)\right] = \frac{R_1 h_r h_z f_{N,0}}{4}$$

$$a_m = 0$$

$$b_m = \frac{\frac{h_z}{2} r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, 0\right)}{h_r}$$

$$c_m = \frac{h_r}{2} R_1 \chi_3 + \frac{\frac{h_r}{2} R_1 k_2\left(R_1, z_{\frac{1}{2}}\right)}{h_z} + \frac{\frac{h_z}{2} r_{N-\frac{1}{2}} k_1\left(r_{N-\frac{1}{2}}, 0\right)}{h_r} + \frac{h_z}{2} R_1 \chi_2$$

$$d_m = 0$$

$$e_m = \frac{\frac{h_r}{2} R_1 k_2\left(R_1, z_{\frac{1}{2}}\right)}{h_z}$$

$$g_m = \frac{R_1 h_r h_z f_{N,0}}{4} - \frac{h_z}{2} R_1 g_2(0) + \frac{h_r}{2} R_1 g_3(R_1)$$

7. При $i = 0, j = N_z$

$$u_{i,j} = g_1(L)$$

$$a_m = 0$$
$$b_m = 0$$
$$c_m = 1$$
$$d_m = 0$$
$$e_m = 0$$
$$g_m = g_1(L)$$

8. При $i = N_r, j = N_z$

$$-\left[-\frac{h_z}{2}R_1\left(\chi_2 u_{N,N} - g_2(L)\right) - \frac{h_z}{2}r_{N-\frac{1}{2}}k_1\left(r_{N-\frac{1}{2}}, L\right)\frac{u_{N,N} - u_{N-1,N}}{h_r} - \frac{h_r}{2}R_1\left(\chi_4 u_{N,N} - g_4(R_1)\right) - \frac{h_r}{2}R_1 k_2\left(R_1, z_{N-\frac{1}{2}}\right)\frac{u_{N,N} - u_{N,N-1}}{h_z}\right] = \frac{R_1 h_r h_z f_{N,N}}{4}$$

$$a_m = \frac{\frac{h_r}{2}R_1 k_2\left(R_1, z_{N-\frac{1}{2}}\right)}{h_z}$$
$$b_m = \frac{\frac{h_z}{2}r_{N-\frac{1}{2}}k_1\left(r_{N-\frac{1}{2}}, L\right)}{h_r}$$
$$c_m = \frac{h_z}{2}R_1\chi_2 + \frac{\frac{h_z}{2}r_{N-\frac{1}{2}}k_1\left(r_{N-\frac{1}{2}}, L\right)}{h_r} + \frac{h_r}{2}R_1\chi_4 + \frac{\frac{h_r}{2}R_1 k_2\left(R_1, z_{N-\frac{1}{2}}\right)}{h_z}$$
$$d_m = 0$$
$$e_m = 0$$
$$g_m = \frac{h_z}{2}R_1 g_2(L) - \frac{h_r}{2}R_1 g_4(R_1) + \frac{R_1 h_r h_z f_{N,N}}{4}$$

Также мы можем сделать матрицу симметричной. В итоге мы получили СЛАУ:

$$Aw = g, \quad A = A^T, \quad (Ay, y) > 0, \quad y \neq 0$$

где A-матрица, w-вектор неизвестных, g-вектор правой части. Решение алгебраической системы проводится метод сопряженных градиентов, для которого необходимо, чтобы матрица A была симметрична и положительно определена.

Решение системы методом сопряженных градиентов

Пусть $w^{(0)}$- произвольное начальное приближение, тогда $Aw - Aw^{(0)} = g - Aw^{(0)}$, что даст нам невязку $r^{(0)} = A(w - w^{(0)})$, предполагается, что у нас есть система из $s^{(i)}$, где i=1,2,…,n, линейно-независимых векторов, тогда можем разложит по базису этих векторов с соответствующими коэффициентами $w - w^{(0)} = \sum_{i=1}^{n} a_i s^{(i)}$, найти коэффициенты можем с помощью СЛАУ $\sum_{i=1}^{n} a_i A s^{(i)} = r^{(0)}$, решение системы сильно упростится, если $\left(As^{(i)}, s^{(i)}\right) = 0$ при $i \neq j$, а при $i = j$, скалярное произведение равнялось не 0 значению, в таком случае мы говорим об артогональности. Из этого мы можем выразить коэффициенты $a_i = \frac{(r^{(0)}, s^{(i)})}{(As^{(i)}, s^{(i)})}$, и выразить решение $w = w^{(0)} + \sum_{i=1}^{n} a_i s^{(i)}$.

Рассмотрим частичную сумму $w^{(n)} = w,\ w^{(n)} = w^{(0)} + \sum_{i=1}^{n} a_i s^{(i)}, w^{(k)} = w^{(0)} + \sum_{i=1}^{k} a_i s^{(i)}$, $w^{(k)} = w^{(k-1)} + a_k A s^{(k)}$, для невязки получим рекуррентное соотношение $r^{(k)} = r^{(k-1)} - a_k A s^{(k)}$.

$$w^{(0)}, \qquad r^{(0)} = \mathrm{g} - Aw^{(0)}, \qquad s^{(1)} = ?$$

$$k = 1, 2, \dots, n, \qquad a_k = \frac{\left(r^{(0)}, s^{(k)}\right)}{\left(As^{(k)}, s^{(k)}\right)}$$

$$w^{(k)} = w^{(k-1)} + a_k s^{(k)}, \qquad r^{(k)} = r^{(k-1)} - a_k A s^{(k)}$$

$$s^{(k+1)} = ?$$

При явном методе сопряженных градиентов $s^{(1)}$ берут равным $r^{(0)}, s^{(k+1)} = r^{(k)} + \beta_k s^{(k)}$, с вводом дополнительного коэффициента $\beta_k = \frac{(r^{(k)}, r^{(k)})}{(r^{(k-1)}, r^{(k-1)})}$ при $\sqrt{(r^{(k)}, r^{(k)})} < \gamma\varepsilon$, явный метод обладает тем свойством что при отсутствии ошибок округления мы можем получить точное решение не позднее чем на n-ом шаге, но возникает двойственность, из-за ошибок округления происходит разрушение аортогональности последовательности s и в результате к неточности, и метод становится итерационным.

Неявный метод

$$\mathrm{A}w = b, \qquad A = A^T, \qquad (Ay, y) > 0, \qquad y \neq 0$$

$$x^{(0)} - \text{произвольное начальное приблидение}$$

$$r^{(0)} = b - Ax^{(0)}, \qquad Bw^{(0)} = r^{(0)}, \qquad s^{(1)} = w^{(0)}, \qquad Bg = b, \qquad \gamma = \sqrt{(g, b)}$$

$$k = 1, 2, \dots, K_{max}$$

$$a_k = \frac{\left(w^{(k-1)}, r^{(k-1)}\right)}{\left(As^{(k-1)}, s^{(k-1)}\right)}$$

$$x^{(k)} = x^{(k-1)} + a_k s^{(k)}, \qquad r^{(k)} = r^{(k-1)} + a_k A s^{(k-1)}$$

$$Bw^{(k)} = r^{(k)}, \sqrt{\left(w^{(k)}, r^{(k)}\right)} < \gamma\varepsilon$$

$$\beta_k = \frac{\left(w^{(k)}, r^{(k)}\right)}{\left(w^{(k-1)}, r^{(k-1)}\right)}, \qquad s^{(k+1)} = w^{(k)} + \beta_k s^{(k)}$$

О выборе матрицы предобусловливания

$$\mathrm{A}w = b, \qquad A = A^T, \qquad (Ay, y) > 0, \qquad y \neq 0$$

$$B = B^T, \qquad (By, y) > 0, \qquad y \neq 0$$

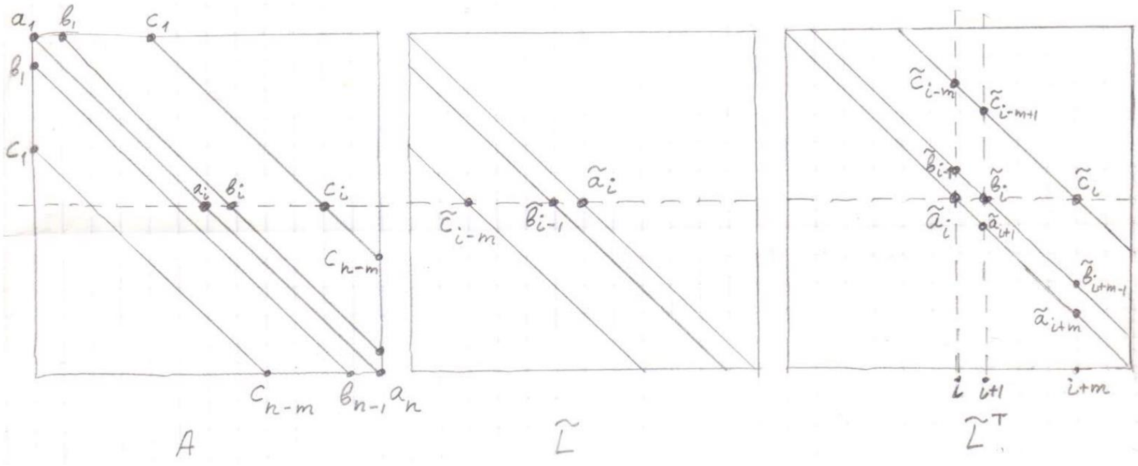
$$B = D, \qquad D = \begin{bmatrix} a_{11} & \_ & \_ \\ \_ & \cdots & \_ \\ \_ & \_ & a_{nn} \end{bmatrix}, \qquad B = \tilde{L}\,\tilde{L}^T$$

$$\tilde{l}_{ij} = 0, \quad i < j$$

$$Bw^{(0)} = r^{(0)}, \qquad \tilde{L}y_0 = r_0, \qquad \tilde{L}^T w_0 = y_0,$$

$$Bw^{(k)} = r^{(k)}, \qquad \tilde{L}y_k = r_k, \qquad \tilde{L}^T w_k = y_k$$

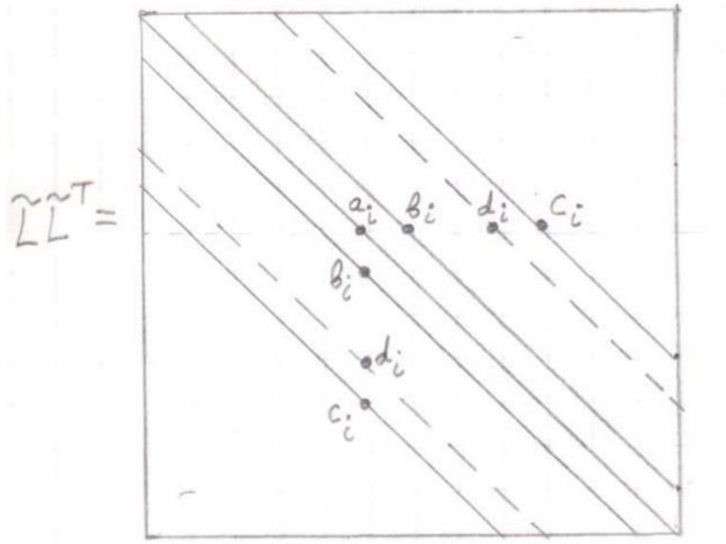
Неполное разложение Холевского

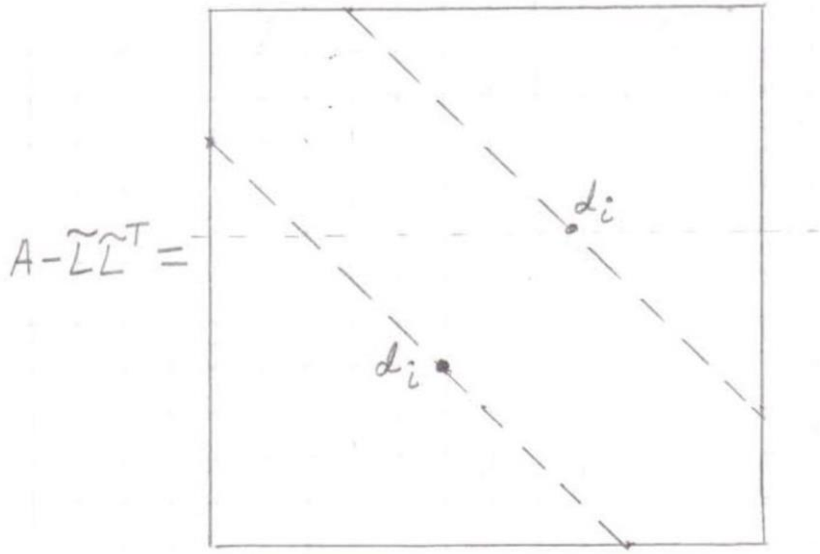


$$a_i = \tilde{a}_i^2 + \tilde{b}_{i-1}^2 + \tilde{c}_{i-m}^2, \qquad b_i = \tilde{a}_i \tilde{b}_i, \qquad c_i = \tilde{a}_i \tilde{c}_i,$$

$$\tilde{a}_i = \sqrt{a_i - \tilde{b}_{i-1}^2 - \tilde{c}_{i-m}^2}, \quad i = 1,2,\ldots,n, \quad \tilde{b}_0 = 0, \quad \tilde{c}_{i-m} = 0, \quad i = 1,2,\ldots,m$$

$$\tilde{b}_i = \frac{b_i}{\tilde{a}_i}, \qquad \tilde{c}_i = \frac{c_i}{\tilde{a}_i},$$





Форма Хранения Матриц

Индексы главных диагональных элементов не хранятся, элементы главной диагонали располагаются в отдельном массиве Diag. В массиве A хранятся ненулевые элементы строго верхней треугольной части матрицы. Так как матрица хранится построчно, то в массиве IC хранятся номера столбцов ненулевых элементов верхнего треугольника матрицы. В массиве IR хранятся указатели на начало каждой строки в массивах A и IC. IR(N+1) содержит количество ненулевых элементов в строго верхнем треугольнике матрицы A плюс один.

Тесты

Для всех тектов:

$$R_0 = 1, R_1 = 2, L = 1$$
$$\chi_2 = 1, \quad \chi_3 = 1, \quad \chi_4 = 1$$

Константный тест

$$k_1 = k_2 = 1$$
$$u = 1$$
$$f = 0$$
$$g_1 = 1, \quad g_2 = 2, \quad g_3 = 1, \quad g_4 = 1$$

Число разбиений Nr, Nz	**Максимальная погрешность**	**Отношение погрешностей**	**Число итераций метода**
4	8.881784197E-16	0	11
8	3.648100533E-08	2.4E-08	29
16	6.796758723E-08	0.5367412	57
32	1.328761980E-07	0.5115106	115
64	4.501897028E-07	0.295156	231
128	6.328432882E-07	0.7113763	461

Линейный тест

$$k_1 = r + 1, \quad k_2 = z + 1$$
$$u = 3r + 2z$$
$$f = -8 - \frac{3}{r}$$
$$g_1(z) = 3 + 2z, \quad g_2(z) = 2z + 15, \quad g_3(r) = 3r - 2, \quad g_4(r) = 3r + 6$$

Число разбиений Nr, Nz	**Максимальная погрешность**	**Отношение погрешностей**	**Число итераций метода**
4	1.953992523E-14	0	16
8	1.664610227E-07	1.2E-07	35
16	3.506694122E-07	0.474695	70
32	7.220940421E-07	0.4856285	141
64	1.049764290E-06	0.687863	279
128	1.941950505E-06	0.5405721	553

Нелинейный тест

$$k_1 = r + z, \qquad k_2 = r + z$$
$$u = r^2 + z^2$$
$$f = -8z - 8r$$
$$g_1(z) = z^2 + 1, \qquad g_2(z) = z^2 + 4z + 12, \qquad g_3(r) = r^2, \qquad g_4(r) = r^2 + 2r + 3$$

Число разбиений Nr, Nz	**Максимальная погрешность**	**Отношение погрешностей**	**Число итераций метода**
4	3.381672021E-02	0	16
8	8.920468110E-03	3.7909132	48
16	2.253769401E-03	3.9580217	97
32	5.648927730E-04	3.9897296	195
64	1.402223429E-04	4.0285504	401
128	3.811650674E-05	3.6787826	796

Вывод

В линейном и константном случаях погрешность аппроксимации отсутствует, ее небольшой рост с увеличением количества разбиений связано с накоплением ошибки округления.

А в нелинейном случае наблюдается уменьшение ошибки в 4 раза при увеличении в 2 раза разбиений по оси r и z. Погрешность решения алгебраической системы мала по сравнению с погрешностью аппроксимации, она возрастает незаметно. Погрешность аппроксимации, в свою очередь, уменьшается, т.к. мы увеличиваем количество разбиений. Причем, согласно теории, при одновременном удвоении числа разбиений погрешность аппроксимации должна уменьшаться в 4 раза, т.к. порядок аппроксимации метода равен 2. Как видим, наблюдаемые результаты очень близок к теоретическому.

Приложение

```
import java.util.Arrays;
import java.util.HashMap;
import java.util.function.Function;

public class Q12 {
    private final static double EPS = 1e-6;
    private static int N = 5;
    private static final double R0 = 1;
    private static final double R1 = 2;
    private static final double L = 1;
    private static final double Chi2 = 1;
    private static final double Chi3 = 1;
    private static final double Chi4 = 1;

    private enum SystemParameters {
        DIAGONAL_A, DIAGONAL_B, DIAGONAL_C, VECTOR_G
    }

    @FunctionalInterface
    public interface FunctionTwoArgs<A, B, R> {
        R apply(A a, B b);
    }

    public static void main(String[] args) {
        System.out.println(" >>>>> Константый случай");
        test(   (r, z) -> 1.0,
                (r, z) -> 1.0,
                (r, z) -> 0.0,
                (z) -> 1.0,
                (z) -> 2.0,
                (r) -> 1.0,
                (r) -> 1.0,
                (r, z) -> 1.0);

        System.out.println(" >>>>>> Линейный случай");
        test(   (r, z) -> r + 1.0,
                (r, z) -> z + 1.0,
                (r, z) -> -8 - 3/r,
                (z) -> 3 + 2 * z,
                (z) -> 2 * z + 15,
                (r) -> 3 * r - 2,
                (r) -> 3 * r + 6,
                (r, z) -> 3 * r + 2 * z);

        System.out.println(" >>>>> Нелинейный случай");
        test(   (r, z) -> r + z,
                (r, z) -> r + z,
                (r, z) -> -8 * z - 8 * r,
                (z) ->  z * z + 1,
                (z) -> z * z + 4 * z + 12,
                (r) -> r * r,
                (r) -> r * r + 2 * r + 3,
                (r, z) -> r * r + z * z);
    }

    private static void test(Q12.FunctionTwoArgs<Double, Double, Double> k1,
                             Q12.FunctionTwoArgs<Double, Double, Double> k2,
                             Q12.FunctionTwoArgs<Double, Double, Double> f,
                             Function<Double, Double> g1,
                             Function<Double, Double> g2,
                             Function<Double, Double> g3,
                             Function<Double, Double> g4,
                             Q12.FunctionTwoArgs<Double, Double, Double> u) {
        HashMap<Q12.SystemParameters, double[]> system;
        N = 5;
        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double r; double z = 0;
        double[] result = new double[N * N];
        system = getSystem(k1, k2, f, g1, g2, g3, g4); for (int i = 0; i < N; ++i) {
            r = R0;
            for (int j = 0; j < N; ++j) {
                result[i * N + j] = u.apply(r, z); r += hR;
            }
```

```
            z += hZ;
        }
        System.out.println("Отклонения от точного решения\n" + Arrays.toString(sub(multiply(system,
result),
                system.get(Q12.SystemParameters.VECTOR_G)))); System.out.println("Ошибка");
        double prevError = 0; double nowError;
        N = 5;
        System.out.println("\tN\tError\tRatio\t");
        for (int i = 2; i <= 8; ++i) {
            N = (int) Math.round(Math.pow(2, i)) + 1; system = getSystem(k1, k2, f, g1, g2, g3, g4);
result = leastGradientMethod(system,
                    system.get(Q12.SystemParameters.VECTOR_G), getEMatrix()); nowError =
getMaxError(result, u);
            System.out.println("\t " + (N - 1) + "\t " + nowError + " \t " + prevError / nowError);
            prevError = nowError;
        }
    }


    private static double[] leastGradientMethod(HashMap<Q12.SystemParameters, double[]> system, double[]
first,
                                                HashMap<Q12.SystemParameters,
                                                        double[]> bMatrix) {
        double[] result = Arrays.copyOf(first, first.length); double[] r =
sub(system.get(Q12.SystemParameters.VECTOR_G),
                multiply(system, first));
        double[] p = solveB(bMatrix, r);
        double[] b = solveB(bMatrix, system.get(Q12.SystemParameters.VECTOR_G)); double[] s =
Arrays.copyOf(p, p.length);
        double alpha; double beta; double[] newR; double[] newP; int k;
        for (k = 1; k <= 10000; k++) {

            alpha = multiply(p, r) / multiply(multiply(system, s), s); result = addition(result,
multiply(alpha, s));
            newR = sub(r, multiply(alpha, multiply(system, s))); newP = solveB(bMatrix, newR);
            double check = Math.sqrt(multiply(newP, newR) / multiply(b,
system.get(Q12.SystemParameters.VECTOR_G)));
            if (check < EPS) {
                ++k;
                break;
            }
            beta = multiply(newP, newR) / multiply(p, r); s = addition(newP, multiply(beta, s));
            r = newR; p = newP;
        }
        System.out.println("K\t" + k);
        return result;
    }


    private static double[] getADiag(Q12.FunctionTwoArgs<Double, Double, Double> k2) {
        double hR = (R1 - R0) / (N - 1);
        double hZ = L / (N - 1);
        double scale = hR / hZ;
        double[] result = new double[N * N];
        double z = hZ;
        double r;

        for (int j = 1; j < N - 1; j++)
        {
            r = R0;
            result[j * N] = -(scale / 2) * r * k2.apply(r, z - hZ / 2);
            r += hR;
            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = -(scale) * r * k2.apply(r, z - hZ / 2);
                r += hR;
            }
            result[j * N + N - 1] = -(scale / 2) * r * k2.apply(r, z - hZ / 2);
            z += hZ;
        }
        return result;
    }

    private static double[] getCDiag(Q12.FunctionTwoArgs<Double, Double, Double> k1,
                                     Q12.FunctionTwoArgs<Double, Double, Double> k2) {

        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double scale = hZ / hR; double z = hZ;
```

```
        double r;
        double[] result = new double[N * N]; for (int i = 0; i < N; i++) {
            result[i] = 1;

        }
        for (int j = 1; j < N - 1; j++) { r = R0;
            result[j * N] = scale * (r + hR / 2) * k1.apply(r + hR / 2, z)
                    + hZ * r * Chi2
                    + (1 / scale / 2) * r * k2.apply(r, z + hZ / 2)
                    + (1 / scale / 2) * r * k2.apply(r, z - hZ / 2);
            r += hR;


            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = scale * (r + hR / 2) * k1.apply(r + hR /2, z)
                        + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                        + (1 / scale) * r * k2.apply(r, z + hZ / 2)
                        + (1 / scale) * r * k2.apply(r, z - hZ / 2);

                r += hR;
            }
            result[j * N + N - 1] = hZ * r * Chi2
                    + scale * (r - hR / 2) * k1.apply(r - hR / 2, z)
                    + (1 / scale / 2) * r * k2.apply(r, z + hZ / 2)
                    + (1 / scale / 2) * r * k2.apply(r, z - hZ / 2);
            z += hZ;
        }
        for (int i = 0; i < N; i++) { result[N * (N - 1) + i] = 1;
        }
        return result;
    }

    private static double[] getDDiag(Q12.FunctionTwoArgs<Double, Double, Double> k1) {

        double hR = (R1 - R0) / (N - 1);
        double hZ = L / (N - 1);
        double scale = hZ / hR;
        double z = hZ;
        double r;
        double[] result = new double[N * N]; for (int j = 1; j < N - 1; j++) {
            r = R0;
            for (int i = 0; i < N - 1; i++) {
                result[j * N + i] = -scale * (r + hR / 2) * k1.apply(r + hR /2, z);
                r += hR;
            }
            z += hZ;
        }
        return result;
    }

    private static double[] getEDiag(Q12.FunctionTwoArgs<Double, Double, Double> k2) {
        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double scale = hR / hZ;
        double[] result = new double[N * N]; double z = hZ;
        double r;
        for (int j = 1; j < N - 1; j++) { r = R0;
            result[j * N] = -scale * r * k2.apply(r, z + hZ / 2) / 2; r += hR;
            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = -scale * r * k2.apply(r, z + hZ / 2); r += hR;
            }
            result[j * N + N - 1] = -scale * r * k2.apply(r, z + hZ / 2) / 2; z += hZ;
        }
        return result;
    }

    private static double[] getVectorG(Q12.FunctionTwoArgs<Double, Double, Double> f,
                                       Function<Double, Double> g1, Function<Double, Double> g2,
Function<Double, Double> g3, Function<Double, Double> g4) {

        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double[] result = new double[N * N]; double z = hZ;
        double r = R0;
        for (int i = 0; i < N; i++) { result[i] = g3.apply(r); r += hR;
        }
        for (int j = 1; j < N - 1; j++) { r = R0;
            result[j * N] = hR * hZ * r * f.apply(r, z) / 2
                    + hZ * r * g1.apply(z);
```

```
            r += hR;
            for (int i = 1; i < N - 1; i++) {
                result[j * N + i] = hR * hZ * r * f.apply(r, z); r += hR;
            }
            result[j * N + N - 1] = hR * hZ * r * f.apply(r, z) / 2
                    + hZ * r * g2.apply(z);
            z += hZ;
        }
        r = R0;
        for (int i = 0; i < N; i++) {
            result[N * (N - 1) + i] = g4.apply(r); r += hR;
        }
        return result;
    }

    private static HashMap<Q12.SystemParameters, double[]> getSystem(Q12.FunctionTwoArgs<Double, Double,
Double> k1,
                                                                     Q12.FunctionTwoArgs<Double, Double,
Double> k2,
                                                                     Q12.FunctionTwoArgs<Double, Double,
Double> f,
                                                                     Function<Double, Double> g1,
                                                                     Function<Double, Double> g2,
                                                                     Function<Double, Double> g3,
                                                                     Function<Double, Double> g4)
    {
        double[] a = getADiag(k2); double[] c = getCDiag(k1, k2); double[] d = getDDiag(k1); double[] e =
getEDiag(k2);
        double[] g = getVectorG(f, g1, g2, g3, g4);

        for (int i = 0; i < N; i++) { g[N + i] -= g[i] * a[N + i]; a[N + i] = 0;
            g[N * (N - 2) + i] -= g[N * (N - 1) + i] * e[N * (N - 2) + i]; e[N * (N - 2) + i] = 0;
        }

        HashMap<Q12.SystemParameters, double[]> system = new HashMap<>();

        system.put(Q12.SystemParameters.DIAGONAL_A, c); system.put(Q12.SystemParameters.DIAGONAL_B, d);
system.put(Q12.SystemParameters.DIAGONAL_C, e); system.put(Q12.SystemParameters.VECTOR_G, g); return
system;
    }

    private static double getMaxError(double[] solve, Q12.FunctionTwoArgs<Double, Double, Double> u) {
        double hR = (R1 - R0) / (N - 1); double hZ = L / (N - 1);
        double z = 0; double r;
        double maxError = 0; double nowError;
        for (int j = 0; j < N; j++) { r = R0;
            for (int i = 0; i < N; i++) {
                nowError = Math.abs(u.apply(r, z) - solve[j * N + i]); if (nowError > maxError) {
                    maxError = nowError;
                }
                r += hR;
            }
            z += hZ;
        }
        return maxError;
    }

    private static HashMap<Q12.SystemParameters, double[]> getBMatrix(HashMap<Q12.SystemParameters,
double[]> system) {
        HashMap<Q12.SystemParameters, double[]> result = new HashMap<>(); int squareN = N * N;
        double[] a = new double[squareN]; double[] b = new double[squareN]; double[] c = new
double[squareN];
        result.put(Q12.SystemParameters.DIAGONAL_A, a); result.put(Q12.SystemParameters.DIAGONAL_B, b);
result.put(Q12.SystemParameters.DIAGONAL_C, c);
        a[0] = Math.sqrt(system.get(Q12.SystemParameters.DIAGONAL_A)[0]); for (int i = 1; i < N; i++) {
            b[i - 1] = system.get(Q12.SystemParameters.DIAGONAL_B)[i - 1] / a[i -
                    1];
            a[i] = Math.sqrt(system.get(Q12.SystemParameters.DIAGONAL_A)[i] -
                    Math.pow(b[i - 1], 2));
        }
        for (int i = N; i < squareN; i++) {
            c[i - N] = system.get(Q12.SystemParameters.DIAGONAL_C)[i - N];
            b[i - 1] = system.get(Q12.SystemParameters.DIAGONAL_B)[i - 1] / a[i -
                    1];
            a[i] = Math.sqrt(system.get(Q12.SystemParameters.DIAGONAL_A)[i] -
                    Math.pow(b[i - 1], 2) - Math.pow(c[i - N], 2));
        }
```

```
        return result;
    }

    private static double[] solveB(HashMap<Q12.SystemParameters, double[]> bMatrix, double[] g) {
        int squareN = N * N;

        double[] y = new double[squareN];
        double[] a = bMatrix.get(Q12.SystemParameters.DIAGONAL_A); double[] b =
bMatrix.get(Q12.SystemParameters.DIAGONAL_B);

        double[] c = bMatrix.get(Q12.SystemParameters.DIAGONAL_C); y[0] = g[0] / a[0];
        for (int i = 1; i < N; i++) {
            y[i] = (g[i] - b[i - 1] * y[i - 1]) / a[i];
        }
        for (int i = N; i < squareN; i++) {
            y[i] = (g[i] - b[i - 1] * y[i - 1] - c[i - N] * y[i - N]) / a[i];
        }

        double[] result = new double[squareN];
        result[squareN - 1] = y[squareN - 1] / a[squareN - 1]; for (int i = squareN - 2; i >= N * (N - 1);
i--) {
            result[i] = (y[i] - b[i] * result[i + 1]) / a[i];
        }
        for (int i = N * (N - 1) - 1; i >= 0; i--) {
            result[i] = (y[i] - b[i] * result[i + 1] - c[i] * result[i + N])/ a[i];

        }
        return result;
    }

    private static HashMap<Q12.SystemParameters, double[]> getEMatrix() { HashMap<Q12.SystemParameters,
double[]> e = new HashMap<>();
        int squareN = N * N;
        double[] a = new double[squareN]; for (int j = 0; j < squareN; j++) {
            a[j] = 1;
        }
        e.put(Q12.SystemParameters.DIAGONAL_A, a); e.put(Q12.SystemParameters.DIAGONAL_B, new
double[squareN]); e.put(Q12.SystemParameters.DIAGONAL_C, new double[squareN]); return e;
    }

    private static double multiply(double[] leftVector, double[] rightVector)
    {
        double result = 0;
        for (int i = 0; i < leftVector.length; i++) { result += leftVector[i] * rightVector[i];
        }
        return result;
    }

    private static double[] multiply(HashMap<Q12.SystemParameters, double[]> system, double[] vector) {
        double[] result = new double[vector.length];
        double[] diagA = system.get(Q12.SystemParameters.DIAGONAL_A); double[] diagB =
system.get(Q12.SystemParameters.DIAGONAL_B); double[] diagC = system.get(Q12.SystemParameters.DIAGONAL_C);
for (int i = 0; i < vector.length; i++) {
            result[i] = diagA[i] * vector[i];
        }
        for (int i = 0; i < vector.length - 1; i++) { result[i] += diagB[i] * vector[i + 1];
        }
        for (int i = 0; i < vector.length - N; i++) { result[i] += diagC[i] * vector[i + N];
        }
        for (int i = 1; i < vector.length; i++) { result[i] += diagB[i - 1] * vector[i - 1];
        }
        for (int i = N; i < vector.length; i++) { result[i] += diagC[i - N] * vector[i - N];

        }
        return result;
    }

    private static double[] multiply(double number, double[] vector) { double[] result = new
double[vector.length];
        for (int i = 0; i < vector.length; i++) { result[i] = vector[i] * number;
        }
        return result;
    }

    private static double[] addition(double[] leftVector, double[] rightVector) {
        double[] result = new double[leftVector.length]; for (int i = 0; i < leftVector.length; i++) {
            result[i] = leftVector[i] + rightVector[i];
```

```
        }
        return result;
    }

    private static double[] sub(double[] leftVector, double[] rightVector) { double[] result = new
double[leftVector.length];
        for (int i = 0; i < leftVector.length; i++) { result[i] = leftVector[i] - rightVector[i];
        }
        return result;
    }
}
```