Linear Temporal Logic

What is Temporal Logic?

- Temporal logic is the logic of time.
- It is a *modal* logic.
- There are different ways of modeling time.
  - linear time vs. branching time
  - time instances vs. time intervals
  - discrete time vs. continuous time
  - past and future vs. future only

First Order Logic

- We have used FOL to express properties of states.
  - $\langle x : 21, y : 49 \rangle \Vdash x < y$
  - $\langle x : 21, y : 7 \rangle \not\Vdash x < y$
- A computation is a sequence of states.
- To express properties of computations, we need to extend FOL.
- This we can do using temporal logic.

In Linear Temporal Logic (LTL) we can describe such properties as, if i is *now*,

- p holds in i and every following point (the future)
- p holds in i and every preceding point (the past)

We will only be concerned with the future.

$$\cdots \longrightarrow \bullet^{p}_{i-2} \longrightarrow \bullet^{p}_{i-1} \longrightarrow \bullet^{p}_{i} \longrightarrow \bullet^{p}_{i+1} \longrightarrow \bullet^{p}_{i+2} \longrightarrow \cdots$$

We extend our first-order language $\mathcal{L}$ to a temporal language $\mathcal{L}_T$ by adding the temporal operators $\Box$, $\Diamond$, $\bigcirc$, U, R and W.

Interpretation

$\Box\varphi$	φ will *always* (in every state) hold
$\Diamond\varphi$	φ will *eventually* (in some state) hold
$\bigcirc\varphi$	φ will hold at the *next* point in time
$\varphi U \psi$	ψ will eventually hold, and *until* that point φ will hold
$\varphi R \psi$	ψ holds until (incl.) the point (if any) where φ holds (*release*)
$\varphi W \psi$	φ will hold until ψ holds (*weak until* or *waiting for*)

We define LTL formulae as follows.

Definition

- $\mathcal{L} \subseteq \mathcal{L}_T$: first-order formulae are also LTL formulae.
- If φ is an LTL formula, so are the following.

$$\Box\varphi \quad \Diamond\varphi \quad \bigcirc\varphi \quad \neg\varphi$$

- If φ and ψ are LTL formulae, so are

$$\varphi U \psi \quad \varphi R \psi \quad (\varphi W \psi)$$
$$(\varphi \vee \psi) \quad (\varphi \wedge \psi) \quad (\varphi \rightarrow \psi) \quad (\varphi \leftrightarrow \psi)$$

- nothing else

Definition

- A path is an infinite sequence

$$\sigma = s_0, s_1, s_2, \ldots$$

  of states.
- σ^k denotes the *path* $s_k, s_{k+1}, s_{k+2}, \ldots$
- σ_k denotes the *state* s_k.
- All computations are paths, but not vice versa.

Definition

We define the notion that an LTL formula φ is true (false) relative to a path σ, written $\sigma \models \varphi$ ($\sigma \not\models \varphi$) as follows.

$$
\begin{array}{lll}
\sigma \models \varphi & \text{iff} & \sigma_0 \Vdash \varphi \text{ when } \varphi \in \mathcal{L} \\
\sigma \models \neg\varphi & \text{iff} & \sigma \not\models \varphi \\
\sigma \models \varphi \vee \psi & \text{iff} & \sigma \models \varphi \text{ or } \sigma \models \psi \\
 & & \\
\sigma \models \Box\varphi & \text{iff} & \sigma^k \models \varphi \text{ for all } k \geq 0 \\
\sigma \models \Diamond\varphi & \text{iff} & \sigma^k \models \varphi \text{ for some } k \geq 0 \\
\sigma \models \bigcirc\varphi & \text{iff} & \sigma^1 \models \varphi
\end{array}
$$

(cont.)

Definition

(cont.)

$$\sigma \models \varphi U \psi \quad \text{iff} \quad \sigma^k \models \psi \text{ for some } k \geq 0, \text{ and } \sigma^i \models \varphi \text{ for every } i \text{ such that } 0 \leq i < k$$

$$\sigma \models \varphi R \psi \quad \text{iff} \quad \text{for every } j \geq 0, \text{ if } \sigma^i \not\models \varphi \text{ for every } i < j \text{ then } \sigma^j \models \psi$$

$$\sigma \models \varphi W \psi \quad \text{iff} \quad \sigma \models \varphi U \psi \text{ or } \sigma \models \Box \varphi$$

Definition

- We say that φ is (temporally) valid, written $\models \varphi$, if
$$\sigma \models \varphi \text{ for all paths } \sigma.$$
- We say that φ and ψ are equivalent, written $\varphi \sim \psi$, if
$$\models \varphi \leftrightarrow \psi \text{ (i.e. } \sigma \models \varphi \text{ iff } \sigma \models \psi \text{, for all } \sigma\text{).}$$

Example

$\Box$ distributes over $\wedge$, while $\Diamond$ distributes over $\vee$.

$$\Box(\varphi \wedge \psi) \sim (\Box\varphi \wedge \Box\psi)$$
$$\Diamond(\varphi \vee \psi) \sim (\Diamond\varphi \vee \Diamond\psi)$$

$\sigma \models \Box p$

$$\bullet_0^p \longrightarrow \bullet_1^p \longrightarrow \bullet_2^p \longrightarrow \bullet_3^p \longrightarrow \bullet_4^p \longrightarrow \ldots$$

$\sigma \models \Diamond p$

$$\bullet_0 \longrightarrow \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3^p \longrightarrow \bullet_4 \longrightarrow \ldots$$

$\sigma \models \bigcirc p$

$$\bullet_0 \longrightarrow \bullet_1^p \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \bullet_4 \longrightarrow \ldots$$

$\sigma \models pUq$ (sequence of p's is finite)

$$\bullet_0^p \longrightarrow \bullet_1^p \longrightarrow \bullet_2^p \longrightarrow \bullet_3^q \longrightarrow \bullet_4 \longrightarrow \ldots$$

$\sigma \models pRq$ (The sequence of qs may be infinite)

$$\bullet_0^q \longrightarrow \bullet_1^q \longrightarrow \bullet_2^q \longrightarrow \bullet_3^{p,q} \longrightarrow \bullet_4 \longrightarrow \ldots$$

$\sigma \models pWq$. The sequence of qs may be infinite.
$(pWq \sim pUq \vee \Box p)$.

$$\bullet_0^p \longrightarrow \bullet_1^p \longrightarrow \bullet_2^p \longrightarrow \bullet_3^q \longrightarrow \bullet_4 \longrightarrow \ldots$$

Observation

- [?] uses pairs (σ, j) of paths and positions instead of just the path σ because they have past-formulae: formulae without future operators (the ones we use) but possibly with past operators, like $\Box^{-1}$ and $\Diamond^{-1}$.

$$
\begin{array}{lcl}
(\sigma, j) \models \Box^{-1}\varphi & \text{iff} & (\sigma, k) \models \varphi \text{ for all } k,\ 0 \leq k \leq j \\
(\sigma, j) \models \Diamond^{-1}\varphi & \text{iff} & (\sigma, k) \models \varphi \text{ for some } k,\ 0 \leq k \leq j
\end{array}
$$

- However, it can be shown that for any formula φ, there is a future-formula (formulae without past operators) ψ such that

$$
(\sigma, 0) \models \varphi \quad \text{iff} \quad (\sigma, 0) \models \psi
$$

Example

What is a future version of $\Box(p \rightarrow \Diamond^{-1} q)$?
$(\sigma, 0) \models \Box(p \rightarrow \Diamond^{-1} q)$

$$\bullet^{p\rightarrow\Diamond^{-1}q} \longrightarrow \bullet^{p\rightarrow\Diamond^{-1}q} \longrightarrow \bullet^{p\rightarrow\Diamond^{-1}q} \longrightarrow \bullet^{p\rightarrow\Diamond^{-1}q} \longrightarrow \bullet \longrightarrow$$

$(\sigma, 0) \models qR(p \rightarrow q)$

$$\bullet^{p\rightarrow q} \longrightarrow \bullet^{p\rightarrow q} \longrightarrow \bullet^{p\rightarrow q, q} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots$$

Example

$\varphi \to \Diamond\psi$: Inf φ holds initially, then ψ holds eventually.

$$\bullet^{\varphi} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet^{\psi} \longrightarrow \bullet \longrightarrow \ldots$$

This formula will also hold in every path where φ does not hold initially.

$$\bullet^{\neg\varphi} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \ldots$$

Example (Response)

$\Box(\varphi \to \Diamond\psi)$

Every φ-position coincides with or is followed by a ψ-position.

$$\bullet \longrightarrow \bullet^{\varphi} \longrightarrow \bullet \longrightarrow \bullet^{\psi} \longrightarrow \bullet \longrightarrow \bullet^{\varphi,\psi} \longrightarrow \dots$$

This formula will also hold in every path where φ never holds.

$$\bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \dots$$

Example

$\Box\Diamond\psi$

There are infinitely many ψ-positions.

$$\bullet^{\varphi} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet^{\varphi} \longrightarrow \bullet \longrightarrow \bullet^{\varphi} \longrightarrow \bullet \longrightarrow \dots$$

This formula can be obtained from the previous one, $\Box(\varphi \to \Diamond\psi)$, by letting $\varphi = \top$: $\Box(\top \to \Diamond\psi)$.

Example

$\Diamond\Box\varphi$

Eventually φ will hold permanently.

$$\bullet \longrightarrow \bullet^{\varphi} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet^{\varphi} \longrightarrow \bullet^{\varphi} \longrightarrow \bullet^{\varphi} \longrightarrow \dots$$

Equivalently: there are finitely many $\neg\varphi$-positions.

Example

$(\neg\varphi) W \psi$

The first φ-position must coincide or be preceded by a ψ-position.

$$\bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\psi} \longrightarrow \bullet^{\varphi} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \ldots$$

φ may never hold

$$\bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow \bullet^{\neg\varphi} \longrightarrow .$$

Example

$\Box(\varphi \to \psi W \chi)$
Every φ-position initiates a sequence of ψ-positions, and if terminated, by a χ-position.

$$\bullet \longrightarrow \bullet^{\varphi,\psi} \longrightarrow \bullet^{\psi} \longrightarrow \bullet^{\psi} \longrightarrow \bullet^{\chi} \longrightarrow \bullet \longrightarrow \bullet^{\varphi,\psi} \longrightarrow \dots$$

The sequence of ψ-positions need not terminate.

$$\bullet \longrightarrow \bullet^{\varphi,\psi} \longrightarrow \bullet^{\psi} \longrightarrow \bullet^{\psi} \longrightarrow \bullet^{\psi} \longrightarrow \bullet^{\psi} \longrightarrow \bullet^{\psi} \longrightarrow \dots$$

Nested waiting-for

A nested waiting-for formula is of the form

$$\Box(\varphi \to (\psi_m W(\psi_{m-1} W \cdots (\psi_1 W \psi_0) \cdots))),$$

where $\varphi, \psi_0, \ldots, \psi_m \in \mathcal{L}$. For the sake of convenience, we write

$$\Box(\varphi \to \psi_m \, W \, \psi_{m-1} \, W \, \cdots \, W \, \psi_1 \, W \, \psi_0).$$

Every φ-position initiates a succession of intervals, beginning with a ψ_m-interval, ending with a ψ_1-interval and possibly terminated by a ψ_0-position. Each interval may be empty or extend to infinity.

$$\ldots \longrightarrow \bullet^{\varphi,\psi_m} \longrightarrow \bullet^{\psi_m} \dashrightarrow \bullet^{\psi_m} \longrightarrow \bullet^{\psi_{m-1}} \cdots\cdots \bullet^{\psi_{m-1}} \longrightarrow \ldots$$

$$\ldots \longrightarrow \bullet^{\psi_2} \cdots\cdots \bullet^{\psi_2} \longrightarrow \bullet^{\psi_1} \cdots\cdots \bullet^{\psi_1} \longrightarrow \bullet^{\psi_0} \longrightarrow \ldots$$

It can be difficult to correctly formalize informally stated requirements in temporal logic.

Example

How does one formalize the informal requirement "φ implies ψ"?

- $\varphi \rightarrow \psi$? $\varphi \rightarrow \psi$ holds in the initial state.
- $\Box(\varphi \rightarrow \psi)$? $\varphi \rightarrow \psi$ holds in every state.
- $\varphi \rightarrow \Diamond\psi$? φ holds in the initial state, ψ will hold in some state.
- $\Box(\varphi \rightarrow \Diamond\psi)$? We saw this earlier.
- None of these is necessarily what is meant.

Definition (Duals)

For binary boolean connectives $\circ$ and $\bullet$, we say that $\bullet$ is the dual of $\circ$ if

$$\neg(\varphi \circ \psi) \sim (\neg\varphi \bullet \neg\psi).$$

Similarly for unary connectives: $\bullet$ is the dual of $\circ$ if $\neg \circ \varphi \sim \bullet\neg\varphi$.

Duality is symmetrical:

- If $\bullet$ is the dual of $\circ$ then
- $\circ$ is the dual of $\bullet$, thus
- we may refer to two connectives as dual.

Which connectives are duals?

- $\wedge$ and $\vee$ are duals:

$$\neg(\varphi \wedge \psi) \sim (\neg\varphi \vee \neg\psi).$$

- $\neg$ is its own dual:

$$\neg\neg\varphi \sim \neg\neg\varphi.$$

- What is the dual of $\rightarrow$? It's $\not\leftarrow$:

$$\begin{aligned}\neg(\varphi \not\leftarrow \psi) &\sim \varphi \leftarrow \psi \\ &\sim \psi \rightarrow \varphi \\ &\sim \neg\varphi \rightarrow \neg\psi\end{aligned}$$

- A set of connectives is complete (for boolean formulae) if every other connective can be defined in terms of them.
- Our set of connectives is complete (e.g., $\not\leftarrow$ can be defined), but also subsets of it, so we don't actually need all the connectives.

Example

$\{\vee, \neg\}$ is complete.

- $\wedge$ is the dual of $\vee$.
- $\varphi \rightarrow \psi$ is equivalent to $\neg\varphi \vee \psi$.
- $\varphi \leftrightarrow \psi$ is equivalent to $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.
- $\top$ is equivalent to $p \vee \neg p$
- $\bot$ is equivalent to $p \wedge \neg p$

We can extend the notions of duality and completeness to temporal formulae.

Duals of temporal operators

- What is the dual of $\Box$? And of $\Diamond$?
- $\Box$ and $\Diamond$ are duals.

$$\neg\Box\varphi \sim \Diamond\neg\varphi$$
$$\neg\Diamond\varphi \sim \Box\neg\varphi$$

- Any other?
- U and R are duals.

$$\neg(\varphi U \psi) \sim (\neg\varphi) R (\neg\psi)$$
$$\neg(\varphi R \psi) \sim (\neg\varphi) U (\neg\psi)$$

We don't need all our temporal operators either.

Proposition

$\{\vee, \neg, U, \bigcirc\}$ *is complete for LTL.*

Proof:

- $\Diamond\varphi \sim \top U \varphi$
  $\Box\varphi \sim \bot R \varphi$
  $\varphi R \psi \sim \neg(\neg\varphi U \neg\psi)$
  $\varphi W \psi \sim \Box\varphi \vee (\varphi U \psi)$

$\square$

We can classify a number of properties expressible in LTL.

Classification

safety	$\Box\varphi$
liveness	$\Diamond\varphi$
obligation	$\Box\varphi \vee \Diamond\psi$
recurrence	$\Box\Diamond\varphi$
persistence	$\Diamond\Box\varphi$
reactivity	$\Box\Diamond\varphi \vee \Diamond\Box\psi$

Definition (Safety)

- A safety formula is of the form

$$\Box\varphi$$

  for some first-order formula φ.
- A conditional safety formula is of the form

$$\varphi \rightarrow \Box\psi$$

  for first-order formulae φ and ψ.
- Safety formulae express *invariance* of some state property φ: that φ holds in every state of the computation.

Example

- *Mutual exclusion* is a safety property. Let C_i denote that process P_i is executing in the critical section. Then

$$\Box\neg(C_1 \wedge C_2)$$

  expresses that it should always be the case that not both P_1 and P_2 are executing in the critical section.
- Observe that the negation of a safety formula is a liveness formula; the negation of the formula above is the liveness formula

$$\Diamond(C_1 \wedge C_2)$$

  which expresses that eventually it *is* the case that both P_1 and P_2 are executing in the critical section.

Definition (Liveness)

- A liveness formula is of the form
$$\Diamond\varphi$$
for some first-order formula φ.
- A conditional liveness formula is of the form
$$\varphi \to \Diamond\psi$$
for first-order formulae φ and ψ.
- Liveness formulae *guarantee* that some event φ eventually happens: that φ holds in at least one state of the computation.

Observation

- Partial correctness is a safety property. Let P be a program and ψ the post condition.

$$\Box(\textit{terminated}(P) \to \psi)$$

- In the case of full partial correctness, where there is a precondition φ, we get a *conditional safety* formula,

$$\varphi \to \Box(\textit{terminated}(P) \to \psi),$$

which we can express as $\{\ \varphi\ \}\ P\ \{\ \psi\ \}$ in Hoare Logic.

Observation

- Total correctness is a liveness property. Let P be a program and ψ the post condition.

$$\Diamond(\mathit{terminated}(P) \wedge \psi)$$

- In the case of full total correctness, where there is a precondition φ, we get a *conditional liveness* formula,

$$\varphi \rightarrow \Diamond(\mathit{terminated}(P) \wedge \psi).$$

Observation

Partial and total correctness are dual.
Let

$$PC(\psi) \triangleq \Box(\mathit{terminated} \rightarrow \psi)$$
$$TC(\psi) \triangleq \Diamond(\mathit{terminated} \wedge \psi)$$

Then

$$\neg PC(\psi) \sim PC(\neg\psi)$$
$$\neg TC(\psi) \sim TC(\neg\psi)$$

Definition (Obligation)

- A simple obligation formula is of the form

$$\Box\varphi \lor \Diamond\psi$$

  for first-order formula φ and ψ.
- An equivalent form is

$$\Diamond\chi \to \Diamond\psi$$

  which states that some state satisfies χ only if some state satisfies ψ.

Proposition

Every safety and liveness formula is also an obligation formula.

Proof: This is because of the following equivalences.

$$\Box\varphi \sim \Box\varphi \vee \Diamond\bot$$
$$\Diamond\varphi \sim \Box\bot \vee \Diamond\varphi$$

and the facts that $\models \neg\Box\bot$ and $\models \neg\Diamond\bot$. $\square$

Definition (Recurrence)

- A recurrence formula is of the form

$$\Box\Diamond\varphi$$

  for some first-order formula φ.
- It states that infinitely many positions in the computation satisfies φ.

Observation

A response formula, of the form $\Box(\varphi \rightarrow \Diamond\psi)$, is equivalent to a recurrence formula, of the form $\Box\Diamond\chi$, if we allow χ to be a past-formula.

$$\Box(\varphi \rightarrow \Diamond\psi) \sim \Box\Diamond(\neg\varphi)W^{-1}\psi$$

Proposition

Weak fairness can be specified as the following recurrence formula.

$$\Box\Diamond(\mathit{enabled}(\tau) \to \mathit{taken}(\tau))$$

Observation

An equivalent form is

$$\Box(\Box\mathit{enabled}(\tau) \to \Diamond\mathit{taken}(\tau)),$$

which looks more like the first-order formula we saw last time.

Definition (Persistence)

- A persistence formula is of the form

$$\Diamond\Box\varphi$$

  for some first-order formula φ.
- It states that all but finitely many positions satisfy φ.
- Persistence formulae are used to describe the eventual stabilization of some state property.

Observation

Recurrence and persistence are duals.

$$\neg(\Box\Diamond\varphi) \sim (\Diamond\Box\neg\varphi)$$
$$\neg(\Diamond\Box\varphi) \sim (\Box\Diamond\neg\varphi)$$

Definition (Reactivity)

- A simple reactivity formula is of the form

$$\Box\Diamond\varphi \vee \Diamond\Box\psi$$

  for first-order formula φ and ψ.
- A very general class of formulae are conjunctions of reactivity formulae.
- An equivalent form is

$$\Box\Diamond\chi \to \Box\Diamond\psi,$$

  which states that if the computation contains infinitely many χ-positions, it must also contain infinitely many ψ-positions.

Proposition

Strong fairness can be specified as the following reactivity formula.

$$\Box\Diamond \mathit{enabled}(\tau) \rightarrow \Box\Diamond \mathit{taken}(\tau)$$

Below is a computation σ of our recurring GCD program.

- a and b are fixed: $\sigma \models \Box(a \doteq 21 \wedge b \doteq 49)$.
- $at(l)$ denotes the formulae $(\pi \doteq \{l\})$.
- *terminated* denotes the formula $at(l_8)$.

P-computation

States are of the form $\langle \pi, x, y, g \rangle$.

$$
\begin{array}{rllll}
\sigma : & \langle l_1, 21, 49, 0 \rangle \rightarrow & \langle l_2^b, 21, 49, 0 \rangle \rightarrow & \langle l_6, 21, 49, 0 \rangle \rightarrow \\
& \langle l_1, 21, 28, 0 \rangle \rightarrow & \langle l_2^b, 21, 28, 0 \rangle \rightarrow & \langle l_6, 21, 28, 0 \rangle \rightarrow \\
& \langle l_1, 21, 7, 0 \rangle \rightarrow & \langle l_2^a, 21, 7, 0 \rangle \rightarrow & \langle l_4, 21, 7, 0 \rangle \rightarrow \\
& \langle l_1, 14, 7, 0 \rangle \rightarrow & \langle l_2^a, 14, 7, 0 \rangle \rightarrow & \langle l_4, 14, 7, 0 \rangle \rightarrow \\
& \langle l_1, 7, 7, 0 \rangle \rightarrow & \langle l_7, 7, 7, 0 \rangle \rightarrow & \langle l_8, 7, 7, 7 \rangle \rightarrow \cdots
\end{array}
$$

Does the following properties hold for σ? And why?

1. $\Box \mathit{terminated}$ (safety)
2. $at(l_1) \rightarrow \mathit{terminated}$
3. $at(l_8) \rightarrow \mathit{terminated}$
4. $at(l_7) \rightarrow \Diamond \mathit{terminated}$ (conditional liveness)
5. $\Diamond at(l_7) \rightarrow \Diamond \mathit{terminated}$ (obligation)
6. $\Box(\mathrm{gcd}(x, y) \doteq \mathrm{gcd}(a, b))$ (safety)
7. $\Diamond \mathit{terminated}$ (liveness)
8. $\Diamond\Box(y \doteq \mathrm{gcd}(a, b))$ (persistence)
9. $\Box\Diamond \mathit{terminated}$ (recurrence)

Exercises

1. Show that the following formulae are (not) LTL-valid.
   1.1 $\Box\varphi \leftrightarrow \Box\Box\varphi$
   1.2 $\Diamond\varphi \leftrightarrow \Diamond\Diamond\varphi$
   1.3 $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
   1.4 $\Box(\Box\varphi \rightarrow \psi) \rightarrow \Box(\Box\psi \rightarrow \varphi)$
   1.5 $\Box(\Box\varphi \rightarrow \psi) \vee \Box(\Box\psi \rightarrow \varphi)$
   1.6 $\Box\Diamond\Box\varphi \rightarrow \Diamond\Box\varphi$
   1.7 $\Box\Diamond\varphi \leftrightarrow \Box\Diamond\Box\Diamond\varphi$
2. A *modality* is a sequence of $\neg$, $\Box$ and $\Diamond$, including the empty sequence ϵ. Two modalities σ and τ are *equivalent* if $\sigma\varphi \leftrightarrow \tau\varphi$ is valid.
   2.1 Which are the non-equivalent modalities in LTL, and
   2.2 what are their relationship (ie. implication-wise)?

[Blackburn et al., 2001] Blackburn, P., de Rijke, M., and Venema, Y. (2001).
Modal Logic.
Cambridge University Press.

[Harel et al., 2000] Harel, D., Kozen, D., and Tiuryn, J. (2000).
Dynamic Logic.
Foundations of Computing. MIT Press.