HW5

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Exercise 1 Let $X_0 = (1, 0, \dots, 0) \in \mathbb{R}^d$ and $X_n \in \mathbb{R}^d$ be defined inductively by choosing X_{n+1}, independently from $X_1, \dots, X_n$, and randomly from the ball of radius $|X_n|$ centered at the origin, that is, $X_{n+1}/|X_n|$ is uniformly distributed on the unit ball.

1. Let $R_n = |X_n|$. Show that R_n, $n \geq 1$, are i.i.d. and characterize the distribution of R_1.

   Hint: for independence, use $\sigma(R_1, \dots, R_n) \subset \sigma(X_1, \dots, X_n)$ and $X_{n+1}/|X_n| \perp \sigma(X_1, \dots, X_n)$.

2. Show that there exists a constant c such that $n^{-1} \log R_n \to c$ a.s. and find c.

Exercise 2 Recall that for independent r.v.'s X_n, $n \geq 1$, the tail σ-algebra is defined by

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(X_m,\ m \geq n).$$

1. Show that $\{\limsup_{n\to\infty} S_n > 0\} \notin \mathcal{T}$.
2. Show that $\{\limsup_{n\to\infty} S_n/c_n > x\} \in \mathcal{T}$ if $c_n \to \infty$.

Exercise 3 Let $X_1, X_2, \dots$ be i.i.d. and not identically 0. Consider the radius of convergence of the random power series $\sum_{n=1}^{\infty} X_n(\omega) t^n$:

$$r(\omega) = \sup\{r > 0 : \sum_{n=1}^{\infty} |X_n(\omega)| r^n < \infty\} = \left(\limsup_{n\to\infty} \sqrt[n]{|X_n(\omega)|}\right)^{-1}.$$

1. Show that $r(\omega) = 1$ a.s. if $\mathsf{E} \log^+ |X_1| < \infty$, where $\log^+ x = \max(\log x, 0)$.
2. Show that $r(\omega) = 0$ a.s. if $\mathsf{E} \log^+ |X_1| = \infty$.

Exercise 4 Let $X_1, X_2, \dots$ be independent with $\mathsf{E}X_n = 0$ and $\mathsf{E}X_n^2 \leq C$ for some $C > 0$. Let $p \in (1/2, 1)$ and $\alpha > 1/(2p-1)$.

1. Show that $S_{n_k}/n_k^p \to 0$, a.s. as $k \to \infty$, where $n_k = [k^\alpha]$.
2. Let $D_k = \max_{n_k \leq n \leq n_{k+1}} |S_n - S_{n_k}|$. Use Kolmogorov's maximal inequality to show that

$$\mathsf{P}\big(\{D_k/k^\beta \geq 1,\ \text{i.o.}\}\big) = 0, \quad \forall \beta \in (\alpha/2, \alpha p).$$

3. Show that $S_n/n^p \to 0$, a.s. as $n \to \infty$.

Exercise 5 We will reprove the independence of collection times in the coupon collector problem *without* any serious computation. Recall that $\xi_1, \xi_2, \dots$ are i.i.d. uniform on $\{1, 2, \dots, n\}$, and

$$\tau_k^n = \min\{m \geq 0 : \big|\{\xi_1, \xi_2, \dots, \xi_m\}\big| \geq k\}, \quad 0 \leq k \leq n,$$

are the *first* time that one collects k *distinct* coupons ($\tau_0^n = 0$). Let $\mathcal{F}_m = \sigma(\xi_1, \dots, \xi_m)$.

Fix $k_0 \in \{1, 2, \dots, n-1\}$ and let $T = \tau_{k_0}^n$. Assume $T < \infty$ a.s. as a fact.

1. Show that $\{T = m\} \in \mathcal{F}_m$ for every $m \geq 1$.
2. Show that
$$\begin{aligned} &\{T = m\} \cap \{\tau_{k_0+1}^n - \tau_{k_0}^n \geq \ell + 1\} \\ &= \bigcup_{|A|=k_0,\ A \subset \{1,\dots,n\}} \Big(\big\{\{\xi_1, \dots, \xi_{m-1}\} \subsetneq \{\xi_1, \dots, \xi_m\} = A\big\} \cap \big\{\xi_{m+1}, \dots, \xi_{m+\ell} \in A\big\}\Big), \end{aligned} \tag{1}$$
and use independence of $\mathcal{F}_m$ and $\sigma(X_\ell,\ \ell \geq m+1)$ to show
$$\mathsf{P}(T = m,\ \tau_{k_0+1}^n - \tau_{k_0}^n \geq \ell + 1) = \mathsf{P}(T = m)\Big(\frac{k_0}{n}\Big)^\ell, \quad \ell \geq 0. \tag{2}$$
3. By summing Eq. (2) over $m \geq 1$, show that $\mathsf{P}(\tau_{k_0+1}^n - \tau_{k_0}^n \geq \ell + 1) = (k_0/n)^\ell$, $\ell \geq 0$.
4. Show that if $B \cap \{T = m\} \in \mathcal{F}_m$ for every $m \geq 1$, then
$$\mathsf{P}(B \cap \{\tau_{k_0+1}^n - \tau_{k_0}^n \geq \ell + 1\}) = \mathsf{P}(B)\Big(\frac{k_0}{n}\Big)^\ell, \quad \ell \geq 0.$$
Hint: one can write $B = \bigcup_{m=1}^{\infty} \big(B \cup \{T = m\}\big)$ *since* $T < \infty$ *a.s.; then use Eq.* (1).
5. For any $\ell_1, \dots, \ell_{k_0}$, show that for every $m \geq 1$,
$$\{\tau_1^n = \ell_1,\ \dots,\ \tau_{k_0}^n = \ell_{k_0}\} \cap \{T = m\} \in \mathcal{F}_m.$$
Conclude that $\tau_{k_0+1}^n - \tau_{k_0}^n$ is independent of $\sigma(\tau_1^n, \dots, \tau_{k_0}^n)$.

Exercise 6 Let X_n, $n \geq 1$, be arbitrary r.v.'s on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\sum_{n=1}^{\infty} \pm X_n$ convergence P-a.s. for all choices of ± 1's. The goal is to show that $\sum_{n=1}^{\infty} X_n^2 < \infty$, a.s.

1. Let ξ_n be i.i.d. r.v.'s on $(\Theta, \mathcal{G}, \mu)$ with $\mu(\xi_n = \pm 1) = \frac{1}{2}$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathsf{P}}) = (\Omega \times \Theta, \mathcal{F} \otimes \mathcal{G}, \mathsf{P} \times \mu)$ be the product space. Using Fubini's theorem, show that
$$\tilde{\mathsf{P}}\Big(\Big\{(\omega, \theta) : \sum_{n=1}^{\infty} \xi_n(\theta) X_n(\omega) \text{ converges}\Big\}\Big) = 1,$$
and hence for P-a.e. ω, $\sum_{n=1}^{\infty} \xi_n(\theta) X_n(\omega)$ converges for μ-a.e. θ.
2. Using Kolmogorov's one-series theorem on $(\Theta, \mathcal{G}, \mu)$ to conclude that for those ω in part 1,
$$\sum_{n=1}^{\infty} |X_n(\omega)|^2 = 2 \sum_{n=1}^{\infty} \operatorname*{Var}_\theta(\xi_n X_n)^2 := 2 \sum_{n=1}^{\infty} \int |\xi_n(\theta) X_n(\omega)|^2 \mu(d\theta) < \infty.$$