

HW3

October 10, 2024

Exercise 1 Use Separation of Variables and Duhamel's principle to solve

$$\begin{cases} \partial_t u = \partial_{xx} u + x(\pi - x), & x \in (0, \pi), \ t > 0, \\ u(0, x) = \sin x, & x \in [0, \pi], \\ u(t, 0) = 0, \ u_x(t, \pi) = -1, & t \geq 0. \end{cases}$$

In what follows, $U \subset \mathbb{R}^d$ will be a bounded domain, $T > 0$, and the parabolic interior U_T and boundary $\partial_p U_T$ are given by

$$U_T = (0, T] \times U, \quad \partial_p U_T = ([0, T] \times \partial U) \cup (\{0\} \times U).$$

Exercise 2 Consider the differential operator

$$(\mathcal{L}u)(t, x) = \partial_t u(t, x) - \sum_{i,j=1}^d a_{ij}(x) \partial_{ij} u(t, x) + \sum_{i=1}^d b_i(x) \partial_i u(t, x),$$

where $a_{ij}, b_i : U \rightarrow \mathbb{R}$ are continuous, and $A(x) = (a_{ij}(x))$ is a positive semi-definite $d \times d$ matrix for every $x \in U$. Show that if $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}(\overline{U_T})$ and $\mathcal{L}u < 0$ in U_T , then

$$\max_{\overline{U_T}} u = \max_{\partial_p U_T} u.$$

Hint. You can use the following fact from linear algebra: if the $d \times d$ matrices (b_{ij}) and (c_{ij}) are both positive semi-definite, then $(b_{ij}c_{ij})$ is also positive semi-definite.

Exercise 3 Consider the differential operator

$$(\mathcal{L}u)(t, x) = \partial_t u(t, x) - \Delta u(t, x) - c(x)u(x),$$

where $c : U \rightarrow [M, +\infty)$ is continuous, $M \geq 0$. The goal is to show that if $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}(\overline{U_T})$ and $\mathcal{L}u \leq 0$ in U_T , then

$$\max_{\overline{U_T}} u \leq \max \left(\max_{\partial_p U_T} u, 0 \right). \quad (1)$$

1. Prove Eq. (1) under the condition $\mathcal{L}u < 0$ in U_T and $M = 0$.
2. Prove Eq. (1) under the condition $\mathcal{L}u \leq 0$ in U_T and $M = 0$.

Hint: consider $u_\varepsilon(t, x) = u(t, x) - t\varepsilon$.

3. Prove Eq. (1) under the condition $\mathcal{L}u \leq 0$ in U_T and $M > 0$.

Hint: consider $v_M(t, x) = e^{-tM}u(t, x)$.

Exercise 4 Let $U = (0, \ell)$.

1. Let $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}^{1,0}(\partial_p U_T)$ satisfy

$$\begin{cases} u_t - u_{xx} \geq 0, & (t, x) \in U_T, \\ u|_{t=0} \geq 0, & x \in U, \\ u(t, 0) \geq 0, & t > 0, \\ u_x(t, \ell) \geq 0, & t > 0. \end{cases}$$

Show that $u \geq 0$ on $\overline{U_T}$.

Hint: you may consider $u_\varepsilon(t, x) = u(t, x) - \varepsilon x$.

2. Let $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}^{1,0}(\partial_p U_T)$ satisfy

$$\begin{cases} u_t - u_{xx} = f, & (t, x) \in U_T, \\ u|_{t=0} = \varphi, & x \in U, \\ u(t, 0) = 0, & t > 0, \\ u_x(t, \ell) = g(t), & t > 0, \end{cases}$$

where f, φ, g are bounded, continuous functions in their domains. Show that

$$\max_{\overline{U_T}} |u| \leq C(|T| + 1)(F + G + \Phi)$$

for some constant C depending only on ℓ , where $F = \sup |f|$, $G = \sup |g|$ and $\Phi = \sup |\varphi|$.

Hint: consider $v(t, x) = tF + Gx + \Phi \pm u(t, x)$ and use part 1.