HW3

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Exercise 1 Use Separation of Variables and Duhamel's principle to solve

$$\begin{cases} \partial_t u = \partial_{xx} u + x(\pi - x), & x \in (0,\pi),\ t > 0, \\ u(0,x) = \sin x, & x \in [0,\pi], \\ u(t,0) = 0,\ u_x(t,\pi) = -1, & t \geq 0. \end{cases}$$

In what follows, $U \subset \mathbb{R}^d$ will be a bounded domain, $T > 0$, and the parabolic interior U_T and boundary $\partial_p U_T$ are given by

$$U_T = (0,T] \times U, \quad \partial_p U_T = \big([0,T] \times \partial U\big) \cup \big(\{0\} \times U\big).$$

Exercise 2 Consider the differential operator

$$(\mathcal{L}u)(t,x) = \partial_t u(t,x) - \sum_{i,j=1}^{d} a_{ij}(x) \partial_{ij} u(t,x) + \sum_{i=1}^{d} b_i(x) \partial_i u(t,x),$$

where $a_{ij}, b_i : U \to \mathbb{R}$ are continuous, and $A(x) = \big(a_{ij}(x)\big)$ is a positive semi-definite $d \times d$ matrix for every $x \in U$. Show that if $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}(\overline{U_T})$ and $\mathcal{L}u < 0$ in U_T, then

$$\max_{\overline{U_T}} u = \max_{\partial_p U_T} u.$$

Hint. You can use the following fact from linear algebra: if the $d \times d$ matrices (b_{ij}) and (c_{ij}) are both positive semi-definite, then $(b_{ij} c_{ij})$ is also positive semi-definite.

Exercise 3 Consider the differential operator

$$(\mathcal{L}u)(t,x) = \partial_t u(t,x) - \Delta u(t,x) + c(x) u(x),$$

where $c : U \to [-M, +\infty)$ is continuous, $M \geq 0$. The goal is to show that if $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}(\overline{U_T})$ and $\mathcal{L}u \geq 0$ in U_T, then

$$\min_{\overline{U_T}} u \geq \max\Big(\min_{\partial_p U_T} u, 0\Big). \tag{1}$$

1. Prove Eq. (1) under the condition $\mathcal{L}u > 0$ in U_T and $M = 0$.
2. Prove Eq. (1) under the condition $\mathcal{L}u \geq 0$ in U_T and $M = 0$.

   Hint: consider $u_\varepsilon(t,x) = u(t,x) - t\varepsilon$.
3. Prove Eq. (1) under the condition $\mathcal{L}u \geq 0$ in U_T and $M > 0$.

   Hint: consider $v(t,x) = e^{\lambda t} u(t,x)$ for an appropriate λ.

Exercise 4 Let $U = (0, \ell)$.

1. Let $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}^{1,0}(\partial_p U_T)$ satisfy
$$\begin{cases} u_t - u_{xx} \geq 0, & (t,x) \in U_T, \\ u\big|_{t=0} \geq 0, & x \in U, \\ u(t,0) \geq 0, & t > 0, \\ u_x(t,\ell) \geq 0, & t > 0. \end{cases}$$
Show that $u \geq 0$ on $\overline{U_T}$.
Hint: you may consider $u_\varepsilon(t,x) = u(t,x) - \varepsilon x$.

2. Let $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}^{1,0}(\partial_p U_T)$ satisfy
$$\begin{cases} u_t - u_{xx} = f, & (t,x) \in U_T, \\ u\big|_{t=0} = \varphi, & x \in U, \\ u(t,0) = 0, & t > 0, \\ u_x(t,\ell) = g(t), & t > 0, \end{cases}$$
where f, φ, g are bounded, continuous functions in their domains. Show that
$$\max_{\overline{U_T}} |u| \leq C(|T| + 1)(F + G + \Phi)$$
for some constant C depending only on ℓ, where $F = \sup |f|$, $G = \sup |g|$ and $\Phi = \sup |\varphi|$.
Hint: consider $v(t,x) = tF + Gx + \Phi \pm u(t,x)$ and use part 1.