

HW2

September 24, 2024

Exercise 1 Let A_n be measurable sets. Show that

$$\mathbb{P}(\limsup_{n \rightarrow \infty} A_n) \geq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n), \quad \mathbb{P}(\liminf_{n \rightarrow \infty} A_n) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n).$$

Hint: use Fatou's lemma and use some results from HW1.

Exercise 2 Show that if $X_n \rightarrow X$ and $Y_n \rightarrow Y$ both in probability, then $X_n \pm Y_n \rightarrow X \pm Y$, $X_n \cdot Y_n \rightarrow X \cdot Y$ all in probability.

Exercise 3

1. Show that $X_n \rightarrow 0$ in probability if and only if $\mathbb{E} \frac{|X_n|}{1+|X_n|} \rightarrow 0$.
2. Let f be a bounded, uniformly continuous function on \mathbb{R} . Show that $X_n \rightarrow 0$ in probability if and only if $\mathbb{E} f(X_n) \rightarrow 0$.

Exercise 4

1. Let f be a continuous function on \mathbb{R} . Show that if $X_n \rightarrow X$ in probability, then $f(X_n) \rightarrow f(X)$ in probability.

Hint: first consider $f_M(x) = (-M) \vee x \wedge M$.

2. Give a counterexample where f is merely Borel measurable.

Exercise 5 Given two r.v.'s X and Y , define

$$\rho_1(X, Y) = \inf\{\varepsilon > 0 : \mathbb{P}(|X - Y| > \varepsilon) \leq \varepsilon\}, \quad \rho_2(X, Y) = \inf\{\mathbb{P}(|X - Y| > \varepsilon) + \varepsilon : \varepsilon > 0\}.$$

1. Show that each of ρ_1, ρ_2 defines a metric on the space of r.v.'s, i.e.,

- $\rho_i(X, Y) \geq 0$, where equality holds if and only if $X = Y$ a.s.
- $\rho_i(X, Y) + \rho_i(Y, Z) \geq \rho_i(X, Z)$.

2. Show that, for $i = 1, 2$, $X_n \rightarrow X$ in probability if and only if $\rho_i(X_n, X) \rightarrow 0$.

Exercise 6 Let X_n be a sequence of r.v.'s. Show that there exists a sequence of constants a_n such that $X_n/a_n \rightarrow 0$ almost surely.

Hint: use Borel–Cantelli Lemma.

Exercise 7

1. Let X_n be a Cauchy sequence in probability, i.e.,

$$\lim_{N \rightarrow \infty} \sup_{n, m \geq N} \mathbf{P}(|X_n - X_m| \geq \varepsilon) = 0, \quad \forall \varepsilon > 0.$$

Show that there exists a r.v. X s.t. $X_n \rightarrow X$ in probability.

2. Let X_n be a Cauchy sequence in L^p , i.e.,

$$\lim_{N \rightarrow \infty} \sup_{n, m \geq N} \mathbf{E}|X_n - X_m|^p = 0.$$

Show that there exists a r.v. X s.t. $X_n \rightarrow X$ in L^p .

Exercise 8 Let X be a r.v. Show that a r.v. Y is a measurable map from $(\Omega, \sigma(X))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if and only if $Y = f(X)$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel measurable function.

Hint: for the “only if” direction, consider the collection \mathcal{H} of all such r.v.’s Y ; then \mathcal{H} contains all simple r.v.’s and behaves well under limits.

Exercise 9 Let X and Y be two independent r.v.’s on $(\Omega, \mathcal{F}, \mathbf{P})$.

1. Show that $h_1(X)$ and $h_2(Y)$ are independent for all Borel measurable functions $h_i : \mathbb{R} \rightarrow \mathbb{R}$.

Hint: choose the definition wisely and the proof can be as short as one line.

2. Let f, g be Borel measurable functions such that $\mathbf{E}|f(X)| < \infty$, $\mathbf{E}|g(Y)| < \infty$. Show that

$$\mathbf{E}f(X)g(Y) = \mathbf{E}f(X) \cdot \mathbf{E}g(Y).$$