Lecture Note for MAT336: PDE (H)

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1 Dirichlet Principle

1.1 Perron's method and Green's function

For a bounded domain U with $\mathcal{C}^2$ boundary and boundary condition $g \in \mathcal{C}(\partial U)$, *Perron's method* gives a unique solution to the Dirichlet problem

$$\begin{cases} -\Delta u = 0, & U, \\ u = g, & \partial U. \end{cases} \tag{1.1}$$

We will explain how to use this to find the *Green's function.*

Let $y \in U$. Recall that the Green's function $G(x, y)$ solves

$$\begin{cases} -\Delta_x G(x, y) = \delta(x - y), & x \in U, \\ G(x, y) = 0, & x \in \partial U. \end{cases} \tag{1.2}$$

The term $\delta(x-y)$ is singular and thus problematic. Fortunately, we can use the fundamental function to remove it. More precesity, the fundamental solution $\Phi(x - y)$ solves

$$-\Delta_x \Phi(x - y) = \delta(x - y),$$

in the sense that $-\Delta(\Phi * f) = f$ for any bounded continuous function f. Therefore, we can write $G(x, y) = \Phi(x - y) - v(y)$, and look for v that solves

$$\begin{cases} \Delta v(x) = 0, & x \in U, \\ v(x) = \Phi(x - y), & x \in \partial U. \end{cases} \tag{1.3}$$

The resulting G be a solution to (1.2) by the *principle of superposition.*

Using the explicit form of Φ, and that fact that $\operatorname{dist}(y, \partial U) > 0$ for $y \in U$, the boundary condition in (1.3) is $\mathcal{C}(\partial U)$. Hence, Perron's method applies and there exists a classical solution $v \in \mathcal{C}^\infty(U) \cap \mathcal{C}(\bar{U})$ to (1.3).

Since $G(x, y) = \Phi(x - y) - v(x)$ and $\Phi(x - y)$ is smooth when $x \neq y$, we immediately know that $G(\cdot, y) \in \mathcal{C}^\infty(U \backslash \{y\})$. Using the equation (1.2) and integration by parts, one can further show that the Green's function is symmetric, that is, $G(x, y) = G(y, x)$. Therefore, $G(x, y) \in \mathcal{C}^\infty(U^2 \setminus \{x = y\})$.

Finally, using the Green's function we can solve the Poisson equation

$$\begin{cases} -\Delta u = f, & U, \\ u = 0, & \partial U, \end{cases}$$

whose solution can be represented as

$$u(x) = \int_U G(x,y)f(y)\,dy, \tag{1.4}$$

as long as the source term f is nice enough so that the integral (1.4) makes sense, e.g., $f \in \mathcal{C}(U) \cap L^\infty(U)$.

1.2 Dirichlet principle

Let I be a functional from $\mathcal{X}_g \coloneqq g + \mathcal{C}_0^2(U)$ to $\mathbb{R}$, defined by

$$I[u] \coloneqq \int_U \frac{1}{2}|\nabla u|^2 - fu, \tag{1.5}$$

where $f \in \mathcal{C}(U) \cap L^2(U)$ and $g \in \mathcal{C}(\partial U)$. Assuming that there exists an extension of g to $\mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$, still denoted by g, we say that $u \in \mathcal{X}_g$ if $u - g \in \mathcal{C}_0^2(U)$.

Here, we will be more cautions about the distinction between $\mathcal{C}_0^k(U)$, the space of functions that *vanish* on ∂U, defined by

$$\mathcal{C}_0^k(U) = \{v \in \mathcal{C}^k(U) : \lim_{x \to \partial U} |v(x)| = 0\},$$

and $\mathcal{C}_c^k(U)$, the space of functions with *compact* support in U, defined by

$$\mathcal{C}_c^k(U) = \{v \in \mathcal{C}^k(U) : \exists \text{ compact } K \subset U \text{ s.t. } u = 0 \text{ in } K^c\}.$$

These two spaces are different; for example, for $U = [-1, 1]$, the function $f = |x| - 1$ is in $\mathcal{C}_0^\infty(U)$ but not $\mathcal{C}_c^\infty(U)$, since $\operatorname{supp} u = [-1, 1] \not\subset (-1, 1)$. Although this distinction will play little important role eventually, there is no harm to be rigorous at the beginning.

The *Dirichlet Principle* states that the "minimizer" of to the variaton problem

$$\inf_{u \in \mathcal{X}_g} I[u] \tag{1.6}$$

will correspond to the solution to the Poisson equation

$$\begin{cases} -\Delta u = f, & U, \\ \quad u = g, & \partial U. \end{cases} \tag{1.7}$$

It is not obvious at all why $I[\cdot]$ has a minimizer in $\mathcal{X}_g$. However, in the rest of section we will explain why the problem of minimizing (1.5) is related to (1.7).

First, $I[\cdot]$ has a unique minimizer in $\mathcal{X}_g$.

We claim that

$$I[\frac{u_1 + u_2}{2}] \le \frac{1}{2}I[u_1] + \frac{1}{2}I[u_2]. \tag{1.8}$$

that is, $I[\cdot]$ is "convex" in some sense. Indeed, writing $w = (u_1 + u_2)/2$, we have

$$\begin{aligned} \frac{1}{2}I[u_1] + \frac{1}{2}I[u_2] - I[w] &= \int_U \frac{1}{4}|\nabla u_1|^2 + \frac{1}{4}|\nabla u_2|^2 - \frac{1}{8}|\nabla u_1 + \nabla u_2|^2 \\ &= \int_U \frac{1}{8}|\nabla u_1 - \nabla u_2|^2 \ge 0. \end{aligned}$$

Moreover, the equality holds only when $|\nabla u_1 - \nabla u_2| \equiv 0$, since $|\nabla u_1 - \nabla u_2|^2$ integrates to 0 and is continuous. Since $u_1 - u_2 = 0$ on ∂U, this implies $u_1 \equiv u_2$ on $\bar{U}$.

Suppose that u_1 and u_2 are two minimizers of $I[\cdot]$ in $\mathcal{X}_g$, that is,

$$I[u_1] = I[u_2] = \inf_{u\in\mathcal{X}_g} I[u].$$

Then, by (1.8), we have $I[w] \le \inf_{\mathcal{X}_g} I[u]$, so w is also a minimizer, and the equality in (1.8) holds. Hence, we have $u_1 \equiv u_2$ on $\bar{U}$, and this is the uniqueness.

Second, if $u \in \mathcal{X}_g$ is a minimizer, then u solves (1.7).

To establish this, we need to understand the "derivative" of $I[\cdot]$, which is the so-called "calculus of variation". Recall that for a $\mathcal{C}^1$ function f, if $f(x_0)$ is the minimum, then by Fermat's lemma $f'(x_0) = 0$. So intuitively, if u is a minimizer of I, then $\frac{dI[u]}{du} = 0$.

But what is $\frac{dI}{du}$? The issue here is that $u \in \mathcal{X}_g$ and $\mathcal{X}_g$ is an infinite dimensional space, so much of our intuition for a function on $\mathbb{R}$ is useless. Let us consider instead a multivariate function $f : \mathbb{R}^d \to \mathbb{R}$. The gradient $\nabla f(x_0)$, is a vector, but it can also be seen as a linear map from $\mathbb{R}^d$ to $\mathbb{R}$, defined by

$$\big(\nabla f(x_0)\big)(h) = \nabla f(x_0)\cdot h = \frac{\partial f}{\partial h}(x_0) = \lim_{\varepsilon\to 0}\frac{f(x_0+\varepsilon h) - f(x_0)}{\varepsilon}.$$

This motivates us to define some kind of "directional derivative" on $\mathcal{X}_g$.

Let $v \in \mathcal{C}_0^2(U)$. Then $u + \varepsilon v \in \mathcal{X}_g$ for every ε. The function v will serve as the "direction".

Let $i(\varepsilon) = I[u + \varepsilon v]$. Let us compute $i'(\varepsilon)$. Note that everything is smooth so we can interchange the integral and differentiation. We have

$$i'(\varepsilon) = \int_U \frac{d}{d\varepsilon}\Big[\frac{1}{2}|\nabla u + \varepsilon\nabla v|^2 - f(u+\varepsilon v)\Big] = \int_U \nabla u\cdot\nabla v + \varepsilon|\nabla v|^2 - fv = \int_U -\Delta u\cdot v + \varepsilon|\nabla v|^2 - fv,$$

where the boundary term $\int_{\partial U}\frac{\partial u}{\partial n}v$ from the integration by parts in the last step is 0 since $v = 0$ on ∂U. Hence,

$$i'(0) = \int_U \Big(-\Delta u - f\Big)v. \tag{1.9}$$

The quantity (1.9) is called the *first variation* of $I[\cdot]$ (with respect to variation v). A necessary condition for u being a minimizer in $\mathcal{X}_g$ is that the first variation vanishes with respect to every variation $v \in \mathcal{C}_0^2(U)$.

Since $-\Delta u - f \in \mathcal{C}(U)$ and the first variation of $I[\cdot]$ is 0 for all v, by Lemma 1.1 below, we have

$$\Delta u(x) + f(x) = 0, \quad \forall x \in U. \tag{1.10}$$

The equation (1.10) is the *Euler–Langrange* equation associated with the variational problem (1.9). To summarize, a necessary condition for u to be a minimizer of a vairation problem is that u solves the corresponding Euler–Langrange equation.

Lemma 1.1 *Let $\varphi \in \mathcal{C}(U)$ be such that*

$$\int_U \varphi(x)v(x)\,dx = 0, \quad \forall v \in C_0^\infty(U).$$

Then $\varphi \equiv 0$ in U.

Proof: We will prove by contradicition. If φ is not identitcally 0, without loss of generality we can assume that $\varphi(x_0) > 0$ for some $x_0 \in U$. Since U is open and φ is continuous, there exist $\varepsilon, \delta > 0$ such that $\varphi(x_0) \ge \varepsilon$ in $B_\delta(x_0) \subset U$. Let

$$v(x) = \delta^{-d}\eta\big(\delta^{-1}(x - x_0)\big), \quad \eta(x) = \begin{cases} e^{-\frac{1}{(1-|x|^2)}}, & |x| < 1,\\ 0, & |x| \ge 1.\end{cases}$$

Then
$$\int_U \varphi(x)v(x)\,dx \geq \varepsilon \int_{B_\delta(x_0)} \delta^{-d}\eta\big(\delta^{-1}(x-x_0)\big) = \varepsilon \int_{B_1(0)} e^{-\frac{1}{1-|x|^2}} > 0,$$
which is a contradiction. □

However, a priori the variation problem (1.6) does not have to possess a minimizer, and even a minimizer exists, it can be out of $\mathcal{X}_g$, since a natural domain for $I[\cdot]$ to be defined should only required $\mathcal{C}^1$ differentiability at most, rather than $\mathcal{C}^2$.

To illustrate, let us consider the variation problem

$$\inf\Big\{\int_0^1 \big((\partial_x u)^2-1\big)^2\,dx : u \in \mathcal{C}^1[0,1],\ u(0)=a,\ u(1)=b\Big\}, \quad a<b<a+1. \tag{1.11}$$

Since $a \leq b < a+1$, the function

$$v(x) = \begin{cases} x+a, & 0 \leq x < \frac{b+1-a}{2}, \\ b+(1-x), & \frac{b+1-a}{2} \leq x \leq 1 \end{cases} \tag{1.12}$$

is well-defined and acheives the smallest possible infimum 0 in (1.11), except that it is not $\mathcal{C}^1$ at $x = x_0 \coloneqq \frac{b+1-a}{2}$. But one can modify in an arbitrary neighborhood around x_0, so that the modification is $\mathcal{C}^1$ and makes (1.11) arbitrarily close to 0. On the other hand, if a function $u \in \mathcal{C}^1$ taking slope ± 1, then by continuity of derivative, $\partial_x u \equiv 1$ or -1, so it cannot satisfy the boundary condition in (1.11). Combining all these together, we can say that (1.11) does not have a $\mathcal{C}^1$ minimizer.

But if we are allowed to include piecewise $\mathcal{C}^1$ functions in the domain for (1.11), the minimizer will not be unique, since one can draw infinitely many polygon curves with slope ± 1 connecting $(0,a)$ and $(1,b)$. So the minimizer will not be unique.

1.3 Weak derivatives and solutions

How do we obtain a minimizer to (1.6)? By definition of the infimum, there exists a sequence $(u_n) \subset \mathcal{X}_g$ such that $I[u_n] \to \inf I[u]$; such sequence is called a “minimizing sequence”. We hope that there exists some limit poiont u_* of the minimizing sequence. However, as we have seen in (1.11), the limit point u_* may fall out of the original domain of the functional, due to lack of continuous derivative.

To overcome the above mentioned issue, we need to generalize our notion of derivatives, as well as our notion of solutions. This is done by the introduction of weak derivatives and weak solutions.

Recall the multi-index notion for derivative:

$$D^\alpha f \coloneqq \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d} f, \quad \alpha = (\alpha_1,\alpha_2,\ldots,\alpha_d).$$

Also recall that $L^1_{loc}(U)$ is the space of functions that are absolutely integrable on any compact sets $K \subset U$; for example, x^{-1} is in $L^1_{loc}(0,1)$ but not $L^1_{loc}(-1,1)$.

Let $u, v \in L^1_{loc}(U)$. We say that $v = D^\alpha u$ in the weak sense, or v is the α-th *weak derivative* of u, if

$$\int_U \varphi v = \int_U (-1)^{|\alpha|}(D^\alpha\varphi)u, \quad \forall \varphi \in \mathcal{C}_c^\infty(U). \tag{1.13}$$

The idea is that (1.13) is just integrabtion by parts (with no boundary terms since φ vanishes at the boundar), if v is a classical derivative of u. For the Poisson equation (1.7), we say that u is a weak solution if $-\Delta u = f$ is satisfied in the weak sense, that it,

$$\int_U (\Delta\varphi)u + \varphi f = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty(U).$$

Example 1.1 Let $u(x) = |x| \in L^1_{loc}(\mathbb{R})$. Then

$$u'(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0, \end{cases}$$

is the first-order weak derivative of u.

But u' is not further differentiable in the weak sense. Otherwise, suppose $v = u'$, then for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\int \varphi(x)v(x)\,dx = -\int \varphi'(x)u'(x)\,dx. \tag{1.14}$$

For $a < b$ and any $n \geq 1$, it is not hard to construct $\varphi_n \in \mathcal{C}_c^\infty(\mathbb{R})$ so that

$$\varphi_n(x) \begin{cases} = 0, & x \notin (a, b), \\ = 1, & x \in [a + 1/n, b - 1/n], \\ \in (0, 1), & \text{otherwise}. \end{cases}$$

The function φ_n will approximate $\mathbb{1}_{(a,b)}$, the indicator function of the interval (a, b). Then taking $\varphi = \varphi_n$ in (1.14) and letting $n \to \infty$, we obtain in the limit

$$\int_a^b v(x)\,dx = -\lim_{n\to\infty} \int_a^{a+1/n} \varphi'(x)u'(x)\,dx + \int_{b-1/n}^b \varphi'(x)u'(x)\,dx = u'(b) - u'(a). \tag{1.15}$$

Now take $(a, b) = (-\varepsilon, \varepsilon)$ and let $\varepsilon \to 0$. On the one hand the right hand side in (1.15) is $1 - (-1) = 2$, on the other hand since $|\mathbb{1}_{(-\varepsilon,\varepsilon)}v| \leq |v|$ and v is locally integrable, by dominated convergence theorem

$$\lim_{\varepsilon\to 0+} \int_{-\varepsilon}^{\varepsilon} v(x)\,dx = \lim_{\varepsilon\to 0+} \int_{\mathbb{R}} \mathbb{1}_{(-\varepsilon,\varepsilon)}(x)v(x)\,dx = \int_{\mathbb{R}} \lim_{\varepsilon\to 0+} \mathbb{1}_{(-\varepsilon,\varepsilon)}(x)v(x)\,dx = \int_{\mathbb{R}} 0\,dx = 0.$$

This gives a contradiction.

In PDE theories, weak solutions allow more flexibility to obtain a solution. One can use other means to show that the so obtained weak solution has the desired smoothness, so that the weak solution becomes the classical solution. Usually, these two parts rely on different sets of tools. The following result is an example.

Proposition 1.2 *If $\Delta u = 0$ in the weak sense, then u is a harmonic function and $\mathcal{C}^\infty$.*

Proof: Let $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ be the standard smooth molifiers. We will use that (η_ε) is also an approximate identity, in the sense that, $\eta_\varepsilon * f \to f$ a.e. and in L^1_{loc} for any $f \in L^1_{loc}$.

Let $u_\varepsilon = u * \eta_\varepsilon$. Then $u_\varepsilon \in \mathcal{C}^\infty$, and moreover, for every $\varphi \in \mathcal{C}_c^\infty$,

$$\int (D^\alpha\varphi)u_\varepsilon = \int D^\alpha\varphi\cdot(u*\eta_\varepsilon) = \int (D^\alpha\varphi*\eta_\varepsilon)\cdot u = \int D^\alpha(\varphi*\eta_\varepsilon)u == \int (-1)^{|\alpha|}(\varphi*\eta_\varepsilon)D^\alpha u = \int (-1)^{|\alpha|}\varphi\cdot(D^\alpha u*\eta_\varepsilon),$$

where we use $\int f(g * h) = \int (f * h)g$. Hence, $D^\alpha u_\varepsilon = (D^\alpha u) * \eta_\varepsilon$ in the weak sense. But $u_\varepsilon \in \mathcal{C}^\infty$, so the weak derivative is strong derivative. In particular, $\Delta u_\varepsilon = 0$ and u_ε is harmonic.

Using the derivative estimate for harmonic function, for any compact set K, there exists $K_1 \supset K$ and constant C depending only on K, K_1, such that

$$\sup_K |u_\varepsilon(x)|, \sup_K |\nabla u_\varepsilon(x)| \leq C|u_\varepsilon|_{L^1(K_1)} \leq C|u|_{L^1(K_1)}.$$

Since u is locally integrable, (u_ε) is uniformly bounded and equi-continuous on K. By Arzelà–Ascoli, there exists a subseqeunce u_{ε_n} and u_* such that $u_{\varepsilon_n} \to u_*$ uniformly on K, and due to the mean-value property for harmonic function, the limiting function u_* is also harmonic. On the other hand, the only possible limit point for (u_ε) is u itself. Therefore, u is harmonic. $\square$

1.4 Sobolev spaces and weak convergence

With the weak derivative, we can define the functional (1.6) on a largest possible domain. This leads to the introduction of certain *Sobolev spaces.*

For $k \geq 0$, let us define

$$H^k(U) = \{u \in L^1_{loc}(U) : D^\alpha u \in L^2(U),\ \forall |\alpha| \leq k\}.$$

There is a natural norm on $H^k(U)$:

$$\|u\|_{H^k(U)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2(U)},$$

and under this norm, $H^k(U)$ becomes a complete space, meaning that every Cauchy sequence under this norm admits a limit in $H^k(U)$.

Next, we try to define the boundary condition on $H^k(U)$. For simplicity we only consider the zero boundary condition. We define

$$H^k_0(U) = \text{closure of } \mathcal{C}^\infty_c \text{ under } \|\cdot\|_{H^k(U}.$$

Note that $\mathcal{C}^\infty_0(U) \subset H^k_0(U)$, but there are more functions. We say that $u \in g + H^k_0(U)$ if $u - g \in H^k_0(U)$, where $g \in \mathcal{C}^k(U) \cap \mathcal{C}(\partial U)$.

The function $I[\cdot]$ in (1.5) will make sense for all $u \in g + H^1_0(U)$, where $f \in L^2(U)$ and $g \in \mathcal{C}^1(U) \cap \mathcal{C}(\partial U)$. The first term $\int |\nabla u|^2$ is defined since ∇u exists in the weak sense and is in $L^2(U)$. For the second term, we have by Cauchy–Schwartz,

$$\Big| \int_U fu \Big| \leq \Big[\int_U f^2 \Big]^{1/2} \Big[\int_U u^2 \Big]^{1/2}.$$

1.4.1 Weak convergence

Now that our functional is defined on the largest possible space. The next problem is how to extract limit points for a minimizing sequence. Recall that a sequence (x_n) in $\mathbb{R}^d$ has a limit point if and only if x_n are bounded. We can rephrase it as "a set $K \in \mathbb{R}^d$ is sequentially pre-compact if and only if K is bounded." One naturally expects similar results in H^k. Unforturnately, this is false.

Example 1.2 Consider $\mathcal{X} = L^2(0, 2\pi) = H^0_0(0, 2\pi)$ and $f_n = \frac{1}{\sqrt{\pi}} \sin(nx)$. Note that f_n are orthornormal, so

$$\|f_n - f_m\|^2 = \int f_n^2 - 2 f_n f_m + f_m^2 = \int f_n^2 + f_m^2 \equiv 2, \quad \forall n \neq m.$$

Hence f_n is bounded in $\mathcal{X}$ but cannot have any limit point since any of its subsequences fails to be Cauchy.

We need a more general notion of convergence. We say that u_n converges to $H^k(U)$ *weakly*, dentoed by $u_n \rightharpoonup u$, if

$$\lim_{n \to \infty} \int_U \varphi D^\alpha u_n = \int_U \varphi D^\alpha u, \quad \forall \varphi \in \mathcal{C}^\infty_c(U),\ \forall |\alpha| \leq k.$$

For weak convergence we have the following powerful result.

Theorem 1.3 *A set in $H^k(U)$ is weakly sequentially pre-compact if and only if it is bounded in the $\|\cdot\|_{H^k(U)}$ norm.*

Example 1.3 In the previous example, $f_n \rightharpoonup 0$. This follows from the *Riemann–Lebesgue Lemma*, which states for any $g \in L^1(\mathbb{R})$,

$$\lim_{n \to \infty} \int g(x) \sin(nx)\, dx = 0.$$

1.4.2 Poincaré inequality

Recall that the $H_0^1(U)$ norm is given by

$$\|f\|_{H_0^1(U)}^2 = \int_U |f(x)|^2 + |\nabla f(x)|^2 \, dx.$$

Theorem 1.4 *Let U is bounded and $u \in H_0^1(U)$. There exists a constant C depending on the diameter of U such that*

$$\int_U |u(x)|^2 \, dx \leq C \int_U |\nabla u|^2 \, dx. \tag{1.16}$$

Proof: It suffices to establish (1.16) for $u \in \mathcal{C}_c^\infty(U)$. Indeed, since $\mathcal{C}_c^\infty(U)$ is dense in $H_0^1(U)$, for any $u \in H_0^1(U)$, there exist $u_n \in \mathcal{C}_c^\infty(U)$ that converge to u in $H_0^1(U)$. Then

$$\|u\|_{L^2(U)} = \lim_{n\to\infty} \|u_n\|_{L^2(U)} \leq C \lim_{n\to\infty} \|\nabla u_n\|_{L^2(U)} = C\|\nabla u\|_{L^2(U)}.$$

Now assume that $u \in \mathcal{C}_c^\infty(U)$. Without loss of generality, we assume that $U \subset [0, L] \times \mathbb{R}^{d-1}$ for some $L > 0$. Then, there exists an extension of u to $\mathbb{R}^d$; we still denote this extension by u. For $x_1 \in (0, L)$, by Cauchy–Schwartz, we have

$$\begin{aligned}
|u(x_1, x_2, \ldots, x_d)|^2 &= |u(x_1, x_2, \ldots, x_d) - u(0, x_2, \ldots, x_d)|^2 \\
&\leq \Big[\int_0^{x_1} \big|(\partial_1 u)(s, x_2, \ldots, x_d)\big| \, ds \Big]^2 \\
&\leq \int_0^{x_1} 1 \, dx \cdot \int_0^{x_1} \big|(\partial_1 u)(s, x_2, \ldots, x_d)\big|^2 \, ds \\
&\leq L \cdot \int_0^L \big|\nabla u(s, x_2, \ldots, x_d)\big|^2 \, ds.
\end{aligned}$$

Integrating over $(x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$, we obtain (1.16) with $C = \sqrt{L}$. $\square$

2 Notations