

HW1 (incomplete)

September 11, 2024

Exercise 1 For every $A \subset \Omega$, its *indicator function* $\mathbb{1}_A : \Omega \rightarrow \{0, 1\}$ is defined by

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

1. Suppose that $A_n \uparrow A$ or $A_n \downarrow A$. Show that

$$\lim_{n \rightarrow \infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega), \quad \forall \omega \in \Omega.$$

2. Let $A_n \subset \Omega$ and

$$I = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m, \quad S = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Show that

$$\liminf_{n \rightarrow \infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_I(\omega), \quad \limsup_{n \rightarrow \infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_S(\omega), \quad \forall \omega \in \Omega.$$

Hint: recall that for a sequence (a_n) , its lower and upper limits are defined by

$$\liminf_{n \rightarrow \infty} a_n = \sup_{n \geq 1} \inf_{m \geq n} a_m, \quad \limsup_{n \rightarrow \infty} a_n = \inf_{n \geq 1} \sup_{m \geq n} a_m;$$

or alternatively, they are the smallest and largest limit points of the set $\{a_n\}$.

Exercise 2 Let $(\Omega, \mathcal{F}_0, \mathbb{P}_0)$ be a probability space. We say that $A \subset \Omega$ is a \mathbb{P}_0 -null set (which may or may not be an element of \mathcal{F}_0), if there exists $N \in \mathcal{F}_0$ such that $A \subset N$ and $\mathbb{P}_0(N) = 0$. Denote by \mathcal{N} the collection of all \mathbb{P}_0 -null sets.

1. Let

$$\mathcal{F} = \{A \subset \Omega : \exists B_1, B_2 \in \mathcal{F}_0 \text{ s.t. } B_1 \subset A \subset B_2, A \setminus B_1, B_2 \setminus A \in \mathcal{N}\}.$$

Show that \mathcal{F} is a σ -algebra, and it is the smallest σ -algebra containing \mathcal{F}_0 and \mathcal{N} .

2. Let $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ be defined by $\mathbb{P}(A) = \mathbb{P}_0(B_1)$ where $A \setminus B_1 \in \mathcal{N}$ and $B_1 \in \mathcal{F}_0$. Show that this definition is independent of the choice of B_1 .
3. Show that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. (This is called the *completion* of $(\Omega, \mathcal{F}_0, \mathbb{P}_0)$.)

Exercise 3 Recall that \mathcal{A} is a π -system if it is closed under intersection, and \mathcal{D} is a Dynkin system if

- $\Omega \in \mathcal{D}$,
- $A, B \in \mathcal{D}, A \subset B \Rightarrow B \setminus A \in \mathcal{D}$,

- $A_n \uparrow A, A_n \in \mathcal{D} \Rightarrow A \in \mathcal{D}.$

Clearly, any intersection of Dynkin systems is still a Dynkin system.

1. Show that \mathcal{A} is a σ -algebra if and only if it is both a π -system and a Dynkin system.
2. Show that if \mathcal{A} is a π -system, then $\sigma(\mathcal{A})$ is the *smallest* Dynkin system containing \mathcal{A} .