HW3

October 6, 2024

Exercise 1 Let X_n, $n \geq 1$, be r.v.'s on $(\Omega, \mathcal{F}, \mathsf{P})$ with $\mathsf{E}|X_n| < \infty$. Let $\mathcal{A}$ be all subsets of $\mathbb{N} = \{1, 2, \dots\}$, and μ be the counting measure on $(\mathbb{N}, \mathcal{A})$, i.e.,

$$\mu(A) = \text{ number of elements in } A, \quad A \in \mathcal{A}.$$

On the product space $(\Omega \times \mathbb{N}, \mathcal{F} \otimes \mathcal{A}, \mathsf{P} \times \mu)$, show that,

1. the map $\mathbf{X}(\omega, n) = X_n(\omega)$ is $(\mathcal{F} \otimes \mathcal{A})$-measurable;
2. if $\sum_{n=1}^{\infty} \mathsf{E}|X_n| < \infty$, then

$$\int_{\Omega \times \mathbb{N}} \mathbf{X}(\omega, n)(\mathsf{P} \times \mu)(d\omega dn) = \int_{\Omega} \sum_{n=1}^{\infty} X_n(\omega)\, \mathsf{P}(d\omega) = \sum_{n=1}^{\infty} \int_{\Omega} X_n(\omega)\, \mathsf{P}(d\omega).$$

   Hint: use Finibi's theorem.

Exercise 2 Let $X \geq 0$ be a r.v. on $(\Omega, \mathcal{F}, \mathsf{P})$. Denote by λ the Lebesgue measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$. On the product space $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathsf{P} \times \lambda)$, show that,

1. $\{(\omega, y) : 0 \leq y \leq X(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$;
   hint: use the measurability of the map $(x, y) \mapsto y - x$;
2. the following equality holds:

$$\int_{\Omega} X(\omega)\, \mathsf{P}(d\omega) = (\mathsf{P} \times \lambda)\big(\{(\omega, y) : 0 \leq y \leq X(\omega)\}\big) = \int_0^{\infty} \mathsf{P}(X \geq y)\, \lambda(dy).$$

Conclude that

$$\mathsf{E}X \leq \sum_{n=0}^{\infty} \mathsf{P}(X \geq n).$$

Exercise 3 Let $f(x_1, x_2)$ be the density of the random vector (X_1, X_2), i.e.,

$$\mathsf{P}\big((X_1, X_2) \in A\big) = \int_A f(x_1, x_2)\, dx_1 dx_2, \quad \forall A \in \mathcal{B}(\mathbb{R}^2).$$

Suppose that $f(x_1, x_2) = g_1(x_1) g_2(x_2)$ where $g_1, g_2 \geq 0$ and are measurable.

1. Using Fubini's theorem to show that $c_i = \int_{\mathbb{R}} g_i(t)\, dt \in (0, \infty)$, $i = 1, 2$.
2. Show that X_1, X_2 are independent continuous r.v.'s with density $(c_i)^{-1} g_i(t)$, $i = 1, 2$.