

HW10

December 19, 2024

In the first two problems, we consider a nearest neighbor random walk on \mathbb{Z} :

$$\mathbb{P}[X_{n+1} = X_n + 1 \mid X_n] = p_n, \quad \mathbb{P}[X_{n+1} = X_n - 1 \mid X_n] = q_n,$$

where $p_n, q_n \geq 0$ and $p_n + q_n = 1$. Recall that $(\mu_n)_{n \in \mathbb{Z}}$ is an invariant measure if $\mu \mathbb{P} = \mu$, that is,

$$\mu_n = p_{n-1}\mu_{n-1} + q_{n+1}\mu_{n+1}, \quad \forall n \in \mathbb{Z}.$$

Exercise 1 Let μ be an invariant measure. We define the “flux” from n to $n + 1$ to be

$$j_n = p_n\mu_n - q_{n+1}\mu_{n+1}.$$

1. Show that j_n is constant.
2. Show that if μ is an invariant distribution, then $j_n \equiv 0$.
3. Show that if $q_{n_0} = 0$ for some $n_0 \in \mathbb{Z}$ (that is, the random walk cannot cross the site n_0 from the right to the left), then $j_n = 0$ for $n \geq n_0$.

Remark: the condition $j_n = 0$ implies that

$$p_n\mu_n = q_{n+1}\mu_{n+1}.$$

This is the “detailed balance” condition.

Exercise 2 Let $q_0 = 0$ and $p_n = \frac{1}{2} - \frac{1}{2n^\alpha}$, $n \geq 1$ for some $\alpha > 0$.

1. Use the detailed balance condition to determine all invariant measures μ .
2. Find the sufficient and necessary condition in terms of α for an invariant *distribution* to exist.

In the next two problems, we assume that X_n is a simple random walk on \mathbb{Z}^d , that is,

$$\mathbb{P}[X_{n+1} = X_n \pm e_i \mid X_n] = \frac{1}{2d}, \quad i \in \{1, 2, \dots, d\}, \quad e_i \text{ unit vectors in } \mathbb{Z}^d.$$

Exercise 3 Let μ be an invariant measure.

1. Show that

$$\mu_m = \frac{1}{2d} \left[\mu_{m+e_1} + \mu_{m-e_1} + \dots + \mu_{m+e_d} + \mu_{m-e_d} \right]. \quad (1)$$

2. Use the first part to deduce that if

$$\mu_{m*} = K = \sup_{m \in \mathbb{Z}^d} \mu_m,$$

then $\mu_m = K$ for all $m \in \mathbb{Z}$.

3. Show that μ cannot be an invariant distribution.

Exercise 4 We change notation and write $f(m) = \mu_m$ for μ satisfying (1). Assume in addition that f is bounded.

1. Show that $f(X_n)$ is a martingale.
2. Show that $f(X_n)$ converges \mathbf{P}^x -a.s. and in $L^1(\mathbf{P}^x)$, if $X_0 = x$.
3. We know that $\lim_{n \rightarrow \infty} f(X_n)$ is measurable to the exchangeable σ -algebra, and thus is a constant. Show that this constant is $f(x)$.
4. Let $\xi_n = X_n - X_{n-1}$. Almost surely we have

$$\lim_{n \rightarrow \infty} f(x + \xi_1 + \xi_2 + \cdots + \xi_n) = f(x).$$

Show that for all $k \geq 1$,

$$f(x + \xi_1 + \cdots + \xi_k) = f(x).$$

5. Conclude that f is a constant.

Remark: This means that the invariant measure of the simple random walk on \mathbb{Z}^d is unique up to a factor.