HW1 (incomplete)

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Exercise 1 For every $A \subset \Omega$, its *indicator function* $\mathbb{1}_A : \Omega \to \{0,1\}$ is defined by

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \in A^c. \end{cases}$$

1. Suppose that $A_n \uparrow A$ or $A_n \downarrow A$. Show that

$$\lim_{n\to\infty} \mathbb{1}_A(\omega) = \mathbb{1}_A(\omega), \quad \forall \omega \in \Omega.$$

2. Let $A_n \subset \Omega$ and

$$I = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m, \quad S = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

   Show that

$$\liminf_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_I(\omega), \quad \limsup_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_S(\omega), \qquad \forall \omega \in \Omega.$$

Hint: recall that for a sequence (a_n), its lower and upper limits are defined by

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{m\geq n} a_m, \quad \limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup_{m\geq n} a_m;$$

or alternatively, they are the smallest and largest limit points of the set $\{a_n\}$.

Exercise 2 Let $(\Omega, \mathcal{F}_0, \mathsf{P}_0)$ be a probability space. We say that $A \subset \Omega$ is a P_0-*null set* (which may or may not be an element of $\mathcal{F}_0$), if there exists $N \in \mathcal{F}_0$ such that $A \subset N$ and $\mathsf{P}(N) = 0$. Denote by $\mathcal{N}$ the collection of all P_0-null sets.

1. Let

$$\mathcal{F} = \{A \subset \Omega : \exists B_1, B_2 \text{ s.t. } B_1 \subset A \subset B_2,\ A \setminus B_1, B_2 \setminus A \in \mathcal{N}\}.$$

   Show that $\mathcal{F}$ is a σ-algebra, and it is the smallest σ-algebra containing $\mathcal{F}_0$ and $\mathcal{N}$.

2. Let $\mathsf{P} : \mathcal{F} \to [0,1]$ be defined by $\mathsf{P}(A) = \mathsf{P}(B_1)$ where $A \setminus B_1 \in \mathcal{N}$. Show that this definition is independent of the choice of B_1.

3. Show that $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space. (This is called the *completion* of $(\Omega, \mathcal{F}_0, \mathsf{P}_0)$.)

Exercise 3 Recall that $\mathcal{A}$ is a π-system if it is closed under intersection, and $\mathcal{D}$ is a Dynkin system if

- $\Omega \in \mathcal{D}$,
- $A, B \in \mathcal{D},\ A \subset B \quad \Rightarrow \quad B \setminus A \in \mathcal{D}$,

- $A_n \uparrow A$, $A_n \in \mathcal{D} \quad \Rightarrow \quad A \in \mathcal{D}$.

Clearly, any intersection of Dynkin systems is still a Dynkin system.

1. Show that $\mathcal{A}$ is a σ-algebra if and only if it is both a π-system and a Dynkin system.
2. Show that if $\mathcal{A}$ is a π-system, then $\sigma(\mathcal{A})$ is the *smallest* Dynkin system containing $\mathcal{A}$.